

A Priori Scale in Classical Scalar and Density Fields

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Abstract

Models involving fields often rely on details of a spatiotemporal scale range that is considered relevant to a particular phenomenon. We propose a generic representation for classical \mathbb{R} -valued scalar and density fields incorporating scale *a priori*. To this end a field is embedded into a causal, versal family with scale as the control parameter, such that no scale bias is introduced and physical content is not affected. Image data can be processed so as to reflect the source field's multiscale differential structure in a well-posed way. Physically relevant scales can be selected *a posteriori*.

Introduction

Empirical assessment of a field is possible only by virtue of interactions at a sensory interface (imaging). Although unconfounding field and sensor *strictu sensu* is clearly not realizable, it does seem legitimate to call for a structural representation which de-emphasises irrelevant details of the latter. In particular, observations are always hampered by a scale bias caused by accumulation of various degrading factors, such as sensor resolution or quantum limitations, graininess of a display medium, or lattice details of a digital reconstruction. Scales introduced in this way obviously limit, but are otherwise not necessarily related to *relevant* scales that govern physical phenomena. In turn, to find the “right” scales in the empirical sciences requires some insight in the phenomenon of interest, and to account for them is a bit of an art. The result is an interesting cornucopia of mathematical techniques for handling scale developed on an *ad hoc* basis. (Illustration: Prandtl's notion of “mixture length” in turbulent motion [1].) Although each such technique may be quite satisfactory within its domain of applicability, it may be advantageous to have a general framework for “scale without physical scale” into which such techniques can be embedded.

Indeed, the significance of scale reaches beyond measurement details and specificities of a physical phenomenon; universal *scale invariance* compels us to account for spacetime scales *a priori*. This has led us to search for a field representation which (i) incorporates scale in a generic and manifest way, and (ii) enables well-posed and operationally well-defined differentiation.

Theory

We consider only classical \mathbb{R} -valued scalar and density fields f in flat, n -dimensional spacetime. Conventional models, in which such fields are represented by functions, respectively n -forms,

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fail in both respects: neither is scale manifest nor is differential structure well-posed or even operationally defined. The fact that unmeasurable field perturbations may throw differential structure into disorder is a conceptual flaw of current field theoretic models—at least of their conventional mathematical *form*. The usual assumption of regularity ($f \in C^\infty(\mathbb{R}^n)$, say) is merely a theoretical hack, as it relies on an unrealistic function topology (notably the behaviour of f at infinitesimal scales). The problem has obvious empirical implications as well, since one always has to allow for finite tolerances.

All aforementioned deficiencies can be played off against each other, and an alternative representation, in a precise sense equivalent to the conventional one, is readily obtained. The appreciation that point mappings are physical nonentities compels us to account for both source field as well as detector device that conspire to produce an observation. (It is not necessary to think of a hardware device; *any* physically feasible aperture allowing a field to be measured in terms of numbers is a conceivable detector.) Denoting the space of fiducial source fields by Σ (“state space”), and that of admissible detectors by Δ (“device space”), a suitable framework is provided by *topological duality*:

$$\Sigma \equiv \Delta'. \quad (1)$$

Recall that Δ' is the space of all \mathbb{R} -valued linear continuous functionals of Δ . The significance of Eq. (1) is a shift of paradigm: instead of modelling a “naked” source field, its degrees of freedom are defined operationally as probes of device space. In principle this enables selective probing of the field, so that one can segregate relevant and irrelevant aspects, depending on physical context (“metamerism”). However, in order to qualify as a viable prior for *general* purposes—our aim v.i.— Δ has to be chosen with care so as not to affect the physical content of the field (i.e. the effective metamers should encapsulate only nonmeasurable entities). This leaves sufficient leeway to endow Δ with a strong topology, e.g. $\Delta \equiv \mathcal{S}(\mathbb{R}^n)$, i.e. the topological vector space of smooth functions of rapid decay, or *Schwartz space* [2]. That this implies no loss of generality follows from the fact that its topological dual $\Sigma = \mathcal{S}'(\mathbb{R}^n)$, the space of *tempered distributions*, is rich enough to convey all function spaces of potential interest, such as $L^1(\mathbb{R}^n)$, as well as Dirac point sources and derivatives, and indeed obviates the need for *ad hoc* regularity conditions on fields and admissible perturbations. Recall the *Riesz representation* formula, which relates distributional and conventional field representations (notation: F maps detectors, f spacetime points):

$$F[\phi] = \int dz f(z) \phi(z), \quad (2)$$

in which f may be of the Dirac type.

In contrast to a Hilbert space formalism, $\mathcal{S}'(\mathbb{R}^n)$ is *not* isomorphic to $\mathcal{S}(\mathbb{R}^n)$. We deliberately decline from interchangeability of sources and detectors (the “crossing” phenomenon, which lies at the heart of the quantum-mechanical bracket formalism), because the very essence of the construct is to probe arbitrary but *finite aspects* of a field by viable apertures. Thus Δ is subject to physical requirements (finite resolution, spatiotemporal confinement), whereas Σ is virtually unconstrained within the boundary conditions imposed by the dynamical system (e.g., source fields may have a fractal structure of infinite depth).

Smoothness of Δ induces smoothness of Σ in a well-posed, distributional, and indeed physically intuitive way via transposition:

$$\nabla F[\phi] \equiv F[\nabla^\dagger \phi]. \quad (3)$$

In the hypothetical case of smooth f , ∇F has Riesz representation ∇f ; transposition by partial

integration brings in a sign factor, so that one is bound to define for the general case

$$\nabla^\dagger \phi = -\nabla \phi. \quad (4)$$

Higher orders follow by trivial extension. The physical significance of this result is that one can take derivatives *prior to* contraction, but not *vice versa*, as there is no way to access to the infinitesimal domain in a physical representation. Unlike classical differential calculus, distribution theory is perfectly compatible with empirics: well-posed differentiation of a field is in fact integration without recourse to infinitesimals or physically void classifications such as $f \in C^\infty(\mathbb{R}^n)$, and has a robust computational realization (linear image processing) within scale limits and tolerances of a particular imaging set-up.

The generality of topological duality is both an asset as well as a drawback. In a way, scale (detector width) is the generic parameter one cannot do away with, so that one may want to map a field $f(x)$ to a uniquely defined, scale-parametrised representation $f(x; \sigma)$ with the help of Eq. (2). This calls for a strong reduction of the ∞ -dimensional device space. To this end, it is natural to conceive of device space as a bundle over the spacetime manifold, and to postulate a stratification into cross-sections parametrised by scale:

$$\Delta \equiv \cup_{x, \sigma} \Delta_{x, \sigma}. \quad (5)$$

That is, at each base point we have a fibre of localised detectors of various widths, which induces a similar structure on fields via Eq. (2). It depends on the physics of the situation—which is of no concern to us here—how to obtain “meaningful” cross-sections of this bundle (scale selection). The fundamental problem we address here is how to obtain an *a priori* stratification which does not affect the physical content of the field (“scale without physical scale” so to speak).

The basic idea is to construct $\Delta_{x, \sigma}$ as a class of derivative operators, with scale explicitly encoded. Without loss of generality we set $x = 0$, omitting all explicit references to this base point, and concentrate on the scale degree of freedom. Then it is natural to (i) arrange the operators by differential order $k \in \mathbb{N}$, a hierarchy naturally induced by Eq. (3) if one postulates a unique, positive definite, zeroth order “point operator” $\phi(z; \sigma)$, and (ii) to enforce the *correspondence principle* $\lim_{\sigma \downarrow 0} \phi(z; \sigma) = \delta(z)$ for the hypothetical zero-scale limit. By virtue of Eq. (3) one could say that such a point operator *defines infinitesimals* of finite resolution in an exact and operational sense.

There are several directions along which one could proceed, all of which, however, tend to arrive at the same conclusion. We select three approaches, all based on linearity, providing complementary insight (the elaboration is of interest in view of conceptual embeddings into diverse physical models). One point of departure is to introduce resolution via a *random walk* process in spacetime, by relating scale σ monotonically to evolution time t . This models the conceptual transition from a fiducial locus x pinpointed with infinite precision to a probabilistic one within a fuzzy σ -neighbourhood of x . In *Tikhonov regularisation* [3], scale enters as a parameter controlling the amount of regularity. That these seemingly different approaches are in fact equivalent is most easily seen by their connection to *isotropic diffusion* (again in the resolution domain), which is itself a viable point of departure.

For the sake of simplicity let us introduce pseudo-Euclidean spacetime parameters by the *ct*-convention: all time-units are multiplied by a formal parameter c carrying the dimension of a velocity. In particular the two independent scale parameters σ and τ for isotropic space and time are then related to a single pseudo-isotropic scale via $\sigma = c\tau$.

Random walk can be described by a path integral based on the Euclidean free-motion functional

$$F[q] = \exp \left\{ -\frac{1}{2} \int_0^t dt' \|\dot{q}(t')\|^2 \right\}, \quad (6)$$

defined for all paths q in spacetime with fixed boundary conditions $q(0) \equiv 0, q(t) \equiv z$. Assuming the t -axis measures scale, the spatiotemporal probability density function for the transition from the fiducial origin at scale zero to a neighbouring base point z at scale t can be expressed by the functional integral

$$\phi_t(z) = \int \mathcal{D}q F[q], \quad (7)$$

subject to given boundary conditions. Using the appropriate Wiener measure, Eq. (7) has a simple and well-known analytical solution

$$\phi_t(z) = \frac{1}{\sqrt{2\pi t^n}} \exp \left\{ -\frac{1}{2} \frac{\|z\|^2}{t} \right\}. \quad (8)$$

The point operator $\phi_t(z)$ represents a “monad” of finite scale $\sigma = \sqrt{t}$ containing the origin. For an arbitrary source f one obtains its σ -representative by superposition according to Eq. (2). Note that this Wiener process is equivalent to (pseudo-)isotropic *diffusion* of f :

$$\begin{cases} \partial_t u &= \frac{1}{2} \Delta u \\ u_{t=0} &= f. \end{cases} \quad (9)$$

As for *Tikhonov regularisation*, consider the extension of Eq. (2) to

$$\mathfrak{F}[u] = F[u] - \sum_{i=0}^{\infty} \frac{t^i}{2^{i+1} i!} \int dz \nabla_{\mu_1 \dots \mu_i} u(z) \nabla^{\mu_1 \dots \mu_i} u(z). \quad (10)$$

(Indices denote covariant derivatives.) Tikhonov regularisation relies on functional minimisation and aims for a “regularised” representation u of f by adding auxiliary terms to the basic source term, similar to the quadratic functional above. Note that at least one scale parameter—again denoted t with a modest amount of foresight—has to appear in the appropriate form to insure dimensional consistency, and that the correspondence principle is trivially satisfied. Furthermore, covariance forces any quadratic, diagonal regularisation term to be proportional to one of the above traces. The actual choice therefore resides entirely in the combinatorics. The Euler-Lagrange equation readily yields

$$u = \exp \left\{ \frac{1}{2} t \Delta \right\} f, \quad (11)$$

which is indeed equivalent to Eq. (9).

Ordinary, i.e. ∞ -resolution differential operators

$$f(z) = \nabla_{\mu_1 \dots \mu_i}^\dagger \delta(z) \quad (12)$$

are diffused into corresponding derivatives $\nabla_{\mu_1 \dots \mu_i}^\dagger \phi_t(z)$ of the zeroth order Green’s function, Eq. (8). Together these constitute a complete family of detectors adequate for probing the differential structure of an arbitrary source field to any order i and at any scale σ . Completeness is immediately obvious by virtue of Taylor’s theorem, but it is also easy to see that the family

fails to be orthogonal. However, with the initial condition given by Eq. (12), a suitable factorisation of $\phi_t(z)$ (*viz.* by extracting a normalised Gaussian amplitude of scale $\sqrt{2t}$) will turn Eq. (9) into a Sturm-Liouville eigenvalue problem formally identical to the stationary Schrödinger equation for a free harmonic oscillator, with eigenvalues identified with differential order [4]. A truncation of the family at finite order is most naturally studied in the framework of local jet bundles.

Implicit in all three views is the *semigroup* property for repeated samplings at one point:

$$\phi_s * \phi_t = \phi_{s+t}, \quad (13)$$

i.e. point operators form a commutative autoconvolution algebra, which is in fact unique in $\mathcal{S}(\mathbb{R}^n)$ under the positivity constraint (easily verified in Fourier space). Thus successive samplings are equivalent to a single one, and increase “blur” according to a Pythagorean sum. More generally, $\mathcal{S}(\mathbb{R}^n)$ is itself a closed algebra, and is thus consistent if field samples are considered potential source degrees of freedom that can be sampled in turn (*ad infinitum*). Another important property is *causality in the resolution domain*, akin to the principle of cartographic generalisation, which entails that coarse scale structure must find its cause in fine scale details, but never the other way around. More precisely, iso-surfaces of solutions to Eq. (9) always end with a convexity towards positive scales (a result of the maximum principle). It has been observed in the context of front-end vision [5] that, given a few plausible symmetries, Eq. (9) is indeed unique with respect to this demand.

Eq. (3) can be generalised, e.g. by considering Lie derivatives:

$$L_v F[\phi] = F[L_v^\dagger \phi]. \quad (14)$$

In general, a group Θ acting on spacetime induces an action on a source or detector instance; if $\theta \in \Theta$ one defines the push forward $\theta_*\phi$ and pull back θ^*F by the usual “carry-along” principle:

$$\theta^* F[\phi] = F[\theta_*\phi]. \quad (15)$$

Push forward and pull back affect the base point of objects in the same way as θ , respectively θ^{-1} does (whence the terminology). If f transforms as a scalar ($L_v f = \nabla_\mu f \cdot v^\mu$ or $\theta^* f = f \circ \theta$), then ϕ transforms as a density in the dual view ($L_v^\dagger \phi = -L_v \phi = -\nabla_\mu(\phi v^\mu)$ or $\theta_*\phi = |\det \nabla \theta^{-1}| \phi \circ \theta^{-1}$), *vice versa*.

Two applications of particular interest are noteworthy. In the context of a conservation law we may take the temporal component of the vector field, v^0 say, equal to unity, and set Eq. (14) equal to zero. Any speedometer type of field v^μ consistent with this—defining one velocity per base point at infinite resolution—can then be represented as a distribution in a similar way as the underlying source field f , *viz.* by defining the corresponding (*in casu* nonlinear) functional

$$V^\mu[\phi] = \frac{F^\mu[\phi]}{F[\phi]}, \quad (16)$$

in which the numerator is the linear functional with Riesz representation $v^\mu f$. Again, the natural way of probing the vector field is by mapping normalised Gaussians, in which case the source current $v^\mu f$ replaces f in Eq. (9), producing a measurable flow field v_t^μ that depends essentially on scale. Note that flow is determined up to a trivial gauge; any flow component along iso-surfaces $f = \text{constant}$ trivially satisfies zero Lie derivative (disambiguation is a matter of physics and requires local or global constraints).

A second application is to utilise Eq. (15) so as to incorporate spacetime topology into Eq. (2). In classical spacetime one can consider the appropriate affine transformation compatible with a flat connection, $\theta(z) = Az + x$ say (typically the spatial part of which is the scale-Euclidean group), and define its carry-along according to Eq. (15). The result is a coherent image of local field samples that reflects the spacetime model, and is a function of the group parameters. If the group comprises spatial rotations (spatial isotropy), spacetime translations (homogeneity), and spacetime scaling (two-fold scale invariance), each $\phi \in \mathcal{S}(\mathbb{R}^n)$ maps to an ensemble of operators, one for each group parameter (*modulo* invariances). Note that the standard Gaussian point operator is rotationally symmetric. This provides a constructive definition of Eq. (5) that could be generalised to account for orientation bundles or nontrivial spacetime topologies.

A final remark concerns canonical parametrisation. By scale invariance a natural scale parameter must be proportional to the logarithm of the natural volume element σ^n . This is consistent with the fact that the information-theoretical entropy $S[\phi] = -\int dz \phi \ln(\phi/m)$ of Eq. (8) as proposed by Shannon [6] and Wiener [7] (for constant measure m) equals $n \ln \sigma$ up to an irrelevant offset.

The *a priori* freedom of scale poses a new challenge, which at the same time pinpoints a shortcoming of contemporary differential geometry: the understanding of *deep structure*, i.e. the unfolding of structure over scale, and its implications for field theories. Catastrophe theory [8] provides a handle, but has to be studied in the context of the p.d.e. constraint Eq. (9). The generic event for 1-parameter families in general (stationary fields, say) is the fold catastrophe, which describes the Morsification of a degenerate critical point (zero gradient and zero Hessian determinant) into a pair of nondegenerate critical points of opposite Hessian signature. It has been shown by Damon [9] that this remains the case given Eq. (9), at least if the source field does not possess any special symmetries. There is, however, an obvious asymmetry in the way topological structure simplifies: when increasing scale, annihilations of Morse critical pairs are typically more frequent than creations (if $n = 1$ creations are altogether forbidden). The genuine spatiotemporal case often brings along an additional constraint on Morsification induced by a conservation law.

Summary and Conclusion

We conclude that, in a precise operational sense, the infinitesimal domain can be enlarged to finite, *a priori* arbitrary scale, enabling the application of differential methods from standard analysis in an exact, physically reasonable and inherently stable way. Plausible arguments suggest a natural way of probing the field's deep structure, based on normalised Gaussians and derivatives. Images of a field can be processed by linear filtering into a similar multiscale format reflecting the field's local jet bundle up to some order within the available scale range. Depending on the details of the underlying field theory one can select physically relevant scales *a posteriori*. By construction, the embedding multiscale framework connects seemingly unrelated field phenomena of different characteristic scales. The crux in this respect is the *semigroup* property Eq. (13) for rescalings—rather than the underlying *group*—from which a source field inherits a continuous tree-like structure that converges (diverges) towards coarse (fine) scales via Eq. (2).

The observations in this article have implications for classical field theories, the empirical sciences relying on field modelling and imaging, and the biophysics of vision.

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