Conclusions and outlook

In this thesis we have presented an explicit construction of quantum chains based on a weak quasi–quantum group $\mathcal{G}$. This way we did arrive at a generalization of the Hopf spin models of [NS97] as well as of the lattice current algebras of [AFFS98], which both were based on ordinary quantum groups. The most important applications are given by choosing $\mathcal{G}$ to be the semisimple quotient of a quantum enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ at roots of unity. The motivation for studying these generalizations is to provide examples of lattice quantum field theories exhibiting a quantum symmetry with non integer statistical dimensions. Also lattice current algebras have been invented as lattice regularizations of WZW-models and should therefore be studied at roots of unity.

We also gave a definition of the quantum double $\mathcal{D}(\mathcal{G})$ of a weak quasi–Hopf algebra $\mathcal{G}$ and showed that $\mathcal{D}(\mathcal{G})$ itself is a weak quasi–Hopf algebra with quasitriangular R-matrix. Moreover we discussed that $\mathcal{D}(\mathcal{G})$ plays a major role in the representation theory of our quantum chains. In particular we proved that the irreducible representations of periodic quantum chains (with any finite number of sites) may be labeled by the irreducible representations of the quantum double $\mathcal{D}(\mathcal{G})$. We also provided localized $\mathcal{D}(\mathcal{G})$-coactions on the local net of open quantum chains of finite length.

These findings strongly suggest that as in [NS97] these localized $\mathcal{D}(\mathcal{G})$–coactions are universal in the sense described in the introduction of this thesis, i.e. incorporate all DHR–sectors. An amplified version of the "field algebra" reconstruction of Mack and Schomerus [MS92, Sch95] may then be described in a very elegant way by using again our diagonal crossed product construction as follows. Let $\mathcal{A}$ denote the observable algebra generated by the local algebras $\mathcal{A}(I)$ ($\cong$ open finite quantum chains). As we have shown $\mathcal{A}$ admits a localized (right) $\mathcal{D}(\mathcal{G})$–coaction $\rho : \mathcal{A} \to \mathcal{A} \otimes \mathcal{D}(\mathcal{G})$. Thus we may define the two–sided crossed product $\mathcal{A}_\rho \rtimes \widehat{\mathcal{D}(\mathcal{G})} \ltimes {}_{\Delta_D}\mathcal{D}(\mathcal{G})$. This algebra has to be interpreted as follows: The universal $\lambda\rho$–intertwiner $\mathbf{\Gamma}$ is a "master" field operator, and $\mathcal{D}(\mathcal{G}) \subset \mathcal{A} \rtimes \widehat{\mathcal{D}(\mathcal{G})} \ltimes \mathcal{D}(\mathcal{G})$ represents the global *quantum symmetry*. The fields *transform covariantly*, which means that

$$[\mathbf{1}_\mathcal{G} \otimes a]\,\mathbf{\Gamma} = \mathbf{\Gamma}\,\lambda(a) \equiv \mathbf{\Gamma}\,\Delta(a), \quad a \in \mathcal{D}(\mathcal{G}), \tag{B.47}$$

whereas the observables $A \in \mathcal{A}$ are $\mathcal{D}(\mathcal{G})$–*invariant*

$$a\,A = A\,a, \quad \forall a \in \mathcal{D}(\mathcal{G}), A \in \mathcal{A}.$$

The linear subspace $\mathcal{F} \equiv \mathcal{A} \rtimes \widehat{\mathcal{D}(\mathcal{G})} := \mathcal{A} \rtimes \widehat{\mathcal{D}(\mathcal{G})} \ltimes \mathbf{1}_\mathcal{G}$ plays the role of the *field "algebra"*. Note that in the quasi–coassociative setting $\mathcal{F}$ is *not* a subalgebra of $\mathcal{A} \rtimes \widehat{\mathcal{D}(\mathcal{G})} \ltimes \mathcal{D}(\mathcal{G})$. But similarly as in [MS92] one may define a new non–associative "product" $\times$ on $\mathcal{F}$ by setting $A \times A' = AA'$ for $A, A' \in \mathcal{A}$ and

$$\mathbf{\Gamma}^{13} \times \mathbf{\Gamma}^{23} := (\Delta \otimes \mathrm{id})(\mathbf{\Gamma}). \tag{B.48}$$

This product is quasi–associative in the sense that

$$[\phi \otimes \mathbf{1}]\left(\mathbf{\Gamma}^{14} \times \mathbf{\Gamma}^{24}\right) \times \mathbf{\Gamma}^{34} = \mathbf{\Gamma}^{14} \times \left(\mathbf{\Gamma}^{24} \times \mathbf{\Gamma}^{34}\right)[\phi \otimes \mathbf{1}],$$

and it satisfies

$$[\mathbf{1}_\mathcal{G} \otimes \mathbf{1}_\mathcal{G} \otimes a]\,(\mathbf{\Gamma}^{13} \times \mathbf{\Gamma}^{23}) = (\mathbf{\Gamma}^{13} \times \mathbf{\Gamma}^{23})\,(\Delta \otimes \mathrm{id})(\Delta(a)),$$

which is the reason why it is called *covariant product* in [MS]. Moreover, since $\mathcal{D}(\mathcal{G})$ is quasitriangular, the field operators satisfy the braiding relations

$$\mathbf{\Gamma}^{13} \times \mathbf{\Gamma}^{23} = R^{12}\,(\mathbf{\Gamma}^{23} \times \mathbf{\Gamma}^{13})\,(R^{-1})^{12}.$$

The difference of this approach with the setting of [MS92, Sch95] lies in the fact that here the field operators appear in the form of irreducible matrix–multiplets

$$F_I^{ij} := (\pi_I^{ij} \otimes \mathrm{id})(\mathbf{\Gamma}), \quad \pi_I \in \mathrm{Irrep}\,\mathcal{D}(\mathcal{G}).$$

Correspondingly, the superselection sectors of the observable algebra $\mathcal{A}$ are given by the *amplimorphisms*

$$\rho_I^{ij}(A) \equiv (\mathrm{id} \otimes \pi^{ij})(\rho(A)) = F_I^{ik}\, A\, (F_I^{kj})^*,$$

which then is equivalent to the defining relation $\rho^{op}(A)\,\mathbf{\Gamma} = \mathbf{\Gamma}\,[\mathbf{1}_{\mathcal{G}} \otimes A]$.

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