Introduction

THIS thesis is essentially concerned with three different C^*–algebras: the algebra of the canonical anticommutation relations (CAR); the Weyl algebra, the exponentiated version of the canonical commutation relations (CCR); and the Cuntz algebra[a]. Common to all three is the fact that each is associated, in a specific way, with an underlying vector space: The CAR algebra is the C^*–Clifford algebra over a real Hilbert space, the Weyl algebra is the C^*–algebra generated by a projective unitary representation of a real symplectic space, and the Cuntz algebra is the universal C^*–algebra generated by a complex Hilbert space. (The vector spaces belonging to the CAR and CCR algebras will always be assumed to be infinite dimensional, as we will be exclusively dealing with systems possessing infinitely many degrees of freedom.)

The CAR and CCR algebras are the most prominent algebras in quantum physics, due to their distinguished rôle in describing systems of Fermions and Bosons. The *canonical anticommutation relations* have been introduced by Jordan and Wigner in 1928 in their analysis of the implications of Pauli's exclusion principle for the Fermi gas [JW28]. As an abstract C^*–algebra, the CAR algebra has a very simple structure: It is an approximately finite dimensional algebra, in a sense a non–commutative analogue of a zero–dimensional topological space, and is isomorphic to an infinite tensor product of copies of the two by two matrices.

Heisenberg's *canonical commutation relations* were found by Born in 1925. They first appeared in Born's joint work with Jordan on the matrix formulation of quantum mechanics [BJ25], but were independently obtained two months later by Dirac [Dir26]. Inspired by group theoretic considerations, H. Weyl discovered the usefulness[b] of replacing Heisenberg's commutation relations (which do not have bounded Hilbert space realizations) with their exponential form [Wey28]. In contrast to the CAR case, the Weyl algebra is a very "large" object (it is not separable), and not much seems to be known about its abstract properties.

The *Cuntz algebras* are the basic examples of infinite C^*–algebras (those containing non–unitary isometries) and are of great importance in the general structure theory of C^*–algebras. They have been introduced by Cuntz in 1977 [Cun77], but their generators ("Hilbert spaces of isometries") had been studied before by Doplicher and Roberts in the context of general quantum field theory [DR72, Rob76a]. The generic appearance of the Cuntz algebras in quantum field theory has been established rather recently by Doplicher and Roberts [DR90].

Each of these algebras possesses a natural class of structure preserving transformations ("*–endomorphisms"), namely those which arise from linear symmetries of the underlying vector spaces. These transformations will in all three cases

[a] The reader who is unfamiliar with operator algebras may think of a C^*–algebra as an algebra of bounded linear operators on some Hilbert space which is closed under taking adjoints and uniform limits. Textbooks on operator algebras, with applications to physics, are [BR81, EK98].

[b] He regarded his relations as the answer to the "Frage nach dem Wesen und der richtigen Definition der kanonischen Variablen" [Wey28].

be termed *quasi–free endomorphisms,* although in the CAR and CCR cases several other names are also used in the literature (linear canonical transformations, Bogoliubov transformations, one–particle transformations, ...). Thus quasi–free endomorphisms correspond to real orthogonal transformations in the CAR case, to real symplectic transformations in the CCR case, and to complex isometries in the Cuntz algebra case. Note that all these transformations need not be invertible (their ranges need not be the whole space) if the underlying vector spaces are infinite dimensional. If they are invertible, then the transformations of the algebras will be called quasi–free *automorphisms.*

It seems that quasi–free automorphisms (of the CCR algebra) first appeared in Bogoliubov's treatment of the Bose gas, although general (not necessarily linear) canonical transformations already occur in the famous "Dreimännerarbeit" of Born, Heisenberg and Jordan [BHJ26]. In the meantime there has been a tremendous work on quasi–free automorphisms of the CCR and CAR algebras, so that it would be hard to say anything new about them. Our interest is mainly in genuine (non–invertible) endomorphisms, which have not been treated systematically in the literature so far. *We develop a complete theory of those quasi–free endomorphisms of the CAR and CCR algebras which are, in the widest sense, related to second quantization*[c]. The second quantization of a genuine endomorphism is however no longer a single unitary operator, but, as follows from the work of Doplicher and Roberts, a whole Hilbert space of isometries on Fock space. This means that there is a representation of a Cuntz algebra associated with each such endomorphism.

But why should a physicist care about endomorphisms of C^*–algebras? Let us give an answer to this question by sketching the history of the theory of superselection sectors.

The Algebraic Theory of Superselection Sectors

The theory of superselection sectors is an important and particularly successful branch of local quantum field theory[d]. It was initiated by the observation of Wick, Wightman and Wigner in 1952 that the validity of the superposition principle in quantum physics is limited by what they called *superselection rules* [WWW52]. For instance, there is no interference between a single–electron state ψ_- and a single–positron state ψ_+. In the state $\psi = \alpha_+\psi_+ + \alpha_-\psi_-$, the relative phase between the $\psi_\pm$–components cannot be measured, but can be arbitrarily changed by applying global gauge transformations $\psi \mapsto \psi' = \alpha_+ e^{i\lambda}\psi_+ + \alpha_- e^{-i\lambda}\psi_-$. Such ψ is not a coherent superposition, but a mixture of the pure states $\psi_\pm$, with weights $|\alpha_\pm|^2$. Accordingly, matrix elements of physical observables between ψ_+ and ψ_- must vanish, observables are gauge invariant, and the physical Hilbert space splits up into invariant "coherent" subspaces, each carrying a definite value of the electric charge. The unobservability of relative phases in such situations led Wick, Wightman and Wigner to the conclusion that the parities of elementary particles with different charges cannot be compared.

Significant progress towards a deeper understanding of the general structure of quantum field theory, and in particular of the concept of superselection rules, was achieved by Haag and Kastler in 1964 [HK64]. Building on earlier ideas of Haag [Haa59], they proposed a C^*–algebraic treatment of quantum field theory. Whereas

[c]It should be noted that parts of our results have already been published [Bin95, Bin97, Bin98].

[d]We refer to Haag's beautiful book [Haa96] for a comprehensive introduction into the subject. We further recommend the lecture notes of Fredenhagen [Fre95], Roberts [Rob90], and Schroer [Sch98b]. The early history of superselection rules has been nicely reviewed by Wightman [Wig95].

"global" C^*–algebra approaches to quantum field theory had previously been advocated by several authors, notably by Araki, Haag, Schroer, and Segal, Haag and Kastler emphasized the importance of the *principle of locality* in field theory, i.e. the absence of actions at a distance. Locality allows to assign to any bounded region O in space–time the C^*–algebra $\mathfrak{A}(O)$ generated by all observables which can be measured within that region, such that the natural partial ordering between regions is preserved. The local algebras provide in some sense a "coordinate–free" description of quantum field theory as opposed e.g. to Wightman's approach [SW68] where one has to make a specific choice among all fields belonging to the same Borchers class. *Einstein causality* requires that algebras belonging to spacelike separated regions have to commute with each other. Haag and Kastler postulated that such a correspondence between space–time regions and algebras of local observables should fix the content of the theory completely. This point of view is plausible because "ultimately all physical processes are analyzed in terms of geometric relations" [HK64], and was supported by the general theory of collision processes that had been developed earlier by Haag and Ruelle [Haa58, Rue62].

The formulation of quantum field theory in terms of local algebras permitted a new look at superselection rules. Haag and Kastler introduced the *quasilocal algebra* $\mathfrak{A}$ as the C^*–algebra generated by all local observables. The algebra $\mathfrak{A}$ does not contain global quantities such as total charge or total energy; these can only be obtained as strong limits in specific representations. Now $\mathfrak{A}$ is expected to possess an abundance of inequivalent irreducible representations, e.g. representations associated with states having different behaviour at spacelike infinity. (Haag and Kastler believed that this was the only mechanism to produce inequivalent representations in quantum field theory, but it was soon recognized that there exist inequivalent representations even among the states with the same asymptotic behaviour; cf. (0.1) below.) That generic C^*–algebras have lots of inequivalent representations had already been discovered by von Neumann in 1939 [vN39][e]. Actually, any simple infinite dimensional C^*–algebra (besides the algebra of compact operators on a separable Hilbert space, which corresponds to the CCR algebra for finitely many degrees of freedom, i.e. to ordinary quantum mechanics) possesses uncountably many inequivalent irreducible representations. It was Haag who realized in the mid–fifties the *need* for inequivalent representations in order to obtain interacting quantum fields ("Haag's Theorem").

The different coherent subspaces are now interpreted as inequivalent irreducible representation spaces of the single algebra $\mathfrak{A}$. Haag and Kastler called (the unitary equivalence class of) an irreducible representation of $\mathfrak{A}$ a *superselection sector.* However, only a small subclass of all representations of $\mathfrak{A}$ can be expected to have a physical interpretation. In quantum field theory one is mainly interested in states which describe local finite–energy excitations of the vacuum. The corresponding sectors are called *charge* superselection sectors (and from now on, a sector will always mean a charge superselection sector). Here the term "charge" is used in a very broad sense: It applies to any quantity which can be used to label the various sectors.

Haag and Kastler argued that already a single sector should comprise all relevant physical information. If one starts e.g. with a state in the vacuum sector, one can create a particle of unit charge together with its antiparticle and then send the antiparticle "behind the moon". The resulting state will deviate, with respect to

[e] He showed this on the example of an infinite tensor product of 2×2 matrix algebras (isomorphic to the CAR algebra) by exhibiting representations of type I_∞, II_1 and II_∞. The field theoretic examples of inequivalent representations found in the 1950s also came from CAR and CCR algebras, e.g. from non–implementable quasi–free automorphisms, and thus are closely related to the subject of this thesis.

local measurements, arbitrarily little from a state in the charge–one sector. Thus any given state belonging to some sector can be approximated by states in each other sector. In the terminology of Haag and Kastler, all sectors are "physically equivalent", which is, by a theorem of Fell, tantamount to all sectors being "equally faithful" (they all have the same kernel). All superselection sectors thus determine the same "abstract" C^*–algebra $\mathfrak{A}$, and the choice of a particular representation of $\mathfrak{A}$ appears essentially as a matter of convenience. This is the solution of the (at that time much discussed) problem of inequivalent representations in quantum field theory offered by Haag and Kastler.

If one takes this philosophy seriously, one faces the basic problem: Given the quasilocal algebra $\mathfrak{A}$ (together with its local structure) in a, say, vacuum sector, how can one extract the interesting physical information? In particular, one would like to determine all charge superselection sectors together with a set of unobservable fields generating these sectors from the vacuum. The charge quantum numbers ascribed to the various sectors and fields are expected to be related to some sort of inner symmetries which act covariantly on the fields and trivially on the observables, and one would like to understand the laws of composition and exchange of charges ("statistics").

In the first step of such an investigation one has to specify which representations of $\mathfrak{A}$ are to be regarded as "local excitations of the vacuum", i.e. as charge superselection sectors. In his pioneering work [Bor65, Bor67a] Borchers proposed to consider all irreducible positive energy representations π which are "strongly locally equivalent" to a given vacuum representation π_0 and fulfill a certain "weak duality" condition[f]. Under these assumptions the unitary operators implementing the strong local equivalence could be interpreted as charged local fields.

However, it soon became clear that Borchers' assumptions were violated in typical examples. Positivity of the energy should of course hold in any reasonable theory, but is in general, due to infrared problems, too weak a condition to allow a complete classification of representations (such a classification is however possible under certain circumstances, as e.g. in conformal field theory [BMT88]). In order to clarify the rôle of Borchers' other assumptions, Doplicher, Haag and Roberts started a careful analysis of the superselection structure of elementary particle physics in 1969 which led to a series of seminal papers [DHR69a, DHR69b, DHR71, DHR74] and was to a certain extent completed some 20 years later by Doplicher and Roberts [DR90].

In [DHR69a] Doplicher, Haag and Roberts considered theories where a compact global gauge group acts on a *given* field algebra such that the observables are precisely the gauge invariant fields. They studied the representations of $\mathfrak{A}$ that are contained in a fixed vacuum representation of the larger field algebra. Among other things, they found that the sectors of $\mathfrak{A}$ occurring in this manner are in a natural way labelled by the inequivalent irreducible representations of the gauge group, and that weak duality implies the failure of strong local equivalence (see also [Rob70]). Therefore Borchers' fields do not exist in this case. Instead Doplicher,

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[f] π is a *positive energy representation* if the space–time translations are unitarily implemented in π such that the relativistic spectrum condition holds (see [BFK96, and references therein] for a local version of the spectrum condition). If one assumes that the local algebras are weakly closed (this can be done in presence of a distinguished vacuum representation), then the restrictions of positive energy representations to the local algebras are known to be unitarily equivalent. It is commonly believed that the von Neumann algebras associated with double cones (see below) are all isomorphic to the unique hyperfinite type III_1 factor [Ara64, Fre85, BDF87]. — π is *strongly locally equivalent* to π_0 if the restrictions of π and π_0 to the relative commutant of any local algebra are equivalent, and *weak duality* means that, in the representation π, the relative commutant is weakly dense in the commutant. Unfortunately, the DHR criterion (0.1) is also sometimes referred to as "strong local equivalence".

Haag and Roberts obtained two properties of these sectors which are closely related to, but much more significant than, strong local equivalence and weak duality. The first is that the various representations of the algebras belonging to the spacelike complement of any bounded region are unitarily equivalent, in symbols

$$\pi|_{\mathfrak{A}(O')} \simeq \pi_0|_{\mathfrak{A}(O')}. \tag{0.1}$$

Here π_0 is the vacuum representation of $\mathfrak{A}$ (corresponding to the trivial representation of the gauge group), π is some superselection sector, and $\mathfrak{A}(O')$ is the C^*–subalgebra of $\mathfrak{A}$ generated by all local observables which can be measured within the spacelike complement O' of the bounded space–time region O. This condition is weaker than Borchers' strong local equivalence. The second property is a strengthening of Einstein causality

$$\pi(\mathfrak{A}(O)) = \pi(\mathfrak{A}(O'))'. \tag{0.2}$$

The prime on the right denotes the commutant (the set of all bounded operators on the representation space of π which commute with all elements of $\pi(\mathfrak{A}(O'))$). Here $\pi(\mathfrak{A}(O))$ is assumed to be weakly closed, cf. footnote (f). Eq. (0.2) can only be expected to hold for particularly simple regions O, e.g. for double cones[g], and henceforth, we will generically take double cones as localization regions. Doplicher, Haag and Roberts showed that Eq. (0.2) holds in this form in all *simple* sectors, i.e. in all sectors corresponding to one–dimensional representations of the gauge group, but not in non–simple sectors. The condition (0.2) is stronger than Borchers' weak duality, but the point is that it is not assumed to hold in *all* sectors.

Eq. (0.1) is commonly called the *DHR selection criterion.* It is supposed to single out most sectors of interest in theories with short range forces. It implies that the states in the representation π look asymptotically like the vacuum, and that the charges distinguishing between π and π_0 can be localized in any bounded region. The DHR criterion is known to hold for all irreducible positive energy representations in conformal field theory [BMT88], but it excludes "topological charges" which appear even in purely massive theories [BF82, FM83], and charges which can be measured at arbitrary distances, e.g. by virtue of Gauß' law. Sectors satisfying the DHR criterion are automatically Poincaré covariant with positive energy under rather general assumptions [GL92], so that covariance does not have to be assumed from the outset.

Following a proposal of Schroer in [FRS89], (0.2) is called *Haag duality* in order to distinguish it from several other concepts of "duality" occurring in physics. It means that the local algebras cannot be enlarged in the representation π without violating Einstein causality. It was originally invented by Haag and Schroer as an expression of underlying relativistic dynamics [HS62, Haa63]. The failure of Haag duality for double cones in the vacuum sector indicates that the theory is in some sense incomplete. It is typically caused by spontaneous breakdown of inner symmetries [Rob76b, BDLR92], but is also generic in two–dimensional quantum field theory (cf. [Müg98]). This phenomenon can be traced back to the existence of operators which are only invariant under the unbroken symmetries, but not under the broken ones, in the first case; and to the existence of "kink operators" in the second case.

In [DHR69b] the converse problem of reconstructing the field algebra and the gauge group from $\mathfrak{A}$ and π_0 was solved for the set of all Poincaré covariant sectors satisfying the DHR criterion and Haag duality (simple sectors). This set of sectors has the structure of a discrete Abelian group, and its Abelian dual can be viewed

[g] Double cones are non–void intersections of suitably situated open backward with forward light cones. They constitute the simplest (yet sufficiently rich) class of causally complete bounded regions in Minkowski space which is closed under Poincaré transformations.

as the gauge group. There is a unique field algebra consisting of Bose and Fermi fields which has $\mathfrak{A}$ as its gauge invariant part and acts irreducibly on the "physical" Hilbert space which contains each simple sector with multiplicity one.

Doplicher, Haag and Roberts extended this analysis to the class of *all* charge superselection sectors conforming with the DHR criterion (0.1) in [DHR71]. It is remarkable how little input is needed for their methods to apply, the deeper reason for that being an underlying general duality theory for compact groups. (The striking analogy of the superselection theory with the representation theory of compact groups had been fully recognized in [DR72], but it took almost 20 years to finish the proof that both structures are really identical [DR90]). It suffices to start with the quasilocal algebra $\mathfrak{A}$, together with a faithful irreducible representation π_0 on a separable Hilbert space $\mathcal{H}_0$, such that Haag duality (0.2) for double cones and Borchers' "property B"[h] hold in π_0. Let us sketch some of their results.

First of all, it is an immediate consequence of (0.1) and (0.2) that any representation π satisfying the DHR criterion is unitarily equivalent to a representation on $\mathcal{H}_0$ of the form $\pi_0 \circ \varrho$ where ϱ is a *localized endomorphism* of $\mathfrak{A}$. Such ϱ has the following properties:

- ϱ is a unital map from $\mathfrak{A}$ into itself which preserves the algebraic structure, the star "$*$" and the norm.
- ϱ is *localized* in some double cone O (and then also in every larger double cone) in the sense that

$$\varrho(a) = a, \quad a \in \mathfrak{A}(O'), \tag{0.3}$$

  and it maps the algebras belonging to larger regions than O into themselves.
- ϱ is *transportable*: If $\hat{O}$ is another double cone, then there is an endomorphism $\hat{\varrho}$ localized in $\hat{O}$ such that the representations $\pi_0 \circ \varrho$ and $\pi_0 \circ \hat{\varrho}$ are equivalent. Such $\hat{\varrho}$ has the form

$$\hat{\varrho}(a) = u\varrho(a)u^*, \quad a \in \mathfrak{A}, \tag{0.4}$$

  where u is a unitary element contained in any $\mathfrak{A}(\check{O})$ with $\check{O} \supset O \cup \hat{O}$ by Haag duality.

Conversely, any localized endomorphism ϱ gives rise to a representation $\pi_0 \circ \varrho$ fulfilling (0.1). Thus the sectors fulfilling the DHR criterion are in one–to–one correspondence with the equivalence classes $[\varrho]$ of irreducible localized endomorphisms ϱ. Here two endomorphisms ϱ, $\hat{\varrho}$ are called *equivalent* if they are related to each other as in (0.4), and ϱ is *irreducible* if the corresponding representation is. Irreducible endomorphisms are not necessarily invertible, it can happen that $\pi_0(\varrho(\mathfrak{A}))'' = \pi_0(\mathfrak{A})''$ but $\varrho(\mathfrak{A}) \subsetneq \mathfrak{A}$. Invertible endomorphisms (= automorphisms) correspond to simple sectors and are characterized by the property that ϱ^2 is irreducible.

Several operations can be performed within the set of localized endomorphisms. The *direct sum* of localized endomorphisms $\varrho_1, \ldots, \varrho_n$ is defined as follows. Take local observables $v_1, \ldots, v_n$ with the properties

$$v_j^* v_l = \delta_{jl} \mathbf{1}, \tag{0.5a}$$

$$\sum_{j=1}^{n} v_j v_j^* = \mathbf{1} \tag{0.5b}$$

[h] Property B means that the local algebras are "almost type III" (any non–trivial local projection is, at least within a slightly larger algebra, equivalent to $\mathbf{1}$). This property was derived from standard assumptions by Borchers [Bor67b]. It implies that any local algebra contains a subalgebra isomorphic to the Cuntz algebra $\mathcal{O}_2$.

(by the way, these are the defining relations of the Cuntz algebra $\mathcal{O}_n$), and set

$$\bigoplus_{j=1}^{n} \varrho_j(a) \equiv \sum_{j=1}^{n} v_j \varrho_j(a) v_j^*, \quad a \in \mathfrak{A}. \tag{0.6}$$

Such v_j exist due to property B. The direct sum $\oplus_j \varrho_j$ is again a localized endomorphism, and its equivalence class does not depend on the choice of the v_j (if $\hat{v}_1, \ldots, \hat{v}_n$ is another collection of local observables fulfilling (0.5), then the equivalence between the two direct sums is established, in the sense of (0.4), by the unitary $u = \sum \hat{v}_j v_j^*$). Similarly, if ϱ is reducible, and $p \in \pi_0(\varrho(\mathfrak{A}))'$ a non–trivial projection, then there exists (by Haag duality and property B) a local observable v such that

$$vv^* = p, \qquad v^*v = \mathbf{1}, \tag{0.7}$$

and the *"subobject"* of ϱ corresponding to p can be defined by $\varrho_p(a) = v^*\varrho(a)v$. The equivalence class of ϱ_p is again independent of the choice of v. Finally, the *composition* $\varrho_1 \circ \varrho_2$ of two localized endomorphisms is a localized endomorphism whose equivalence class depends only on the classes of ϱ_1 and ϱ_2, and in particular not on the order of the factors, because endomorphisms which are localized in mutually spacelike double cones commute with each other.

Thus direct sums and subrepresentations of representations fulfilling the DHR criterion also fulfill this criterion, and one has a well–defined commutative product ("fusion") of equivalence classes of such representations, corresponding to the composition of charges and given by the composition of the associated endomorphisms. The availability of this product of sectors is the main advantage of working with endomorphisms.

These observations provide the basis for an intrinsic understanding of statistics, which is independent of a possible particle interpretation of the theory, and for the reconstruction of gauge symmetries and charged fields from observable data only.

The *statistics* of a sector $[\varrho]$ describes the effect of exchanging identical charges (remember that every sector carries a specific charge). It is determined by the *statistics operator* ε_ϱ, which can be defined as follows. Pick a unitary "charge transporter" u as in (0.4) such that ϱ and the corresponding $\hat{\varrho}$ are localized in spacelike separated double cones. Then

$$\varepsilon_\varrho \equiv u^*\varrho(u) \tag{0.8}$$

is a unitary operator which commutes with all elements of $\varrho^2(\mathfrak{A})$, and its definition is independent of the particular choice of u. (This is not true in two–dimensional Minkowski space, where the spacelike complement of a double cone has two connected components. There one can have two different choices of ε_ϱ, one the other's inverse, depending on whether $\hat{\varrho}$ is localized to the left or right of ϱ. This possibility had already been observed by Streater and Wilde in 1970 [SW70].) The statistics operator ε_ϱ fulfills the algebraic relations

$$\varepsilon_\varrho \varrho(\varepsilon_\varrho) \varepsilon_\varrho = \varrho(\varepsilon_\varrho) \varepsilon_\varrho \varrho(\varepsilon_\varrho), \tag{0.9}$$

$$\varepsilon_\varrho^2 = \mathbf{1}, \tag{0.10}$$

so that the operators $\varrho^n(\varepsilon_\varrho)$, $n \geq 0$, fulfill the characteristic relations of elementary permutations (transpositions) which exchange n and $n+1$. Thus canonically associated with any sector is a unitary representation of the infinite permutation group. (Relation (0.10) gets in general lost in two dimensions, so that one obtains representations of the infinite *braid group* instead. The first examples of sectors with Abelian braid group statistics were again given in [SW70].) This permutation group representation is analogous to the action of the permutation group on wave functions in quantum mechanics. It describes permutations of factors in products of localized state vectors (or permutations of identical particles, if the theory has a particle content; see [DHR74]).

The analysis of the statistics of the sector $[\varrho]$ now proceeds with the help of left inverses. A *left inverse* ϕ_ϱ of a non–invertible endomorphism ϱ is a substitute for the inverse of an automorphism. It is a unital positive linear map from $\mathfrak{A}$ into itself which is not multiplicative on the whole algebra $\mathfrak{A}$, but satisfies

$$\phi_\varrho(a\varrho(b)) = \phi_\varrho(a)b, \qquad a, b \in \mathfrak{A}.$$

It follows from this that $\phi_\varrho \circ \varrho = \mathrm{id}$, that $\varrho \circ \phi_\varrho$ is a conditional expectation from $\mathfrak{A}$ onto $\varrho(\mathfrak{A})$, and that ϕ_ϱ enjoys the same localization properties (0.3) as ϱ. Such ϕ_ϱ always exists, and corresponds to the physical operation of transferring the charge of ϱ to spacelike infinity. By applying a left inverse ϕ_ϱ to the statistics operator ε_ϱ one obtains the *statistics parameter* $\lambda_{[\varrho]}$,

$$\lambda_{[\varrho]} \equiv \phi_\varrho(\varepsilon_\varrho).$$

The statistics parameter is a scalar because ϕ_ϱ maps $\varrho^2(\mathfrak{A})'$ into $\varrho(\mathfrak{A})'$, and the latter contains only scalars by Schur's Lemma. $\lambda_{[\varrho]}$ is a numerical invariant[i] of the sector $[\varrho]$. It can be used to classify the statistics of $[\varrho]$. In the case of "finite statistics"[j] (i.e. $\lambda_{[\varrho]} \neq 0$), the left inverse ϕ_ϱ is unique, and one obtains the *statistics phase* $\eta_{[\varrho]}$, a complex number of modulus one, and the *statistics dimension* $d_{[\varrho]} \geq 1$ by polar decomposition:

$$\lambda_{[\varrho]} = \frac{\eta_{[\varrho]}}{d_{[\varrho]}}.$$

Simple sectors are precisely the sectors with $d_{[\varrho]} = 1$. The statistics dimension can be viewed as a measure for the deviation from Haag duality in the sector $[\varrho]$. It can also be defined for reducible endomorphisms and coincides with the square root of the minimal index [Jon83, Kos86] of the inclusion $\varrho(\mathfrak{A}(O)) \subset \mathfrak{A}(O)$ [Lon89]. It is additive on direct sums and multiplicative on products of endomorphisms. Any localized endomorphism with finite statistics is a finite direct sum of irreducible endomorphisms with finite statistics.

Doplicher, Haag and Roberts classified the possible statistics in Minkowski space of dimension greater than 2 [DHR71]. There the statistics phase is just a sign $\eta_{[\varrho]} = \pm 1$, and the statistics dimension $d_{[\varrho]}$ is a natural number (in the infinite statistics case one sets $d_{[\varrho]} = \infty$). These numbers characterize the unitary representation of the permutation group: Depending on the sign $\eta_{[\varrho]}$, a sector obeys either *para–Bose* or *para–Fermi* statistics of order $d_{[\varrho]}$. This means that all representations of the permutation group occur whose Young tableaux have columns resp. rows up to length $d_{[\varrho]}$. In a sector with infinite statistics, all irreducible representations of the permutation group occur. Moreover, for every sector $[\varrho]$ with finite statistics, there exists a unique *conjugate sector* $[\bar{\varrho}]$ which is determined by the property that the product $[\varrho\bar{\varrho}]$ contains the vacuum sector as a subrepresentation. A sector and its conjugate have the same statistics: $\lambda_{[\varrho]} = \lambda_{[\bar{\varrho}]}$ ("particle–antiparticle symmetry"). The conjugate sector can be viewed as arising from the state induced by applying the left inverse to the vacuum state.

For the program of reconstructing the field algebra and the gauge group from the observables, it proved instructive to reinvestigate the situation with given field algebra and gauge group [DR72]. As mentioned above, the sectors occurring under these circumstances correspond to the various irreducible representations of the gauge group and satisfy the DHR criterion (0.1), hence can be described by localized

[i] See [FRS89, FRS92, KMR90] for a discussion of uniqueness of λ in two dimensions.

[j] The case $\lambda_{[\varrho]} = 0$ does not occur in theories with particle–antiparticle symmetry and is usually disregarded. Reasonable examples of sectors with infinite statistics dimension have been given by Fredenhagen [Fre94]. See also [BCL97].

endomorphisms (one also needs a duality assumption on the fields for this). They all have finite statistics. In [DR72] Doplicher and Roberts assigned to a localized endomorphism ϱ the closed linear subspace $H(\varrho)$ of local field operators Ψ which induce ϱ:

$$H(\varrho) \equiv \{\Psi \mid \Psi a = \varrho(a)\Psi \quad \text{for all } a \in \mathfrak{A}\}. \tag{0.11}$$

Because any field which commutes with all quasilocal observables is a multiple of the identity, each $\Psi \in H(\varrho)$ is a multiple of an isometry

$$\Psi^*\Psi = \|\Psi\|^2\mathbf{1}.$$

Likewise, $\Psi^*\Psi'$ is proportional to $\mathbf{1}$ for any two $\Psi, \Psi' \in H(\varrho)$, defining an inner product in $H(\varrho)$:

$$\langle\Psi, \Psi'\rangle\mathbf{1} \equiv \Psi^*\Psi'.$$

This inner product induces the usual operator norm. Thus $H(\varrho)$ is a *Hilbert space of isometries* [Rob76a] inside the field algebra. It has the property that the joint kernel of all Ψ^* vanishes: $\cap \ker \Psi^* = \{0\}$. Moreover, the dimension of $H(\varrho)$ is equal to the statistics dimension of ϱ:

$$\dim H(\varrho) = d_{[\varrho]}, \tag{0.12}$$

and the gauge action restricts to a continuous unitary representation of the gauge group on $H(\varrho)$. In this way one obtains a concrete equivalence between the "category" of localized endomorphisms with finite statistics (whose morphisms are the intertwiners between endomorphisms), and the category of finite dimensional continuous unitary representations of the compact gauge group (with morphisms the intertwiners between representations). This equivalence preserves irreducibility and direct sums. Since $H(\varrho_1) \otimes H(\varrho_2)$ is canonically isomorphic to $H(\varrho_1\varrho_2) = H(\varrho_1) \cdot H(\varrho_2)$, the composition of endomorphisms corresponds to taking tensor products of representations. The permutation symmetry is related to changing the order of factors in tensor powers, and charge conjugation corresponds to passing to the complex conjugate representation.

Since our own work will be concerned with the description of the Hilbert spaces $H(\varrho)$ associated with quasi–free endomorphisms ϱ, let us add the remark that any orthonormal basis $\Psi_1, \dots, \Psi_{d_{[\varrho]}}$ in $H(\varrho)$ fulfills Cuntz' relations (0.5) and implements ϱ in the sense that

$$\varrho(a) = \sum_{j=1}^{d_{[\varrho]}} \Psi_j a \Psi_j^*, \qquad a \in \mathfrak{A}. \tag{0.13}$$

This concept of *implementation of endomorphisms by Hilbert spaces of isometries* reduces, in the case $d_{[\varrho]} = 1$, to the familiar unitary implementation of automorphisms. Since $H(\varrho)$ is a representation space of the gauge group, the Ψ_j transform like a tensor under gauge transformations. Indeed, Doplicher and Roberts have shown that the elements of $H(\varrho)$ are the "typical elements" of the field algebra in the sense that any irreducible tensor $\Phi_1, \dots, \Phi_d$ of local fields is of the form $\Phi_j = a\Psi_j$ with $a \in \mathfrak{A}$ and $\Psi_j \in H(\varrho)$, for some irreducible localized endomorphism ϱ. It follows that the linear span of all $\Psi \in H(\varrho)$, where ϱ runs through all endomorphisms localized in a double cone O, is weakly dense in the von Neumann algebra of fields localized in O. We would also like to note that, associated with $H(\varrho)$, there is a "Bosonized" version $\hat{\varepsilon}_\varrho$ of the statistics operator, obtained by setting

$$\hat{\varepsilon}_\varrho \equiv \sum_{j,l=1}^{d_{[\varrho]}} \Psi_j \Psi_l \Psi_j^* \Psi_l^*. \tag{0.14}$$

This is the operator which effects the exchange of factors in a tensor product, since it fulfills

$$\hat{\varepsilon}_\varrho \Psi \Psi' = \Psi' \Psi, \qquad \Psi, \Psi' \in H(\varrho).$$

There is also a simple formula for the left inverse of ϱ in terms of the Ψ_j:

$$\phi_\varrho(a) = \frac{1}{d_{[\varrho]}} \sum_{j=1}^{d_{[\varrho]}} \Psi_j^* a \Psi_j, \qquad a \in \mathfrak{A}. \tag{0.15}$$

The corresponding "Bosonized" statistics parameter is of course given by

$$\phi_\varrho(\hat{\varepsilon}_\varrho) = \frac{1}{d_{[\varrho]}}; \tag{0.16}$$

that is, the information about the statistics phase $\eta_{[\varrho]}$ is lost. The full field theoretic statistics operator ε_ϱ has instead the form

$$\varepsilon_\varrho = \sum_{j,k,l=1}^{d_{[\varrho]}} \Psi_j \hat{\Psi}_j^* \Psi_k \hat{\Psi}_l \Psi_l^* \Psi_k^*,$$

where the $\hat{\Psi}_j \equiv u\Psi_j$ are an orthonormal basis in the Hilbert space $H(\hat{\varrho})$ implementing the endomorphism $\hat{\varrho}$ (cf. (0.8) and (0.4)). This expression is obtained by substituting ϱ in (0.8) by the formula (0.13), and by writing the unitary charge transporter u in terms of the $\Psi_j, \hat{\Psi}_j$ as $u = \sum_j \hat{\Psi}_j \Psi_j^*$. Using the asymptotic commutation relations of the fields, which are in the present case of Bose or Fermi type ($\Psi_k \hat{\Psi}_l = \pm \hat{\Psi}_l \Psi_k$), one gets that $\varepsilon_\varrho = \pm \hat{\varepsilon}_\varrho$.

These observations led Doplicher and Roberts to the conjecture that the (finite statistics) superselection structure described in [DHR71], and valid in at least 3 space–time dimensions, should always be equivalent to the representation theory of a unique compact group. The proof of this conjecture was completed in the late 1980s [DR90] via an extension [DR88, DR89a, DR89b] of the Tannaka–Krein duality theory of compact groups. The *Tannaka–Krein theory* allows to recover a compact group from its "concrete dual", i.e. from the collection of finite dimensional unitary representation spaces together with the intertwiners between representations. The group elements can then be identified with certain functions assigning to each representation space a unitary operator on that space. Doplicher and Roberts instead characterized the *abstract* duals of compact groups. They found that any category which has essentially all the properties shared by the category of localized endomorphisms, namely a composition law with permutation symmetry, and the existence of subobjects, direct sums and conjugates, is equivalent to a category of finite dimensional continuous unitary representations of a unique compact group.

The construction of field algebra and gauge group from the observables and localized endomorphisms now amounts to the construction of a concrete group dual from an abstract one. The field algebra can be obtained as the "cross product" of $\mathfrak{A}$ by the semigroup of localized endomorphisms. This is a C^*–algebra which contains $\mathfrak{A}$ and, for each endomorphism ϱ, a finite dimensional Hilbert space $H(\varrho)$ inducing ϱ, with certain relations between the elements of $\mathfrak{A}$ and the elements of the algebras generated by the $H(\varrho)$. The gauge group can be identified with the compact group of all automorphisms of the field algebra which leave $\mathfrak{A}$ pointwise fixed.

Summarizing, the main result of Doplicher and Roberts states that, in Minkowski space of dimension greater than two, the observable algebra can always be embedded into a larger field algebra on which a compact gauge group acts in such a way that the observables are precisely the gauge invariant fields. The fields are local relative to the observables and act irreducibly on a Hilbert space which contains each superselection sector ϱ with multiplicity $d_{[\varrho]}$. The charge quantum numbers are in one–to–one correspondence with the equivalence classes of irreducible representations of the gauge group. This construction is unique up to unitary equivalence, if one requires that fields commute or anticommute at spacelike separations ("normal commutation relations").

Thus the program of reconstructing charged fields and gauge symmetries from local observables has been carried through successfully, at least in the case of strictly localizable charges (cf. (0.1)) in space–time of dimension greater than two. This is of course a strong confirmation of the central idea behind local quantum field theory, that all physical information should be encoded in the relative position of the algebras of local observables.

The picture is however less complete in low dimensional quantum field theory. An analysis of the low dimensional superselection structure based on the DHR selection criterion (0.1) and on Haag duality (0.2) in the vacuum sector has been worked out by Fredenhagen, Rehren and Schroer [FRS89, FRS92]. The methods of Doplicher, Haag and Roberts can be adapted to this situation, and, as already indicated above, one finds a somewhat richer structure in this case. The statistics phase can be an arbitrary element of the circle group $\mathbb{T}$, and the statistics dimension need not be an integer, but can take almost any value that is allowed by Jones' list [Jon83] of indices of subfactors. Statistics is governed by the braid group instead of the permutation group, and the gauge symmetries (the "dual object" of the superselection structure) do not form a group in general. However, there are only partial results concerning the quantum symmetry problem (e.g. [MS92, FK93, Reh96, BNS98, NSW98]), the classification of the occurring braid group representations (see e.g. [FRS89, Lon90]), and the reconstruction of charged fields (e.g. the "reduced field bundle" [FRS92], a bounded version of the conformal "exchange algebras" of Rehren and Schroer [RS89], or [Sch95]). It is interesting to note that many structural peculiarities such as braid group statistics, non–integer dimensions, Verlinde's modular algebra etc., which had been found previously in conformal field theory, and which were thought to be consequences of conformal invariance, could be shown to be generic features of low dimensional quantum field theory, independent of conformal invariance; see e.g. [Reh90, FRS92].

Whereas Haag duality automatically holds in conformally invariant theories on the circle (due to space–time compactification, see [BSM90, BGL93, FG93]), it is not such a reasonable assumption in two–dimensional Minkowski space. The basic mechanism for the violation of Haag duality in the vacuum sector is the following: If O is a given double cone, then there exist operators in $\mathfrak{A}(O)'$ which create a charge in the left spacelike complement of O and annihilate a charge of the same type in the right spacelike complement. Such operators cannot be approximated by observables in $\mathfrak{A}(O')$, so that (0.2) fails.

A preliminary analysis of the two–dimensional situation without assuming Haag duality from the outset has recently been attempted by Müger [Müg98]. Similar as Doplicher, Haag and Roberts in [DHR69a], he starts from a field algebra with normal commutation relations from which the observables are selected by a gauge principle. The fields are assumed to satisfy a certain duality property, which would imply Haag duality for the observables in higher dimensions, but entails only a weaker form of duality ("essential duality") in two dimensions; and a specific form of causal independence, which is believed to hold in massive theories. One can then enlarge the local field algebras by adding certain non–local "kink" or "disorder" operators to the fields which act like the identity on one half of the spacelike complement of some double cone, and like a global gauge transformation on the other half. One can also introduce a system of enlarged local observable algebras on the vacuum Hilbert space of the original observables, which fulfills Haag duality and is therefore called the "dual net"[k], by setting

$$\mathfrak{A}^d(O) \equiv \mathfrak{A}(O')'.$$

It turns out that this dual net is just the fixed point net of the enlarged field algebra under the gauge group G; and that, if G is finite, there is a natural action of a certain Hopf algebra containing G (the "quantum double" of G) on the enlarged field algebra such that the original observables are the fixed points under the action of the whole quantum double. In this sense the violation of Haag duality in two–dimensional Minkowski space is also related to symmetry breaking.

Let us finally comment on the status of the DHR selection criterion (0.1). As we already pointed out, all superselection sectors which can be reached by applying local fields to the vacuum, and all positive energy representations in conformal field theory, fulfill this criterion. But even in purely massive theories it is not true that all positive energy representations can be localized in bounded space–time regions. Buchholz and Fredenhagen proved that for a primary positive energy representation π whose energy–momentum spectrum starts with an isolated mass shell, there is a unique vacuum representation relative to which π can be localized around "semi–infinite strings" extending from one point to spacelike infinity [BF82]. (In two dimensions there are possibly two inequivalent vacua associated with π, and one can have *soliton sectors* interpolating between these vacua; see [Frö76, Fre90, Fre93, Sch96a, Reh97].) One can perform an analysis of superselection sectors having this weaker localization property relative to a fixed vacuum. Such an analysis is technically more involved than in the case of the DHR criterion, but the resulting structure is very similar. In particular, superselection sectors can still be described with the help of endomorphisms. Braid group statistics arises already in three dimensions; see [BF82, DR90, FGM90, MS95, FGR96] for details.

Even weaker localization properties must be expected for charged sectors in the presence of long range forces, e.g. in QED, where the asymptotic direction of the electric flux at spacelike infinity has to be taken into account. There is some hope that the localization of charged states can be improved by comparing them with an "infravacuum" state (a certain radiation background) instead of a vacuum state, so that the criterion of Buchholz and Fredenhagen would apply [Buc82, Kun97]. Another mechanism which could make the methods of superselection theory applicable to charges obeying Gauß' law has recently been proposed in [BDM$^+$96]. These authors showed on the example of a free massless scalar field that such charges can be described by automorphisms ϱ which violate the Buchholz–Fredenhagen condition on $\mathfrak{A}$, but fulfill the stronger DHR criterion (0.1) relative to a smaller subalgebra. Although one can no longer conclude from Haag duality that the charge transporters u entering the definition (0.8) of the statistics operators ε_ϱ are contained in $\mathfrak{A}$ (cf. (0.4)), it is nevertheless possible to define $\varrho(u)$ unambiguously. Then the statistics operators are well–defined, and one can discuss the statistics of the model in the usual way.

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[k] "Essential duality" means that the so–defined dual net satisfies Einstein causality [Rob76b]: $\mathfrak{A}^d(O') \subset \mathfrak{A}^d(O)'$. Essential duality is known to hold if the local algebras are generated by Wightman fields [BW75]. The passage from a non–Haag dual theory to the dual net is the customary way of restoring Haag duality. Its value lies in the fact that, in higher dimensions and under the assumption of essential duality, the dual net possesses precisely the same superselection sectors as the original theory [Rob80]. This is however not true in two dimensions, so that the significance of the dual net is somewhat limited in this case. Under Müger's assumptions, the dual net has in fact *no* superselection sectors fulfilling the DHR criterion. It seems that superselection sectors of the original theory extend at best to soliton sectors of the dual net [Müg97]. — Similar questions have been investigated for conformally invariant theories on the real line in [GLW97].

There are many other important topics in the theory of superselection sectors that we cannot touch upon in this introductory survey. Let us mention the deep connection with the theory of von Neumann algebras, whose powerful tools as e.g. the modular theory of Tomita and Takesaki (leading for instance to a new approach to the just mentioned localization problem [Sch97b, Sch97a]) and the techniques of the theory of subfactors have found a lot of applications in local quantum physics. It is even true that some of these developments had been anticipated in physics before they were established in greater generality in mathematics. As these techniques will not be applied in our work, we refer to the original literature [Lon89, Lon90, FRS89] and to the reviews [Bor95, Wie96, Sch96b, Sch98a], and references quoted there, for these matters.

What We Have Done

Review of the perspective. It should have become clear from the preceding excursion to non–perturbative quantum field theory that endomorphisms of C^*–algebras do play an important rôle in quantum physics. Starting from a theory of local observables, for which clear physical principles can be formulated, localized endomorphisms are the basic tools for an intrinsic construction of unobservable charged fields, for the derivation of their localization properties, commutation relations and symmetries. Since one has less intuition concerning the physical properties of unobservable quantities, it is gratifying that one does not have to postulate e.g. the commutation relations of fields, but can deduce them from the principles of locality and causality. It is equally remarkable that the gauge symmetries can be derived from the interrelations between local observables, which are by their very nature gauge invariant.

But, as it sometimes happens if one tries to put a physical theory on a sound mathematical basis, it is difficult to identify the general structures, whose existence is predicted by the abstract mathematical analysis, in concrete models. This applies in particular to quantum field theory where mathematically rigorous models beyond free fields are still lacking in four–dimensional space–time. Let us have a look at some field theoretical models whose superselection structure has been puzzled out.

Localized automorphisms of the *free charged Klein–Gordon field,* with gauge group $\mathbb{T}$, can be constructed as follows [Fre73, BLOT90]. One smears the field with a smooth real test function which has support in a compact region O and whose Fourier transform does not vanish identically on the positive mass shell. Then one obtains a unitary operator by polar decomposition of the smeared field which implements an automorphism localized in O and which carries one unit charge. (One cannot use quasi–free automorphisms of the CCR algebra for this purpose because they are all neutral.)

Localized automorphisms of the *free Majorana field,* with gauge group $\mathbb{Z}_2$, are even simpler to get. Here one can take the Majorana field itself, smeared out with a suitable localized real test function, as unitary implementer. This amounts to an especially simple choice of a quasi–free automorphism (a reflection).

The situation is a bit different in the case of the *free Dirac field*, with gauge group $\mathbb{T}$. Since the field operators are no longer injective, one cannot build automorphisms and unitary implementers directly out of the field operators. It is then natural to look for localized automorphisms among the class of quasi–free automorphisms of the CAR algebra. In [Bin93] we exhibited a family of charge–carrying localized quasi–free automorphisms which are induced by certain unitary multipliers on the single particle space. This construction works however only in two dimensions, and it is unlikely that unitary multipliers yield charge–carrying implementable automorphisms in higher dimensions. Our construction has been generalized by admitting kink–like multipliers by Adler [Adl96]. The corresponding automorphisms then show Abelian braid group statistics and extend to solitons of the dual net.

The *current algebra* derived from the massless free scalar field has *no* superselection sectors in dimension greater than two, but exhibits spontaneous symmetry breaking [Str74, BDLR92]. On the contrary, it possesses a *continuum* ($\cong \mathbb{R}^2$) of superselection sectors in two dimensions, due to its peculiar infrared properties. The associated localized automorphisms correspond to displacements of the fields [SW70]. The superselection structure of the conformal current algebra on the circle and of its local extensions (gauge group $\mathbb{Z}_{2N}$) has been studied in [BMT88].

Of greater interest is the case of genuine endomorphisms, corresponding to *non–simple* sectors. The only explicit examples of genuine localized endomorphisms constructed so far are, to the best of our knowledge, the ones leading to the non–simple sectors of the chiral conformal $so(N)$ WZW models at level one. They all belong to the class of quasi–free endomorphisms of the CAR algebra. The first example appeared in the treatment of the conformal Ising model by Mack and Schomerus [MS90]. This model has one non–simple sector, with highest weight $\frac{1}{16}$ and statistics dimension $\sqrt{2}$, and Mack and Schomerus offered a candidate of a localized endomorphism which was conjectured to describe this sector. They actually did not use this localized endomorphism in their computations, but worked with a global endomorphism throughout. This makes the analysis technically simpler, but also somewhat questionable, because e.g. the concept of statistics depends crucially on locality. The ideas of Mack and Schomerus were soon generalized by Fuchs, Ganchev and Vecsernyés to the level one $so(N)$ WZW models, which also have a Fermionic realization [FGV92]. These models possess one non–simple sector, with highest weight $\frac{N}{16}$ and statistics dimension $\sqrt{2}$, if N is odd; the case $N = 1$ reproduces the Ising model. But these authors also used global endomorphisms. This state of affairs was subsequently improved by Böckenhauer who constructed localized endomorphisms, among them the candidate of Mack and Schomerus, which are equivalent to the global ones of [MS90, FGV92], and which imply the same fusion rules [Böc94, Böc96]. Note that, due to the non–integer statistics dimension, none of these endomorphisms can have the properties predicted by Doplicher and Roberts (cf. (0.11)–(0.13)). These endomorphisms are not related to group symmetries, but to genuine quantum symmetries.

In this connection, we should also mention the attempts of A. Wassermann [Was] and Recknagel [Rec93, Rec96] to substitute localized endomorphisms by certain other structures. In [Was] the fusion of positive energy representations of the loop groups $LSU(N)$ is described. These representations can be constructed using implementers of certain quasi–free automorphisms of the CAR and CCR algebras over $L^2(\mathbb{T})$ [PS86], and are closely related to the $su(N)$ WZW models. In [Was], their fusion is not performed with the help of endomorphisms, but uses an equivalent technique, the tensor product of bimodules over von Neumann algebras [Con94] ("Connes fusion"). In [Rec93] it is proposed to replace endomorphisms of algebras by endomorphisms of some associated K_0–groups which are in principle much easier to handle. Though this heuristic approach is plagued with some serious shortcomings, it was possible on its basis to reproduce the fusion rules of the $su(2)$ WZW model. In [Rec96] it is tried to reach the sectors of some minimal models by "amplimorphisms" of certain associated path algebras. A characteristic feature of [Rec93, Rec96] is the complete lack of locality. (There is also the reformulation of the DHR theory given by Fredenhagen, where representations and endomorphisms are replaced by states and completely positive maps [Fre92]. The usefulness of this approach has been demonstrated on some subtheories of the conformal current algebra [Fre94].)

Summing up, one is confronted with a scarcity of field theoretic models whose localized endomorphisms are explicitly known, and there are in fact *no* known examples of endomorphisms which fit completely into the scheme of Doplicher and Roberts.

Quasi–free endomorphisms of CAR and CCR algebras. Ever since the invention of quantum field theory in the late 1920s [Dir27, JW28], the CAR and CCR algebras have been the dominating algebras in this field. In view of the above remarks it is natural to ask whether one can find endomorphisms of these algebras which share all the properties predicted by the theory of Doplicher and Roberts (see the discussion after Eq. (0.11)).

Specifically, the questions we are facing are the following: Do there exist quasi–free endomorphisms of the CAR and CCR algebras which can be implemented on Fock space by Hilbert spaces of isometries in the sense of Eq. (0.13)? If yes, how can such endomorphisms be characterized? What are their possible charge quantum numbers? And can one construct the corresponding Hilbert spaces of isometries (the "charged fields") explicitly?

Very briefly, the answers implied by our work can be summarized as follows. Each algebra possesses a rich semigroup of quasi–free endomorphisms having the desired properties. These semigroups are the natural generalizations of the well–known restricted orthogonal and symplectic groups. A quasi–free endomorphism belongs to one of them if and only if its associated one–particle operator fulfills a certain Hilbert–Schmidt condition. There are detailed formulas for the corresponding charged fields on Fock space which have a well–defined meaning as infinite sums converging strongly on a dense domain. The Hilbert spaces of isometries spanned by these fields can in a natural way be regarded as Fock spaces over some auxiliary space. (These auxiliary spaces can have finite or infinite dimension. Be aware that these "Fock spaces of isometries" are not contained in the original Fock space, but consist of operators acting on the latter.) This Fock space structure is compatible with the action of the gauge symmetries, and provides the key to the determination of the charge quantum numbers. Genuine endomorphisms are always reducible; the possible values of their statistics dimensions are the powers of 2 (CAR) resp. ∞ (CCR). They induce representations of the Cuntz algebras $\mathcal{O}_{2^n}$ and $\mathcal{O}_\infty$.

For the convenience of the reader who finds the preceding remarks too condensed we would like to give now a detailed exposition of the material contained in the central chapter of this thesis.

Section 1. Here we review the Cuntz algebras $\mathcal{O}(H)$ and their basic properties. This section is intended as a supplement to the main text, and its content is not essential for an understanding of the remainder.

After stating the definition of $\mathcal{O}(H)$, we quote Evans' Fock space construction of $\mathcal{O}(H)$ as a quotient of the Cuntz–Toeplitz algebra. Some general properties of $\mathcal{O}(H)$, mostly due to Cuntz as e.g. its K–theory, are mentioned, and endomorphisms of $\mathcal{O}(H)$ are discussed. Quasi–free endomorphisms and quasi–free states of $\mathcal{O}(H)$ are closely related to the structure of the gauge invariant subalgebra of $\mathcal{O}(H)$, and have been studied by Evans et al. Quasi–free group actions on $\mathcal{O}(H)$ are an important element in the theory of Doplicher and Roberts.

There has recently been some interest, e.g. in connection with Powers' E_0–semigroups, in the relation between representations of $\mathcal{O}(H)$ on a Hilbert space $\mathcal{H}$ and endomorphisms of $\mathfrak{B}(\mathcal{H})$. Note that the Cuntz algebras enter our analysis in exactly the same way: We study endomorphisms of the CAR and CCR algebras which give rise to representations of the Cuntz algebras on Fock space via Eq. (0.11) and (0.13). Thus our results also provide interesting examples for the representation theory of $\mathcal{O}(H)$.

Finally some remarks concerning the rôle of the Cuntz algebras in the general theory of C^*–algebras are made, but there is no room to discuss these important topics in greater detail.

Section 2. This section contains a thorough analysis of the semigroup of all quasi–free endomorphisms of the CAR algebra which can be implemented by Hilbert spaces of isometries in a fixed Fock representation. Essentially all results which go beyond the case of automorphisms are new. Most of them have already been published in [Bin95, Bin97], but the presentation given here is in several respects superior to the one in [Bin95]. This analysis is completely general in that we do not make specific assumptions on the structure of the real Hilbert space underlying the CAR algebra. The price that one has to pay for this generality is that the concept of locality is not incorporated at this level, but has to be discussed separately in a more restrictive setting. As a consequence, we cannot apply the methods of the theory of superselection sectors which hinge upon the principle of locality. Our methods are instead taken from the representation theory of the CAR algebra and from general functional analysis (e.g. Fredholm theory), and are largely independent of the Doplicher–Roberts theory. (Analogous remarks are valid for the treatment of the CCR case in Section 3.)

Section 2.1. The basic objects and facts that will be needed later on are introduced. Araki's "selfdual" CAR algebra formalism is used throughout which amounts to complexification of the underlying real Hilbert space. Quasi–free endomorphisms are in one–to–one correspondence with their restrictions ("Bogoliubov operators") to this space. Bogoliubov operators are isometric. The fundamental invariant of a Bogoliubov operator V is its Fredholm index, a non–positive integer (or ∞), and we find that this index is related to the statistics dimension d_V of the corresponding endomorphism ϱ_V by the formula

$$d_V = 2^{-\frac{1}{2}\operatorname{ind} V}. \tag{0.17}$$

Here the statistics dimension d_V is defined to be the square root of the Watatani index of the range of ϱ_V. This is a purely C^*–algebraic notion which does not depend on representations. Automorphisms are characterized by $d_V = 1$. By a somewhat surprising result in [Bin97], any ϱ_V with statistics dimension $d_V = \sqrt{2}$ induces in a canonical way an isomorphism from the CAR algebra onto its even subalgebra. These isomorphisms can be used to study the even subalgebra, which models the algebra of observables in various physical systems.

The class of quasi–free states is defined, and the main technical tool to be used later, the quasi–equivalence criterion for quasi–free states of Powers and Størmer and Araki, is stated. Fock states are the pure quasi–free states. The Powers–Størmer–Araki criterion can be simplified if one of the states involved is a Fock state. This has been observed by Powers, but we arrived independently at the same conclusion, by an argument which can be found in the preprint version of [Bin95].

Associated with every Fock representation is a second (equivalent) representation which we call the "twisted Fock representation". The twisted Fock representation provides a convenient way to describe "twisted duality", the analogue of Haag duality in the presence of Fermi fields, and is always useful if one has to deal with commutants of "local" subalgebras. The consequent use of the twisted Fock representation will lead to some simplifications in the construction of the charged fields in Section 2.4.

Section 2.2. This section is concerned with the representations $\pi \circ \varrho$ that are obtained by composing a Fock representation with a quasi–free endomorphism. (Recall from p. 6 that the representations occurring in the theory of superselection sectors have such a form.) We start with a discussion of the implementation problem for endomorphisms of arbitrary C^*–algebras. The conclusion to be drawn from this general discussion is that an endomorphism ϱ is implementable in an irreducible representation π if and only if the representations π and $\pi \circ \varrho$ are quasi–equivalent.

If π is the Fock representation of the CAR algebra induced by a Fock state ω, and if ϱ is a quasi–free endomorphism, we show that $\pi \circ \varrho$ is a multiple of the GNS representation of the quasi–free state $\omega \circ \varrho$. (We do actually a little bit more because we give an explicit decomposition of Fock space into invariant subspaces. This detailed description will be used in Section 2.4 to prove the completeness relation (0.5b) for the charged fields.) The multiplicity is some power of two or infinite. The question of implementability is thereby reduced to the question of quasi–equivalence of quasi–free states, and we can derive our basic implementability condition from the Powers–Størmer–Araki criterion. This condition generalizes the well–known Shale–Stinespring condition which is restricted to the case of automorphisms. Both conditions are formally the same. The Fredholm index of the Bogoliubov operator corresponding to an implementable endomorphism is always even, so that the statistics dimensions of implementable endomorphisms are integers (or ∞) as they should (cf. (0.12)).

The second half of this section deals with representations $\pi \circ \varrho$ where π is a given Fock representation, but ϱ an arbitrary (non–implementable) quasi–free endomorphism with finite index. (The analysis would be trivial in the implementable case, because $\pi \circ \varrho$ is then unitarily equivalent to $d_V \cdot \pi$.) This analysis is e.g. relevant for the endomorphisms describing the non–simple sectors of the WZW models (cf. the remarks made on p. 15). We derive criteria for unitary equivalence of two such representations, describe the quasi–free states of the form $\omega \circ \varrho$ where ω is the Fock state corresponding to π, and characterize the endomorphisms ϱ for which the states $\omega \circ \varrho$ are pure or "almost pure" (mixtures of two inequivalent pure states. This is in some sense the best one can get for endomorphisms ϱ_V with $\operatorname{ind} V$ odd.) These preparatory results are then applied to give an alternative proof a theorem of Böckenhauer, namely that $\pi \circ \varrho_V$ is a multiple of another Fock representation, with multiplicity d_V, if $\operatorname{ind} V$ is even; and that it is a multiple of two "pseudo Fock representations", with multiplicity $d_V/\sqrt{2}$, if $\operatorname{ind} V$ is odd. Invoking our isomorphism onto the even subalgebra from Section 2.1, we obtain analogous results for the restrictions of these representations to the even subalgebra. It was observed by Szlachányi and Böckenhauer, but regarded as a curiosity, that the restriction of $\pi \circ \varrho_V$ to the even subalgebra behaves like a representation $\pi \circ \varrho_{V'}$ of the whole CAR algebra, where V' is a Bogoliubov operator with $\operatorname{ind} V' = \operatorname{ind} V - 1$. Our approach gives a natural explanation for this phenomenon.

Section 2.3. Having established the necessary and sufficient condition for implementability in Section 2.2, we study here the structure of the topological semigroup of all quasi–free endomorphisms which can be implemented in a fixed Fock representation. This semigroup is an extension of the restricted orthogonal group. One of our achievements is the proof that this semigroup can be written as a product of a small subgroup consisting of automorphisms which are close to the identity, and the sub–semigroup of endomorphisms which leave the given Fock state invariant. What is more, we are able to make a definite choice of the two factors in which an implementable endomorphism decomposes.

There are some results involved in the proof of this product decomposition which are of independent interest. The first is a useful parameterization of the class of all Fock states which are unitarily equivalent to the given one. This parameterization is done in terms of certain pairs $(T, \mathfrak{h})$ consisting of an antisymmetric Hilbert–Schmidt operator T and a finite dimensional subspace $\mathfrak{h}$ of the kernel of T, and is adapted to the structure of the cyclic vectors in Fock space which induce the states. The next result is a canonical (up to a finite dimensional part related to $\mathfrak{h}$) choice of a quasi–free automorphism belonging to the small subgroup mentioned above which transforms a given equivalent Fock state into the original one.

Associated with any quasi–free endomorphism ϱ_V there is a "partial" Fock state, viz. a Fock state of the CAR algebra over the range of V. Using the above parameterization, we can extend this partial Fock state (in the implementable case) to a proper Fock state, say ω_V, of the whole CAR algebra. (This procedure is reminiscent of the construction of the conjugate sector, with the help of the left inverse, in quantum field theory; cf. p. 8.) The choice of ω_V is made definite by minimizing both "parameters" T and $\mathfrak{h}$ in an appropriate way. We can thus assign, in an unambiguous way, to each implementable quasi–free endomorphism ϱ_V a Fock state ω_V which is equivalent to the original Fock state ω, and such that $\omega_V \circ \varrho_V = \omega$. By the above, we get in addition a quasi–free automorphism ϱ_U in the small subgroup which also has the property that $\omega_V \circ \varrho_U = \omega$. The announced product decomposition of ϱ_V is then obtained by setting

$$\varrho_V = \varrho_U \varrho_W, \qquad W \equiv U^{-1}V. \tag{0.18}$$

There is an ambiguity in the definition of ϱ_U which cannot be resolved. It amounts to the freedom in the choice of an orthonormal basis in the "minimized" space $\mathfrak{h}_V$ associated with the Fock state ω_V.

The fact that any implementable quasi–free endomorphism can be written as a product of two very simple factors is then used to determine the connected components of the semigroup of implementable endomorphisms. It is well–known that the restricted orthogonal group (the group of implementable automorphisms) has two components which are distinguished by the Araki–Evans index, i.e. by the parity of the dimension of the space $\mathfrak{h}_V$. We find that the Araki–Evans index is not an invariant of genuine endomorphisms, and that each set of endomorphisms ϱ_V with fixed nonzero value of $\operatorname{ind} V$ is connected. The product decomposition of endomorphisms will further be used in Section 2.4 to reduce the proof of the completeness relation (0.5b) for the charged fields to the simpler case of endomorphisms which leave ω invariant.

Section 2.4. We show in this section how to construct an orthonormal basis in the Hilbert space of isometries $H(\varrho_V)$ which implements a given quasi–free endomorphism ϱ_V on Fock space. The implementers can be expressed in terms of annihilation and creation operators. This will make it necessary to employ special Fock space techniques, which have been avoided in the previous sections. The strategy underlying this construction is the following: We first generalize the known methods of constructing unitary implementers for automorphisms to the case of genuine endomorphisms[l]. This generalization makes essential use of the Fock state ω_V introduced in Section 2.3 and permits us to define *one* isometric implementer $\Psi_0(V)$ for ϱ_V. $\Psi_0(V)$ is characterized by the property that its value on the Fock vacuum Ω reproduces the cyclic vector inducing the state ω_V. We then construct a complete set of implementers by multiplying $\Psi_0(V)$ with suitable partial isometries from the left. This approach is suggested by the observation that, if one has an orthonormal basis $\Psi_\alpha(V)$ in $H(\varrho_V)$, then the operators $\Psi_\alpha(V)\Psi_0(V)^*$ are partial isometries in the commutant of the range of ϱ_V, and the $\Psi_\alpha(V)$ can be reconstructed from these

[l] It is well–known that an automorphism ϱ_U is implementable if and only if there exists a unit vector Ω_U in Fock space which lies in the joint kernel of all transformed annihilation operators. This vector Ω_U induces the state $\omega \circ {\varrho_U}^{-1}$. Once Ω_U is known, the unitary implementer $\Psi(U)$ for ϱ_U can be constructed essentially by setting $\Psi(U)\pi(a)\Omega = \pi(\varrho_U(a))\Omega_U$, where π denotes the Fock representation and Ω the original Fock vacuum vector. In the case of endomorphisms, the state ω_V plays the rôle of $\omega \circ {\varrho_U}^{-1}$. But note that the cyclic vector Ω_V associated with ω_V is no longer uniquely determined by the requirement that it be destroyed by the transformed annihilation operators. In fact, any element of the Hilbert space $H(\varrho_V)\Omega$ has the latter property. This corresponds to the fact that ω_V is not the only possible extension of the partial Fock state mentioned above. What remains true is that any implementer is uniquely determined by its value on Ω. Also note that, given ω_V, it is still not obvious that the construction of $\Psi_0(V)$ goes through in the known way: In the case of automorphisms, both relations $U^*U = \mathbf{1}$ and $UU^* = \mathbf{1}$ usually enter the construction on the same footing; whereas in the case of endomorphisms, the second relation is lost, and one has to take this into account with care.

operators (together with $\Psi_0(V)$) by setting

$$\Psi_\alpha(V) \equiv \Big(\Psi_\alpha(V)\Psi_0(V)^*\Big)\Psi_0(V). \tag{0.19}$$

The first step in this program is to give an appropriate definition of "bilinear Hamiltonians". This will be achieved by combining the algebraic approach of Araki with the more analytic approach of Ruijsenaars. As is well–known, the CAR algebra contains the spin group, the universal covering group of the connected component of the identity of the group of Bogoliubov operators which induce inner quasi–free automorphisms, together with its Lie algebra. The elements of this Lie algebra are called "bilinear Hamiltonians" because they are bilinear expressions in the generators of the CAR algebra, and can be identified with certain trace class operators H on the underlying Hilbert space. But we need a more general definition of bilinear Hamiltonians if we want to cover the case of general implementable transformations. A natural extension of this Lie algebra is the current algebra, the Lie algebra of skew–adjoint operators on Fock space whose exponentials implement one–parameter groups of quasi–free automorphisms. The current algebra can be identified with a larger class of operators H, namely with the Lie algebra of the restricted orthogonal group (if one allows for the occurrence of Schwinger terms). But even in the case of quasi–free automorphisms implementers can in general not be obtained as exponentials of these currents, so that this class is still too narrow.

The way out is to consider *Wick ordered* exponentials. Such Wick ordered exponentials can be defined, a priori as quadratic forms on a dense domain in Fock space, for Wick ordered bilinear Hamiltonians induced by arbitrary bounded operators H. Under a certain Hilbert–Schmidt condition on H, these Wick ordered exponentials are the quadratic forms of densely defined, in general unbounded, operators. The commutation relations of these operators with creation and annihilation operators can be explicitly computed, and are used to determine all operators H such that the Wick ordered exponential of the bilinear Hamiltonian induced by H has the "correct" intertwining properties relative to a given quasi–free endomorphism ϱ_V. The operators H with this property are in one–to–one correspondence with certain operators T obtained during the study of the state extension problem in Section 2.3. The minimal choice T_V made in that section leads then to a unique operator H_V associated with ϱ_V, and the isometric implementer $\Psi_0(V)$ is obtained as a finite sum of terms, each involving the Wick ordered exponential of the bilinear Hamiltonian induced by H_V plus some additional operators which essentially fill up the "Dirac sea" corresponding to the finite dimensional subspace $\mathfrak{h}_V$.

A complete orthonormal basis in $H(\varrho_V)$ is then constructed from $\Psi_0(V)$ by the method outlined above. Here it is used that the commutant of the range of ϱ_V can be easily described with the help of the twisted Fock representation. It is also straightforward to obtain partial isometries in this commutant since any Fermionic creation or annihilation operator is already (a multiple of) a partial isometry, by Pauli's principle. It is less obvious how to characterize partial isometries which contain the range of $\Psi_0(V)$ in their initial spaces, as is required by (0.19). This can be done with the help of the Fock state ω_V and its associated "parameter" T_V.

We finally arrive at the following scenario. There is a certain subspace $\mathfrak{k}_V$ of the kernel of V^* which has dimension $-\frac{1}{2}\operatorname{ind} V$. We choose an orthonormal basis in $\mathfrak{k}_V$. The representors of these basis vectors in the twisted Fock representation then behave like creation operators relative to the implementer $\Psi_0(V)$. That is, we obtain an orthonormal basis in $H(\varrho_V)$ by multiplying $\Psi_0(V)$ from the left with all possible ordered monomials in these operators. It follows that $H(\varrho_V)$ is isomorphic to the antisymmetric Fock space over $\mathfrak{k}_V$, so that

$$\dim H(\varrho_V) = d_V$$

(cf. (0.17)). The basis of implementers is chosen in such a way that the value of any one of them on the Fock vacuum Ω is the state vector of some Fock state. Moreover, the choice of implementers is compatible with the product decomposition (0.18). Roughly speaking, the factor ϱ_U carries the exponential term plus the operators corresponding to $\mathfrak{h}_V$, whereas ϱ_W is responsible for the additional partial isometries. The completeness of implementers (0.5b) is proved by showing that the ranges of the implementers of the factor ϱ_W are equal to the invariant subspaces which appeared in the decomposition of the representation $\pi \circ \varrho_W$ in Section 2.2.

Let us finally remark that the formulas given for the implementers are not in "normal form" in the strict sense, i.e. they are not completely Wick ordered. There are two reasons for this. The first is the use of the twisted Fock representation, which involves the second quantization of $-\mathbf{1}$ as a factor. These factors could be avoided by incorporating them into the bilinear Hamiltonian of H_V, but the formulas would become less transparent then, and the combinatorics would be more complicated. The second reason is that the additional partial isometries with which $\Psi_0(V)$ is multiplied contain an annihilation part which should be moved to the right, with the help of the commutation relations, in order to get a completely Wick ordered expression. We have not done so because the Fock space structure of $H(\varrho_V)$ would then no longer be visible.

Section 2.5. Here we derive formulas for the Bosonized statistics operators of quasi–free endomorphisms with finite statistics. The basic observation is that partial isometries of the form $\Psi_\alpha(V)\Psi_\beta(V)^*$ have an explicit representation as monomials in the basis vectors of $\mathfrak{k}_V$. Special examples are the operators $\Psi_\alpha(V)\Psi_0(V)^*$ appearing in (0.19), and the operators $\Psi_\alpha(V)\Psi_\alpha(V)^*$, the projections onto the ranges of the $\Psi_\alpha(V)$.

Recall from (0.14) that the Bosonized statistics operator is a certain polynomial in the implementers of the endomorphism. The knowledge of the operators $\Psi_\alpha(V)\Psi_\beta(V)^*$ and of the intertwiner properties of the $\Psi_\alpha(V)$ suffices to identify this polynomial with an element of the even CAR algebra. As a consistency check, we compute the "Bosonized statistics parameter" by applying the C^*–algebraic left inverses that were introduced in Section 2.1 to the Bosonized statistics operators. The result is the inverse of the statistics dimension, in accordance with (0.16).

Section 3. This section contains an analogous analysis of quasi–free endomorphisms of the CCR (or Weyl) algebra. It is essentially based on [Bin98]. The general remarks made above in the introduction to Section 2 apply here as well. The methods are now borrowed from the representation theory of the CCR algebra, and the theory of Fredholm operators will again be of importance. The survey of Section 3 will be comparatively short, and we will try to emphasize the differences between the CAR and CCR cases.

Section 3.1. The basic notions are established. We find it again convenient to use Araki's "selfdual" formulation. In the CCR case it is necessary to start with a distinguished "reference" Fock state ω in order to get a Hilbert space topology on the underlying symplectic space. A certain dichotomy will arise in the following from the fact that the algebraic relations are dictated by the symplectic form, whereas the analytic aspects refer to the Hilbert space inner product. Relevant topics discussed here are Araki's duality relation and a statement about the affiliation of sums of creation and annihilation operators to "local" Weyl algebras.

Section 3.2. Quasi–free endomorphisms are introduced. They are given by Bogoliubov operators V acting on the symplectic space. Bogoliubov operators preserve the symplectic form and have well–defined Fredholm indices. In contrast to the CAR case, $\operatorname{ind} V$ is always even, irrespective of implementability.

As in the CAR case, we reduce the question of implementability to the question of quasi–equivalence of quasi–free states, by showing that the representation induced by a quasi–free endomorphism ϱ in the given Fock representation is a multiple of the GNS representation of the state $\omega \circ \varrho$. The multiplicity is either 1 or ∞. Invariant subspaces are explicitly described.

The derivation of the necessary and sufficient condition for implementability is based on the criterion for quasi–equivalence of quasi–free states in the form given by Araki and Yamagami. Some work has to be done to get rid of the square roots appearing in this criterion. We do this with the help of an inequality of Araki and Yamagami; this inequality enables us to reduce the problem to the CAR case via polar decomposition of V, because the isometric part of V is a CAR Bogoliubov operator. The resulting condition is a generalization of the well–known condition of Shale which covers the case of automorphisms. The two conditions do *not* have the same form, in contrast to the CAR case. The statistics dimensions are now given by

$$d_V = \begin{cases} 1, & \operatorname{ind} V = 0, \\ \infty, & \operatorname{ind} V \neq 0. \end{cases} \tag{0.20}$$

The Fredholm index is therefore a finer invariant than the algebraic index.

Section 3.3. We study the semigroup of implementable quasi–free endomorphisms, an extension of the restricted symplectic group. Again we aim at showing that this semigroup can be written as a product of a subgroup of automorphisms close to the identity and a sub–semigroup of endomorphisms which leave the Fock state ω unchanged. To this end we consider the set of Fock states which are equivalent to ω. We can parameterize this set similar to the CAR case. There is however no counterpart of the spaces[m] $\mathfrak{h}$, and the only parameter that is needed is an element Z of the infinite dimensional open unit disk. This parameter Z characterizes the cyclic vector in Fock space which induces the corresponding state, and has properties similar to the operator T occurring in the Fermionic case. For any Fock state equivalent to ω there is a canonical choice of an automorphism in the small subgroup which transforms this state into ω. Note that, with $\mathfrak{h}$, also the ambiguity in the choice of this automorphism has disappeared.

To obtain the product decomposition of a quasi–free endomorphism ϱ_V as in (0.18), we must again solve the problem of extending a certain "partial" Fock state associated with ϱ_V to a proper Fock state ω_V. Recall that this problem was solved in the CAR case by "minimizing" the parameters T and $\mathfrak{h}$ in some sense. In particular, it is possible to define the operator T_V as a function of (the components of) V. However, we could not find a similar prescription for the operators Z (and presumably such a prescription does not exist). This complication is caused by the fact that Z has to fulfill an additional requirement related to the positivity of the state, viz. its norm has to be smaller than one. (There is no such restriction on the operators T; the admissible T form in fact a Hilbert space.) Instead we discovered a canonical method, based on spectral theory, how to extend the partial state *directly,* i.e. without having recourse to the parameter Z. This state extension ω_V is then used to *define* the parameter Z_V; remember that Z_V will be needed later for the construction of implementers. Having assigned a Fock state ω_V to ϱ_V so that $\omega_V \circ \varrho_V = \omega$ holds, there is then an unambiguous choice of an automorphism ϱ_U in

[m] The canonical anticommutation relations are symmetric in creation and annihilation operators, but the canonical commutation relations are not. Thus there exist endomorphisms of the CAR algebra which interchange creation and annihilation operators, and this is the origin of the spaces $\mathfrak{h}$. This possibility is absent in the CCR case.

the small subgroup such that $\omega_V \circ \varrho_U = \omega$, and the desired product decomposition is finally obtained as in (0.18).

As a corollary, we determine the connected components of the semigroup. It turns out that any subset of endomorphisms ϱ_V with $\operatorname{ind} V$ constant is connected.

Section 3.4. The construction of a complete set of implementers for a given endomorphism ϱ_V is performed. We start by defining Wick ordered Bosonic bilinear Hamiltonians, and Wick ordered exponentials thereof, on Fock space. These are in general quadratic forms, but determine densely defined operators under some conditions on the associated operators H. One can again compute commutation relations of Wick ordered exponentials with creation and annihilation operators, and select the operators H with the property that the corresponding Wick ordered exponential fulfills appropriate intertwiner relations with respect to ϱ_V. These operators H are in one–to–one correspondence with the operators Z parameterizing the Fock states which solve the extension problem from Section 3.3.

One implementer $\Psi_0(V)$ is then obtained as the (normalized) Wick ordered exponential of the bilinear Hamiltonian induced by the operator H_V which corresponds to Z_V. The value of $\Psi_0(V)$ on the Fock vacuum Ω is the cyclic vector associated with the state ω_V. To get a complete set of implementers, we choose a certain basis in a subspace $\mathfrak{k}_V$ of the kernel of the symplectic adjoint of V. The dimension of this subspace is $-\frac{1}{2} \operatorname{ind} V$. Polar decomposition of the representors of these basis elements yields a set of isometries which commute with each other and with the elements of the range of ϱ_V. One can then show that the operators $\Psi_\alpha(V)$ obtained by multiplying $\Psi_0(V)$ from the left with all possible ordered monomials in these isometries satisfy the relations of (an essential representation of) the Cuntz algebra $\mathcal{O}_\infty$. Since these isometries behave like (isometric parts of) creation operators with respect to the "vacuum" $\Psi_0(V)$, one finds that the Hilbert space $H(\varrho_V)$ is canonically isomorphic to the symmetric Fock space over $\mathfrak{k}_V$.

Our choice of implementers is again compatible with the product decomposition $\varrho_V = \varrho_U \varrho_W$. Roughly, the factor ϱ_U is responsible for the Wick ordered exponential, and the factor ϱ_W for the additional isometries. The completeness of implementers follows from the fact that the ranges of the implementers of the factor ϱ_W coincide with the invariant subspaces for the representation $\pi \circ \varrho_W$ described in Section 3.2. The expressions for the implementers are again not completely Wick ordered, because the additional isometries will in general contain an annihilation part. But strict Wick ordering would hide the inherent Fock space structure of $H(\varrho_V)$.

Section 4. The general theory of the implementation of quasi–free endomorphisms of the CAR and CCR algebras has been completely developed in Sections 2 and 3. The detailed knowledge of the structure of the implementing Hilbert spaces will now be used to gain insight into the charge structure of quasi–free endomorphisms and to determine the possible charge quantum numbers.

The setting will be tailored to the situation implied by the Doplicher–Roberts theory. That is, we will consider the CAR and CCR algebras as field algebras which contain the observables as fixed points under the action of a given global gauge group G. The gauge group will be assumed to consist of quasi–free automorphisms which leave a fixed Fock state, the vacuum state of the field algebra, invariant. Therefore G acts by usual second quantization on Fock space.

As a consequence, quasi–free endomorphisms are a priori endomorphisms of the field algebra. (Note that by the results of Doplicher and Roberts, localized endomorphisms of the observable algebra have a natural extension to the field algebra in terms of their implementers, by replacing the observable a in (0.13) by elements of the field algebra.) Thus we have to single out a subset of quasi–free endomorphisms which restrict to endomorphisms of the gauge invariant subalgebra.

The relevant subset is the semigroup of *gauge invariant endomorphisms,* i.e. of endomorphisms ϱ commuting with G, because these are precisely the endomorphisms whose implementing Hilbert spaces $H(\varrho)$ carry a representation of G. By the discussion following Eq. (0.11), the determination of the charge quantum numbers of ϱ is equivalent to the determination of the representation of G on $H(\varrho)$. This representation is further equivalent to the representation of G on the Hilbert space $H(\varrho)\Omega$, which is easier to handle than $H(\varrho)$ itself (here Ω denotes the Fock vacuum vector).

It should be noted that our assumptions are satisfied in models like the N–component Dirac field with gauge group $U(N)$.

Section 4.1. We compute the charge quantum numbers of gauge invariant quasi–free endomorphisms ϱ_V of the CAR algebra. It turns out that they are essentially determined by the subspaces $\mathfrak{h}_V$ and $\mathfrak{k}_V$ introduced in Section 2.

We have to study the behaviour of the implementers under gauge transformations. The values of the implementers of ϱ_V on the vacuum vector Ω have the following structure: To the left stands a product of partial isometries associated with the subspace $\mathfrak{k}_V$, followed by the "filled Dirac sea" corresponding to the finite dimensional subspace $\mathfrak{h}_V$, and finally the "pure creation part" of the Wick ordered exponential of the bilinear Hamiltonian of H_V, applied to Ω. We show that the subspaces $\mathfrak{h}_V$ and $\mathfrak{k}_V$ are representation spaces of G, and that the operators related to these subspaces transform linearly under G. The exponential term on the other hand is invariant. This follows from the fact that the operators T_V can be expressed as a function of the components of V, and confirms that the "minimal" choice of T_V made in Section 2.3 is a reasonable one.

The transformation law of implementers implies that the representation $\mathcal{U}_V$ of G on $H(\varrho_V)$ has the form

$$\mathcal{U}_V \simeq \det{}_{\mathfrak{h}_V} \otimes \Lambda_{\mathfrak{k}_V}. \tag{0.21}$$

Here $\det_{\mathfrak{h}_V}$ is the one–dimensional representation obtained by taking the determinant on $\mathfrak{h}_V$ of the Bogoliubov operators in G, and $\Lambda_{\mathfrak{k}_V}$ is the d_V–dimensional representation of G on the antisymmetric Fock space over $\mathfrak{k}_V$. (Recall that $H(\varrho_V)$ is isomorphic to this Fock space.)

It follows that $\mathcal{U}_V$, and hence ϱ_V, is reducible if $\operatorname{ind} V \neq 0$, because $\Lambda_{\mathfrak{k}_V}$ is reducible. The representation $\Lambda_{\mathfrak{k}_V}$ contains together with the representation on $\mathfrak{k}_V$ all the higher antisymmetric tensor powers thereof. The "least reducible" case is obtained if $\mathfrak{k}_V$ is irreducible.

A special case worth mentioning is the case $G = \mathbb{T}$ and $\operatorname{ind} V = 0$, i.e. the case of the *restricted unitary group.* It is well–known from the work on the external field problem that the charge of elements of the restricted unitary group is given by a certain Fredholm index $\operatorname{ind} V_{++}$ (which has nothing to do with the index of V). This fact can be easily derived from our much more general result: The factor $\Lambda_{\mathfrak{k}_V}$ in (0.21) becomes trivial, and the factor $\det_{\mathfrak{h}_V}$ yields the index of V_{++}.

We study the question which representations of G can possibly occur on the subspaces $\mathfrak{h}_V$ and $\mathfrak{k}_V$. In typical cases, any representation of G that is realized on Fock space appears as a subrepresentation on some $H(\varrho_V)$. Then we compare our findings with the generic superselection structure of quantum field theory. The semigroup of gauge invariant endomorphisms is not closed under taking subobjects or direct sums. It is closed under taking conjugates if one makes natural assumptions on the action of G. Under these assumptions, one can assign to each gauge invariant Bogoliubov operator V another such operator V^c such that $\mathfrak{h}_{V^c}$ and $\mathfrak{k}_{V^c}$

are antiunitarily equivalent to $\mathfrak{h}_V$ and $\mathfrak{k}_V$, so that the representation $\mathcal{U}_{V^c}$ is unitarily equivalent to the complex conjugate representation of $\mathcal{U}_V$.

Finally we give an explicit example of a localized implementable gauge invariant endomorphism with statistics dimension 2^N of the free massless N–component Dirac field in two dimensions. The construction rests on the use of "local" Fourier bases for the chiral components, and is in this respect similar to the known examples of localized endomorphisms in conformal field theory.

Section 4.2. Here we compute the charge quantum numbers of gauge invariant quasi–free endomorphisms ϱ_V of the CCR algebra. Since the spaces $\mathfrak{h}_V$ are absent in the CCR case, the charge quantum numbers are entirely determined by the representation of G on $\mathfrak{k}_V$.

We derive the transformation law of the implementers of ϱ_V. Recall that the implementers are obtained by multiplying the distinguished implementer $\Psi_0(V)$ from the left with certain isometries which are associated with the space $\mathfrak{k}_V$. $\Psi_0(V)$ itself is a Wick ordered exponential of a bilinear Hamiltonian which is characterized by the operator Z_V. Similar to the CAR case, this Wick ordered exponential is gauge invariant, and we take this as a confirmation that we gave the "correct" definition of Z_V in Section 3.3. Though there is no explicit formula for Z_V in terms of V, it is still true that Z_V is in some sense a function of V, and this suffices to prove the invariance of $\Psi_0(V)$.

The subspace $\mathfrak{k}_V$ is again G–invariant. But in contrast to the CAR case, the isometries associated with $\mathfrak{k}_V$ do *not* transform linearly under G. One can however show that they obey a linear transform law when restricted to the range of $\Psi_0(V)$, and that is essentially all we need.

We conclude that the representation $\mathcal{U}_V$ of the gauge group G on the Hilbert space $H(\varrho_V)$ is unitarily equivalent to the representation on the symmetric Fock space over $\mathfrak{k}_V$ induced by the representation on $\mathfrak{k}_V$. Thus one can say that the endomorphisms in the CCR case are even "more" reducible than in the CAR case, because $\mathcal{U}_V$ (and hence ϱ_V) always splits into an *infinite* direct sum of irreducibles if $\operatorname{ind} V \neq 0$. Another consequence is that automorphisms ($\operatorname{ind} V = 0$) carry no charge.

Finally, we investigate which representations can be realized on $\mathfrak{k}_V$, and under which conditions charge conjugation is ensured. The remarks made in this context in the Fermionic case apply here as well.