Endomorphisms of CAR and CCR Algebras

For the sake of completeness, we would like to include a brief survey on the Cuntz algebras before discussing the main subject of our thesis. The content of Section 1 is in no way essential for an understanding of Sections 2 – 4, so that this section may be skipped at a first reading of the text.

We would further like to remark that proofs of statements in Sections 2 and 3 that already appeared in our papers [Bin95, Bin97, Bin98] will occasionally only be sketched here.

1. The Cuntz Algebras $\mathcal{O}(H)$

The universal C^*–algebras $\mathcal{O}(H)$ generated by separable complex Hilbert spaces H have been introduced by Cuntz in 1977 [Cun77]. (Hilbert spaces inside operator algebras had before been considered by Doplicher and Roberts [DR72, Rob76a]). They provided new examples of C^*–algebras with unexpected properties, but also play an important role in the general structure theory of C^*–algebras. See [EK98] for an account of $\mathcal{O}(H)$ in a textbook.

Let H be a separable complex Hilbert space of dimension[n] ≥ 2, with orthonormal basis $\{s_j\}_{j\in J}$. The unital *–algebra generated by the elements of H, with relations (cf. (0.5))

$$s^*t = \langle s,t\rangle \mathbf{1}, \quad s,t\in H$$

and

$$\sum_{j\in J} s_j s_j^* = \mathbf{1}$$

(resp. $\sum_{j\in J_0} s_j s_j^* < \mathbf{1}$ for any finite subset $J_0 \subset J$, should H be infinite dimensional), possesses a unique C^*–norm. Its completion in this norm, the *Cuntz algebra* $\mathcal{O}(H)$, is a separable, simple, nuclear C^*–algebra [Cun77]. It can be constructed from the full Fock space $\mathcal{F}(H)$ over H as follows [Eva80b]. Let $\tau(s)$ be the operator of norm $\|s\|$ which acts on $\mathcal{F}(H)$ by tensoring on the left with $s\in H$. Then

$$\tau(s)^*\tau(t) = \langle s,t\rangle\mathbf{1}, \qquad \sum_{j\in J}\tau(s_j)\tau(s_j)^* = \mathbf{1} - p_\Omega \quad \text{(strongly)}$$

where p_Ω denotes the rank–one projection onto the Fock vacuum Ω. The C^*–algebra $\mathcal{T}(H)$ generated by all $\tau(s)$, $s\in H$, is the *Cuntz–Toeplitz algebra* over H. If H is finite dimensional, then $\mathcal{T}(H)$ contains the ideal generated by p_Ω, the compact operators on $\mathcal{F}(H)$. In this case $\mathcal{O}(H)$ is isomorphic to the quotient of $\mathcal{T}(H)$ by this ideal. If H is infinite dimensional, then $\mathcal{T}(H)$ is simple, and in fact isomorphic to $\mathcal{O}(H)$.

The isomorphism class of $\mathcal{O}(H)$ depends only on the dimension of H, thus it is also customary to write $\mathcal{O}_n$ instead of $\mathcal{O}(H)$ ($n \equiv \dim H$). One has the following

[n] If $\dim H = 1$, then the C^*–algebra defined by these relations is the Abelian algebra generated by a single unitary operator s, isomorphic to the algebra of continuous functions on the spectrum of s. The Cuntz–Toeplitz algebra $\mathcal{T}(H)$ introduced below then reduces to the C^*–algebra generated by the unilateral shift, the Toeplitz extension of $C(\mathbb{T})$ by the compacts [Cob67].

inclusions, since e.g. the elements $s_1^2, s_1s_2, \dots, s_1s_n, s_2, \dots, s_n$ generate a copy of $\mathcal{O}_{2n-1}$ in $\mathcal{O}_n$

$$\mathcal{O}_n \supset \mathcal{O}_{n+m(n-1)}, \qquad m \le \infty.$$

In particular, $\mathcal{O}_2$ contains copies of all other $\mathcal{O}_n$. Algebras $\mathcal{O}_m$, $\mathcal{O}_n$ with $m \neq n$ cannot be isomorphic as their K_0–groups differ

$$K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1), \qquad K_0(\mathcal{O}_\infty) = \mathbb{Z}.$$

Any orthogonal projection in $\mathcal{O}(H)$ is equivalent to a projection of the form $\sum_{\text{finite}} s_j s_j^*$ or $\mathbf{1} - \sum_{\text{finite}} s_j s_j^*$. The unitary group $\mathcal{U}(\mathcal{O}(H))$ of $\mathcal{O}(H)$ is connected, thus one has $K_1(\mathcal{O}(H)) = \{0\}$ [Cun81]. If $\dim H$ is finite, $\mathcal{U}(\mathcal{O}(H))$ is homeomorphic to the semigroup of all unital *–endomorphisms of $\mathcal{O}(H)$: given $u \in \mathcal{U}(\mathcal{O}(H))$, there is (by the universality of $\mathcal{O}(H)$) a unique endomorphism ϱ such that $\varrho(s) = us$, $s \in H \subset \mathcal{O}(H)$; conversely, an endomorphism ϱ determines the unitary $u = \sum_{j \in J} \varrho(s_j) s_j^* \in \mathcal{O}(H)$ [Cun80].

Special examples of endomorphisms of $\mathcal{O}(H)$ have been given by Izumi [Izu93], using fusion rules of sectors (cf. the definition (0.6) of direct sums of endomorphisms). Another class of endomorphisms of $\mathcal{O}(H)$ are the *quasi–free endomorphisms,* namely those which leave H invariant. Non–surjective quasi–free endomorphisms exist only in the case $\dim H = \infty$. Quasi–free *auto*morphisms are extensions of unitary operators on H. As a first step towards their duality theory for compact groups, Doplicher and Roberts studied the quasi–free action of a closed subgroup G of the special unitary group of H ($\dim H < \infty$) on $\mathcal{O}(H)$ [DR87], and in particular the relation between $\mathcal{O}(H)$ and its fixed point subalgebra $\mathcal{O}(H)^G$ (serving as a prototype for the relation between field algebra and observable algebra). They showed that $\mathcal{O}(H)^G$ is simple, with trivial relative commutant in $\mathcal{O}(H)$, and that any automorphism of $\mathcal{O}(H)$ which acts trivially on $\mathcal{O}(H)^G$ is given by an element of G. Moreover, if $\varrho_H(a) = \sum s_j a s_j^*$ is the endomorphism of $\mathcal{O}(H)$ induced by H, then the G–invariant intertwiners between powers of $\varrho_H|_{\mathcal{O}(H)^G}$ are the same as the intertwiners between tensor products of the representation of G (the tensor powers of H are canonically embedded in $\mathcal{O}(H)$), and these intertwiners generate $\mathcal{O}(H)^G$. These results have been extended to the case of Hopf algebra actions by Cuntz [Cun91].

An important tool in the study of $\mathcal{O}(H)$ is to consider $\mathcal{O}(H)^{\mathbb{T}}$, the fixed point algebra under the quasi–free action of the circle group $\mathbb{T}$. As a Banach space, $\mathcal{O}(H)^{\mathbb{T}}$ is generated by monomials $s_{i_1} \cdots s_{i_m} s_{j_m}^* \cdots s_{j_1}^*$ (same number of s and s^*). If $n \equiv \dim H$ is finite, then the monomials of the above form, with m fixed, constitute a system of $n^m \times n^m$ matrix units, hence span an algebra isomorphic to $M(n^m, \mathbb{C})$. Since the embedding $s_{i_1} \cdots s_{i_m} s_{j_m}^* \cdots s_{j_1}^* \mapsto \sum_j s_{i_1} \cdots s_{i_m} s_j s_j^* s_{j_m}^* \cdots s_{j_1}^*$ corresponds to the usual embedding $M(n^m, \mathbb{C}) \to M(n^{m+1}, \mathbb{C})$, $A \mapsto A \otimes \mathbf{1}_n$, it follows that $\mathcal{O}(H)^{\mathbb{T}}$ is an UHF algebra of type n^∞, canonically isomorphic to the infinite tensor product of copies of $M(n, \mathbb{C})$. In particular, ${\mathcal{O}_2}^{\mathbb{T}}$ is isomorphic to the CAR algebra. If $\dim H$ is infinite, then the monomials with fixed length $m \ge 1$ generate an algebra isomorphic to the compact operators $\mathcal{J}_\infty(H)$ on H, and $\mathcal{O}(H)^{\mathbb{T}}$ is isomorphic to a non–simple AF–subalgebra of the infinite tensor product of copies (with unit adjoined) of $\mathcal{J}_\infty(H)$ [Cun77].

The representation of $\mathcal{O}(H)^{\mathbb{T}}$ as (a subalgebra of) an infinite tensor product allows to define *quasi–free states* over $\mathcal{O}(H)$ as gauge invariant extensions of product states to $\mathcal{O}(H)$ [Eva80b], by utilizing the canonical conditional expectation $\mathcal{O}(H) \to \mathcal{O}(H)^{\mathbb{T}}$. Specifically, any sequence of positive trace class operators $\{K_j\}$ on H with

$\operatorname{tr} K_j = 1$ (resp. $\operatorname{tr} K_j \leq 1$ if $\dim H = \infty$) yields a quasi–free state $\omega_{\{K_j\}}$

$$\omega_{\{K_j\}}(f_1 \cdots f_k g_l^* \cdots g_1^*) = \delta_{kl} \sum_{j=1}^{k} \langle f_j, K_j g_j \rangle, \quad f_j, g_j \in H.$$

Quasi–free automorphisms restrict to product automorphisms on $\mathcal{O}(H)^{\mathbb{T}}$, and ϱ_H restricts to the unilateral shift.

Quasi–free automorphisms and quasi–free states of $\mathcal{O}(H)$ have been studied by Evans et al. [Eva80b, ACE84]. Among their results are conditions for existence and uniqueness of KMS and ground states for one–parameter groups of quasi–free automorphisms, characterizations of primary quasi–free states, and criteria for implementability of quasi–free automorphisms in quasi–free states. For instance, they showed that $\mathcal{O}(H)$ has no inner quasi–free automorphisms besides the identity. Equivalence of quasi–free states over $\mathcal{O}_\infty$ has been studied by Laca [Lac93b].

Representations of $\mathcal{O}(H)$ are closely related to endomorphisms of $\mathfrak{B}(\mathcal{H})$. A nondegenerate representation of $\mathcal{O}(H)$ on a separable Hilbert space $\mathcal{H}$ gives rise, via the endomorphism ϱ_H induced by H, to a unital *–endomorphism of $\mathfrak{B}(\mathcal{H})$ (if $\dim H = \infty$, the representation π of $\mathcal{O}(H)$ has to be *essential* [Lac93a], i.e. $\sum \pi(s_j)\pi(s_j)^* = \mathbf{1}$ in the strong topology). Conversely, since $\mathfrak{B}(\mathcal{H})$ has only one normal representation up to quasi–equivalence, each unital *–endomorphism of $\mathfrak{B}(\mathcal{H})$ is inner. The representation of $\mathcal{O}(H)$ corresponding to an endomorphism of $\mathfrak{B}(\mathcal{H})$ is, however, only determined modulo quasi–free automorphisms. To illustrate how properties of representations of $\mathcal{O}(H)$ are linked to properties of endomorphisms of $\mathfrak{B}(\mathcal{H})$, one has e.g. that the commutant of an essential representation π of $\mathcal{O}(H)$ is equal to the algebra of fixed points of the corresponding endomorphism ϱ_π, and that the commutant of the restriction of π to $\mathcal{O}(H)^{\mathbb{T}}$ is equal to the intersection of the ranges of all powers of ϱ_π. Thus π is irreducible if and only if ϱ_π is ergodic, and $\pi|_{\mathcal{O}(H)^{\mathbb{T}}}$ is irreducible if and only if ϱ_π is a shift [Lac93a, BJP96]. A classification of certain ergodic endomorphisms of $\mathfrak{B}(\mathcal{H})$ up to conjugacy has been achieved by Laca and Fowler, by describing all extensions of pure states over $\mathcal{O}(H)^{\mathbb{T}}$ to $\mathcal{O}(H)$ [FL97].

Interest in the connection between representations of $\mathcal{O}(H)$ and endomorphisms of $\mathfrak{B}(\mathcal{H})$ arose from the theory of E_0–semigroups of endomorphisms which was initiated by Powers [Pow88], with contributions by Arveson, Bratteli, Jørgensen, Laca, Price, Robinson and others (see the review [Arv94] and references therein). One has an index theory for E_0–semigroups which gives partial results for the classification of E_0–semigroups up to outer conjugacy. The basic examples of E_0–semigroups are semigroups of Bogoliubov endomorphisms of CAR and CCR algebras.

Representations of $\mathcal{O}(H)$ related to wavelet theory have been studied by Bratteli et. al. (see [BJ96, BJKW97] and references therein). They obtained the decomposition of a special class of such representations (and of their restrictions to $\mathcal{O}(H)^{\mathbb{T}}$) into irreducibles via number theory.

The Cuntz algebras can be regarded as elementary building blocks of infinite C^*–algebras (algebras containing non–unitary isometries). Any unital simple infinite C^*–algebra contains copies of all $\mathcal{O}_n$ as subquotients (quotients of subalgebras) [Cun77]. By Kirchberg's results [Kir94a, Kir94b, Kir95, KP97], any separable unital exact C^*–algebra is isomorphic to a subalgebra of $\mathcal{O}_2$ (and to a subquotient of the CAR algebra); any separable unital nuclear (s.u.n.) C^*–algebra is isomorphic to the range of a conditional expectation of $\mathcal{O}_2$; and any simple s.u.n. C^*–algebra which contains a central sequence of copies of $\mathcal{O}_2$ is itself isomorphic to $\mathcal{O}_2$ (cf. footnote (h) on page 6). An important open question concerning Elliott's classification program for nuclear C^*–algebras [Ell95, EK98] is whether any simple s.u.n. C^*–algebra with vanishing K_0 and K_1–groups is already isomorphic to $\mathcal{O}_2$. Inductive limits of matrix algebras over $\mathcal{O}_n$ have been classified by Rørdam [Rør93].

2. Quasi–free Endomorphisms of the CAR Algebra

Basics on the CAR algebra can be found in the textbooks [BR81, EK98, PR94] and will be cited here only as far as necessary. Because of its convenience for handling Bogoliubov transformations, Araki's formalism of selfdual CAR algebras [Ara68, Ara71, Ara87] will be used throughout. A selfdual CAR algebra is simply the complexification of a real Clifford algebra.

Most of the results in this section are contained in [Bin95]. However, it has become clear in the meantime that a different arrangement of the material would be desirable. Thus the present setup deviates from the one in [Bin95], most significantly in Section 2.3. Minor improvements can be found throughout the text.

2.1. Quasi–free endomorphisms and quasi–free states. Let $\mathcal{K}$ be an infinite dimensional separable complex Hilbert space, equipped with a complex conjugation $f \mapsto f^*$. The *(selfdual) CAR algebra* $\mathfrak{C}(\mathcal{K})$ over $\mathcal{K}$ is the unique (simple) C^*–algebra generated by $\mathbf{1}$ and the elements of $\mathcal{K}$, subject to the anticommutation relation

$$\{f^*, g\} \equiv f^*g + gf^* = \langle f, g\rangle \mathbf{1}, \qquad f, g \in \mathcal{K}.$$

$\mathfrak{C}(\mathcal{K})$ is the complexified (or C^*–) Clifford algebra over the real Hilbert space

$$\operatorname{Re}\mathcal{K} \equiv \{f \in \mathcal{K} \mid f^* = f\},$$

a UHF algebra of type 2^∞. The C^*–norm on $\mathfrak{C}(\mathcal{K})$ extends the norm on $\operatorname{Re}\mathcal{K}$ (but not the norm on $\mathcal{K}$) up to a factor $\sqrt{2}$. $\mathcal{K}$ will henceforth be viewed as a subspace of $\mathfrak{C}(\mathcal{K})$.

Quasi–free endomorphisms (or *Bogoliubov transformations*) are precisely the unital $*$–endomorphisms of $\mathfrak{C}(\mathcal{K})$ that leave $\mathcal{K}$ invariant. Put differently, every isometry V on $\mathcal{K}$ that commutes with complex conjugation (and therefore restricts to a real–linear isometry of $\operatorname{Re}\mathcal{K}$) extends to a unital isometric $*$–endomorphism ϱ_V of $\mathfrak{C}(\mathcal{K})$:

$$\varrho_V(f) = Vf, \qquad f \in \mathcal{K}.$$

Such isometries V are called *Bogoliubov operators*, and the semigroup of Bogoliubov operators is denoted by

$$\mathcal{I}(\mathcal{K}) \equiv \{V \in \mathfrak{B}(\mathcal{K}) \mid V^*V = \mathbf{1},\ \overline{V} = V\} \tag{2.1}$$

where the bar indicates complex conjugation

$$\overline{A}f \equiv A(f^*)^*, \qquad f \in \mathcal{K} \tag{2.2}$$

for bounded linear operators $A \in \mathfrak{B}(\mathcal{K})$. The map $V \mapsto \varrho_V$ is a unital isomorphism from $\mathcal{I}(\mathcal{K})$ onto the semigroup of quasi–free endomorphisms; for fixed $a \in \mathfrak{C}(\mathcal{K})$, the map $V \mapsto \varrho_V(a)$ is continuous with respect to the strong topology on $\mathcal{I}(\mathcal{K})$ and the norm topology on $\mathfrak{C}(\mathcal{K})$.

Let $V \in \mathcal{I}(\mathcal{K})$. Since $\operatorname{ran} V$ is closed and $\ker V = \{0\}$, V and V^* are semi–Fredholm operators [Kat66] and have well–defined Fredholm indices. The map

$$\mathcal{I}(\mathcal{K}) \to \mathbb{N} \cup \{\infty\}, \quad V \mapsto -\operatorname{ind} V \equiv \dim\ker V^*$$

is a surjective homomorphism of semigroups ($0 \in \mathbb{N}$ by convention). Additivity of the Fredholm index follows in this special case simply from $\ker(VW)^* = \ker V^* \oplus V(\ker W^*)$. The semigroup $\mathcal{I}(\mathcal{K})$ is the disjoint union of subsets

$$\mathcal{I}(\mathcal{K}) = \bigcup_{n \in \mathbb{N}\cup\{\infty\}} \mathcal{I}^n(\mathcal{K}), \qquad \mathcal{I}^n(\mathcal{K}) \equiv \{V \in \mathcal{I}(\mathcal{K}) \mid \operatorname{ind} V = -n\}. \tag{2.3}$$

Note that ϱ_V is an automorphism if and only if V belongs to $\mathcal{I}^0(\mathcal{K})$, the group of unitary Bogoliubov operators. $\mathcal{I}^0(\mathcal{K})$ acts on $\mathcal{I}(\mathcal{K})$ by left multiplication, the orbits of this action are the sets $\mathcal{I}^n(\mathcal{K})$, and the stabilizer of $V \in \mathcal{I}^n(\mathcal{K})$ is isomorphic to $O(n)$ (the orthogonal group of an n–dimensional real Hilbert space).

The quasi–free automorphism $\varrho_{-\mathbf{1}}$ induces a $\mathbb{Z}_2$–grading on $\mathfrak{C}(\mathcal{K})$:

$$\begin{aligned}&\mathfrak{C}(\mathcal{K}) = \mathfrak{C}(\mathcal{K})_0 \oplus \mathfrak{C}(\mathcal{K})_1,\\ &\mathfrak{C}(\mathcal{K})_g \cdot \mathfrak{C}(\mathcal{K})_{g'} \subset \mathfrak{C}(\mathcal{K})_{g+g'},\\ &\mathfrak{C}(\mathcal{K})_g \equiv \{a \mid \varrho_{-\mathbf{1}}(a) = (-1)^g a\}, \qquad g, g' \in \mathbb{Z}_2 = \{0,1\}.\end{aligned}$$

Doplicher and Powers proved that the *even subalgebra* $\mathfrak{C}(\mathcal{K})_0$ is a simple C^*–algebra [DP68]. Størmer sharpened this result by showing that $\mathfrak{C}(\mathcal{K})_0$ is UHF of type 2^∞, hence *-isomorphic to $\mathfrak{C}(\mathcal{K})$ itself [Stø70]. We found that any $V \in \mathcal{I}^1(\mathcal{K})$ gives rise to an isomorphism from $\mathfrak{C}(\mathcal{K})$ onto $\mathfrak{C}(\mathcal{K})_0$ in the following way [Bin97]. Let f_V be the unique (up to a sign) unitary skew–adjoint element in $\ker V^* \subset \mathfrak{C}(\mathcal{K})$, and let $u_V \equiv \frac{1}{\sqrt{2}}(\mathbf{1} + f_V)$. Then u_V is unitary, $u_V^2 = f_V$, and the map

$$\sigma_V : a \mapsto u_V \varrho_V(a) u_V{}^* \tag{2.4}$$

defines a unital *-isomorphism from $\mathfrak{C}(\mathcal{K})$ onto $\mathfrak{C}(\mathcal{K})_0$. σ_V acts on even elements like ϱ_V, and on odd elements like ϱ_V followed by left multiplication with f_V. A similar construction has been given independently by [Rob93].

Next we describe the set of states we are interested in. A state ω over $\mathfrak{C}(\mathcal{K})$ is called *quasi–free* if it is invariant under $\varrho_{-\mathbf{1}}$ (i.e. it vanishes on $\mathfrak{C}(\mathcal{K})_1$), and if its even n–point functions have the form [Ara71]

$$\omega(f_1 \cdots f_{2m}) = (-1)^{\frac{m(m-1)}{2}} \sum_\sigma \operatorname{sign}\sigma \cdot \omega\Big(f_{\sigma(1)} f_{\sigma(m+1)}\Big) \cdots \omega\Big(f_{\sigma(m)} f_{\sigma(2m)}\Big)$$

where the sum runs over all permutations σ satisfying $\sigma(1) < \ldots < \sigma(m)$ and $\sigma(j) < \sigma(j+m)$, $j = 1, \ldots, m$. Therefore quasi–free states are completely determined by their two–point functions, and one has a bijection

$$S \mapsto \omega_S, \quad \omega_S(f^* g) = \langle f, Sg\rangle$$

between the convex set

$$\mathfrak{Q}(\mathcal{K}) \equiv \{S \in \mathfrak{B}(\mathcal{K}) \mid 0 \le S \le \mathbf{1},\ \overline{S} = \mathbf{1} - S\}$$

and the set of quasi–free states. A (non–trivial) convex combination of two distinct quasi–free states $\omega_S, \omega_{S'}$ is quasi–free if and only if $S - S'$ has rank two [Wol75]. Quasi–free endomorphisms act from the right on quasi–free states according to

$$\omega_S \circ \varrho_V = \omega_{V^* S V}. \tag{2.5}$$

Any *–automorphism which maps the set of quasi–free states onto itself is known to be quasi–free [Wol75].

Projections in $\mathfrak{Q}(\mathcal{K})$ are called *basis projections*, and the corresponding states are called *Fock states;* the latter are precisely the *pure* quasi–free states [MRT69]. The group of quasi–free automorphisms acts transitively on the set of Fock states, because $\mathcal{I}^0(\mathcal{K})$ acts transitively on the set of basis projections. Note that for a basis projection P, the complementary (basis) projection is simply given by $\overline{P}$. Since $\omega_P(f^*f) = 0$ if $f \in \overline{P}(\mathcal{K})$, the elements of $\overline{P}(\mathcal{K})$ (resp. $P(\mathcal{K})$) correspond to annihilation (resp. creation) operators in the state ω_P. A (faithful and irreducible) GNS representation π_P for ω_P is given by

$$\pi_P(f) \equiv a(Pf)^* + a(P(f^*)) \tag{2.6}$$

on the antisymmetric Fock space $\mathcal{F}_a(P(\mathcal{K}))$ over $P(\mathcal{K})$, with the usual Fock vacuum Ω_P as cyclic vector and annihilation operators $a(g)$, $g \in P(\mathcal{K})$. In a Fock representation π_P, a quasi–free endomorphism ϱ_V induces the transformation

$$a(g) \mapsto a(PVPg) + a(PV\overline{P}(g^*))^*, \quad g \in P(\mathcal{K}),$$

which shows the connection to the (state–dependent) description of Bogoliubov transformations by pairs of operators as preferred by some authors. GNS representations of arbitrary quasi–free states can be obtained, by the "doubling procedure" of Powers and Størmer [PS70, Ara71], as restrictions of Fock representations of an enlarged CAR algebra.

The grading automorphism $\varrho_{-\mathbf{1}}$ is not inner in $\mathfrak{C}(\mathcal{K})$. However, since every quasi–free state is invariant under this automorphism, $\varrho_{-\mathbf{1}}$ is canonically implemented in any quasi–free state. Let P be a basis projection, and let $\Psi_P(-\mathbf{1})$ be the self–adjoint unitary on Fock space given by

$$\Psi_P(-\mathbf{1})\pi_P(a)\Omega_P = \pi_P(\varrho_{-\mathbf{1}}(a))\Omega_P, \qquad a \in \mathfrak{C}(\mathcal{K}). \tag{2.7}$$

Then $\Theta_P(-\mathbf{1}) \equiv \frac{1}{\sqrt{2}}\big(\mathbf{1} - i\Psi_P(-\mathbf{1})\big)$ is also unitary. Define a new representation ψ_P of $\mathfrak{C}(\mathcal{K})$ on $\mathcal{F}_a(P(\mathcal{K}))$, which is unitarily equivalent to π_P

$$\psi_P(a) \equiv \Theta_P(-\mathbf{1})\pi_P(a)\Theta_P(-\mathbf{1})^*. \tag{2.8}$$

We will call ψ_P the *twisted Fock representation* induced by P. Then one has

$$\psi_P|_{\mathfrak{C}(\mathcal{K})_0} = \pi_P|_{\mathfrak{C}(\mathcal{K})_0}, \tag{2.9}$$

$$\psi_P(a) = i\pi_P(a)\Psi_P(-\mathbf{1}), \qquad a \in \mathfrak{C}(\mathcal{K})_1, \tag{2.10}$$

$$[\pi_P(f)^*, \psi_P(g)] = i\langle f, g\rangle \Psi_P(-\mathbf{1}), \qquad f, g \in \mathcal{K}, \tag{2.11}$$

and, as shown by Foit [Foi83], *twisted duality* holds for all *–invariant subspaces $\mathcal{H} \subset \mathcal{K}$, an adaptation of Haag duality (see p. 5) to Fermi fields:

$$\pi_P(\mathfrak{C}(\mathcal{H}))' = \psi_P(\mathfrak{C}(\mathcal{H}^\perp))''. \tag{2.12}$$

Here $\mathfrak{C}(\mathcal{H})$ is the C^*–subalgebra of $\mathfrak{C}(\mathcal{K})$ generated by the elements of $\mathcal{H}$, and similar for $\mathfrak{C}(\mathcal{H}^\perp)$.

Given a basis projection P, a state over $\mathfrak{C}(\mathcal{K})$ is said to be ($\mathbb{T}$–) *gauge invariant* if it is invariant under the one–parameter group of quasi–free automorphisms $(\varrho_{U_\lambda})_{\lambda\in\mathbb{R}}$ with

$$U_\lambda \equiv e^{i\lambda}P + e^{-i\lambda}\overline{P} \in \mathcal{I}^0(\mathcal{K}). \tag{2.13}$$

A quasi–free state ω_S is gauge invariant if and only if $[P, S] = 0$.

The so–called *central state* $\omega_{1/2}$ [SS64, Man70, Ara71] is the unique tracial state over $\mathfrak{C}(\mathcal{K})$. By uniqueness, $\omega_{1/2}$ is invariant under all unital *–endomorphisms of $\mathfrak{C}(\mathcal{K})$. It can be used to define conditional expectations on $\mathfrak{C}(\mathcal{K})$. If V is a Bogoliubov operator with $-\operatorname{ind} V < \infty$, then there is a unique minimal (faithful) conditional expectation E_V from $\mathfrak{C}(\mathcal{K})$ onto $\varrho_V(\mathfrak{C}(\mathcal{K}))$, determined by $E_V(ab) = a\omega_{1/2}(b)$ if $a \in \mathfrak{C}(\operatorname{ran} V) = \varrho_V(\mathfrak{C}(\mathcal{K}))$, $b \in \mathfrak{C}(\ker V^*)$ (see [Bin95]). Using an explicit "quasi–basis" for E_V, we computed the Watatani index [Wat90] of E_V in [Bin95]

$$\operatorname{ind} E_V = 2^{-\operatorname{ind} V}.$$

Thus we found the fundamental index formula

$$d_V \equiv [\mathfrak{C}(\mathcal{K}) : \varrho_V(\mathfrak{C}(\mathcal{K}))]^{\frac{1}{2}} = 2^{M_V}, \qquad M_V \equiv -\frac{1}{2}\operatorname{ind} V \tag{2.14}$$

which relates the Fredholm index of V to the Watatani index $[\mathfrak{C}(\mathcal{K}) : \varrho_V(\mathfrak{C}(\mathcal{K}))]$ of ϱ_V. We shall take (2.14) as the definition of the numbers d_V and M_V also in the case $-\operatorname{ind} V = \infty$. d_V may be regarded as the *statistics dimension* of the quasi–free endomorphism ϱ_V, cf. p. 8. One obviously has $M_{VV'} = M_V + M_{V'}$ and $d_{VV'} = d_V d_{V'}$. Also note that the conditional expectations E_V allow to define *left inverses* (see [Haa96]) $\varrho_V^{-1} \circ E_V$ for quasi–free endomorphisms. Explicitly, for a quasi–free endomorphism ϱ_V, a left inverse ϕ_V is given by

$$\phi_V(ab) \equiv \varrho_V^{-1}(a)\omega_{1/2}(b) \qquad \text{if } a \in \mathfrak{C}(\operatorname{ran} V),\ b \in \mathfrak{C}(\ker V^*). \tag{2.15}$$

Not surprisingly, the conditional expectations from $\mathfrak{C}(\mathcal{K})$ onto $\mathfrak{C}(\mathcal{K})_0$ that are obtained in this way from the isomorphisms σ_V of (2.4) ($d_V = \sqrt{2}$) are equal to the mean over the action of $\mathbb{Z}_2$.

The von Neumann algebra generated by $\mathfrak{C}(\mathcal{K})$ in the central state $\omega_{1/2}$ is the hyperfinite II_1 factor. In general, the types of quasi–free factor states can be computed from spectral properties of the associated operators $S \in \mathfrak{Q}(\mathcal{K})$. See [MY95] for a complete classification (extending earlier results in [Del68, Rid68, PS70]), including the fine classification of type III.

Of uppermost importance for our study of implementable quasi–free endomorphisms are the criteria for quasi–equivalence of quasi–free states. First results in this direction were obtained by Shale and Stinespring [SS65]. These authors showed that a quasi–free automorphism ϱ_U, $U \in \mathcal{I}^0(\mathcal{K})$, is unitarily implementable in a Fock representation π_P if and only if

$$[P, U] \text{ is Hilbert–Schmidt.} \tag{2.16}$$

Equivalently, two Fock states $\omega_P, \omega_{P'}$ (i.e. their GNS representations) are unitarily equivalent if and only if $P - P'$ is Hilbert–Schmidt. A sufficient condition for quasi–equivalence of gauge invariant quasi–free states followed from the work of Dell'Antonio [Del68] and Rideau [Rid68]. Powers and Størmer proved this condition also to be necessary [PS70], and Araki extended the result to arbitrary quasi–free states [Ara71]. One has

$$\omega_S \approx \omega_{S'} \iff S^{1/2} - S'^{1/2} \text{ is Hilbert–Schmidt.} \tag{2.17}$$

Here "$\approx$" means "quasi–equivalent". It has been observed by Powers [Pow87] that this criterion can be simplified if one of the operators S, S' is a projection. Namely, if P is a basis projection, then

$$\omega_P \approx \omega_S \iff \operatorname{tr} \overline{P} S \overline{P} < \infty. \tag{2.18}$$

Quasi–equivalence of the restrictions of gauge invariant quasi–free states to gauge invariant CAR algebras $\mathfrak{C}(\mathcal{K})^G$ (now with respect to the quasi–free action of an arbitrary compact group G) has been investigated by Matsui [Mat87b], extending results of Araki and Evans for the case $G = \mathbb{Z}_2$ [AE83], and of Baker and Powers for the groups $\mathbb{Z}_2$, $\mathbb{T}$ and $SU(2)$ [BP83b, BP83a]. If P, P' are basis projections commuting with the action of G, then one has [Mat87b]

$$\omega_P|_{\mathfrak{C}(\mathcal{K})^G} \simeq \omega_{P'}|_{\mathfrak{C}(\mathcal{K})^G} \iff P - P' \text{ is Hilbert–Schmidt,} \tag{2.19}$$

$$\det{}_{P(\mathcal{K}) \cap \overline{P'}(\mathcal{K})}(g) = 1 \text{ for all } g \in G. \tag{2.20}$$

Here "$\simeq$" means "unitarily equivalent", and $P(\mathcal{K}) \cap \overline{P'}(\mathcal{K})$ is a finite dimensional G–invariant subspace if $P - P'$ is Hilbert–Schmidt. The condition on the determinant is a generalization of the $\mathbb{Z}_2$–index of Araki and Evans [AE83, Ara87, EK98]. If the GNS representations of ω_P and $\omega_{P'}$ are both realized on $\mathcal{F}_a(P(\mathcal{K}))$ (this is possible under the Hilbert–Schmidt condition), then the transformation law of the cyclic vector $\Omega_{P'}$ is exactly given by the character $\det_{P(\mathcal{K}) \cap \overline{P'}(\mathcal{K})}(g)$. We will rediscover this character in Section 4.1. On the other hand, if $S, S' \in \mathfrak{Q}(\mathcal{K})$ commute with the action of G and have trivial kernels, then one has [Mat87b]

$$\omega_S|_{\mathfrak{C}(\mathcal{K})^G} \approx \omega_{S'}|_{\mathfrak{C}(\mathcal{K})^G} \iff S^{\frac{1}{2}} - S'^{\frac{1}{2}} \text{ is Hilbert–Schmidt.}$$

2.2. Representations of the form $\pi \circ \varrho$. As mentioned in the introduction on p. 6, the representations describing superselection sectors in the algebraic approach have the form $\pi_0 \circ \varrho$ where π_0 is a vacuum representation and ϱ is some localized endomorphism of the observable algebra. Here we study representations

$\pi_P \circ \varrho_V$ of $\mathfrak{C}(\mathcal{K})$ where π_P is a Fock representation and ϱ_V a quasi–free endomorphism, and in particular the decomposition of such representations into cyclic and irreducible subrepresentations. Among the results are a necessary and sufficient condition for implementability of quasi–free endomorphisms (a generalization of the Shale–Stinespring condition), and alternative proofs of results of Böckenhauer [Böc96].

Let us first repeat what is meant by "implementability of endomorphisms".

DEFINITION 2.1.
A *–endomorphism ϱ of a C^*–algebra $\mathfrak{A}$ is *implementable* in a representation $(\pi, \mathcal{H})$ if there exists a (possibly finite) sequence $(\Psi_n)_{n\in I}$ in $\mathfrak{B}(\mathcal{H})$ with relations

$$\Psi_m^* \Psi_n = \delta_{mn} \mathbf{1}, \qquad \sum_{n\in I} \Psi_n \Psi_n^* = \mathbf{1},^{o} \tag{2.21}$$

which implements ϱ via

$$\pi(\varrho(a)) = \sum_{n\in I} \Psi_n \pi(a) \Psi_n^*,^{o} \qquad a \in \mathfrak{A}. \tag{2.22}$$

$\mathcal{H}$ then decomposes into the orthogonal direct sum of the ranges of the isometries Ψ_n, and $\pi \circ \varrho$ decomposes into subrepresentations $\pi \circ \varrho|_{\operatorname{ran}\Psi_n}$, each of them unitarily equivalent to π. But the converse is also true, so ϱ is implementable in π if and only if $\pi \circ \varrho$ is equivalent to a multiple of π. For irreducible π this means

$$\varrho \text{ is implementable in } \pi \iff \pi \circ \varrho \approx \pi. \tag{2.23}$$

The isometries $(\Psi_n)_{n\in I}$ constitute an orthonormal basis of the Hilbert space $H \equiv \overline{\operatorname{span}}(\Psi_n)$ in $\mathfrak{B}(\mathcal{H})$, with scalar product given by $\Psi^*\Psi' = \langle \Psi, \Psi' \rangle \mathbf{1}$. Every element Ψ of H is an intertwiner from π to $\pi \circ \varrho$:

$$\Psi \pi(a) = \pi(\varrho(a)) \Psi, \quad a \in \mathfrak{A}. \tag{2.24}$$

H coincides with the space of intertwiners from π to $\pi \circ \varrho$ if and only if π is irreducible. If π is reducible, there may exist several Hilbert spaces implementing ϱ, mutually related by unitaries in $\pi(\varrho(\mathfrak{A}))'$. More precisely, if $(\Psi_n)_{n\in I}$ and $(\Psi'_n)_{n\in I}$ both implement ϱ in π, then $\Psi \equiv \sum_n \Psi'_n \Psi_n^*$ is a unitary in $\pi(\varrho(\mathfrak{A}))'$, and $\Psi'_n = \Psi\Psi_n$. Conversely, given $(\Psi_n)_{n\in I}$ and a unitary $\Psi \in \pi(\varrho(\mathfrak{A}))'$, $(\Psi\Psi_n)_{n\in I}$ is a set of implementing isometries.

An implementable endomorphism ϱ gives rise to normal *–endomorphisms $\varrho_H(a) \equiv \sum_{n\in I} \Psi_n a \Psi_n^*$ of $\mathfrak{B}(\mathcal{H})$, with index [Lon89]

$$[\mathfrak{B}(\mathcal{H}) : \varrho_H(\mathfrak{B}(\mathcal{H}))] = (\dim H)^2,$$

where $\dim H$ does not depend on the choice of $H = \overline{\operatorname{span}}(\Psi_n)$. Let us outline the computation of the index in the algebraic setting of Watatani [Wat90], for the case $\dim H < \infty$. $\phi_H(a) \equiv (\dim H)^{-1} \sum_n \Psi_n^* a \Psi_n$ is a left inverse for ϱ_H (cf. (0.15)), yielding the minimal conditional expectation $E_H \equiv \varrho_H \circ \phi_H$ from $\mathfrak{B}(\mathcal{H})$ onto $\varrho_H(\mathfrak{B}(\mathcal{H}))$. $(\sqrt{\dim H} \cdot \Psi_n^*)_{n=1,\dots,\dim H}$ is a quasi–basis for E_H, hence $\operatorname{ind} E_H = \dim H \cdot \sum_n \Psi_n^* \Psi_n = (\dim H)^2$. Of course, we will see that $\dim H = d_V$ (defined by (2.14)) if ϱ_V is a quasi–free endomorphism which is implementable in some Fock representation.

Let us add a last remark on the general situation. Suppose we are given a set of implementers $(\Psi_n)_{n\in I}$. Then for $m, n \in I$, $\Psi_m \Psi_n^* \in \pi(\varrho(\mathfrak{A}))'$ is a partial isometry containing $\operatorname{ran} \Psi_n$ in its initial space, and $\Psi_m = (\Psi_m \Psi_n^*) \Psi_n$. This suggests to construct a complete set of implementing isometries by multiplying one isometry

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^{o}in the strong topology if I is infinite. In the terminology of Laca [Lac93a], we only consider essential representations of Cuntz algebras. See Section 1.

Ψ fulfilling (2.24) with certain partial isometries in $\pi(\varrho(\mathfrak{A}))'$. We shall employ this idea later in Section 2.4.

After this digression we concentrate on Bogoliubov transformations again. Inspection of (2.23) leads one to study the representations $\pi_P \circ \varrho_V$; as will turn out, they are quasi–equivalent to GNS representations associated with the states $\omega_P \circ \varrho_V$ (a similar observation has been made, in a different setting, by Rideau [Rid68]). Thus the question of quasi–equivalence of such representations can be traced back to the question of quasi–equivalence of the corresponding states.

Let P be a basis projection, let $V \in \mathcal{I}(\mathcal{K})$, and regard

$$v \equiv PVV^*P \tag{2.25}$$

as an operator on $P(\mathcal{K})$. The direct sum decomposition $P(\mathcal{K}) = \ker v \oplus \overline{\operatorname{ran} v}$ induces a tensor product decomposition of Fock space: $\mathcal{F}_a(P(\mathcal{K})) \cong \mathcal{F}_a(\ker v) \otimes \mathcal{F}_a(\overline{\operatorname{ran} v})$. Choose an orthonormal basis $(f_j)_{j=1,\dots,N_V}$ for $\ker v = P(\mathcal{K}) \cap \ker V^*$, where

$$N_V \equiv \dim \ker v \leq M_V \tag{2.26}$$

(the inequality follows from $\ker v \oplus \ker \overline{v} \subset \ker V^*$). Let ψ_P be the twisted Fock representation defined in (2.8), and let I_{N_V} be the set of 2^{N_V} multi–indices $\alpha = (\alpha_1, \dots, \alpha_l)$, $\alpha_j \in \mathbb{N}$, $l < \infty$, obeying

$$0 \leq l \leq N_V, \quad 1 \leq \alpha_1 < \cdots < \alpha_l \leq N_V \quad (\alpha \equiv 0 \text{ if } l = 0). \tag{2.27}$$

For $\alpha \in I_{N_V}$, set

$$\begin{aligned} \psi_\alpha &\equiv \psi_P(f_{\alpha_1} \cdots f_{\alpha_l}) \quad (\psi_0 \equiv \mathbf{1}), \\ \phi_\alpha &\equiv \psi_\alpha \Omega_P, \\ \mathcal{F}_\alpha &\equiv \overline{\pi_P(\varrho_V(\mathfrak{C}(\mathcal{K})))\phi_\alpha}, \\ \pi_\alpha &\equiv \pi_P \circ \varrho_V|_{\mathcal{F}_\alpha}. \end{aligned} \tag{2.28}$$

Note that, by the CAR and (2.12), the ψ_α are partial isometries in $\pi_P(\varrho_V(\mathfrak{C}(\mathcal{K})))'$.

PROPOSITION 2.2.
Each of the 2^{N_V} cyclic subrepresentations $(\pi_\alpha, \mathcal{F}_\alpha, \phi_\alpha)$ induces the state $\omega_P \circ \varrho_V$, and $\pi_P \circ \varrho_V$ splits into their direct sum:

$$\pi_P \circ \varrho_V = \bigoplus_{\alpha \in I_{N_V}} \pi_\alpha.$$

Proof. It is clear by definition that $\mathcal{F}_\alpha$ is an invariant subspace for $\pi_P \circ \varrho_V$ with cyclic vector ϕ_α. Since $\psi_\alpha \in \pi_P(\varrho_V(\mathfrak{C}(\mathcal{K})))'$ and $\psi_\alpha^*\psi_\alpha\Omega_P = \Omega_P$, we have $\langle \phi_\alpha, \pi_\alpha(a)\phi_\alpha\rangle = \langle \Omega_P, \pi_P(\varrho_V(a))\Omega_P\rangle = \omega_P(\varrho_V(a))$, $a \in \mathfrak{C}(\mathcal{K})$. Thus $(\pi_\alpha, \mathcal{F}_\alpha, \phi_\alpha)$ is a GNS representation for $\omega_P \circ \varrho_V$ (and the representations π_α are mutually unitarily equivalent).

Next we show $\mathcal{F}_\alpha \perp \mathcal{F}_\beta$ for $\alpha \neq \beta$. Since at least one of the vectors $\psi_\alpha^*\psi_\beta\Omega_P$, $\psi_\beta^*\psi_\alpha\Omega_P$ vanishes if $\alpha \neq \beta$, we have for $a, b \in \mathfrak{C}(\mathcal{K})$

$$\langle \pi_P(\varrho_V(a))\phi_\alpha, \pi_P(\varrho_V(b))\phi_\beta\rangle = \langle \psi_\alpha\Omega_P, \pi_P(\varrho_V(a^*b))\psi_\beta\Omega_P\rangle = 0,$$

implying orthogonality of $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$.

Finally we have to prove $\mathcal{F}_a(P(\mathcal{K})) = \oplus_\alpha \mathcal{F}_\alpha$. Using $\pi_P(\varrho_V(f)) = a(PVf)^* + a(PVf^*)$, $f \in \mathcal{K}$, one can show by induction on the particle number

$$\mathcal{F}_0 = \overline{\pi_P(\varrho_V(\mathfrak{C}(\mathcal{K})))\Omega_P} = \mathcal{F}_a(\overline{\operatorname{ran} PV}) = \mathcal{F}_a(\overline{\operatorname{ran} v}).$$

Since the ϕ_α form an orthonormal basis for $\mathcal{F}_a(\ker v)$, the assertion follows. ☐

The decomposition of these cyclic representations into irreducibles will be examined after stating the implementability condition. Remember that $\overline{P} = \mathbf{1} - P$.

THEOREM 2.3.
A quasi–free endomorphism ϱ_V is implementable in a Fock representation π_P if and only if $PV\overline{P}$ is a Hilbert–Schmidt operator.

Proof. In view of (2.23) and Proposition 2.2, ϱ_V is implementable in π_P if and only if $\omega_P \circ \varrho_V \approx \omega_P$. Since $\omega_P \circ \varrho_V = \omega_{V^*PV}$ by (2.5), the Powers–Størmer–Araki criterion in the form (2.18) implies that $\omega_P \circ \varrho_V \approx \omega_P$ if and only if $\operatorname{tr} \overline{P}V^*PV\overline{P} < \infty$. The latter condition is clearly equivalent to $PV\overline{P}$ being Hilbert–Schmidt. □

Note that $PV\overline{P}$ is Hilbert–Schmidt if and only if $[P,V] = PV\overline{P} - \overline{P}VP$ is, so the Shale–Stinespring condition (2.16) remains valid. We denote the semigroup of Bogoliubov operators fulfilling this condition by

$$\mathfrak{I}_P(\mathcal{K}) \equiv \{V \in \mathfrak{I}(\mathcal{K}) \mid PV\overline{P} \text{ is Hilbert–Schmidt}\}.$$

Since $PV\overline{P}$ and $\overline{P}VP$ are compact for $V \in \mathfrak{I}_P(\mathcal{K})$, $PVP + \overline{P}V\overline{P} = V - PV\overline{P} - \overline{P}VP$ is semi–Fredholm, and

$$M_V = -\operatorname{ind} PVP \in \mathbb{N} \cup \{\infty\}.$$

Thus we have a decomposition (cf. (2.3))

$$\mathfrak{I}_P(\mathcal{K}) = \bigcup_{m \in \mathbb{N} \cup \{\infty\}} \mathfrak{I}_P^{2m}(\mathcal{K}), \qquad \mathfrak{I}_P^{2m}(\mathcal{K}) \equiv \{V \in \mathfrak{I}_P(\mathcal{K}) \mid M_V = m\}.$$

The group $\mathfrak{I}_P^0(\mathcal{K})$ is usually called the *restricted orthogonal group* [PS86]. Note that the "statistics dimension" d_V defined by (2.14) is contained in $\mathbb{N} \cup \{\infty\}$ if $V \in \mathfrak{I}_P(\mathcal{K})$. Note also that non–surjective quasi–free endomorphisms cannot be inner in $\mathfrak{C}(\mathcal{K})$ since the CAR algebra, being AF and thus finite, does not contain non–unitary isometries.

In the course of constructing localized endomorphisms for the conformal WZW models, J. Böckenhauer described the decomposition of representations $\pi_P \circ \varrho_V$ and of their restrictions to the even subalgebra $\mathfrak{C}(\mathcal{K})_0$ into irreducibles [Böc96] (see also [Szl94]). His methods work only for Bogoliubov operators with finite index, i.e. those belonging to the sub–semigroup

$$\mathfrak{I}^{\text{fin}}(\mathcal{K}) \equiv \{V \in \mathfrak{I}(\mathcal{K}) \mid M_V < \infty\}.$$

We shall now present alternative proofs of his results which have the merit of being completely basis–independent.

For $V \in \mathfrak{I}(\mathcal{K})$, let Q_V be the orthogonal projection onto $\ker V^*$, and let S_V be the operator characterizing the quasi–free state $\omega_P \circ \varrho_V$

$$Q_V \equiv [V^*, V] = \mathbf{1} - VV^*, \qquad S_V \equiv V^*PV \in \mathfrak{Q}(\mathcal{K}).$$

The operators Q_V and $S_V\overline{S_V}$ have finite rank if $V \in \mathfrak{I}^{\text{fin}}(\mathcal{K})$.

Let us first determine when two representations of the form $\pi_P \circ \varrho_V$ (P fixed) are unitarily equivalent.

LEMMA 2.4.
Let $V, V' \in \mathfrak{I}^{\text{fin}}(\mathcal{K})$. Then the following conditions are equivalent:
a) $\pi_P \circ \varrho_V$ and $\pi_P \circ \varrho_{V'}$ are unitarily equivalent;
b) there exists $U \in \mathfrak{I}_P^0(\mathcal{K})$ with $V' = UV$;
c) $\operatorname{ind} V = \operatorname{ind} V'$, and $S_V - S_{V'}$ is Hilbert–Schmidt.

Proof. We first show a) ⇒ c). By Proposition 2.2, $\pi_P \circ \varrho_V \simeq \pi_P \circ \varrho_{V'}$ implies $\omega_{S_V} = \omega_P \circ \varrho_V \approx \omega_P \circ \varrho_{V'} = \omega_{S_{V'}}$. Hence by (2.17), $S_V^{\frac{1}{2}} - S_{V'}^{\frac{1}{2}}$ is Hilbert–Schmidt (HS) which is, for $V, V' \in \mathfrak{I}^{\text{fin}}(\mathcal{K})$, equivalent to $S_V - S_{V'}$ being HS (see [Böc96]). Moreover, equivalent representations have isomorphic commutants. We have by (2.12) $\pi_P(\varrho_V(\mathfrak{C}(\mathcal{K})))' = \psi_P(\mathfrak{C}(\ker V^*))'' \simeq \pi_P(\mathfrak{C}(\ker V^*))''$. Hence the commutants have dimensions $d_V^2 = 2^{-\operatorname{ind} V}$ resp. $d_{V'}^2 = 2^{-\operatorname{ind} V'}$, and the indices of V and V' must be equal.

Next we show c) $\Rightarrow$ b). Let u be a partial isometry with initial space $\ker V^*$, final space $\ker V'^*$, and $u = \overline{u}$ (such u exists due to $*$–invariance and equality of dimensions of the kernels). Then $U \equiv V'V^* + u$ is an element of $\mathcal{I}^0(\mathcal{K})$ and fulfills $V' = UV$. We have to prove that $PU\overline{P}$ is HS. But u has finite rank, so it suffices to show that $A \equiv \overline{P}VS_{V'}V^*\overline{P}$ is of trace class. Since $S_V\overline{S_V}$ and $S_{V'}\overline{S_{V'}}$ have finite rank, $S_{V'}\overline{S_V} + S_V\overline{S_{V'}} = (S_{V'} - S_V)(\overline{S_V} - \overline{S_{V'}}) + S_V\overline{S_V} + S_{V'}\overline{S_{V'}}$ is trace class. So the same is true for $A = AQ_V + AVV^* = AQ_V + \overline{P}V(S_{V'}\overline{S_V} + S_V\overline{S_{V'}})V^* + \overline{P}Q_VPV\overline{S_{V'}}V^*$.

b) $\Rightarrow$ a) is obvious. $\square$

In order to apply part c) of the lemma, we need information about the operators S_V. An orthogonal projection E on $\mathcal{K}$ is called a *partial basis projection* [Ara71] if $E\overline{E} = 0$. By definition, the *codimension* of E is the dimension of $\ker(E + \overline{E})$. For instance, VPV^* is a partial basis projection with codimension $2M_V = -\operatorname{ind} V$ for any $V \in \mathcal{I}(\mathcal{K})$. The following lemma holds for arbitrary $S \in \mathcal{Q}(\mathcal{K})$ (except for the formula for the codimension, of course) as long as $S\overline{S}$ has finite rank.

LEMMA 2.5.
Let $V \in \mathcal{I}^{\text{fin}}(\mathcal{K})$, and let E_V denote the orthogonal projection onto $\ker S_V\overline{S_V}$. Then $S_VE_V = E_VS_V$ is a partial basis projection with finite codimension $2(M_V - N_V)$. Moreover, there exist $\lambda_1, \dots, \lambda_r \in (0, \frac{1}{2})$, $r \le M_V - N_V$, partial basis projections $E_1, \dots, E_r$, and an orthogonal projection $E_{\frac{1}{2}} = \overline{E_{\frac{1}{2}}}$ such that

$$E_V + E_{\frac{1}{2}} + \sum_{j=1}^{r}(E_j + \overline{E_j}) = \mathbf{1},$$

$$S_V = S_VE_V + \frac{1}{2}E_{\frac{1}{2}} + \sum_{j=1}^{r}\Big(\lambda_jE_j + (1 - \lambda_j)\overline{E_j}\Big). \tag{2.29}$$

Proof. Since $S_V\overline{S_V} = S_V - S_V^2$, S_V commutes with E_V and fulfills $S_VE_V = S_V^2E_V$ and $(S_VE_V)(\overline{S_VE_V}) = S_V\overline{S_V}E_V = 0$. Hence S_VE_V is a partial basis projection. The dimension of $\ker(S_VE_V + \overline{S_VE_V}) = \ker E_V$ (the codimension of S_VE_V) equals the rank of $S_V\overline{S_V}$. By $S_V\overline{S_V} = V^*PQ_VPV$, the rank of $S_V\overline{S_V}$ is equal to $\dim V^*P(\ker V^*)$. Now consider the decomposition

$$\ker V^* = \ker v \oplus \ker \overline{v} \oplus \Big(\ker V^* \ominus (\ker v \oplus \ker \overline{v})\Big)$$

with v given by (2.25). V^*P vanishes on $\ker v \oplus \ker \overline{v}$, but the restriction of V^*P to $\ker V^* \ominus (\ker v \oplus \ker \overline{v})$ is one to one since $V^*Pk = 0 = V^*k$ implies $V^*\overline{P}k = 0$, i.e. $k \in \ker v \oplus \ker \overline{v}$. Hence the codimension of S_VE_V equals $\dim(\ker V^* \ominus (\ker v \oplus \ker \overline{v})) = -\operatorname{ind} V - 2N_V$.

Let s_V denote the restriction of S_V to $\operatorname{ran} S_V\overline{S_V}$. s_V is a positive operator on a finite dimensional Hilbert space and has a complete set of eigenvectors with eigenvalues in $(0, 1)$. If λ is an eigenvalue of s_V, then $1 - \lambda$ is also an eigenvalue (with the same multiplicity) due to $s_V + \overline{s_V} = \mathbf{1} - E_V$. Thus there exist $\lambda_1, \dots, \lambda_r \in (0, \frac{1}{2})$ and spectral projections $E_{\frac{1}{2}}, E_1, \dots, E_r$ with $\overline{E_{\frac{1}{2}}} = E_{\frac{1}{2}}$, $E_j\overline{E_j} = 0$ such that $E_{\frac{1}{2}} + \sum_{j=1}^r(E_j + \overline{E_j}) = \mathbf{1} - E_V$ and $s_V = \frac{1}{2}E_{\frac{1}{2}} + \sum_{j=1}^r(\lambda_jE_j + (1 - \lambda_j)\overline{E_j})$. $\square$

As a consequence, operators S_V with $M_V = \frac{1}{2}$ necessarily have the form $S_V = S_VE_V + \frac{1}{2}E_{\frac{1}{2}}$ where $E_{\frac{1}{2}} = \mathbf{1} - E_V$ has rank one. By taking direct sums of $V \in \mathcal{I}^1(\mathcal{K})$ with operators $V(\varphi)$ from Example 1 below, we see that any combination of eigenvalues and multiplicities that is allowed by Lemma 2.5 actually occurs for some $S_{V'}$. Therefore any $S \in \mathfrak{Q}(\mathcal{K})$ such that $S\overline{S}$ has finite rank is of the form $S = S_V$ for some $V \in \mathfrak{I}^{\mathrm{fin}}(\mathcal{K})$.

We further remark that a quasi–free state ω_S with S of the form (2.29) is a product state[p] as defined by Powers [Pow67] (see also [MRT69, Man70]) with respect to the decomposition $\mathcal{K} = \ker S\overline{S} \oplus \operatorname{ran} E_{\frac{1}{2}} \oplus \bigoplus_j \operatorname{ran}(E_j + \overline{E_j})$. Clearly, the restriction of ω_S to $\mathfrak{C}(\ker S\overline{S})$ is a Fock state, the restriction to $\mathfrak{C}(\operatorname{ran} E_{\frac{1}{2}})$ the central state.

Example 1.
Let $(f_n)_{n\in\mathbb{N}}$ be an orthonormal basis for $P(\mathcal{K})$, and set $E_n \equiv f_n\langle f_n, .\rangle$, with $f_n^+ \equiv (f_n + f_n^*)/\sqrt{2}$, $f_n^- \equiv i(f_n - f_n^*)/\sqrt{2}$. Then $(f_n^s)_{s=\pm,\, n\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{K}$ consisting of $*$–invariant vectors. For $\varphi \in \mathbb{R}$, define a Bogoliubov operator

$$V(\varphi) \equiv (f_0^+ \cos\varphi + f_1^- \sin\varphi)\langle f_0^+, .\rangle + (f_0^- \sin\varphi - f_1^+ \cos\varphi)\langle f_0^-, .\rangle + \sum_{s=\pm,\, n\geq 1} f_{n+1}^s \langle f_n^s, .\rangle.$$

Then $V(\varphi) \in \mathfrak{I}^2(\mathcal{K})$, and the eigenvalue $\lambda_\varphi = \frac{1}{2}(1 + \sin 2\varphi)$ of $S_{V(\varphi)} = \lambda_\varphi E_0 + (1 - \lambda_\varphi)\overline{E_0} + \sum_{n\geq 1} E_n$ assumes any value in $[0,1]$ as φ varies over $[-\pi/4, \pi/4]$.

Next we characterize the Bogoliubov operators V for which S_V takes a particularly simple form. A distinction arises between the cases of even and odd Fredholm index. Note that the parity of $-\operatorname{ind} V$ is equal to the parity of $\dim\ker(S_V - \frac{1}{2})$ (it is well–known that quasi–free states ω_S with $\dim\ker(S - \frac{1}{2})$ even resp. odd behave differently (cf. [MY95])).

Lemma 2.6.
a) Let $W \in \mathfrak{I}(\mathcal{K})$. Then the following conditions are equivalent:
- *(i) $\omega_P \circ \varrho_W$ is a pure state;*
- *(ii) S_W is a basis projection;*
- *(iii) $[P, WW^*] = 0$.*

If any of these conditions is fulfilled, then $M_W = N_W$ and $\pi_P \circ \varrho_W \simeq d_W \cdot \pi_{S_W}$.
b) For any basis projection P' and $m \in \mathbb{N} \cup \{\infty\}$, there exists $W \in \mathfrak{I}^{2m}(\mathcal{K})$ with $S_W = P'$.
c) Let $W \in \mathfrak{I}^{\mathrm{fin}}(\mathcal{K})$. Then the following conditions are equivalent:
- *(i) $\omega_P \circ \varrho_W$ is a mixture of two disjoint pure states;*
- *(ii) $S_W E_W$ is a partial basis projection with codimension 1;*
- *(iii) $[P, WW^*]$ has rank 2;*
- *(iv) $M_W = N_W + \frac{1}{2}$.*

d) For any partial basis projection P' with codimension 1 and $m \in \mathbb{N} \cup \{\infty\}$, there exists $W \in \mathfrak{I}^{2m}(\mathcal{K})$ with $S_W E_W = P'$.

Proof. a) We know from Section 2.1 that $\omega_P \circ \varrho_W$ is pure if and only if S_W is a projection. We have

$$S_W^2 = S_W \iff W^*PQ_WPW = 0 \iff Q_WPWW^* = 0 \iff [P, WW^*] = 0.$$

If this is fulfilled, then $\ker WW^* = \ker(PWW^*P) \oplus \ker(\overline{P}WW^*\overline{P})$ has dimension $2M_W = 2N_W$. By Proposition 2.2, $\pi_P \circ \varrho_W$ is the direct sum of $2^{N_W} = d_W$ irreducible subrepresentations, each equivalent to the Fock representation π_{S_W}.

b) Let m and P' be given. There clearly exists $W' \in \mathfrak{I}^{2m}(\mathcal{K})$ with $[P, W'] = 0$. Since $\mathfrak{I}^0(\mathcal{K})$ acts transitively on the set of basis projections, we may choose $U \in \mathfrak{I}^0(\mathcal{K})$ with $U^*PU = P'$. Then $W \equiv W'U$ has the desired properties.

[p] A state ω is a *product state* with respect to a decomposition $\mathcal{K} = \oplus_j \mathcal{K}_j$ of $\mathcal{K}$ into closed, $*$–invariant subspaces if $\omega(ab) = \omega(a)\omega(b)$ whenever $a \in \mathfrak{C}(\mathcal{K}_j)$, $b \in \mathfrak{C}(\mathcal{K}_j^\perp)$.

c) (ii) $\Leftrightarrow$ (iii) follows from the facts that the codimension of $S_W E_W$ equals the rank of WW^*PQ_W (cf. the proof of Lemma 2.5) and that $[P, WW^*] = Q_W PWW^* - WW^*PQ_W$. (ii) is equivalent to (iv) by virtue of Lemma 2.5. (ii) $\Rightarrow$ (i) has been shown by Araki [Ara71]. To prove (i) $\Rightarrow$ (iv), assume that $M_W > N_W + \frac{1}{2}$ (if $M_W = N_W$, then S_W is a basis projection and $\omega_P \circ \varrho_W$ pure). Then one can show that $\omega_P \circ \varrho_W$ is a mixture of two quasi–equivalent orthogonal states (see [Bin95]), hence cannot be a mixture of two disjoint pure states. This proves part c).

d) Let $(f_n)_{n\in\mathbb{N}}$ be an orthonormal basis for $P(\mathcal{K})$, $(g_n)_{n\geq 1}$ an orthonormal basis for $P'(\mathcal{K})$, and g_0 a unit vector in $\ker(P' + \overline{P'})$. Set

$$V \equiv f_0^+\langle g_0, .\,\rangle + \sum_{s=\pm,\, n\geq 1} f_n^s \langle g_n^s, .\,\rangle$$

(we used the notation of Example 1). Then $V \in \mathcal{I}^1(\mathcal{K})$ and $S_V = \frac{1}{2} g_0\langle g_0, .\,\rangle + P'$. This implies $S_V E_V = P'$, and if we choose W' as in the proof of b), then $W \equiv W'V$ has the desired properties. ☐

Now we are in a position to discuss the decomposition of representations $\pi_P \circ \varrho_V$ with $V \in \mathcal{I}^{\mathrm{fin}}(\mathcal{K})$. If $-\operatorname{ind} V$ is even (resp. odd), then $S_V E_V$ is a partial basis projection with even (odd) codimension by Lemma 2.5, and there exists a basis projection (partial basis projection with codimension 1) P' such that $P' - S_V$ is Hilbert–Schmidt (we may choose P' to coincide with $S_V E_V$ on $\ker S_V \overline{S_V}$; then $P' - S_V$ has finite rank). By Lemma 2.6, there exists W with $\operatorname{ind} W = \operatorname{ind} V$ and $S_W E_W = P'$, and Lemma 2.4 implies $\pi_P \circ \varrho_V \simeq \pi_P \circ \varrho_W$. The latter representation splits into 2^{N_W} copies of the GNS representation π_{S_W} for the state $\omega_P \circ \varrho_W$ by Proposition 2.2. If $-\operatorname{ind} V$ is even, then $\pi_{S_W} = \pi_{P'}$ and $2^{N_W} = d_V$. If $-\operatorname{ind} V$ is odd, then $\pi_{S_W} = \pi^+ \oplus \pi^-$ where $\pi^\pm$ are mutually inequivalent, irreducible, so–called *pseudo Fock representations* by virtue of a lemma of Araki (see [Ara71] for details), and $2^{N_W} = 2^{-\frac{1}{2}} d_V$.

Summarizing, we rediscover Böckenhauer's first result on representations of the CAR algebra [Böc96]:

THEOREM 2.7.
Let P be a basis projection and $V \in \mathcal{I}^{\mathrm{fin}}(\mathcal{K})$. If $-\operatorname{ind} V$ is even, then there exist basis projections P' such that $P' - S_V$ is Hilbert–Schmidt, and for each such P'

$$\pi_P \circ \varrho_V \simeq d_V \cdot \pi_{P'}.$$

If $-\operatorname{ind} V$ is odd, there exist partial basis projections P' with codimension 1 such that $P' - S_V$ is Hilbert–Schmidt. For each such P',

$$\pi_P \circ \varrho_V \simeq 2^{-\frac{1}{2}} d_V \cdot (\pi_{P'}^+ \oplus \pi_{P'}^-)$$

where $\pi_{P'}^\pm$ are the (inequivalent, irreducible) pseudo Fock representations induced by P'.

Using the isomorphisms $\sigma_{V_1} : \mathfrak{C}(\mathcal{K}) \to \mathfrak{C}(\mathcal{K})_0$, $V_1 \in \mathcal{I}^1(\mathcal{K})$, from (2.4), we can immediately see how the restriction of $\pi_P \circ \varrho_V$ to $\mathfrak{C}(\mathcal{K})_0$ decomposes. From

$$\pi_P \circ \varrho_V(\mathfrak{C}(\mathcal{K})_0) = \pi_P \circ \varrho_V \circ \sigma_{V_1}(\mathfrak{C}(\mathcal{K})) \simeq \pi_P \circ \varrho_{VV_1}(\mathfrak{C}(\mathcal{K}))$$

and $M_{VV_1} = M_V + \frac{1}{2}$, $d_{VV_1} = \sqrt{2} d_V$, we infer Böckenhauer's second result [Böc96]:

COROLLARY 2.8.
Let P be a basis projection and $V \in \mathcal{I}^{\mathrm{fin}}(\mathcal{K})$. If $-\operatorname{ind} V$ is even, then $\pi_P \circ \varrho_V|_{\mathfrak{C}(\mathcal{K})_0}$ is equivalent to a multiple of the direct sum of two inequivalent irreducible representations, with multiplicity d_V. If $-\operatorname{ind} V$ is odd, then $\pi_P \circ \varrho_V|_{\mathfrak{C}(\mathcal{K})_0}$ is equivalent to a multiple of an irreducible representation, with multiplicity $\sqrt{2} d_V$.

2.3. The semigroup of implementable endomorphisms. From now on we choose a fixed basis projection P_1, and we set $P_2 \equiv \mathbf{1} - P_1 = \overline{P_1}$. Let us introduce the following notation. The components of an operator $A \in \mathfrak{B}(\mathcal{K})$ are denoted by

$$A_{mn} \equiv P_m A P_n, \quad m, n = 1, 2$$

and are regarded as operators from $\mathcal{K}_n \equiv P_n(\mathcal{K})$ to $\mathcal{K}_m$. Thus $\ker A_{mn}$, $(\ker A_{mn})^\perp$, $(\operatorname{ran} A_{nm})^\perp$ etc. are viewed as subspaces of $\mathcal{K}_n$, and relations like

$$A_{mn}{}^* = A^*{}_{nm}, \quad \overline{A_{11}} = A_{22}, \quad \text{etc.}$$

will frequently be used. We also use matrix notation $A = \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right)$ with respect to the decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$. A is called *antisymmetric* if

$$A^\tau \equiv \overline{A^*} = -A. \tag{2.30}$$

If $\mathcal{H}$ is a subspace of $\mathcal{K}$, then $\mathcal{H}^*$ will denote the complex conjugate space (and not the dual space)

$$\mathcal{H}^* \equiv \{f^* \mid f \in \mathcal{H}\}.$$

Thus one has e.g. $\mathcal{K}_1{}^* = \mathcal{K}_2$. The reader is kindly asked to pay attention to the various meanings of the star "*". The correct one should always be clear from the context.

Let $V \in \mathcal{I}_{P_1}(\mathcal{K}) = \{V \in \mathcal{I}(\mathcal{K}) \mid V_{12} \text{ is Hilbert–Schmidt}\}$. The relation $V^*V = \mathbf{1}$ reads in components

$$V_{11}{}^*V_{11} + V_{21}{}^*V_{21} = P_1, \tag{2.31a}$$
$$V_{22}{}^*V_{22} + V_{12}{}^*V_{12} = P_2, \tag{2.31b}$$
$$V_{11}{}^*V_{12} + V_{21}{}^*V_{22} = 0, \tag{2.31c}$$
$$V_{22}{}^*V_{21} + V_{12}{}^*V_{11} = 0, \tag{2.31d}$$

and the relation $\overline{V} = V$ entails that

$$\overline{V_{11}} = V_{22}, \quad \overline{V_{12}} = V_{21}.$$

$V_{11}{}^*V_{12}$, $V_{21}{}^*V_{22}$, $V_{22}{}^*V_{21}$ and $V_{12}{}^*V_{11}$ are antisymmetric by (2.31c) and (2.31d).

Since V_{12} is a Hilbert–Schmidt operator by Theorem 2.3, $V_{22}{}^*V_{22}$ is Fredholm (with vanishing index) by (2.31b). This implies

$$L_V \equiv \dim\ker V_{22} = \dim\ker V_{22}{}^*V_{22} < \infty. \tag{2.32}$$

Note that $V_{12}|_{\ker V_{22}}$ is isometric and, by (2.31c),

$$V_{12}(\ker V_{22}) \subset \ker V_{11}{}^* \cap \operatorname{ran} V. \tag{2.33}$$

As mentioned after Theorem 2.3, V_{11} is semi–Fredholm with $-\operatorname{ind} V_{11} = M_V$. These observations imply

$$\dim(\ker V_{11}{}^* \ominus V_{12}(\ker V_{22})) = M_V. \tag{2.34}$$

The semigroup of implementable quasi–free endomorphisms is isomorphic to the semigroup $\mathcal{I}_{P_1}(\mathcal{K})$. The latter is a topological semigroup relative to the metric (cf. [Ara87])

$$\delta_{P_1}(V, V') \equiv \|V - V'\| + \|V_{12} - V'_{12}\|_{\mathrm{HS}},$$

where $\|\ \|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm. $\mathcal{I}_{P_1}(\mathcal{K})$ contains the closed sub–semigroup of diagonal Bogoliubov operators (which are also called *gauge invariant,* because they commute with the $\mathbb{T}$–action (2.13)):

$$\mathcal{I}_{\mathrm{diag}}(\mathcal{K}) \equiv \{W \in \mathcal{I}(\mathcal{K}) \mid [P_1, W] = 0\}. \tag{2.35}$$

$\mathcal{I}_{\mathrm{diag}}(\mathcal{K})$ is isomorphic to the semigroup of isometries of $\mathcal{K}_1$, via the map $V \mapsto V_{11}$.

The restricted orthogonal group $\mathcal{I}^0_{P_1}(\mathcal{K})$ has a natural normal subgroup

$$\mathcal{I}_{\mathrm{HS}}(\mathcal{K}) \equiv \{U \in \mathcal{I}(\mathcal{K}) \mid U - \mathbf{1} \text{ is Hilbert–Schmidt}\}.$$

An automorphism ϱ_U, $U \in \mathcal{I}^0_{P_1}(\mathcal{K})$, is implementable in *all* Fock representations if and only if $U \in \mathcal{I}_{\mathrm{HS}}(\mathcal{K})$ or $-U \in \mathcal{I}_{\mathrm{HS}}(\mathcal{K})$ [SS65, Ara71]. The group $\mathcal{I}_{\mathrm{HS}}(\mathcal{K})$ is also related to the group of quasi–free automorphisms which are weakly inner in the representation associated with the central state $\omega_{\frac{1}{2}}$ [Bla58].

We shall prove that each $V \in \mathcal{I}_{P_1}(\mathcal{K})$ can be written as a product

$$V = UW, \quad \text{with } U \in \mathcal{I}_{\mathrm{HS}}(\mathcal{K}),\ W \in \mathcal{I}_{\mathrm{diag}}(\mathcal{K}) \tag{2.36}$$

(it is known that each $V \in \mathcal{I}^0_{P_1}(\mathcal{K})$ has this form [CO83]). Suppose for the moment that such U and W exist. Then $P \equiv UP_1U^*$ is a basis projection which extends the partial basis projection VP_1V^* such that

$$P_1 - P \text{ is Hilbert–Schmidt}, \qquad V^*PV = P_1. \tag{2.37}$$

The corresponding Fock state ω_P is unitarily equivalent to ω_{P_1} and fulfills $\omega_P \circ \varrho_V = \omega_{P_1}$. The proof of the product decomposition $V = UW$ involves the construction of such basis projections P. So let us start with a parameterization[q] of the set

$$\mathfrak{P}_{P_1} \equiv \{P \in \mathfrak{Q}(\mathcal{K}) \mid P^2 = P,\ P_1 - P \text{ is Hilbert–Schmidt}\} \tag{2.38}$$

of basis projections of $\mathcal{K}$ which differ from P_1 only by a Hilbert–Schmidt operator. $\mathfrak{P}_{P_1}$ is isomorphic to the set of all Fock states which are equivalent to ω_{P_1}. Let $\mathfrak{H}_{P_1}$ be the Hilbert space of all antisymmetric (see (2.30)) Hilbert–Schmidt operators from $\mathcal{K}_1$ to $\mathcal{K}_2$

$$\mathfrak{H}_{P_1} \equiv \{T \in \mathfrak{B}(\mathcal{K}_1, \mathcal{K}_2) \mid T^\tau = -T,\ T \text{ is Hilbert–Schmidt}\}, \tag{2.39}$$

let $\mathfrak{F}_{P_1}$ be the set of all finite dimensional subspaces of $\mathcal{K}_1$

$$\mathfrak{F}_{P_1} \equiv \{\mathfrak{h} \subset \mathcal{K}_1 \mid \mathfrak{h} \text{ is a finite dimensional subspace}\},$$

and let

$$\tilde{\mathfrak{P}}_{P_1} \equiv \{(T, \mathfrak{h}) \in \mathfrak{H}_{P_1} \times \mathfrak{F}_{P_1} \mid \mathfrak{h} \subset \ker T\}. \tag{2.40}$$

Then the following holds.

PROPOSITION 2.9.
The map

$$P \mapsto (P_{21}{P_{11}}^{-1}, \ker P_{11})$$

is a bijection from $\mathfrak{P}_{P_1}$ *onto* $\tilde{\mathfrak{P}}_{P_1}$*, with inverse given by*

$$(T, \mathfrak{h}) \mapsto P_{(T,\mathfrak{h})} \equiv P_T - p_{\mathfrak{h}} + \overline{p_{\mathfrak{h}}}. \tag{2.41}$$

Here ${P_{11}}^{-1} \in \mathfrak{B}(\mathcal{K}_1)$ *is defined as the inverse of* P_{11} *on the closed subspace* $\operatorname{ran} P_{11}$ *and as zero on the finite dimensional subspace* $\ker P_{11}$. P_T *is the (basis) projection onto* $\operatorname{ran}(P_1 + T)$

$$P_T \equiv (P_1 + T)(P_1 + T^*T)^{-1}(P_1 + T^*), \tag{2.42}$$

and $p_{\mathfrak{h}}$ *is the (partial basis) projection onto* $\mathfrak{h} \subset \mathcal{K}_1$.

Proof. Let us rewrite the conditions on P to be a basis projection

$$P = P^* = \mathbf{1} - \overline{P} = P^2$$

[q] Similar ideas can be found in the book of Pressley and Segal [PS86].

in components:

$$P_{11} = P_{11}{}^* = P_1 - \overline{P_{22}}, \tag{2.43a}$$
$$P_{22} = P_{22}{}^* = P_2 - \overline{P_{11}}, \tag{2.43b}$$
$$P_{21} = P_{12}{}^* = -P_{21}{}^\tau, \tag{2.43c}$$
$$P_{11} - P_{11}{}^2 = P_{21}{}^* P_{21}, \tag{2.43d}$$
$$P_{22} - P_{22}{}^2 = P_{12}{}^* P_{12}, \tag{2.43e}$$
$$(P_1 - P_{11})P_{12} = P_{12}P_{22}, \tag{2.43f}$$
$$(P_2 - P_{22})P_{21} = P_{21}P_{11}, \tag{2.43g}$$

and note that, because $P_1 - P = \overline{P_{22}} - P_{21} - P_{21}{}^* - P_{22}$,

$$P_1 - P \text{ is Hilbert–Schmidt (HS)} \iff P_2 P \text{ is HS.}$$

Now let $P \in \mathfrak{P}_{P_1}$. Then P_{21} is HS, P_{22} is of trace class, and P_{11} is Fredholm with index zero by (2.43a). Therefore $\operatorname{ran} P_{11}$ is closed and $\ker P_{11}$ has finite dimension. It follows that $P_{11}{}^{-1}$ is a well–defined bounded operator such that $P_{11}P_{11}{}^{-1} = P_{11}{}^{-1}P_{11}$ is the projection onto $\operatorname{ran} P_{11}$. We have $\ker P_{11} \subset \ker P_{21}$ by (2.43d), hence by (2.43a)–(2.43c) and (2.43g)

$$\begin{aligned}
P_{21}P_{11}{}^{-1} + (P_{21}P_{11}{}^{-1})^\tau &= P_{21}P_{11}{}^{-1} - \overline{P_{11}}{}^{-1}P_{21} \\
&= \overline{P_{11}}{}^{-1}\big(\overline{P_{11}}P_{21} - P_{21}P_{11}\big)P_{11}{}^{-1} \\
&= \overline{P_{11}}{}^{-1}\big((P_2 - P_{22})P_{21} - P_{21}P_{11}\big)P_{11}{}^{-1} \\
&= 0,
\end{aligned}$$

so $P_{21}P_{11}{}^{-1}$ is antisymmetric in the sense of (2.30). This proves $P_{21}P_{11}{}^{-1} \in \mathfrak{H}_{P_1}$. Since by definition $\ker P_{11}{}^{-1} = \ker P_{11}$, it follows that $(P_{21}P_{11}{}^{-1}, \ker P_{11}) \in \tilde{\mathfrak{P}}_{P_1}$.

Conversely, let $(T, \mathfrak{h}) \in \tilde{\mathfrak{P}}_{P_1}$ be given. We associate with T the bounded operator

$$X \equiv (P_T)_{11} = (P_1 + T^*T)^{-1} \tag{2.44}$$

so that, by (2.42)

$$P_T = (P_1 + T)X(P_1 + T^*) = \begin{pmatrix} X & XT^* \\ TX & TXT^* \end{pmatrix}.$$

P_T is a projection because $(P_1 + T^*)(P_1 + T) = X^{-1}$. To prove that $P_T + \overline{P_T} = \mathbf{1}$ holds, note that $TX^{-1} = \overline{X}^{-1}T$ and therefore $\overline{X}T = TX$, $XT^* = T^*\overline{X}$. It follows that

$$\begin{aligned}
P_T + \overline{P_T} &= (P_1 + T)X(P_1 + T^*) + (P_2 - T^*)\overline{X}(P_2 - T) \\
&= X + TX + XT^* + TT^*\overline{X} + \overline{X} - XT^* - TX + T^*TX \\
&= (P_1 + T^*T)X + (P_2 + TT^*)\overline{X} \\
&= P_1 + P_2 \\
&= \mathbf{1}.
\end{aligned}$$

Thus P_T is a basis projection. It is obvious that $\operatorname{ran} P_T = \operatorname{ran}(P_1 + T)$. Since $P_T P_2$ is HS, P_T belongs to $\mathfrak{P}_{P_1}$. The condition $T\mathfrak{h} = 0$ ensures that $p_{\mathfrak{h}} \subset P_T$. Therefore $P_{(T,\mathfrak{h})} = P_T - p_{\mathfrak{h}} + \overline{p_{\mathfrak{h}}}$ is a basis projection. It is also contained in $\mathfrak{P}_{P_1}$ because $p_{\mathfrak{h}}$ has finite rank.

To show that the two maps $(T, \mathfrak{h}) \mapsto P_{(T,\mathfrak{h})}$ and $P \mapsto (P_{21}P_{11}{}^{-1}, \ker P_{11})$ are each other's inverse, let first $(T, \mathfrak{h}) \in \tilde{\mathfrak{P}}_{P_1}$ be given and set $P \equiv P_{(T,\mathfrak{h})}$. Since

$X = (P_T)_{11}$ is bijective and since $T\mathfrak{h} = 0$, we have

$$\begin{aligned}\ker P_{11} &= \ker X^{-1}P_{11} = \ker X^{-1}(X - p_{\mathfrak{h}}) = \ker\big(P_1 - (P_1 + T^*T)p_{\mathfrak{h}}\big)\\ &= \ker(P_1 - p_{\mathfrak{h}}) = \mathfrak{h}.\end{aligned}$$

By $T\mathfrak{h} = 0$, and because $(X - p_{\mathfrak{h}})(X - p_{\mathfrak{h}})^{-1}$ is the projection onto $\operatorname{ran} P_{11} = \mathfrak{h}^{\perp}$,

$$P_{21}{P_{11}}^{-1} = (P_T)_{21}\big((P_T)_{11} - p_{\mathfrak{h}}\big)^{-1} = TX(X - p_{\mathfrak{h}})^{-1} = T(X - p_{\mathfrak{h}})(X - p_{\mathfrak{h}})^{-1} = T.$$

Conversely, let $P \in \mathfrak{P}_{P_1}$ be given and set $T \equiv P_{21}{P_{11}}^{-1}$, $\mathfrak{h} \equiv \ker P_{11}$. Then we have, using (2.43d),

$$P_1 + T^*T = {P_{11}}^{-1} + P_1 - {P_{11}}^{-1}P_{11} = {P_{11}}^{-1} + p_{\mathfrak{h}} = (P_{11} + p_{\mathfrak{h}})^{-1}.$$

Together with $T\mathfrak{h} = 0$, (2.43c) and (2.43b) we get

$$\begin{aligned}P_{(T,\mathfrak{h})} &= (P_1 + T)(P_1 + T^*T)^{-1}(P_1 + T^*) - p_{\mathfrak{h}} + \overline{p_{\mathfrak{h}}}\\ &= (P_1 + T)(P_{11} + p_{\mathfrak{h}})(P_1 + T^*) - p_{\mathfrak{h}} + \overline{p_{\mathfrak{h}}}\\ &= P_{11} + TP_{11} + P_{11}T^* + TP_{11}T^* + p_{\mathfrak{h}} - p_{\mathfrak{h}} + \overline{p_{\mathfrak{h}}}\\ &= P_{11} + P_{21} + P_{12} + TT^*\overline{P_{11}} + \overline{p_{\mathfrak{h}}}\\ &= P - P_{22} + TT^*\overline{P_{11}} + \overline{p_{\mathfrak{h}}}\\ &= P - (P_2 - \overline{p_{\mathfrak{h}}}) + \overline{P_{11}}^{-1}\overline{P_{11}}\\ &= P.\end{aligned}$$

This completes the proof. ☐

Remark. Note that

$$\ker P_{11} = \mathcal{K}_1 \cap \ker P. \tag{2.45}$$

The basis projections of the form $P_{(0,\mathfrak{h})}$, $\mathfrak{h} \in \mathfrak{F}_{P_1}$, are precisely the elements of $\mathfrak{P}_{P_1}$ which commute with P_1, and the projections of the form $P_T = P_{(T,\{0\})}$, $T \in \mathfrak{H}_{P_1}$, are precisely the elements $P \in \mathfrak{P}_{P_1}$ with $\ker P_{11} = \{0\}$. For a general element $P_{(T,\mathfrak{h})} \in \mathfrak{P}_{P_1}$, it is well known that the unique (up to a phase) cyclic vector in $\mathcal{F}_a(\mathcal{K}_1)$ which induces the state $\omega_{P_{(T,\mathfrak{h})}}$ is proportional to

$$a(e_1)^* \cdots a(e_L)^* \exp(\tfrac{1}{2}\overline{T}a^*a^*)\Omega_{P_1} \tag{2.46}$$

where the e_j form an orthonormal basis in $\mathfrak{h}$, and the exponential term will be explained later on.

LEMMA 2.10.
For $T \in \mathfrak{H}_{P_1}$, define a Bogoliubov operator

$$U_T \equiv (P_1 + T)(P_1 + T^*T)^{-\frac{1}{2}} + (P_2 - T^*)(P_2 + TT^*)^{-\frac{1}{2}} \in \mathcal{I}_{\mathrm{HS}}(\mathcal{K}).$$

For $\mathfrak{h} \in \mathfrak{F}_{P_1}$, choose an orthonormal basis $\{e_1, \ldots, e_L\}$ in $\mathfrak{h}$, define the partial isometry $u_{\mathfrak{h}} \equiv \sum_{r=1}^{L} e_r^*\langle e_r, .\,\rangle$ with initial space $\mathfrak{h}$ and final space $\mathfrak{h}^*$, and define a self-adjoint Bogoliubov operator

$$U_{\mathfrak{h}} \equiv \mathbf{1} - \begin{pmatrix} p_{\mathfrak{h}} & \overline{u_{\mathfrak{h}}} \\ u_{\mathfrak{h}} & \overline{p_{\mathfrak{h}}} \end{pmatrix} \in \mathcal{I}_{\mathrm{HS}}(\mathcal{K}). \tag{2.47}$$

Then one has, in the notation of Proposition 2.9,

$$U_T P_1 U_T^* = P_T, \qquad U_{\mathfrak{h}} P_1 U_{\mathfrak{h}}^* = P_{(0,\mathfrak{h})}.$$

If $(T, \mathfrak{h}) \in \tilde{\mathfrak{P}}_{P_1}$, one has in addition

$$[U_T, U_{\mathfrak{h}}] = 0, \qquad U_T U_{\mathfrak{h}} P_1 U_{\mathfrak{h}}^* U_T^* = P_{(T,\mathfrak{h})}.$$

It follows that the action $P \mapsto UPU^*$ of the restricted orthogonal group $\mathcal{I}^0_{P_1}(\mathcal{K})$ on $\mathfrak{P}_{P_1}$ restricts to a transitive action of $\mathcal{I}_{\mathrm{HS}}(\mathcal{K})$.

Proof. Let $T \in \mathfrak{H}_{P_1}$ be given. The unitary U_T results from polar decomposition of $\mathbf{1} + T + \overline{T} = U_T|\mathbf{1} + T + \overline{T}|$. With X defined by (2.44), U_T can be written as

$$U_T = \begin{pmatrix} X^{\frac{1}{2}} & \overline{T}\,\overline{X}^{\frac{1}{2}} \\ TX^{\frac{1}{2}} & \overline{X}^{\frac{1}{2}} \end{pmatrix}.$$

It is straightforward to see that U_T is a Bogoliubov operator which transforms P_1 into P_T. To prove that $U_T - \mathbf{1}$ is HS, note that

$$X^{\frac{1}{2}} - P_1 = X^{\frac{1}{2}}(P_1 - X^{-1})(P_1 + X^{-\frac{1}{2}})^{-1} = -X^{\frac{1}{2}}T^*T(P_1 + X^{-\frac{1}{2}})^{-1}$$

is of trace class. Therefore $(U_T - \mathbf{1})P_1 = (P_1 + T)X^{\frac{1}{2}} - P_1 = X^{\frac{1}{2}} - P_1 + TX^{\frac{1}{2}}$ is HS, which implies that $U_T - \mathbf{1} = (U_T - \mathbf{1})P_1 + \overline{(U_T - \mathbf{1})P_1}$ is HS.

Now let $\mathfrak{h} \in \mathfrak{F}_{P_1}$. Then $U_\mathfrak{h}$ is clearly a Bogoliubov operator with $U_\mathfrak{h} - \mathbf{1}$ of finite rank and with $U_\mathfrak{h} P_1 U_\mathfrak{h}^* = P_{(0,\mathfrak{h})}$. $U_\mathfrak{h}$ is therefore contained in $\mathfrak{I}_{\mathrm{HS}}(\mathcal{K})$. It is self–adjoint because $u_\mathfrak{h}$ is symmetric: $u_\mathfrak{h}^\tau = u_\mathfrak{h}$. (Actually, self–adjointness of $U_\mathfrak{h}$ or symmetry of $u_\mathfrak{h}$ will not be needed in the sequel. The above definitions aim at reducing the ambiguity in the choice of $U_\mathfrak{h}$. $u_\mathfrak{h}$ is now determined up to the action of the unitary group of $\mathfrak{h}$.)

If $(T, \mathfrak{h}) \in \tilde{\mathfrak{P}}_{P_1}$, then one has $Tp_\mathfrak{h} = 0$ and, by functional calculus, $X^{\frac{1}{2}}p_\mathfrak{h} = p_\mathfrak{h} X^{\frac{1}{2}} = p_\mathfrak{h}$. Using this, one gets by straightforward computations

$$U_T U_\mathfrak{h} = U_\mathfrak{h} U_T = U_T + U_\mathfrak{h} - \mathbf{1} = \begin{pmatrix} X^{\frac{1}{2}}p_{\mathfrak{h}^\perp} & -T^*\overline{X}^{\frac{1}{2}}\overline{p_{\mathfrak{h}^\perp}} - \overline{u_\mathfrak{h}} \\ TX^{\frac{1}{2}}p_{\mathfrak{h}^\perp} - u_\mathfrak{h} & \overline{X}^{\frac{1}{2}}\overline{p_{\mathfrak{h}^\perp}} \end{pmatrix}, \tag{2.48}$$

$$U_T U_\mathfrak{h} P_1 U_\mathfrak{h}^* U_T^* = U_T P_{(0,\mathfrak{h})} U_T^* = U_\mathfrak{h} P_T U_\mathfrak{h}^* = P_{(T,\mathfrak{h})}. \tag{2.49}$$

Here $p_{\mathfrak{h}^\perp}$ denotes the orthogonal projection onto $\mathfrak{h}^\perp \subset \mathcal{K}_1$. It is obvious that $\mathfrak{I}^0_{P_1}(\mathcal{K})$ acts transitively from the left on $\mathfrak{P}_{P_1}$ as indicated. By (2.49) and by $U_T U_\mathfrak{h} \in \mathfrak{I}_{\mathrm{HS}}(\mathcal{K})$, $\mathfrak{I}_{\mathrm{HS}}(\mathcal{K})$ acts already transitively. □

Remark. T and $\mathfrak{h}$ can be recovered from $U \equiv U_T U_\mathfrak{h}$ as $\mathfrak{h} = \ker U_{11}$, $T = U_{21} {U_{11}}^{-1}$ (see below for the definition of ${U_{11}}^{-1}$).

Next we would like to assign to each $V \in \mathfrak{I}_{P_1}(\mathcal{K})$ distinguished operators P_V, U_V and W_V having the properties stated in (2.37) and (2.36). Note that any basis projection P fulfilling (2.37) has the form

$$P = VP_1V^* + q$$

where q is a partial basis projection (see page 38) with

$$q + \overline{q} = Q_V \equiv \mathbf{1} - VV^*.$$

It follows from this and from (2.33) that necessarily

$$V_{12}(\ker V_{22}) \subset \ker P_{11}. \tag{2.50}$$

In general, $\dim \ker P_{11}$ can take any value between L_V (defined in (2.32)) and $L_V + M_V$. Similarly, if U is a Bogoliubov operator such that $[U^*V, P_1] = 0$ (cf. (2.36)), then

$$V_{12}(\ker V_{22}) \subset \ker {U_{11}}^* \tag{2.51}$$

(this follows from $0 = P_1U^*VP_2 = {U_{11}}^*V_{12} + {U_{21}}^*V_{22}$). We shall choose U_V and P_V such that equality holds in (2.50) and (2.51).

Now let ${V_{11}}^{-1} \in \mathfrak{B}(\mathcal{K}_1)$ be defined as the inverse of V_{11} on the closed subspace $\operatorname{ran} V_{11}$ and as zero on $\ker {V_{11}}^*$ (and define V_{22} analogously). For a closed subspace

$\mathcal{H} \subset \mathcal{K}$, let $p_{\mathcal{H}}$ be the orthogonal projection onto $\mathcal{H}$. Then

$$\operatorname{ran} V_{11}{}^{-1} = \operatorname{ran} V_{11}{}^*, \qquad V_{11}V_{11}{}^{-1} = p_{\operatorname{ran} V_{11}}, \tag{2.52}$$

$$\ker V_{11}{}^{-1} = \ker V_{11}{}^*, \qquad V_{11}{}^{-1}V_{11} = p_{\operatorname{ran} V_{11}{}^*}. \tag{2.53}$$

Define $(T_V, \mathfrak{h}_V) \in \tilde{\mathfrak{P}}_{P_1}$ by

$$T_V \equiv V_{21}V_{11}{}^{-1} - V_{22}{}^{-1*}V_{12}{}^* p_{\ker V_{11}{}^*}, \tag{2.54}$$

$$\mathfrak{h}_V \equiv V_{12}(\ker V_{22}). \tag{2.55}$$

By Proposition 2.9, any basis projection $P \in \mathfrak{P}_{P_1}$ with $\ker P_{11} = V_{12}(\ker V_{22})$ has the form $P = P_{(T,\mathfrak{h}_V)}$ for some $T \in \mathfrak{H}_{P_1}$. The possible choices of T such that (2.37) holds are determined in

LEMMA 2.11.
Let $\mathfrak{h}_V$ be defined by (2.55). A basis projection $P_{(T,\mathfrak{h}_V)} \in \mathfrak{P}_{P_1}$ satisfies

$$V^* P_{(T,\mathfrak{h}_V)} V = P_1, \tag{2.56}$$

i.e. extends the partial basis projection VP_1V^*, if and only if

$$T = T_V + T', \tag{2.57}$$

where $T' \in \mathfrak{H}_{P_1}$ is an operator from $\ker V_{11}{}^* \ominus \mathfrak{h}_V$ to $\ker V_{22}{}^* \ominus (\mathfrak{h}_V)^*$. For such T one has

$$\|T\|_{\mathrm{HS}}^2 = \|T_V\|_{\mathrm{HS}}^2 + \|T'\|_{\mathrm{HS}}^2. \tag{2.58}$$

Proof. The formula

$$p_{\mathfrak{h}_V} = V_{12} p_{\ker V_{22}} V_{12}{}^*$$

entails that $V^* p_{\mathfrak{h}_V} V = V_{12}{}^* V_{12} p_{\ker V_{22}} V_{12}{}^* V_{12} = p_{\ker V_{22}}$ by (2.31b), (2.33). It follows from (2.41) that

$$V^* P_{(T,\mathfrak{h}_V)} V = V^* P_T V - p_{\ker V_{22}} + p_{\ker V_{11}}.$$

Therefore (2.56) is equivalent to

$$V^* P_T V = P_{(0,\ker V_{11})} \equiv p_{\operatorname{ran} V_{11}{}^*} + p_{\ker V_{22}}$$

or, by Lemma 2.6a, to $P_T V = V P_{(0,\ker V_{11})}$. This is further equivalent to $0 = \overline{P_T} V P_{(0,\ker V_{11})}$ and, since $\ker \overline{P_T} = \ker(P_2 - T)$, to $0 = (P_2 - T) V P_{(0,\ker V_{11})} = V_{21} p_{\operatorname{ran} V_{11}{}^*} - T V_{11} - T V_{12} p_{\ker V_{22}}$. Looking at the components, we finally obtain the following conditions, which are equivalent to (2.56)

$$\mathfrak{h}_V \subset \ker T, \qquad T V_{11} = V_{21} p_{\operatorname{ran} V_{11}{}^*}. \tag{2.59}$$

(Of course, the first relation of (2.59) is also necessary for having $(T, \mathfrak{h}_V) \in \tilde{\mathfrak{P}}_{P_1}$, cf. (2.40).) Let us show that T_V is a special solution of this problem. T_V is Hilbert–Schmidt since V_{21} and V_{12} are. It is antisymmetric because

$$\begin{aligned}
T_V + T_V^{\tau} &= V_{21}V_{11}{}^{-1} - V_{22}{}^{-1*}V_{12}{}^* p_{\ker V_{11}{}^*} + V_{22}{}^{-1*}V_{12}{}^* - p_{\ker V_{22}{}^*} V_{21} V_{11}{}^{-1} \\
&= p_{\operatorname{ran} V_{22}} V_{21} V_{11}{}^{-1} + V_{22}{}^{-1*} V_{12}{}^* p_{\operatorname{ran} V_{11}} \\
&= V_{22}{}^{-1*}(V_{22}{}^* V_{21} + V_{12}{}^* V_{11}) V_{11}{}^{-1} \\
&= 0
\end{aligned}$$

(we used (2.31d) and (2.52)). Thus T_V belongs to $\mathfrak{H}_{P_1}$. One clearly has $T_V V_{11} = V_{21} p_{\operatorname{ran} V_{11}{}^*}$ and $T_V \mathfrak{h}_V = 0$ (use (2.33), (2.52) and (2.31b) to see the latter). Thus T_V solves (2.59).

If $T \in \mathfrak{H}_{P_1}$ is another solution of (2.59), then $T' \equiv T - T_V$ is contained in $\mathfrak{H}_{P_1}$. (2.59) is equivalent to $\operatorname{ran} V_{11} \oplus \mathfrak{h}_V \subset \ker T'$, i.e. to $(\ker T')^{\perp} \subset \ker V_{11}{}^* \ominus \mathfrak{h}_V$. By antisymmetry of T', this is also equivalent to $\operatorname{ran} T' \subset \ker V_{22}{}^* \ominus (\mathfrak{h}_V)^*$.

Finally, (2.58) holds because T_V and T' are orthogonal as elements of the Hilbert space $\mathfrak{H}_{P_1}$. □

Remark. T_V takes the simpler form $T_V = V_{21}{V_{11}}^{-1}$, which is well–known from the case of automorphisms ($\operatorname{ind} V = 0$), whenever $[P_1, VV^*] = 0$, i.e. whenever the state $\omega_{P_1} \circ \varrho_V$ is pure (cf. Lemma 2.6a). One can show that $T_V = 0$ if and only if $V_{21}{V_{11}}^* = 0$ if and only if one (and hence, in view of (2.31a), (2.31b), all) of the operators $V_{11}, V_{21}, V_{12}, V_{22}$ is a partial isometry.

Having specified $(T_V, \mathfrak{h}_V)$ in (2.54), (2.55), let us now associate with $V \in \mathcal{I}_{P_1}(\mathcal{K})$ the following operators:

$$P_V \equiv P_{(T_V, \mathfrak{h}_V)}, \tag{2.60}$$

$$U_V \equiv U_{T_V} U_{\mathfrak{h}_V}, \tag{2.61}$$

$$W_V \equiv {U_V}^* V, \tag{2.62}$$

and let us collect their properties.

PROPOSITION 2.12.
P_V belongs to $\mathfrak{P}_{P_1}$ and satisfies $V^* P_V V = P_1$. It is chosen such that $\ker(P_V)_{11}$ and $\|(P_V)_{21}{(P_V)_{11}}^{-1}\|_{\mathrm{HS}}$ are minimal (cf. Proposition 2.9 and (2.50), (2.58)). U_V belongs to $\mathcal{I}_{\mathrm{HS}}(\mathcal{K})$ and W_V to $\mathcal{I}_{\mathrm{diag}}(\mathcal{K})$; their definition depends on the choice of an orthonormal basis in $\mathfrak{h}_V$. The operators U_V and W_V fulfill

$$M_{U_V} = 0, \qquad T_{U_V} = T_V, \qquad \mathfrak{h}_{U_V} = \mathfrak{h}_V, \qquad P_{U_V} = P_V; \tag{2.63}$$

$$M_{W_V} = M_V, \qquad T_{W_V} = 0, \qquad \mathfrak{h}_{W_V} = \{0\}, \qquad P_{W_V} = P_1, \tag{2.64}$$

$$W_V = \begin{pmatrix} (P_1 + {T_V}^* T_V)^{\frac{1}{2}} V_{11} - \overline{u_{\mathfrak{h}_V}} V_{21} & 0 \\ 0 & (P_2 + T_V {T_V}^*)^{\frac{1}{2}} V_{22} - u_{\mathfrak{h}_V} V_{12} \end{pmatrix}, \tag{2.65}$$

and

$$V = U_V W_V. \tag{2.66}$$

If $M_V = 0$, then the formulas reduce to

$$U_V = \begin{pmatrix} |{V_{11}}^*| & V_{12}{v_{22}}^* - \overline{u_{\mathfrak{h}_V}} \\ V_{21}{v_{11}}^* - u_{\mathfrak{h}_V} & |{V_{22}}^*| \end{pmatrix}, \quad W_V = \begin{pmatrix} v_{11} - \overline{u_{\mathfrak{h}_V}} V_{21} & 0 \\ 0 & v_{22} - u_{\mathfrak{h}_V} V_{12} \end{pmatrix} \tag{2.67}$$

where v_{11} and v_{22} are the partial isometries appearing in the polar decomposition of $V_{11} = v_{11}|V_{11}|$ and $V_{22} = v_{22}|V_{22}|$. On the other hand, if $V \in \mathcal{I}_{\mathrm{diag}}(\mathcal{K})$, then $U_V = \mathbf{1}$ and $W_V = V$.

Remark. Equipped with the metric induced by $\|\ \|_{\mathrm{HS}}$, $\mathfrak{P}_{P_1}$ becomes a topological space. It consists of two connected components which are distinguished by the parity of $\dim\ker P_{11}$. It follows that the map $V \mapsto P_V$ is *not* continuous on $\mathcal{I}_{P_1}(\mathcal{K})$, because $\ker(P_V)_{11} = \mathfrak{h}_V$, and $(-1)^{\dim \mathfrak{h}_V}$ is not constant on the connected components of $\mathcal{I}_{P_1}(\mathcal{K})$ (see Corollary 2.15 and Example 2 below).

Proof. The assertions made about P_V have been proved in Proposition 2.9 and Lemma 2.11. The claims concerning U_V are implied by Lemma 2.10 and the remark following that lemma. W_V is diagonal because $P_1 W_V = P_1 {U_V}^* V = {U_V}^* P_V V = {U_V}^* V P_1 = W_V P_1$, where we used Lemma 2.10 and an argument from the proof of Lemma 2.11. The formula (2.65) can be derived from (2.48) together with the relation $V_{11} + {T_V}^* V_{21} = X_V^{-1} V_{11}$, $X_V \equiv (P_1 + {T_V}^* T_V)^{-1}$. $V = U_V W_V$ holds by definition of W_V and entails that $\operatorname{ind} W_V = \operatorname{ind} V$. The remaining statements on W_V clearly follow from $W_V \in \mathcal{I}_{\mathrm{diag}}(\mathcal{K})$.

If $\operatorname{ind} V = 0$, then one has $\mathfrak{h}_V = \ker V_{11}{}^*$ (cf. (2.34)) and $X_V = p_{\ker V_{11}{}^*} + |V_{11}{}^*|^2$ so that $X_V^{\frac{1}{2}} p_{\mathfrak{h}_V^\perp} = |V_{11}{}^*|$. (2.67) then follows by straightforward computation from (2.48).

If V is diagonal, then $T_V = 0$ and $\mathfrak{h}_V = \{0\}$, so that obviously $U_V = \mathbf{1}$ and $W_V = V$. □

Remark. A different product decomposition $V = \tilde{U}\tilde{W}$ was established in [Bin95], essentially by polar decomposition of V_{11}:

$$\tilde{W} = \begin{pmatrix} v_{11} & V_{12} p_{\ker V_{22}} \\ V_{21} p_{\ker V_{11}} & v_{22} \end{pmatrix}, \qquad \tilde{U} = V\tilde{W}^* + \cdots$$

$\tilde{W}$ is obviously not diagonal (unless $\ker V_{11} = \{0\}$), but has off–diagonal components of finite rank. (However, using the operators $U_{\mathfrak{h}}$ from (2.47), the definition of $\tilde{W}$ could easily be modified to make $\tilde{W}$ diagonal.) $\tilde{W}\tilde{W}^*$ commutes with P_1. $\tilde{U}$ is in general not contained in $\mathfrak{I}_{\mathrm{HS}}(\mathcal{K})$, but only in $\mathfrak{I}^0_{P_1}(\mathcal{K})$. These operators have nevertheless some useful properties, similar to U_V and W_V:

$$\begin{array}{llll} M_{\tilde{U}} = 0, & T_{\tilde{U}} = T_V, & \mathfrak{h}_{\tilde{U}} = \{0\}, & P_{\tilde{U}} = P_{T_V}; \\ M_{\tilde{W}} = M_V, & T_{\tilde{W}} = 0, & \mathfrak{h}_{\tilde{W}} = \mathfrak{h}_V, & P_{\tilde{W}} = P_{(0,\mathfrak{h}_V)}. \end{array}$$

Corollary 2.13.
$\mathfrak{I}_{P_1}(\mathcal{K}) = \mathfrak{I}_{\mathrm{HS}}(\mathcal{K}) \cdot \mathfrak{I}_{\mathrm{diag}}(\mathcal{K})$. The $\mathfrak{I}^0_{P_1}(\mathcal{K})$–orbits in $\mathfrak{I}_{P_1}(\mathcal{K})$ with respect to left multiplication are precisely the sets $\mathfrak{I}^{2m}_{P_1}(\mathcal{K})$, $m \in \mathbb{N} \cup \{\infty\}$.

Proof. The product decomposition of $\mathfrak{I}_{P_1}(\mathcal{K})$ has been obtained above. To show that $\mathfrak{I}^0_{P_1}(\mathcal{K})$ acts transitively on each $\mathfrak{I}^{2m}_{P_1}(\mathcal{K})$, let $V, V' \in \mathfrak{I}^{2m}_{P_1}(\mathcal{K})$ be given, with decompositions $V = UW$, $V' = U'W'$ as in (2.66). Since P_1 leaves $\ker W'^*$ and $\ker W^*$ invariant, we can choose a partial isometry $\hat{u}$ with initial space $\ker W'^*$ and final space $\ker W^*$ such that $\overline{\hat{u}} = \hat{u}$ and $[P_1, \hat{u}] = 0$. Then $\hat{U} \equiv WW'^* + \hat{u} \in \mathfrak{I}^0_{P_1}(\mathcal{K})$ fulfills $\hat{U}W' = W$. This implies that $(U\hat{U}U'^*)V' = V$. □

These results can be used to determine the connected components of the semigroup $\mathfrak{I}_{P_1}(\mathcal{K})$. It is known that the restricted orthogonal group $\mathfrak{I}^0_{P_1}(\mathcal{K}) \subset \mathfrak{I}_{P_1}(\mathcal{K})$ has two connected (and simply connected) components $\mathfrak{I}^0_{P_1}(\mathcal{K})^\pm$ [Car84, PS86, Ara87]. Namely,

$$\chi(U) \equiv (-1)^{\dim \ker U_{11}} = (-1)^{\dim \mathfrak{h}_U}$$

defines a continuous character χ on $\mathfrak{I}^0_{P_1}(\mathcal{K})$, and $\chi|_{\mathfrak{I}^0_{P_1}(\mathcal{K})^\pm} = \pm 1$. (This character is equal to the Araki–Evans index of the pair of basis projections (P_1, UP_1U^*) [AE83, Ara87].) We shall see that $\mathfrak{I}^{2m}_{P_1}(\mathcal{K})$ is connected if $m > 0$, and that the map $\chi : V \mapsto (-1)^{\dim \ker V_{11}} = (-1)^{\dim \mathfrak{h}_V}$ remains neither multiplicative nor continuous when extended to the whole semigroup $\mathfrak{I}_{P_1}(\mathcal{K})$. We need the following preparatory result.

Lemma 2.14.
The set of all isometries with a given fixed index on an infinite dimensional complex Hilbert space is arcwise connected in the uniform topology.

Proof. Let V, V' be two isometries with $\operatorname{ind} V = \operatorname{ind} V'$. Since $\dim \ker V^* = \dim \ker V'^*$, there exists a unitary operator U with $V' = UV$ (choose a partial isometry u with initial space $\ker V^*$ and final space $\ker V'^*$, and set $U \equiv V'V^* + u$). Since the unitary group is arcwise connected, there exists a continuous curve $U(t)$ of unitary operators with $U(0) = \mathbf{1}$ and $U(1) = U$. Then $V(t) \equiv U(t)V$ is a continuous curve of isometries with $V(0) = V$ and $V(1) = V'$. □

COROLLARY 2.15.
The connected components of $\mathcal{I}_{P_1}(\mathcal{K})$ *are precisely the subsets* $\mathcal{I}^0_{P_1}(\mathcal{K})^\pm$ *and* $\mathcal{I}^{2m}_{P_1}(\mathcal{K})$, $1 \le m \le \infty$.

Proof. Let $V, V' \in \mathcal{I}^{2m}_{P_1}(\mathcal{K})$ with $\chi(V) = \chi(V')$ be given, and let

$$V = UW, \quad V' = U'W'$$

be decompositions as in (2.66). It follows from (2.63) that

$$\chi(U) = \chi(V) = \chi(V') = \chi(U')$$

so that there exists a continuous curve in $\mathcal{I}^0_{P_1}(\mathcal{K})$ connecting U to U'. Since W and W' are both diagonal and have index $-2m$, there exists a continuous curve in $\mathcal{I}^{2m}_{P_1}(\mathcal{K})$ connecting W to W' by Lemma 2.14. It follows that $UW = V$ and $U'W' = V'$ can be connected by a continuous curve in $\mathcal{I}^{2m}_{P_1}(\mathcal{K})$. Therefore either of the two subsets

$$\mathcal{I}^{2m}_{P_1}(\mathcal{K})^\pm \equiv \{V \in \mathcal{I}^{2m}_{P_1}(\mathcal{K}) \mid \chi(V) = \pm 1\}$$

is arcwise connected. Below, we give an example of a continuous curve in $\mathcal{I}^{2m}_{P_1}(\mathcal{K})$ which connects $\mathcal{I}^{2m}_{P_1}(\mathcal{K})^+$ to $\mathcal{I}^{2m}_{P_1}(\mathcal{K})^-$. Hence $\mathcal{I}^{2m}_{P_1}(\mathcal{K})$ itself is connected. □

EXAMPLE 2.
Let $V(\varphi)$ be the Bogoliubov operator introduced in Example 1 in Section 2.2 (with $P = P_1$). Then $V(\varphi) \in \mathcal{I}^2_{P_1}(\mathcal{K})$ since $V(\varphi)_{12}{}^* V(\varphi)_{12} = (1 - \lambda_\varphi)\overline{E_0}$ has finite rank, and $\varphi \mapsto V(\varphi)$ is a continuous curve in $\mathcal{I}^2_{P_1}(\mathcal{K})$. We have $\ker V(\varphi)_{11} = \ker(\lambda_\varphi E_0 + \sum_{n\ge1} E_n)$, hence

$$\chi(V(\varphi)) = \begin{cases} 1, & \varphi \notin (4\mathbb{Z}+3)\frac{\pi}{4} \\ -1, & \varphi \in (4\mathbb{Z}+3)\frac{\pi}{4} \end{cases}.$$

Now let $V \in \mathcal{I}^{2m-2}_{P_1}(\mathcal{K})$ with $[P_1, V] = 0$. Then $\chi(VV(\varphi)) = \chi(V(\varphi))$ since V_{11} is isometric, so $\varphi \mapsto VV(\varphi)$ is a continuous curve in $\mathcal{I}^{2m}_{P_1}(\mathcal{K})$ which connects $\mathcal{I}^{2m}_{P_1}(\mathcal{K})^+$ to $\mathcal{I}^{2m}_{P_1}(\mathcal{K})^-$. This completes the proof of Corollary 2.15.

$V(\varphi)$ may also serve to illustrate that χ is not multiplicative on $\mathcal{I}_{P_1}(\mathcal{K})$. Define a Bogoliubov operator

$$\begin{aligned} U \equiv\; & \tfrac{1}{\sqrt{2}} f_0^+ \langle f_0^+ + f_1^-, .\rangle - \tfrac{1}{\sqrt{2}} f_1^+ \langle f_0^- - f_1^+, .\rangle + \tfrac{1}{\sqrt{2}} f_0^- \langle f_0^- + f_1^+, .\rangle \\ & - \tfrac{1}{\sqrt{2}} f_1^- \langle f_0^+ - f_1^-, .\rangle + \sum_{n\ge2} (E_n + \overline{E_n}). \end{aligned}$$

Then $U \in \mathcal{I}^0_{P_1}(\mathcal{K})$, and a calculation shows that $U_{11} = \frac{1}{\sqrt{2}}(E_0 + E_1) + \sum_{n\ge2} E_n$ and $UV(\frac{3\pi}{4}) = V(\frac{\pi}{2})$. This entails

$$1 = \chi\Big(UV(\tfrac{3\pi}{4})\Big) \neq \chi(U)\chi\Big(V(\tfrac{3\pi}{4})\Big) = -1$$

since $\ker U_{11} = \ker(V(\frac{\pi}{2})_{11}) = \{0\}$, but $\ker(V(\frac{3\pi}{4})_{11}) = \mathbb{C}f_0$. We finally note that the eigenvalues $\pm(1 - \lambda_\varphi)$ of $P_1 - S_{V(\varphi)} = (1 - \lambda_\varphi)(E_0 - \overline{E_0})$ have multiplicity one if $\lambda_\varphi \neq 1$, in contrast to the unitary case where the multiplicities of eigenvalues in $(0, 1)$ are always even [AE83, Ara87, EK98].

2.4. Normal form of implementers. Unitary operators which implement quasi–free automorphisms of the CAR algebra have been constructed by several authors, notably by Friedrichs [Fri53], Berezin [Ber66], Schroer, Seiler and Swieca [SSS70], Labonté [Lab74], Fredenhagen [Fre77], Klaus and Scharf [KS77], Ruijsenaars [Rui77, Rui78]. Our construction of isometric implementers for quasi–free endomorphisms follows Ruijsenaars' approach in [Rui78] which is to our knowledge the most complete treatment of the implementation problem for quasi–free automorphisms. Another advantage of [Rui78] for our purposes is the (implicit) use of Araki's "selfdual" formalism.

Let us begin with a generalization of the definition of "bilinear Hamiltonians" from the case of trace class operators to the case of bounded operators. Bilinear Hamiltonians have been introduced by Araki [Ara68] as infinitesimal generators of one–parameter groups of inner Bogoliubov automorphisms. More specifically, one may assign to a finite rank operator $H = \sum_j f_j \langle g_j, . \rangle$ on $\mathcal{K}$ the bilinear Hamiltonian

$$b(H) \equiv \sum_j f_j g_j^*$$

and extend b by continuity to a linear map from the ideal of all trace class operators on $\mathcal{K}$ to $\mathfrak{C}(\mathcal{K})$. If a trace class operator H satisfies $\overline{H} = H$ and $H^\tau = -H$, then $\frac{1}{2}b(H)$ is the generator of the one–parameter group $(\varrho_{\exp(tH)})_{t\in\mathbb{R}}$:

$$\varrho_{\exp(tH)}(a) = \exp\big(\tfrac{1}{2}tb(H)\big)a \exp\big(-\tfrac{1}{2}tb(H)\big), \qquad a \in \mathfrak{C}(\mathcal{K}).$$

The map $H \mapsto \frac{1}{2}b(H)$ is an isomorphism from the Lie algebra formed by all such H onto the Lie algebra of the spin group. See [Ara71, Ara87] for details.

Since the elements $f \in \mathcal{K}_1$ correspond to creation operators in the Fock representation π_{P_1}, we may write

$$\pi_{P_1}\big(b(H)\big) = H_{11}a^*a + H_{12}a^*a^* + H_{21}aa + H_{22}aa^*$$

where the terms on the right are defined by $H_{11}a^*a \equiv \pi_{P_1}(b(H_{11}))$ etc. Introducing *Wick ordering* by $:a(f)a(g)^*: \,= -a(g)^*a(f)$, we get

$$:H_{22}aa^*: \,= -{H_{22}}^\tau a^*a = H_{22}aa^* - (\operatorname{tr} H_{22})\mathbf{1}, \tag{2.68}$$

$$:\pi_{P_1}\big(b(H)\big): \,= (H_{11} - {H_{22}}^\tau)a^*a + H_{12}a^*a^* + H_{21}aa. \tag{2.69}$$

According to [Rui78, CR87], one can define such Wick ordered expressions for *bounded* H as follows. Assume from now on that (without loss of generality)

$$\mathcal{K}_1 = L^2(\mathbb{R}^d),$$

and let $\mathfrak{S} \subset \mathcal{F}_a(\mathcal{K}_1)$ be the dense subspace consisting of finite particle vectors ϕ with n–particle wave functions $\phi^{(n)}$ in the Schwartz space $\mathfrak{S}(\mathbb{R}^{dn})$. For $p \in \mathbb{R}^d$, the unsmeared annihilation operator $a(p)$ with (invariant) domain $\mathfrak{S}$ is defined by

$$\big(a(p)\phi\big)^{(n)}(p_1, \ldots, p_n) \equiv \sqrt{n+1}\,\phi^{(n+1)}(p, p_1, \ldots, p_n).$$

Since $a(p)$ is not closable, one defines $a(p)^*$ as the quadratic form adjoint of $a(p)$ on $\mathfrak{S} \times \mathfrak{S}$. Then Wick ordered monomials $a(q_m)^* \cdots a(q_1)^* a(p_1) \cdots a(p_n)$ are well–defined quadratic forms on $\mathfrak{S} \times \mathfrak{S}$, and for $\phi, \phi' \in \mathfrak{S}$,

$$\langle \phi, a(q_m)^* \cdots a(q_1)^* a(p_1) \cdots a(p_n)\phi' \rangle \equiv \langle a(q_1) \cdots a(q_m)\phi, a(p_1) \cdots a(p_n)\phi' \rangle$$

is a function in $\mathfrak{S}(\mathbb{R}^{d(m+n)})$ to which tempered distributions may be applied. For example, one has in the quadratic form sense

$$a(f) = \int \overline{f(p)}a(p)\,dp,$$

$$a(f)^* = \int f(p)a(p)^*\,dp, \quad f \in \mathcal{K}_1.$$

Now let H be a bounded operator on $\mathcal{K}$ which is antisymmetric in the sense of (2.30)[r]. Then there exist tempered distributions $H_{mn}(p,q)$, $m,n = 1,2$, given by

$$\begin{aligned}
\langle f, H_{11} g\rangle &= \int \overline{f(p)} H_{11}(p,q) g(q)\, dp\, dq,\\
\langle f, H_{12} g^*\rangle &= \int \overline{f(p)} H_{12}(p,q) \overline{g(q)}\, dp\, dq,\\
\langle f^*, H_{21} g\rangle &= \int f(p) H_{21}(p,q) g(q)\, dp\, dq,\\
\langle f^*, H_{22} g^*\rangle &= \int f(p) H_{22}(p,q) \overline{g(q)}\, dp\, dq, \quad f,g \in \mathfrak{S}(\mathbb{R}^d) \subset \mathcal{K}_1.
\end{aligned}$$

Hence the following expressions are quadratic forms on $\mathfrak{S} \times \mathfrak{S}$

$$\begin{aligned}
H_{11} a^* a &\equiv \int a(p)^* H_{11}(p,q) a(q)\, dp\, dq,\\
H_{12} a^* a^* &\equiv \int a(p)^* H_{12}(p,q) a(q)^*\, dp\, dq,\\
H_{21} aa &\equiv \int a(p) H_{21}(p,q) a(q)\, dp\, dq,\\
{:}H_{22} aa^*{:} &\equiv -\int a(q)^* H_{22}(p,q) a(p)\, dp\, dq \;=\; H_{11} a^* a.
\end{aligned}$$

Wick ordering of $H_{22}aa^*$ is necessary to make this expression well–defined. The last equality follows from antisymmetry of H:

$$H_{11}(p,q) = -H_{22}(q,p), \quad H_{12}(p,q) = -H_{12}(q,p), \quad H_{21}(p,q) = -H_{21}(q,p).$$

The Wick ordered bilinear Hamiltonian induced by H is then defined in analogy to (2.69) as

$${:}b(H){:} \equiv H_{12} a^* a^* + 2H_{11} a^* a + H_{21} aa;$$

it is linear in H. We define its Wick ordered powers as

$${:}b(H)^l{:} \equiv l! \sum_{\substack{l_1,l_2,l_3=0\\ l_1+l_2+l_3=l}}^{l} \frac{1}{l_1! l_2! l_3!} (H_{12})^{l_1} (2H_{11})^{l_2} (H_{21})^{l_3} a^{*2l_1+l_2} a^{l_2+2l_3} \tag{2.70}$$

where the terms on the right hand side are quadratic forms on $\mathfrak{S} \times \mathfrak{S}$ (cf. [Rui78])

$$\begin{aligned}
&(H_{12})^{l_1} (H_{11})^{l_2} (H_{21})^{l_3} a^{*2l_1+l_2} a^{l_2+2l_3}\\
&\quad\equiv \int H_{12}(p_1,q_1) \cdots H_{12}(p_{l_1},q_{l_1}) H_{11}(p'_1,q'_1) \cdots H_{11}(p'_{l_2},q'_{l_2})\\
&\qquad \cdot H_{21}(p''_1,q''_1) \cdots H_{21}(p''_{l_3},q''_{l_3}) a(p_1)^* \cdots a(p_{l_1})^* a(q_{l_1})^* \cdots a(q_1)^*\\
&\qquad \cdot a(p'_1)^* \cdots a(p'_{l_2})^* a(q'_{l_2}) \cdots a(q'_1) a(p''_1) \cdots a(p''_{l_3}) a(q''_{l_3}) \cdots a(q''_1)\\
&\qquad \cdot dp_1\, dq_1 \ldots dp_{l_1}\, dq_{l_1}\, dp'_1\, dq'_1 \ldots dp'_{l_2}\, dq'_{l_2}\, dp''_1\, dq''_1 \ldots dp''_{l_3}\, dq''_{l_3}.
\end{aligned} \tag{2.71}$$

Finally, we define the Wick ordered exponential

$${:}\exp\big(\tfrac{1}{2} b(H)\big){:} \equiv \sum_{l=0}^{\infty} \frac{1}{l! 2^l} {:}b(H)^l{:} \tag{2.72}$$

which is also a well–defined quadratic form on $\mathfrak{S} \times \mathfrak{S}$, since the sum in (2.72) is finite when applied to vectors $\phi, \phi' \in \mathfrak{S}$.

By Ruijsenaars' result [Rui78], ${:}\exp\big(\frac{1}{2} b(H)\big){:}$ is the quadratic form of a unique linear operator, defined on the dense subspace $\mathfrak{D}$ of algebraic tensors in $\mathcal{F}_a(\mathcal{K}_1)$, provided that H_{12} is Hilbert–Schmidt. (This is equivalent to $\overline{H_{12}} \in \mathfrak{H}_{P_1}$.) In

[r]The bilinear Hamiltonian corresponding to a symmetric operator vanishes.

this case, the series (2.72) converges strongly on $\mathfrak{D}$, $:\exp\bigl(\frac{1}{2}b(H)\bigr):$ (viewed as an operator) maps $\mathfrak{D}$ into the dense subspace of C^∞–vectors for the number operator, and

$$:\exp\bigl(\tfrac{1}{2}b(H)\bigr):\Omega_{P_1} = \exp(\tfrac{1}{2}H_{12}a^*a^*)\Omega_{P_1}, \tag{2.73}$$

$$\bigl\|\exp(\tfrac{1}{2}H_{12}a^*a^*)\Omega_{P_1}\bigr\| = \bigl(\det_{\mathcal{K}_1}(P_1 + H_{12}H_{12}{}^*)\bigr)^{1/4}. \tag{2.74}$$

Let us compute the commutation relations of the operators $:\exp\bigl(\frac{1}{2}b(H)\bigr):$ with creation and annihilation operators.

Lemma 2.16.
Let $H \in \mathfrak{B}(\mathcal{K})$ be antisymmetric with H_{12} Hilbert–Schmidt. For $f, g \in \mathcal{K}_1$, the following relations hold on $\mathfrak{D}$

$$\begin{aligned}
[\,:\exp\bigl(\tfrac{1}{2}b(H)\bigr):, a(f)^*] &= a(H_{11}f)^* :\exp\bigl(\tfrac{1}{2}b(H)\bigr): + :\exp\bigl(\tfrac{1}{2}b(H)\bigr): a\bigl((H_{21}f)^*\bigr),\\
[\,:\exp\bigl(\tfrac{1}{2}b(H)\bigr):, a(g)] &= a(H_{12}g^*)^* :\exp\bigl(\tfrac{1}{2}b(H)\bigr): - :\exp\bigl(\tfrac{1}{2}b(H)\bigr): a(H_{11}{}^*g).
\end{aligned}$$

Proof. It is a lengthy but straightforward exercise in anticommutation relations to calculate the commutation relations of Wick monomials of the form (2.71) with creation and annihilation operators:

$$\begin{aligned}
[H_{l_1,l_2,l_3}, a(f)^*] &= l_2 a(H_{11}f)^* H_{l_1,l_2-1,l_3} + 2l_3 H_{l_1,l_2,l_3-1} a\bigl((H_{21}f)^*\bigr),\\
[H_{l_1,l_2,l_3}, a(g)] &= 2l_1 a(H_{12}g^*)^* H_{l_1-1,l_2,l_3} - l_2 H_{l_1,l_2-1,l_3} a(H_{11}{}^*g),
\end{aligned}$$

where $H_{l_1,l_2,l_3} \equiv (H_{12})^{l_1}(H_{11})^{l_2}(H_{21})^{l_3} a^{*2l_1+l_2} a^{l_2+2l_3}$. From this one obtains

$$\begin{aligned}
[\,:\exp\bigl(\tfrac{1}{2}b(H)\bigr):, a(f)^*] &= \sum_{l=0}^\infty 2^{-l} \sum_{l_1+l_2+l_3=l} \frac{2^{l_2}}{l_1!l_2!l_3!}[H_{l_1,l_2,l_3}, a(f)^*]\\
&= a(H_{11}f)^* \sum_{l=1}^\infty 2^{-(l-1)} \sum_{l_1+l_2+l_3=l} \frac{2^{l_2-1}}{l_1!(l_2-1)!l_3!} H_{l_1,l_2-1,l_3}\\
&\quad + \sum_{l=1}^\infty 2^{-(l-1)} \sum_{l_1+l_2+l_3=l} \frac{2^{l_2}}{l_1!l_2!(l_3-1)!} H_{l_1,l_2,l_3-1} a\bigl((H_{21}f)^*\bigr)\\
&= a(H_{11}f)^* :\exp\bigl(\tfrac{1}{2}b(H)\bigr): + :\exp\bigl(\tfrac{1}{2}b(H)\bigr): a\bigl((H_{21}f)^*\bigr)
\end{aligned}$$

and

$$\begin{aligned}
[\,:\exp\bigl(\tfrac{1}{2}b(H)\bigr):, a(g)] &= \sum_{l=0}^\infty 2^{-l} \sum_{l_1+l_2+l_3=l} \frac{2^{l_2}}{l_1!l_2!l_3!}[H_{l_1,l_2,l_3}, a(g)]\\
&= a(H_{12}g^*)^* \sum_{l=1}^\infty 2^{-(l-1)} \sum_{l_1+l_2+l_3=l} \frac{2^{l_2}}{(l_1-1)!l_2!l_3!} H_{l_1-1,l_2,l_3}\\
&\quad - \sum_{l=1}^\infty 2^{-(l-1)} \sum_{l_1+l_2+l_3=l} \frac{2^{l_2-1}}{l_1!(l_2-1)!l_3!} H_{l_1,l_2-1,l_3} a(H_{11}{}^*g)\\
&= a(H_{12}g^*)^* :\exp\bigl(\tfrac{1}{2}b(H)\bigr): - :\exp\bigl(\tfrac{1}{2}b(H)\bigr): a(H_{11}{}^*g).
\end{aligned}$$

□

From now on let a fixed $V \in \mathfrak{I}_{P_1}(\mathcal{K})$ be given. To construct implementers for ϱ_V, we have to look for antisymmetric operators H with $\overline{H_{12}} \in \mathfrak{H}_{P_1}$ and (cf. (2.24))

$$:\exp\bigl(\tfrac{1}{2}b(H)\bigr): \pi_{P_1}(f) = \pi_{P_1}(Vf) :\exp\bigl(\tfrac{1}{2}b(H)\bigr):, \qquad f \in \mathcal{K}_1 \oplus \operatorname{ran} V_{22}{}^* \tag{2.75}$$

on $\mathfrak{D}$. Note that (2.75) cannot be fulfilled for nonzero $f \in \ker V_{22}$ since for such f, $\pi_{P_1}(f) = a(f^*)$ is an annihilation operator and $\pi_{P_1}(Vf) = a(V_{12}f)^*$ a creation operator, so that the left hand side (but not the right hand side) of (2.75) vanishes on Ω_{P_1}. This defect can be cured by "filling up the Dirac sea" corresponding to $\mathfrak{h}_V$ (cf. (2.46) and (2.85)) if we impose the following relation for vectors in $\ker V_{22}$:

$$:\exp\big(\tfrac{1}{2}b(H)\big):\pi_{P_1}(g^*) = 0, \qquad g \in \ker V_{22}. \tag{2.76}$$

It turns out that the solutions H of (2.75) and (2.76) are in one–to–one correspondence with the operators T described in Lemma 2.11:

LEMMA 2.17.
The antisymmetric solutions H of (2.75) *and* (2.76) *with H_{12} Hilbert–Schmidt are precisely the operators of the form*

$$H = \begin{pmatrix} V_{11} - P_1 + T^*V_{21} & -T^* \\ (V_{22}{}^* - V_{12}{}^*T^*)V_{21} & P_2 - V_{22}{}^* + V_{12}{}^*T^* \end{pmatrix}, \tag{2.77}$$

where $T \in \mathfrak{H}_{P_1}$ fulfills (2.59), *i.e. is of the form* (2.57).

Proof. First note that a Wick ordered expression of the form $a(f)^* :\exp\big(\tfrac{1}{2}b(H)\big): + :\exp\big(\tfrac{1}{2}b(H)\big): a(g)$ vanishes if and only if f and g both vanish. In fact, application to the vacuum gives $a(f)^* \exp(\tfrac{1}{2}H_{12}a^*a^*)\Omega_{P_1}$ which is zero if and only if $f = 0$ (to see this, look for instance at the one–particle component). Similarly, $:\exp\big(\tfrac{1}{2}b(H)\big): a(g)a(g)^*\Omega_{P_1} = \|g\|^2_{\mathcal{K}} \exp(\tfrac{1}{2}H_{12}a^*a^*)\Omega_{P_1}$ vanishes if and only if $g = 0$.

Hence we get all solutions of (2.75) and (2.76) if we write these equations in Wick ordered form and then compare term by term. We have by Lemma 2.16 and by the definition (2.6) of π_{P_1}, with the shorthand $\eta_H \equiv\ :\exp\big(\tfrac{1}{2}b(H)\big):$,

$$\begin{aligned} \eta_H\pi_{P_1}(f) &= a\big((P_1 + H_{11})f\big)^*\eta_H + \eta_H a\big((P_1 + \overline{H_{21}})f^*\big), \\ \pi_{P_1}(Vf)\eta_H &= a\big((P_1 - H_{12})Vf\big)^*\eta_H + \eta_H a\big((P_1 - \overline{H_{22}})Vf^*\big), \\ \eta_H\pi(f)^* &= a\big((P_1 + H_{11})f^*\big)^*\eta_H + \eta_H a\big(\overline{H_{21}}f\big), \qquad f \in \mathcal{K}. \end{aligned}$$

Thus (2.75) is equivalent to

$$P_1 + H_{11} + (H_{12} - P_1)V(P_1 + p_{\operatorname{ran} V_{22}{}^*}) = 0, \tag{2.78a}$$

$$p_{\operatorname{ran} V_{22}{}^*} + H_{21} + (H_{22} - P_2)V(P_1 + p_{\operatorname{ran} V_{22}{}^*}) = 0, \tag{2.78b}$$

and (2.76) is equivalent to

$$(P_1 + H_{11})p_{\ker V_{11}} = 0, \tag{2.78c}$$

$$H_{21}p_{\ker V_{11}} = 0, \tag{2.78d}$$

where complex conjugation was occasionally applied. Note that (2.78a)–(2.78d) are equivalent to the single equation

$$\mathbf{1} - V + H(P_1 + P_2V) + (V_{12} + \overline{H})p_{\ker V_{22}} = 0, \tag{2.79}$$

which is a generalization of Eq. (4.4) in [Rui78].

Let us show that each solution H of (2.79) is completely determined by its component H_{12}. Given H_{12}, H_{11} is fixed by (2.78a):

$$H_{11} = V_{11} - P_1 - H_{12}V_{21}, \tag{2.80}$$

H_{22} is fixed by antisymmetry:

$$H_{22} = -H_{11}{}^\tau = P_2 - V_{22}{}^* - V_{12}{}^*H_{12}, \tag{2.81}$$

and H_{21} is determined by (2.78b):

$$H_{21} = (P_2 - H_{22})V_{21} = (V_{22}{}^* + V_{12}{}^*H_{12})V_{21}. \tag{2.82}$$

Therefore H must have the form (2.77), with $\overline{T} \equiv H_{12}$.

It remains to determine the admissible components H_{12}. (2.78a) implies that

$$H_{12}V_{22} = V_{12}p_{\operatorname{ran} V_{22}^*}.$$

Inserting (2.80), (2.78c) is equivalent to

$$V_{21}(\ker V_{11}) \subset \ker H_{12},$$

and under this condition, (2.78d) holds automatically. Thus $T \equiv \overline{H_{12}}$ has to fulfill (2.59). Conversely, it is straightforward to verify that via (2.77), any $T \in \mathfrak{H}_{P_1}$ obeying (2.59) gives rise to a solution H of (2.79). In fact, one only has to check that H_{21} defined by (2.82) is antisymmetric and that $p_{\operatorname{ran} V_{22}^*} + (H_{22} - P_2)V_{22} = 0$ (the rest is clear by construction). By antisymmetry of H_{12} and by (2.31d)

$$H_{21} + {H_{21}}^{\tau} = ({V_{22}}^* + {V_{12}}^* H_{12})V_{21} + {V_{12}}^*(V_{11} - H_{12}V_{21}) = 0,$$

so H_{21} is antisymmetric. By (2.81), (2.59) and (2.31a),

$$\begin{aligned} p_{\operatorname{ran} V_{22}^*} + (H_{22} - P_2)V_{22} &= p_{\operatorname{ran} V_{22}^*} - {V_{22}}^*V_{22} - {V_{12}}^*H_{12}V_{22} \\ &= (P_1 - {V_{22}}^*V_{22} - {V_{12}}^*V_{12})p_{\operatorname{ran} V_{22}^*} \\ &= 0. \end{aligned}$$

□

Now we can proceed to exhibit the normal form of a complete set of implementers for ϱ_V. Let H_V be defined by (2.77), with $T = T_V$ (see (2.54)). Using $p_{\ker V_{22}^*} = P_2 - V_{22}{V_{22}}^{-1}$, one computes that

$$\begin{aligned} (H_V)_{11} &= {V_{11}}^{-1*} - P_1 - p_{\ker V_{11}^*}V_{12}{V_{22}}^{-1}V_{21}, \\ (H_V)_{12} &= V_{12}{V_{22}}^{-1} - {V_{11}}^{-1*}{V_{21}}^*p_{\ker V_{22}^*}, \\ (H_V)_{21} &= ({V_{22}}^{-1} - {V_{12}}^*{V_{11}}^{-1*}{V_{21}}^*p_{\ker V_{22}^*})V_{21}, \\ (H_V)_{22} &= P_2 - {V_{22}}^{-1} + {V_{12}}^*{V_{11}}^{-1*}{V_{21}}^*p_{\ker V_{22}^*}. \end{aligned}$$

H_V is the analogue of Ruijsenaars' "associate" Λ [Rui78].

Furthermore let $\{e_1, \dots, e_{L_V}\}$ be the orthonormal basis in $\mathfrak{h}_V = V_{12}(\ker V_{22})$ that was already used to define U_V in (2.61) (cf. Lemma 2.10), let $\{e'_1, \dots, e'_{L_V}\}$ be the orthonormal basis in $\ker V_{22}$ given by

$$e'_r \equiv V^*e_r = {V_{12}}^*e_r, \qquad r = 1, \dots, L_V,$$

and let $\{g_1, \dots, g_{M_V}\}$, $M_V \equiv -\frac{1}{2}\operatorname{ind} V$, be an orthonormal basis in

$$\mathfrak{k}_V \equiv P_V(\ker V^*). \tag{2.83}$$

(Note that P_V commutes with VV^* and therefore restricts to a basis projection of $\ker V^*$.) Recall that the statistics dimension d_V (2.14) of ϱ_V is given by

$$d_V = 2^{M_V}$$

and that the twisted Fock representation ψ_{P_1} defined in (2.8) fulfills

$$\pi_{P_1}\Big(\varrho_V\big(\mathfrak{C}(\mathcal{K})\big)\Big)' = \psi_{P_1}\big(\mathfrak{C}(\ker V^*)\big)''. \tag{2.84}$$

One has

$$\psi_{P_1}(e_r) = ia(e_r)^*\Psi(-\mathbf{1}), \qquad \psi_{P_1}(e'_r) = ia({e'_r}^*)\Psi(-\mathbf{1})$$

(for notational convenience, we shall drop the index P_1 on implementers like $\Psi(-\mathbf{1})$ from now on, cf. (2.7)). Finally define the following operators on $\mathfrak{D}$, with range contained in the space of C^∞–vectors for the number operator

$$\begin{aligned}\Psi_\alpha(V) \equiv \big(\det_{\mathcal{K}_1}(P_1 + T_V{}^*T_V)\big)^{-1/4}\psi_{P_1}(g_{\alpha_1}\cdots g_{\alpha_l})\\ \cdot \sum_{(\sigma,s)\in\mathcal{P}_{L_V}} (-1)^{L_V-s}\operatorname{sign}\sigma\,\psi_{P_1}(e_{\sigma(1)}\cdots e_{\sigma(s)})\ {:}\exp\big(\tfrac12 b(H_V)\big){:}\\ \cdot\psi_{P_1}(e'_{\sigma(s+1)}\cdots e'_{\sigma(L_V)}).\end{aligned} \tag{2.85}$$

Here $\alpha = (\alpha_1,\dots,\alpha_l) \in I_{M_V}$ is a multi–index as in (2.27), and $\mathcal{P}_{L_V}$ is the index set consisting of all pairs (σ, s) with $s \in \{0,\dots,L_V\}$ and σ a permutation of order L_V satisfying $\sigma(1) < \cdots < \sigma(s)$ and $\sigma(s+1) < \cdots < \sigma(L_V)$. $\mathcal{P}_{L_V}$ is canonically isomorphic to the power set $\tilde{\mathcal{P}}_{L_V}$ of $\{1,\dots,L_V\}$ through identification of (σ,s) with $\{\sigma(1),\dots,\sigma(s)\}$, hence its cardinality is 2^{L_V}. Note that

$$\begin{aligned}\Psi_\alpha(V)\Omega_{P_1} = \big(\det_{\mathcal{K}_1}(P_1 + T_V{}^*T_V)\big)^{-1/4}\psi_{P_1}(g_{\alpha_1}\cdots g_{\alpha_l})\\ \cdot\psi_{P_1}(e_1\cdots e_{L_V})\exp(\tfrac12\overline{T_V}a^*a^*)\Omega_{P_1}\end{aligned} \tag{2.86}$$

by (2.73) and because the $\psi_{P_1}(e'_r)$ annihilate the vacuum.

THEOREM 2.18.
Let $V \in \mathcal{I}_{P_1}(\mathcal{K})$. Then the d_V operators $\Psi_\alpha(V)$, $\alpha \in I_{M_V}$, have continuous extensions to isometries on $\mathcal{F}_a(\mathcal{K}_1)$ (henceforth denoted by the same symbols) which implement ϱ_V in π_{P_1} in the sense of Definition 2.1.

Proof. 1. We first show that the following intertwiner relation holds on $\mathfrak{D}$

$$\Psi_\alpha(V)\pi_{P_1}(f) = \pi_{P_1}(Vf)\Psi_\alpha(V), \quad f \in \mathcal{K}. \tag{2.87}$$

Note that it suffices to prove (2.87) for $\alpha = 0$ because

$$\Psi_\alpha(V) = \psi_{P_1}(g_{\alpha_1}\cdots g_{\alpha_l})\Psi_0(V) \tag{2.88}$$

and because the $\psi_{P_1}(g_j)$ belong to $\pi_{P_1}(\varrho_V(\mathfrak{C}(\mathcal{K})))'$.

Let first $f \in \operatorname{ran} V_{11}{}^* \oplus \operatorname{ran} V_{22}{}^*$. Then it follows from (2.11) that

$$[\psi_{P_1}(e'_r), \pi_{P_1}(f)] = 0 = [\psi_{P_1}(e_r), \pi_{P_1}(Vf)]$$

so that (2.87) is a consequence of (2.75).

To prove (2.87) for $f \in \ker V_{11} \oplus \ker V_{22}$, note that for fixed r, the bijection

$$\begin{aligned}\{\mathcal{M} \in \tilde{\mathcal{P}}_{L_V} \mid r \in \mathcal{M}\} \to \{\mathcal{M}' \in \tilde{\mathcal{P}}_{L_V} \mid r \notin \mathcal{M}'\},\\ \mathcal{M} \mapsto \mathcal{M}\setminus\{r\}\end{aligned}$$

induces a bijection $(\sigma,s)\mapsto(\sigma',s')$ from $\{(\sigma,s)\in\mathcal{P}_{L_V} \mid r \in \{\sigma(1),\dots,\sigma(s)\}\}$ onto $\{(\sigma',s')\in\mathcal{P}_{L_V} \mid r \notin \{\sigma'(1),\dots,\sigma'(s')\}\}$ with

$$s = s'+1, \quad (-1)^s\operatorname{sign}\sigma = (-1)^r\operatorname{sign}\sigma', \quad \sigma^{-1}(r) + \sigma'^{-1}(r) = r+s. \tag{2.89}$$

Now let $D_V \equiv (\det_{\mathcal{K}_1}(P_1 + T_V{}^*T_V))^{-1/4}$, and consider the case $f = e'_r \in \ker V_{22}$. We have on $\mathfrak{D}$, by virtue of

$$\begin{aligned}\psi_{P_1}(e'_r)\pi_{P_1}(e'_r) = 0 = \pi_{P_1}(e_r)\psi_{P_1}(e_r),\\ \{\psi_{P_1}(h), \Psi(-\mathbf{1})\} = 0 = [\,{:}\exp\big(\tfrac12 b(H_V)\big){:}\,, \Psi(-\mathbf{1})]\end{aligned}$$

and by (2.11), (2.10) and (2.89), where terms under the sign "$\widehat{\ }$" are to be omitted

$$\begin{aligned}\Psi_0(V)\pi_{P_1}(e'_r) &= D_V\psi_{P_1}(e_r) \sum_{\substack{(\sigma,s)\in\mathcal{P}_{L_V}\\ r\in\{\sigma(1),\dots,\sigma(s)\}}} (-1)^{L_V-s+\sigma^{-1}(r)-1}\operatorname{sign}\sigma \\ &\quad\cdot\psi_{P_1}(e_{\sigma(1)}\cdots\widehat{e_r}\cdots e_{\sigma(s)}) :\exp\left(\tfrac12 b(H_V)\right): \\ &\quad\cdot\psi_{P_1}(e'_{\sigma(s+1)})\cdots i\Psi(-\mathbf{1})\psi_{P_1}(e'_r)\cdots\psi_{P_1}(e'_{\sigma(L_V)}) \\ &= D_V\pi_{P_1}(e_r)\sum_{\substack{(\sigma',s')\in\mathcal{P}_{L_V}\\ r\notin\{\sigma'(1),\dots,\sigma'(s')\}}} (-1)^{L_V-s'}\operatorname{sign}\sigma'\psi_{P_1}(e_{\sigma'(1)}\cdots e_{\sigma'(s')}) \\ &\quad\cdot :\exp\left(\tfrac12 b(H_V)\right):\psi_{P_1}(e'_{\sigma'(s'+1)}\cdots e'_{\sigma'(L_V)}) \\ &= \pi_{P_1}(Ve'_r)\Psi_0(V).\end{aligned}$$

The remaining case $f = {e'_r}^* \in \ker V_{11}$ is similarly obtained with the help of (2.75) and (2.76)

$$\pi_{P_1}(e^*_r) :\exp\left(\tfrac12 b(H_V)\right): \;=\; :\exp\left(\tfrac12 b(H_V)\right): \pi_{P_1}({e'_r}^*) = 0, \tag{2.90}$$

in connection with

$$[\pi_{P_1}(e^*_r),\psi_{P_1}(e_s)] = [\pi_{P_1}({e'_r}^*),\psi_{P_1}(e'_s)] = i\delta_{rs}\Psi(-\mathbf{1})$$

(cf. (2.11)):

$$\begin{aligned}\pi_{P_1}(V{e'_r}^*)\Psi_0(V) &= \pi_{P_1}(e^*_r)\Psi_0(V) \\ &= i\Psi(-\mathbf{1})D_V \sum_{\substack{(\sigma,s)\in\mathcal{P}_{L_V}\\ r\in\{\sigma(1),\dots,\sigma(s)\}}} (-1)^{L_V-s+\sigma^{-1}(r)-1}\operatorname{sign}\sigma \\ &\quad\cdot\psi_{P_1}(e_{\sigma(1)}\cdots\widehat{e_r}\cdots e_{\sigma(s)}) :\exp\left(\tfrac12 b(H_V)\right): \\ &\quad\cdot\psi_{P_1}(e'_{\sigma(s+1)}\cdots e'_{\sigma(L_V)}) \\ &= i\Psi(-\mathbf{1})D_V \sum_{\substack{(\sigma',s')\in\mathcal{P}_{L_V}\\ r\notin\{\sigma'(1),\dots,\sigma'(s')\}}} (-1)^{L_V-s'+\sigma'^{-1}(r)}\operatorname{sign}\sigma' \\ &\quad\cdot\psi_{P_1}(e_{\sigma'(1)}\cdots e_{\sigma'(s')}) :\exp\left(\tfrac12 b(H_V)\right): \\ &\quad\cdot\psi_{P_1}(e'_{\sigma'(s'+1)}\cdots\widehat{e'_r}\cdots e'_{\sigma'(L_V)}) \\ &= \Psi_0(V)\pi_{P_1}({e'_r}^*).\end{aligned}$$

This completes the proof of (2.87).

2. To show that $\Psi_\alpha(V)$ is isometric, note that one has on $\mathfrak{D}$

$$\psi_{P_1}(g)^*\Psi_0(V) = 0, \qquad g\in\mathfrak{k}_V = P_V(\ker V^*). \tag{2.91}$$

To see this, remember that the basis projection P_V has the form (2.41) $P_V = P_{T_V} - p_{\mathfrak{h}_V} + \overline{p_{\mathfrak{h}_V}}$. Since $\mathfrak{h}_V$ is contained in $\operatorname{ran} V$, we have

$$P_V Q_V = P_{T_V} Q_V \tag{2.92}$$

where $Q_V = \mathbf{1} - VV^*$ is the projection onto $\ker V^*$. It follows that $\mathfrak{k}_V = \ker V^* \cap \operatorname{ran} P_{T_V} = \ker V^* \cap \operatorname{ran}(P_1 + T_V)$. We thus get from $\operatorname{ran}(P_1+T_V) = \ker(P_2 - T_V)$ that $P_2 g = T_V P_1 g$. By Lemma 2.16, this entails

$$\pi_{P_1}(g)^* :\exp\left(\tfrac12 b(H_V)\right): \Omega_{P_1} = 0, \qquad g\in\mathfrak{k}_V,$$

and we get, using (2.87), (2.10) and (2.11), for $f_1, \dots, f_n \in \mathcal{K}$

$$\begin{aligned} &\psi_{P_1}(g)^* \Psi_0(V) \pi_{P_1}(f_1) \cdots \pi_{P_1}(f_n) \Omega_{P_1} \\ &\qquad = -i \pi_{P_1}(Vf_1) \cdots \pi_{P_1}(Vf_n) \Psi(-\mathbf{1}) \pi_{P_1}(g)^* \Psi_0(V) \Omega_{P_1} \\ &\qquad = -i D_V \pi_{P_1}(Vf_1) \cdots \pi_{P_1}(Vf_n) \Psi(-\mathbf{1}) \\ &\qquad\quad \cdot \psi_{P_1}(e_1 \cdots e_{L_V}) \pi_{P_1}(g)^* :\!\exp\!\big(\tfrac{1}{2} b(H_V)\big)\!: \Omega_{P_1} \\ &\qquad = 0 \end{aligned}$$

which proves (2.91).

Since the $\psi_{P_1}(g_j)$ are partial isometries whose source and range projections sum up to $\mathbf{1}$ by the CAR

$$\psi_{P_1}(g_j)^* \psi_{P_1}(g_j) + \psi_{P_1}(g_j) \psi_{P_1}(g_j)^* = \mathbf{1}$$

and because these projections mutually commute for different values of j, it follows that $\psi_{P_1}(g_{\alpha_1}) \cdots \psi_{P_1}(g_{\alpha_l})$ is a partial isometry in $\pi_{P_1}(\varrho_V(\mathfrak{C}(\mathcal{K})))'$ which contains $\operatorname{ran} \Psi_0(V)$ in its initial space. Therefore $\Psi_\alpha(V)$ will be isometric provided that $\Psi_0(V)$ is. We have from (2.86), (2.90), (2.74) and the CAR

$$\begin{aligned} \|\Psi_0(V)\Omega_{P_1}\|^2 &= D_V^2 \langle :\!\exp\!\big(\tfrac{1}{2} b(H_V)\big)\!: \Omega_{P_1}, \psi_{P_1}(e_{L_V}^* \cdots e_1^*) \\ &\qquad \cdot \psi_{P_1}(e_1 \cdots e_{L_V}) :\!\exp\!\big(\tfrac{1}{2} b(H_V)\big)\!: \Omega_{P_1} \rangle \\ &= D_V^2 \| :\!\exp\!\big(\tfrac{1}{2} b(H_V)\big)\!: \Omega_{P_1}\|^2 \\ &= 1. \end{aligned}$$

Using the CAR and the fact that $\Psi_0(V)\Omega_{P_1}$ serves as a vacuum for the transformed annihilation operators, this implies for arbitrary $f_1, \dots, f_m, h_1, \dots, h_n \in \mathcal{K}$

$$\begin{aligned} &\langle \Psi_0(V) \pi_{P_1}(f_1 \cdots f_m) \Omega_{P_1}, \Psi_0(V) \pi_{P_1}(h_1 \cdots h_n) \Omega_{P_1} \rangle \\ &\qquad = \langle \Psi_0(V)\Omega_{P_1}, \pi_{P_1}\big(\varrho_V(f_m^* \cdots f_1^* h_1 \cdots h_n)\big) \Psi_0(V) \Omega_{P_1} \rangle \\ &\qquad = \langle \Omega_{P_1}, \pi_{P_1}(f_m^* \cdots f_1^* h_1 \cdots h_n) \Omega_{P_1} \rangle. \end{aligned}$$

Therefore $\Psi_0(V)$ is isometric on $\mathfrak{D}$ and has a continuous extension to an isometry which satisfies (2.87) on $\mathcal{F}_a(\mathcal{K}_1)$. By the above, the same holds true for the $\Psi_\alpha(V)$.

3. It remains to show that the $\Psi_\alpha(V)$ fulfill the Cuntz relations (2.21) (or (0.5)). Since ψ_{P_1} is a representation of the CAR and by (2.88), (2.91), the $\Psi_\alpha(V)$ are orthonormal:

$$\Psi_\alpha(V)^* \Psi_\beta(V) = \Psi_0(V)^* \psi_{P_1}(g_{\alpha_l}^* \cdots g_{\alpha_1}^*) \psi_{P_1}(g_{\beta_1} \cdots g_{\beta_m}) \Psi_0(V) = \delta_{\alpha\beta} \mathbf{1}$$

for α as above and $\beta = (\beta_1, \dots, \beta_m) \in I_{M_V}$.

The proof of the completeness relation

$$\sum_{\alpha \in I_{M_V}} \Psi_\alpha(V) \Psi_\alpha(V)^* = \mathbf{1} \tag{2.93}$$

relies on the product decomposition $V = U_V W_V$ from Section 2.3. Recall that the orthonormal basis $\{e_1, \dots, e_{L_V}\}$ in $\mathfrak{h}_V = \mathfrak{h}_{U_V}$ was used to define U_V. By Proposition 2.12, U_V maps $\ker {W_V}^*$ unitarily onto $\ker V^*$ and fulfills $U_V P_1 = P_V U_V$. Therefore U_V restricts to a unitary isomorphism from $\mathfrak{k}_{W_V} = P_1(\ker {W_V}^*)$ onto $\mathfrak{k}_V$. We choose

$$f_j \equiv (-1)^{L_V} {U_V}^* g_j, \qquad j = 1, \dots, M_V \quad (M_V = M_{W_V}) \tag{2.94}$$

as orthonormal basis in $\mathfrak{k}_{W_V}$. Applying (2.85) to W_V, we obtain an orthonormal set of isometries satisfying (2.87) with respect to W_V

$$\begin{aligned} &\Psi_\alpha(W_V) = \psi_{P_1}(f_{\alpha_1} \cdots f_{\alpha_l}) :\!\exp\!\big(\tfrac{1}{2} b(H_{W_V})\big)\!:\,, \\ &\text{with } H_{W_V} = \begin{pmatrix} (W_V)_{11} - P_1 & 0 \\ 0 & P_2 - {(W_V)_{22}}^* \end{pmatrix}. \end{aligned}$$

Let us show that this set of isometries is complete. We have

$$\Psi_\alpha(W_V)\Omega_{P_1} = \psi_{P_1}(f_{\alpha_1}\cdots f_{\alpha_l})\Omega_{P_1}. \tag{2.95}$$

Comparing with (2.28), we see that $\operatorname{ran}\Psi_\alpha(W_V)$ is exactly the cyclic (in fact, irreducible) subspace $\mathcal{F}_\alpha(W_V)$ for the representation $\pi_{P_1}\circ\varrho_{W_V}$. Completeness for the $\Psi_\alpha(W_V)$ thus follows from Proposition 2.2:

$$\oplus_\alpha \operatorname{ran}\Psi_\alpha(W_V) = \oplus_\alpha \mathcal{F}_\alpha(W_V) = \mathcal{F}_a(\mathcal{K}_1).$$

Now let $\Psi(U_V)$ be the unitary implementer for ϱ_{U_V} given by (2.85). Then the isometries $\Psi(U_V)\Psi_\alpha(W_V)$ obviously constitute a complete set of implementers for ϱ_V. We are going to show that actually

$$\Psi(U_V)\Psi_\alpha(W_V) = \Psi_\alpha(V) \tag{2.96}$$

holds under the above assumptions. Since each implementer is completely determined by its value on Ω_{P_1} (this follows from (2.24)), it suffices to prove (2.96) when applied to Ω_{P_1}. Note that $\Psi(U_V)\Psi(-\mathbf{1}) = (-1)^{L_V}\Psi(-\mathbf{1})\Psi(U_V)$ so that

$$\Psi(U_V)\psi_{P_1}(f) = (-1)^{L_V}\psi_{P_1}(U_V f)\Psi(U_V), \qquad f\in\mathcal{K}.$$

Hence we obtain from (2.95), (2.86) and Proposition 2.12

$$\begin{aligned}\Psi(U_V)\Psi_\alpha(W_V)\Omega_{P_1} &= (-1)^{lL_V}\psi_{P_1}(U_V f_{\alpha_1}\cdots U_V f_{\alpha_l})\Psi(U_V)\Omega_{P_1}\\ &= D_V\psi_{P_1}(g_{\alpha_1}\cdots g_{\alpha_l})\psi_{P_1}(e_1\cdots e_{L_V})\\ &\quad\cdot\exp(\tfrac{1}{2}\overline{T_V}a^*a^*)\Omega_{P_1}\\ &= \Psi_\alpha(V)\Omega_{P_1}.\end{aligned}$$

Therefore (2.96) holds. Since (2.22) follows from (2.21) and (2.24), the theorem is proven. □

Remark. The proof of Theorem 2.18 shows (cf. (2.96), (2.95), (2.94), (2.5) and Prop. 2.9) that the vectors $\Psi_\alpha(V)\Omega_{P_1}$ are cyclic vectors inducing certain Fock states, viz. the Fock states corresponding to the basis projections

$$P_V^\alpha \equiv P_V - p_{\mathfrak{k}_V^\alpha} + \overline{p_{\mathfrak{k}_V^\alpha}}.$$

Here $\mathfrak{k}_V^\alpha$ is the subspace of $\mathfrak{k}_V$ spanned by the vectors $g_{\alpha_1},\ldots,g_{\alpha_l}$ if l is the length of α. This is a non–trivial fact because linear combinations of such vectors will in general not induce quasi–free states.

Since ψ_{P_1} is a representation of the CAR and by (2.91), the Hilbert space $H(\varrho_V)$ generated by the $\Psi_\alpha(V)$ carries a Fock space structure:

Corollary 2.19.
The map

$$\Psi_\alpha(V)\mapsto a(g_{\alpha_1})^*\cdots a(g_{\alpha_l})^*\Omega, \qquad \alpha=(\alpha_1,\ldots,\alpha_l)\in I_{M_V}$$

extends to a unitary isomorphism from $H(\varrho_V)$ onto the antisymmetric Fock space $\mathcal{F}_a(\mathfrak{k}_V)$ over $\mathfrak{k}_V$.

Here $a(g)^*$ and Ω denote the creation operators and the Fock vacuum in $\mathcal{F}_a(\mathfrak{k}_V)$. We shall see in Section 4.1 that, if V is gauge invariant, then the isomorphism depicted in Corollary 2.19 is not only an isomorphism of (graded) Hilbert spaces but (up to a character of the gauge group) also an isomorphism of G–modules.

2.5. Bosonized statistics. Though the formula (2.85) for $\Psi_\alpha(V)$ looks quite complicated, it is not difficult to write the "Bosonized statistics operator" $\hat{\varepsilon}_V$ associated with $V \in \mathfrak{I}_{P_1}(\mathcal{K})$ as a polynomial in g_j, g_j^* if ϱ_V has finite statistics dimension. Recall from the introduction (see (0.14)) that $\hat{\varepsilon}_V$ is defined in terms of the implementers as

$$\hat{\varepsilon}_V = \sum_{\alpha,\beta\in I_{M_V}} \Psi_\alpha(V)\Psi_\beta(V)\Psi_\alpha(V)^*\Psi_\beta(V)^*. \tag{2.97}$$

Let us first derive a simple formula for the operators $\Psi_\alpha(V)\Psi_\beta(V)^*$. (These operators are matrix units for the commutant $\pi_{P_1}(\varrho_V(\mathfrak{C}(\mathcal{K})))'$.) We will use the following notation for multi–indices $\alpha, \beta \in I_{M_V}$, which is suggested by the identification $\alpha \equiv (\alpha_1, \dots, \alpha_l) \mapsto \{\alpha_1, \dots, \alpha_l\}$ of I_{M_V} with the power set of $\{1, \dots, M_V\}$. $l_\alpha \equiv l$ will denote the length of α, and $\alpha \cap \beta \in I_{M_V}$ will denote the multi–index whose entries are the elements of the intersection of the entries of α and β. $\alpha^c \in I_{M_V}$ will be the "complementary" multi–index whose entries are the elements of $\{1, \dots, M_V\} \setminus \{\alpha_1, \dots, \alpha_{l_\alpha}\}$. We further set

$$g_\alpha \equiv g_{\alpha_1} \cdots g_{\alpha_{l_\alpha}},$$
$$\Gamma_{\alpha\beta} \equiv g_\alpha (g_{\alpha^c\cap\beta^c})^* g_{\alpha^c\cap\beta^c} {g_\beta}^* = {\Gamma_{\beta\alpha}}^*, \qquad \alpha, \beta \in I_{M_V}.$$

LEMMA 2.20.
Let $V \in \mathfrak{I}_{P_1}(\mathcal{K})$ with $-\operatorname{ind} V < \infty$, and let $\alpha, \beta \in I_{M_V}$. Then

$$\Psi_\alpha(V)\Psi_\beta(V)^* = \psi_{P_1}(\Gamma_{\alpha\beta}).$$

Proof. Let $A_\alpha \equiv \psi_{P_1}(g_\alpha {g_{\alpha^c}}^* g_{\alpha^c})$. One has, by the CAR and by (2.91), (2.88)

$$A_\alpha \Psi_0(V)\Psi_0(V)^* = \psi_{P_1}(g_\alpha)\Psi_0(V)\Psi_0(V)^* = \Psi_\alpha(V)\Psi_0(V)^*.$$

If $\alpha' \neq 0$ is another multi–index in I_{M_V}, then

$$A_\alpha \Psi_{\alpha'}(V) = A_\alpha \psi_{P_1}(g_{\alpha'})\Psi_0(V) = 0$$

because $g_j^2 = 0,\ j = 1, \dots, M_V$. Hence we obtain from (2.93)

$$A_\alpha = A_\alpha \sum_{\beta\in I_{M_V}} \Psi_\beta(V)\Psi_\beta(V)^* = A_\alpha \Psi_0(V)\Psi_0(V)^* = \Psi_\alpha(V)\Psi_0(V)^*.$$

This entails, for arbitrary $\alpha, \beta \in I_{M_V}$,

$$\begin{aligned}\Psi_\alpha(V)\Psi_\beta(V)^* &= \Psi_\alpha(V)\Psi_0(V)^*\big(\Psi_\beta(V)\Psi_0(V)^*\big)^* \\ &= A_\alpha {A_\beta}^* \\ &\equiv \psi_{P_1}(g_\alpha {g_{\alpha^c}}^* g_{\alpha^c} {g_{\beta^c}}^* g_{\beta^c} {g_\beta}^*).\end{aligned}$$

Now, by the CAR, g_α commutes with ${g_{\alpha^c}}^* g_{\alpha^c}$, ${g_\beta}^*$ commutes with ${g_{\beta^c}}^* g_{\beta^c}$, and one has $g_\alpha (g_{\beta^c\cap\alpha})^* g_{\beta^c\cap\alpha} = g_\alpha$ and $(g_{\alpha^c\cap\beta})^* g_{\alpha^c\cap\beta} {g_\beta}^* = {g_\beta}^*$. Thus we finally get $\Psi_\alpha(V)\Psi_\beta(V)^* = \psi_{P_1}\big(g_\alpha (g_{\alpha^c\cap\beta^c})^* g_{\alpha^c\cap\beta^c} {g_\beta}^*\big) = \psi_{P_1}(\Gamma_{\alpha\beta})$. ☐

Remark. 1. As a special case, one obtains the projections onto $\operatorname{ran} \Psi_\alpha(V)$

$$\Psi_\alpha(V)\Psi_\alpha(V)^* = \pi_{P_1}(g_\alpha {g_\alpha}^* {g_{\alpha^c}}^* g_{\alpha^c}),$$

(we used (2.9)) from which one directly sees that (2.93) holds

$$\sum_\alpha \Psi_\alpha(V)\Psi_\alpha(V)^* = \pi_{P_1}\big((g_1 {g_1}^* + {g_1}^* g_1) \cdots (g_{M_V} {g_{M_V}}^* + {g_{M_V}}^* g_{M_V})\big) = \mathbf{1}.$$

2. If $\operatorname{ind} V = -\infty$, then one still has

$$\Psi_\alpha(V)\Psi_\beta(V)^* = \psi_{P_1}(g_\alpha)\Psi_0(V)\Psi_0(V)^*\psi_{P_1}({g_\beta}^*),$$

where $\Psi_0(V)\Psi_0(V)^*$ can be obtained as a strong limit

$$\Psi_0(V)\Psi_0(V)^* = \operatorname*{s-lim}_{n\to\infty} \pi_{P_1}({g_1}^* g_1 \cdots {g_n}^* g_n).$$

But this projection is no longer contained in $\mathfrak{C}(\mathcal{K})$.

PROPOSITION 2.21.
Let V be as in Lemma 2.20. Then the Bosonized statistics operator $\hat{\varepsilon}_V$ defined by (2.97) can be written as

$$\begin{aligned} \hat{\varepsilon}_V &= \pi_{P_1}(\tilde{\varepsilon}_V), \\ \tilde{\varepsilon}_V &= \sum_{\alpha,\beta\in I_{M_V}} (-1)^{(l_\alpha+l_\beta)(l_\beta+L_V)}\Gamma_{\alpha\beta}\varrho_V(\Gamma_{\beta\alpha}) \in \mathfrak{C}(\mathcal{K})_0. \end{aligned} \tag{2.98}$$

Proof. It follows from (2.85) that

$$\Psi(-\mathbf{1})\Psi_\alpha(V)\Psi(-\mathbf{1}) = (-1)^{(l_\alpha+L_V)}\Psi_\alpha(V). \tag{2.99}$$

Therefore one has for $f \in \mathcal{K}$, using (2.10)

$$\begin{aligned} \Psi_\alpha(V)\psi_{P_1}(f) &= i\Psi_\alpha(V)\pi_{P_1}(f)\Psi(-\mathbf{1}) \\ &= i(-1)^{(l_\alpha+L_V)}\pi_{P_1}(Vf)\Psi(-\mathbf{1})\Psi_\alpha(V) \\ &= (-1)^{(l_\alpha+L_V)}\psi_{P_1}(Vf)\Psi_\alpha(V), \end{aligned}$$

and hence

$$\psi_{P_1}(g_\beta)\Psi_\alpha(V)^* = (-1)^{(l_\alpha+L_V)l_\beta}\Psi_\alpha(V)^*\psi_{P_1}(\varrho_V(g_\beta)), \quad \alpha,\beta \in I_{M_V}.$$

Thus one gets from Lemma 2.20

$$\begin{aligned} \hat{\varepsilon}_V &= \sum_{\alpha,\beta}\Psi_\alpha(V)\Psi_\beta(V)\Psi_\alpha(V)^*\Psi_\beta(V)^* \\ &= \sum_{\alpha,\beta}\Psi_\alpha(V)\psi_{P_1}(\Gamma_{\beta\alpha})\Psi_\beta(V)^* \\ &= \sum_{\alpha,\beta}(-1)^{(l_\beta+L_V)(l_\alpha+l_\beta)}\Psi_\alpha(V)\Psi_\beta(V)^*\psi_{P_1}\big(\varrho_V(\Gamma_{\beta\alpha})\big) \\ &= \sum_{\alpha,\beta}(-1)^{(l_\beta+L_V)(l_\alpha+l_\beta)}\psi_{P_1}\big(\Gamma_{\alpha\beta}\varrho_V(\Gamma_{\beta\alpha})\big) \\ &= \psi_{P_1}(\tilde{\varepsilon}_V), \end{aligned}$$

with $\tilde{\varepsilon}_V$ as above. But $\tilde{\varepsilon}_V$ is even so that $\psi_{P_1}(\tilde{\varepsilon}_V) = \pi_{P_1}(\tilde{\varepsilon}_V)$ by (2.9). □

To check the consistency of our constructions, let us finally calculate the "Bosonized statistics parameter"

$$\hat{\lambda}_V \equiv \pi_{P_1}(\phi_V(\tilde{\varepsilon}_V))$$

which is associated with the Bosonized statistics operator and with the left inverse ϕ_V from Section 2.1 (see (2.15)). Recall that ϕ_V is given by

$$\phi_V(ab) = \varrho_V^{-1}(a)\omega_{1/2}(b) \quad \text{if } a \in \mathfrak{C}(\operatorname{ran} V),\ b \in \mathfrak{C}(\ker V^*),$$

where $\omega_{\frac{1}{2}}$ is the trace on $\mathfrak{C}(\mathcal{K})$.

COROLLARY 2.22.
The "Bosonized statistics parameter" $\hat{\lambda}_V$ of $V \in \mathfrak{I}_{P_1}(\mathcal{K})$ with $-\operatorname{ind} V < \infty$ equals

$$\hat{\lambda}_V = \frac{1}{d_V}\mathbf{1}.$$

Proof. It follows from $\Gamma_{\alpha\beta} \in \mathfrak{C}(\ker V^*)$ that $\Gamma_{\alpha\beta}\varrho_V(\Gamma_{\beta\alpha}) = (-1)^{l_\alpha+l_\beta}\varrho_V(\Gamma_{\beta\alpha})\Gamma_{\alpha\beta}$ and that $\phi_V(\varrho_V(\Gamma_{\beta\alpha})\Gamma_{\alpha\beta}) = \Gamma_{\beta\alpha}\omega_{\frac{1}{2}}(\Gamma_{\alpha\beta})$. We claim that

$$\omega_{\frac{1}{2}}(\Gamma_{\alpha\beta}) = \frac{1}{d_V}\cdot\delta_{\alpha\beta}.$$

Consider first a term of the form $\omega_{\frac{1}{2}}(g_\alpha{}^* g_\alpha)$. Since the g_j are mutually orthogonal, one has $\omega_{\frac{1}{2}}(g_\alpha{}^* g_\alpha) = \omega_{\frac{1}{2}}(g_{\alpha_1}{}^* g_{\alpha_1}) \cdots \omega_{\frac{1}{2}}(g_{\alpha_{l_\alpha}}{}^* g_{\alpha_{l_\alpha}}) = 2^{-l_\alpha}$. Next assume that $\alpha \neq \beta$. Without loss of generality, assume that α_1 does not occur in β. Then there exists a quasi–free automorphism ϱ which maps g_{α_1} to $-g_{\alpha_1}$ and leaves all other g_j unchanged. It follows that $\varrho(g_\alpha{}^* g_\beta) = -g_\alpha{}^* g_\beta$ so that $\omega_{\frac{1}{2}}(g_\alpha{}^* g_\beta) = 0$, because $\omega_{\frac{1}{2}}$ is invariant under ϱ. This yields

$$\begin{aligned}\omega_{\frac{1}{2}}(\Gamma_{\alpha\beta}) &= \omega_{\frac{1}{2}}(g_\beta{}^* g_\alpha (g_{\alpha^c\cap\beta^c})^* g_{\alpha^c\cap\beta^c}) = \omega_{\frac{1}{2}}(g_\beta{}^* g_\alpha)\omega_{\frac{1}{2}}((g_{\alpha^c\cap\beta^c})^* g_{\alpha^c\cap\beta^c}) \\ &= 2^{-M_V}\delta_{\alpha\beta} = d_V^{-1}\delta_{\alpha\beta}\end{aligned}$$

as claimed. Hence we obtain

$$\begin{aligned}\phi_V(\tilde{\varepsilon}_V) &= \sum_{\alpha,\beta}(-1)^{(l_\alpha+l_\beta)(l_\beta+L_V)}\phi_V\big(\Gamma_{\alpha\beta}\varrho_V(\Gamma_{\beta\alpha})\big) \\ &= \sum_{\alpha,\beta}(-1)^{(l_\alpha+l_\beta)(l_\beta+L_V+1)}\Gamma_{\beta\alpha}\cdot\omega_{\frac{1}{2}}(\Gamma_{\alpha\beta}) \\ &= \frac{1}{d_V}\sum_\alpha \Gamma_{\alpha\alpha} \\ &= \frac{1}{d_V}\mathbf{1}.\end{aligned}$$

□

It is clear that one gets the same result by applying the left inverse $\phi_{H(\varrho_V)}$ from Section 2.2 (or from the Introduction, Eq. (0.15)) to $\hat{\varepsilon}_V$. Recall that $\phi_{H(\varrho_V)}$ is a left inverse for the normal extension of ϱ_V to $\mathfrak{B}(\mathcal{F}_a(\mathcal{K}_1))$, given by

$$\phi_{H(\varrho_V)}(x) = \frac{1}{d_V}\sum_\alpha \Psi_\alpha(V)^* x \Psi_\alpha(V), \qquad x \in \mathfrak{B}(\mathcal{F}_a(\mathcal{K}_1)).$$

One can show, by similar computations as above, that $\phi_{H(\varrho_V)}$ extends ϕ_V:

$$\phi_{H(\varrho_V)}(\pi_{P_1}(a)) = \pi_{P_1}(\phi_V(a)), \qquad a \in \mathfrak{C}(\mathcal{K}).$$

3. Quasi–free Endomorphisms of the CCR Algebra

This section contains an analysis of the semigroup of quasi–free endomorphisms of the CCR algebra similar to the analysis done in Section 2 for the CAR algebra. Generally speaking, the CCR case is algebraically simpler, but the analytic aspects are more involved. For an introduction to the CCR algebra see the textbooks [BR81, Pet90].

3.1. The selfdual CCR algebra. Let $\mathcal{K}^0$ be an infinite dimensional complex linear space, equipped with a nondegenerate hermitian sesquilinear form κ and an antilinear involution $f \mapsto f^*$, such that

$$\kappa(f^*, g^*) = -\kappa(g, f), \qquad f, g \in \mathcal{K}^0.$$

One should think of $\mathcal{K}^0$ as being the complexification of the real linear space

$$\mathrm{Re}\,\mathcal{K}^0 \equiv \{f \in \mathcal{K}^0 \mid f^* = f\},$$

together with the canonical conjugation on $\mathcal{K}^0 = \mathbb{C}\otimes_{\mathbb{R}} \mathrm{Re}\,\mathcal{K}^0$. $-i\kappa$ should be viewed as the sesquilinear extension of a nondegenerate symplectic form on $\mathrm{Re}\,\mathcal{K}^0$.

The *(selfdual) CCR algebra* $\mathfrak{C}(\mathcal{K}^0, \kappa)$ [AS72, Ara72, AY82] over $(\mathcal{K}^0, \kappa)$ is the simple *–algebra which is generated by $\mathbf{1}$ and elements $f \in \mathcal{K}^0$, subject to the commutation relation

$$[f^*, g] = \kappa(f, g)\mathbf{1}, \qquad f, g \in \mathcal{K}^0. \tag{3.1}$$

We henceforth assume the existence of a distinguished Fock state over $\mathfrak{C}(\mathcal{K}^0, \kappa)$. As in the CAR case, Fock states correspond to basis projections. A linear operator P_1, defined on the whole of $\mathcal{K}^0$, is a *basis projection* of $(\mathcal{K}^0, \kappa)$ if it satisfies for $f, g \in \mathcal{K}^0$

$$\begin{aligned} P_1^2 &= P_1, \qquad && \kappa(f, P_1 g) = \kappa(P_1 f, g), \\ P_1 + \overline{P_1} &= \mathbf{1}, \qquad && \kappa(f, P_1 f) > 0 \quad \text{if } P_1 f \neq 0. \end{aligned} \tag{3.2}$$

Here we used the notation (2.2)

$$\overline{P_1} f \equiv P_1(f^*)^*$$

for the complex conjugate operator. Let

$$P_2 \equiv \mathbf{1} - P_1, \quad C \equiv P_1 - P_2, \quad \langle f, g\rangle_{P_1} \equiv \kappa(f, Cg).$$

The positive definite inner product $\langle\ ,\ \rangle_{P_1}$ turns $\mathcal{K}^0$ into a pre–Hilbert space. We assume that the completion $\mathcal{K}$ is separable. By continuity, the involution "$*$" extends to a conjugation on $\mathcal{K}$, P_1 and P_2 to orthogonal projections, C to a self–adjoint unitary, and κ to a nondegenerate hermitian form. These extensions will be denoted by the same symbols. Setting

$$\mathcal{K}_n \equiv P_n(\mathcal{K}), \qquad n = 1, 2,$$

we get a direct sum decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ which is orthogonal with respect to both κ and $\langle\ ,\ \rangle_{P_1}$. The following notations will frequently be used for $A \in \mathfrak{B}(\mathcal{K})$:

$$\begin{aligned} A_{mn} &\equiv P_m A P_n, \quad m, n = 1, 2, \\ A^\dagger &\equiv C A^* C \\ A^\tau &\equiv \overline{A^*}. \end{aligned}$$

The components A_{mn} of A are regarded as operators from $\mathcal{K}_n$ to $\mathcal{K}_m$, and A will sometimes be written as a matrix $\left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right)$. $A^\dagger$ is the adjoint of A relative to κ, while A^* is the Hilbert space adjoint. Thus one has relations like

$$\overline{P_2} = P_1 = P_1^\dagger = P_1^*, \quad \overline{C} = -C, \quad {A_{12}}^\dagger = {A^\dagger}_{21} = -{A_{12}}^*, \quad \overline{A_{11}} = \overline{A}_{22}, \quad \text{etc.}$$

The *Fock state* ω_{P_1} is the unique state[s] which is annihilated by all $f \in \operatorname{ran} P_2$:

$$\omega_{P_1}(f^*f) = 0 \quad \text{if } P_1 f = 0.$$

(In the conventional setting mentioned above, ω_{P_1} is the Fock state corresponding to the complex structure iC on $\operatorname{Re}\mathcal{K}$.) Let $\mathcal{F}_s(\mathcal{K}_1)$ be the symmetric Fock space over $\mathcal{K}_1$ and let $\mathfrak{D}$ be the dense subspace of algebraic tensors. A GNS representation π_{P_1} for ω_{P_1} is provided by

$$\pi_{P_1}(f) = a^*(P_1 f) + a(P_1 f^*), \quad f \in \mathcal{K}$$

where $a^*(g)$ and $a(g)$, $g \in \mathcal{K}_1$, are the usual (Bosonic) creation and annihilation operators on $\mathfrak{D}$. The cyclic vector inducing the state ω_{P_1} is Ω_{P_1}, the Fock vacuum. The operators $\pi_{P_1}(a)$, $a \in \mathfrak{C}(\mathcal{K},\kappa)$, have invariant domain $\mathfrak{D}$, are closable, and one has $\pi_{P_1}(a^*) \subset \pi_{P_1}(a)^*$. In particular, if $f \in \operatorname{Re}\mathcal{K}$, then $\pi_{P_1}(f)$ is essentially self–adjoint on $\mathfrak{D}$, and the unitary *Weyl operator* $w(f)$ is defined as the exponential of the closure of $i\pi_{P_1}(f)$. Its vacuum expectation value is

$$\omega_{P_1}(w(f)) \equiv \langle \Omega_{P_1}, w(f)\Omega_{P_1}\rangle = e^{-\frac{1}{4}\|f\|^2_{P_1}},$$

and the *Weyl relations* hold

$$w(f)w(g) = e^{-\frac{1}{2}\kappa(f,g)}w(f+g), \quad f,g \in \operatorname{Re}\mathcal{K}.$$

The Weyl operators generate a simple C^*–algebra $\mathfrak{W}(\mathcal{K},\kappa)$ which acts irreducibly on $\mathcal{F}_s(\mathcal{K}_1)$. If $\mathcal{H}$ is a subspace of $\mathcal{K}$ with $\mathcal{H} = \mathcal{H}^*$, then the C^*–algebra generated by all $w(f)$ with $f \in \operatorname{Re}\mathcal{H}$ will be denoted by $\mathfrak{W}(\mathcal{H})$. If $\mathcal{H}^\sharp$ is the orthogonal complement of $\mathcal{H}$ with respect to κ, then *duality* holds [Ara63, AS72, AY82]:

$$\mathfrak{W}(\mathcal{H})' = \mathfrak{W}(\mathcal{H}^\sharp)''. \tag{3.3}$$

LEMMA 3.1.
For $f \in \mathcal{K}$, let $\mathcal{H}_f$ be the subspace spanned by f and f^*. Then the closure of $\pi_{P_1}(f)$ is affiliated with $\mathfrak{W}(\mathcal{H}_f)''$.

Proof. Let T be the closure of $\pi_{P_1}(f)$, with domain $D(T)$. We have to show that, for any $A \in \mathfrak{W}(\mathcal{H}_f)'$

$$A(D(T)) \subset D(T), \qquad AT = TA \text{ on } D(T).$$

Now by virtue of the CCR (3.1), $\|T\phi\|^2 = \|T^*\phi\|^2 + \kappa(f,f)\|\phi\|^2$ for $\phi \in \mathfrak{D}$. Hence, for a given Cauchy sequence $\phi_n \in \mathfrak{D}$, $T\phi_n$ converges if and only if $T^*\phi_n$ does. This implies that

$$D(T) = D(T^*).$$

Let $f^\pm \in \operatorname{Re}\mathcal{H}_f$ be defined as $f^+ \equiv \frac{1}{2}(f+f^*)$, $f^- \equiv \frac{i}{2}(f-f^*)$, and let $T^\pm$ be the (self–adjoint) closure of $\pi_{P_1}(f^\pm)$. We claim that

$$D(T) = D(T^+) \cap D(T^-), \qquad T = T^+ - iT^- \text{ on } D(T).$$

For if $\phi \in D(T)$, then there exists a sequence $\phi_n \in \mathfrak{D}$ converging to ϕ such that $\pi_{P_1}(f)\phi_n$ and $\pi_{P_1}(f^*)\phi_n$ converge. Thus ϕ belongs to the domain of the closure of $\pi_{P_1}(f^\pm)$. Conversely, if $\phi \in D(T^+) \cap D(T^-)$, then there exists a sequence $\phi_n \in \mathfrak{D}$ converging to ϕ such that both $\pi_{P_1}(f+f^*)\phi_n$ and $\pi_{P_1}(f-f^*)\phi_n$ converge (this follows from the detailed description of the domains of such operators given in [Rui78]). Therefore $\pi_{P_1}(f)\phi_n$ is also convergent, i.e. ϕ is contained in $D(T)$, and $T\phi = (T^+ - iT^-)\phi$.

Now if $A \in \mathfrak{W}(\mathcal{H}_f)'$, then A commutes with the one–parameter unitary groups $w(tf^\pm) = \exp(itT^\pm)$. As a consequence, A leaves $D(T^\pm)$ invariant and commutes with $T^\pm$ on $D(T^\pm)$. It follows that $A(D(T)) \subset D(T)$ and $AT = TA$ on $D(T)$ as was to be shown. □

[s] A *state* ω over $\mathfrak{C}(\mathcal{K},\kappa)$ is a linear functional with $\omega(\mathbf{1}) = 1$ and $\omega(a^*a) \geq 0$, $a \in \mathfrak{C}(\mathcal{K},\kappa)$.

3.2. Implementability of quasi–free endomorphisms. *Quasi–free endomorphisms* are the unital *–endomorphisms of $\mathfrak{C}(\mathcal{K},\kappa)$ which map $\mathcal{K}$, viewed as a subspace of $\mathfrak{C}(\mathcal{K},\kappa)$, into itself. They are completely determined by their restrictions to $\mathcal{K}$ which are called *Bogoliubov operators.* Hence $V \in \mathfrak{B}(\mathcal{K})$ is a Bogoliubov operator[t] if and only if it commutes with complex conjugation and preserves the hermitian form κ. Bogoliubov operators form a unital semigroup which we denote by

$$\mathcal{S}(\mathcal{K},\kappa) \equiv \{V \in \mathfrak{B}(\mathcal{K}) \mid \overline{V} = V,\ V^\dagger V = \mathbf{1}\}.$$

Each $V \in \mathcal{S}(\mathcal{K},\kappa)$ extends to a unique quasi–free endomorphism of $\mathfrak{C}(\mathcal{K},\kappa)$ and to a unique *–endomorphism of $\mathfrak{W}(\mathcal{K},\kappa)$. By abuse of notation, both endomorphisms are denoted by ϱ_V, so that $\varrho_V(f) = Vf,\ f \in \mathcal{K}$, and $\varrho_V(w(g)) = w(Vg),\ g \in \operatorname{Re}\mathcal{K}$.

The condition $V^\dagger V = \mathbf{1}$ entails that V is injective and V^* surjective; hence $\operatorname{ran} V$ is closed, and V is a semi–Fredholm operator [Kat66]. We claim that the Fredholm index $-\operatorname{ind} V = \dim\ker V^\dagger$ cannot be odd, in contrast to the CAR case (cf. (2.3)). For let $f \in \ker V^\dagger$ such that $0 = \kappa(f,g) \equiv \langle f, Cg\rangle_{P_1}\ \forall g \in \ker V^\dagger$. Then $f \in (C\ker V^\dagger)^\perp = (\ker V^*)^\perp = \operatorname{ran} V$, but $\operatorname{ran} V \cap \ker V^\dagger = \{0\}$ due to $V^\dagger V = \mathbf{1}$, so f has to vanish. This shows that the restriction of κ to $\ker V^\dagger$ stays nondegenerate. It follows that $\dim\ker V^\dagger$ cannot be odd (there is no nondegenerate symplectic form on an odd dimensional space).

On the other hand, each even number (and ∞) occurs as $\dim\ker V^\dagger$ for some V. Hence we have an epimorphism of semigroups

$$\mathcal{S}(\mathcal{K},\kappa) \to \mathbb{N} \cup \{\infty\}, \quad V \mapsto -\tfrac{1}{2}\operatorname{ind} V = \tfrac{1}{2}\dim\ker V^\dagger$$

(remember that $0 \in \mathbb{N}$). Let

$$\mathcal{S}^n(\mathcal{K},\kappa) \equiv \{V \in \mathcal{S}(\mathcal{K},\kappa) \mid \operatorname{ind} V = -2n\}, \quad n \in \mathbb{N}\cup\{\infty\}.$$

$\mathcal{S}^0(\mathcal{K},\kappa)$ is the group of quasi–free automorphisms (isomorphic to the symplectic group of $\operatorname{Re}\mathcal{K}$). It acts on $\mathcal{S}(\mathcal{K},\kappa)$ by left multiplication. Analogous to the CAR case, the orbits under this action are the subsets $\mathcal{S}^n(\mathcal{K},\kappa)$, and the stabilizer of $V \in \mathcal{S}^n(\mathcal{K},\kappa)$ is isomorphic to the symplectic group $\operatorname{Sp}(n)$.

We are interested in endomorphisms ϱ_V which can be implemented by Hilbert spaces of isometries on $\mathcal{F}_s(\mathcal{K}_1)$. This means that there exist isometries Ψ_j on $\mathcal{F}_s(\mathcal{K}_1)$ which fulfill the Cuntz algebra relations (2.21) and implement ϱ_V according to (2.22):

$$\varrho_V(w(f)) = \sum_j \Psi_j w(f) \Psi_j^*, \quad f \in \operatorname{Re}\mathcal{K}.$$

As explained in Section 2.2, such isometries exist if and only if ϱ_V, viewed as a representation of $\mathfrak{W}(\mathcal{K},\kappa)$ on $\mathcal{F}_s(\mathcal{K}_1)$, is quasi–equivalent to the defining (Fock) representation.

To study ϱ_V as a representation, for fixed $V \in \mathcal{S}(\mathcal{K},\kappa)$, let us decompose it into cyclic subrepresentations. Let $f_1, f_2, \ldots$ be an orthonormal basis in $\mathcal{K}_1 \cap \ker V^\dagger$ and let $\alpha = (\alpha_1, \ldots, \alpha_l)$ be a multi–index with $\alpha_j \leq \alpha_{j+1}$. Such α has the form

$$\alpha = (\underbrace{\alpha_1', \ldots, \alpha_1'}_{l_1}, \underbrace{\alpha_2', \ldots, \alpha_2'}_{l_2}, \ldots, \underbrace{\alpha_r', \ldots, \alpha_r'}_{l_r}) \tag{3.4}$$

[t]We may disregard unbounded Bogoliubov operators V (defined on $\mathcal{K}^0$) since the topologies induced by the corresponding states $\omega_{P_1} \circ \varrho_V$ on $\mathcal{K}^0$ differ from the one induced by ω_{P_1}. Hence these states cannot be quasi–equivalent to ω_{P_1} (cf. [Ara72, AY82]), and ϱ_V cannot be implemented.

with $\alpha'_1 < \alpha'_2 < \cdots < \alpha'_r$ and $l_1 + \cdots + l_r = l$. Let

$$\begin{aligned} c_\alpha &\equiv (l_1! \cdots l_r!)^{-\frac{1}{2}} \\ \phi_\alpha &\equiv c_\alpha a^*(f_{\alpha_1}) \cdots a^*(f_{\alpha_l})\Omega_{P_1}, \\ \mathfrak{F}_\alpha &\equiv \overline{\mathfrak{W}(\operatorname{ran} V)\phi_\alpha}, \\ \pi_\alpha &\equiv \varrho_V|_{\mathfrak{F}_\alpha}. \end{aligned} \tag{3.5}$$

The Bosonic analogue of Proposition 2.2 is

PROPOSITION 3.2.
One has $\varrho_V = \oplus_\alpha \pi_\alpha$, where the sum extends over all multi–indices α as above, including $\alpha = 0$ ($\phi_0 \equiv \Omega_{P_1}$). Each $(\pi_\alpha, \mathfrak{F}_\alpha, \phi_\alpha)$ is a GNS representation for $\omega_{P_1} \circ \varrho_V$ (regarded as a state over $\mathfrak{W}(\mathcal{K}, \kappa)$).

Proof. By definition, the ϕ_α constitute an orthonormal basis for $\mathfrak{F}_s(\mathcal{K}_1 \cap \ker V^\dagger)$, and $(\pi_\alpha, \mathfrak{F}_\alpha, \phi_\alpha)$ is a cyclic representation of $\mathfrak{W}(\mathcal{K}, \kappa)$. Since the closures of $a^*(f_j)$ and $a(f_j)$ are affiliated with $\mathfrak{W}(\ker V^\dagger)'' = \mathfrak{W}(\operatorname{ran} V)'$ (see Lemma 3.1 and (3.3)), one obtains for $f \in \operatorname{Re}\mathcal{K}$

$$\begin{aligned} \langle \phi_\alpha, \pi_\alpha(w(f))\phi_\alpha\rangle &= c_\alpha^2 \langle a^*(f_{\alpha_1}) \cdots a^*(f_{\alpha_l})\Omega_{P_1}, w(Vf) a^*(f_{\alpha_1}) \cdots a^*(f_{\alpha_l})\Omega_{P_1}\rangle \\ &= c_\alpha^2 \langle \Omega_{P_1}, w(Vf)\, \underbrace{a(f_{\alpha_l}) \cdots a(f_{\alpha_1}) a^*(f_{\alpha_1}) \cdots a^*(f_{\alpha_l})\Omega_{P_1}}_{c_\alpha^{-2}\Omega_{P_1}}\rangle \\ &= \langle \Omega_{P_1}, w(Vf)\Omega_{P_1}\rangle. \end{aligned}$$

This proves that $(\pi_\alpha, \mathfrak{F}_\alpha, \phi_\alpha)$ is a GNS representation for $\omega_{P_1} \circ \varrho_V$. Similarly, one finds that $\langle \phi_\alpha, w(Vf)\phi_{\alpha'}\rangle = 0$ for $\alpha \neq \alpha'$, so the $\mathfrak{F}_\alpha$ are mutually orthogonal.

It remains to show that $\oplus_\alpha \mathfrak{F}_\alpha = \mathfrak{F}_s(\mathcal{K}_1)$. We claim that $\mathfrak{F}_0$ equals $\mathfrak{F}_s(\overline{\operatorname{ran} P_1 V})$, the symmetric Fock space over the closure of $\operatorname{ran} P_1 V$. The inclusion $\mathfrak{F}_0 \subset \mathfrak{F}_s(\overline{\operatorname{ran} P_1 V})$ holds because vectors of the form $w(Vf)\Omega_{P_1} = \exp i\big(a^*(P_1 V f) + a(P_1 V f)\big)\Omega_{P_1} \in \mathfrak{F}_s(\overline{\operatorname{ran} P_1 V})$ are total in $\mathfrak{F}_0$. The converse inclusion may be proved inductively. Assume that $a^*(g_1) \cdots a^*(g_m)\Omega_{P_1}$ is contained in $\mathfrak{F}_0$ for all $m \leq n$, $g_1, \ldots, g_m \in \operatorname{ran} P_1 V$. Then, for $f \in V(\operatorname{Re}\mathcal{K})$ and $g_1, \ldots, g_n \in \operatorname{ran} P_1 V$, $\frac{1}{i}\frac{w(tf) - \mathbf{1}}{t} a^*(g_1) \cdots a^*(g_n)\Omega_{P_1}$ has a limit $a^*(P_1 f) a^*(g_1) \cdots a^*(g_n)\Omega_{P_1} + a(P_1 f) a^*(g_1) \cdots a^*(g_n)\Omega_{P_1}$ in $\mathfrak{F}_0$ as $t \searrow 0$. By assumption, the second term lies in $\mathfrak{F}_0$, and so does the first. Since each $g \in \operatorname{ran} P_1 V$ is a linear combination of such $P_1 f$, it follows that $a^*(g_1) \cdots a^*(g_{n+1})\Omega_{P_1}$ is contained in $\mathfrak{F}_0$ for arbitrary $g_j \in \operatorname{ran} P_1 V$, and, by induction, for arbitrary $n \in \mathbb{N}$. But such vectors span a dense subspace in $\mathfrak{F}_s(\overline{\operatorname{ran} P_1 V})$, so $\mathfrak{F}_0 = \mathfrak{F}_s(\overline{\operatorname{ran} P_1 V})$ as claimed.

Finally, $\mathcal{K}_1 \cap \ker V^\dagger$ equals $\ker V^* P_1$, where $V^* P_1$ is regarded as an operator from $\mathcal{K}_1$ to $\mathcal{K}$. Thus we have $\mathcal{K}_1 = \overline{\operatorname{ran} P_1 V} \oplus (\mathcal{K}_1 \cap \ker V^\dagger)$ and $\mathfrak{F}_s(\mathcal{K}_1) \cong \mathfrak{F}_0 \otimes \mathfrak{F}_s(\mathcal{K}_1 \cap \ker V^\dagger)$. Under this isomorphism, $\mathfrak{F}_\alpha$ is identified with $\mathfrak{F}_0 \otimes (\mathbb{C}\phi_\alpha)$. Since the ϕ_α form an orthonormal basis for $\mathfrak{F}_s(\mathcal{K}_1 \cap \ker V^\dagger)$, the desired result $\oplus_\alpha \mathfrak{F}_\alpha = \mathfrak{F}_s(\mathcal{K}_1)$ follows. □

As a consequence, the representation ϱ_V is quasi–equivalent to the GNS representation associated with the quasi–free state $\omega_{P_1} \circ \varrho_V$. So ϱ_V is implementable if and only if $\omega_{P_1} \circ \varrho_V$ and ω_{P_1} are quasi–equivalent. Now the two–point function of $\omega_{P_1} \circ \varrho_V$ (as a state over $\mathfrak{C}(\mathcal{K}, \kappa)$) is given by

$$\omega_{P_1} \circ \varrho_V(f^* g) = \kappa(f, Sg) = \langle f, \tilde{S} g\rangle_{P_1}, \quad f, g \in \mathcal{K},$$

with

$$S \equiv V^\dagger P_1 V, \qquad \tilde{S} \equiv V^* P_1 V.$$

The latter operators contain valuable information about $\omega_{P_1} \circ \varrho_V$. For example, it can be shown (cf. [MV68]) that $\omega_{P_1} \circ \varrho_V$ is a *pure* state over $\mathfrak{W}(\mathcal{K}, \kappa)$ if and only if S is a basis projection, that is, if and only if S is idempotent (the remaining conditions in (3.2) are automatically fulfilled). This is further equivalent to $[P_1, VV^\dagger] = 0$, by the following argument:

$$\begin{aligned}
S^2 = S \quad &\Leftrightarrow \quad 0 = S\overline{S} && (\text{since } \overline{S} = \mathbf{1} - S) \\
&\Leftrightarrow \quad 0 = V^* P_1 V C V^* P_2 V && \\
&\Leftrightarrow \quad 0 = P_1 V C V^* P_2 && (\text{since } \operatorname{ran} V^* P_2 V = \operatorname{ran} V^* P_2 \\
& && \text{and } \ker V^* P_1 V = \ker P_1 V) \\
&\Leftrightarrow \quad 0 = P_1 V V^\dagger P_2 && \\
&\Leftrightarrow \quad 0 = [P_1, VV^\dagger]. &&
\end{aligned}$$

On the other hand, the criterion for quasi–equivalence of quasi–free states, in the form given by Araki and Yamagami [AY82] (see also [AS72, Ara72, vD71]), yields that $\omega_{P_1} \circ \varrho_V$ is quasi–equivalent to ω_{P_1} if and only if $P_1 - \tilde{S}^{\frac{1}{2}}$ is a Hilbert–Schmidt operator on $\mathcal{K}$. Using Theorem 2.3, this condition can be simplified:

THEOREM 3.3.
Let a Bogoliubov operator $V \in \mathcal{S}(\mathcal{K}, \kappa)$ be given. Then there exists a Hilbert space of isometries $H(\varrho_V)$ which implements the endomorphism ϱ_V in the Fock representation determined by the basis projection P_1 if and only if $[P_1, V]$ (or, equivalently, V_{12}) is a Hilbert–Schmidt operator. The dimension of $H(\varrho_V)$ is 1 if $\operatorname{ind} V = 0$, otherwise ∞.

Proof. First note that $[P_1, V] = V_{12} - V_{21} = V_{12} - \overline{V_{12}}$ is Hilbert–Schmidt (HS) if and only if V_{12} is.

By the preceding discussion, ϱ_V is implementable if and only if $P_1 - \tilde{S}^{\frac{1}{2}}$ is HS. In this case, $P_2(P_1 - \tilde{S}^{\frac{1}{2}})^2 P_2 = P_2 \tilde{S} P_2 = V_{12}{}^* V_{12}$ is of trace class, hence V_{12} is HS.

Conversely, assume V_{12} to be HS. Let $V = V'|V|$ be the polar decomposition of V. Then $|V| = \overline{|V|}$ is a bounded bijection with a bounded inverse, and $|V| - \mathbf{1} = (|V|^2 - \mathbf{1})(|V| + \mathbf{1})^{-1} = (V^* - V^\dagger)V(|V| + \mathbf{1})^{-1} = 2(V_{12}{}^* + V_{21}{}^*)V(|V| + \mathbf{1})^{-1}$ is HS. Thus, by a corollary [AY82] of an inequality of Araki and Yamagami [AY81], $(|V|A|V|)^{\frac{1}{2}} - A^{\frac{1}{2}}$ is HS for any positive $A \in \mathfrak{B}(\mathcal{K})$. Applying this to $A = V'^* P_1 V'$, we get that

$$\tilde{S}^{\frac{1}{2}} - (V'^* P_1 V')^{\frac{1}{2}} \text{ is HS.} \tag{3.6}$$

Now V' is an isometry with $\overline{V'} = V'$, i.e. a CAR Bogoliubov operator (see (2.1)). Since $[P_1, V]$ and $[P_1, |V|^{-1}] = |V|^{-1}\big[|V|, P_1\big]|V|^{-1} = |V|^{-1}\big[|V| - \mathbf{1}, P_1\big]|V|^{-1}$ are HS, the same holds true for $[P_1, V'] = [P_1, V|V|^{-1}]$. So V' fulfills the implementability condition for CAR Bogoliubov operators derived in Theorem 2.3, and, as shown there, this forces $P_1 - (V'^* P_1 V')^{\frac{1}{2}}$ to be HS. This, together with (3.6), implies that $P_1 - \tilde{S}^{\frac{1}{2}}$ is HS as claimed.

It remains to prove the statement about $\dim H(\varrho_V)$. Let $\tilde{\varrho}_V$ be the normal extension of ϱ_V to $\mathfrak{B}(\mathcal{F}_s(\mathcal{K}_1))$. Then $\mathfrak{B}(H(\varrho_V)) \cong \tilde{\varrho}_V(\mathfrak{B}(\mathcal{F}_s(\mathcal{K}_1)))' = \varrho_V(\mathfrak{W}(\mathcal{K}, \kappa))' = \mathfrak{W}(\operatorname{ran} V)' = \mathfrak{W}(\ker V^\dagger)''$. The latter (and hence $H(\varrho_V)$) is one–dimensional if $\ker V^\dagger = \{0\}$ and infinite dimensional if $\ker V^\dagger \neq \{0\}$. □

Remark. Shale's original result [Sha62] asserts that a quasi–free automorphism ϱ_V, $V \in \mathcal{S}^0(\mathcal{K}, \kappa)$, is implementable if and only if $|V| - \mathbf{1}$ is HS. This condition is equivalent to $[P_1, V]$ being HS, not only for $V \in \mathcal{S}^0(\mathcal{K}, \kappa)$, but for all $V \in \mathcal{S}(\mathcal{K}, \kappa)$ with $-\operatorname{ind} V < \infty$. However, the two conditions are *not* equivalent for $V \in \mathcal{S}^\infty(\mathcal{K}, \kappa)$, as the following example shows. Let $\mathcal{K}_1 = \mathcal{H} \oplus \mathcal{H}'$ be a decomposition into infinite dimensional subspaces. Choose an operator V_{12} from $\mathcal{K}_2$ to $\mathcal{H}$ with $\operatorname{tr}|V_{12}|^4 < \infty$, but $\operatorname{tr}|V_{12}|^2 = \infty$. Let $V_{21} \equiv \overline{V_{12}}$ and $|V_{11}| \equiv (P_1 + |V_{21}|^2)^{\frac{1}{2}}$. Choose an isometry v_{11}

from $\mathcal{K}_1$ to $\mathcal{H}'$ and set $V_{11} \equiv v_{11}|V_{11}|$, $V_{22} \equiv \overline{V_{11}}$. These components define a Bogoliubov operator $V \in \mathcal{S}^\infty(\mathcal{K}, \kappa)$ (cf. (3.8a)–(3.8d) below) which violates the condition of Theorem 3.3. But it fulfills Shale's condition since $|V|^2 - \mathbf{1} = 2(|V_{12}|^2 + |V_{21}|^2)$ is HS and since $|V| - \mathbf{1} = (|V|^2 - \mathbf{1})(|V| + \mathbf{1})^{-1}$.

Let $V \in \mathcal{S}(\mathcal{K}, \kappa)$ with V_{12} compact. Due to stability under compact perturbations [Kat66], V_{11} and $V_{22} = \overline{V_{11}}$ are semi–Fredholm with

$$\operatorname{ind} V_{11} = \operatorname{ind} V_{22} = \frac{1}{2} \operatorname{ind} V. \tag{3.7}$$

We will occasionally use the relation $V^\dagger V = \mathbf{1}$ componentwise:

$$V_{11}{}^* V_{11} - V_{21}{}^* V_{21} = P_1, \tag{3.8a}$$
$$V_{22}{}^* V_{22} - V_{12}{}^* V_{12} = P_2, \tag{3.8b}$$
$$V_{11}{}^* V_{12} - V_{21}{}^* V_{22} = 0, \tag{3.8c}$$
$$V_{22}{}^* V_{21} - V_{12}{}^* V_{11} = 0. \tag{3.8d}$$

Since V_{11} is injective by (3.8a) and has closed range, we may define a bounded operator $V_{11}{}^{-1}$ as the inverse of V_{11} on $\operatorname{ran} V_{11}$ and as zero on $\ker V_{11}{}^*$ (the same applies to V_{22}). These operators will be needed later. Note that $\dim \ker V_{11}{}^* = -\frac{1}{2} \operatorname{ind} V$.

3.3. The semigroup of implementable endomorphisms. According to Theorem 3.3, the semigroup of implementable quasi–free endomorphisms is isomorphic to the following semigroup of Bogoliubov operators:

$$\mathcal{S}_{P_1}(\mathcal{K}, \kappa) \equiv \{V \in \mathcal{S}(\mathcal{K}, \kappa) \mid V_{12} \text{ is Hilbert–Schmidt}\}.$$

$\mathcal{S}_{P_1}(\mathcal{K}, \kappa)$ is a topological semigroup with respect to the metric $\delta_{P_1}(V, V') \equiv \|V - V'\| + \|V_{12} - V'_{12}\|_{\mathrm{HS}}$, where $\|\ \|_{\mathrm{HS}}$ denotes Hilbert–Schmidt norm. It contains the closed sub–semigroup of diagonal Bogoliubov operators

$$\mathcal{S}_{\mathrm{diag}}(\mathcal{K}, \kappa) = \{V \in \mathcal{S}(\mathcal{K}, \kappa) \mid [P_1, V] = 0\}.$$

One has (cf. (2.35))

$$\mathcal{S}_{\mathrm{diag}}(\mathcal{K}, \kappa) = \mathcal{I}_{\mathrm{diag}}(\mathcal{K}) = \mathcal{S}(\mathcal{K}, \kappa) \cap \mathcal{I}(\mathcal{K}).$$

The Fredholm index yields a decomposition

$$\mathcal{S}_{P_1}(\mathcal{K}, \kappa) = \bigcup_{n \in \mathbb{N} \cup \{\infty\}} \mathcal{S}^n_{P_1}(\mathcal{K}, \kappa), \qquad \mathcal{S}^n_{P_1}(\mathcal{K}, \kappa) \equiv \mathcal{S}_{P_1}(\mathcal{K}, \kappa) \cap \mathcal{S}^n(\mathcal{K}, \kappa).$$

The group $\mathcal{S}^0_{P_1}(\mathcal{K}, \kappa)$ is usually called the *restricted symplectic group* [Sha62, Seg81]. It has a natural normal subgroup

$$\mathcal{S}_{\mathrm{HS}}(\mathcal{K}, \kappa) \equiv \{V \in \mathcal{S}(\mathcal{K}, \kappa) \mid V - \mathbf{1} \text{ is Hilbert–Schmidt}\} \subset \mathcal{S}^0_{P_1}(\mathcal{K}, \kappa).$$

As in the CAR case, we will eventually show that each $V \in \mathcal{S}_{P_1}(\mathcal{K}, \kappa)$ can be written as a product $V = UW$ with $U \in \mathcal{S}_{\mathrm{HS}}(\mathcal{K}, \kappa)$ and $W \in \mathcal{S}_{\mathrm{diag}}(\mathcal{K}, \kappa)$. Assume that such U and W exist. Then $P_V \equiv U P_1 U^\dagger$ is a basis projection extending the "partial basis projection" $V P_1 V^\dagger$ such that

$$P_1 - P_V \text{ is Hilbert–Schmidt}, \qquad V^\dagger P_V V = P_1, \tag{3.9}$$

so the corresponding Fock state ω_{P_V} is unitarily equivalent to ω_{P_1} and fulfills $\omega_{P_V} \circ \varrho_V = \omega_{P_1}$. In order to construct such basis projections, let us investigate the set $\mathfrak{P}_{P_1}$ (not to be confused with $\mathfrak{P}_{P_1}$ from Section 2) of basis projections of $(\mathcal{K}, \kappa)$ which differ from P_1 only by a Hilbert–Schmidt operator:

$$\mathfrak{P}_{P_1} \equiv \{P \mid P \text{ is a basis projection}, P_1 - P \text{ is Hilbert–Schmidt}\}.$$

$\mathfrak{P}_{P_1}$ is isomorphic to the set of all Fock states which are unitarily equivalent to ω_{P_1}. Further let $\mathfrak{E}_{P_1}$ be the infinite dimensional analogue of the open unit disk [Sie64, Seg81], consisting of all symmetric Hilbert–Schmidt operators Z from $\mathcal{K}_1$ to $\mathcal{K}_2$ with norm less than 1

$$\mathfrak{E}_{P_1} \equiv \{Z \in \mathfrak{B}(\mathcal{K}_1, \mathcal{K}_2) \mid Z = Z^\tau,\ \|Z\| < 1,\ Z \text{ is Hilbert–Schmidt}\} \tag{3.10}$$

(the condition $\|Z\| < 1$ is equivalent to $P_1 + Z^\dagger Z$ being positive definite on $\mathcal{K}_1$). Then the following is more or less well–known (cf. [Seg81]).

PROPOSITION 3.4.
The map $P \mapsto P_{21}{P_{11}}^{-1}$ defines a bijection from $\mathfrak{P}_{P_1}$ onto $\mathfrak{E}_{P_1}$, with inverse

$$Z \mapsto P_Z \equiv (P_1 + Z)(P_1 + Z^\dagger Z)^{-1}(P_1 + Z^\dagger). \tag{3.11}$$

The restricted symplectic group $\mathcal{S}^0_{P_1}(\mathcal{K}, \kappa)$ acts transitively on either set, in a way compatible with the above bijection, through the formulas

$$P \mapsto UPU^\dagger \tag{3.12}$$

$$Z \mapsto (U_{21} + U_{22}Z)(U_{11} + U_{12}Z)^{-1}. \tag{3.12$'$}$$

The restrictions of these actions to the subgroup $\mathcal{S}_{\mathrm{HS}}(\mathcal{K}, \kappa)$ remain transitive, as follows from the fact that, for $Z \in \mathfrak{E}_{P_1}$,

$$U_Z \equiv (P_1 + Z)(P_1 + Z^\dagger Z)^{-\frac{1}{2}} + (P_2 - Z^\dagger)(P_2 + ZZ^\dagger)^{-\frac{1}{2}} \tag{3.13}$$

lies in $\mathcal{S}_{\mathrm{HS}}(\mathcal{K}, \kappa)$ and fulfills $U_Z P_1 U_Z^\dagger = P_Z$ (equivalently, under the action (3.12$'$), U_Z takes $0 \in \mathfrak{E}_{P_1}$ to Z).

Proof. Having made $\mathcal{K}$ into a Hilbert space, the conditions (3.2) on P to be a basis projection may be rewritten as

$$P = P^\dagger = \mathbf{1} - \overline{P} = P^2, \qquad CP \text{ is positive definite on } \operatorname{ran} P; \tag{3.14}$$

or, in components:

$$P_{11} = {P_{11}}^* = P_1 - \overline{P_{22}}, \tag{3.15a}$$

$$P_{22} = {P_{22}}^* = P_2 - \overline{P_{11}}, \tag{3.15b}$$

$$P_{21} = \overline{{P_{21}}^*} = -{P_{12}}^*, \tag{3.15c}$$

$${P_{11}}^2 - P_{11} = {P_{21}}^* P_{21}, \tag{3.15d}$$

$${P_{22}}^2 - P_{22} = {P_{12}}^* P_{12}, \tag{3.15e}$$

$$(P_1 - P_{11})P_{12} = P_{12}P_{22}, \tag{3.15f}$$

$$(P_2 - P_{22})P_{21} = P_{21}P_{11}, \tag{3.15g}$$

$$\begin{pmatrix} P_{11} & P_{12} \\ -P_{21} & -P_{22} \end{pmatrix} \text{ is positive definite on } \operatorname{ran} P. \tag{3.15h}$$

Moreover, $P_1 - P$ is Hilbert–Schmidt if and only if $P_2 P$ is.

Now let $P \in \mathfrak{P}_{P_1}$. Then $P_{22} \le 0$ by (3.15h), hence, by (3.15a),

$$P_{11} = P_1 - \overline{P_{22}} \ge P_1,$$

so that P_{11} has a bounded inverse. Thus $Z \equiv P_{21}{P_{11}}^{-1}$ is a well–defined Hilbert–Schmidt operator. By (3.15a)–(3.15c) and (3.15g),

$$\begin{aligned} Z - \overline{Z^*} &= P_{21}{P_{11}}^{-1} - \overline{{P_{11}}^{-1}{P_{21}}^*} \\ &= \overline{{P_{11}}^{-1}}\big((P_2 - P_{22})P_{21} - P_{21}P_{11}\big){P_{11}}^{-1} \\ &= 0, \end{aligned}$$

so Z is symmetric in the sense of (3.10). Furthermore, by (3.15d),

$$\begin{aligned} P_1 - Z^*Z &= P_1 - {P_{11}}^{-1}{P_{21}}^*P_{21}{P_{11}}^{-1} \\ &= P_1 - {P_{11}}^{-1}({P_{11}}^2 - P_{11}){P_{11}}^{-1} \\ &= {P_{11}}^{-1} \end{aligned} \tag{3.16}$$

is positive definite on $\mathcal{K}_1$, which proves $Z \in \mathfrak{E}_{P_1}$.

Next let $Z \in \mathfrak{E}_{P_1}$ and let P_Z be given by (3.11). We associate with Z an operator

$$Y \equiv (P_1 + Z^\dagger Z)^{-1} = (P_1 - Z^*Z)^{-1} \tag{3.17}$$

which is bounded by assumption. Then $P_Z = P_Z^\dagger = P_Z^2$ since $(P_1 + Z^\dagger)(P_1 + Z) = Y^{-1}$. To prove that $P_Z + \overline{P_Z} = \mathbf{1}$ holds, note that $ZY^{-1} = \overline{Y}^{-1}Z$ and therefore $\overline{Y}Z = ZY$, $YZ^\dagger = Z^\dagger\overline{Y}$. It follows that

$$\begin{aligned} P_Z + \overline{P_Z} &= (P_1 + Z)Y(P_1 + Z^\dagger) + (P_2 - Z^\dagger)\overline{Y}(P_2 - Z) \\ &= Y + ZY + YZ^\dagger + ZZ^\dagger\overline{Y} + \overline{Y} - YZ^\dagger - ZY + Z^\dagger ZY \\ &= Y^{-1}Y + \overline{Y}^{-1}\overline{Y} \\ &= P_1 + P_2 \\ &= \mathbf{1}. \end{aligned}$$

Since P_2P_Z is clearly HS and since

$$CP_Z = (P_1 - Z)Y(P_1 - Z^*) \tag{3.18}$$

is positive definite on $\operatorname{ran} P_Z = \operatorname{ran}(P_1 + Z)$, we get that $P_Z \in \mathfrak{P}_{P_1}$ as desired.

To show that these two maps are mutually inverse, let first $Z \in \mathfrak{E}_{P_1}$. Then $(P_Z)_{21}{(P_Z)_{11}}^{-1} = ZYY^{-1} = Z$. Conversely, let $P \in \mathfrak{P}_{P_1}$ be given and set $Z \equiv P_{21}{P_{11}}^{-1}$. Then $ZP_{11} = P_{21}$ and $P_{11}Z^\dagger = {P_{21}}^\dagger = P_{12}$. By (3.16) and (3.17), $Y = P_{11}$, hence $P_{11}Z^\dagger = Z^\dagger\overline{P_{11}}$. Thus we get

$$\begin{aligned} P - P_Z &= P - (P_1 + Z)P_{11}(P_1 + Z^\dagger) \\ &= P - P_{11} - ZP_{11} - P_{11}Z^\dagger - ZP_{11}Z^\dagger \\ &= P - P_{11} - P_{21} - P_{12} - ZZ^\dagger\overline{P_{11}} \\ &= P_{22} - ZZ^\dagger\overline{P_{11}} \\ &= P_2 - (P_2 + ZZ^\dagger)\overline{P_{11}} \text{ (by (3.15b))} \\ &= 0. \end{aligned}$$

It remains to prove the statements about the group actions. It is fairly obvious that $\mathcal{S}^0_{P_1}(\mathcal{K}, \kappa)$ acts on $\mathfrak{P}_{P_1}$ via (3.12). The proof that U_Z is a Bogoliubov operator which takes P_1 to P_Z is also straightforward. To show that $U_Z \in \mathcal{S}_{\mathrm{HS}}(\mathcal{K}, \kappa)$, let Y be given by (3.17). Then

$$Y^{\frac{1}{2}} - P_1 = Y^{\frac{1}{2}}(P_1 - Y^{-1})(P_1 + Y^{-\frac{1}{2}})^{-1} = Y^{\frac{1}{2}}Z^*Z(P_1 + Y^{-\frac{1}{2}})^{-1}$$

is of trace class. Therefore $(U_Z - \mathbf{1})P_1 = (P_1 + Z)Y^{\frac{1}{2}} - P_1 = Y^{\frac{1}{2}} - P_1 + ZY^{\frac{1}{2}}$ is HS, which implies $U_Z \in \mathcal{S}_{\mathrm{HS}}(\mathcal{K}, \kappa)$.

Finally we have to show that the action (3.12) on $\mathfrak{P}_{P_1}$ carries over to the action (3.12′) on $\mathfrak{E}_{P_1}$. Thus, for given $Z \in \mathfrak{E}_{P_1}$ and $U \in \mathcal{S}^0_{P_1}(\mathcal{K}, \kappa)$, we have to compute the operator $Z' = P'_{21}{P'_{11}}^{-1}$ which corresponds to $P' = UP_ZU^\dagger$. By definition,

$$\begin{aligned} P'_{21} &= (U_{21} + U_{22}Z)Y(U_{11} + U_{12}Z)^*, \\ P'_{11} &= (U_{11} + U_{12}Z)Y(U_{11} + U_{12}Z)^*. \end{aligned} \tag{3.19}$$

Suppose that $(U_{11}+U_{12}Z)f=0$ for some $f\in\mathcal{K}_1$. Then $\|f\|_{P_1}=\|U_{11}{}^{-1}U_{12}Zf\|_{P_1}$. Since $\|U_{11}{}^{-1}U_{12}\|^2 = \|U_{12}{}^*U_{11}{}^{-1*}U_{11}{}^{-1}U_{12}\| = \|U_{12}{}^*(P_1+U_{12}U_{12}{}^*)^{-1}U_{12}\| = \|U_{12}\|^2/(1+\|U_{12}\|^2)<1$ and $\|Z\|<1$, it follows that $f=0$. Hence $U_{11}+U_{12}Z$ is injective, and, as a Fredholm operator with vanishing index (3.7), it has a bounded inverse. So we get from (3.19) that $Z'=P'_{21}P'_{11}{}^{-1}=(U_{21}+U_{22}Z)(U_{11}+U_{12}Z)^{-1}$ as claimed. □

Remark. It is known that the unique (up to a phase) cyclic vector in $\mathcal{F}_s(\mathcal{K}_1)$ inducing the Fock state ω_{P_Z} is proportional to $\exp(\frac{1}{2}Z^\dagger a^*a^*)\Omega_{P_1}$.

The following construction will enable us to assign, in an unambiguous way, to each Bogoliubov operator $V\in\mathcal{S}_{P_1}(\mathcal{K},\kappa)$ a basis projection P_V such that (3.9) holds.

LEMMA 3.5.
*Let $\mathcal{H}\subset\mathcal{K}$ be a closed *–invariant subspace such that $\kappa|_{\mathcal{H}\times\mathcal{H}}$ is nondegenerate and such that $[P_1,E]$ is Hilbert–Schmidt, where E is the orthogonal projection onto $\mathcal{H}$. Let $A\equiv ECE$ be the self–adjoint operator, invertible on $\mathcal{H}$, such that $\kappa(f,g)=\langle f,Ag\rangle_{P_1}$, $f,g\in\mathcal{H}$, and let $A_\pm$ be the unique positive operators such that $A=A_+-A_-$ and $A_+A_-=0$. Further let A^{-1} be defined as the inverse of A on $\mathcal{H}$ and as zero on $\mathcal{H}^\perp$, and similarly for $A_\pm^{-1}$. Then $A^{-1}C$ is the κ–orthogonal projection onto $\mathcal{H}$, $p_+\equiv A_+^{-1}C$ is a basis projection of $(\mathcal{H},\kappa|_{\mathcal{H}\times\mathcal{H}})$, and P_2p_+ is Hilbert–Schmidt. Moreover, $p_+=P_1E$ if and only if $[P_1,E]=0$.*

Proof. Let $E'\equiv\mathbf{1}-E$. Since ECE' and $E'CE$ are compact by assumption, $C-ECE'-E'CE=A+E'CE'$ is a Fredholm operator on $\mathcal{K}$ with vanishing index. Hence A is Fredholm on $\mathcal{H}$ with $\operatorname{ind}A=0$. A is injective since κ is nondegenerate on $\mathcal{H}$. It is therefore a bounded bijection on $\mathcal{H}$ with a bounded inverse (the same holds true for $A_\pm$ as operators on $\operatorname{ran}A_\pm$). Thus $Q\equiv A^{-1}C$ is well–defined. It fulfills $Q^2=A^{-1}(ECE)A^{-1}C=Q$ and $Q^\dagger=C(CA^{-1})C=Q$. So Q is a projection, self–adjoint with respect to κ. Since its range equals $\operatorname{ran}A^{-1}=\mathcal{H}$, it is the κ–orthogonal projection onto $\mathcal{H}$.

By a similar argument, p_+ is also a κ–orthogonal projection. It is straightforward to see that $p_+=P_1E$ if and only if $[P_1,E]=0$. To show that p_+ is actually a basis projection of $\mathcal{H}$ (cf. (3.14)), note that $\overline{A_+}=A_-$ because of $\overline{A}=-A$ (and uniqueness of $A_\pm$). This implies $p_++\overline{p_+}=A_+^{-1}C-A_-^{-1}C=A^{-1}C=\mathbf{1}_{\mathcal{H}}$. Positive definiteness of Cp_+ on $\operatorname{ran}p_+$ follows from $\langle f,Cp_+f\rangle_{P_1}=\|A_+^{-1/2}Cf\|^2_{P_1}$.

To prove that P_2p_+ is HS, let $D\equiv EP_1E-A_+$. Since $EP_1E-EP_2E=A=A_+-A_-$, we have $D=\overline{D}$. We claim that D is of trace class. Since ECE' is HS,

$$\begin{aligned}ECE'CE&=EC(\mathbf{1}-E)CE\\&=E-(ECE)^2\\&=E-A^2\\&=(E+|A|)(E-|A|)\end{aligned}$$

is of trace class. Since $E+|A|$ has a bounded inverse (as an operator on $\mathcal{H}$) and since $|A|=A_++A_-$, it follows that $E-|A|=EP_1E+EP_2E-A_+-A_-=D+\overline{D}=2D$ is of trace class as claimed. As a consequence, $A_+P_2=(EP_1E-D)P_2$ is HS (P_1EP_2 is HS by assumption). By boundedness of A_+^{-1}, $p_+P_2=-A_+^{-2}(A_+P_2)$ and $P_2p_+=(p_+P_2)^\dagger$ are also HS. This completes the proof. □

Now let $V\in\mathcal{S}_{P_1}(\mathcal{K},\kappa)$. We already showed in Section 3.2 that the restriction of κ to $\ker V^\dagger$ is nondegenerate. We also showed in the proof of Theorem 3.3 that $[P_1,V']$ is Hilbert–Schmidt where V' is the isometry arising from polar decomposition of

V. Hence $[P_1, E]$ is Hilbert–Schmidt where

$$E \equiv C(\mathbf{1} - V'V'^*)C \tag{3.20}$$

is the orthogonal projection onto $\ker V^\dagger$. Thus Lemma 3.5 applies to $\mathcal{H} = \ker V^\dagger$.

Definition 3.6.
For $V \in \mathcal{S}_{P_1}(\mathcal{K}, \kappa)$, let $p_V \equiv p_+$ be the basis projection of $(\ker V^\dagger, \kappa|_{\ker V^\dagger \times \ker V^\dagger})$ given by Lemma 3.5, and set

$$P_V \equiv VP_1V^\dagger + p_V \qquad \in \mathfrak{P}_{P_1}, \tag{3.21}$$

$$Z_V \equiv (P_V)_{21}(P_V)_{11}{}^{-1} \in \mathfrak{E}_{P_1} \tag{3.22}$$

(cf. Proposition 3.4). Further let $U_V \in \mathcal{S}_{\mathrm{HS}}(\mathcal{K}, \kappa)$ be the Bogoliubov operator associated with Z_V according to (3.13), and define $W_V \equiv U_V^\dagger V \in \mathcal{S}_{\mathrm{diag}}(\mathcal{K}, \kappa)$.

P_V clearly is a basis projection which satisfies (3.9). Actually, any basis projection P fulfilling $V^\dagger PV = P_1$ or, equivalently, $PV = VP_1$, is of the form $P = VP_1V^\dagger + q$ where q is some basis projection of $(\ker V^\dagger, \kappa|_{\ker V^\dagger \times \ker V^\dagger})$. What had to be proved above is that q can be chosen such that $P_2 q$ is Hilbert–Schmidt, which is not obvious in the case $\dim \ker V^\dagger = \infty$. In fact, any such extension of $VP_1V^\dagger$ would suffice for what follows.

The condition $V^\dagger P_V V = P_1$ translates into the condition

$$Z_V V_{11} = V_{21} \tag{3.23}$$

for Z_V. Again, each $Z \in \mathfrak{E}_{P_1}$ fulfilling (3.23) would do, but we prefer to have a definite choice. It follows from symmetry (3.10) that any Z which solves (3.23) must have the form

$$Z = V_{21}V_{11}{}^{-1} + V_{22}{}^{-1*}V_{12}{}^* p_{\ker V_{11}{}^*} + Z' \tag{3.24}$$

where $p_{\mathcal{H}}$ denotes the orthogonal projection onto some closed subspace $\mathcal{H} \subset \mathcal{K}$, $V_{11}{}^{-1}$ and $V_{22}{}^{-1}$ have been defined below (3.8), and Z' is a symmetric Hilbert–Schmidt operator from $\ker V_{11}{}^*$ to $\ker V_{22}{}^*$. The freedom in the choice of Z' corresponds to the freedom in the choice of q. Note that Z can be written, with respect to the decompositions $\mathcal{K}_1 = \operatorname{ran} V_{11} \oplus \ker V_{11}{}^*$, $\mathcal{K}_2 = \operatorname{ran} V_{22} \oplus \ker V_{22}{}^*$, as

$$Z = \begin{pmatrix} p_{\operatorname{ran} V_{22}} V_{21}V_{11}{}^{-1} & V_{22}{}^{-1*}V_{12}{}^* p_{\ker V_{11}{}^*} \\ p_{\ker V_{22}{}^*} V_{21}V_{11}{}^{-1} & Z' \end{pmatrix}. \tag{3.25}$$

The Hilbert–Schmidt norm of Z is minimized by choosing $Z' = 0$, but there are examples in which this choice violates the condition $\|Z\| < 1$, i.e. it does not always define an element of $\mathfrak{E}_{P_1}$. This is in contrast to the CAR case where the choice analogous to $Z' = 0$ appeared to be natural (cf. (2.54)).

The operators U_V and W_V constitute the product decomposition of V that was announced earlier, generalizing a construction given by Maaß [Maa71] to the infinite dimensional case. W_V is diagonal because $P_1 W_V = P_1 U_V^\dagger V = U_V^\dagger P_V V = U_V^\dagger V P_1 = W_V P_1$. Explicitly, one computes that

$$W_V = \begin{pmatrix} (P_1 + Z_V^\dagger Z_V)^{\frac{1}{2}} V_{11} & 0 \\ 0 & (P_2 + Z_V Z_V^\dagger)^{\frac{1}{2}} V_{22} \end{pmatrix}$$

with respect to the decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$. Let us summarize the properties of these operators.

Proposition 3.7.
Definition 3.6 establishes a product decomposition of $V \in \mathcal{S}_{P_1}(\mathcal{K}, \kappa)$,

$$V = U_V W_V,$$

where $U_V \in \mathcal{S}_{\mathrm{HS}}(\mathcal{K},\kappa)$ *and* $W_V \in \mathcal{S}_{\mathrm{diag}}(\mathcal{K},\kappa)$ *have the properties*

$$\begin{aligned} \operatorname{ind} U_V &= 0, & Z_{U_V} &= Z_V, & P_{U_V} &= P_V, \\ \operatorname{ind} W_V &= \operatorname{ind} V, & Z_{W_V} &= 0, & P_{W_V} &= P_1. \end{aligned}$$

In particular, if $V \in \mathcal{S}^0_{P_1}(\mathcal{K},\kappa)$*, then*

$$U_V = \begin{pmatrix} |V_{11}{}^*| & V_{12}v_{22}{}^* \\ V_{21}v_{11}{}^* & |V_{22}{}^*| \end{pmatrix}, \qquad W_V = \begin{pmatrix} v_{11} & 0 \\ 0 & v_{22} \end{pmatrix}$$

where $v_{11} \equiv V_{11}|V_{11}|^{-1}$ *and* $v_{22} = \overline{v_{11}}$ *are the unitary parts of* V_{11} *and* V_{22}*; whereas if* $V \in \mathcal{S}_{\mathrm{diag}}(\mathcal{K},\kappa)$*, then* $U_V = \mathbf{1}$ *and* $W_V = V$*.*

The well–known fact that the restricted symplectic group $\mathcal{S}^0_{P_1}(\mathcal{K},\kappa)$ is connected [Seg81, Car84] entails for $\mathcal{S}_{P_1}(\mathcal{K},\kappa)$

COROLLARY 3.8.
$\mathcal{S}_{P_1}(\mathcal{K},\kappa) = \mathcal{S}_{\mathrm{HS}}(\mathcal{K},\kappa)\cdot\mathcal{S}_{\mathrm{diag}}(\mathcal{K},\kappa)$. *The orbits of the action of* $\mathcal{S}^0_{P_1}(\mathcal{K},\kappa)$ *on* $\mathcal{S}_{P_1}(\mathcal{K},\kappa)$ *are the subsets* $\mathcal{S}^n_{P_1}(\mathcal{K},\kappa)$, $n \in \mathbb{N}\cup\{\infty\}$*. They coincide with the connected components of* $\mathcal{S}_{P_1}(\mathcal{K},\kappa)$*.*

3.4. Normal form of implementers. The first step in the construction of implementers consists in a generalization of the definition of "bilinear Hamiltonians" [Ara72] from the finite rank case to the case of bounded operators. If H is a finite rank operator on $\mathcal{K}$ such that $H = H^\tau = -H^*$, then e^{HC} belongs to $\mathcal{S}_{\mathrm{HS}}(\mathcal{K},\kappa)$. Expanding $H = \sum f_j\langle g_j,.\rangle_{P_1}$, one obtains a skew–adjoint element $b_0(H) \equiv \sum f_j g_j^*$ in $\mathfrak{C}(\mathcal{K},\kappa)$ which is a linear function of H, independent of the choice of $f_j, g_j \in \mathcal{K}$. Then $\pi_{P_1}\big(b_0(H)\big)$ is essentially skew–adjoint on $\mathfrak{D}$, and, if $b(H)$ denotes its closure, $\exp\big(\frac{1}{2}b(H)\big)$ is a unitary which implements the automorphism induced by e^{HC} [Ara72, AY82].

Using Wick ordering, the definition of bilinear Hamiltonians can be extended to arbitrary bounded symmetric[u] operators H:

$$H_{11} = H_{22}{}^\tau, \qquad H_{12} = H_{12}{}^\tau, \qquad H_{21} = H_{21}{}^\tau. \tag{3.26}$$

Without loss of generality, we henceforth assume that $\mathcal{K}_1 = L^2(\mathbb{R}^d)$. Then let $\mathfrak{S} \subset \mathcal{F}_s(\mathcal{K}_1)$ be the dense subspace consisting of finite particle vectors ϕ with n–particle wave functions $\phi^{(n)}$ in the Schwartz space $\mathfrak{S}(\mathbb{R}^{dn})$. The unsmeared annihilation operator $a(p)$ with (invariant) domain $\mathfrak{S}$ is defined as usual

$$(a(p)\phi)^{(n)}(p_1,\ldots,p_n) \equiv \sqrt{n+1}\,\phi^{(n+1)}(p,p_1,\ldots,p_n).$$

Let $a^*(p)$ be its quadratic form adjoint on $\mathfrak{S}\times\mathfrak{S}$. Then Wick ordered monomials $a^*(q_1)\cdots a^*(q_m)a(p_1)\cdots a(p_n)$ make sense as quadratic forms on $\mathfrak{S}\times\mathfrak{S}$ [GJ71, RS75], and, for $\phi,\phi' \in \mathfrak{S}$,

$$\langle\phi, a^*(q_1)\cdots a^*(q_m)a(p_1)\cdots a(p_n)\phi'\rangle \equiv \langle a(q_1)\cdots a(q_m)\phi, a(p_1)\cdots a(p_n)\phi'\rangle$$

is a Schwartz function to which tempered distributions can be applied. In particular, the distributions $H_{jk}(p,q)$, $j,k = 1,2$, given by

$$\begin{aligned} \langle f, H_{11}g\rangle_{P_1} &= \int \overline{f(p)}H_{11}(p,q)g(q)\,dp\,dq, \\ \langle f, H_{12}g^*\rangle_{P_1} &= \int \overline{f(p)}H_{12}(p,q)\overline{g(q)}\,dp\,dq, \\ \langle f^*, H_{21}g\rangle_{P_1} &= \int f(p)H_{21}(p,q)g(q)\,dp\,dq, \\ \langle f^*, H_{22}g^*\rangle_{P_1} &= \int f(p)H_{22}(p,q)\overline{g(q)}\,dp\,dq \end{aligned}$$

[u] The bilinear Hamiltonian corresponding to an antisymmetric operator ($H = -H^\tau$) vanishes.

for $f, g \in \mathfrak{S}(\mathbb{R}^d) \subset \mathcal{K}_1$, give rise to the following quadratic forms on $\mathfrak{S} \times \mathfrak{S}$:

$$\begin{aligned} H_{11}a^*a &\equiv \int a(p)^* H_{11}(p,q)a(q)\,dp\,dq \\ H_{12}a^*a^* &\equiv \int a(p)^* H_{12}(p,q)a(q)^*\,dp\,dq \\ H_{21}aa &\equiv \int a(p)H_{21}(p,q)a(q)\,dp\,dq \\ :H_{22}aa^*: &\equiv \int a(q)^* H_{22}(p,q)a(p)\,dp\,dq = H_{11}a^*a. \end{aligned}$$

Wick ordering of $H_{22}aa^*$ is necessary to make this expression well–defined. The last equality follows from symmetry of H:

$$H_{11}(p,q) = H_{22}(q,p), \quad H_{12}(p,q) = H_{12}(q,p), \quad H_{21}(p,q) = H_{21}(q,p).$$

We next define $:b(H):$ and its Wick ordered powers as quadratic forms on $\mathfrak{S} \times \mathfrak{S}$:

$$\begin{aligned} :b(H): &\equiv H_{12}a^*a^* + 2H_{11}a^*a + H_{21}aa, \\ :b(H)^l: &\equiv l! \sum_{\substack{l_1,l_2,l_3=0 \\ l_1+l_2+l_3=l}}^{l} \frac{2^{l_2}}{l_1!l_2!l_3!} H_{l_1,l_2,l_3}, \qquad l \in \mathbb{N}, \end{aligned}$$

with
$$\begin{aligned} H_{l_1,l_2,l_3} \equiv \int & H_{12}(p_1,q_1)\cdots H_{12}(p_{l_1},q_{l_1})H_{11}(p'_1,q'_1)\cdots H_{11}(p'_{l_2},q'_{l_2}) \\ &\cdot H_{21}(p''_1,q''_1)\cdots H_{21}(p''_{l_3},q''_{l_3})a^*(p_1)\cdots a^*(p_{l_1})a^*(q_1)\cdots a^*(q_{l_1}) \\ &\cdot a^*(p'_1)\cdots a^*(p'_{l_2})a(q'_1)\cdots a(q'_{l_2})a(p''_1)\cdots a(p''_{l_3})a(q''_1)\cdots a(q''_{l_3}) \\ &\cdot dp_1\,dq_1\ldots dp_{l_1}\,dq_{l_1}\,dp'_1\,dq'_1\ldots dp'_{l_2}\,dq'_{l_2}\,dp''_1\,dq''_1\ldots dp''_{l_3}\,dq''_{l_3}. \end{aligned}$$

The Wick ordered exponential of $\frac{1}{2}b(H)$ is also well-defined on $\mathfrak{S} \times \mathfrak{S}$, since only a finite number of terms contributes when applied to vectors from $\mathfrak{S}$:

$$:\exp\left(\tfrac{1}{2}b(H)\right): \equiv \sum_{l=0}^{\infty} \frac{1}{l!2^l} :b(H)^l: .$$

The important point is that these quadratic forms are actually the forms of uniquely determined linear operators, defined on the dense subspace $\mathfrak{D}$ and mapping $\mathfrak{D}$ into the domain of (the closure of) any creation or annihilation operator, provided that [Rui78]

$$\|H_{12}\| < 1, \qquad H_{12} \text{ is Hilbert–Schmidt.} \tag{3.27}$$

These operators will be denoted by the same symbols as the quadratic forms. The analogue of Lemma 2.16 is

LEMMA 3.9.
Let $H \in \mathfrak{B}(\mathcal{K})$ satisfy (3.26) *and* (3.27). *Then the following commutation relations hold on $\mathfrak{D}$, for $f \in \mathcal{K}_1$:*

$$\begin{aligned} [H_{l_1,l_2,l_3}, a(f)^*] &= l_2 a(H_{11}f)^* H_{l_1,l_2-1,l_3} + 2l_3 H_{l_1,l_2,l_3-1} a\big((H_{21}f)^*\big), \\ [a(f), H_{l_1,l_2,l_3}] &= 2l_1 a(H_{12}f^*)^* H_{l_1-1,l_2,l_3} + l_2 H_{l_1,l_2-1,l_3} a(H_{11}{}^*f), \end{aligned}$$

implying that

$$\begin{aligned} [\,:\exp\left(\tfrac{1}{2}b(H)\right):, a(f)^*] &= a(H_{11}f)^* :\exp\left(\tfrac{1}{2}b(H)\right): + :\exp\left(\tfrac{1}{2}b(H)\right): a\big((H_{21}f)^*\big), \\ [a(f), :\exp\left(\tfrac{1}{2}b(H)\right):\,] &= a(H_{12}f^*)^* :\exp\left(\tfrac{1}{2}b(H)\right): + :\exp\left(\tfrac{1}{2}b(H)\right): a(H_{11}{}^*f). \end{aligned}$$

Proof. Compute as in [Rui78, Bin95]. □

For given $V \in \mathcal{S}_{P_1}(\mathcal{K}, \kappa)$, we are now looking for bounded symmetric operators H which satisfy (3.27) and the following intertwiner relation on $\mathfrak{D}$

$$:\!\exp\big(\tfrac{1}{2}b(H)\big)\!: \pi_{P_1}(f) = \pi_{P_1}(Vf) :\!\exp\big(\tfrac{1}{2}b(H)\big)\!:, \quad f \in \mathcal{K} \tag{3.28}$$

(taking the closure of $\pi_{P_1}(Vf)$ is tacitly assumed here). This problem turns out to be equivalent to the determination of the operators Z done in (3.23), (3.24).

LEMMA 3.10.
Each $Z \in \mathfrak{C}_{P_1}$ fulfilling (3.23) gives rise to a unique solution H of the above problem through the formula

$$H = \begin{pmatrix} V_{11} - P_1 + Z^\dagger V_{21} & Z^\dagger \\ (V_{22}{}^* + V_{12}{}^* Z^\dagger)V_{21} & V_{22}{}^* - P_2 + V_{12}{}^* Z^\dagger \end{pmatrix},$$

and each solution arises in this way.

Proof. Let us abbreviate $\eta_H \equiv \;:\!\exp\big(\tfrac{1}{2}b(H)\big)\!:$. Choosing $f \in \mathcal{K}_2$ resp. $f \in \mathcal{K}_1$ and inserting the definition of π_{P_1}, one finds that (3.28) is equivalent to

$$\eta_H a(g) = \big(a(V_{11}g) + a^*(V_{12}g^*)\big)\eta_H, \quad \eta_H a^*(g) = \big(a^*(V_{11}g) + a(V_{12}g^*)\big)\eta_H$$

for $g \in \mathcal{K}_1$. Using the commutation relations from Lemma 3.9, these equations may be brought into Wick ordered form:

$$\begin{aligned} 0 &= a^*\big((V_{12} + H_{12}V_{22})g^*\big)\eta_H + \eta_H a\Big(\big((P_1 + H_{11}{}^*)V_{11} - P_1\big)g\Big), \\ 0 &= a^*\big((P_1 + H_{11} - V_{11} - H_{12}V_{21})g\big)\eta_H + \eta_H a\Big(\big(\overline{H_{21}} - (P_1 + H_{11}{}^*)V_{12}\big)g^*\Big). \end{aligned}$$

As in the CAR case (see the proof of Lemma 2.17), these equations hold for all $g \in \mathcal{K}_1$ if and only if

$$0 = V_{12} + H_{12}V_{22}, \tag{3.29a}$$
$$0 = P_1 + H_{11} - V_{11} - H_{12}V_{21}, \tag{3.29b}$$
$$0 = H_{21} - (P_2 + H_{22})V_{21}, \tag{3.29c}$$
$$0 = P_2 - (P_2 + H_{22})V_{22} \tag{3.29d}$$

(we applied complex conjugation and used $H_{11}{}^\tau = H_{22}$).

Now assume that H solves the above problem. It is then obvious from (3.26), (3.27) and (3.29a) that $Z \equiv H_{12}{}^\dagger$ belongs to $\mathfrak{C}_{P_1}$ and fulfills (3.23).

Conversely, let $Z \in \mathfrak{C}_{P_1}$ satisfy (3.23). If there exists a solution H with $H_{12} = Z^\dagger$, then H_{11} is fixed by (3.29b), H_{22} must equal $H_{11}{}^\tau$, and H_{21} is determined by (3.29c). Thus there can be at most one solution corresponding to Z, and it is necessarily of the form stated in the proposition.

It remains to prove that the so–defined H has all desired properties, i.e. that H_{21} is symmetric and that (3.29d) holds, the rest being clear by construction. The first claim follows from (3.8d):

$$H_{21} - \overline{H_{21}{}^*} = (V_{22}{}^* + V_{12}{}^* Z^\dagger)V_{21} - V_{12}{}^*(V_{11} + Z^\dagger V_{21}) = 0,$$

and the second from (3.23) and (3.8b):

$$(P_2 + H_{22})V_{22} = (V_{22}{}^* - V_{12}{}^*\overline{Z})V_{22} = V_{22}{}^* V_{22} - V_{12}{}^* V_{12} = P_2.$$

□

Inserting the formula (3.24) for Z, one obtains

$$\begin{aligned}
H_{11} &= V_{11}{}^{-1*} - P_1 - p_{\ker V_{11}{}^*} V_{12} V_{22}{}^{-1} V_{21} + Z'^{\dagger} V_{21},\\
H_{12} &= -V_{12} V_{22}{}^{-1} - V_{11}{}^{-1*} V_{21}{}^* p_{\ker V_{22}{}^*} + Z'^{\dagger},\\
H_{21} &= (V_{22}{}^{-1} - V_{12}{}^* V_{11}{}^{-1*} V_{21}{}^* p_{\ker V_{22}{}^*}) V_{21} + V_{12}{}^* Z'^{\dagger} V_{21},\\
H_{22} &= V_{22}{}^{-1} - P_2 - V_{12}{}^* V_{11}{}^{-1*} V_{21}{}^* p_{\ker V_{22}{}^*} + V_{12}{}^* Z'^{\dagger}.
\end{aligned}$$

H corresponds to Ruijsenaars' "associate" Λ [Rui78]. If one compares the above formula for H with Ruijsenaars' formula for Λ in the case of automorphisms ($\ker V_{jj}{}^* = \{0\}$, $j = 1, 2$; $Z' = 0$), one finds that the off–diagonal components carry opposite signs. This is due to the fact that Ruijsenaars constructs implementers for the transformation induced by CVC rather than V, cf. (3.27) and (3.29) in [Rui78].

Note that

$$:\exp\big(\tfrac{1}{2}b(H)\big):\Omega_{P_1} = \exp(\tfrac{1}{2}H_{12}a^*a^*)\Omega_{P_1}. \tag{3.30}$$

By Ruijsenaars' computation [Rui78] (see also [Seg81]), the norm of such vectors is

$$\Big\| :\exp\big(\tfrac{1}{2}b(H)\big):\Omega_{P_1}\Big\| = \Big(\det_{\mathcal{K}_1}(P_1 + H_{12}H_{12}{}^{\dagger})\Big)^{-1/4}.$$

Definition 3.11.
Let $V \in \mathcal{S}_{P_1}(\mathcal{K}, \kappa)$, and let P_V, Z_V and H_V be the operators associated with V according to Definition 3.6 and Lemma 3.10. Choose a κ–orthonormal basis $g_1, g_2, \dots$ in

$$\mathfrak{k}_V \equiv P_V(\ker V^{\dagger}), \tag{3.31}$$

i.e. a basis such that $\kappa(g_j, g_k) = \delta_{jk}$ (this is possible because the restriction of κ to $\mathfrak{k}_V$ is positive definite). Note that $\dim \mathfrak{k}_V = -\frac{1}{2}\operatorname{ind} V$. Let ψ_j be the isometry obtained by polar decomposition of the closure of $\pi_{P_1}(g_j)$. Then define operators $\Psi_\alpha(V)$ on $\mathfrak{D}$, for any multi–index $\alpha = (\alpha_1, \dots, \alpha_l)$ with $\alpha_j \le \alpha_{j+1}$ (or $\alpha = 0$) as in (3.4), as

$$\Psi_\alpha(V) \equiv \Big(\det_{\mathcal{K}_1}(P_1 + Z_V^{\dagger} Z_V)\Big)^{\frac{1}{4}} \psi_{\alpha_1}\cdots\psi_{\alpha_l} :\exp\big(\tfrac{1}{2}b(H_V)\big):. \tag{3.32}$$

Theorem 3.12.
The $\Psi_\alpha(V)$ extend continuously to isometries (denoted by the same symbols) on the symmetric Fock space $\mathcal{F}_s(\mathcal{K}_1)$ such that

$$\Psi_\alpha(V)^*\Psi_\beta(V) = \delta_{\alpha\beta}\mathbf{1}, \qquad \sum_\alpha \Psi_\alpha(V)\Psi_\alpha(V)^* = \mathbf{1}, \tag{3.33}$$

and such that, for any element w of the Weyl algebra $\mathfrak{W}(\mathcal{K}, \kappa)$,

$$\varrho_V(w) = \sum_\alpha \Psi_\alpha(V) w \Psi_\alpha(V)^*. \tag{3.34}$$

The infinite sums converge in the strong topology.

Proof. By (3.1) we have $\pi_{P_1}(g_j)^*\pi_{P_1}(g_j) = \mathbf{1} + \pi_{P_1}(g_j)\pi_{P_1}(g_j)^*$ on $\mathfrak{D}$, so the closure of $\pi_{P_1}(g_j)$ is injective, and ψ_j is isometric. It is also easy to see, using (3.28), the CCR and $\|\Psi_\alpha(V)\Omega_{P_1}\| = 1$, that for $f_1, \dots, f_m, h_1, \dots, h_n \in \mathcal{K}$

$$\begin{aligned}
\langle \Psi_\alpha(V)\pi_{P_1}(f_1\cdots f_m)\Omega_{P_1}, \Psi_\alpha(V)\pi_{P_1}(h_1\cdots h_n)\Omega_{P_1}\rangle&\\
= \langle \pi_{P_1}(f_1\cdots f_m)\Omega_{P_1}, \pi_{P_1}(h_1\cdots h_n)\Omega_{P_1}\rangle&.
\end{aligned}$$

Hence $\Psi_\alpha(V)$ is isometric on $\mathfrak{D}$ and has a continuous extension to an isometry on $\mathcal{F}_s(\mathcal{K}_1)$.

Let $\mathcal{H}_j \equiv \operatorname{span}(g_j, g_j^*)$, so that $\psi_j \in \mathfrak{W}(\mathcal{H}_j)''$ by virtue of Lemma 3.1. Since $\mathcal{H}_j \subset \ker V^\dagger$, the duality relation (3.3) implies that $\mathfrak{W}(\mathcal{H}_j) \subset \mathfrak{W}(\operatorname{ran} V)'$. Now let $f \in \operatorname{Re}\mathcal{K}$ and $\phi \in \mathfrak{D}$. Since ϕ is an entire analytic vector for $\pi_{P_1}(f)$ [AS72], since $\mathfrak{D}$ is invariant under $\pi_{P_1}(f)$, and since $\overline{\pi_{P_1}(Vf)}$ is affiliated with $\mathfrak{W}(\operatorname{ran} V)$ by Lemma 3.1 (the bar denotes closure), it follows from (3.28) that

$$\begin{aligned}
\Psi_\alpha(V) w(f)\phi &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \Psi_\alpha(V)(\pi_{P_1}(f))^n \phi \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} \psi_{\alpha_1} \cdots \psi_{\alpha_l} \big(\overline{\pi_{P_1}(Vf)}\big)^n \Psi_0(V)\phi \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} \big(\overline{\pi_{P_1}(Vf)}\big)^n \Psi_\alpha(V)\phi \\
&= w(Vf)\Psi_\alpha(V)\phi.
\end{aligned}$$

By continuity, this entails

$$\Psi_\alpha(V) w = \varrho_V(w)\Psi_\alpha(V), \qquad w \in \mathfrak{W}(\mathcal{K}, \kappa). \tag{3.35}$$

We next claim that

$$\psi_j^* \Psi_0(V) = 0 \tag{3.36}$$

or, equivalently, that $\pi_{P_1}(g_j)^*\Psi_0(V) = 0$. To see this, apply Lemma 3.9 and write $\pi_{P_1}(g_j)^*\Psi_0(V)$ in Wick ordered form:

$$\pi_{P_1}(g_j)^*\Psi_0(V) = a\big((P_1 + H_{12})g_j^*\big)^*\Psi_0(V) + \Psi_0(V)a\big((P_1 + {H_{11}}^*)g_j\big)$$

on $\mathfrak{D}$, with $H \equiv H_V$. Then (3.36) holds if and only if

$$(P_1 + H_{12})g_j^* = 0, \qquad (P_1 + {H_{11}}^*)g_j = 0. \tag{3.36$'$}$$

Now $g_j \in \operatorname{ran} P_V$ is equivalent to $g_j^* \in \ker P_V = \ker CP_V = \ker(P_1 + H_{12})$ (we used (3.18)). This proves the first equation in (3.36′). It also shows that ${H_{12}}^* g_j = -P_2 g_j$. Hence by Lemma 3.10,

$$(P_1 + {H_{11}}^*)g_j = ({V_{11}}^* + {V_{21}}^*{H_{12}}^*)g_j = ({V_{11}}^* - {V_{21}}^*)g_j = P_1 V^\dagger g_j = 0$$

which proves the second equation in (3.36′) and therefore (3.36).

The orthogonality relation $\Psi_\alpha(V)^*\Psi_\beta(V) = 0$ ($\alpha \neq \beta$) now follows from (3.36) and from $\mathfrak{W}(\mathcal{H}_j) \subset \mathfrak{W}(\mathcal{H}_k)'$ ($j \neq k$) which in turn is a consequence of $\kappa(\mathcal{H}_j, \mathcal{H}_k) = 0$ and (3.3).

The proof of the completeness relation $\sum \Psi_\alpha(V)\Psi_\alpha(V)^* = \mathbf{1}$ is facilitated by invoking the product decomposition $V = U_V W_V$ from Proposition 3.7. Set $f_j \equiv U_V^\dagger g_j$ to obtain a κ–orthonormal basis $f_1, f_2, \ldots$ in $\mathfrak{k}_{W_V} = P_1(\ker W_V^\dagger)$. Let ψ_j' be the isometric part of $a(f_j)^*$. An application of Definition 3.11 to W_V yields implementers $\Psi_\alpha(W_V) = \psi_{\alpha_1}' \cdots \psi_{\alpha_l}' \Psi_0(W_V)$ for W_V. $Z_{W_V} = 0$ entails that

$$\Psi_\alpha(W_V)\Omega_{P_1} = \psi_{\alpha_1}' \cdots \psi_{\alpha_l}' \Omega_{P_1}. \tag{3.37}$$

One computes, using the CCR, that

$$\psi_{\alpha_1}' \cdots \psi_{\alpha_l}' \Omega_{P_1} = \phi_\alpha', \tag{3.38}$$

where the ϕ_α' are the cyclic vectors associated with the pure state $\omega_{P_1} \circ \varrho_{W_V} = \omega_{P_1}$ as in Proposition 3.2. Let $\mathcal{F}_\alpha'$ be the closure of $\mathfrak{W}(\operatorname{ran} W_V)\phi_\alpha'$. Since the $\mathcal{F}_\alpha'$ are irreducible subspaces for $\mathfrak{W}(\operatorname{ran} W_V)$ by Proposition 3.2, they must coincide with the irreducible subspaces $\operatorname{ran}\Psi_\alpha(W_V)$. $\oplus\mathcal{F}_\alpha' = \mathcal{F}_s(\mathcal{K}_1)$ then implies completeness of the $\Psi_\alpha(W_V)$.

The proof will be completed by showing that

$$\Psi_\alpha(V) = \Psi(U_V)\Psi_\alpha(W_V) \tag{3.39}$$

holds where $\Psi(U_V)$ is the unitary implementer for U_V given by Definition 3.11. It suffices to show that (3.39) holds on Ω_{P_1} since any bounded operator fulfilling (3.35) is already determined by its value on Ω_{P_1}. Because of $Z_{U_V} = Z_V$ we have

$$\Psi_0(V)\Omega_{P_1} = \Psi(U_V)\Omega_{P_1}, \tag{3.40}$$

so it remains to show that $\psi_{\alpha_1}\cdots\psi_{\alpha_l}\Psi(U_V)\Omega_{P_1} = \Psi(U_V)\psi'_{\alpha_1}\cdots\psi'_{\alpha_l}\Omega_{P_1}$. We claim that

$$\psi_j\Psi(U_V) = \Psi(U_V)\psi'_j. \tag{3.41}$$

For let T (resp. T') be the closure of $\pi_{P_1}(g_j)$ (resp. $\pi_{P_1}(f_j)$). It suffices to prove that $\Psi(U_V)\big(D(T')\big) = D(T)$ and that

$$T\Psi(U_V) = \Psi(U_V)T'. \tag{3.42}$$

(3.42) clearly holds on $\mathfrak{D}$. Now let $\phi \in D(T')$, and choose $\phi_n \in \mathfrak{D}$ with $\phi_n \to \phi$ and $T'\phi_n \to T'\phi$. Then $\Psi(U_V)\phi_n \in \mathfrak{D}$ converges to $\Psi(U_V)\phi$, and $T\Psi(U_V)\phi_n = \Psi(U_V)T'\phi_n$ converges to $\Psi(U_V)T'\phi$. It follows that $\Psi(U_V)\phi \in D(T)$ and $T\Psi(U_V)\phi = \Psi(U_V)T'\phi$, i.e. that $T\Psi(U_V) \supset \Psi(U_V)T'$. In the same way one obtains that $T'\Psi(U_V)^* \supset \Psi(U_V)^*T$, so that (3.42) and (3.41) hold. (An alternative proof of (3.41) goes as follows. Let $T^\pm$ (resp. $T'^\pm$) be the self–adjoint operators corresponding to T (resp. T') as in the proof of Lemma 3.1. Then one has $D(T) = D(T^+) \cap D(T^-)$ and $T = T^+ - iT^-$, and similar for T'. There holds $\Psi(U_V)\exp(itT'^\pm)\Psi(U_V)^* = \exp(itT^\pm)$, $t \in \mathbb{R}$. Therefore $\Psi(U_V)$ maps $D(T'^\pm)$ onto $D(T^\pm)$, and $\Psi(U_V)T'^\pm\Psi(U_V)^* = T^\pm$. Consequently, $\Psi(U_V)\big(D(T')\big) = D(T)$ and $\Psi(U_V)T'\Psi(U_V)^* = T$. This implies that $\Psi(U_V)\psi'_j\Psi(U_V)^* = \psi_j$ as claimed.)

The proof is complete since (3.33) and (3.35) together imply (3.34). □

Corollary 3.13.
There is a unitary isomorphism from $H(\varrho_V)$, the Hilbert space generated by the $\Psi_\alpha(V)$, onto the symmetric Fock space $\mathcal{F}_s(\mathfrak{k}_V)$ over $\mathfrak{k}_V$, which maps $\Psi_\alpha(V)$ to $c_\alpha a^*(g_{\alpha_1})\cdots a^*(g_{\alpha_l})\Omega$, where the normalization factor c_α is defined in (3.5)*, and $a^*(g_j)$ and Ω are now creation operators and the Fock vacuum in $\mathcal{F}_s(\mathfrak{k}_V)$.*

We shall see in Section 4.2 that, for gauge invariant V, the isomorphism described above is not only an isomorphism of graded Hilbert spaces but also of modules of the gauge group.

4. Superselection Sectors Reached by Gauge Invariant Quasi–free Endomorphisms

In the present section we will apply our results from Sections 2 and 3 to the theory of superselection sectors. We are especially interested in the possible "charge quantum numbers" that can be realized by quasi–free endomorphisms. We will consider situations where the theory of Doplicher and Roberts applies, i.e. where the observable algebra $\mathfrak{A}$ consists of the invariant elements of a field algebra (given in its vacuum representation) under a group G of gauge automorphisms of the first kind. As mentioned in the introduction, the charge quantum numbers of a localized endomorphism ϱ then are labels for the unitary representation of G which is realized on the Hilbert space $H(\varrho)$.

The CAR and CCR algebras will play the rôle of the field algebra, so that quasi–free endomorphisms are from the outset endomorphisms of the field algebra rather than the observable algebra. The following simple observation shows that one has to restrict attention to *gauge invariant* endomorphisms, i.e. to endomorphisms which commute with all gauge transformations.

Proposition 4.1.
Let ϱ be an endomorphism of the field algebra which is implemented by a Hilbert space $H(\varrho)$ of isometries. Then $H(\varrho)$ is invariant under G if and only if ϱ is gauge invariant.

Proof. Assume first that $H(\varrho)$ is invariant under G. Let R be the representation of G on $H(\varrho)$:

$$R(\gamma) \equiv \gamma|_{H(\varrho)}, \quad \gamma \in G.$$

R is clearly unitary because one has for any $\gamma \in G$

$$\langle R(\gamma)\Psi, R(\gamma)\Psi'\rangle\mathbf{1} = \gamma(\Psi^*\Psi') = \gamma(\langle\Psi,\Psi'\rangle\mathbf{1}) = \langle\Psi,\Psi'\rangle\mathbf{1}, \quad \Psi,\Psi' \in H(\varrho).$$

Writing $R(\gamma)$ as a matrix with respect to the orthonormal basis (Ψ_j), one gets $\gamma(\Psi_j) = \sum_k R(\gamma)_{kj}\Psi_k$ and $\sum_j R(\gamma)_{kj}\overline{R(\gamma)_{lj}} = \delta_{kl}$, so that

$$\begin{aligned}\gamma(\varrho(F)) &= \sum_j \gamma(\Psi_j)\gamma(F)\gamma(\Psi_j)^* \\ &= \sum_{k,l}\Big(\sum_j R(\gamma)_{kj}\overline{R(\gamma)_{lj}}\Big)\Psi_k\gamma(F)\Psi_l^* \\ &= \sum_k \Psi_k\gamma(F)\Psi_k^* \\ &= \varrho(\gamma(F))\end{aligned}$$

for any field F. Therefore ϱ is gauge invariant.

Conversely, assume that ϱ is gauge invariant. Since the field algebra is irreducibly represented, the Hilbert space $H(\varrho)$ consists of all field operators Ψ which intertwine between the vacuum representation and the representation induced by ϱ:

$$\Psi \in H(\varrho) \iff \Psi F = \varrho(F)\Psi \text{ for all local fields } F \tag{4.1}$$

(cf. (2.24)). Now let $\Psi \in H(\varrho)$ and $\gamma \in G$. Then one has for any F

$$\gamma(\Psi)F = \gamma\big(\Psi\gamma^{-1}(F)\big) = \gamma\big(\varrho(\gamma^{-1}(F))\Psi\big) = \varrho(F)\gamma(\Psi),$$

so that $\gamma(\Psi) \in H(\varrho)$ by (4.1). □

We are thus led to consider the following setting. We assume that a distinguished basis projection P_1 of $\mathcal{K}$ (CAR) resp. of $(\mathcal{K},\kappa)$ (CCR) is given. The global gauge group G will consist of diagonal Bogoliubov operators (resp. of the corresponding automorphisms)

$$G \subset \mathcal{I}^0_{P_1}(\mathcal{K}) \cap \mathcal{I}_{\mathrm{diag}}(\mathcal{K}) \cong U(\mathcal{K}_1) \qquad \text{(CAR)}$$
$$G \subset \mathcal{S}^0_{P_1}(\mathcal{K},\kappa) \cap \mathcal{S}_{\mathrm{diag}}(\mathcal{K},\kappa) \cong U(\mathcal{K}_1) \quad \text{(CCR)}.$$

Gauge transformations leave the Fock state ω_{P_1} invariant. The usual second quantization of $U \in G$ (or, more precisely, of U_{11}) will be denoted by $\Gamma(U)$. This is the same as $\Psi(U)$ defined by (2.85) resp. (3.32); the map $U \mapsto \Gamma(U)$ is strongly continuous. The gauge automorphism corresponding to U on Fock space will be denoted by γ_U:

$$\gamma_U(F) \equiv \Gamma(U) F \Gamma(U)^*,$$

where F can now be any bounded operator on Fock space. The charge structure of all implementable gauge invariant quasi–free endomorphisms ϱ_V will be unraveled in the next sections, i.e. of all ϱ_V with V contained in either of the following semigroups

$$\mathcal{I}_{P_1}(\mathcal{K})^G \equiv \{V \in \mathcal{I}_{P_1}(\mathcal{K}) \mid [V,U]=0 \text{ for all } U \in G\} \qquad \text{(CAR)}$$
$$\mathcal{S}_{P_1}(\mathcal{K},\kappa)^G \equiv \{V \in \mathcal{S}_{P_1}(\mathcal{K},\kappa) \mid [V,U]=0 \text{ for all } U \in G\} \quad \text{(CCR)}.$$

Bogoliubov operators commuting with G will be called *gauge invariant.* The notations $\mathcal{I}^n_{P_1}(\mathcal{K})^G$ resp. $\mathcal{S}^n_{P_1}(\mathcal{K},\kappa)^G$ will be used to denote the subsets of V with $\operatorname{ind} V = -n$. At this stage of generality, it is not necessary to assume that G is a (strongly) compact topological group. However, if G is "too large", then it can happen that $\mathcal{I}_{P_1}(\mathcal{K})^G$ and $\mathcal{S}_{P_1}(\mathcal{K},\kappa)^G$ become trivial and consist only of the operators of the form $e^{i\lambda}P_1 + e^{-i\lambda}P_2,\ \lambda \in \mathbb{R}$. On the other hand, if G is compact, then $\mathcal{I}_{P_1}(\mathcal{K})^G$ and $\mathcal{S}_{P_1}(\mathcal{K},\kappa)^G$ can be described more explicitly as follows. The representation of G on $\mathcal{K}$ can be brought into the form

$$\mathcal{K} = \bigoplus_{\xi} (\mathcal{K}_\xi \otimes \mathfrak{h}_\xi) \tag{4.2}$$

where the sum extends over all equivalence classes ξ of irreducible representations of G realized on $\mathcal{K}$, $\mathfrak{h}_\xi$ is a finite dimensional subspace carrying a representation $\mathcal{U}_\xi$ of class ξ, and $\mathcal{K}_\xi$ is a Hilbert space with dimension equal to the multiplicity of ξ. G acts on $\mathcal{K}_\xi \otimes \mathfrak{h}_\xi$ like $\mathbf{1}_{\mathcal{K}_\xi} \otimes \mathcal{U}_\xi$. Gauge invariant Bogoliubov operators then have the form $V = \oplus_\xi (V_\xi \otimes \mathbf{1}_{\mathfrak{h}_\xi})$, and there exist non–surjective gauge invariant Bogoliubov operators if and only if at least one $\mathcal{K}_\xi$ is infinite dimensional. Also note that one should require on physical grounds to have $-\mathbf{1} \in G$, but we do not need this assumption at this point.

The analysis of the representations of G on the implementing Hilbert spaces $H(\varrho_V)$ is facilitated by the following lemma.

LEMMA 4.2.
Let V be an element of $\mathcal{I}_{P_1}(\mathcal{K})^G$ or $\mathcal{S}_{P_1}(\mathcal{K},\kappa)^G$. Then the representation of G on $H(\varrho_V)$ (obtained by restricting γ_U to $H(\varrho_V)$) is canonically unitarily equivalent to the representation on $H(\varrho_V)\Omega_{P_1}$ (obtained by restricting $\Gamma(U)$ to $H(\varrho_V)\Omega_{P_1}$), via the map $\Psi \mapsto \Psi\Omega_{P_1}$.

Proof. Obvious from gauge invariance of the vacuum. □

EXAMPLE 3 (THE FREE DIRAC FIELD WITH GAUGE SYMMETRY).
The setting specified above is abstracted from field theoretic examples of the following kind. Let $\mathcal{H} = L^2(\mathbb{R}^{2n-1}, \mathbb{C}^{2^n})$ be the single particle space of the free time–zero Dirac field in $2n$ spacetime dimensions. Let $H = -i\vec{\alpha}\vec{\nabla} + \beta m$ be the free Dirac Hamiltonian, with spectral projections $p_\pm$ corresponding to the positive resp. negative part of the spectrum of H. Tensored with $\mathbf{1}_N$, these operators act on the

space $\mathcal{H}' \equiv \mathcal{H} \otimes \mathbb{C}^N$. The gauge group $U(N)$ also acts naturally on $\mathcal{H}'$. In the selfdual CAR formalism, one sets

$$\mathcal{K} \equiv \mathcal{H}' \oplus \mathcal{H}'^*, \tag{4.3}$$

where $\mathcal{H}'^*$ is the Hilbert space conjugate to $\mathcal{H}'$. There is a natural conjugation $f \mapsto f^*$ on $\mathcal{K}$ which is inherited from the antiunitary identity map $\mathcal{H}' \to \mathcal{H}'^*$. The basis projection P_1 corresponding to the vacuum representation of the field is then given by

$$P_1 \equiv p'_+ \oplus \overline{p'_-}$$

with $p'_\pm = p_\pm \otimes \mathbf{1}_N$. Gauge transformations act like $U = (\mathbf{1}_\mathcal{H} \otimes u) \oplus (\mathbf{1}_{\mathcal{H}^*} \otimes \overline{u})$, $u \in U(N)$, on $\mathcal{K}$. They commute with P_1. With respect to the decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ induced by P_1, U has the form

$$U = \begin{pmatrix} p_+ \otimes u + \overline{p_-} \otimes \overline{u} & 0 \\ 0 & \overline{p_+} \otimes \overline{u} + p_- \otimes u \end{pmatrix}. \tag{4.4}$$

The field operators φ_t at time t are given by

$$\varphi_t(f) \equiv \pi_{P_1}(e^{itH'} f) = a(p'_+ e^{itH'} f)^* + a(\overline{p'_-} e^{-it\overline{H'}} f^*)$$

where $H' \equiv H \otimes \mathbf{1}_N$, $f \in \mathcal{H}'$. They are solutions of the Dirac–Schrödinger equation

$$-i\frac{d}{dt}\varphi_t(f) = \varphi_t(H'f), \qquad f \in D(H')$$

(the minus sign is due to the fact that our field operators $\varphi_t(f)$ are complex linear in f). If O is a double cone with base $B \subset \mathbb{R}^{2n-1}$ at time t, then the local field algebra associated with O is generated by all $\varphi_t(f)$ with $\operatorname{supp} f \subset B$. The local observable algebra $\mathfrak{A}(O)$ is the fixed point subalgebra of this local field algebra under the gauge action. (The whole net of local algebras is generated from these special ones by applying Lorentz transformations.) Bogoliubov operators in $\mathcal{I}_{P_1}(\mathcal{K})^{U(N)}$ which act like the identity on functions f with $\operatorname{supp} f \cap B = \emptyset$ induce endomorphisms of $\mathfrak{A}$ which are localized in the double cone O, cf. Prop. 4.8 below. (The charge carried by such localized endomorphisms can be read off from our formulas in the following section.) All gauge invariant Bogoliubov operators V have the form

$$V = (v \otimes \mathbf{1}_N) \oplus (\overline{v} \otimes \overline{\mathbf{1}_N}) \tag{4.5}$$

with respect to the decomposition (4.3) where v is some isometry of $\mathcal{H}$. This holds because (4.3), with $\mathcal{H}' = \mathcal{H} \otimes \mathbb{C}^N$, is the decomposition of $\mathcal{K}$ analogous to (4.2). Thus $\mathcal{I}_{P_1}(\mathcal{K})^{U(N)}$ is isomorphic to the semigroup of all isometries of $\mathcal{H}$. This fact remains true if the group $U(N)$ is replaced by $SU(N)$, except for the case of $SU(2)$. In the latter case (and only in that case), the defining representation of the group is equivalent to its complex conjugate representation, since one has $JuJ^* = \overline{u}$, $u \in SU(2)$, with $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Therefore (4.3) is *not* the decomposition of $\mathcal{K}$ as in (4.2), and there exist Bogoliubov operators in $\mathcal{I}_{P_1}(\mathcal{K})^{SU(2)}$ which do not have the form (4.5).

Similar constructions work for the free charged Klein–Gordon field.

4.1. The charge of gauge invariant endomorphisms of the CAR algebra. In this section we will compute the behaviour of the implementers $\Psi_\alpha(V)$ defined by (2.85) under gauge transformations for arbitrary gauge invariant V. Recall from (2.86) that the value of $\Psi_\alpha(V)$ on the Fock vacuum is given by

$$\Psi_\alpha(V)\Omega_{P_1} = D_V \psi_{P_1}(g_{\alpha_1} \cdots g_{\alpha_l} e_1 \cdots e_{L_V}) \exp(\tfrac{1}{2}\overline{T_V} a^* a^*)\Omega_{P_1} \tag{4.6}$$

where D_V is a numerical constant, $\{g_1, \ldots, g_{M_V}\}$ is an orthonormal basis in $\mathfrak{k}_V = P_V(\ker V^*)$, $\{e_1, \ldots, e_{L_V}\}$ is an orthonormal basis in $\mathfrak{h}_V = V_{12}(\ker V_{22})$, and T_V is the antisymmetric Hilbert–Schmidt operator defined in (2.54). We have to calculate the transformed vectors $\Gamma(U)\Psi_\alpha(V)\Omega_{P_1}$, $U \in G$.

LEMMA 4.3.
$\Gamma(U)$ implements the gauge automorphism ϱ_U in the twisted Fock representation ψ_{P_1}, i.e. one has

$$\Gamma(U)\psi_{P_1}(a) = \psi_{P_1}(\varrho_U(a))\Gamma(U), \qquad a \in \mathfrak{C}(\mathcal{K}).$$

Proof. It suffices to consider the case that a belongs to $\mathcal{K}$. Recall from (2.10) that $\psi_{P_1}(f) = i\pi_{P_1}(f)\Psi(-\mathbf{1})$ for $f \in \mathcal{K}$. Since U is diagonal, the implementer $\Gamma(U)$ is even:

$$[\Gamma(U), \Psi(-\mathbf{1})] = 0.$$

This implies $\Gamma(U)\psi_{P_1}(f)\Gamma(U)^* = \psi_{P_1}(Uf)$. □

It turns out that the transformation properties of the exponential term in (4.6) are also easily obtained.

LEMMA 4.4.
If $V \in \mathfrak{I}_{P_1}(\mathcal{K})^G$, then $\exp(\frac{1}{2}\overline{T_V}a^*a^*)\Omega_{P_1}$ is invariant under all gauge transformations $\Gamma(U)$, $U \in G$.

Proof. If $T \in \mathfrak{H}_{P_1}$ (see (2.39)) has finite rank, then one readily verifies that

$$\Gamma(U)(\tfrac{1}{2}\overline{T}a^*a^*)\Gamma(U)^* = \tfrac{1}{2}(U\overline{T}U^*)a^*a^*.$$

Approximating T_V by finite rank operators from $\mathfrak{H}_{P_1}$ relative to Hilbert–Schmidt norm (cf. [CR87]), one convinces oneself that

$$\Gamma(U)(\tfrac{1}{2}\overline{T_V}a^*a^*)^n\Omega_{P_1} = \big(\tfrac{1}{2}(U\overline{T_V}U^*)a^*a^*\big)^n\Omega_{P_1}, \qquad n \in \mathbb{N},$$

because $\Gamma(U)\Omega_{P_1} = \Omega_{P_1}$. It follows that

$$\begin{aligned}\Gamma(U)\exp(\tfrac{1}{2}\overline{T_V}a^*a^*)\Omega_{P_1} &= \sum_{n=0}^{\infty}\frac{1}{n!}\Gamma(U)(\tfrac{1}{2}\overline{T_V}a^*a^*)^n\Omega_{P_1}\\ &= \exp\big(\tfrac{1}{2}(U\overline{T_V}U^*)a^*a^*\big)\Omega_{P_1}.\end{aligned}$$

Since U commutes with P_1, P_2 and V, it also commutes with all components of V and V^*, including the operators ${V_{11}}^{-1}$, $p_{\ker V_{11}{}^*}$ etc. T_V is by definition (2.54) a bounded function of these operators, so that

$$[U, T_V] = 0, \qquad U \in G. \tag{4.7}$$

Hence we get $\Gamma(U)\exp(\frac{1}{2}\overline{T_V}a^*a^*)\Omega_{P_1} = \exp(\frac{1}{2}\overline{T_V}a^*a^*)\Omega_{P_1}$ as claimed. □

Thus we arrive at the following formula

$$\Gamma(U)\Psi_\alpha(V)\Omega_{P_1} = D_V\psi_{P_1}(Ug_{\alpha_1}\cdots Ug_{\alpha_l})\psi_{P_1}(Ue_1\cdots Ue_{L_V})\exp(\tfrac{1}{2}\overline{T_V}a^*a^*)\Omega_{P_1} \tag{4.8}$$

which enables us to derive the "charge" carried by ϱ_V. This is best described by using the Fock space structure on $H(\varrho_V)$ established in Corollary 2.19.

THEOREM 4.5.
Let P_1 be a basis projection of $\mathcal{K}$, let G be a group consisting of diagonal Bogoliubov operators, and let $V \in \mathfrak{I}_{P_1}(\mathcal{K})^G$. Further let $\mathfrak{h}_V \subset \mathcal{K}_1$ be the L_V–dimensional subspace defined in (2.55), $L_V < \infty$, and let $\mathfrak{k}_V \subset \mathcal{K}$ be the M_V–dimensional subspace defined in (2.83), $M_V = -\frac{1}{2}\operatorname{ind} V$. Then both $\mathfrak{h}_V$ and $\mathfrak{k}_V$ are invariant under G. Let $\Lambda_{\mathfrak{k}_V}$ be the unitary representation of G on the antisymmetric Fock space $\mathcal{F}_a(\mathfrak{k}_V)$ over $\mathfrak{k}_V$ that is obtained by taking antisymmetric tensor powers of the representation on $\mathfrak{k}_V$. Then the unitary representation $\mathfrak{U}_V$ of G on the Hilbert space of isometries $H(\varrho_V)$ which implements ϱ_V in the Fock representation π_{P_1} is unitarily equivalent to $\Lambda_{\mathfrak{k}_V}$, tensored with the one–dimensional representation $\det_{\mathfrak{h}_V}(U) \equiv \det(U|_{\mathfrak{h}_V})$:

$$\mathfrak{U}_V \simeq \det\nolimits_{\mathfrak{h}_V} \otimes \Lambda_{\mathfrak{k}_V}. \tag{4.9}$$

Proof. The finite dimensional subspace $\mathfrak{h}_V = V_{12}(\ker V_{22})$ is invariant under G because the elements of G commute with the components of V. Since the e_r form an orthonormal basis in $\mathfrak{h}_V$, it follows from the CAR that

$$U(e_1)\cdots U(e_{L_V}) = \det\left(U|_{\mathfrak{h}_V}\right)\cdot e_1\cdots e_{L_V}, \qquad U\in G.$$

Similarly, the elements of G commute with the basis projection P_V (cf. (4.7) and (2.60)) and leave $\ker V^*$ invariant, so that $\mathfrak{k}_V = P_V(\ker V^*)$ is also left invariant. It then follows from the CAR that $g_{\alpha_1}\cdots g_{\alpha_l}$ transforms like the l–fold antisymmetric tensor product of $g_{\alpha_1},\dots,g_{\alpha_l}$ under G.

Thus we see from (4.8) that the representation of G on $H(\varrho_V)\Omega_{P_1}$ is unitarily equivalent to $\det_{\mathfrak{h}_V}\otimes\Lambda_{\mathfrak{k}_V}$. By Lemma 4.2, the same holds true for the representation on $H(\varrho_V)$. □

Theorem 4.5 shows that genuine (i.e. non–surjective) quasi–free endomorphisms ϱ_V are always *reducible* in the sense that the representation $\mathcal{U}_V$ of G on $H(\varrho_V)$ or, equivalently, the representation of the gauge invariant “observable” algebra $\mathfrak{C}(\mathcal{K})^G$ induced by ϱ_V on the subspace of $\Gamma(G)$–invariant vectors in $\mathcal{F}_a(\mathcal{K}_1)$, is reducible. In fact, each “n–particle” subspace of $H(\varrho_V)$ (i.e. the closed linear span of all $\Psi_\alpha(V)$ with α of length n) is invariant under G. Let $\mathcal{U}_V^{(n)}$ be the restriction of $\mathcal{U}_V$ to this subspace. Closest to irreducibility is the case that at least $\mathcal{U}_V^{(1)}$ is irreducible. In typical situations, the remaining representations $\mathcal{U}_V^{(n)}$ will then also be irreducible. This happens for instance if $G\cong U(N)$ or $G\cong SU(N)$, and $\mathfrak{k}_V$ carries the defining representation of G. In the $U(N)$ case, the $\mathcal{U}_V^{(n)}$ are not only irreducible, but also mutually inequivalent. In the $SU(N)$ case, the representations $\mathcal{U}_V^{(0)},\dots,\mathcal{U}_V^{(N-1)}$ are mutually inequivalent, but $\mathcal{U}_V^{(N)}$ is equivalent to $\mathcal{U}_V^{(0)}$. In general, it can nevertheless happen that $\mathcal{U}_V^{(1)}$ is irreducible but some $\mathcal{U}_V^{(n)}$ are not, as is the case if $G\cong SO(N)$ ($N>2$ even) and $\mathfrak{k}_V$ carries the defining representation of G (cf. [Wey46, Boe70]). If $\mathcal{U}_V^{(1)}$ is reducible, then one has an additional Clebsch–Gordan type splitting according to the unitary isomorphism $\mathcal{F}_a(\mathfrak{k}_1\oplus\mathfrak{k}_2)\cong\mathcal{F}_a(\mathfrak{k}_1)\otimes\mathcal{F}_a(\mathfrak{k}_2)$ where $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are G–invariant subspaces of $\mathfrak{k}_V$.

In the special case that ϱ_V is an automorphism ($\operatorname{ind}V=0$), there survives only the factor $\det_{\mathfrak{h}_V}$ in Theorem 4.5. This is consistent with Matsui's result (2.19), (2.20) on the equivalence of Fock states over $\mathfrak{C}(\mathcal{K})^G$ (G compact). For let $V\in\mathcal{I}^0_{P_1}(\mathcal{K})^G$ and set $P\equiv VP_1V^*$. Then the GNS representation π_P for the Fock state ω_P can be realized on $\mathcal{F}_a(\mathcal{K}_1)$ as $\pi_P=\pi_{P_1}\circ{\varrho_V}^{-1}$, with cyclic vector Ω_{P_1}. The unitary implementer $\Psi(V)$ intertwines the representations $\pi_P|_{\mathfrak{C}(\mathcal{K})^G}$ and $\pi_{P_1}|_{\mathfrak{C}(\mathcal{K})^G}$. $\Psi(V)$ restricts to a unitary isomorphism between the closed cyclic subspaces $(\pi_P|_{\mathfrak{C}(\mathcal{K})^G}\Omega_{P_1})^-$ and $(\pi_{P_1}|_{\mathfrak{C}(\mathcal{K})^G}\Omega_{P_1})^-$ if and only if $[\Psi(V),\Gamma(G)]=0$, i.e. if and only if $\det_{\mathfrak{h}_V}(G)=1$. Now one has $\mathfrak{h}_V=V_{12}(\ker V_{22})=\ker {V_{11}}^*=\ker P_{11}=\mathcal{K}_1\cap\overline{P}(\mathcal{K})$, where we used (2.34) and (2.45). Therefore $\det_{\mathfrak{h}_V}(G)=1$ is equivalent to Matsui's condition (2.20). However, Matsui's result applies more generally because a basis projection $P\in\mathfrak{P}_{P_1}$ (see (2.38)) which commutes with G is not necessarily of the form $P=VP_1V^*$ with $V\in\mathcal{I}^0_{P_1}(\mathcal{K})^G$. In general, there exist $V\in\mathcal{I}^0_{P_1}(\mathcal{K})$ such that $[VP_1V^*,G]=0$ but $[V,G]\neq0$. For such V, one does not have $\Gamma(U)\Psi(V)\Gamma(U)^*\sim\Psi(V)$ but only $\gamma(U)\Psi(V)\Omega_{P_1}\sim\Psi(V)\Omega_{P_1}$. If, for a given basis projection P, the representation $\mathcal{U}$ of G on $\ker P_{11}$ is equivalent to its complex conjugate representation $\mathcal{U}^*$, then there exists a gauge invariant V with $VP_1V^*=P$. If $\mathcal{U}$ and $\mathcal{U}^*$ are disjoint but both contained in $\mathcal{K}_1$ with infinite multiplicity then V with $VP_1V^*=P$ can also be chosen to be gauge invariant. (These statements follow from part (a) of Proposition 4.6 together with Proposition 2.9.)

Another case of interest is obtained by specializing further to the case of automorphisms and $G \cong \mathbb{T}$. Assume that $\mathbb{T}$ acts on $\mathcal{K}$ via $U_\lambda = e^{i\lambda}P + e^{-i\lambda}\overline{P}$, $\lambda \in \mathbb{R}$, where P is a basis projection of $\mathcal{K}$ which commutes with the given basis projection P_1. Then all gauge invariant Bogoliubov operators commute with P, and the semigroup of (not necessarily implementable) gauge invariant Bogoliubov operators is isomorphic to the semigroup of all isometries of $P(\mathcal{K})$. Let

$$p_+ \equiv PP_1, \qquad p_- \equiv PP_2. \tag{4.10}$$

(In the situation of the free Dirac field outlined in Example 3, $p_\pm$ are just the spectral projections of the Dirac Hamiltonian.) If V is gauge invariant, then the implementability condition (V_{12} Hilbert–Schmidt) is equivalent to the condition that the components V_{+-} and V_{-+} be Hilbert–Schmidt, because one has

$$\begin{aligned} V_{11} &= V_{++} + \overline{V_{--}}, & V_{12} &= V_{+-} + \overline{V_{-+}}, \\ V_{21} &= V_{-+} + \overline{V_{+-}}, & V_{22} &= V_{--} + \overline{V_{++}}, \end{aligned} \tag{4.11}$$

with $V_{\epsilon\epsilon'} \equiv p_\epsilon V p_{\epsilon'}$, $\epsilon, \epsilon' = \pm$. These relations also entail that, for $V \in \mathcal{I}_{P_1}(\mathcal{K})^{\mathbb{T}}$,

$$\mathfrak{h}_V = V_{12}(\ker V_{22}) = V_{+-}(\ker V_{--}) \oplus \overline{V_{-+}}(\ker \overline{V_{++}}). \tag{4.12}$$

The subgroup $\mathcal{I}^0_{P_1}(\mathcal{K})^{\mathbb{T}}$ of gauge invariant implementable Bogoliubov operators with index zero can be identified with the *restricted unitary group* of $P(\mathcal{K})$, i.e. with the group of all unitaries on $P(\mathcal{K})$ whose $(+-)$ and $(-+)$ components are Hilbert–Schmidt, through the restriction map $V \mapsto PVP$. To describe the charge corresponding to $V \in \mathcal{I}^0_{P_1}(\mathcal{K})^{\mathbb{T}}$, we have to compute the determinant of $U_\lambda|_{\mathfrak{h}_V}$, $\lambda \in \mathbb{R}$. By $\operatorname{ind} V = 0$, V_{+-} maps $\ker V_{--}$ isometrically onto $\ker V_{++}{}^*$, and V_{-+} maps $\ker V_{++}$ isometrically onto $\ker V_{--}{}^*$ (cf. (2.33), (2.34)). Hence we get from (4.12)

$$\begin{aligned} \det{}_{\mathfrak{h}_V}(U_\lambda) &= \exp\big(i\lambda(\dim\ker V_{--} - \dim\ker V_{++})\big) \\ &= \exp(i\lambda \operatorname{ind} V_{--}) \\ &= \exp(-i\lambda \operatorname{ind} V_{++}). \end{aligned}$$

The charge corresponding to an element V of the restricted unitary group is therefore equal to $-\operatorname{ind} V_{++}$. The implementer $\Psi(V)$ maps the charge–q sector in $\mathcal{F}_a(\mathcal{K}_1)$, $q \in \mathbb{Z}$, onto the sector with charge $q - \operatorname{ind} V_{++}$. Of course, $\operatorname{ind} V_{++}$ can only be nonzero if p_+ and p_- both have infinite rank. The fact that the charge created by a gauge invariant quasi–free automorphism ϱ_V is in this way related to the Fredholm index of V_{++} was implicit in the literature of the 1970s on the external field problem (see e.g. [Lab74, Lab75, Fre77, KS77, Rui77, Rui78, Sei78]) but has apparently first been pointed out explicitly by Carey, Hurst and O'Brien [CHO82]. These authors showed that the connected components of the restricted unitary group $\mathcal{I}^0_{P_1}(\mathcal{K})^{\mathbb{T}}$ are precisely labelled by $\operatorname{ind} V_{++}$. Computations of $\operatorname{ind} V_{++}$ for certain classes of unitary operators V can be found in more recent publications [CR87, Mat87a, Rui89a, Rui89b, Mat90, BH92].

Let us return to the general situation. So far we have analyzed the representation of G on $H(\varrho_V)$ in terms of the given representations on $\mathfrak{h}_V$ and $\mathfrak{k}_V$. But we can also characterize the representations of G which can possibly occur on $\mathfrak{h}_V$ and $\mathfrak{k}_V$. Note that, if $\mathcal{K}_1' \subset \mathcal{K}_1$ is a G–invariant subspace carrying a representation of class ξ, then the complex conjugate space $(\mathcal{K}_1')^* \subset \mathcal{K}_2$ carries a representation of the complex conjugate class ξ^*.

PROPOSITION 4.6.
a) Let $\mathfrak{h} \subset \mathcal{K}_1$ be a finite dimensional G–invariant subspace carrying a representation of class ξ. If ξ is self–conjugate (i.e. $\xi = \xi^*$), then there exists $V \in \mathcal{I}^0_{P_1}(\mathcal{K})^G$ with $\mathfrak{h}_V = \mathfrak{h}$ such that $V - \mathbf{1}$ has finite rank. If ξ and ξ^* are

disjoint, then there exists $V \in \mathcal{I}^0_{P_1}(\mathcal{K})^G$ *with* $\mathfrak{h}_V = \mathfrak{h}$ *if and only if both* ξ *and* ξ^* *are contained in* $\mathcal{K}_1$ *with infinite multiplicity.*

b) Let $\mathfrak{k} \subset \mathcal{K}_1$ *be a closed* G*–invariant subspace carrying a representation of class* ξ*. If* ξ *is contained in* $\mathcal{K}_1$ *with infinite multiplicity, then there exists a diagonal Bogoliubov operator* $V \in \mathcal{I}_{P_1}(\mathcal{K})^G$ *with* $\mathfrak{k}_V = \mathfrak{k}$*. If* $\mathfrak{k}$ *is irreducible and if* ξ *is contained in* $\mathcal{K}_1$ *with finite multiplicity, then there exists* $V \in \mathcal{I}_{P_1}(\mathcal{K})^G$ *with* $\mathfrak{k}_V = \mathfrak{k}$ *if and only if* $\mathfrak{k}$ *is finite dimensional and* ξ^* *is contained in* $\mathcal{K}_1$ *with infinite multiplicity. A class* ξ' *which is only contained in* $\mathcal{K}_2$ *but not in* $\mathcal{K}_1$ *cannot be realized on any* $\mathfrak{k}_V$*,* $V \in \mathcal{I}_{P_1}(\mathcal{K})^G$*.*

Proof. The proof will be constructive.

a) 1. Assume first that ξ is self–conjugate. Then there exists a partial isometry u from $\mathcal{K}_2$ to $\mathcal{K}_1$ with initial space $\mathfrak{h}^*$ and final space $\mathfrak{h}$ which commutes with G. Let $p_{\mathfrak{h}^\perp}$ be the orthogonal projection onto $\mathfrak{h}^\perp \subset \mathcal{K}_1$. Then

$$V \equiv \begin{pmatrix} p_{\mathfrak{h}^\perp} & u \\ \overline{u} & \overline{p_{\mathfrak{h}^\perp}} \end{pmatrix}$$

is a unitary Bogoliubov operator with $[V, G] = 0$, $\mathfrak{h}_V = V_{12}(\ker V_{22}) = u(\mathfrak{h}^*) = \mathfrak{h}$, and $V - \mathbf{1}$ has finite rank.

2. Next assume that ξ and ξ^* are disjoint. If ξ and ξ^* are contained in $\mathcal{K}_1$ with infinite multiplicity, one has a decomposition

$$\mathcal{K}_1 \cong \ell^2(\mathfrak{h}) \oplus \ell^2(\mathfrak{h}') \oplus \mathcal{H}$$

where $\mathfrak{h}$ is the given subspace carrying a representation of class ξ, $\mathfrak{h}'$ is a subspace carrying a representation of the complex conjugate class ξ^*, $\ell^2(\mathfrak{h})$ is the Hilbert space of square summable sequences of elements of $\mathfrak{h}$, and $\mathcal{H}$ is the orthogonal complement of $\ell^2(\mathfrak{h}) \oplus \ell^2(\mathfrak{h}')$ in $\mathcal{K}_1$. Then let s resp. s' be the (G–invariant) unilateral shift on $\ell^2(\mathfrak{h})$ resp. $\ell^2(\mathfrak{h}')$, given by $(f_1, f_2, \dots) \mapsto (0, f_1, f_2, \dots)$. Define a Fredholm operator V_{11} on $\mathcal{K}_1$ with index zero by

$$V_{11} \equiv s \oplus s'^* \oplus p_{\mathcal{H}}$$

so that $\ker V_{11} = \mathfrak{h}'$, $\ker V_{11}{}^* = \mathfrak{h}$. (Here $\mathfrak{h}$ is identified with $(\mathfrak{h}, 0, 0, \dots)$, and similar for $\mathfrak{h}'$.) Furthermore let V_{12} be a G–invariant partial isometry from $\mathcal{K}_2$ to $\mathcal{K}_1$ with initial space $\mathfrak{h}'^*$ and final space $\mathfrak{h}$ (such V_{12} exists because $\mathfrak{h}'^*$ and $\mathfrak{h}$ both belong to the class ξ). Then V_{11} and V_{12} define a Bogoliubov operator $V \in \mathcal{I}^0_{P_1}(\mathcal{K})^G$ with $\mathfrak{h}_V = \mathfrak{h}$. (One can show that V fulfills in addition $VP_1V^* = P_{(0,\mathfrak{h})}$, in the notation of Proposition 2.9.)

Conversely, if there exists $V \in \mathcal{I}^0_{P_1}(\mathcal{K})^G$ with $\mathfrak{h}_V = \mathfrak{h}$, then ξ^* must be contained in $\mathcal{K}_1$ since $\ker V_{11}$ belongs to this class (recall that V_{12} restricts to a unitary intertwiner between $\ker V_{22} = (\ker V_{11})^*$ and $\mathfrak{h}_V$). Since ξ and ξ^* are disjoint, one has a decomposition

$$\mathcal{K}_1 = \ker V_{11}{}^* \oplus \ker V_{11} \oplus \mathcal{H} \tag{4.13}$$

with $\mathcal{H} \equiv \operatorname{ran} V_{11} \cap \operatorname{ran} V_{11}{}^*$. Viewing V_{11} as a bounded bijection from $\operatorname{ran} V_{11}{}^*$ onto $\operatorname{ran} V_{11}$, one gets a unitary intertwiner from $\operatorname{ran} V_{11}{}^*$ onto $\operatorname{ran} V_{11}$ by polar decomposition of V_{11}. Therefore the representations of G on $\operatorname{ran} V_{11}{}^*$ and $\operatorname{ran} V_{11}$ must be unitarily equivalent. One has by (4.13)

$$\operatorname{ran} V_{11}{}^* = \ker V_{11}{}^* \oplus \mathcal{H}, \qquad \operatorname{ran} V_{11} = \ker V_{11} \oplus \mathcal{H}.$$

Since $\ker V_{11}{}^*$ belongs to the class ξ and $\ker V_{11}$ to the (disjoint) class ξ^*, the representations on $\operatorname{ran} V_{11}{}^*$ and $\operatorname{ran} V_{11}$ can only be equivalent if ξ and ξ^* are both contained in $\mathcal{H}$ with infinite multiplicity.

b) 1. Assume first that ξ appears in $\mathcal{K}_1$ infinitely often. Then $\mathcal{K}_1$ has the form

$$\mathcal{K}_1 \cong \ell^2(\mathfrak{k}) \oplus \mathcal{H}$$

where $\mathfrak{k}$ belongs to the class ξ. If s is the shift on $\ell^2(\mathfrak{k})$ as above, then

$$V \equiv \begin{pmatrix} s \oplus p_{\mathcal{H}} & 0 \\ 0 & \overline{s \oplus p_{\mathcal{H}}} \end{pmatrix}$$

is an element of $\mathcal{I}_{P_1}(\mathcal{K})^G$ with $\mathfrak{k}_V = \ker {V_{11}}^* = \mathfrak{k}$.

2. Next assume that $\mathfrak{k} \subset \mathcal{K}_1$ is irreducible of class ξ and that ξ is contained in $\mathcal{K}_1$ with multiplicity $M < \infty$. Let first $\mathfrak{k}$ be finite dimensional and ξ^* be contained in $\mathcal{K}_1$ with multiplicity ∞. Then ξ and ξ^* are necessarily disjoint. Without loss of generality, we may restrict attention to the closed *–invariant subspace of $\mathcal{K}$ which comprises all representations of the classes ξ and ξ^*, because a Bogoliubov operator having the desired properties on this subspace can be canonically extended to $\mathcal{K}$ by letting it act like the identity on the complement of that subspace (and because all other relevant operators leave this space invariant). Thus we will assume that $\mathcal{K}$ is of the form

$$\mathcal{K} = \ell^2(\mathfrak{k}) \oplus \ell^2(\mathfrak{k})^*$$

with $\ell^2(\mathfrak{k}) \cap \mathcal{K}_1 = \{(f_1, \ldots, f_M, 0, 0, \ldots) \mid f_j \in \mathfrak{k}\}$. If P denotes the basis projection onto $\ell^2(\mathfrak{k})$, then we are in a situation similar to the one discussed on p. 82. P commutes with P_1, and if we define $p_\pm$ as in (4.10): $p_+ \equiv PP_1$, $p_- \equiv PP_2$, then p_+ has finite rank since $M < \infty$ by assumption. Any gauge invariant Bogoliubov operator V has to commute with P and is therefore of the form (4.11). Such V is automatically implementable because p_+ has finite rank. Now let s again be the unilateral shift on $\ell^2(\mathfrak{k})$, and set

$$V \equiv s \oplus \overline{s}.$$

Then V lies in $\mathcal{I}_{P_1}(\mathcal{K})^G$. We claim that $\mathfrak{k}_V = \mathfrak{k}$. Note that, by (2.92), $\mathfrak{k}_V \equiv P_V(\ker V^*) = P_{T_V}(\ker V^*)$ where the basis projection P_{T_V} is defined in (2.42) and (2.54). By definition of V, one finds that V_{21} is a partial isometry so that $T_V = 0$ by the remark on p. 47. As a consequence, $P_{T_V} = P_1$ and $\mathfrak{k}_V = P_{T_V}(\ker V^*) = P_1(\mathfrak{k} \oplus \mathfrak{k}^*) = \mathfrak{k}$ as claimed.

Conversely, assume that $V \in \mathcal{I}_{P_1}(\mathcal{K})^G$ exists with $\mathfrak{k}_V = \mathfrak{k}$. Since all $V^n(\mathfrak{k}_V)$, $n \in \mathbb{N}$, are mutually orthogonal and belong to the class ξ, ξ must be contained in $\mathcal{K}$ with infinite multiplicity. This means that ξ^* must appear in $\mathcal{K}_1$ with infinite multiplicity, because the multiplicity of ξ in $\mathcal{K}_1$ is finite by assumption. Then consider again the closed *–invariant subspace $\mathcal{K}'$ of $\mathcal{K}$ which contains all the representations of class ξ and ξ^*. Since ξ and ξ^* are disjoint, we can write

$$\mathcal{K}' = \ell^2(\mathfrak{k}) \oplus \ell^2(\mathfrak{k})^* \cong \big(\ell^2(\mathbb{C}) \otimes \mathfrak{k}\big) \oplus \big(\ell^2(\mathbb{C}) \otimes \mathfrak{k}\big)^*.$$

Any operator A on $\mathcal{K}$ which commutes with G leaves $\mathcal{K}'$ invariant, and its restriction to $\mathcal{K}'$ has the form $(A' \otimes \mathbf{1}_\mathfrak{k}) \oplus (\overline{A'' \otimes \mathbf{1}_\mathfrak{k}})$. In particular, the projections $p_\pm$ introduced above can be written as $p_\pm = p'_\pm \otimes \mathbf{1}_\mathfrak{k}$, and the component V_{12} of a gauge invariant Bogoliubov operator restricts to an operator of the form $(V'_{+-} \otimes \mathbf{1}_\mathfrak{k}) \oplus (\overline{V'_{-+} \otimes \mathbf{1}_\mathfrak{k}})$ on $\mathcal{K}'$ (cf. (4.11)). Since V fulfills the implementability condition, V_{12} is Hilbert–Schmidt. If $\mathfrak{k}$ were infinite dimensional, this would entail that V'_{+-} and V'_{-+} had to vanish. But then the restrictions of P_1 and V to $\mathcal{K}'$ would commute, so that the powers of V acting on $\mathfrak{k}_V$ would produce infinitely many mutually orthogonal copies of $\mathfrak{k}$ in $\mathcal{K}_1$. This contradicts the assumption, and therefore $\mathfrak{k}$ has to be finite dimensional.

3. Finally, $\mathfrak{k}_V$ cannot be a subspace of $\mathcal{K}_2$ because one has $\mathfrak{k}_V = P_{T_V}(\ker V^*)$ (see above), but $\mathcal{K}_2 \cap \operatorname{ran} P_T = \{0\}$ for any operator $T \in \mathfrak{H}_{P_1}$ (cf. the remark on p. 44). □

One could also ask which combinations of representations can occur simultaneously on $\mathfrak{h}_V$ and $\mathfrak{k}_V$, for a single V. Let us only remark here that, if a fixed $V \in \mathcal{I}_{P_1}(\mathcal{K})^G$ is multiplied with a unitary $V' \in \mathcal{I}^0_{P_1}(\mathcal{K})^G$, then the corresponding representation changes according to $\mathcal{U}_{V'V} \simeq \det_{\mathfrak{h}_{V'}} \cdot \mathcal{U}_V$ so that the determinant factor in (4.9) can be modified almost arbitrarily without affecting the factor $\Lambda_{\mathfrak{k}_V}$.

Also note that, in the generic field theoretic situation described in Example 3, $\mathcal{K}_1$ has the form $\mathcal{K}_1 = \mathcal{K}_+ \oplus \mathcal{K}_-{}^*$ where $\mathcal{K}_+$ and $\mathcal{K}_-$ both carry the defining representation of G with multiplicity ∞. Hence the defining representation and its complex conjugate are both realized on some $\mathfrak{h}_V$ and $\mathfrak{k}_V$.

These remarks complete our general discussion of the charge structure of quasi–free endomorphisms of the CAR algebra. We would like to devote the rest of this section to a comparison between the semigroup of gauge invariant quasi–free endomorphisms discussed above and the semigroup of localized endomorphisms appearing in the theory of superselection sectors. First of all, superselection sectors are by definition *irreducible* so that the question arises how to obtain the irreducible *"subobjects"* of gauge invariant quasi–free endomorphisms ϱ_V. Suppose that $\{\Psi_1, \ldots, \Psi_m\}$ is an orthonormal set in $H(\varrho_V)$ which transforms irreducibly under G. According to the general theory, there should exist a gauge invariant isometry Φ on Fock space with $\operatorname{ran}\Phi = \oplus_{j=1}^m \operatorname{ran}\Psi_j$ (cf. (0.7)). The corresponding irreducible endomorphism ϱ (which is not quasi–free) would then be given by $\varrho(a) \equiv \sum_{j=1}^m \Phi^*\Psi_j a \Psi_j^* \Phi$. However, the construction of such gauge invariant isometries Φ is at present unclear.

Similarly, the *direct sum* of quasi–free endomorphisms (which should be defined as in (0.6)) can in general not be quasi–free. This is evident from the index formula (2.14) and from the additivity of the statistics dimension on direct sums. The statistics dimension and the Bosonized statistics operator from (2.97) and (2.98) are the only invariants related to the *statistics* of a superselection sector which can be unambiguously ascribed to a quasi–free endomorphism in this general setting.

Thus $\mathcal{I}_{P_1}(\mathcal{K})^G$ is only closed under composition and not under taking direct sums or subobjects. Furthermore, the existence of *conjugates* is only guaranteed if one makes additional assumptions on the action of G on $\mathcal{K}$. Specifically, one needs charge conjugation already on the level of first quantization:

PROPOSITION 4.7.
Assume that there exists a further basis projection P of $\mathcal{K}$ which commutes with P_1 and with G, and let $p_\pm$ be defined as in (4.10)*:*

$$p_+ \equiv PP_1, \qquad p_- \equiv PP_2.$$

Assume further that there exists a unitary operator $\mathcal{C}_{+-}$ from $\mathcal{K}_- \equiv p_-(\mathcal{K})$ onto $\mathcal{K}_+ \equiv p_+(\mathcal{K})$ which commutes with G, and let $\mathcal{C}$ be the unique Bogoliubov operator which commutes with P and which is given on $P(\mathcal{K}) = \mathcal{K}_+ \oplus \mathcal{K}_-$ by the matrix

$$P\mathcal{C}P \equiv \begin{pmatrix} 0 & \mathcal{C}_{+-} \\ \mathcal{C}_{+-}{}^* & 0 \end{pmatrix}.$$

Then $\mathcal{C}$ is gauge invariant, unitary and self–adjoint, and the map

$$V \mapsto V^c \equiv \mathcal{C}V\mathcal{C} \tag{4.14}$$

is an involutive automorphism of $\mathcal{I}_{P_1}(\mathcal{K})^G$ which preserves the statistics dimension. In addition, one has

$$\mathfrak{h}_{V^c} = \mathcal{C}(\mathfrak{h}_V{}^*), \qquad \mathfrak{k}_{V^c} = \mathcal{C}(\mathfrak{k}_V{}^*)$$

so that the representation $\mathcal{U}_{V^c}$ associated with V^c according to (4.9) *is unitarily equivalent to the complex conjugate of $\mathcal{U}_V$* ("charge conjugation").

Proof. $\mathcal{C} = P\mathcal{C}P + \overline{P\mathcal{C}P}$ is clearly a gauge invariant self–adjoint element of $\mathcal{I}^0(\mathcal{K})$. Its components with respect to P_1, P_2 are

$$\mathcal{C}_{11} = 0, \qquad \mathcal{C}_{12} = \begin{pmatrix} 0 & \mathcal{C}_{+-} \\ \mathcal{C}_{+-}{}^\tau & 0 \end{pmatrix}.$$

Thus one has ${V^c}_{12} = \mathcal{C}_{12} V_{21} \mathcal{C}_{12}$ so that V^c belongs to $\mathcal{I}_{P_1}(\mathcal{K})^G$ if V does. $(V^c)^c = V$ follows from $\mathcal{C}^2 = \mathbf{1}$. The map (4.14) is consequently an involutive automorphism of $\mathcal{I}_{P_1}(\mathcal{K})^G$ (it is unital and multiplicative). It preserves the statistics dimension because $\operatorname{ind} \mathcal{C} = 0$.

Now let $V \in \mathcal{I}_{P_1}(\mathcal{K})^G$. One has ${V^c}_{22} = \mathcal{C}_{21} V_{11} \mathcal{C}_{12}$ so that

$$\begin{aligned}
\mathfrak{h}_{V^c} &\equiv {V^c}_{12}(\ker {V^c}_{22}) \\
&= \mathcal{C}_{12} V_{21} \mathcal{C}_{12}(\ker \mathcal{C}_{21} V_{11} \mathcal{C}_{12}) \\
&= \mathcal{C}_{12} V_{21}(\ker V_{11}) \\
&= \mathcal{C}_{12}(\mathfrak{h}_V{}^*) \\
&= \mathcal{C}(\mathfrak{h}_V{}^*).
\end{aligned}$$

Similarly, one finds that the antisymmetric Hilbert–Schmidt operators T_V, T_{V^c} associated with V, V^c through (2.54) are related to each other by

$$T_{V^c} = \mathcal{C}_{21} \overline{T_V} \mathcal{C}_{21}$$

since

$$\begin{aligned}
T_{V^c} &\equiv {V^c}_{21} {{V^c}_{11}}^{-1} - {{V^c}_{22}}^{-1*} {V^c}_{12}{}^* p_{\ker {V^c}_{11}{}^*} \\
&= \mathcal{C}_{21} V_{12} \mathcal{C}_{21} \cdot \mathcal{C}_{12} {V_{22}}^{-1} \mathcal{C}_{21} - \mathcal{C}_{21} {V_{11}}^{-1*} \mathcal{C}_{12} \cdot \mathcal{C}_{21} {V_{21}}^* \mathcal{C}_{21} \cdot \mathcal{C}_{12} p_{\ker V_{22}{}^*} \mathcal{C}_{21} \\
&= \mathcal{C}_{21}(V_{12} {V_{22}}^{-1} - {V_{11}}^{-1*} {V_{21}}^* p_{\ker V_{22}{}^*}) \mathcal{C}_{21} \\
&= \mathcal{C}_{21} \overline{T_V} \mathcal{C}_{21}.
\end{aligned}$$

One obtains for the corresponding basis projections (cf. (2.42))

$$\begin{aligned}
P_{T_{V^c}} &\equiv (P_1 + T_{V^c})(P_1 + T_{V^c}{}^* T_{V^c})^{-1}(P_1 + T_{V^c}{}^*) \\
&= (P_1 + \mathcal{C}_{21} \overline{T_V} \mathcal{C}_{21})(P_1 + \mathcal{C}_{12} T_V T_V{}^* \mathcal{C}_{21})^{-1}(P_1 + \mathcal{C}_{12} T_V{}^\tau \mathcal{C}_{12}) \\
&= \mathcal{C}(P_2 + \overline{T_V}) \mathcal{C}_{21} \cdot \mathcal{C}_{12}(P_2 + T_V T_V{}^*)^{-1} \mathcal{C}_{21} \cdot \mathcal{C}_{12}(P_2 + T_V{}^\tau) \mathcal{C} \\
&= \mathcal{C} \overline{P_{T_V}} \mathcal{C}.
\end{aligned}$$

This entails further that

$$\mathfrak{k}_{V^c} \equiv P_{T_{V^c}}(\ker V^{c*}) = \mathcal{C} \overline{P_{T_V}} \mathcal{C}(\mathcal{C} \ker V^*) = \mathcal{C}(\mathfrak{k}_V{}^*)$$

and finally

$$\mathcal{U}_{V^c} \simeq \det{}_{\mathfrak{h}_{V^c}} \otimes \Lambda_{\mathfrak{k}_{V^c}} \simeq (\det{}_{\mathfrak{h}_V})^* \otimes (\Lambda_{\mathfrak{k}_V})^* \simeq \mathcal{U}_V{}^*.$$

□

Note that the assumptions of the proposition amount to a decomposition of the single–particle space $\mathcal{K}_1$ into the direct sum of two antiunitarily equivalent G–modules $\mathcal{K}_+$ and $\mathcal{K}_-{}^*$. These assumptions are of course satisfied in Example 3. It is also fairly obvious that, if these assumptions are violated, there will in general exist operators V in $\mathcal{I}_{P_1}(\mathcal{K})^G$ which do *not* admit conjugates in $\mathcal{I}_{P_1}(\mathcal{K})^G$ (i.e. there is no $V' \in \mathcal{I}_{P_1}(\mathcal{K})^G$ with $\mathcal{U}_{V'} \simeq \mathcal{U}_V{}^*$; cf. Proposition 4.6). But also note that, if $V \in \mathcal{I}_{P_1}(\mathcal{K})^G$ has finite index and if G acts on both subspaces $\mathfrak{h}_V$ and $\mathfrak{k}_V$ with determinant 1, then ϱ_V is automatically *self–conjugate* in the sense that $\mathcal{U}_V \simeq \mathcal{U}_V{}^*$. Recall from Theorem 4.5 that $\mathcal{U}_V$ is then equivalent to the representation $\Lambda_{\mathfrak{k}_V}$ on the antisymmetric Fock space $\mathcal{F}_a(\mathfrak{k}_V)$ over $\mathfrak{k}_V$. Let $\Lambda^{(n)}_{\mathfrak{k}_V}$ be the restriction of $\Lambda_{\mathfrak{k}_V}$ to the n–particle subspace of $\mathcal{F}_a(\mathfrak{k}_V)$ so that

$$\Lambda_{\mathfrak{k}_V} = \bigoplus_{n=0}^{M_V} \Lambda^{(n)}_{\mathfrak{k}_V}.$$

The character $\chi^{(n)}(U)$, $U \in G$, of $\Lambda^{(n)}_{\mathfrak{k}_V}$ is equal to the nth elementary symmetric function $\sum_{j_1<\cdots<j_n} \lambda_{j_1}\cdots\lambda_{j_n}$ of the eigenvalues $\lambda_1,\dots,\lambda_{M_V}$ of $U|_{\mathfrak{k}_V}$ (cf. [Mur62]). It follows that

$$\chi^{(n)} = \chi^{(M_V)}\overline{\chi^{(M_V-n)}}$$

and that

$$\Lambda^{(n)}_{\mathfrak{k}_V} \simeq \det\nolimits_{\mathfrak{k}_V} \otimes \Lambda^{(M_V-n)*}_{\mathfrak{k}_V}, \qquad n = 0,\dots,M_V$$

(this holds for any unitary representation on $\mathfrak{k}_V$, the condition on the determinant is not needed here). Assuming now that $\det_{\mathfrak{h}_V} = \det_{\mathfrak{k}_V} = 1$, we get that

$$\mathcal{U}^{(n)}_V \simeq \mathcal{U}^{(M_V-n)*}_V$$

so that $\mathcal{U}_V$ is self–conjugate as claimed. (This remark applies in particular to the Dirac field with gauge group $SU(N)$, see Example 3.)

The main reason for our inability to mimic the generic superselection structure more closely is of course the complete lack of *locality* in our preceding considerations. If one could find, in a specific model, a localized implementable quasi–free endomorphism, then it is clear that our methods would apply and could be used to construct the corresponding charged local fields and to determine their charge quantum numbers. It is however *not* clear from the outset that localization and implementability are compatible with each other. Known results in this direction only deal with the case of automorphisms. Building on the work of Carey and Ruijsenaars [CR87] and others, we constructed in [Bin93] a family of (implementable and transportable) localized automorphisms, carrying arbitrary $\mathbb{T}$–charges, for the free Dirac field in two spacetime dimensions with arbitrary mass. The Bogoliubov operators V belonging to these automorphisms are given by two $\mathbb{T}$–valued functions which are equal to 1 at spacelike infinity, and the charge $-\operatorname{ind} V_{++}$ (cf. p. 83) created by ϱ_V is equal to the difference of the winding numbers of these functions. However, in contrast to the two–dimensional case, there are no known examples of implementable charge–carrying (with respect to $G = \mathbb{T}$) Bogoliubov automorphisms in four spacetime dimensions. (Non–zero charge seems to be compatible with V_{12} compact, but not with V_{12} Hilbert–Schmidt, cf. [Mat87a, Rui89a, Mat90, BH92].)

As a slight generalization of a result in [Bin93], we can characterize localized gauge invariant endomorphisms of the free Dirac field as follows. Recall from Example 3 that all gauge invariant Bogoliubov operators for the N–component Dirac field with $U(N)$ gauge symmetry have the form (4.5)

$$V = (v \otimes \mathbf{1}_N) \oplus (\overline{v} \otimes \overline{\mathbf{1}_N}) \tag{4.15}$$

where $\mathcal{K} = \mathcal{H}' \oplus \mathcal{H}'^*$, $\mathcal{H}' = \mathcal{H} \otimes \mathbb{C}^N$, $\mathcal{H} = L^2(\mathbb{R}^{2n-1}, \mathbb{C}^{2^n})$, and v is an isometry of $\mathcal{H}$. We restrict to the time zero situation.

Proposition 4.8.
Let O be a double cone with base $B \subset \mathbb{R}^{2n-1}$ at time zero. Let $V \in \mathcal{I}_{P_1}(\mathcal{K})^{U(N)}$ and let v be the isometry of $\mathcal{H}$ associated with V via (4.15). Then ϱ_V is localized in O in the sense of $(0.3)^v$ if and only if there exists, for each connected component Δ of $\mathbb{R}^{2n-1} \setminus B$, a phase factor $\tau_\Delta \in \mathbb{T}$ such that

$$v(f) = \tau_\Delta f \qquad \text{for all } f \in \mathcal{H} \text{ with } \operatorname{supp} f \subset \Delta. \tag{4.16}$$

Proof. Assume that ϱ_V is localized in O. Let $b_1,\dots,b_N$ be the standard basis in $\mathbb{C}^N$, let Δ be a component of the complement of B, and let $f, g \in \mathcal{H}$ with $\operatorname{supp} f, g \subset \Delta$. Then

$$a(f,g) \equiv \sum_{j=1}^{N} (f \otimes b_j)(g \otimes b_j)^* \in \mathfrak{C}(\mathcal{K})^{U(N)}$$

[v] More precisely, the normal extension of ϱ_V in the representation π_{P_1} is localized in O.

is gauge invariant, and $\pi_{P_1}(a(f,g))$ is a local observable in $\mathfrak{A}(O')$. Since ϱ_V is localized in O, one has $a(f,g) = \varrho_V(a(f,g)) = \sum_j (vf \otimes b_j)(vg \otimes b_j)^*$. Since the b_j are linearly independent, it follows that

$$(f \otimes b_j)(g \otimes b_j)^* = (vf \otimes b_j)(vg \otimes b_j)^*, \qquad j = 1, \ldots, N. \tag{4.17}$$

Now let P' be the basis projection onto $\mathcal{H}' \subset \mathcal{K}$, and let $\omega_{P'}$ be the corresponding Fock state. One has

$$\omega_{P'}\Big((f \otimes b_j)^*(f \otimes b_j)(f \otimes b_j)^*(f \otimes b_j)\Big) = \|f\|_{\mathcal{H}}^4,$$

and, since $(vf \otimes b_j)^*$ belongs to the Gelfand (or annihilator) ideal of $\omega_{P'}$,

$$\omega_{P'}\Big((f\otimes b_j)^*(vf\otimes b_j)(vf\otimes b_j)^*(f\otimes b_j)\Big) = \langle vf, f\rangle \omega_{P'}\big((f\otimes b_j)^*(vf\otimes b_j)\big) = |\langle vf, f\rangle|^2.$$

Therefore one gets from (4.17), in the special case $f = g$, that $\|f\|_{\mathcal{H}}^2 = |\langle vf, f\rangle|$. It follows that there exists $\tau_f \in \mathbb{T}$ such that $v(f) = \tau_f f$. By the same argument, $v(g) = \tau_g g$ for some $\tau_g \in \mathbb{T}$. Then (4.17) yields that $\tau_f = \tau_g$. Therefore these phase factors depend only on Δ and not on the functions.

Conversely, assume that (4.16) holds. Let $\tilde{O} \subset O'$ be a symmetric double cone with base $\tilde{B}$ at time zero, and let Δ be the connected component of $\mathbb{R}^{2n-1} \setminus B$ which contains $\tilde{B}$. Then ϱ_V acts on the local field algebra belonging to $\tilde{O}$ like the gauge transformation induced by $\tau_\Delta \in \mathbb{T} \subset U(N)$, and it consequently acts like the identity on $\mathfrak{A}(\tilde{O})$. Since the algebra $\mathfrak{A}(O')$ is generated by such local algebras $\mathfrak{A}(\tilde{O})$, it follows that ϱ_V is localized in O. □

Of course, $\mathbb{R}^{2n-1} \setminus B$ is connected if $n > 1$, but has two connected components if $n = 1$. Recall that this is the basic reason for the generic violation of Haag duality and for the possible occurrence of braid group statistics and soliton sectors in two–dimensional Minkowski space.

We would like to close this section with a demonstration that at least the free massless Dirac field in two spacetime dimensions possesses non–surjective implementable localized quasi–free endomorphisms. It suffices to consider one chiral component of the field. Thus consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ with Dirac Hamiltonian $-i\frac{d}{dx}$. It is convenient to transform to the Hilbert space $L^2(\mathbb{T})$ via the Cayley transform $\varkappa$

$$\varkappa : \mathbb{R} \cup \{\infty\} \to \mathbb{T}, \qquad x \mapsto -e^{2i \arctan x} = \frac{x-i}{x+i}$$

(cf. [CR87, Rui89b]). $\varkappa$ induces a unitary transformation $\tilde{\varkappa}$

$$\tilde{\varkappa} : L^2(\mathbb{T}) \to L^2(\mathbb{R}), \qquad (\tilde{\varkappa} f)(x) = \pi^{-\frac{1}{2}} \frac{f(\varkappa(x))}{x+i}.$$

The important point is that the spectral projections $p_\pm$ of $-i\frac{d}{dx}$ are transformed into the Hardy space projections (cf. [Dou72]): Set $q_\pm \equiv \tilde{\varkappa}^{-1} p_\pm \tilde{\varkappa}$, then

$$q_+ = \sum_{n \geq 0} e_n \langle e_n, .\rangle, \quad q_- = \sum_{n<0} e_n \langle e_n, .\rangle, \quad e_n(z) \equiv z^n \ (z \in \mathbb{T}, n \in \mathbb{Z}). \tag{4.18}$$

Our task is to construct an isometry v of $L^2(\mathbb{T})$ with $q_+ v q_-$, $q_- v q_+$ Hilbert–Schmidt (implementability), with $\operatorname{ind} v = -1$ (close to irreducibility, cf. p. 82), and such that $vf = f$ for all $f \in L^2(\mathbb{T})$ with $\operatorname{supp} f \subset \mathbb{T} \setminus I$ where $I \subset \mathbb{T}$ is a fixed interval (localization). As localization region we shall choose the interval

$$I \equiv \{e^{i\lambda} \mid \tfrac{\pi}{2} \leq \lambda \leq \tfrac{3\pi}{2}\}$$

which corresponds, by the inverse Cayley transform, to the interval $\varkappa^{-1}(I) = [-1,1]$ in $\mathbb{R}$. We need the following orthonormal basis $(f_m)_{m\in\mathbb{Z}}$ in $L^2(I) \subset L^2(\mathbb{T})$

$$f_m(z) \equiv \sqrt{2}(-1)^m z^{2m}\chi_I(z), \quad z \in \mathbb{T}$$

where χ_I is the characteristic function of I. The isometry v is now defined by

$$v \equiv \mathbf{1} + \sum_{m\geq 0}(f_{m+1} - f_m)\langle f_m, .\rangle. \tag{4.19}$$

Note that v acts like the identity on functions with support in $\mathbb{T}\setminus I$, that $vf_m = f_m$ if $m < 0$, and that v acts like the unilateral shift on the remaining f_m: $vf_m = f_{m+1}$ if $m \geq 0$.

The author gratefully acknowledges some private lessons in estimating infinite series given to him by P. Grinevich which were indispensable for the proof of the following lemma.

Lemma 4.9.
q_+vq_- and q_-vq_+ are Hilbert–Schmidt.

Proof. The following inner products are easily computed

$$\langle e_l, f_m\rangle = \frac{(-1)^m}{\pi\sqrt{2}}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{i(2m-l)\lambda}d\lambda = \begin{cases} \frac{(-1)^m}{\sqrt{2}}\delta_{2m,l}, & l \text{ even} \\ \frac{\sqrt{2}}{\pi}\frac{(-1)^{\frac{l-1}{2}}}{2m-l}, & l \text{ odd.}\end{cases}$$

1. This yields for the Hilbert–Schmidt norm of q_+vq_-, using (4.18) and (4.19)

$$\begin{aligned}
\|q_+vq_-\|^2_{\mathrm{HS}} &= \sum_{n<0}\left\|q_+ve_n\right\|^2 \\
&= \sum_{n<0}\Big\|\sum_{l,m\geq 0} e_l\langle e_l, f_{m+1} - f_m\rangle\langle f_m, e_n\rangle\Big\|^2 \\
&= \sum_{n<0}\sum_{l\geq 0}\Big|\sum_{m\geq 0}\langle e_l, f_{m+1} - f_m\rangle\langle f_m, e_n\rangle\Big|^2 \\
&= \frac{4}{\pi^4}\sum_{\substack{n<0,\\ n\,\mathrm{odd}}}\sum_{\substack{l>0,\\ l\,\mathrm{odd}}}\left(\sum_{m\geq 0}\left(\frac{1}{2m+2-l} - \frac{1}{2m-l}\right)\frac{1}{2m-n}\right)^2 \\
&\quad + \frac{1}{\pi^2}\sum_{\substack{n<0,\\ n\,\mathrm{odd}}}\sum_{\substack{l\geq 0,\\ l\,\mathrm{even}}}\left(\sum_{m\geq 0}\left((-1)^{m+1}\delta_{2m+2,l} - (-1)^m\delta_{2m,l}\right)\frac{1}{2m-n}\right)^2 \\
&= \frac{16}{\pi^4}\sum_{\substack{n>0,\\ n\,\mathrm{odd}}}\sum_{\substack{l>0,\\ l\,\mathrm{odd}}}\left(\sum_{m\geq 0}\frac{1}{(2m+2-l)(2m-l)(2m+n)}\right)^2 \\
&\quad + \frac{1}{\pi^2}\sum_{\substack{n>0,\\ n\,\mathrm{odd}}}\sum_{\substack{l\geq 0,\\ l\,\mathrm{even}}}\left(\frac{1}{l-2+n} - \frac{1}{l+n}\right)^2 \\
&= \frac{16}{\pi^4}\sum_{n,l\geq 0}\left(\sum_{m\geq 0}\frac{1}{(2m-2l+1)(2m-2l-1)(2m+2n+1)}\right)^2 \\
&\quad + \frac{4}{\pi^2}\sum_{n,l\geq 0}\left(\frac{1}{(2l+2n-1)(2l+2n+1)}\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{16}{\pi^4} \sum_{n,l\geq 0} \left(\sum_{m\geq 0} \frac{1}{\big(4(m-l)^2-1\big)(2m+2n+1)} \right)^2 \\
&\quad + \frac{4}{\pi^2} \sum_{n,l\geq 0} \frac{1}{\big(4(l+n)^2-1\big)^2} \\
&= \frac{16}{\pi^4} \sum_{n,l\geq 0} \left(\sum_{m\geq -l} \frac{1}{(4m^2-1)(2m+a_{ln})} \right)^2 + \frac{4}{\pi^2} \sum_{n\geq 0} \frac{1}{(4n^2-1)^2} \\
&\quad + \frac{4}{\pi^2} \sum_{n\geq 0} \sum_{l\geq 1} \frac{1}{\big(4(l+n)^2-1\big)^2}, \qquad \text{with } a_{ln} \equiv 2l+2n+1 \\
&< \frac{16}{\pi^4} \sum_{n,l\geq 0} \left(\sum_{m\geq -l} \frac{1}{4m^2-1} \left(\frac{1}{a_{ln}} - \left(\frac{1}{a_{ln}} - \frac{1}{2m+a_{ln}} \right) \right) \right)^2 \\
&\quad + \frac{4}{\pi^2} \Bigg(1 + \underbrace{\sum_{n\geq 1} \frac{1}{4n^2-1}}_{=\frac{1}{2}} + \underbrace{\sum_{n\geq 0} \frac{1}{4(n+1)^2-1}}_{=\frac{1}{2}} \underbrace{\sum_{l\geq 1} \frac{1}{4l^2-1}}_{=\frac{1}{2}} \Bigg) \\
&= \frac{7}{\pi^2} + \frac{16}{\pi^4} \sum_{n,l\geq 0} \Bigg(\frac{1}{a_{ln}} \Bigg(\underbrace{\sum_{m\in\mathbb{Z}} \frac{1}{4m^2-1}}_{=0} - \sum_{m<-l} \frac{1}{4m^2-1} \Bigg) \\
&\qquad\qquad - \frac{1}{a_{ln}} \sum_{m\geq -l} \frac{2m}{(4m^2-1)(2m+a_{ln})} \Bigg)^2 \\
&= \frac{7}{\pi^2} + \frac{16}{\pi^4} \sum_{n,l\geq 0} \frac{1}{a_{ln}{}^2} \Bigg(\sum_{m>l} \frac{1}{4m^2-1} + \sum_{m>l} \frac{2m}{(4m^2-1)(2m+a_{ln})} \\
&\qquad\qquad + \sum_{m=-l}^{l} \frac{2m}{(4m^2-1)(2m+a_{ln})} \Bigg)^2 \\
&= \frac{7}{\pi^2} + \frac{16}{\pi^4} \sum_{n,l\geq 0} \frac{1}{a_{ln}{}^2} \Bigg(\sum_{m>l} \frac{1}{4m^2-1} \underbrace{\left(1 + \frac{2m}{2m+a_{ln}} \right)}_{<2} \\
&\qquad\qquad + \sum_{m=1}^{l} \underbrace{\frac{2m}{4m^2-1} \left(\frac{1}{2m+a_{ln}} + \frac{1}{2m-a_{ln}} \right)}_{= \underbrace{\frac{8m^2}{4m^2-1}}_{\leq \frac{8}{3}} \underbrace{\frac{1}{4m^2-a_{ln}{}^2}}_{<0}} \Bigg)^2 \\
&< \frac{7}{\pi^2} + \frac{64}{\pi^4} \sum_{n,l\geq 0} \frac{1}{a_{ln}{}^2} \left(\sum_{m>l} \frac{1}{4m^2-1} + \frac{4}{3} \sum_{m=1}^{l} \frac{1}{a_{ln}{}^2-4m^2} \right)^2 \\
&< \frac{7}{\pi^2} + \frac{64}{\pi^4} \underbrace{\sum_{n\geq 0} \frac{1}{(2n+1)^2}}_{=\frac{\pi^2}{8}} \underbrace{\left(\sum_{m>0} \frac{1}{4m^2-1} \right)^2}_{=\frac{1}{2}}
\end{aligned}
$$

$$+\frac{64}{\pi^4}\sum_{n\geq 0}\sum_{l\geq 1}\frac{1}{{a_{ln}}^2}\left(\int_l^\infty \frac{dm}{(2m-1)^2}+\frac{4}{3}\int_1^l \frac{dm}{(2m-a_{ln})^2}\right)^2$$

$$=\frac{9}{\pi^2}+\frac{64}{\pi^4}\sum_{n\geq 0}\sum_{l\geq 1}\frac{1}{{a_{ln}}^2}\left(\frac{1}{2(2l-1)}+\frac{4}{3}\frac{l-1}{(a_{ln}-2l)(a_{ln}-2)}\right)^2$$

$$=\frac{9}{\pi^2}+\frac{16}{\pi^4}\sum_{n,l\geq 0}\Bigg(\underbrace{\frac{1}{(a_{ln}+2)(2l+1)}}_{<\frac{1}{(2n+1)(2l+1)}}+\frac{8}{3}\underbrace{\frac{l}{(a_{ln}+2)(2n+1)a_{ln}}}_{<\frac{1}{(2n+1)(2l+1)}}\Bigg)^2$$

$$<\frac{9}{\pi^2}+\frac{16}{\pi^4}\left(\frac{11}{3}\right)^2\underbrace{\left(\sum_{n\geq 0}\frac{1}{(2n+1)^2}\right)^2}_{=\frac{\pi^2}{8}}$$

$$=\frac{9}{\pi^2}+\left(\frac{11}{6}\right)^2<\infty.$$

This proves that q_+vq_- is Hilbert–Schmidt. The Hilbert–Schmidt norm of q_-vq_+ can be estimated as follows:

$$\|q_-vq_+\|_{\mathrm{HS}}^2=\sum_{n\geq 0}\left\|q_-ve_n\right\|^2$$

$$=\sum_{n\geq 0}\sum_{l<0}\left|\sum_{m\geq 0}\langle e_l, f_{m+1}-f_m\rangle\langle f_m, e_n\rangle\right|^2$$

$$=\frac{1}{\pi^2}\sum_{\substack{n\geq 0,\\ n\text{ even}}}\sum_{\substack{l<0,\\ l\text{ odd}}}\left(\sum_{m\geq 0}\left(\frac{1}{2m+2-l}-\frac{1}{2m-l}\right)(-1)^m\delta_{2m,n}\right)^2$$

$$+\frac{4}{\pi^4}\sum_{\substack{n>0,\\ n\text{ odd}}}\sum_{\substack{l<0,\\ l\text{ odd}}}\left(\sum_{m\geq 0}\left(\frac{1}{2m+2-l}-\frac{1}{2m-l}\right)\frac{1}{2m-n}\right)^2$$

$$=\frac{1}{\pi^2}\sum_{n,l\geq 0}\left(\frac{1}{a_{ln}+2}-\frac{1}{a_{ln}}\right)^2\qquad\text{(with } a_{ln}\text{ as above)}$$

$$+\frac{4}{\pi^4}\sum_{n,l\geq 0}\left(\sum_{m\geq 0}\left(\frac{1}{2m+2l+3}-\frac{1}{2m+2l+1}\right)\frac{1}{2m-2n-1}\right)^2$$

$$=\frac{4}{\pi^2}\sum_{n,l\geq 0}\left(\frac{1}{a_{ln}(a_{ln}+2)}\right)^2$$

$$+\frac{16}{\pi^4}\sum_{n,l\geq 0}\left(\sum_{m\geq 0}\frac{1}{(2m+2l+1)(2m+2l+3)(2m-2n-1)}\right)^2$$

$$<\frac{4}{\pi^2}\sum_{n,l\geq 0}\frac{1}{(2n+1)^2}\frac{1}{(2l+1)^2}$$

$$+\frac{16}{\pi^4}\sum_{n\geq 0}\sum_{l\geq 1}\left(\sum_{m\geq 0}\frac{1}{\big(4(l+m)^2-1\big)(2m-2n-1)}\right)^2$$

$$<\frac{\pi^2}{16}+\frac{16}{\pi^4}\sum_{n\geq 0}\sum_{l\geq 1}\left(\left|\sum_{m=-n}^{n+1}\frac{1}{\big(4(l+m+n)^2-1\big)(2m-1)}\right|\right.$$

$$
\begin{aligned}
& \qquad + \sum_{m\geq 2n+2} \frac{1}{\bigl(4(l+m)^2-1\bigr)(2n+3)}\Biggr)^2 \\
&= \frac{\pi^2}{16} + \frac{16}{\pi^4}\sum_{n\geq 0}\sum_{l\geq 1}\Biggl(\Biggl|\sum_{m=1}^{n+1}\frac{1}{\bigl(4(l+m+n)^2-1\bigr)(2m-1)} \\
& \qquad + \sum_{m=1}^{n+1}\frac{1}{\bigl(4(l+1-m+n)^2-1\bigr)(1-2m)}\Biggr| \\
& \qquad + \frac{1}{2n+3}\sum_{m\geq 2n+l+2}\frac{1}{4m^2-1}\Biggr)^2 \\
&< \frac{\pi^2}{16} + \frac{16}{\pi^4}\sum_{n\geq 0}\sum_{l\geq 1}\Biggl(\sum_{m=1}^{n+1}\Biggl|\frac{1}{4(l+m+n)^2-1} \\
& \qquad\qquad - \frac{1}{4(l+1-m+n)^2-1}\Biggr|\cdot\frac{1}{2m-1} \\
& \qquad + \frac{1}{2n+3}\int_{2n+l+1}^{\infty}\frac{dm}{(2m-1)^2}\Biggr)^2 \\
&= \frac{16}{\pi^4}\sum_{n\geq 0}\sum_{l\geq 1}\Biggl(\sum_{m=1}^{n+1}\frac{4a_{ln}}{\underbrace{\bigl(4(l+m+n)^2-1\bigr)}_{=\underbrace{(2(l+m+n)-1)}_{>a_{ln}}(2(l+m+n)+1)}\bigl(4(l+1-m+n)^2-1\bigr)} \\
& \qquad + \frac{1}{2(2n+3)(4n+2l+1)}\Biggr)^2 + \frac{\pi^2}{16} \\
&< \frac{\pi^2}{16} + \frac{16}{\pi^4}\sum_{n\geq 0}\sum_{l\geq 1}\Biggl(\sum_{m=0}^{n}\frac{4}{\bigl(2(l+m+n)+3\bigr)\bigl(4(l-m+n)^2-1\bigr)} \\
& \qquad + \frac{1}{2(2n+3)(4n+2l+1)}\Biggr)^2 \\
&< \frac{\pi^2}{16} + \frac{16}{\pi^4}\sum_{n\geq 0}\sum_{l\geq 1}\Biggl(\frac{4}{2l+2n+3}\sum_{m=l}^{l+n}\frac{1}{4m^2-1} \\
& \qquad + \frac{1}{2(2n+3)(4n+2l+1)}\Biggr)^2 \\
&< \frac{\pi^2}{16} + \frac{16}{\pi^4}\sum_{n\geq 0}\sum_{l\geq 1}\Biggl(\frac{4}{2l+2n+3}\Biggl(\frac{1}{4l^2-1} + \underbrace{\int_l^{l+n}\frac{dm}{4m^2-1}}_{\substack{<\int_l^{l+n}\frac{dm}{(2m-1)^2} \\ =\frac{n}{(2l+2n-1)(2l-1)} \\ <\frac{1}{2(2l-1)}}}\Biggr) \\
& \qquad + \frac{1}{2(2n+3)(4n+2l+1)}\Biggr)^2
\end{aligned}
$$

$$
\begin{aligned}
&< \frac{\pi^2}{16} + \frac{16}{\pi^4} \sum_{n \geq 0} \sum_{l \geq 1} \Bigg(\frac{4}{2l+2n+3} \cdot \underbrace{\frac{2l+3}{2(2l+1)}}_{<1} \cdot \frac{1}{2l-1} \\
&\qquad\qquad\qquad\qquad + \frac{1}{2(2n+3)(4n+2l+1)} \Bigg)^2 \\
&< \frac{\pi^2}{16} + \frac{16}{\pi^4} \sum_{n,l \geq 1} \left(\frac{4}{(2l+2n+1)(2l-1)} + \frac{1}{2(2n+1)(4n+2l-3)} \right)^2 \\
&< \frac{\pi^2}{16} + \frac{16}{\pi^4} \sum_{n,l \geq 1} \left(\frac{2}{(n+l)l} + \frac{2}{n(n+l)} \right)^2 \\
&= \frac{\pi^2}{16} + \frac{64}{\pi^4} \sum_{n,l \geq 1} \left(\frac{n+l}{(n+l)nl} \right)^2 \\
&= \frac{\pi^2}{16} + \frac{64}{\pi^4} \left(\sum_{n \geq 1} \frac{1}{n^2} \right)^2 \\
&= \frac{\pi^2}{16} + \frac{16}{9}.
\end{aligned}
$$

□

Thus the Bogoliubov operator V induced by v by (4.15) yields a $U(N)$–gauge invariant implementable localized endomorphism ϱ_V of the chiral Dirac field. Since $M_V = \dim \mathfrak{k}_V = N$ by construction, it is clear that $\mathfrak{k}_V$ carries either the defining representation of $U(N)$ or its complex conjugate. By the discussion on p. 82, the irreducible constituents of ϱ_V correspond to the irreducible, mutually inequivalent representations $\mathcal{U}_V^{(n)}$, $n = 0, \ldots, N$. It would be interesting to find a manageable description of the Cayley–transformed operator $\tilde{\varkappa} v \tilde{\varkappa}^{-1}$ on $L^2(\mathbb{R})$, which would perhaps give an idea how to obtain implementable localized endomorphisms in the two–dimensional massive case.

4.2. The charge of gauge invariant endomorphisms of the CCR algebra. Our discussion of the charge structure of gauge invariant endomorphisms of the CCR algebra will be short compared to the CAR case. Recall from the beginning of Section 4 that we consider the following situation: We have an infinite dimensional vector space $\mathcal{K}$ together with a nondegenerate hermitian sesquilinear form κ and a fixed basis projection P_1. $\mathcal{K}$ is assumed to be complete with respect to the inner product induced by P_1. The gauge group G acts by diagonal Bogoliubov operators on $\mathcal{K}$ and can be identified with a subgroup of the unitary group of $\mathcal{K}_1 \equiv P_1(\mathcal{K})$. Our interest is in the representations of G on the Hilbert spaces $H(\varrho_V)$ which implement gauge invariant quasi–free endomorphisms ϱ_V, $V \in \mathcal{S}_{P_1}(\mathcal{K}, \kappa)^G$.

By Lemma 4.2, it suffices to look at the values of the implementers on the Fock vacuum. We have by (3.30), (3.32)

$$
\Psi_\alpha(V)\Omega_{P_1} = D_V \psi_{\alpha_1} \cdots \psi_{\alpha_l} \exp\left(\tfrac{1}{2} Z_V^\dagger a^* a^*\right) \Omega_{P_1} \tag{4.20}
$$

where D_V is a numerical constant, Z_V is the symmetric Hilbert–Schmidt operator defined in (3.22), $g_1, g_2, \ldots$ is a κ–orthonormal basis in $\mathfrak{k}_V \equiv P_V(\ker V^\dagger)$, and ψ_j is the isometry obtained by polar decomposition of the closure of $\pi_{P_1}(g_j)$. In contrast to the Fermionic case (cf. Lemma 4.3), there is no simple transformation law for the ψ_j under G. This difficulty can however be circumvented by rewriting $\Psi_\alpha(V)\Omega_{P_1}$

with the help of (3.37)–(3.40) and (3.5) as follows

$$\begin{aligned}\Psi_\alpha(V)\Omega_{P_1} &= \Psi(U_V)\Psi_\alpha(W_V)\Omega_{P_1} \\ &= c_\alpha \Psi(U_V)\pi_{P_1}\big(U_V^\dagger(g_{\alpha_1})\cdots U_V^\dagger(g_{\alpha_l})\big)\Omega_{P_1} \\ &= c_\alpha \overline{\pi_{P_1}(g_{\alpha_1})}\cdots\overline{\pi_{P_1}(g_{\alpha_l})}\Psi_0(V)\Omega_{P_1} \\ &= c_\alpha D_V \overline{\pi_{P_1}(g_{\alpha_1})}\cdots\overline{\pi_{P_1}(g_{\alpha_l})}\exp\Big(\tfrac{1}{2}Z_V^\dagger a^* a^*\Big)\Omega_{P_1}\end{aligned} \tag{4.21}$$

(the bar denotes closure). The behaviour of the $\overline{\pi_{P_1}(g_j)}$ under gauge transformations is obvious. It remains to consider the exponential term. It is a salient feature of Definition 3.6 that this term is gauge invariant:

LEMMA 4.10.
Let $V \in \mathcal{S}_{P_1}(\mathcal{K},\kappa)^G$ be given, and let $Z_V \in \mathfrak{C}_{P_1}$ be defined by (3.22). Then $\exp(\frac{1}{2}Z_V^\dagger a^* a^*)\Omega_{P_1}$ is invariant under all gauge transformations $\Gamma(U)$, $U \in G$.

Proof. Let E be the orthogonal projection onto $\mathcal{H} \equiv \ker V^\dagger = C\ker V^*$, $C = P_1 - P_2$, as in (3.20). Then E commutes with G since V and P_1 do so. Let $A \equiv ECE$ be the self–adjoint operator introduced in Lemma 3.5. Then A and all its spectral projections commute with G. It follows that the positive part A_+ of A and the operator ${A_+}^{-1}$ defined in Lemma 3.5 also commute with G. Therefore $P_V \equiv VP_1V^\dagger + {A_+}^{-1}C$ and $Z_V \equiv (P_V)_{21}{(P_V)_{11}}^{-1}$ commute with G as well.

Arguing as in the proof of Lemma 4.4, one finds for $U \in G$

$$\Gamma(U)\Big(\tfrac{1}{2}Z_V^\dagger a^* a^*\Big)^n\Omega_{P_1} = \Big(\tfrac{1}{2}(UZ_V^\dagger U^\dagger)a^* a^*\Big)^n\Omega_{P_1}$$

and finally

$$\Gamma(U)\exp\Big(\tfrac{1}{2}Z_V^\dagger a^* a^*\Big)\Omega_{P_1} = \exp\Big(\tfrac{1}{2}(UZ_V^\dagger U^\dagger)a^* a^*\Big)\Omega_{P_1} = \exp\Big(\tfrac{1}{2}Z_V^\dagger a^* a^*\Big)\Omega_{P_1}.$$

□

Invoking the isomorphism $H(\varrho_V) \cong \mathcal{F}_s(\mathfrak{k}_V)$ from Corollary 3.13, we thus deduce

THEOREM 4.11.
Let P_1 be a basis projection of $(\mathcal{K},\kappa)$, let G be a group consisting of diagonal Bogoliubov operators, and let $V \in \mathcal{S}_{P_1}(\mathcal{K},\kappa)^G$. Let $\mathfrak{k}_V = P_V(\ker V^\dagger)$ be the subspace of $\mathcal{K}$ defined in Definition 3.11, with $\dim\mathfrak{k}_V = -\frac{1}{2}\operatorname{ind}V$. Then $\mathfrak{k}_V$ is invariant under G, and the unitary representation $\mathcal{U}_V$ of G on the Hilbert space of isometries $H(\varrho_V)$ which implements ϱ_V in the Fock representation π_{P_1} is unitarily equivalent to the representation on $\mathcal{F}_s(\mathfrak{k}_V)$ that is obtained by taking symmetric tensor powers of the representation on $\mathfrak{k}_V$.

Proof. $\mathfrak{k}_V$ is invariant under G because $\ker V^\dagger$ is invariant and because G commutes with P_V (see the proof of Lemma 4.10). The assertion hence follows from (4.21), Lemma 4.10 and Lemma 4.2. □

Theorem 4.11 shows that genuine quasi–free endomorphisms of the CCR algebra are even "more reducible" than endomorphisms of the CAR algebra in that they are always *infinite* direct sums of irreducibles, a fact which explains the generic occurrence of infinite statistics in the CCR case (cf. Theorem 3.3). Again, each closed subspace of $H(\varrho_V)$ spanned by all $\Psi_\alpha(V)$ with length of α fixed is invariant under G. The special case of quasi–free automorphisms is of little interest in the CCR case because they are all neutral ($\mathcal{U}_V$ is the trivial representation of G if $\operatorname{ind}V = 0$).

There is also less freedom in the choice of the representation of G on $\mathfrak{k}_V$ (cf. Proposition 4.6):

Proposition 4.12.

a) If $\mathfrak{k} \subset \mathcal{K}_1$ is a closed G–invariant subspace carrying a representation of class ξ and if ξ is contained in $\mathcal{K}_1$ with infinite multiplicity, then there exists a diagonal Bogoliubov operator $V \in \mathcal{S}_{P_1}(\mathcal{K},\kappa)^G$ with $\mathfrak{k}_V = \mathfrak{k}$.

b) A class ξ of irreducible representations of G is realized on some $\mathfrak{k}_V$, $V \in \mathcal{S}_{P_1}(\mathcal{K},\kappa)^G$, if and only if ξ is contained in $\mathcal{K}_1$ with infinite multiplicity.

Proof. The proof of a) is the same as the proof of Proposition 4.6 b) 1. because $\mathcal{S}_{\rm diag}(\mathcal{K},\kappa) = \mathcal{I}_{\rm diag}(\mathcal{K})$.

To prove b), assume that ξ is contained in $\mathcal{K}_1$ with finite (or zero) multiplicity. (Recall that G commutes with P_1 so that any irreducible class ξ contained in $\mathcal{K}$ is contained in $\mathcal{K}_1$ or $\mathcal{K}_2$.) Assume further that $V \in \mathcal{S}_{P_1}(\mathcal{K},\kappa)^G$ exists such that $\mathfrak{k}_V$ belongs to the class ξ. Then let q_n be the κ–orthogonal projection onto $V^n(\mathfrak{k}_V)$

$$q_n \equiv V^n P_V(\mathbf{1} - VV^\dagger)V^{\dagger n}.$$

Since $q_m q_n = 0$ $(m \neq n)$, the $V^n(\mathfrak{k}_V)$ are mutually κ–orthogonal, and their direct sum $\mathcal{K}_V \equiv \oplus_{n=0}^{\infty} V^n(\mathfrak{k}_V) \subset \mathcal{K}$ carries the representation ξ with infinite multiplicity. Therefore ξ must be contained in $\mathcal{K}_2$ with multiplicity ∞. However, the restriction of κ to $\mathcal{K}_V$ is positive definite, whereas the restriction to $\mathcal{K}_2$ is negative definite. This is a contradiction and shows that no V with $\mathfrak{k}_V$ of class ξ can exist. On the other hand, if ξ is contained in $\mathcal{K}_1$ with infinite multiplicity, then we are back in the situation of a). □

The remarks made in Section 4.1 after Proposition 4.6 on the relation to the generic superselection structure apply here as well. In particular, one needs additional assumptions to ensure the existence of conjugates (cf. Proposition 4.7):

Proposition 4.13.

Assume that there exists an orthogonal projection[w] *P of $\mathcal{K}$ with $P + \overline{P} = \mathbf{1}$ which commutes with P_1 and G. Let $p_\pm$ be defined by*

$$p_+ \equiv PP_1, \qquad p_- \equiv PP_2,$$

and assume that there exists a unitary operator $\mathcal{C}_{+-}$ from $\mathcal{K}_- \equiv p_-(\mathcal{K})$ onto $\mathcal{K}_+ \equiv p_+(\mathcal{K})$ which commutes with G. Let $\mathcal{C}$ be the unique operator on $\mathcal{K}$ which commutes with P, which fulfills $\overline{\mathcal{C}} = \mathcal{C}$, and which is given on $P(\mathcal{K}) = \mathcal{K}_+ \oplus \mathcal{K}_-$ by the matrix

$$P\mathcal{C}P \equiv \begin{pmatrix} 0 & \mathcal{C}_{+-} \\ \mathcal{C}_{+-}{}^* & 0 \end{pmatrix}.$$

Then $\mathcal{C}$ is gauge invariant, unitary and self–adjoint, it fulfills $\{\mathcal{C}, C\} = 0$ and $\mathcal{C}^\dagger = -\mathcal{C}$, and the map

$$V \mapsto V^c \equiv \mathcal{C}V\mathcal{C} \tag{4.22}$$

is an involutive automorphism of $\mathcal{S}_{P_1}(\mathcal{K},\kappa)^G$. One has

$$\mathfrak{k}_{V^c} = \mathcal{C}(\mathfrak{k}_V{}^*) \tag{4.23}$$

so that the representation $\mathcal{U}_{V^c}$ of G on $H(\varrho_{V^c})$ is unitarily equivalent to the complex conjugate of $\mathcal{U}_V$.

Proof. The asserted properties of $\mathcal{C}$ are readily verified. It only remains to prove (4.23). Let us first show that

$$P_{V^c} = \mathcal{C}\overline{P_V}\mathcal{C} \tag{4.24}$$

[w] Such P is a CAR basis projection of $\mathcal{K}$, but not a basis projection of $(\mathcal{K},\kappa)$ since κ is not positive definite on $\operatorname{ran} P$.

where P_V, P_{V^c} are the basis projections associated with V, V^c according to Definition 3.6:

$$P_V = VP_1V^\dagger + p_V, \qquad P_{V^c} = V^cP_1V^{c\dagger} + p_{V^c}. \tag{4.25}$$

Let E resp. E^c be the orthogonal projections onto $\ker V^\dagger$ resp. $\ker V^{c\dagger}$, and let $A \equiv ECE$, $A^c \equiv E^cCE^c$ be the corresponding operators as in Lemma 3.5, with positive/negative parts $A_\pm$, ${A^c}_\pm$. Since $\ker V^{c\dagger} = \mathfrak{C}(\ker V^\dagger)$, one has $E^c = \mathfrak{C}E\mathfrak{C}$ and

$$A^c = \mathfrak{C}E(\mathfrak{C}C\mathfrak{C})E\mathfrak{C} = -\mathfrak{C}ECE\mathfrak{C} = -\mathfrak{C}A\mathfrak{C}$$

so that ${A^c}_+ = \mathfrak{C}A_-\mathfrak{C}$. Since $A_- = \overline{A_+}$ (see the proof of Lemma 3.5), since $\{\mathfrak{C}, C\} = 0$ and $\overline{C} = -C$, this implies

$$p_{V^c} \equiv {A^c}_+{}^{-1}C = \mathfrak{C}\overline{A_+}^{-1}\mathfrak{C}C = \mathfrak{C}\overline{{A_+}^{-1}C}\mathfrak{C} = \mathfrak{C}\overline{p_V}\mathfrak{C}.$$

Thus we get from (4.25), using $\mathfrak{C}P_1\mathfrak{C} = P_2$ and $\mathfrak{C}^\dagger = -\mathfrak{C}$

$$P_{V^c} = V^cP_1V^{c\dagger} + p_{V^c} = \mathfrak{C}VP_2V^\dagger\mathfrak{C} + \mathfrak{C}\overline{p_V}\mathfrak{C} = \mathfrak{C}\overline{P_V}\mathfrak{C}$$

which proves (4.24). It follows that (4.23) holds:

$$\mathfrak{k}_{V^c} = P_{V^c}(\ker V^{c\dagger}) = \mathfrak{C}\overline{P_V}\mathfrak{C}(\mathfrak{C}\ker V^\dagger) = \mathfrak{C}({\mathfrak{k}_V}^*).$$

Therefore the representation of G on $\mathfrak{k}_{V^c}$ is unitarily equivalent to the complex conjugate of the representation on $\mathfrak{k}_V$, so that $\mathcal{U}_{V^c} \simeq {\mathcal{U}_V}^*$ by Theorem 4.11. $\square$