Introduction

With the help of the theory of differential forms this dissertation presents some well-known and also new results about quasiregular mappings on Riemannian manifolds. We consider some recently introduced classes of differential forms, the $\mathcal{WT}$-classes, and show that elements of these classes are solutions of a quasilinear elliptic equation. Some differential forms composed of the components of a quasiregular mapping can be used as an example for these classes. In the second part we give a new definition for the Hölder continuity of differential forms on Riemannian manifolds and show that one of our classes fulfills this requirement.

Quasiregular mappings in n dimensions, $n \geq 3$, were first introduced and studied by Yu.G. Reshetnyak in a series of articles since 1966, they are defined in the same way as quasiconformal mappings, absent the requirement of the global homeomorphy. Quasiconformal or quasiregular mappings in higher dimensions were analytically first defined as the solutions of special partial differential equations. Later this definition was extended to mappings with less regularity. The investigation of quasiregular mappings has been continued and deepened especially by O. Martio, S. Rickman and J. Väisälä . For a broad survey and also for details, see the monographs [Re], [GR], [Vu], and [Ri]. In this connection we remark that the theory of quasiregular mappings gives a natural generalization of the geometric aspects of the theory of analytic functions in the plane to Riemannian n-manifolds. Quasiregular mappings occur in many different contexts, but their distinctive property is always conformality with respect to certain metrics. The theory of quasiregular mappings in higher dimensions is essentially nonlinear, it is connected with some nonlinear partial differential equations and with nonlinear potential theory [HKM], [AH].

In the first chapters we introduce the basis and methods of the theory of Riemannian manifolds and the theory of differential forms on $I\!R^n$ and on Riemannian manifolds, which lead to our results. Differential forms have often also applications in physics and engineering. For even more details concerning especially Riemannian manifolds see [Jo].

We define four classes of differential forms, named $\mathcal{WT}$-classes. Investigations which enabled to develop these classes are done in [Mi]. The $\mathcal{WT}$-classes were first presented in 1995 at the 16th Rolf Nevanlinna Colloquium in Joensuu, Finland, by O. Martio, V.M. Miklyukov and M. Vuorinen [MMV2].

Let $\mathcal{M}$ be an n-dimensional Riemannian manifold and let

$$\omega \in L^p_{\mathrm{loc}}(\mathcal{M}),\ \deg \omega = k,\ 0 \leq k \leq n,\ p > 1,$$

be a weakly closed differential form on $\mathcal{M}$. For the concept of a weakly closed differential form we refer to Definition 3.13. The differential form ω is said to be of the class $\mathcal{WT}_2$ on $\mathcal{M}$, if there exists a weakly closed differential form

$$\theta \in L^q_{\mathrm{loc}}(\mathcal{M}),\ \deg \theta = n - k,\ \frac{1}{p} + \frac{1}{q} = 1,$$

such that almost everywhere on $\mathcal{M}$ the conditions

$$\nu_1 \, |\omega|^p \leq \ \langle \omega, \star \theta \rangle$$

and

$$|\theta| \leq \nu_2 \, |\omega|^{p-1}$$

are satisfied, with two positive constants ν_1 and ν_2. Here $\star\theta$ denotes the orthogonal complement of the differential form θ. We further introduce the quasilinear elliptic equation

$$d^* A(m, d\omega) = 0 \ . \tag{0.1}$$

For almost every $m \in \mathcal{M}$ the mapping A is defined on the k-vector tangent space $\Lambda^k(T_m(\mathcal{M}))$. For details see Chapter 5, an example for A can be found in (7.19). The exterior derivative operator is denoted by d, the formal adjoint operator to d, the so called Hodge codifferential, by d^*. We show that if the differential form $d\omega$ is an element of the class $\mathcal{WT}_2$, then ω is a generalized solution of (0.1).

As an important example for an element of the class $\mathcal{WT}_2$ and as well for a solution of this quasilinear elliptic equation we use the local components of quasiregular mappings and compose differential forms. For example if the mapping $f \in W^{1,n}_{\mathrm{loc}}(\mathcal{M})$ with the local components $f^1, \ldots, f^n$ is a quasiregular mapping then we compose the differential form

$$\omega = f^k df^1 \wedge \ldots \wedge df^{k-1}$$

of degree $k-1$, $1 \leq k < n$. We show that the differential form $d\omega$ of degree k is an element of the class $\mathcal{WT}_2$. This result is presented in two different versions (Theorem 8.1 and 8.2), also the proofs base on two different ideas. These and similar differential forms were previously investigated by Yu.A. Zhuravleva [Zh] (1992), T. Iwaniec and G. Martin [Iw1] (1992) and [IM] (1993), O. Martio, V.M. Miklyukov and M. Vuorinen [MMV1] (1995) and R. Wisk and myself [FW] (1995).

A central concept of this paper is the new definition of Hölder continuity for differential forms on Riemannian manifolds. Let $\mathcal{M}$ be an n-dimensional Riemannian manifold and $\Gamma(m_1, m_2)$ the family of locally rectifiable arcs $\gamma \in \mathcal{M}$ joining the points m_1 and m_2. Here $d(m_1, m_2)$ is the geodesic distance between the points m_1 and m_2, the element of length on $\mathcal{M}$ is denoted by $ds_{\mathcal{M}}$. Let ω be a differential form of degree k and D a compact subset of $\mathcal{M}$. The differential form ω satisfies the Hölder condition at a point $m_1 \in D$ with index α, $0 < \alpha \leq 1$, and with the coefficient $C(m_1)$, if the estimation

$$\inf_{\gamma \in \Gamma(m_1, m_2)} \int_{\gamma} |d\omega| ds_{\mathcal{M}} \leq C(m_1)\, d^{\alpha}(m_1, m_2) \tag{0.2}$$

is valid for all $m_2 \in D$ sufficiently close to m_1. One says that ω is Hölder continuous with index α on D if (0.2) is satisfied for all $m_1 \in D$. If the differential form ω is continuous and of degree zero, i.e. if ω is a continuous function, our definition coincides with the standard definition of the Hölder continuity. An important point in this definition is that from the Hölder continuity of a differential form we normally cannot state the same for its coefficients.

The main result in this paper is that if $d\omega$ is of the class $\mathcal{WT}_2$ the Hölder continuity of ω follows. The proof is based on Morrey's Lemma on Riemannian manifolds, presented by O. Martio, V.M. Miklyukov and M. Vuorinen [MMV3] (1996), which we could here extend to differential forms. Then the estimation of the energy integral of a differential form $d\omega \in \mathcal{WT}_2$, namely

$$\int_{B(a,r)} |d\omega|^p dv_{\mathcal{M}} \leq c\, r^{n-p+\beta} \,,$$

with the constants c and β, lead to success. Here $B(a, r)$ denotes the geodesic ball with center $a \in \mathcal{M}$ and radius $r > 0$. The n-dimensional volume element on $\mathcal{M}$ is denoted by $dv_{\mathcal{M}}$.

In the last chapter we give some examples for Hölder continuous differential forms. First of course the differential forms associated with a quasiregular mapping f are an example, but also differential forms associated with mappings of bounded length distortion. We also prove the Hölder continuity of harmonic differential forms.