11 Hölder continuity for differential forms

We give a new definition of the Hölder continuity for differential forms, based on the definition for functions.

Let $\mathcal{M}$ be an n-dimensional Riemannian manifold and $\Gamma = \Gamma(m_1, m_2)$ the family of locally rectifiable arcs $\gamma \in \mathcal{M}$ joining the points m_1 and m_2. Here $d = d(m_1, m_2)$ is the geodesic distance between the points m_1 and m_2. We denote by $ds_{\mathcal{M}}$ the element of length on $\mathcal{M}$.

11.1. Definition. *Let ω be a differential form of degree k and D a compact subset of $\mathcal{M}$. The differential form ω satisfies the Hölder condition at a point $m_1 \in D$ with index α, where $0 < \alpha \leq 1$, and with the coefficient $C(m_1)$ if*

$$\inf_{\gamma \in \Gamma} \int_\gamma |d\omega| ds_{\mathcal{M}} \leq C(m_1)\, d^\alpha \tag{11.2}$$

for all $m_2 \in D$ sufficiently close to m_1. One says that ω is Hölder continuous with index α on D if (11.2) is satisfied for all $m_1 \in D$. If $C = \sup_{m_1 \in D} C(m_1) < \infty$ the differential form ω is called uniformly Hölder continuous on D.

If the differential form ω is continuous and of degree zero, i.e. ω is a function, then

$$|\omega(m_1) - \omega(m_2)| \leq \inf_{\gamma \in \Gamma} \int_\gamma |\nabla \omega| ds_{\mathcal{M}} = \inf_{\gamma \in \Gamma} \int_\gamma |d\omega| ds_{\mathcal{M}}$$

and the well-known definition for Hölder continuity follows.

For example a quasiregular mapping $f : \Omega \to I\!R^n$ is Hölder continuous on compact subsets of Ω, $\Omega \subset I\!R^n$. The best index α is known to be $K_I(f)^{1/1-n}$, see [Vu] §11.

If the differential form ω is of degree greater than zero and if ω_0 is a closed differential form we have

$$\inf_{\gamma \in \Gamma} \int_\gamma |d(\omega + \omega_0)| ds_{\mathcal{M}} = \inf_{\gamma \in \Gamma} \int_\gamma |d\omega| ds_{\mathcal{M}}\ .$$

This means, if ω is Hölder continuous in a compact set D then also $\omega + \omega_0$. The Hodge Decomposition Theorem [IM] §6 for differential forms says that for every differential form $\omega : \mathbb{R}^n \to \Lambda^k(\mathbb{R}^n)$, $1 < p < \infty$, there exist differential forms

$$\alpha \in \ker d^* \cap L_1^p(\mathbb{R}^n) \quad \text{and} \quad \beta \in \ker d \cap L_1^p(\mathbb{R}^n)$$

such that

$$\omega = d\alpha + d^*\beta \; . \tag{11.3}$$

The differential forms $d\alpha$ and $d^*\beta$ are unique. Back to the Hölder condition, we see that (11.2) in this case only gives an estimate for the coclosed part of ω. This idea holds local also on manifolds.

Now we formulate the Hölder continuity for differential forms of the class $\mathcal{WT}_2$. Again for $D \subset\subset \mathcal{M}$ we set $\delta(D)$ as in (9.19).

11.4. Theorem. *Let D be a compact subset of the n-dimensional Riemannian manifold $\mathcal{M}$. We assume that at every point $a \in D$ the manifold $\mathcal{M}$ satisfies the conditions I),II), and III) with a constant $\delta \le \delta(D)/2$ and that for every pair of points $a_1, a_2 \in D$ satisfying (9.20) also (9.21) holds.*

Let the differential form $d\omega$ be of the class $\mathcal{WT}_2$. If $\beta = \beta(a, \delta) > 0$ for all $a \in D$, then for all $a_1, a_2 \in D$ with $d = d(a_1, a_2) < \delta$ the inequality

$$\inf_{\gamma \in \Gamma(a_1, a_2)} \int_\gamma |d\omega| ds_{\mathcal{M}} \le \frac{c_6}{c_7}\, d^{\frac{\beta}{p}}$$

holds. This is the Hölder continuity of ω in D with Hölder index $\alpha = \beta/p$ and coefficient c_6/c_7.

11.5. Remark. Here c_6 is the constant from Lemma 9.7 with c_5 from Lemma 10.9. Further β is the constant from (10.4).

Proof. The theorem follows directly with Theorem 10.9 and Theorem 9.22.

□