12 Examples

Let $\mathcal{M}$ and $\mathcal{N}$ be Riemannian manifolds of dimension n and f a mapping between these two manifolds. Locally we write the mapping f again in the components $f^1, \ldots, f^n$. Further let D be a compact subset of $\mathcal{M}$ and $\mathcal{M}$ and D satisfy the conditions of Theorem 11.4. If the mapping $f \in W^{1,n}_{\mathrm{loc}}(\mathcal{M})$ is quasiregular, we can take the differential form $\omega = f^1 df^2 \wedge \ldots \wedge df^k$, $k < n$, or more general the differential form $u_I = f^{i_k} df^{i_1} \wedge \ldots \wedge df^{i_{k-1}}$ (see 7.2) with the multi-index $I = (i_1, \ldots, i_k) \in \mathcal{I}(k, n)$. Because of Theorem 8.1 we know that the differential forms $d\omega$ and du_I are of the class $\mathcal{WT}_2$. It follows that ω and u_I are Hölder continuous in D. In this case we also know that the differential forms f^k, $1 \leq k \leq n$, of degree zero are Hölder continuous, in general if we have a differential form ω which satisfies the Hölder condition we can state nothing about the coefficients of ω.

If the mapping $f \in W^{1,1}_{\mathrm{loc}}(\mathcal{M})$ is L-BLD (see Example 6.6) we know that f is also quasiregular. It follows as above that for example the differential form $\omega = f^1 df^2 \wedge \ldots \wedge df^k$, $k < n$, is Hölder continuous in D.

Let now ω be a differential form of the class $L^2_{\mathrm{loc}}(\mathcal{M})$ with $\deg \omega = k$, $1 \leq k \leq n$. We say that ω is a harmonic differential form or a harmonic field if it is simultaneously weakly closed and weakly coclosed, that is

$$d\omega = d^*\omega = 0 .$$

In particular, if $f \in C^2(\mathcal{M})$, then the differential form df of degree 1 is harmonic if and only if $\Delta f = 0$.

The Laplace-Beltrami operator for differential forms $\Delta : \Lambda^k(T_m(\mathcal{M})) \to \Lambda^k(T_m(\mathcal{M}))$ is defined by

$$\Delta := dd^* + d^*d .$$

12.1. Lemma. *Let ω be a differential form of the class $W^{1,p}_{\mathrm{loc}}(\mathcal{M})$, $1 \leq p \leq \infty$. We have $\Delta\omega = 0$ if and only if $d\omega = 0$ and $d^*\omega = 0$.*

Proof. If $d\omega = 0$ and $d^*\omega = 0$ it follows obviously that $\Delta\omega = 0$.
Conversely if $\Delta\omega = 0$ we have

$$(\Delta\omega, \omega) = (dd^*\omega, \omega) + (d^*d\omega, \omega) = (d^*\omega, d^*\omega) + (d\omega, d\omega) .$$

Since both terms on the right hand side are nonnegative and vanish only if $d\omega = d^*\omega = 0$, $\Delta\omega = 0$ implies $d\omega = d^*\omega = 0$. $\square$

12.2. Corollary. *Let ω be a differential form of the class $L^2_{\mathrm{loc}}(\mathcal{M})$, $\deg\omega = k$, $1 \le k \le n$. If ω is a harmonic differential form, then ω is of the class $\mathcal{WT}_2$ with the structure constants $p = 2$, $\nu_1 = \nu_2 = 1$.*

Proof. If we choose $\theta = \star^{-1}\omega \in L^2_{\mathrm{loc}}(\mathcal{M})$ we have

$$\langle\omega, \star\theta\rangle = \langle\omega, \omega\rangle = |\omega|^2$$

and $|\theta| = |\omega|$. The differential form $\star^{-1}\omega$ is weakly closed, because $\star^{-1}\omega = (-1)^{k(n-k)} \star \omega$. Therefore the conditions (4.6) and (4.7) hold with the constants $p = 2$, $\nu_1 = \nu_2 = 1$. $\square$

If the differential form $\omega \in W^{1,2}_{\mathrm{loc}}$ is harmonic then obviously also the differential form $d\omega$. With Theorem 11.4 we can follow that if $d\omega$ is a harmonic differential form then ω satisfies a Hölder condition in a compact subset D of the n-dimensional Riemannian manifold $\mathcal{M}$.