3 Differential forms on Riemannian manifolds

Let $\mathcal{M}$ be an orientable compact Riemannian manifold of dimension n and of the class C^3. Let $x^1, \dots, x^n$ be local coordinates in the neighborhood of a point $m \in \mathcal{M}$. The square of a line element on $\mathcal{M}$ has the following expression in terms of the local coordinates $x^1, \dots, x^n$

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}(x) dx^i \, dx^j \, .$$

Each section ω of the bundle $\Lambda^k(T(\mathcal{M}))$ is a differential form of degree k on the manifold $\mathcal{M}$. The differential form ω can be written in terms of the local coordinates $x^1, \dots, x^n$ (see (1.2)) as the linear combination

$$\omega = \sum_{I \in \mathcal{I}(k,n)} \omega_I \, dx^I = \sum_{1 \le i_1 < \ldots < i_k \le n} \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \, . \tag{3.1}$$

Let ω be a differential form defined on an open set $D \subset \mathcal{M}$. If $\mathcal{F}(D)$ is a class of functions defined on D, then we say that the differential form ω is in this class provided that all coefficients ω_I, $I \in \mathcal{I}(k,n)$, are in this class.

For example $\omega \in L^p(D)$, $1 \le p \le \infty$, if all coefficients ω_I belong to $L^p(D)$. Endowed with the norm

$$\|\omega\|_{p,D} = \left(\int_D |\omega(m)|^p dv_{\mathcal{M}} \right)^{1/p} = \left(\int_D \left(\sum_{I \in \mathcal{I}(k,n)} |\omega_I(m)|^2 \right)^{p/2} dv_{\mathcal{M}} \right)^{1/p} \tag{3.2}$$

$L^p(D)$ is a Banach space. Here $dv_{\mathcal{M}}$ denotes the n-dimensional volume element on $\mathcal{M}$. The space $L^p_1(D)$ consists of all differential forms ω with

$$\|\omega\|_{L^p_1(D)} = \left(\int_D \left(\sum_{I \in \mathcal{I}(k,n)} \sum_{i=1}^{n} \left| \frac{\partial \omega_I(m)}{\partial x^i} \right|^2 \right)^{p/2} dv_{\mathcal{M}} \right)^{1/p} < \infty \, . \tag{3.3}$$

The norm (3.3) is only a semi-norm. The Sobolev space $W^{1,p}(\mathcal{M})$, $1 \le p < \infty$, is defined by

$$W^{1,p}(\mathcal{M}) = L^p(\mathcal{M}) \cap L^p_1(\mathcal{M})$$

with the norm $\|\omega\|_{W^{1,p}(\mathcal{M})} = \|\omega\|_p + \|\omega\|_{L^p_1(\mathcal{M})}$. The local spaces $L^p_{\mathrm{loc}}(\mathcal{M})$ and $W^{1,p}_{\mathrm{loc}}(\mathcal{M})$ are defined in the usual way.

The Sobolev embeddings in Euclidean spaces (see for example [Re] §2) are valid for compact manifolds. For the following theorem and proof see [He] §3.3.

3.4. Theorem. *Let $\mathcal{M}$ be a compact Riemannian manifold of dimension n. For every p, $1 \le p < n$, and every $q \ge 1$ such that $q < np/(n-p)$, the embedding of $W^{1,p}(\mathcal{M})$ in $L^q(\mathcal{M})$ is compact.*

For all differential forms $\alpha \in L^p(D)$ and $\beta \in L^q(D)$ with $1 \le p, q \le \infty$, $1/p + 1/q = 1$, the inner product is defined by

$$(\alpha, \beta) = \int_D \langle \alpha(x), \beta(x) \rangle \, dv_{\mathcal{M}} \; . \tag{3.5}$$

The orthogonal complement of a differential form ω on a Riemannian manifold $\mathcal{M}$ will be denoted by $\star\,\omega$, where the linear operator $\star$ is the Hodge star operator of (1.5). If $\deg \omega = 1$, then in the local orthonormal system of coordinates $x^1, \ldots, x^n$ at m we can write

$$\star\,\omega(m) = \star \sum_{i=1}^{n} \omega_i(m)\, dx^i = \sum_{i=1}^{n} (-1)^{i-1} \omega_i(m)\, dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n \, ,$$

where the sign $\widehat{}$ means that the expression under $\widehat{}$ is omitted.

We shall make extensive use of the exterior derivative operator d. If ω, $\deg \omega = k$, $0 \le k \le n$, is a differential form whose coefficients are in $C^1(\mathcal{M})$, then $d\omega$, $\deg(d\omega) = k+1$, denotes its differential defined by

$$d\omega = \sum_{1 \le i_1 < \ldots < i_k \le n} d\omega_{i_1 \ldots i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} = \sum_{I \in \mathcal{I}(k,n)} d\omega_I \wedge dx^I \; . \tag{3.6}$$

The exterior derivative operator is a linear operator. For arbitrary differential forms α and β, differentiable in a domain $D \subset \mathcal{M}$, the following properties hold

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \; , \tag{3.7a}$$

$$d(d\alpha) = d(d\beta) = 0 \; , \tag{3.7b}$$

where k is the degree of the differential form α.

The formal adjoint operator to d, the so called Hodge codifferential d^*, is defined by the help of the exterior derivative operator and the Hodge star operator. For a differential form ω of degree k we define

$$d^*\omega = (-1)^k \star^{-1} d \star \omega \ . \tag{3.8}$$

It follows that $d^*\omega$ is of degree $k-1$ with the representation

$$d^*\omega = \sum_{1\le i_1<\ldots<i_k\le n} \sum_{\nu=1}^{k} (-1)^{\nu-1} \frac{\partial \omega_{i_1\ldots i_k}}{\partial x^{i_\nu}} \, dx^{i_1} \wedge \ldots \wedge \widehat{dx^{i_\nu}} \wedge \ldots \wedge dx^{i_k} \ .$$

Observe that the application of the exterior derivative to a differential form of degree n is always zero, the same is true for the codifferential applied to a differential form of degree zero. From (3.8) it follows that $d^*(d^*\omega) = 0$.

In the previous chapter we already defined orientable manifolds, with the help of differential forms we can say it in other words.

3.9. Lemma. *A differentiable manifold $\mathcal{M}$, $\dim \mathcal{M} = n$, is orientable if and only if there exists a differential form of degree n, everywhere non-vanishing.*

For the proof see [Au] §9.

Let $\mathcal{M}$ and $\mathcal{N}$ be orientable Riemannian manifolds of dimension n and $f : \mathcal{M} \to \mathcal{N}$ a mapping of the Sobolev class $W^{1,p}_{\text{loc}}(\mathcal{M})$, $p \ge 1$. Concerning local coordinates $x^1, \ldots, x^n$ we can write the mapping f locally in the components $f^1, \ldots, f^n$. Then f induces a homomorphism $f^* : C^\infty(\mathcal{M}) \to L^p_{\text{loc}}(\mathcal{M})$ on differential forms of degree k, called the pull-back. More precisely, for a differential form $\alpha = \sum_{I\in\mathcal{I}(k,n)} \alpha_I dx^I \in C^\infty(\mathcal{M})$, $\deg \alpha = k$, we get

$$\begin{aligned} (f^*\alpha)(m) &= \sum_{I\in\mathcal{I}(k,n)} \alpha_I(f(m))\, df^{i_1} \wedge \ldots \wedge df^{i_k} \\ &= \sum_{I\in\mathcal{I}(k,n)} \alpha_I(f(m))\, df^I \ . \end{aligned} \tag{3.10}$$

The pull-back f^* can be interpreted as a coordinate transformation of differential forms. The operator f^* applied on differential forms of degree k

with constant coefficients is easily recognized as the kth exterior power of the linear transformation $D^t f(m)$. That is

$$(f^*\alpha)(m) = [D^t f(m)]_\# \, \alpha \; . \tag{3.11}$$

For the theory of differential forms on Riemannian manifolds and especially for the following statements we refer to [Rh].

If $\mathcal{M}$ is a compact n-dimensional orientable Riemannian manifold with nonempty piecewise smooth boundary $\partial\mathcal{M}$, the following Stokes formula holds for an arbitrary differential form $\omega \in C^1(\mathcal{M})$, $\deg\omega = n-1$,

$$\int\limits_{\partial\mathcal{M}} \omega = \int\limits_{\mathcal{M}} d\omega \; . \tag{3.12}$$

3.13. Definition. *A differential form α, $\deg\alpha = k$, on the manifold $\mathcal{M}$ with coefficients $\alpha_I \in L^p_{\mathrm{loc}}(\mathcal{M})$, $I \in \mathcal{I}(k,n)$, is called weakly closed, if for each differential form β, $\deg\beta = k+1$, with*

$$\operatorname{supp}\beta \cap \partial\mathcal{M} = \emptyset, \quad \operatorname{supp}\beta = \overline{\{m \in \mathcal{M} : \beta \neq 0\}} \subset \mathcal{M},$$

and with coefficients in the class $W^{1,q}_{\mathrm{loc}}(\mathcal{M})$, $1/p + 1/q = 1$, $1 \le p, q \le \infty$, we have

$$\int\limits_{\mathcal{M}} \langle \alpha, d^*\beta \rangle \, dv_{\mathcal{M}} = 0 \; . \tag{3.14}$$

The following lemma shows that for smooth differential forms α, condition (3.14) agrees with the usual condition of closedness $d\alpha = 0$, see [Rh] §25. Let $\mathcal{M}$ be an orientable Riemannian manifold with nonempty piecewise smooth boundary.

3.15. Lemma. *Let $\alpha, \beta \in C^1(\mathcal{M})$ with $\deg\alpha = k$ and $\deg\beta = k+1$. If either α or β has compact support in $\mathcal{M}$, then*

$$\int\limits_{\mathcal{M}} \langle d\alpha, \beta \rangle \, dv_{\mathcal{M}} = \int\limits_{\mathcal{M}} \langle \alpha, d^*\beta \rangle \, dv_{\mathcal{M}} \; . \tag{3.16}$$

Proof. With (1.9) and property (3.7a) we know that

$$\begin{aligned}
\int\limits_{\mathcal{M}} \langle d\alpha, \beta \rangle \, dv_{\mathcal{M}} &= \int\limits_{\mathcal{M}} d\alpha \wedge \star\beta \\
&= \int\limits_{\mathcal{M}} d(\alpha \wedge \star\beta) + (-1)^{k+1} \int\limits_{\mathcal{M}} \alpha \wedge d\star\beta \; .
\end{aligned}$$

Because α or β has compact support on $\mathcal{M}$, the first integral on the right side is zero by Stokes formula for differential forms. Thus and with (3.8) it follows

$$\begin{aligned}\int\limits_{\mathcal{M}} d\alpha \wedge \star\beta &= (-1)^{k+1} \int\limits_{\mathcal{M}} \alpha \wedge \star\star^{-1}\, d\star\beta = \int\limits_{\mathcal{M}} \alpha \wedge \star d^*\beta \\ &= \int\limits_{\mathcal{M}} \langle \alpha, d^*\beta\rangle \, dv_{\mathcal{M}} \,.\end{aligned}$$

□

We next introduce the following very useful theorem.

3.17. Theorem. *Let α and β be differential forms, β with a compact support, and $\alpha \in W^{1,p}_{\mathrm{loc}}(\mathcal{M})$, $\beta \in W^{1,q}(\mathcal{M})$, $1 \le p, q \le \infty$, $\deg\alpha + \deg\beta = n-1$, $1/p + 1/q = 1$. Then*

$$\int\limits_{\mathcal{M}} d\alpha \wedge \beta = (-1)^{\deg\alpha + 1} \int\limits_{\mathcal{M}} \alpha \wedge d\beta \,. \tag{3.18}$$

In particular, the differential form α is weakly closed if and only if $d\alpha = 0$ a.e. on $\mathcal{M}$.

Proof. Fix α and β with the stated properties. Because the coefficients of the differential form α are in the class $W^{1,p}_{\mathrm{loc}}(\mathcal{M})$, there exists a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of differential forms with coefficients in the class $C^1(\mathcal{M})$ converging in the $W^{1,p}$-norm to the coefficients of the differential form α on every compact set $K \subset \mathrm{int}\mathcal{M}$.

Let $\{\beta_n\}_{n=1}^{\infty}$ be a sequence of differential forms, $\deg\beta_n = \deg\beta$, in the class $C^1(\mathcal{M})$ having compact supports and converging in the norm of $W^{1,q}$ to the differential form β. We may assume that there exists a smooth submanifold $U \subset\subset \mathcal{M}$ such that $\operatorname{supp}\beta_n \subset U$ for all integers n.

The differential forms $\alpha_n \wedge \beta_n$ have compact supports contained in U. Stokes formula yields

$$\int\limits_{\mathcal{M}} d(\alpha_n \wedge \beta_n) = \int\limits_{U} d(\alpha_n \wedge \beta_n) = 0 \,,$$

and hence

$$\int\limits_{U} d\alpha_n \wedge \beta_n + (-1)^{\deg\alpha} \int\limits_{U} \alpha_n \wedge d\beta_n = 0 \,.$$

We have

$$\int_U d\alpha\wedge\beta-\int_U d\alpha_n\wedge\beta_n=\int_U (d\alpha-d\alpha_n)\wedge\beta+\int_U d\alpha_n\wedge(\beta-\beta_n)\ .$$

Therefore, using the Hölder inequality (1.10) we obtain

$$\begin{aligned}\Big|\int_U d\alpha\wedge\beta \ &- \int_U d\alpha_n\wedge\beta_n\Big|\\
&\le \int_U |d(\alpha-\alpha_n)\wedge\beta|dv_{\mathcal{M}}+\int_U |d\alpha_n\wedge(\beta-\beta_n)|dv_{\mathcal{M}}\\
&\le C\int_U |d(\alpha-\alpha_n)|\,|\beta|dv_{\mathcal{M}}+C\int_U |d\alpha_n|\,|\beta-\beta_n|dv_{\mathcal{M}}\\
&\le C\|d(\alpha-\alpha_n)\|_{L^p(U)}\,\|\beta\|_{L^q(U)}+C\|d\alpha_n\|_{L^p(U)}\,\|\beta-\beta_n\|_{L^q(U)}\ ,\end{aligned}$$

where $C=(C_{k+1,l})^{1/2}$ is the constant of (1.10) with $k=\deg\alpha$ and $l=\deg\beta$. Similarly we obtain

$$\begin{aligned}\Big|\int_U \alpha\wedge d\beta \ &- \int_U \alpha_n\wedge d\beta_n\Big|\\
&\le C_1\|\alpha\|_{L^p(U)}\,\|d(\beta-\beta_n)\|_{L^q(U)}\ +C_1\|\alpha-\alpha_n\|_{L^p(U)}\,\|d\beta\|_{L^q(U)}\ ,\end{aligned}$$

where $C_1=(C_{k,l+1})^{1/2}$. These inequalities easily yield (3.18).

If $d\alpha=0$ a.e. on $\mathcal{M}$, then by (3.18)

$$\int_{\mathcal{M}}\alpha\wedge d\beta=0 \tag{3.19}$$

for an arbitrary differential form $\beta\in W^{1,q}$ with compact support. This, obviously, implies (3.14). On the other hand, if we take a weakly closed differential form $\alpha\in W^{1,p}_{\mathrm{loc}}(\mathcal{M})$, then by (3.18) one has

$$\int_{\mathcal{M}} d\alpha\wedge\beta=0\quad\text{for all }\ \beta\in W^{1,q}(\mathcal{M})\quad\text{with}\quad \operatorname{supp}\beta\subset\mathcal{M}\ .$$

We fix an arbitrary point $m\in\mathcal{M}$ and pass to the local coordinates on $\mathcal{M}$ in a neighborhood of this point. We see that almost everywhere in a neighborhood of the point m the coefficients of the differential form $d\alpha$ are zero. Hence the theorem has been proved. □