4 The $\mathcal{WT}$-classes of differential forms

In this section we introduce several classes of differential forms with generalized derivatives, they were first presented in [MMV1] and [MMV2]. These classes are used to study the associated classes of quasilinear elliptic partial differential equations.

Let $\mathcal{M}$ be a Riemannian manifold of class C^3, $\dim \mathcal{M} = n$, with a boundary or without boundary and let

$$\omega \in L^p_{\mathrm{loc}}(\mathcal{M}),\ \deg \omega = k,\ 0 \le k \le n,\ p > 1, \tag{4.1}$$

be a weakly closed differential form on $\mathcal{M}$.

4.2. Definition. *A differential form ω (4.1) is said to be of the class $\mathcal{WT}_1$ on $\mathcal{M}$ if there exists a weakly closed differential form*

$$\theta \in L^q_{\mathrm{loc}}(\mathcal{M}),\ \deg \theta = n - k,\ \frac{1}{p} + \frac{1}{q} = 1, \tag{4.3}$$

such that almost everywhere on $\mathcal{M}$ we have

$$\nu_0\ |\theta|^q \le\ \langle \omega, \star\, \theta \rangle \tag{4.4}$$

where ν_0 is a positive constant.

4.5. Definition. *The differential form (4.1) is said to be of the class $\mathcal{WT}_2$ on $\mathcal{M}$, if there exists a differential form (4.3) such that almost everywhere on $\mathcal{M}$ the conditions*

$$\nu_1\, |\omega|^p \le\ \langle \omega, \star\, \theta \rangle \tag{4.6}$$

and

$$|\theta| \le \nu_2\, |\omega|^{p-1} \tag{4.7}$$

are satisfied, with constants $\nu_1, \nu_2 > 0$. It is clear that we have $\nu_1 \le \nu_2$.

For an arbitrary simple differential form of degree k

$$\omega = \omega_1 \wedge \ldots \wedge \omega_k$$

we set

$$\|\omega\| = \Big(\sum_{i=1}^{k} |\omega_i|^2\Big)^{1/2}.$$

For a simple differential form ω we have

$$|\omega| \le \prod_{i=1}^{k} |\omega_i|$$

and thus, using the inequality between geometric and arithmetic means

$$\Big(\prod_{i=1}^{k} |\omega_i|\Big)^{1/k} \le \frac{1}{k}\sum_{i=1}^{k} |\omega_i| \le \Big(\frac{1}{k}\sum_{i=1}^{k} |\omega_i|^2\Big)^{1/2},$$

we obtain

$$|\omega| \le k^{-\frac{k}{2}} \|\omega\|^k. \tag{4.8}$$

4.9. Definition. *The simple differential form of degree k*

$$\omega = \omega_1 \wedge \ldots \wedge \omega_k,\ \omega_i \in L^p_{\text{loc}}(\mathcal{M}),\ 1 \le i \le k,$$

is said to be of the class $\mathcal{WT}_3$ on $\mathcal{M}$, if there exists a differential form (4.3) such that almost everywhere on $\mathcal{M}$ the inequality (4.7) holds and

$$\nu_3 \, \|\omega\|^{kp} \le k^{\frac{kp}{2}} \ \langle \omega, \star\theta \rangle. \tag{4.10}$$

4.11. Definition. *The simple differential form of degree k*

$$\omega = \omega_1 \wedge \ldots \wedge \omega_k,\ \omega_i \in L^p_{\text{loc}}(\mathcal{M}),\ 1 \le i \le k,$$

is said to be of the class $\mathcal{WT}_4$ on $\mathcal{M}$, if there exists a simple differential form (4.3) such that the inequality (4.10) holds almost everywhere on $\mathcal{M}$ and

$$(n-k)^{\frac{-(n-k)}{2}} \|\theta\|^{n-k} \le \nu_4 \, |\omega|^{p-1}. \tag{4.12}$$

4.13. Remark. Because every differential form of degree 1 is simple, for $k = 1$ the class $\mathcal{WT}_2$ coincides with the class $\mathcal{WT}_3$ while for $k = n-1$ the class $\mathcal{WT}_3$ coincides with $\mathcal{WT}_4$.

4.14. Theorem. *The following inclusions hold between these $\mathcal{WT}$-classes*

$$\mathcal{WT}_4 \subset \mathcal{WT}_3 \subset \mathcal{WT}_2 \subset \mathcal{WT}_1.$$

Proof. The first two relations follow in an obvious way from (4.8). For the proof of the last one it is enough to observe that

$$|\theta|^q = |\theta|^{\frac{p}{p-1}} \leq (\nu_2^{\frac{1}{p-1}}|\omega|)^p \leq \nu_2^{\frac{p}{p-1}}\nu_1^{-1}\ \langle\omega, \star\theta\rangle.$$

□

5 Quasilinear elliptic equations

Let $\mathcal{M}$ be a Riemannian manifold and let

$$A : \Lambda^k(T(\mathcal{M})) \to \Lambda^k(T(\mathcal{M}))$$

be a mapping defined almost everywhere on the k-vector tangent bundle $\Lambda^k(T(\mathcal{M}))$. We assume that for almost every $m \in \mathcal{M}$ the mapping A is defined on the k-vector tangent space $\Lambda^k(T_m(\mathcal{M}))$, that is for almost every $m \in \mathcal{M}$ the mapping

$$A(m,\ .\ ) : \Lambda^k(T_m(\mathcal{M})) \to \Lambda^k(T_m(\mathcal{M}))$$

is defined and continuous. We assume that the mapping $m \mapsto A_m(X)$ is measurable for all measurable k-vector fields X. Suppose that for almost every $m \in \mathcal{M}$ and for all $\xi \in \Lambda^k(T_m(\mathcal{M}))$ the properties

$$\nu_1|\xi|^p \leq \langle\xi, A(m,\xi)\rangle\ , \tag{5.1}$$

$$|A(m,\xi)| \leq \nu_2\,|\xi|^{p-1} \tag{5.2}$$

hold for some constants $\nu_1, \nu_2 > 0$, where $p > 1$. Also here it is clear that $\nu_1 \leq \nu_2$.

For the case $A : T(\mathcal{M}) \to T(\mathcal{M})$ see [HKM] §3 and [HR].

5.3. Definition. *A differential form $\omega \in W^{1,p}_{\mathrm{loc}}(\mathcal{M})$ is said to be A-harmonic if it is a solution of the A-harmonic equation*

$$d^*A(m, d\omega) = 0\ , \tag{5.4}$$