4.14. Theorem. *The following inclusions hold between these $\mathcal{WT}$-classes*

$$\mathcal{WT}_4 \subset \mathcal{WT}_3 \subset \mathcal{WT}_2 \subset \mathcal{WT}_1.$$

Proof. The first two relations follow in an obvious way from (4.8). For the proof of the last one it is enough to observe that

$$|\theta|^q = |\theta|^{\frac{p}{p-1}} \le (\nu_2^{\frac{1}{p-1}} |\omega|)^p \le \nu_2^{\frac{p}{p-1}} \nu_1^{-1} \ \langle \omega, \star\theta \rangle.$$

□

5 Quasilinear elliptic equations

Let $\mathcal{M}$ be a Riemannian manifold and let

$$A : \Lambda^k(T(\mathcal{M})) \to \Lambda^k(T(\mathcal{M}))$$

be a mapping defined almost everywhere on the k-vector tangent bundle $\Lambda^k(T(\mathcal{M}))$. We assume that for almost every $m \in \mathcal{M}$ the mapping A is defined on the k-vector tangent space $\Lambda^k(T_m(\mathcal{M}))$, that is for almost every $m \in \mathcal{M}$ the mapping

$$A(m, \ . \ ) : \Lambda^k(T_m(\mathcal{M})) \to \Lambda^k(T_m(\mathcal{M}))$$

is defined and continuous. We assume that the mapping $m \mapsto A_m(X)$ is measurable for all measurable k-vector fields X. Suppose that for almost every $m \in \mathcal{M}$ and for all $\xi \in \Lambda^k(T_m(\mathcal{M}))$ the properties

$$\nu_1 |\xi|^p \le \langle \xi, A(m, \xi) \rangle \ , \tag{5.1}$$

$$|A(m, \xi)| \le \nu_2 \, |\xi|^{p-1} \tag{5.2}$$

hold for some constants $\nu_1, \nu_2 > 0$, where $p > 1$. Also here it is clear that $\nu_1 \le \nu_2$.

For the case $A : T(\mathcal{M}) \to T(\mathcal{M})$ see [HKM] §3 and [HR].

5.3. Definition. *A differential form $\omega \in W^{1,p}_{\text{loc}}(\mathcal{M})$ is said to be A-harmonic if it is a solution of the A-harmonic equation*

$$d^* A(m, d\omega) = 0 \ , \tag{5.4}$$

understood in the weak sense, that is

$$\int_{\mathcal{M}} \langle A(m, d\omega), d\Phi \rangle \, dv_{\mathcal{M}} = 0 \tag{5.5}$$

for all differential forms $\Phi \in W^{1,q}_{\mathrm{loc}}(\mathcal{M})$ *with* $\operatorname{supp} \Phi \cap \partial\mathcal{M} = \emptyset$, $1/p + 1/q = 1$.

5.6. Theorem. *A differential form* $\omega \in W^{1,p}_{\mathrm{loc}}(\mathcal{M})$ *is A-harmonic with properties (5.1) and (5.2) if and only if* $d\omega \in \mathcal{WT}_2$.

Proof. Let ω, $\deg \omega = k$, be a solution of (5.4) understood in the weak sense. Let the differential form $\alpha(m)$ be associated with the vector field $A(m, d\omega)$ at the point m and set $\theta = \star\,\alpha$. The differential form ω is weakly closed because of (5.5) and the weak closedness of θ follows from

$$\begin{aligned}
(-1)^{nk+1} \int_{\mathcal{M}} \langle \theta, d^*\psi \rangle \, dv_{\mathcal{M}} &= \int_{\mathcal{M}} \langle \star\,\alpha, \star\, d \star \psi \rangle \, dv_{\mathcal{M}} \\
&= \int_{\mathcal{M}} \langle \alpha, d \star \psi \rangle \, dv_{\mathcal{M}} \\
&= \int_{\mathcal{M}} \langle A(m, d\omega), d\phi \rangle \, dv_{\mathcal{M}} \;=\; 0
\end{aligned}$$

for all $\psi = \star^{-1}\phi \in W^{1,q}(\mathcal{M})$ with $\operatorname{supp} \psi \cap \partial\mathcal{M} = \emptyset$. It remains to verify (4.6) and (4.7). From (5.1) it follows that

$$\nu_1 |d\omega|^p \le \langle d\omega, A(m, d\omega) \rangle = \langle d\omega, \star\,\theta \rangle$$

and from (5.2)

$$|\theta| = |\star\,\alpha| = |A(m, d\omega)| \le \nu_2 |d\omega|^{p-1} \,.$$

Conversely, if $d\omega \in \mathcal{WT}_2$, then there exists a weakly closed differential form θ (see (4.3)) such that (4.6) and (4.7) are satisfied. With the vector field $a : \mathcal{M} \to \Lambda_k(I\!\!R)$ associated to the differential form $\alpha = \star\,\theta$ we define

$$A(m, \xi) = \begin{cases} a(m) & \text{for } \xi = d\omega(m) \,, \\ \xi |\xi|^{p-2} & \text{for } \xi \ne d\omega(m) \,. \end{cases}$$

The weak closedness of θ ensures that ω is a solution of (5.4) understood in the weak sense. The conditions (5.1) and (5.2) for A are satisfied because of (4.6) and (4.7). □