The norm of the tangent vector $\xi \in T_m(\mathcal{M})$ at $m \in \mathcal{M}$ with respect to this metric is defined by $|\xi|_G := \langle G(m)\xi, \xi\rangle^{1/2}$. Every K-quasiconformal mapping f induces a metric tensor on $\mathcal{M}$, namely

$$G(m) := J_f(m)^{-2/n} D^t f(m) Df(m) \tag{6.9}$$

if $J_f(m) \neq 0$, and $G(m) = \text{Id}$ if $J_f(m) = 0$. It is clear that f is conformal with respect to this metric. We refer to $G(m)$ as the matrix dilatation of f at $m \in \mathcal{M}$. The following lemma ensures the inequalities in (6.8). For the proof see Lemma 7.9 in the case $k = 1$.

6.10. Lemma. *Let $f \in W^{1,p}(\mathcal{M})$, $1 \leq p \leq n$, be weakly K-quasiregular, then the equation*

$$K^{\frac{1}{n}-1}|\xi| \leq \langle G(m)\xi, \xi\rangle^{\frac{1}{2}} \leq K^{1-\frac{1}{n}}|\xi| \tag{6.11}$$

holds for almost every $m \in \mathcal{M}$ and for all $\xi \in T_m(\mathcal{M})$.

Quasiregular mappings are weak solutions of the differential system

$$D^t f(m) Df(m) = J_f(m)^{2/n} G(m) , \tag{6.12}$$

commonly called the n-dimensional Beltrami equation.

7 A-harmonic differential forms and quasiregular mappings

This chapter connects quasilinear elliptic equations with quasiregular mappings. Similar results in Euclidean spaces are shown in [Iw1], [IM] and [FW].

Let $\mathcal{M}$ and $\mathcal{N}$ be orientable Riemannian manifolds of dimension n and $f : \mathcal{M} \to \mathcal{N}$ a mapping of Sobolev class $W^{1,s}_{\text{loc}}(\mathcal{M})$, $1 \leq s \leq n$. We fix an ordered multi-index $I = (i_1, \ldots, i_k) \in \mathcal{I}(k, n)$ and its complementary multi-index $J = (j_1, \ldots, j_{n-k}) \in \mathcal{I}(n-k, n)$ (see also (1.3)), ordered in such a way that

$$dx^I = \star\, dx^J . \tag{7.1}$$

Again we use local systems of coordinates $x^1, \ldots, x^n$ because we want to calculate with the components of the mapping f. Suppose $s \geq \max\{k, n-k\}$. To each pair (I, J) we assign locally the differential form

$$u_I = f^{i_k} df^{i_1} \wedge \ldots \wedge df^{i_{k-1}} \quad \in L^{\frac{n}{n-1}}_{\text{loc}}(\mathcal{M}) \tag{7.2}$$

of degree $k-1$ and the conjugate differential form

$$v_J = (-1)^{n+1} \star f^{j_1} df^{j_2} \wedge \ldots \wedge df^{j_{n-k}} \quad \in L^{\frac{n}{n-1}}_{\text{loc}}(\mathcal{M}) \tag{7.3}$$

of degree $k+1$. The degree of local integrability is verified by Sobolev embedding Theorem 3.4, which can be used because u_I and v_J are of the Sobolev class $W^{1,s}_{\text{loc}}(\mathcal{M})$. It follows that $u_I, v_J \in L^{s'}_{\text{loc}}(\mathcal{M})$, with $s' = \frac{sn}{n-s}$. Because of $\frac{sn}{n-s} > \frac{n}{n-1}$ we have $u_I, v_J \in L^{n/n-1}_{\text{loc}}(\mathcal{M})$.

The differential forms du_I and $d^* v_J$, both of degree k, are regular distributions, more explicitly

$$\begin{aligned} du_I &= df^{i_k} \wedge df^{i_1} \wedge \ldots \wedge df^{i_{k-1}} \\ &= (-1)^{k-1} df^{i_1} \wedge \ldots \wedge df^{i_k} \quad \in L^1_{\text{loc}}(\mathcal{M}) \end{aligned} \tag{7.4}$$

and with (3.8)

$$\begin{aligned} d^* v_J &= (-1)^{nk+1} \star d \star v_J \\ &= (-1)^{n+1} (-1)^{nk+1} \star d \star \star f^{j_1} df^{j_2} \wedge \ldots \wedge df^{j_{n-k}} \\ &= (-1)^{k+1} \star df^{j_1} \wedge \ldots \wedge df^{j_{n-k}} \quad \in L^1_{\text{loc}}(\mathcal{M}) \, . \end{aligned} \tag{7.5}$$

Now suppose that $f \in W^{1,s}_{\text{loc}}(\mathcal{M})$, $s = \max\{k, n-k\}$, is weakly K-quasiregular with the matrix dilatation $G(m)$. We recall that $G(m) : T_m(\mathcal{M}) \to T_{f(m)}(\mathcal{N})$ induces for a simple differential form a linear mapping $G_{\#}(m) : \Lambda^k(T_m(\mathcal{M}) \to \Lambda^k(T_{f(m)}(\mathcal{M}))$ called the kth exterior power of $G(m)$ (see (1.11)). Directly from the representation (6.9) it follows that $G(m)$ is symmetric with determinant equal to one.

If $0 < \lambda_1(m) \leq \ldots \leq \lambda_n(m)$ denote the eigenvalues of $G(m)$ at the point $m \in \mathcal{M}$, then the eigenvalues of $G_{\#}(m)$ are the products $\lambda_{l_1}(m) \ldots \lambda_{l_k}(m)$ corresponding to all ordered systems $(l_1, \ldots, l_k) \in \mathcal{I}(k, n)$.

Every linear mapping $A : I\!R^n \to I\!R^n$ with $A \in \mathrm{GL}(n)$ maps the n-dimensional unit ball to an ellipsoid $E(A)$ centered at the origin. Through the identification $T_m(\mathcal{M}) \simeq I\!R^n$ we can use this statement also for $Df(m) \in \mathrm{GL}(n)$. We denote by $\gamma_1 \leq \ldots \leq \gamma_n$ the lengths of the half-axes of $E(Df(m))$. They also are the positive quadratic roots of the eigenvalues of the mapping $Df(m)D^t f(m)$. We deduce that

$$\gamma_n = \max_{|\xi|=1} |Df(m)\xi| \quad \text{and} \quad \gamma_1 = \min_{|\xi|=1} |Df(m)\xi|$$

for $\xi \in T_m(\mathcal{M})$. We denote by $K = \gamma_n/\gamma_1$ the linear dilatation of a quasiregular mapping f and we get

$$\left(\frac{\lambda_i}{\lambda_j}\right)^{1/2} = \frac{\gamma_i}{\gamma_j} \tag{7.6}$$

for all $1 \leq i, j \leq n$, see also [Vä] §2.

7.7. Lemma. *Suppose that $f \in W^{1,s}_{\mathrm{loc}}(\mathcal{M})$ is weakly K-quasiregular and that $0 < \lambda_1(m) \leq \ldots \leq \lambda_n(m)$ are the eigenvalues of the matrix dilatation $G(m)$. Then the dilatation condition for f at the point $m \in \mathcal{M}$ reads as*

$$\lambda_i(m) \leq K^2 \lambda_j(m) \tag{7.8}$$

for all $1 \leq i, j \leq n$.

Proof. We have

$$\lambda_1(m)\lambda_i(m) \leq \lambda_1(m)\lambda_n(m) \leq \lambda_j(m)\lambda_n(m)$$

and with (7.6) it follows that

$$\lambda_i(m) \leq \frac{\lambda_n(m)}{\lambda_1(m)}\lambda_j(m) = \left(\frac{\gamma_n(m)}{\gamma_1(m)}\right)^2 \lambda_j(m) = K^2\lambda_j(m)\ .$$

□

7.9. Lemma. *The metric tensor $G_\#$ induces a scalar product on $\Lambda^k(T_m(\mathcal{M}))$. The corresponding norm is equivalent to the norm of a differential form, i.e. the following estimation holds*

$$K^{\frac{k(k-n)}{n}}|\xi| \leq \langle G_\#(m)\xi, \xi\rangle^{\frac{1}{2}} \leq K^{\frac{k(n-k)}{n}}|\xi| \tag{7.10}$$

for all simple differential forms $\xi \in \Lambda^k(T_m(\mathcal{M}))$.

Proof. With the representation of the matrix dilatation (6.9) it follows that

$$\begin{aligned} \langle G_\#(m)\xi,\xi\rangle^{\frac{1}{2}} &= J_f(m)^{-\frac{k}{n}}\langle\,[Df(m)]_\#\xi,[Df(m)]_\#\xi\rangle^{\frac{1}{2}} \\ &= J_f(m)^{-\frac{k}{n}}|\,[Df(m)]_\#\xi| \\ &= J_f(m)^{-\frac{k}{n}}|\xi|\max_{|\xi|=1}|\,[Df(m)]_\#\xi|\ . \end{aligned}$$

Further, it is enough to proof that

$$J_f(m)^{-\frac{k}{n}}\max_{|\xi|=1}|\,[Df(m)]_\#\xi| \le K^{\frac{k(n-k)}{n}}\ .$$

Because of $J_f(m)^2 = \det(Df(m)D^tf(m))$ it follows with (7.6) and Lemma 7.7 that

$$\begin{aligned} J_f(m)^{-k}\left(\max_{|\xi|=1}|\,[Df(m)]_\#\xi|\right)^n &= J_f(m)^{-k}\left(\gamma_{n-k+1}(m)\dots\gamma_n(m)\right)^n \\ &= \frac{\left(\gamma_{n-k+1}(m)\dots\gamma_n(m)\right)^n}{\left(\gamma_1(m)\dots\gamma_n(m)\right)^k} \\ &\le \frac{\left(\lambda_{n-k+1}(m)\dots\lambda_n(m)\right)^{\frac{n-k}{2}}}{\left(\lambda_1(m)\dots\lambda_{n-k}(m)\right)^{\frac{k}{2}}} \\ &\le K^{k(n-k)}\ . \end{aligned}$$

Since $\lambda_i/\lambda_j \le K^2$, it follows that also $\lambda_j/\lambda_i \ge 1/K^2$. Thus we get the lower estimation of (7.10) in the same way. □

For simple differential forms of degree k we notate the linear mapping $H(m) : \Lambda^k(T_{f(m)}(\mathcal{N})) \to \Lambda^k(T_m(\mathcal{M}))$ by setting $H(m) := G_\#^{-1}(m)$.

7.11. Lemma. *For the simple differential forms $du_I, d^*v_J \in L^1_{\mathrm{loc}}(\mathcal{M})$ of degree k we have*

$$H(m)du_I = J_f(m)^{\frac{2k}{n}-1}d^*v_J\ . \tag{7.12}$$

Proof. With the definition of the pull-back f^* (3.10) and with (3.11) we get

$$du_I = (-1)^{k-1}f^*dx_I = (-1)^{k-1}[D^tf(m)]_\#dx_I \tag{7.13}$$

and

$$d^* v_J = (-1)^{k+1} \star f^* dx_J = (-1)^{k+1} \star [D^t f(m)]_\# dx_J \ . \tag{7.14}$$

Through the identification $T_m(\mathcal{M}) \simeq I\!R^n$ we can use Lemma 1.15 also for differential forms on Riemannian manifolds. If we apply to $D^t f(m)$ (1.16) for differential forms of degree $n-k$, it follows with (7.1) that

$$\star [D^t f(m)]_\# = J_f(m)^{1-\frac{2k}{n}} G_\#^{-1}(m) [D^t f(m)]_\# \star \ ,$$

and with (7.13) and (7.14)

$$\begin{aligned} H^{-1}(m) J_f(m)^{\frac{2k}{n}-1} d^* v_J &= J_f(m)^{\frac{2k}{n}-1} G_\#(m)\, d^* v_J \\ &= J_f(m)^{\frac{2k}{n}-1} G_\#(m)\, (-1)^{k+1} \star\, [D^t f(m)]_\# dx^J \\ &= (-1)^{k+1} G_\#(m)\, G_\#^{-1}(m)\, [D^t f(m)]_\# \star dx^J \\ &= (-1)^{k-1} [D^t f(m)]_\# dx^I \\ &= du_I \ . \end{aligned}$$

This completes the proof. □

7.15. Lemma. *For the Jacobian $J_f(m)$ of $f \in W^{1,s}_{\mathrm{loc}}(\mathcal{M})$, $s = \max\{k, n-k\}$, we can write*

$$J_f(m) = \langle du_I, d^* v_J \rangle \ . \tag{7.16}$$

Proof. Again with (3.10) and (3.11) we get

$$\begin{aligned} J_f(m) &= \det(Df(m)) \star \star \mathbb{1} \ = \ \star \det(D^t f(m)) \star \mathbb{1} \\ &= \star [D^t f(m)]_\# \star \mathbb{1} \ = \ \star (df^1 \wedge \ldots \wedge df^n) \\ &= \star f^* \star \mathbb{1} \ . \end{aligned}$$

Now with (1.9) we have

$$dx^I \wedge \star\star\, dx^J = \langle dx^I, \star\, dx^J \rangle \star \mathbb{1} = \langle dx^I, dx^I \rangle \star \mathbb{1} = \star \mathbb{1} \ .$$

Both together with (7.13) and (7.14) yields

$$\begin{aligned} J_f(m) &= \star f^*(dx^I \wedge \star\star\, dx^J) \ = \ \star (f^* dx^I \wedge f^* \star\star\, dx^J) \\ &= \star (f^* dx^I \wedge \star\star\, f^* dx^J) \ = \ \star (du_I \wedge \star\, d^* v_J) \\ &= \star \langle du_I, d^* v_J \rangle \star \mathbb{1} \ = \ \langle du_I, d^* v_J \rangle \ . \end{aligned}$$

□

7.17. Lemma. *The Jacobian $J_f(m)$ of the mapping $f \in W^{1,s}_{\mathrm{loc}}(\mathcal{M})$, $s = \max\{k, n-k\}$, has the representation*

$$J_f(m) = |du_I|^p_H = |d^* v_J|^q_{H^{-1}} \ , \tag{7.18}$$

with $p = \frac{n}{k}$ and $q = \frac{n}{n-k}$.

Proof. With (7.12) and (7.6) we find that

$$\begin{aligned}\langle H(m) du_I, du_I\rangle &= J_f(m)^{\frac{2k}{n}-1}\langle d^* v_J, du_I\rangle \\ &= J_f(m)^{\frac{2k}{n}-1}\langle du_I, d^* v_J\rangle \\ &= J_f(m)^{\frac{2k}{n}}\end{aligned}$$

and therefore

$$\langle H(m) du_I, du_I\rangle^{\frac{n}{2k}} = |du_I|^p_H = J_f(m)$$

for $p = \frac{n}{k}$. With the same calculation we get

$$\langle H^{-1}(m) d^* v_J, d^* v_J\rangle^{\frac{n}{2(n-k)}} = |d^* v_J|^q_{H^{-1}} = J_f(m)$$

for $q = \frac{n}{n-k}$. □

Now we introduce a nonlinear Lebesgue measurable mapping $A : \mathcal{M} \times \Lambda^k(T_m(\mathcal{M})) \to \Lambda^k(T_m(\mathcal{M}))$ by

$$A(m, \xi) = \langle H(m)\xi, \xi\rangle^{\frac{p-2}{2}} H(m)\xi \tag{7.19}$$

for $p = \frac{n}{k}$ and the conjugate mapping $A^{-1} : \mathcal{M} \times \Lambda^k(T_m(\mathcal{M})) \to \Lambda^k(T_m(\mathcal{M}))$ by

$$A^{-1}(m, \xi) = \langle H^{-1}(m)\xi, \xi\rangle^{\frac{q-2}{2}} H^{-1}(m)\xi \tag{7.20}$$

with $q = \frac{n}{n-k}$, $1/p + 1/q = 1$. Both A and A^{-1} are defined for almost every $m \in \mathcal{M}$ and for all $\xi \in \Lambda^k(T_m(\mathcal{M}))$.

7.21. Lemma. *If $f \in W^{1,s}_{\mathrm{loc}}(\mathcal{M})$, $s = \max\{k, n-k\}$, is weakly K-quasiregular, then the differential form u_I (7.2) of degree $k-1$ is A-harmonic.*

Proof. We have to show that the differential form u_I is a solution of the A-harmonic equation (5.4). We use A in the form of (7.19). With the help of (7.12) and (7.18) we obtain for du_I

$$\begin{aligned} A(m, du_I) &= \langle H(m) du_I, du_I \rangle^{\frac{p-2}{2}} H(m) du_I \\ &= |du_I|_H^{p-2} J_f(m)^{\frac{2k}{n}-1} d^* v_J \\ &= J_f(m)^{1-\frac{2}{p}} J_f(m)^{\frac{2}{p}-1} d^* v_J \end{aligned}$$

and therefore

$$A(m, du_I) = d^* v_J \; . \tag{7.22}$$

Applying the Hodge codifferential d^*, it follows the (quasilinear elliptic) A-harmonic equation

$$d^* A(m, du_I) = 0 \; ,$$

for du_I. □

Analogously we get for the differential form $d^* v_J$

$$A^{-1}(m, d^* v_J) = du_I \; .$$

and arrive at the A^{-1}-harmonic equation for $d^* v_J$

$$dA^{-1}(m, d^* v_J) \; = \; 0 \; . \tag{7.23}$$

7.24. Example. For $n = 2k = 2$ and $K = 1$ we get $H(m) = \text{Id}$. This implies $du_I = d^* v_J$. For $I = \{2\}$ and $J = \{1\}$ the Cauchy-Riemann differential equations

$$\frac{\partial f^2}{\partial x^1} = -\frac{\partial f^1}{\partial x^2} \quad \text{and} \quad \frac{\partial f^1}{\partial x^1} = \frac{\partial f^2}{\partial x^2}$$

of an analytic function in 2 dimensions follow (local).

7.25. Lemma. *With the two constants* $0 < \nu_1, \nu_2 < \infty$ *we have*

$$\nu_1 \, |du_I|^p \leq \langle du_I, A(m, du_I) \rangle \tag{7.26}$$

and

(7.27) $$|A(m, du_I)| \le \nu_2 \, |du_I|^{p-1} \, ,$$

for $p > 1$ and for all differential forms $du_I \in L^1_{\mathrm{loc}}(\mathcal{M})$ of degree k, $0 \le k \le n$.

Proof. Because of Lemma 7.9 the norm of du_I on the manifold $\mathcal{M}$ and the norm generated by $H(m)$ are equivalent and thus

$$\nu \, |du_I| \;\le\; |du_I|_H \; .$$

With (7.16), (7.18) and (7.22) we get

$$\begin{aligned} \nu_1 \, |du_I|^p \;&\le\; |du_I|^p_H \;=\; J_f(m) \\ &=\; \langle du_I, d^* v_J \rangle \;=\; \langle du_I, A(m, du_I) \rangle \; . \end{aligned}$$

The second estimation follows directly from the definition of the mapping $A(m, \xi)$

$$\begin{aligned} |A(m, du_I)| \;&=\; |\langle H(m) du_I, du_I \rangle^{\frac{p-2}{2}} H(m) du_I| \;\le\; |H(m)|^{\frac{p}{2}} |du_I|^{p-1} \\ &=\; \nu_2 \, |du_I|^{p-1} \; . \end{aligned}$$

□

8 Quasiregular mappings and $\mathcal{WT}$-classes

In this chapter we want to consider the connection of quasiregular mappings and the $\mathcal{WT}$-classes of differential forms.

8.1. Theorem. *If $f \in W^{1,s}_{\mathrm{loc}}(\mathcal{M})$, $s = \max\{k, n-k\}$, is weakly K-quasiregular, then the differential form du_I (7.4), $\deg du_I = k$, is of the class $\mathcal{WT}_2$.*

Proof. This result follows direct with the Lemmas 7.21 and 7.25 together with Theorem 5.6. □

We want to show now a different approach, based more on the properties of differential forms of the $\mathcal{WT}$-classes. Here we follow [MMV1] and [FMMVW].