main difference is that we use now the inequality between geometric and arithmetic means (8.7) and (8.9) and with the help of (8.10) we get

$$\Big(\sum_{i=1}^{k}|df^i|^2 \;+\; \sum_{i=k+1}^{n}|df^i|^2\Big)^{n/2}$$
$$\leq\; k^{-n/2}(n-k)^{-(n-k)/2}n^{n/2}K_O\Big(\sum_{i=1}^{k}|df^i|^2\Big)^{k/2}\Big(\sum_{i=k+1}^{n}|df^i|^2\Big)^{(n-k)/2}.$$

From this point the proofs follow the concept of the proof of Theorem 8.2. For details of the proofs of Theorem 8.12 and 8.13 see [FMMVW] §6. Our slightly better constants ν_3 and ν_4 follow directly from the definitions of the classes $\mathcal{WT}_3$ and $\mathcal{WT}_4$.

There exist some differences between the Theorems 8.1 and 8.2. In the first theorem the mapping f is only weakly quasiregular. This could be weakened by a theorem from T. Iwaniec ([Iw1] §11) which says that a weakly K-quasiregular mapping $f \in W^{1,p}_{\text{loc}}$, $p < n$, is also K-quasiregular, if p is close enough to n, here p depends only on n and K, see also [FW] §9. The theorem depends on a Caccioppoli-type estimate, which recently was refined in [Iw2].

The differential form du_I (7.4) depends on a multi-index, we have more possibilities for a differential form of the class $\mathcal{WT}_2$. The differential form $u^*w_{\mathcal{A}}$ in Theorem 8.2 is fixed, but we gave concreter constants ν_1 and ν_2.

9 Morrey's Lemma on manifolds

In this chapter we follow mostly the considerations of [MMV3]. Let $\mathcal{M}$ be a Riemannian manifold of dimension n and without boundary. We assume that $\mathcal{M}$ is orientable and of the class C^3. Let $d(m_1, m_2)$ be the geodesic distance between the points $m_1, m_2 \in \mathcal{M}$. We denote by

$$B(a,t) = \{m \in \mathcal{M} : d(a,m) < t\}$$
$$\Sigma(a,t) = \{m \in \mathcal{M} : d(a,m) = t\}$$

the geodesic ball and the geodesic sphere, respectively, with center $a \in \mathcal{M}$ and radius $t > 0$.

In the following we make use of the co-area formula or the Kronrod-Federer formula [Fe] §3.2. We give this formula in the form needed, see for example [GT] §16.5.

9.1. Theorem. *Let ϕ be a nonnegative Borel measurable set in a domain $D \subset \mathcal{M}$ and u a local Lipschitz function on D. Then*

$$\int_D \phi(m)|\nabla u(m)|dv_{\mathcal{M}} = \int_0^\infty dt \int_{E_t} \phi(m)dH \tag{9.2}$$

where H is the surface measure on $E_t = \{m \in \mathcal{M} : |u(m)| = t\}$.

To ensure that the local structure of the manifold $\mathcal{M}$ is uniformly euclidean, we need the following three properties. Hereby we assume that in these properties the constants $\delta, c_1, ..., c_4$ and the function h are independent of the point $a \in \mathcal{M}$.

I) For $a \in \mathcal{M}$ the radius of injectivity $r_{\text{inj}}(a)$ satisfies $0 < \delta < r_{\text{inj}}(a)$. Thus, the geodesic ball $B(a, \delta)$ admits polar coordinates (r, θ), $0 \leq r \leq \delta$, $\theta \in S^{n-1}$, with the volume element

$$dv_{\mathcal{M}} = G_a(r, \theta)drd\theta \tag{9.3}$$

where $G_a(r, \theta) > 0$ is a continuous function, compare with [BC] §11.10.

II) The function $G_a(r, \theta)$ satisfies

$$c_1\, h(r) \leq G_a(r, \theta) \leq c_2\, h(r) \tag{9.4}$$

for all $0 < r < \delta$ and $\theta \in S^{n-1}$ with the continuous function $h(r) > 0$.

III) The area of the geodesic sphere $\Sigma(a, r)$

$$S(a, r) = \int_{\Sigma(a,r)} dH^{n-1} = \int_{S^{n-1}} G(r, \theta)d\theta \tag{9.5}$$

for $r \in (0, \delta)$ is an increasing function on $(0, \delta)$. For the derivative of $S(a, r)$ with respect to r the following inequality holds

$$c_3\, r^{n-2} \leq S'(a, r) \leq c_4\, r^{n-2} \tag{9.6}$$

for all $r \in (0, \delta)$.

For an arbitrary pair of points $m_1, m_2 \in \mathcal{M}$ we denote by $\Gamma = \Gamma(m_1, m_2)$ the family of locally rectifiable curves $\gamma \subset \mathcal{M}$ of the class C^k, $k \geq 2$, joining the points m_1 and m_2.

9.7. Lemma. *Suppose that the manifold $\mathcal{M}$ satisfies properties I), II), and III) with the constant $\delta > 0$. Let $m_1, m_2 \in \mathcal{M}$ with $d = d(m_1, m_2) \leq \delta$ and let the function $\rho \in L^p_{\mathrm{loc}}(\mathcal{M})$, $p \geq 1$, be nonnegative. If there exist constants $\alpha, c_5 > 0$, such that*

$$\int\limits_{B(a_k,r)} \rho^p dv_{\mathcal{M}} \leq c_5\, r^{n-p+\alpha} \tag{9.8}$$

for $r \in (0, d)$, $k = 1, 2$, then

$$\inf_{\gamma \in \Gamma(a_1,a_2)} \int\limits_{\gamma} \rho\, ds_{\mathcal{M}} \leq c_6\, \frac{d^{n+\frac{\alpha}{p}}}{\mathrm{mes}_n(B(a_1, d) \cap B(a_2, d))}\ . \tag{9.9}$$

We can choose

$$c_6 = \Big(\frac{c_2}{c_1}\Big)^2 \frac{2}{n + \alpha/p} \Big(1 + \frac{n-1}{\alpha/p} \Big(\frac{c_4}{c_3}\Big)^2\Big)\, c_5^{\frac{1}{p}} \Big(\frac{c_4}{n(n-1)}\Big)^{\frac{p-1}{p}}$$

with the constants c_j, $j = 1, \ldots, 4$ from (9.4) and (9.6).

Proof. First we consider the case $p = 1$. Let $Q = B(a_1, d) \cap B(a_2, d)$. For $k = 1, 2$ let $l_k(m)$ be a geodesic segment joining the point a_k to a point $m \in Q$. Since $r_{\mathrm{inj}}(a_k) > d$, these geodesic segments $l_k(m)$ are the shortest curves joining the mentioned points.

We have

$$\inf_{\gamma \in \Gamma(a_1,a_2)} \int\limits_{\gamma} \rho ds_{\mathcal{M}} \leq \inf_{m \in Q} \left(\int\limits_{l_1(m)} \rho ds_{\mathcal{M}} + \int\limits_{l_2(m)} \rho ds_{\mathcal{M}} \right) = \mathcal{R}(\Gamma) \tag{9.10}$$

and hence

$$\begin{aligned} \mathcal{R}(\Gamma) \int\limits_{Q} dv_{\mathcal{M}} &\leq \int\limits_{Q} dv_{\mathcal{M}} \int\limits_{l_1(m)} \rho ds_{\mathcal{M}} + \int\limits_{Q} dv_{\mathcal{M}} \int\limits_{l_2(m)} \rho ds_{\mathcal{M}} \\ &\leq \int\limits_{B(a_1,d)} dv_{\mathcal{M}} \int\limits_{l_1(m)} \rho ds_{\mathcal{M}} + \int\limits_{B(a_2,d)} dv_{\mathcal{M}} \int\limits_{l_2(m)} \rho ds_{\mathcal{M}} \\ &= I_1 + I_2\ . \end{aligned} \tag{9.11}$$

Here we need to estimate the integral I_1 only, the integral I_2 can be estimated similarly.

Applying the Kronrod-Federer formula (9.2) and observing that

$$|\nabla_m d(a_k, m)| = 1 \quad \text{in } B(a_k, d)\,,$$

we obtain from (9.3) that

$$\begin{aligned} I_1 &= \int_0^d dr \int_{\Sigma(a_1,r)} dH^{n-1} \int_{l_1(m)} \rho ds_{\mathcal{M}} \\ &= \int_0^d dr \int_{S^{n-1}} G_1(r,\theta)d\theta \int_0^r \rho(t,\theta)dt, \end{aligned} \tag{9.12}$$

where $G_1(r,\theta) = G_{a_1}(r,\theta)$. Now (9.4) yields

$$\begin{aligned} I_1 &\le c_2 \int_0^d h(r)dr \int_{S^{n-1}} d\theta \int_0^r \rho(t,\theta)dt \\ &= c_2 \int_0^d h(r)dr \int_0^r dt \int_{S^{n-1}} \rho(t,\theta)d\theta. \end{aligned} \tag{9.13}$$

If we set

$$J(r) = \int_{B(a_1,r)} \rho dv_{\mathcal{M}} = \int_0^r dt \int_{S^{n-1}} G_1(t,\theta)\rho(t,\theta)d\theta\,,$$

then for almost every $r \in [0, d)$, we have by (9.4)

$$J'(r) = \int_{S^{n-1}} G_1(r,\theta)\rho(r,\theta)d\theta \ge c_1\, h(r) \int_{S^{n-1}} \rho(r,\theta)d\theta\,.$$

Now we obtain from (9.13)

$$I_1 \le c_2 \int_0^d h(r)dr \int_0^r \frac{J'(t)}{c_1 h(t)}dt = \frac{c_2}{c_1} \int_0^d h(r)dr \int_0^r \frac{J'(t)}{h(t)}dt\,.$$

However, the inequality (9.4) implies

$$\frac{1}{c_2\omega_{n-1}}S_1(r) \le h(r) \le \frac{1}{c_1\omega_{n-1}}S_1(r)\,,$$

where $S_1(r) = S(a_1, r)$ and ω_{n-1} is the surface area of the unit sphere S^{n-1} of R^n. Thus from the preceding inequality we get

$$I_1 \le \Big(\frac{c_2}{c_1}\Big)^2 \int_0^d S_1(r)dr \int_0^r \frac{J'(t)}{S_1(t)}dt\,. \tag{9.14}$$

The last integral has the value

$$\begin{aligned} \int_0^r \frac{J'(t)}{S_1(t)}dt &= \frac{J(t)}{S_1(t)}\Big|_0^r + \int_0^r \frac{J(t)}{S_1^2(t)}S_1'(t)dt \\ &= \frac{J(r)}{S_1(r)} + \int_0^r \frac{J(t)}{S_1^2(t)}S_1^{'}(t)dt \end{aligned} \tag{9.15}$$

since the conditions imply that

$$\frac{J(t)}{S_1(t)} \le ct^\alpha \to 0 \text{ as } t \to 0\,.$$

From (9.14) and (9.15) we obtain

$$\Big(\frac{c_1}{c_2}\Big)^2 I_1 \le \int_0^d J(r)dr + \int_0^d S_1(r)dr \int_0^r \frac{J(t)}{S_1^2(t)}S_1'(t)dt\,. \tag{9.16}$$

The condition (9.8) yields

$$\int_0^d J(r)dr \le \frac{c_5}{n+\alpha}\, d^{n+\alpha}\,. \tag{9.17}$$

We conclude from (9.6) and (9.8) that

$$\begin{aligned} \int_0^d S_1(r)dr \int_0^r \frac{J(t)}{S_1^2(t)}S_1'(t)dt &\le \frac{c_4}{n-1}\int_0^d r^{n-1}dr \int_0^r \frac{c_5 t^{n-1+\alpha}}{(\frac{c_3}{n-1}t^{n-1})^2}c_4 t^{n-2}dt \\ &= \Big(\frac{c_4}{c_3}\Big)^2 c_5 \frac{n-1}{\alpha(n+\alpha)}d^{n+\alpha}\,. \end{aligned}$$

This inequality together with the estimates (9.16) and (9.17), leads us to the inequality

$$\begin{aligned}\Big(\frac{c_1}{c_2}\Big)^2 I_1 &\le \frac{c_5}{n+\alpha}\, d^{n+\alpha} + \Big(\frac{c_4}{c_3}\Big)^2 c_5 \frac{n-1}{\alpha(n+\alpha)}\, d^{n+\alpha} \\ &= \frac{c_5}{n+\alpha}\Big(1+\Big(\frac{c_4}{c_3}\Big)^2\,\frac{n-1}{\alpha}\Big)\, d^{n+\alpha}\,.\end{aligned}$$

Since a similar estimate is valid for I_2, we obtain from (9.11)

$$\mathcal{R}\,(\Gamma)\,\mathrm{mes}_n Q \le \Big(\frac{c_2}{c_1}\Big)^2 \frac{2c_5}{n+\alpha}\Big(1+\Big(\frac{c_4}{c_3}\Big)^2\,\frac{n-1}{\alpha}\Big)\, d^{n+\alpha}\,, \tag{9.18}$$

and this inequality together with (9.10) finishes the proof of the lemma for $p=1$.

The case $p>1$ can be reduced to $p=1$. By the Hölder inequality we have for $k=1,2$

$$\int\limits_{B(a_k,r)} \rho dv_{\mathcal{M}} \le (\mathrm{mes}_n B(a_k,r))^{\frac{p-1}{p}} \Bigg(\int\limits_{B(a_k,r)} \rho^p dv_{\mathcal{M}}\Bigg)^{\frac{1}{p}}\,.$$

Using (9.2) and (9.6) we obtain

$$\begin{aligned}\mathrm{mes}_n B(a_k,r) &= \int\limits_0^r dt \int\limits_{\Sigma(a_k,t)} \frac{dH^{n-1}}{|\nabla d(a_k,m)|} \\ &= \int\limits_0^r S(a_k,t)dt \le \frac{c_4}{n(n-1)}\, r^n\,.\end{aligned}$$

With this relation and with (9.8) we arrive to the estimate

$$\int\limits_{B(a_k,r)} \rho dv_{\mathcal{M}} \le \Big(\frac{c_4}{n(n-1)}\Big)^{\frac{p-1}{p}} c_5^{\frac{1}{p}} r^{n-1+\frac{\alpha}{p}}.$$

Now we can use the lemma for $p=1$ and get (9.9) in the general case. □

For a subdomain $D \subset\subset \mathcal{M}$ we set

$$\delta(D) = \inf_{\{m_k\}} \liminf_{k\to\infty} d(m_k, D) \tag{9.19}$$

where the infimum is taken over all possible sequences $\{m_k\}$, $m_k \in \mathcal{M}$, not having accumulation points in $\mathcal{M}$. For the domain D we assume that there exists a constant $c_7 > 0$, such that

$$\operatorname{mes}_n(B(a_1, d) \cap B(a_2, d)) \geq c_7\, d^n \tag{9.20}$$

for all points $a_1, a_2 \in D$, satisfying the condition

$$d = d(a_1, a_2) \leq \frac{1}{2}\delta(D)\ . \tag{9.21}$$

Now we deduce the well-known form of Morrey's lemma for differential forms on Riemannian manifolds. For the special case of functions compare with [GT] §12.1 and [Re] §2.1.

9.22. Theorem. *Suppose that the manifold $\mathcal{M}$ satisfies the properties I), II), and III) with the constant $\delta > 0$. Let $D \subset\subset \mathcal{M}$ be a domain such that $\delta \leq \delta(D)/2$ and (9.20) holds. Let $\omega \in W^{1,p}_{\mathrm{loc}}(\mathcal{M})$ be a differential form of degree k, $0 \leq k \leq n$, $p \geq 1$. If for every point $a \in D$ and for every $r \leq \delta(D)/2$ the inequality*

$$\int\limits_{B(a,r)} |d\omega|^p dv_{\mathcal{M}} \leq c_5\, r^{n-p+\alpha} \tag{9.23}$$

holds, then the differential form ω can be redefined on a set of measure zero such that for all $a_1, a_2 \in D$, $d(a_1, a_2) < \delta$, we get

$$\inf_{\gamma \in \Gamma(a_1,a_2)} \int\limits_{\gamma} |d\omega| ds_{\mathcal{M}} \leq \frac{c_6}{c_7}\, d^{\frac{\alpha}{p}}\ , \tag{9.24}$$

where c_6 is the constant from Lemma 9.7.

Proof. If we replace in Lemma 9.7 the function ρ by the value of the differential form $d\omega$, the theorem follows directly with the help of (9.20). □

10 Estimate for the energy integral

Here we present an estimate for the energy integral of the differential form $d\omega \in \mathcal{WT}_2$.