Chapter 2

Regularity results for interface problems for the Laplacian

2.1 Outline

In this chapter we discuss piecewise H^s-regularity of interface problems for the Laplacian which holds independently of the number and shape of the subdomains on which the diffusion coefficient k is constant. Our main interest are H^s-regularity results for $s > 1$ which hold independently of the bound $\delta \leq k \leq \delta^{-1}$ of the diffusion coefficient k. Such regularity holds only in the class of diffusion coefficients satisfying the quasi-monotonicity condition introduced in [24].

Additionally we give regularity results in Sobolev spaces H^s where s *explicitly* depends on the bounds of the diffusion coefficient and we show that these results are sharp. Most of the results given in this chapter were already shown in our recent article [50].

The interface problem will be posed in section 2.2. We restrict ourselves to piecewise regularity on subdomains because global regularity is limited to $H^{3/2-\varepsilon}(\Omega)$ for any $\varepsilon > 0$ (section 2.3.2).

In the first part of this chapter we discuss regularity in 2D. We give a short review of the connection of regularity, singular functions and a Sturm-Liouville eigenvalue problem (section 2.4.1, 2.4.2). Known regularity results are reviewed in section 2.4.3. Further we give examples of problems with regularity H^s, where s, depending on the geometry, tends to the values $s_0 = 1$ or $s_0 = 2$ if $\delta \to 0$. This demonstrates that large variations of the diffusion coefficient may lead but do not necessarily lead to low regularity.

We show, that quasi-monotonicity (section 2.5.1) is necessary and sufficient to yield regularity $H^{1+1/4}$ that is independent of the global bounds of k and without restrictions on the number of subdomains. We prove further that this result is optimal. For showing this we need a lower bound of the eigenvalues of a Sturm-Liouville eigenvalue problem. This bound is derived by investigating the structure of according eigenfunctions (section 2.5.2). In some special situations the bound for the eigenvalues can be improved further, see section 2.5.3.

The main result, where $H^{5/4}$-regularity is shown in the quasi-monotone case, is given in section 2.5.4. We show that the situations covered by known results are special cases

of our approach. The reader interested in the regularity results may skip the preceding sections 2.4.1, 2.5.2, 2.5.3.
In section 2.6 we derive piecewise H^s-regularity results, where s depends on the global bounds $\delta \leq k \leq \delta^{-1}$ of the diffusion coefficient k. For these "worst case" results there are no restrictions on the structure of the diffusion coefficient imposed, like the quasi-monotonicity condition. The main result, where regularity $H^{1+\delta/(2\pi)}$ is shown, is given in section 2.6.2. Further we give slightly stronger regularity results being *sharp with respect to* δ. Sharpness is shown by giving the explicit definition of a special singular function ψ_2 defined on for checkerboard-like pattern of diffusion coefficients δ, δ^{-1}. This means that the singular function ψ_2 has the lowest H^s-regularity among all other singular functions arising from interface problems with the only assumption $\delta \leq k \leq \delta^{-1}$. We are able to establish a link between the regularity theory for the quasi-monotone case and between the theory for the "worst case" introducing additional parameters depending on the diffusion coefficient (section 2.6.3).
In the second part of this chapter we address regularity in 3D (section 2.7). This result is based on a decomposition theorem [17]. In 3D-problems vertex and edge singular functions occur (section 2.7.2 and 2.7.3). As the 3D vertex singular functions are closely related to 2D singularities, we can use the results derived in 2D in section 2.4.
The bounds on the eigenvalues for the Laplace interface problem are directly applicable to Maxwell interface problems [17].

2.2 The interface problem for the Laplacian

In this section we will define the *interface problem* for the Laplacian. Interface problems for the Laplacian are also known as *transmission problems* or in the literature coming from Numerical Mathematics as *problems with discontinuous diffusion coefficients*.
Let an open, Lipschitz domain $\Omega \subset R^d, d = 2, 3$ which is polygonal (polyhedral) be given. That means that its boundary $\partial\Omega$ is piecewise plane. Let its boundary be decomposed into parts $\partial\Omega = \Gamma_D \cup \Gamma_N, \Gamma_D \cap \Gamma_N = \emptyset, meas_{d-1}(\Gamma_D) > 0$, corresponding to Dirichlet and Neumann boundary conditions. In this chapter we restrict ourselves to homogeneous boundary conditions. Let $f \in L^2(\Omega)$ be given.
Let us define the space $V := \left\{ v \in H^1(\Omega) : v|_{\Gamma_D} = 0 \right\}$. We pose the interface problem in variational form: seek $u \in V$ satisfying:

(2.2.1) $$\int_\Omega k(x)\, \nabla u(x)\, \nabla v(x)\; dx \;=\; \int_\Omega f(x)\; v(x)\; dx \quad \forall v \in V \quad .$$

We make the following assumptions on the diffusion coefficient k: Ω can be partitioned in disjoint, open, polygonal (polyhedral) Lipschitz subdomains $\Omega_i, i = 1, .., n$, on which the diffusion coefficient has the constant value k_i. Additionally we impose the global bound

(2.2.2) $$\delta \leq k(x) \leq \delta^{-1} \;\; , \;\; x \in \Omega \;\; ,$$

for a constant $\delta > 0$. Multiplying k by a constant one can assure that both bounds in (2.2.2) are sharp.

For a measurable subset $\Omega' \subset \Omega$ we will use the Sobolev (semi-)norms

$$|u|_{H^1(\Omega')} := \|\nabla u\|_{L^2(\Omega')} \quad , \quad |v|^2_{kH^1(\Omega')} := \int_{\Omega'} k(x)\,(\nabla v(x))^2 \; dx \quad .$$

As $meas_{d-1}(\Gamma_D) > 0$ relation (2.2.2) imply that $|\cdot|_{kH^1(\Omega)}$ is a norm on V which is equivalent up to a factor δ with $\|v\|_{H^1(\Omega)}$ and hence existence and uniqueness of the solution of (2.2.1) follow from Riesz's Theorem [28].

2.3 Notation

We will use Sobolev Spaces of fractional order $H^s, s \in R$, as defined in [1] [30] [40]. If $s \in Z$, then the Sobolev Space H^s coincides with the usual Sobolev Space defined for integer exponents.
In this chapter we will use the shorter term *coefficient* instead of diffusion coefficient.
The Sobolev Space $H^s(\Omega), s > 0, s \notin N$, can be defined as the space of all distributions with finite norm:

$$\|v\|^2_{H^s(\Omega)} := \|v\|^2_{H^m(\Omega)} + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha v(x) - D^\alpha v(y)|^2}{|x-y|^{d+2\sigma}} \, dx \, dy \quad , \tag{2.3.1}$$

where $s = m + \sigma, m \geq 0, \sigma \in (0,1)$ and D^α denotes the derivatives with respect to the multi-index $\alpha = (\alpha_1, .., \alpha_d)$ [30].
We define the interface $\Gamma := Cl(\bigcup_i \partial\Omega_i / \partial\Omega)$.

2.3.1 Notation in 2D

We can assume, that for subdomains Ω_i, Ω_m with $meas_{d-1}(\partial\Omega_i \cap \partial\Omega_m) > 0$ it yields $k_i \neq k_m$. Otherwise define a new subdomains by the union of Ω_i and Ω_m. Here $meas_d(\cdot)$ denotes the d-dimensional Lebesgue measure.
To discuss regularity we introduce so-called *singular points*, which will be subdivided into *homogeneous* singular points and *heterogeneous* singular points, depending on whether the diffusion k is constant in a small neighbourhood or not.

Definition 2.1 *A point $x \in \partial\Omega$ is a* homogeneous singular point *if in a neighbourhood of x the diffusion coefficient k is constant and one of the following two conditions holds:*

- *the interior angle of Ω at x is greater than π*
- *the boundary conditions change in x and the interior angle of Ω at x is greater than $\pi/2$.*

Definition 2.2 *A point on the interface $x \in \Gamma$ is a* heterogeneous singular point *if*

- *either x is an interior point $x \in \Gamma/\partial\Omega$ and in any neighbourhood of x the interface is not a straight line*
- *or x lies on the boundary $x \in \partial\Omega$.*

Definition 2.3 *If x is a homogeneous or a heterogeneous singular point we call x a singular point.*

Interior heterogeneous singular points are also called *crosspoints*. In Figure 2.1 several singular points are depicted.

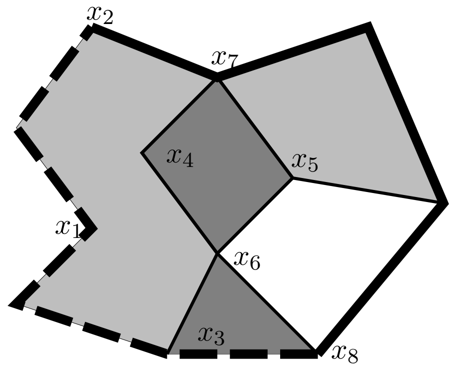

Figure 2.1: Subdomains are shaded with different levels of grey, Dirichlet and Neumann boundaries are shaded differently, $x_l, l = 1, 2$ are homogeneous singular points, $x_l, l > 2$ are heterogeneous singular points (not all singular points are depicted)

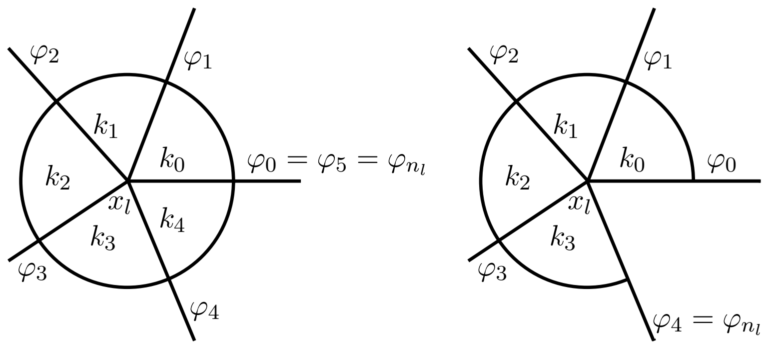

Figure 2.2: Subdomains $\Omega_{l,i}$ coincide with cones $C_{l,i}$ in a neighbourhood of an interior (left figure) and a boundary (right figure) heterogeneous singular point x_l

Let x_l be a heterogeneous singular point. We introduce polar coordinates (r, φ) with respect to x_l. We identify the unit sphere with the interval $[0, 2\pi)$. Similarly the interval $[\varphi_1, \varphi_2]$ denotes the cone containing all rays φ in between φ_1, φ_2, where positive orientation is assumed. For instance any two intervals $[\varphi_i, \varphi_j), [\varphi_j, \varphi_i)$ cover the sphere $[0, 2\pi)$ in a natural way.
Number the subdomains sharing the singular point x_l with $\Omega_{l,i}, i = 0, .., n_l - 1$, and choose a radius $r_l > 0$ such that $\Omega_{l,i} \cap B_{r_l}(x_l)$ coincides with a cone $C_{l,i}$. The cones $C_{l,i}$ are given by the rays φ_i and $\varphi_{i+1}, i = 0, .., n_l - 1$ where $\varphi_0 < \varphi_1 < .. < \varphi_{n_l-1}$. This notation is illustrated with help of Figure 2.2. If x_l is an interior point we see $\varphi_{n_l} = \varphi_0$. If not, the rays φ_0, φ_{n_l} coincide with a part of $\partial\Omega$. By the sequence $\varphi_0 < \varphi_1 < .. < \varphi_{n_l-1}$ we describe the *geometry* around the singular point x_l.

We denote by $k_{l,i}$ the value of k on $\Omega_{l,i} \cap B_{x_l}(r_l)$. Now let us define the *local diffusion coefficient* $k_{x_l}(\varphi)$ on the interval $[\varphi_0, \varphi_{n_l}]$ that takes the value $k_{l,i}$ on the interval $(\varphi_i, \varphi_{i+1}), i = 0, .., n_l - 1$. For simplification, we may drop the sub-indices l when choosing a singular point x_l.

The notation is valid also for homogeneous singular points. There $n_l = 1$.

2.3.2 Restriction to piecewise regularity

In section 2.2 we have shown that the solution of problem (2.2.1) belongs to the Sobolev space $H^1(\Omega)$.

We want to discuss regularity of the solution u of problem (2.2.1). First we observe, that the normal derivatives have a jump discontinuity across the interface. To see this, choose two adjacent subdomains Ω_i, Ω_j and let n_i, n_j be the outward normals to the interface. Note that $n_i = -n_j$.

Due to (2.2.1) the solution u fulfills

(2.3.2)
$$k_i \frac{\partial u}{\partial n_i}|_{\partial\Omega_i \cap \partial\Omega_j} = k_j \frac{\partial u}{\partial n_j}|_{\partial\Omega_i \cap \partial\Omega_j} \quad ,$$

where the equality holds in the distributional sense. Since $k_i \neq k_j$ the normal derivatives are discontinuous. Therefore, $u \notin H^{3/2}(\Omega)$.

In the following we restrict ourselves to piecewise regularity $u \in H^s(\Omega_i)$ for $i = 0, .., n - 1$. Piecewise regularity is important for instance in Finite Element applications.

The following simple lemma establishes a connection between piecewise and global regularity. Usually from regularity on subdomains $H^{1+\lambda}(\Omega_i), i = 1, 2$ does not follow regularity on the union of these subdomains $H^{1+\lambda}(\Omega_1 \cup \Omega_2)$. This may be true only for $0 \leq \lambda < 1/2$.

Lemma 2.1 *Let the polygonal (polyhedral) Lipschitz domain Ω be decomposed into disjoint polygonal (polyhedral) Lipschitz subdomains Ω_1, Ω_2. Let $0 \leq \lambda < 1/2$, $v \in H^{1+\lambda}(\Omega_i), i = 1, 2$ and $v \in H^1(\Omega)$. Then $v \in H^{1+\lambda}(\Omega)$.*

PROOF. The proof follows from Definition 1.2.4 and Theorem 1.2.16 of [30]. It suffices to prove that $\nabla v \in \left(H^\lambda(\Omega)\right)^2$. Denote by $v_{j,i} = \frac{\partial v}{\partial x_j}$ the partial derivatives of v in Ω_i. Since $v_{j,i} \in H^\lambda(\Omega_i), i = 1, 2$ and due to the implication given below [30, Thm 1.2.16] one can extend $v_{j,i}$ by zero to $v^+_{j,i} \in H^\lambda(\Omega)$. By Gauss' theorem one checks $\frac{\partial v}{\partial x_j} = v^+_{j,1} + v^+_{j,2}$ and hence $\frac{\partial v}{\partial x_j} \in H^\lambda(\Omega)$. ■

The maximum piecewise regularity one can expect under the condition $f \in L^2(\Omega)$ is $u \in H^2(\Omega_i)$. In general such regularity does not hold for solutions of problem (2.2.1).

2.4 Regularity in 2D

2.4.1 The Sturm-Liouville eigenvalue problem and regularity

Choose a singular point x. We regard the self-adjoint and positive definite Sturm-Liouville eigenvalue problem given by

$$-s(\varphi)'' = \lambda^2 s(\varphi) \ , \ \ \varphi \in (\varphi_i, \varphi_{i+1}) \ , \ \ i = 0, .., n-1 \ , \tag{2.4.1}$$

with the interface conditions on lines $\varphi = \varphi_i$ that coincide with a part of the interface

$$\begin{aligned} s(\varphi_i - 0) &= s(\varphi_i + 0) \\ k_{i-1} s(\varphi_i - 0)' &= k_i s(\varphi_i + 0)' \end{aligned} \tag{2.4.2}$$

and, in case $x \in \partial\Omega$, with the boundary conditions

$$\begin{aligned} &\text{either} \quad s(\varphi_0 + 0) = 0 \quad \text{or} \quad s(\varphi_0 + 0)' = 0 \\ &\text{either} \quad s(\varphi_n - 0) = 0 \quad \text{or} \quad s(\varphi_n - 0)' = 0 \quad . \end{aligned} \tag{2.4.3}$$

Here we denote by $s(\varphi_i - 0), s(\varphi_i + 0)$ the left resp. right hand side limit of the function s in the point φ_i.
If x is an interior singular point, the problem is posed in $W = H^1_{per}([0, 2\pi])$, the subspace of $H^1([0, 2\pi])$ with periodic boundary conditions. In the case of $x \in \partial\Omega$ define W as a subspace of $H^1([\varphi_0, \varphi_n])$ with appropriate homogeneous Dirichlet boundary conditions, depending on whether the line $\varphi = \varphi_0$ or $\varphi = \varphi_n$ coincides with a part of Γ_D or Γ_N.
We conclude that the eigenvalues are real and that the spectrum has no point of density from R. We denote by λ the positive square root of λ^2.
The above eigenvalue problem can be rewritten in a simpler form. The general solution of equation (2.4.1) on an interval $[\varphi_i, \varphi_{i+1}]$ has the form

$$e_i \cos(\lambda\varphi) + f_i \sin(\lambda\varphi) \ , \ \ e_i, f_i \in R \ ,$$

and can be written as

$$b_i \cos(\lambda(\varphi - c_i))$$

for some $b_i, c_i \in R$. We conclude from (2.4.1), that the Sturm-Liouville eigenvalue problem (2.4.1), (2.4.2), (2.4.3) is equivalent to the following problem. There are real numbers $b_i, c_i, i = 0, .., n-1$, such that

$$s(\varphi) = b_i \cos(\lambda(\varphi - c_i)) \text{ for } \ \varphi \in [\varphi_i, \varphi_{i+1}], i = 0, .., n-1 \quad .$$

The interface condition reads for i such that the angle $\varphi = \varphi_i$ coincides with a part of the interface

$$\begin{aligned} b_i \cos(\lambda(\varphi_{i+1} - c_i)) &= b_{i+1} \cos(\lambda(\varphi_{i+1} - c_{i+1})) \ , \\ k_i b_i \sin(\lambda(\varphi_{i+1} - c_i)) &= k_{i+1} b_{i+1} \sin(\lambda(\varphi_{i+1} - c_{i+1})) \quad . \end{aligned}$$

For singular points $x \in \partial\Omega$ the boundary conditions read

$$\begin{aligned} &\text{either} \quad b_0 \cos(\lambda(\varphi_0 - c_0)) = 0 \ \text{ or } \ - b_0 \sin(\lambda(\varphi_0 - c_0)) = 0 \\ &\text{either} \quad b_{n-1} \cos(\lambda(\varphi_n - c_{n-1})) = 0 \ \text{ or } \ - b_{n-1} \sin(\lambda(\varphi_n - c_{n-1})) = 0 \quad . \end{aligned}$$

2.4.2 A decomposition theorem

The next lemma establishes a connection between the above Sturm-Liouville eigenvalue problem and regularity.

Theorem 2.2 *For any singular point x_l denote by $\lambda^2_{l,j}, j = 1, .., m_l$ all eigenvalues from the interval $(0, 1]$ of the respective Sturm-Liouville eigenvalue problem (2.4.1), (2.4.2), (2.4.3) and suppose $\lambda^2_{l,j} \neq 1, j = 1, 2, .., m_l$. Denote with $s_{l,j}(\varphi)$ the according eigenfunctions.*
The solution u of (2.2.1) admits a decomposition

$$u = w + \sum_{x_l} \sum_{j=1}^{m_l} c_{l,j}\, \eta(r_l) \;\; r^{\lambda_{l,j}}\, s_{l,j}(\varphi) \;\; , \tag{2.4.4}$$

where $w \in H^2(\Omega_i), i = 0, .., n_i - 1$, and the sum is over all singular points x_l. Here $c_{l,j} \in R$ and $\eta(r_l)$ is a smooth cut-off function vanishing outside a neighbourhood of each singular point. We call a $r^{\lambda_{l,j}} s_{l,j}(\varphi)$ a singular function *for the point x_l.*

PROOF. The proof of the representation (2.4.4) follows from [33, Thm 1] and section 3 of [33] with $s = 0$. The representation is also given in [43, Thm 2.27] [45]. ■

We see that the regularity of u is restricted by the regularity of the singular functions $r^{\lambda_{l,j}} s_{l,j}(\varphi) \notin H^{1+\lambda_{l,j}}(\Omega_i), i = 0, .., n_l - 1$. Furthermore $r^{\lambda_{l,j}} s_{l,j}(\varphi) \in H^{1+\lambda_{l,j}-\varepsilon}(\Omega_i), i = 0, .., n_l - 1$ for any $\varepsilon > 0$. To show this one can use [30, Thm 1.2.18].
The probably first decomposition theorem for the case of a smooth diffusion coefficient k can be found in [35].

Corollary 2.1 *Let $\gamma \in (0, 1)$ be given and let $\lambda^2 > \gamma^2$ for all nonzero eigenvalues λ^2 of the Sturm-Liouville eigenvalue problem (2.4.1), (2.4.2), (2.4.3) for any singular point x. Then $u \in H^{1+\gamma}(\Omega_i), i = 0, .., n - 1$.*

PROOF. The corollary follows directly from Theorem 2.2 if all eigenvalues are $\lambda^2_{l,j}$ different from 1. If there is an eigenvalue $\lambda^2_{l,j} = 1$ then one can rely on [43]. Using the notation of [43, Cor 2.28] set $p_0 := 2/(2 - \gamma)$. As $p_0 < 2$ we see that $f \in L^2(\Omega) \subset L^{p_0}(\Omega)$. Further $\gamma = 2 - 2/p_0 < \lambda$ and the assumptions of [43, Cor 2.28] are fulfilled. We conclude that $u \in W^{2,p_0}(\Omega_i \cap U)$, where W^{2,p_0} is the Sobolev space of functions having all their derivatives (in distributional sense) up to order 2 integrable with the power of p_0. Use of the continuous embedding $W^{2,p_0}(\Omega_i \cap U) \subset H^{1+\gamma}(\Omega_i \cap U)$ [29, Thm 1.4.4.1] finishes the proof. ■

2.4.3 Known regularity results

In this subsection we want to briefly review known regularity results and point out some open questions.
We conclude from corollary 2.1 that regularity is a local property. In a neighbourhood U containing no singular point the regularity is $u \in H^2(U \cap \Omega_i)$ and in a neighbourhood of a singular point x the regularity depends on the local diffusion coefficient $k_x(\varphi)$, that

means on the geometry around the singular point and the values of k_i. Most of the known regularity results rely on decomposition theorems like Theorem 2.2.
A simple conclusion of Theorem 2.2 is the following lemma.

Lemma 2.3 *Let u be a solution of problem (2.2.1). Then the solution has regularity $u \in H^{1+\varepsilon(k)}(\Omega)$ where $\varepsilon(k) > 0$ depends on k.*

PROOF. The proof follows directly from Theorem 2.2. See also [33]. ■

The dependence of ε on k will be given in section 2.6. A similar result covering the case of more general subdomains can be found in [31].
We classify different geometrical situations for several singular points in Figure 2.1. For each of the depicted singular points x_i in Figure 2.1 we will now discuss regularity.

Regularity for homogeneous singular points

For homogeneous singular points (i.e. points x_1, x_2 in Figure 2.1) the following result is well known:

Lemma 2.4 *Let $k = 1$ and u be the solution of problem (2.2.1). Then for any neighbourhood of a singular point $x \in U_x$, such that U_x contains no other singular points the solution has regularity $u \in H^{1+1/2}(U_x \cap \Omega)$, if the boundary conditions do not change in x, and $u \in H^{1+1/4}(U_x \cap \Omega)$, if they do.*

PROOF. This is [30, Cor 2.4.4]. ■

The strongest singularity is of type $r^{1/4}\cos(\lambda/4)$ and occurs in a slit domain with mixed boundary conditions [30].

Regularity for heterogeneous singular points

Our concern is the regularity for heterogeneous singular points. To get more detailed results we choose a heterogeneous singular point x and classify the geometrical situations according to the number of subdomains neighbouring on this singular point. Let us denote by n the number of domains to whose boundary x belongs and by m the number of types of boundary conditions. That means that $m = 0$ if x is an interior point (points x_4, x_5, x_6 in Figure 2.1). We set $m = 1$ if $x \in \partial\Omega$ and the boundary conditions do not change in x (points x_3, x_7 from Figure 2.1). If they change, then $m = 2$ (point x_8).
The following results are known:

Lemma 2.5 *Let u be the solution of problem (2.2.1). Let x be a heterogeneous singular point with a neighbourhood U containing no other singular points.*
Then, if

$$n \leq 3 - m$$

u has regularity $u \in H^{1+1/4}(\Omega_i), i = 0, .., n-1$. If $n > 3 - m$ then $u \in H^1(\Omega_i)$.
If x is an interior singular point and $n = 2$ then $u \in H^{1+1/2}(\Omega_i), i = 0, 1$.
These regularity bounds are sharp in the respective class of problems where no restrictions on k and the geometry are made in the sense that the solution u does not belong to more regular Sobolev Spaces H^s.

PROOF. For the case of $n = 2, m = 0$ see [34] [33] or [56]. The case of $n = 3, m = 0$ and $n = 2, m = 1$ has been studied in [36].
Let us discuss the case $n = 4 - m$. For the case of $n = 4, m = 0$ Kellogg gives an explicit solution u_ε with regularity $u_\varepsilon \notin H^{1+\varepsilon}(\Omega)$ for any $\varepsilon > 0$ (see [34]). The solution u_ε is discussed in more detail in the following subsection. For the case $n = 3, m = 1$ and $n = 2, m = 2$ a problem can be constructed by restricting the domain of definition of u_ε. For the case $n > 4 - m$ one can define a function which will be a slight modification of the function u_ε.
All of the assertions have been shown recently also in [17]. ■

Related results are given in [61] [18] [39] and for the case of two Lipschitz subdomains where a different technique has been used in [51].

Examples of singular functions for interior heterogeneous singular points

We want to discuss in more detail the case of an interior heterogeneous singular point with two and with four adjacent subdomains.
Denote as before by (r, φ) the polar coordinates with respect to the singular point located at the origin.
In the first example the interface will be an angle [56]. See point x_4 of Figure 2.1 for an example. Let $\Omega = (-1, 1) \times (-1, 1)$ be decomposed to $\Omega_2 := \{(x, y) \in \Omega : 0 < \varphi(x, y) < \theta\}$, and $\Omega_1 := \Omega/\Omega_2$. The diffusion coefficient is piecewise constant:

$$k(x,y) := \begin{cases} 1, & \text{for} \quad (x,y) \in \Omega_1 \\ k_2, & \text{for} \quad (x,y) \in \Omega_2 \end{cases} .$$

If $k_2 > 1$, the singular function is given by

$$\psi_1 := r^\lambda \begin{cases} \cos(\lambda(\varphi - \theta/2)) & \text{for} \qquad (x,y) \in \Omega_2 \\ \beta \cos(\lambda(\pi - |\varphi - \theta/2|)) & \text{otherwise} \end{cases} , \tag{2.4.5}$$

where λ, β depend on k_2. If $k_2 \leq 1$ the function ψ_1 will be regular, i.e. $\psi_1|_{\Omega_i} \in H^2(\Omega_i)$ and the singular function is given by a different function.
A calculation shows that $\psi_1 \in H^{1+s}(\Omega_i)$ for $s < \lambda$ [30, Thm 1.2.18].
In the case of a interface angle $\theta = \pi/2$, the coefficients λ and β are given by:

$$\lambda = \frac{4}{\pi} \arctan\left(\sqrt{\frac{3 + k_2}{1 + 3k_2}}\right) \quad , \quad \beta = -k_2 \frac{\sin(\lambda\frac{\pi}{4})}{\sin(\lambda\frac{3\pi}{4})} \quad .$$

In the case $\theta \neq \pi/2$ they are given implicitly [39]. See [39] [44] [45] for numerically calculated λ.

Example 2.1 *Set $\theta = \pi/2$. Let $\Omega = (-1, 1) \times (-1, 1)$ and ψ_1 be defined as in (2.4.5). As above set $k_1 = 1$ and take k_2 as parameter.*

The singular function $\psi_1(r, \varphi)$ is illustrated in Figure 2.3 a) for $k_2 = 100$. Here $\lambda \approx 0.69$. According to the above formula for λ one sees that λ is monotonically decreasing with k_2 to the value $3/4$. Therefore, the solution of the interface problem belongs to $H^{1+3/4}(\Omega)$ independently of the jump discontinuity of the diffusion coefficient.

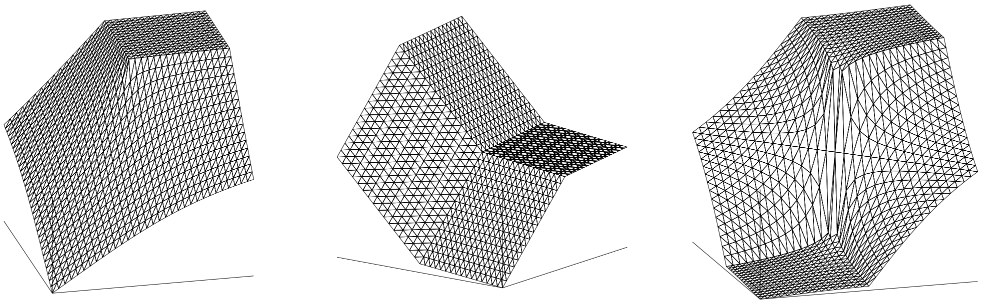

Figure 2.3: Different singular functions with $\delta = 0.1$, regularity depends on the geometry, a) $\lambda \approx 0.69$, b) $\lambda \approx 0.99$, c) $\lambda \approx 0.1$

Now we show an example where the best regularity result one could get independently of k is $u \in H^1(\Omega)$. Such situations can occur if in 2D there are more than three subdomains sharing an interior point (see point x_5 from Figure 2.1).
Suppose that the interface consists in the vicinity of an interior heterogeneous singular point of two intersecting lines. Set $\Omega = (-1,1) \times (-1,1)$ and define $\Omega_1 := \{(x,y) \in \Omega : 0 < \varphi < \theta \text{ or } \pi < \varphi < \theta + \pi\}$ and $\Omega_2 = \Omega/\Omega_1$.
Define $\psi_2(r,\varphi) = r^\lambda s_2(\varphi)$ where

$$s_2(\varphi) := \begin{cases} \cos(\lambda(\pi-\theta-c))\,\cos(\lambda(\varphi-\theta+b)) & \text{for } 0 \le \varphi \le \theta \\ \cos(\lambda b)\,\cos(\lambda(\varphi-\pi+c)) & \text{for } \theta \le \varphi \le \pi \\ \cos(\lambda c)\,\cos(\lambda(\varphi-\pi-b)) & \text{for } \pi \le \varphi \le \pi+\theta \\ \cos(\lambda(\theta-b))\,\cos(\lambda(\varphi-\theta-\pi-c)) & \text{for } \pi+\theta \le \varphi \le 2\pi \end{cases}, \tag{2.4.6}$$

see [34]. With given coefficients k_1, k_2, k_3, k_4 the parameters $\lambda > 0, b, c$ are chosen in such a way to satisfy the interface conditions (2.4.2). If $0 < \lambda < 1$ then ψ_2 defines a singular function.
Conversely with given parameters $0 < \lambda \le 1, b, c$ the coefficients k_1, k_2, k_3, k_4 can be defined (up to a multiplicative constant) by the interface conditions (2.4.2).

Example 2.2 *Take $\Omega = (-1,1) \times (-1,1)$ and $\psi_2 = r^\lambda s_2(\varphi)$, where s_2 is defined in (2.4.6). Set $\theta = \pi/2$, $b = 0.5\theta, c = \pi/2(1+\frac{1}{\lambda}) - b$ and vary λ as a parameter within $(0,1]$. Then $k_1 = k_3 = -\tan(\lambda c) = \tan(\lambda b)^{-1}$ and $k_2 = k_4 = \tan(\lambda b)$.*

The singular function $\psi_2(r,\varphi)$ is illustrated in Figure 2.3 c) for $\lambda = 0.1$. Here the ratio of the maximum and the minimum value of k is $k_{\max}/k_{\min} \approx 100$. This function has been defined in [17] too.
In the setting of example 2.2 we see $\lim_{\lambda\to 0} k_2/(\lambda\frac{\pi}{4}) = 1$ and $k_2 = k_1^{-1}$.
Now we want to demonstrate that in general a large ratio $k_{\max}/k_{\min}$ not necessarily implies low regularity. We can construct a singular function with smoothness $H^{2-\varepsilon}$ where the ratio $k_{\max}/k_{\min}$ increases to infinity as ε goes to 0:

Example 2.3 *Let $\varepsilon > 0$ be given. Take $\Omega = (-1,1) \times (-1,1)$ and $\psi = r^\lambda s_2(\varphi)$, where s_2 is defined in (2.4.6) . Choose $\varepsilon > 0$. Set $\lambda = 1-\varepsilon$, $\theta = \pi/2$, $b = \varepsilon$ and $c = \pi/2(1+\frac{1}{\lambda}) - b$.*

The coefficients in example 2.3 fulfill $k_2 < k_1 = k_3 < k_4$. An example of such a function is depicted in Figure 2.3 b) for $\lambda = 0.99$. Also in this case $k_{\max}/k_{\min} \approx 100$.

Remark 2.1 *Examples 2.2 and 2.3 show that the regularity parameter λ where $u \in H^{1+s}(\Omega_i)$ for any s fulfilling $0 < s < \lambda \leq 1$ may tend to a value $\lambda_0 = 0$ or to a value $\lambda_0 = 1$ if $\delta^{-1} \to \infty$.*

2.4.4 Open questions

An open question is whether there are conditions on k such that regularity H^s for some $s > 1$ is guaranteed and s does not depend on the bounds of k or the geometry. Lemma 2.5 implies the necessity of certain conditions on k in order to guarantee regularity $H^s(\Omega)$ for an $s > 1$ independent of the bounds of k. Such conditions will be introduced in the next section 2.3.
A further question is about the lower bound of ε from Lemma 2.3 in terms of the global bounds of k. We will give an answer to that question in section 2.6.

2.5 The quasi-monotone case

2.5.1 The quasi-monotonicity condition

We define the quasi-monotonicity condition for the diffusion coefficient k. This condition has been introduced in [24] in the context of Finite Elements. Remember that we assumed that $k_{l,i} \neq k_{l,i+1}$.
Roughly speaking the quasi-monotonicity condition means that the local diffusion coefficient $k_{x_l}(\varphi)$ has only one local maximum. Since $k_{x_l}(\varphi)$ is a function being piecewise constant on intervals $(\varphi_i, \varphi_{i+1})$, it has infinitely many maxima. But we agree to identify all maxima lying in the same interval $(\varphi_i, \varphi_{i+1})$. If $x_l \in \bar{\Gamma}_D$, we demand alternatively that each maximum touches the Dirichlet boundary Γ_D.

Definition 2.4 *Let a heterogeneous singular point x be given. The distribution of the coefficients $k_i, i = 0, .., n-1$ will be called quasi-monotone with respect to the singular point x, if the following conditions are fulfilled:*
Denote by N_i the indices of cones C_j that are neighbours of the cone C_i that is $N_i := \{j : meas_1(\bar{C}_j \cap \bar{C}_i) > 0, j \neq i\}$. The following condition holds

- *if $x \in \bar{\Omega}/\bar{\Gamma}_D$, there is only one index i such that $k_i > \max_{j \in N_i} \{k_j\}$*
- *if $x \in \bar{\Gamma}_D$, for each index i such that $k_i > \max_{j \in N_i} \{k_j\}$ the measure $meas_1(\bar{C}_i \cap \Gamma_D \cap B_x(r))$ is positive.*

Definition 2.5 *The diffusion coefficient k is quasi-monotone if for all singular points x the distribution of coefficients $k_i, i = 0, .., n-1$ is quasi-monotone.*

Remark 2.2 *The quasi-monotonicity of k is necessary for $H^s(\Omega_i)$-regularity of solutions of problem (2.2.1), where $s > 1$ is independent of k. In the case of an interior heterogeneous singular point this follows from the local diffusion coefficient defined in example 2.2 that has two local maxima and is not quasi-monotone. In order define a local diffusion coefficient which*

has any given number (greater than 2) of local maxima slightly perturb the diffusion coefficient given example 2.2 by enlarging it in parts of the domain, where it takes the lower value. This will change the singular function from example 2.2 and its low regularity only a little.
In case of a heterogeneous singular point on the boundary a singular function can be constructed by restricting the domain of definition of the singular function defined in example 2.2.

We give conditions for the quasi-monotonicity conditions to hold without restrictions on k but with restrictions on the maximum number of subdomains that share singular points.

Remark 2.3 *Choose a heterogeneous singular point x and denote as in section 2.4.3 by n the number of subdomains Ω_i to whose boundary x belongs and by m the number of types of boundary conditions. Then if*

$$n \leq 3 - m \quad ,$$

then for any values of $k_i, i = 0, .., n-1$ the distribution of the coefficients $k_i, i = 0, .., n-1$ is quasi-monotone with respect to x.
Observe that for exactly these restrictions on the maximum number of subdomains regularity results with piecewise regularity $H^s, s > 1$, with s independent of δ, are known (Lemma 2.5).

Thus the distribution of the coefficients $k_{l,i}, i = 0, .., n_l - 1$ is always quasi-monotone for points x_1, x_2, x_3, x_4, x_5 from Figure 2.1. For points x_6, x_7, x_8 from Figure 2.1 quasi-monotonicity depends on k. For instance coefficients $k_{6,0} = k_{6,2} = 1$ and $k_{6,1} = k_{6,3} = 100$ are not quasi-monotonically distributed with respect to the singular point x_6.

2.5.2 Quasi-monotonicity bounds eigenvalues from below

In this section we show that if the diffusion coefficient k is quasi-monotone, the eigenvalues of the Sturm-Liouville eigenvalue problem are bounded from below. We precede the proof of this fact by two technical lemmas.

Lemma 2.6 *Let functions $t_i(\varphi) = b_i \cos(\varphi - c_i), i = 1, 2$, be given that fulfill conditions*

$$\begin{aligned} t_1(\varphi_1) &= t_2(\varphi_1) \text{ and} \\ k_1\, t_1'(\varphi_1) &= k_2\, t_2'(\varphi_1) \quad , \end{aligned} \tag{2.5.1}$$

for some $\varphi_1, k_i > 0, b_i > 0, i = 1, 2$.
If one of the following conditions is fulfilled

a) $t_1'(\varphi_1) < t_2'(\varphi_1)$

b) $k_1 < k_2$ and ($t_1'(\varphi_1) < 0$ or $t_2'(\varphi_1) < 0$)

then $t_1(\varphi) \leq t_2(\varphi), \varphi_1 \leq \varphi \leq \varphi_1 + \pi$ and $t_2(\varphi) \leq t_1(\varphi), \varphi_1 - \pi \leq \varphi \leq \varphi_1$.

PROOF. Observe that $t_2 - t_1 = b_3 \cos(\varphi - c_3)$ for some b_3, c_3. It is not hard to see that $c_3 \in \{\varphi_1 - \pi/2, \varphi_1 + \pi/2\}$ and we choose $c_3 = \varphi_1 - \pi/2$. Then $b_3 = (t_2 - t_1)'(\varphi_1)$ and it remains to show $0 < b_3 = (t_2 - t_1)'(\varphi_1)$.

a) If $t_1'(\varphi_1) < t_2'(\varphi_1)$ this is obviously true.
b) In this case we conclude from equation (2.5.1)

$$\frac{t_1'(\varphi_1)}{t_2'(\varphi_1)} = \frac{k_2}{k_1} > 1 \quad .$$

and that $t_1'(\varphi_1) < 0$ and $t_2'(\varphi_1) < 0$. This shows $t_1'(\varphi_1) < t_2'(\varphi_1)$. ■

Lemma 2.7 *Let numbers* $0 = \varphi_0 < \varphi_1 < ... < \varphi_n < \pi/2$ *and* $k_i, i = 0,..,n-1$ *with* $0 < k_0 \le k_1 \le ... \le k_{n-1}$ *be given. Denote by* $\chi_{[\varphi_i,\varphi_{i+1})}$ *the characteristic function of the interval* $[\varphi_i, \varphi_{i+1})$*. Further let numbers* $c_i, b_i, i = 0, ..., n-1$ *be given such that the function*

$$s(\varphi) = \sum_{i=0}^{n-1} b_i \cos(\varphi - c_i)\, \chi_{[\varphi_i,\varphi_{i+1})} \quad , \tag{2.5.2}$$

is continuous and its derivatives weighted with k_i *are also continuous:*

$$b_i \cos(\varphi_{i+1} - c_i) \;= b_{i+1} \cos(\varphi_{i+1} - c_{i+1}) \;\;,\;\; i = 0,..,n-2 \;\;, \tag{2.5.3}$$

$$k_i b_i \sin(\varphi_{i+1} - c_i) \;= k_{i+1} b_{i+1} \sin(\varphi_{i+1} - c_{i+1}) \;\;,\;\; i = 0,..,n-2 \quad . \tag{2.5.4}$$

Assume $c_0 = 0$ *and* $b_0 > 0$*. Then* $s(\varphi) > 0$ *for all* $\varphi \in [0, \varphi_n]$.

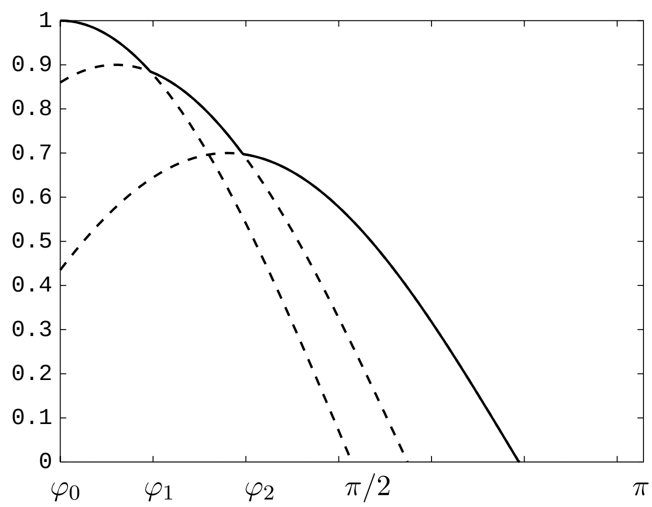

Figure 2.4: Function s from equation (2.5.2) is the upper envelope and depicted with a continuous line in case of decreasing k_i, functions t_i are indicated by a dashed line

PROOF. Define auxiliary functions $t_i(\varphi) := b_i \cos(\varphi - c_i)$. These functions are illustrated in Figure 2.4. The idea is that if $k_{j+1} > k_j$ function t_{j+1} will dominate the function t_j on a interval of length π starting from the point where t_{j+1} and t_j intersect.
Multiplying the function $s(\varphi)$ by a constant we can assure $b_0 = 1$. We want to prove

$$\begin{aligned} &0 < \cos(\varphi) = t_0(\varphi) \le .. \le t_j(\varphi) \quad , \quad \varphi_j \le \varphi \le \varphi_n < \pi/2 \\ &t_j'(\varphi_j) \le 0 \end{aligned} \tag{2.5.5}$$

with help of Lemma 2.6 by induction over $j = 0,..,n-1$.
For $j = 0$ inequality (2.5.5) is clearly fulfilled.
Suppose $i > 0$ and inequality (2.5.5) is fulfilled for $j = i-1$. Observe that $t'_{i-1}(\varphi_{i-1}) \leq 0$ and $t_{i-1}(\varphi) > 0$ for $\varphi_{i-1} \leq \varphi \leq \varphi_i$ implies $t'_{i-1}(\varphi_i) < 0$. Condition (2.5.4) then gives $t'_i(\varphi_i) < 0$.
Setting in terms of Lemma 2.6 $t_1 = t_{i-1}, t_2 = t_i$ and $\varphi_1 = \varphi_i$ we see that assumption b) from Lemma 2.6 is fulfilled and obtain $t_{i-1} \leq t_i$ for $\varphi \in [\varphi_i, \varphi_i + \pi]$. From assumption $0 \leq \varphi_i < \pi/2$ we see $[\varphi_i, \varphi_n] \subset [\varphi_i, \varphi_i + \pi]$. This together with $t'_i(\varphi_i) < 0$ finishes the proof of the induction step (2.5.5) for $j = i$. ■

Remark 2.4 *Lemma 2.7 could be sharpened to hold also for $\varphi_n \leq \pi/2$ if $n > 1$. To show this use $k_0 < k_1$ and show that $0 < c_1$.*

Theorem 2.8 *Let an interior heterogeneous singular point $x \in \Omega$ be given and let the distribution of the coefficients $k_i, i = 0,..,n-1$ be quasi-monotone with respect to x. Then the smallest non-vanishing eigenvalue of the associated Sturm-Liouville eigenvalue problem (2.4.1), (2.4.2), (2.4.3) is greater than $(1/4)^2$. This bound is sharp.*

PROOF. We choose an eigenfunction of the associated Sturm-Liouville eigenvalue problem with eigenvalue λ^2. The eigenfunction has the representation

$$s(\varphi) = \sum_{i=0}^{n-1} b_i \cos(\lambda(\varphi - c_i))\ \chi_{[\varphi_i, \varphi_{i+1})} \quad , \tag{2.5.6}$$

where $b_i, c_i \in [0, 2\pi), i = 0,..,n-1$ are real numbers. The eigenfunction $s(\varphi)$ fulfills the interface conditions

$$b_i \cos(\lambda(\varphi_{i+1} - c_i)) = b_{i+1} \cos(\lambda(\varphi_{i+1} - c_{i+1})) \quad , \tag{2.5.7}$$

$$k_i b_i \sin(\lambda(\varphi_{i+1} - c_i)) = k_{i+1} b_{i+1} \sin(\lambda(\varphi_{i+1} - c_{i+1})) \quad . \tag{2.5.8}$$

The idea of the proof is to show that there is an index $j \in 0,..,n-1$ such that $s(\varphi) \geq b_j \cos(\lambda(\varphi - c_j)) > 0$ on the interval $[c_j, c_j + \pi/(2\lambda))$. Here we need the quasi-monotonicity condition. Since $s(\varphi)$ vanishes in some points the length of the interval $[c_j, c_j + \pi/(2\lambda))$ is bounded by 2π. This yields the bound $\lambda > 1/4$.
Let us have a closer look onto $s(\varphi)$. This periodic function is continuous and therefore achieves a minimum at a point $\varphi_{\min}$ and a maximum at $\varphi_{\max}$.
Choose j such that $\varphi_{\max} \in [\varphi_j, \varphi_{j+1})$. Possibly substituting c_j with $c_j + \pi/\lambda$ we can assume $b_j \geq 0$. The case $b_j = 0$ can be excluded since then the interface condition imply $s \equiv 0$ and hence $\lambda = 0$. If $\varphi_{\max}$ lies $(\varphi_j, \varphi_{j+1})$ we conclude from $b_j > 0$ that $\varphi_{\max} = 2l\pi/\lambda + c_j$ for a number $l \in N$. Possibly redefining c_j we may set $\varphi_{\max} = c_j$ and we see $s(\varphi_{\max}) > 0$.
If $\varphi_{\max} = \varphi_j$ proceed as follows. Since $\varphi_{\max}$ is a maximum it is clear that $s(\varphi_j - 0)' \geq 0$ and $s(\varphi_j + 0)' \leq 0$. Condition (2.5.8) implies on the other hand that $s(\varphi_j - 0)'$ and $s(\varphi_j + 0)'$ cannot have different signs. Hence $s(\varphi_j - 0)' = s(\varphi_j + 0)' = 0$ and in this case too it holds $c_j = \varphi_{\max}$. From this follows $s(\varphi_{\max}) > 0$.

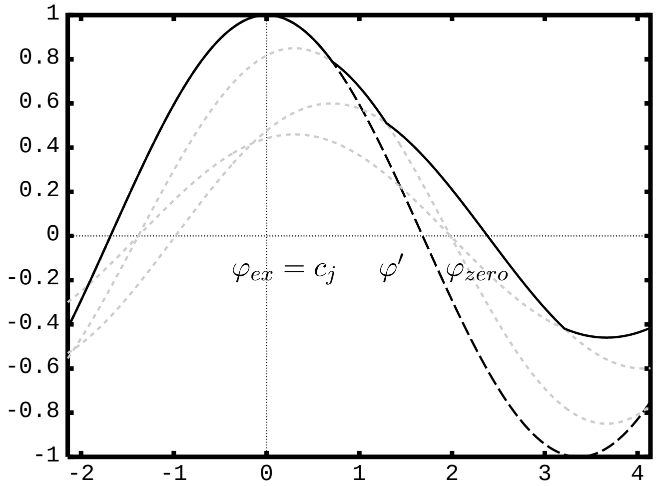

Figure 2.5: eigenfunction $s_\lambda(\varphi)$ (black line) for quasi-monotonically distributed k and a function $b_j \cos(\lambda(\varphi - c_j))$ (black dashed line); both functions coincide at the maximum φ_{ex} of $s_\lambda(\varphi)$; for points $\varphi_{zero}, \varphi' := c_j + \pi/(2\lambda)$ holds $s_\lambda(\varphi_{zero}) = 0,\ b_j \cos(\lambda(\varphi' - c_j)) = 0$; since k increases on $[\varphi_{ex}, \varphi_{zero}]$ it holds $\varphi' < \varphi_{zero}$ and we see $c_j < \varphi' < \varphi_{zero} < c_j + 2\pi$

Similarly $s(\varphi_{\min}) < 0$ and we conclude that there are at least two points $\varphi_{zero,1}$ and $\varphi_{zero,2}$ with $s(\varphi_{zero,1}) = s(\varphi_{zero,2}) = 0$. Without loss of generality we assume

$$\varphi_{zero,1} < \varphi_{\max} < \varphi_{zero,2} < \varphi_{\min} \quad .$$

Now we exploit the quasi-monotonicity condition. We want to show that the following property **P** holds:

There is a extremum φ_{ex} *from* $\{\varphi_{\min}, \varphi_{\max}\}$ *and a point* φ_{zero} *from* $\{\varphi_{zero,1}, \varphi_{zero,2}\}$ *such that* $k_x(\varphi)$ *does not decrease when going from* φ_{ex} *to* φ_{zero}. This means we want to show that $k_x(\varphi)$ is increasing on $[\varphi_{ex}, \varphi_{zero}]$ or decreasing on $[\varphi_{zero}, \varphi_{ex}]$.

To do so denote by $[\varphi_{i_{\min}}, \varphi_{i_{\min}+1}), [\varphi_{i_{\max}}, \varphi_{i_{\max}+1})$ the intervals where $k_x(\varphi)$ reaches the minimum and maximum. The sphere $[0, 2\pi)$ is then decomposed into two intervals $I_{decr} := [\varphi_{i_{\max}}, \varphi_{i_{\min}})$ and $I_{incr} := [\varphi_{i_{\min}}, \varphi_{i_{\max}})$ on which $k_x(\varphi)$ is monotone as depicted in Figure 2.6.

There are two possibilities: a) either there are three points from $\{\varphi_{zero,1}, \varphi_{\max}, \varphi_{zero,2}, \varphi_{\min}\}$ contained in I_{decr} or I_{incr} or b) there are two points from $\{\varphi_{zero,1}, \varphi_{\max}, \varphi_{zero,2}, \varphi_{\min}\}$ contained in I_{decr} and I_{incr}. In Figure 2.6 a possible distribution of the points $\varphi_{zero,1}, \varphi_{\max}, \varphi_{zero,2}, \varphi_{\min}$ in the intervals I_{decr} and I_{incr} in case a) is shown. One notices that in the depicted distribution the diffusion is decreasing on $[\varphi_{zero,1}, \varphi_{\max}]$ and property **P** is fulfilled. For the other (essentially three) possible distributions of points $\varphi_{zero,1}, \varphi_{\max}, \varphi_{zero,2}, \varphi_{\min}$ it is easy to check that property **P** holds too.

In case b) points from $\{\varphi_{zero,1}, \varphi_{\max}, \varphi_{zero,2}, \varphi_{\min}\}$ could be distributed like $\varphi_{zero,1}, \varphi_{\max} \in I_{decr}$ and $\varphi_{zero,2}, \varphi_{\min} \in I_{incr}$. In this special case the function $k_x(\varphi)$ is decreasing on $[\varphi_{zero,1}, \varphi_{\max}]$ and property **P** is fulfilled. Other distributions of points in case b) are checked in the same way to satisfy property **P**.

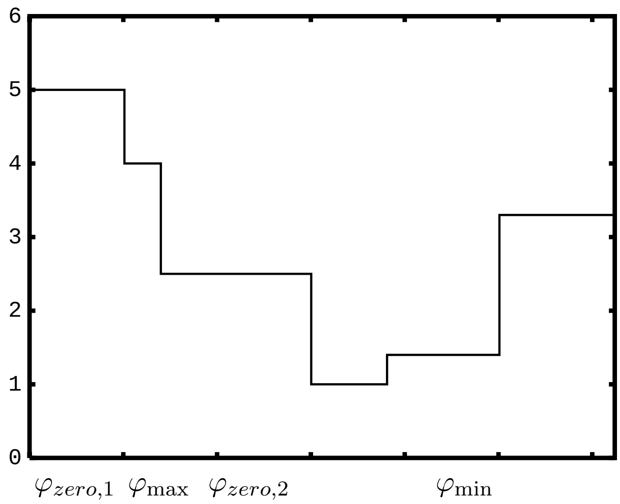

Figure 2.6: local diffusion coefficient $k_x(\varphi)$ is piecewise monotone on two intervals covering the sphere $[0, 2\pi)$; possible location of points $\varphi_{zero,1}$, $\varphi_{\max}$, $\varphi_{zero,2}$, $\varphi_{\min}$; $k_x(\varphi)$ decreases on $[\varphi_{zero,1}, \varphi_{\max}]$

Multiplying by -1 in (2.5.6), rotating the polar coordinate system and possibly reflecting it on the x-axis we can assure

(2.5.9) $$0 = \varphi_{ex} = \varphi_{\max} < \varphi_{zero} < 2\pi \quad .$$

Remember that $k_x(\varphi)$ increases on $[\varphi_{ex}, \varphi_{zero}]$. If the function $s_\lambda(\varphi)$ vanishes on $[\varphi_{\max}, \varphi_{zero})$ in some point(s), choose φ'_{zero} to be the minimum of these points and re-define $\varphi_{zero} := \varphi'_{zero}$.
Choose j such that $\varphi_{ex} \in [\varphi_j, \varphi_{j+1})$. The function $b_j \cos(\lambda(\varphi - c_i))$ is depicted in Figure 2.5 with a dashed line. We show as before $c_j = \varphi_{ex} = 0$. Further since φ_{ex} is a maximum $b_j > 0$. Renumbering the points φ_i we may assume $j = 0$.
Now we are nearly done with the proof. In the last step we restrict the function s_λ to the interval $[\varphi_{ex}, \varphi_{zero}]$ and apply a homogeneous scaling to transform functions

$$b_i \cos(\lambda(\varphi - c_i)) \text{ to functions } \widehat{b}_i \cos(\widehat{\varphi} - \widehat{c}_i)$$

which satisfy similar interface conditions and apply Lemma 2.7 to the transformed functions.
Choose the largest m such that $\varphi_{m-1} < \varphi_{zero}$. We introduce an homogeneous transformation

(2.5.10) $$F : [0 = \varphi_{ex}, \varphi_{zero}] \to [0, \lambda\varphi_{zero}] \text{ with } F(\varphi) := \lambda\varphi$$

and define $s_F(F(\varphi)) = s(\varphi), \varphi \in [\varphi_{ex}, \varphi_{zero}]$.
Under this transformation we define a sequence $\widehat{\varphi}_0 < \widehat{\varphi}_1 < \ldots < \widehat{\varphi}_m$ where $\widehat{\varphi}_0 := F(\varphi_{ex}) = 0$, $\widehat{\varphi}_i := F(\varphi_i), 0 < i < m - 1$ and $\widehat{\varphi}_m := F(\varphi_{zero})$.
It follows that s_F fulfills

$$s_F(\varphi) = \sum_{i=0}^{n-1} \widehat{b}_i \ \cos(\varphi - \widehat{c}_i) \ \chi_{[\widehat{\varphi}_i, \widehat{\varphi}_{i+1})} \quad ,$$

and

$$\begin{aligned} \widehat{b}_i \cos(\widehat{\varphi}_{i+1} - \widehat{c}_i) &= \widehat{b}_{i+1} \cos(\widehat{\varphi}_{i+1} - \widehat{c}_{i+1}) \\ \widehat{k}_i \widehat{b}_i \sin(\widehat{\varphi}_{i+1} - \widehat{c}_i) &= \widehat{k}_{i+1} \widehat{b}_{i+1} \sin(\widehat{\varphi}_{i+1} - \widehat{c}_{i+1}) \ , \end{aligned}$$

for $\widehat{c}_i = \lambda\, c_i, \widehat{b}_i = b_i, \widehat{k}_i = k_i$ with $i = 0, .., m-1$. Due to the choice of $\varphi_{ex}, \varphi_{zero}$ we have $\widehat{c}_0 = 0$ and $s_F(\widehat{\varphi}_m) = 0$ with $\widehat{\varphi}_m = \lambda \varphi_{zero} < \lambda 2\pi$. Further $\widehat{k}_i \le \widehat{k}_{i+1}, i = 0, .., m-1$.
Suppose $\lambda \le 1/4$. Thus $\widehat{\varphi}_m < \lambda 2\pi \le \pi/2$ and s_F defined on $[\widehat{\varphi}_0, \widehat{\varphi}_m]$ with the sequence $0 < \widehat{\varphi}_1 < ... < \widehat{\varphi}_m < \pi/2$ fulfills the assumption of Lemma 2.7. We conclude from Lemma 2.7 that s_F does not vanish on $[0, \widehat{\varphi}_m]$. But this is a contradiction with $s_F(\widehat{\varphi}_m) = 0$ and hence $1/4 < \lambda$.

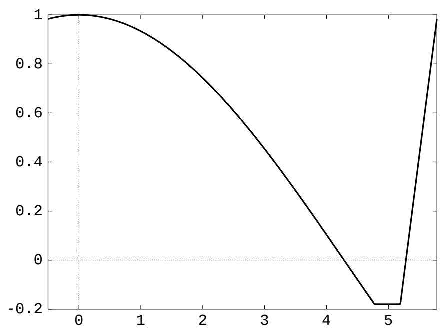

Figure 2.7: eigenfunction $s_3(\varphi)$ for $\varepsilon = 0.5$ defined piecewise by functions $b_i \cos(\lambda(\varphi - c_i))$, $i = 0, 1, 2$; here λ is close to $1/4$

From the above proof it is not hard to see how to construct an eigenfunction $s_3(\varphi)$ with eigenvalue λ^2 arbitrarily close to $(1/4)^2$. We see in the following example that $\lambda \to 1/4$ when the interior angle of a subdomain tends to 2π.
Choose $\varepsilon > 0$ and $n = 3$. The idea is to construct an eigenfunction $s_3(\varphi)$ as depicted in Figure 2.5.2. The interval (φ_0, φ_1) will have length of order $2\pi - O(\varepsilon)$ and $k_0 = 1$. The other two intervals will have length $O(\varepsilon)$ and $k_1 = O(\varepsilon^{-1})$, $k_2 = O(\varepsilon)$. The constructed eigenfunction will have the eigenvalue λ^2 where $\lambda := \frac{\pi/2}{2\pi - 4\varepsilon} \to 1/4$ as $\varepsilon \to 0$.
For the interested reader we will give the details below: Define $\varphi_0 = -\varepsilon$, $\varphi_1 = 2\pi - 3\varepsilon$. The remaining parameter φ_2 will be defined below. The aim is define a function that achieves a maximum at $\varphi = 0$ and vanishes at $\varphi = 2\pi - 4\varepsilon$ and $\varphi = 2\pi - 2\varepsilon$. Furthermore a minimum is attained in $\varphi \in (\varphi_1, \varphi_2)$.
To do so set $c_0 = 0, b_0 = 1$. Define c_2, b_2 in such a way that $\cos(\lambda(\varphi - c_2))$ vanishes in $\varphi = 2\pi - 2\varepsilon$ and that $b_2 \cos(\lambda(\varphi - c_2)) = \cos(\lambda\varphi)$ for $\varphi = \varphi_0$.
Further choose $\varphi_2 > \varphi_1$ such that $\cos(\lambda\,(\varphi_2 - c_2)) = \cos(\lambda\,(\varphi_1 - c_0))$. Set $c_1 = 0.5\,(\varphi_1 + \varphi_2)$ and $b_1 = -\cos(\lambda(\varphi_1 - c_0))/\cos(\lambda(\varphi_1 - c_1))$.
The definition of the eigenfunction s_3 is finished by setting $k_0 = 1$ and choosing k_1, k_2 in such a way that interface conditions for the derivatives (2.5.8) are fulfilled. ■

Theorem 2.9 *Let a heterogeneous singular point $x \in \partial\Omega$ on the boundary be given and let the distribution of the coefficients $k_i, i = 0, .., n-1$ be quasi-monotone with respect to x.*
Then the smallest non-vanishing eigenvalue λ^2 of the associated Sturm-Liouville eigenvalue problem (2.4.1), (2.4.2), (2.4.3) fulfills $(\frac{1}{4})^2 < \lambda^2$. This bound is sharp.

PROOF. The proof runs similar to that of Theorem 2.8. The eigenfunction of the associated Sturm-Liouville eigenvalue problem with eigenvalue λ^2 has the representation

$$s(\varphi) = \sum_{i=0}^{n-1} b_i \cos(\lambda(\varphi - c_i))\ \chi_{[\varphi_i,\varphi_{i+1})} \quad ,$$

where $\chi_{[\varphi_i,\varphi_{i+1})}$ denotes the characteristic function of the interval $[\varphi_i, \varphi_{i+1})$ and $b_i, c_i, i = 0, .., n-1$ are real numbers. The eigenfunction $s(\varphi)$ has to fulfill the interface conditions for $i = 0, .., n-2$

$$b_i \cos(\lambda(\varphi_{i+1} - c_i)) \ = b_{i+1} \cos(\lambda(\varphi_{i+1} - c_{i+1})) \tag{2.5.11}$$

$$k_i b_i \sin(\lambda(\varphi_{i+1} - c_i)) \ = k_{i+1} b_{i+1} \sin(\lambda(\varphi_{i+1} - c_{i+1})) \tag{2.5.12}$$

and some boundary conditions that will be specified later.
Since we deal with two different boundary conditions there are three possibilities how to combine them. We will treat each case separately. In any case $s(\varphi)$ is not a constant function. Denote by F_1, F_2 parts of the boundary on both sides of $x \in \partial\Omega \cap B_x(r)$.
Case I. $F_1 \subset \Gamma_D, F_2 \subset \Gamma_D$
We deduce that there exists a local extremum φ_{ex} of the function $s(\varphi)$ and multiplying $s(\varphi)$ by -1 we may assume that φ_{ex} is maximum. We choose j such that $\varphi_{ex} \in [\varphi_j, \varphi_{j+1})$ and show as in the proof of Theorem 2.8 that $c_j = \varphi_{ex}$. The quasi-monotonicity condition implies now that $k_x(\varphi)$ is monotonically increasing on $[\varphi_{ex}, \varphi_n]$ or decreasing on $[\varphi_0, \varphi_{ex}]$. We may suppose without loss of generality that $k_x(\varphi)$ is increasing on $[\varphi_{ex}, \varphi_n]$ and since $s(\varphi_n) = 0$ we may define $\varphi_{zero} := \varphi_n$.
By rotation of the coordinate system we can assume

$$0 = \varphi_{ex} < \varphi_{zero} \le \theta < 2\pi \quad , \tag{2.5.13}$$

where θ is the interior angle of Ω at x. Possibly redefining φ_{zero} we can assure $s(\varphi) > 0$ for $\varphi \in [\varphi_{ex}, \varphi_{zero}]$.
Choose m such that $\varphi_{j+m-1} < \varphi_{zero}$.
We transform the sequence $\varphi_{ex} < \varphi_{j+1} < ... < \varphi_{j+m-1} < \varphi_{zero} < \theta$ with the affine transformation defined in (2.5.10) and obtain a new sequence $\widehat{\varphi}_0 < \widehat{\varphi}_1 < ... < \widehat{\varphi}_m$ where $\widehat{\varphi}_0 = 0, \widehat{\varphi}_i = F(\varphi_{i+j}) = \lambda\varphi_{i+j}, 1 < i < m-1$ and $\widehat{\varphi}_m = F(\varphi_{zero}) = \lambda\varphi_{zero}$.
Suppose $\lambda \le \frac{1}{4}$. Defining $s_F(F(\varphi)) := s(\varphi)$ we obtain a scaled function which fulfills the modified conditions (2.5.11) and (2.5.12) that is

$$\begin{aligned} \widehat{b}_i \cos(\widehat{\varphi}_{i+1} - \widehat{c}_i) \ &= \widehat{b}_{i+1} \cos(\widehat{\varphi}_{i+1} - \widehat{c}_{i+1}) \\ \widehat{k}_i \widehat{b}_i \sin(\widehat{\varphi}_{i+1} - \widehat{c}_i) \ &= \widehat{k}_{i+1} \widehat{b}_{i+1} \sin(\widehat{\varphi}_{i+1} - \widehat{c}_{i+1}) \quad , \end{aligned}$$

for $\widehat{c}_i = \lambda\, c_i, \widehat{b}_i = b_{i+j}$ and $\widehat{k}_i = k_{i+j}$ with $i = 0, .., m-1$. Further $\widehat{c}_0 = 0$, $s_F(0) > 0$ and $s_F(\widehat{\varphi}_m) = 0$ with $\widehat{\varphi}_m \le \lambda\theta \le \frac{1}{4}\theta < \pi/2$ and $\widehat{k}_i \le \widehat{k}_{i+1}, i = 0, .., m-1$.

Hence s_F fulfills the assumption of Lemma 2.7 and it follows that s_F does not vanish on $[0, \widehat{\varphi}_m]$. But this is a contradiction since s_F vanishes at $\widehat{\varphi}_m$.
Case II. $F_1 \subset \Gamma_N, F_2 \subset \Gamma_D$
Suppose that the Dirichlet conditions are set on the angle φ_n. Define $\varphi_{ex} = \varphi_0$ and $\varphi_{zero} = \varphi_n$. The quasi-monotonicity condition implies that the local diffusion coefficient $k_x(\varphi)$ has not more than one local maximum $[\varphi_i, \varphi_{i+1}]$ and this local maximum is achieved for $i = n-1$. Hence $k_x(\varphi)$ is monotone increasing on $[\varphi_{ex}, \varphi_{zero}]$.
It follows from $k\frac{\partial u}{\partial n} = 0$ at $\varphi = \varphi_{ex}$ that $c_j = \varphi_{ex}$. Using remark 2.4 we show as in the case I that $\frac{1}{4} < \lambda$.
Case III. $F_1 \subset \Gamma_N, F_2 \subset \Gamma_N$
Set $\varphi_{ex,1} = \varphi_0$ and $\varphi_{ex,2} = \varphi_n$. As in case II we conclude $c_0 = 0$ and $c_{n-1} = \varphi_n$. Denote by φ_{zero} a point where $s(\varphi)$ vanishes. The quasi-monotonicity condition implies that the local diffusion coefficient $k_x(\varphi)$ has not more than one local maximum $[\varphi_j, \varphi_{j+1}]$.
Using the quasi-monotonicity property we show that there is an number $\varphi_{ex} \in \{\varphi_{ex,1}, \varphi_{ex,2}\}$ such that $k_x(\varphi)$ increases monotonically, when going from φ_{ex} to φ_{zero}. If $\varphi_{zero} < \varphi_j$ then $k_x(\varphi)$ is monotonically increasing on $[\varphi_0, \varphi_{zero}]$ and $\varphi_{ex} := \varphi_0$. Otherwise $k_x(\varphi)$ is monotone decreasing on $[\varphi_{zero}, \varphi_n]$ and $\varphi_{ex} := \varphi_n$. We may suppose that the first case holds. The remainder of the proof is similar to case II and we show $\lambda > \frac{1}{4}$.
To prove sharpness we use the example from the proof of Theorem 2.8. Denote by I the closure of the support $\max\{0, s_3(\varphi)\}$. We define the eigenfunction $s_4(\varphi) := s_3(\varphi)$ on I. This eigenfunction has the eigenvalue $\lambda = \frac{\pi/2}{2\pi - 4\varepsilon}$. ■

Remark 2.5 *Denote by θ the interior angle of Ω at $x \in \partial\Omega$. Under assumptions of Theorem 2.9 and using the bound $\theta < 2\pi$ in inequality (2.5.13) it is not hard to show the improved bound $\left(\frac{2\pi}{4\theta}\right)^2 < \lambda^2$. Similarly one could derive better estimates for the lowest non-vanishing eigenvalue in Theorem 2.8 if one substitutes in equation (2.5.9) 2π by θ, where θ is the length of the largest interval on which $k_x(\varphi)$ is monotone.*

2.5.3 Special cases

As pointed out in remark 2.5 one can use the above techniques to derive in special situations sharper bounds of the minimum eigenvalue of the Sturm-Liouville problem. We illustrate this for the case of three subdomains sharing an interior heterogeneous singular point and the one of four subdomains and an interface consisting of two intersecting lines.
Although the interesting case of an interior heterogeneous singular point that belongs to three subdomains has been already studied independently in [36] [17] we give a proof which is simple and which slightly extends these results.

Lemma 2.10 *Let a heterogeneous singular point $x \in \bar{\Omega}$ be given such that if $x \in \Omega$, there are at most three subdomains to whose boundary x belongs. In case of $x \in \partial\Omega$ there are only two subdomains $x \in \partial\Omega_i$ and the boundary conditions do not change in x . Let the maximum interior angle of these subdomains be smaller than θ.*
Then the smallest non vanishing eigenvalue of the associated Sturm-Liouville eigenvalue problem is greater then $(\pi/(2\theta))^2$. This bound is sharp.

PROOF. The proof is a special case of Theorem 2.8 and Theorem 2.9. Let us first consider the case $x \in \Omega$. Since there are four points

$$\varphi_{zero,1} < \varphi_{\max} < \varphi_{zero,2} < \varphi_{\min}$$

and only three subdomains we conclude that there is an interval $[\varphi_j, \varphi_{j+1}]$ containing an extremum $\varphi_{ex} \in \{\varphi_{\min}, \varphi_{\max}\}$ and a point $\varphi_{zero} \in \{\varphi_{zero,1}, \varphi_{zero,1}\}$.
As before we can assume $0 = \varphi_{ex} = \varphi_{\max} < \varphi_{zero} < \theta$. Further $c_j = \varphi_{ex} = 0$.
Thus $s(\varphi) = b_j \cos(\lambda\varphi), \varphi \in [\varphi_{ex}, \varphi_{zero})$ and $s(\varphi_{zero}) = 0$. Possibly redefining φ_{zero} we see that $\lambda\varphi_{zero} = \pi/2$. This together with $\varphi_{zero} < \theta$ implies $\pi/(2\theta) < \lambda$.
The case $x \in \partial\Omega$ is dealt with similarly.
Sharpness of the bound follows by rescaling functions $s_3(\varphi)$, $s_4(\varphi)$ defined in the proof of Theorem 2.8 and 2.9, respectively. ■

Another interesting situation is the special case of an interior heterogeneous singular point, where the interface consists of two lines intersecting with angle ψ. This situation has been considered also in [34]. Let the coefficients k_i be distributed quasi-monotonically with respect to this singular point.
As before, the idea is to show that there is an extremum φ_{ex} and a zero point φ_{zero} of $s(\varphi)$ such that, either $k_x(\varphi)$ is monotonically increasing on $[\varphi_{ex}, \varphi_{zero}]$ and the length of the interval $[\varphi_{ex}, \varphi_{zero}]$ is smaller than π or $k_x(\varphi)$ is monotonically decreasing on $[\varphi_{zero}, \varphi_{ex}]$ and the length of the interval $[\varphi_{zero}, \varphi_{ex}]$ is smaller than π. This is easily checked. Now we can show

Lemma 2.11 *Let a heterogeneous singular point $x \in \Omega$ be given such that the interface consists of two intersecting lines. Let the coefficients $k_i, i = 0, .., 3$, be distributed quasi-monotonically with respect to x.*
Then the smallest non-vanishing eigenvalue of the associated Sturm-Liouville eigenvalue problem is greater than $(1/2)^2$. This bound is sharp.

PROOF. The proof follows from the above considerations. To see that the bound is sharp, regard the special case $k_1 = k_2 = k_3$, that means the case of two subdomains only. Define an eigenfunction as done in equation (3.17) in [34]. ■

A special case of this lemma is the case of two subdomains sharing a singular point. In this case we get the same bounds as [34].

2.5.4 Regularity results in the quasi-monotone case

Here we present our main results.

Theorem 2.12 *Let the distribution of coefficients $k_i, i = 0, .., n-1$ be quasi-monotone with respect to all singular points x. The solution of (2.2.1) fulfills $u \in H^{1+1/4}(\Omega)$. This is the maximum regularity independent of the bounds of k.*

PROOF. The assertion follows with corollary 2.1 from Theorem 2.8, 2.9, Lemma 2.4 and Lemma 2.1 . ■

Note that we get in principle the same regularity as for $k = 1$ were regularity $H^{1+1/4}$ is the maximal regularity at a reentrant corner with changing boundary conditions.

Corollary 2.2 *Quasi-monotonicity of the diffusion coefficient k is necessary and sufficient for regularity of the solution of problem (2.2.1) $u \in H^{1+1/4}(\Omega_i), i = 0, .., n-1$, which is independent of the bounds of k.*

PROOF. Necessity follows from remark 2.2. The regularity result follows from Theorem 2.12. ■

If one does not want to impose restrictions on k one has to restrict the number of subdomains n_l to whose boundary the singular point x_l belongs.
Additionally in some special cases sharper bounds are possible if one introduces further parameters depending on the geometry.

Theorem 2.13 *Let a singular point $x \in \Omega$ be given. Denote by n the number of subdomains that meet in x and let U be a neighbourhood containing no other singular points. Denote by θ the maximum interior angle of subdomains $\Omega_i, i = 0, .., n-1$ at x.*
If x is an interior singular point let $n \leq 3$. If $x \in \partial\Omega$ then let $n \leq 2$ and additionally the boundary conditions do not change in x.
Then the solution u of (2.2.1) fulfills $u \in H^{1+1/4}(\Omega \cap U)$.
Further $u \in H^{1+\max(1,\pi/(2\theta))}(\Omega_i \cap U), i = 0, .., n-1$.
The restrictions on n are sharp.

PROOF. One checks that under the above restrictions on n the diffusion coefficient k is quasi-monotone. The first part follows from Theorem 2.12. The second part follows from Lemma 2.10 together with corollary 2.1. To see that the restrictions on n are sharp we refer to example 2.2. ■

For heterogeneous singular points on the boundary with quasi-monotonically distributed coefficients k_i the results could be sharpened:

Corollary 2.3 *Let x an boundary heterogeneous singular point and U a neighbourhood containing no other singular points. Denote by θ the interior angle at $x \in \partial\Omega$. Assume that the distribution of coefficients $k_i, i = 0, .., n-1$ is quasi-monotone with respect to x. Then the solution u of problem (2.2.1) has regularity $u \in H^{1+\max(1,\pi/(2\theta))}(\Omega_i)$.*

PROOF. For a proof use remark 2.5. ■

The special case, where the interface consists locally of two intersecting lines, has been already studied in [34]. We state a regularity result for the quasi-monotone case.

Theorem 2.14 *Let an interior heterogeneous singular point $x \in \Omega$ be given and let U be a neighbourhood of x containing no other singular points. The interface consists in a neighbourhood of x of two intersecting lines. Let the distribution of coefficients $k_i, i = 0, .., 3$, be quasi-monotone with respect to x.*
Then the solution of problem (2.2.1) fulfills $u \in H^{1+1/2}(\Omega_i \cap U), i = 0, .., 3$. This bound is sharp.

PROOF. The assertion follows from corollary 2.1 and Lemma 2.11, Lemma 2.4. To prove sharpness use the singular function ψ_1 defined in example 2.1. ■

Our regularity results are sharp and in this sense regularity results from [34] [45] [61] [18] [36] are a special case of Theorem 2.13 or Theorem 2.14.

Remark 2.6 *One notices that Lemma 2.7 is the key ingredient for deriving lower bounds for the eigenfunctions of the Sturm-Liouville problem. It uses explicitly that the eigenfunctions of the Sturm-Liouville problem are piecewise scaled and shifted cosines. One could prove a similar result by using only concavity of the positive part of the cosine function. In such a way extensions to other problems are possible.*
Our consideration were restricted to problems with piecewise constant diffusion coefficients. But the definition of the quasi-monotonicity condition and the idea of the proof of the lower bound for the eigenvalues of the Sturm-Liouville eigenvalue problem may be applicable to a more general class of coefficients too.

2.6 The general case

2.6.1 Eigenvalue bounds in the general case

We conclude from corollary 2.2 that in the case of a non-quasi-monotone diffusion coefficient, the regularity may go down to H^1. This may happen if δ^{-1} becomes large.
In this section we derive explicit bounds of the regularity depending on δ. We show that $u \in H^{1+\delta/(2\pi)}$. Moreover, we can derive slightly better results, which are sharp.
To our knowledge a result which gives *explicit* H^s-regularity where s depends on k is new.
The following technical lemma is the equivalent of Lemma 2.7. Before formulating the lemma we will sketch its content. We have given a piecewise constant function $k(\varphi)$ defined on $[0, 2\pi)$ fulfilling $k(0) = 1$ and $k(\varphi) \geq k_{\min}$, where $0 < k_{\min} < 1$ is a given constant. The function k defines a continuous function $s(\varphi)$ which has piecewise the form $b_i \cos(\varphi - c_i)$ and whose derivatives satisfy interface conditions of type $[ks'] = 0$. We demand $s'(0) = 0, s(0) = 1$.
Let $\varphi_{\min}$ be the infimum of all roots φ_{zero} of these functions $s(\varphi)$. *The question is about the dependence of* $\varphi_{\min}$ *on* $k_{\min}$. To answer the question we look for the function k which defines the function $s(\varphi)$ that has $\varphi_{\min}$ as a root. This function k is defined by $k = 1$ on $[0, \varphi_{\min}/2)$ and $k = k_{\min}$ on $[\varphi_{\min}/2, \varphi_{\min})$ where $\varphi_{\min} = 2\,\arctan(k_{\min}^{1/2})$ (see figure 2.9).

Lemma 2.15 *Let a number* $0 < k_{\min} < 1$ *and numbers* $0 = \varphi_0 < \varphi_1 < ... < \varphi_n = 2\,\arctan(k_{\min}^{1/2})$ *be given. Further there are coefficients* k_i *given where* $k_0 > 0$ *and* $k_i/k_0 \geq k_{\min}, i = 0, .., n-1$. *Denote by* $\chi_{[\varphi_i, \varphi_{i+1})}$ *the characteristic function of the interval* $[\varphi_i, \varphi_{i+1})$. *Let numbers* $c_i, b_i \geq 0, i = 0, ..., n-1$, *be given which define a function*

$$s(\varphi) = \sum_{i=0}^{n-1} b_i \cos(\varphi - c_i)\; \chi_{[\varphi_i, \varphi_{i+1})} \tag{2.6.1}$$

that is continuous and whose derivatives weighted with k_i are also continuous:

(2.6.2) $$b_i \cos(\varphi_{i+1} - c_i) = b_{i+1} \cos(\varphi_{i+1} - c_{i+1}) \ , \quad i = 0,..,n-2 \ ,$$

(2.6.3) $$k_i b_i \sin(\varphi_{i+1} - c_i) = k_{i+1} b_{i+1} \sin(\varphi_{i+1} - c_{i+1}) \ , \quad i = 0,..,n-2 \ ,$$

Assume $c_0 = 0, b_0 = 1$. *Then* $s(\varphi) > 0, 0 \le \varphi < \varphi_n$.

PROOF. We define $t_i(\varphi) := b_i \cos(\varphi - c_i)$. Dividing k_i by k_0 we may set $k_0 = 1$. But in order to make the dependence on k clear we will use in the proof the notation k_0 remembering $k_0 = 1$. We first assume $k_0 > k_1$. Otherwise regard the discussion at the end of the proof.
The proof is done in three steps.
The idea is to bound function t_i from below by functions t_{j_i}. Here we write j_i to denote the dependence of j on i. Then we show that the function t_{j_i} is greater than a function u_{j_i}. In the last step we discuss the functions u_{j_i}.
In the first step our goal is to show that for $i = 1,..,n-1$ there is a index $0 \le j \le n-1$ and a number φ_j^- fulfilling

(2.6.4) $$t_j(\varphi_j^-) = t_0(\varphi_j^-), 0 < \varphi_j^- \le \varphi_i \ \text{ and } t_j(\varphi) \le t_0(\varphi), \varphi_j^- \le \varphi \le \varphi_n$$
$$t_j(\varphi) \le t_i(\varphi), \varphi_i \le \varphi \le \varphi_n \quad .$$

First Step. The proof of the first step is somewhat technical. We show equation (2.6.4) by induction with respect to $i = 1,..,n-1$.
Initial step $i = 1$**.** Simply define $\varphi_{j_1}^- := \varphi_1$ and $j_1 = 1$. As $k_0 > k_1$ Lemma 2.6 implies $t_{j_1}(\varphi) \le t_0(\varphi), \varphi_{j_1}^- \le \varphi \le \varphi_n$. We showed equation (2.6.4) for $i = 1$.

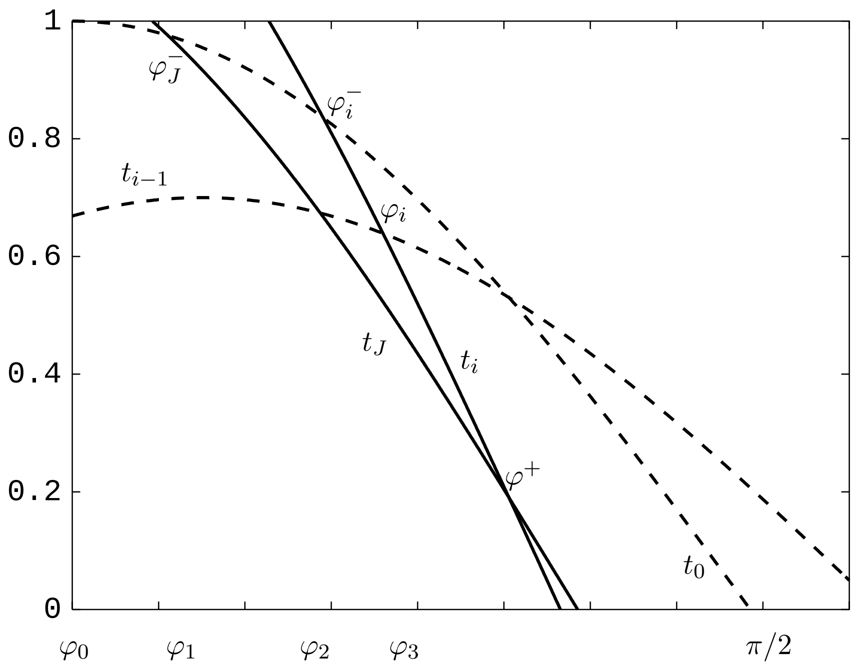

Figure 2.8: Step $i = 3$ is illustrated, here $J = 1$, note $\varphi_J \le \varphi \le \varphi_i \le \varphi^+$

Induction for $i > 1$**.** Set $J := j_{i-1}$. There are two cases.
In the first case $t_J(\varphi) \le t_i(\varphi), \varphi_i \le \varphi \le \varphi_n$. We define $j_i := J$ and proved (2.6.4).
In the second case we define $j_i := i$. This case is illustrated in Figure 2.8. There is a $\varphi^+ \in (\varphi_i, \varphi_n]$ with

$$t_J(\varphi^+) = t_i(\varphi^+) \quad .$$

Further due to equations (2.6.2), (2.6.4) $t_J(\varphi_i) \le t_{i-1}(\varphi_i) = t_i(\varphi_i)$. The last equations imply $0 \le (t_J - t_i)'(\varphi^+)$. We may use Lemma 2.6 to show $t_J(\varphi) \le t_i(\varphi), 0 \le \varphi \le \varphi^+$. From equation (2.6.4) and from $\varphi_J^- < \varphi_i < \varphi^+$ follows

$$t_0(\varphi_J^-) = t_J(\varphi_J^-) \le t_i(\varphi_J^-) \quad .$$

We conclude that there is a number φ_i^- fulfilling $\varphi_J^- \le \varphi_i^- \le \varphi_i$ with $t_0(\varphi_i^-) = t_i(\varphi_i^-)$. It is not hard to see that $t_i(\varphi) \le t_0(\varphi), \varphi_i^- \le \varphi_i \le \varphi \le \varphi_n$ and hence we proved (2.6.4).
Second Step. For $i = 1, .., n-1$ set $j = j_i$ and define u_j by

$$u_j = a_j \cos(\varphi - d_j) \quad , \tag{2.6.5}$$

where a_j, d_j are chosen in such a way that the following interface conditions are fulfilled

$$\begin{aligned} t_0(\varphi_j^-) &= u_j(\varphi_j^-) \\ k_0 t_0'(\varphi_j^-) &= k_{\min}\, k_0\, u_j(\varphi_j^-) \quad , \end{aligned} \tag{2.6.6}$$

for $\varphi_j^- \in [0, \varphi_n]$. Remember $k_0 = 0$. Since $u_j(\varphi_j^-) = t_0(\varphi_j^-) = t_j(\varphi_j^-)$ and $k_{\min} < 1$ we conclude with help of Lemma 2.6 that $u_j(\varphi) \le t_j(\varphi), \varphi_j^- \le \varphi_j \le \varphi \le \varphi_n \le \pi/2$. This yields together with equation (2.6.4)

$$u_j(\varphi) \le t_j(\varphi) \le t_i(\varphi) \quad , \quad \varphi_i \le \varphi \le \varphi_n \quad . \tag{2.6.7}$$

Third Step. We want to show $0 < u_j(\varphi), \varphi \in [0, \varphi_n), i = 0, .., n-1$ by showing that $d_j \in (\varphi_n - \pi/2, 0)$.
Therefore, we choose $\varphi = \varphi_j^-, d := d_j$ and rewrite (2.6.6)

$$\begin{aligned} \cos(\varphi) &= a_j \cos(\varphi - d) \\ k_0 \sin(\varphi) &= k_{\min}\, k_0\, a_j \sin(\varphi - d) \quad . \end{aligned} \tag{2.6.8}$$

Now we look for the minimum value of d depending on φ. Dividing the two equations by each other we obtain

$$d(\varphi) = \varphi - \arctan(k_{\min}^{-1} \tan(\varphi)) \quad .$$

Differentiating with respect to φ reveals that minimum is attained for $\tan(\varphi^\star) = k_{\min}^{1/2}$. Insertion of the minimum leads to

$$d(\varphi^\star) = \arctan(k_{\min}^{1/2}) - \arctan(k_{\min}^{-1/2}) = 2 \arctan(k_{\min}^{1/2}) - \pi/2 = \varphi_n - \pi/2 \quad .$$

Here we used the trigonometric relation $\arctan(x) + \arctan(x^{-1}) = \pi/2$. Thus we finished the proof of the third step.

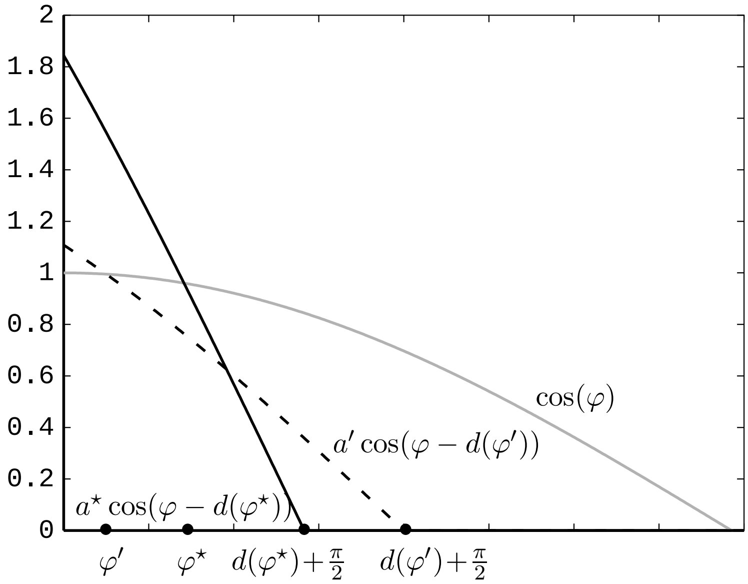

Figure 2.9: the black functions $a' \cos(\varphi - d(\varphi'))$, $a^\star \cos(\varphi - d(\varphi^\star))$ fulfill interface conditions (2.6.8) with diffusion coefficients $k_0 = 1, k_{\min} \approx 0.08$ at points $\varphi \in \left\{\varphi', \varphi^\star = \arctan k_{\min}^{1/2}\right\}$, they vanish at $d(\varphi') + \pi/2, d(\varphi^\star) + \pi/2$; for the black function depicted with a continuous line the value $d(\varphi^\star) + \pi/2$ is the minimal one for all possible values of φ', we show $d(\varphi^\star) + \pi/2 = \varphi_n = 2 \arctan k_{\min}^{1/2} = 2\varphi^\star$

Now we collect the results from the previous three steps to obtain from inequality (2.6.7)

$$0 < u_j(\varphi) \leq t_j(\varphi) \leq t_i(\varphi) \quad , \quad \varphi_i \leq \varphi \leq \varphi_{i+1}, i = 1, .., n-1 \quad .$$

This shows the assertion for the case $l = 0$.
If there is an index $l > 0$ such that $k_0 < k_1 < ... < k_l$, we use Lemma 2.7 to prove that functions $t_i, i = 0, .., l$ do not vanish on $[\varphi_0, \varphi_n]$. It remains to prove the assertion for functions $t_i, i > l$. From the relation $k_0 < k_1 < ... < k_l$ follows $0 = c_0 < c_1 < ... < c_l$. We shift the functions $t_i, i > l$ to the left by $c_l > 0$ and prove the assertion for the shifted functions as in the case $l = 0$. ■

2.6.2 A "worst case" regularity result

We use Lemma 2.15 to derive bounds for the eigenvalues of the associated Sturm-Liouville eigenvalue problem. Comparing these bounds with the function defined in example 2.2 we can show that our bounds are sharp. The main result in this section is Theorem 2.17.

Remark 2.7 *The singular function defined in example 2.2 fulfills the conditions*

(2.6.9) $$\delta = \tan(\lambda\pi/4) \leq k \leq \tan(\lambda\pi/4)^{-1} \text{ for any number } 0 < \lambda < 1 \quad .$$

Recall that λ^2 is the eigenvalue of the associated Sturm-Liouville eigenvalue problem. Rewriting (2.6.9) we get

$$\lambda = \frac{4}{\pi}\arctan\delta \quad .$$

Theorem 2.16 *Let a heterogeneous singular point $x \in \bar{\Omega}$ be given and let $\delta \le k_i \le \delta^{-1}, i = 0, .., n-1$. Denote by λ^2 the smallest non-vanishing eigenvalue of the associated Sturm-Liouville eigenvalue problem.*
If x is an interior heterogeneous singular point, then it holds

$$\left(\frac{2\,\delta}{\pi}\right)^2 < \left(\frac{4}{\pi}\arctan\delta\right)^2 \le \lambda^2 \tag{2.6.10}$$

and the bound $\left(\frac{4}{\pi}\arctan\delta\right)^2 \le \lambda^2$ is sharp with respect to δ.
If x is a boundary heterogeneous singular point, denote by θ the interior angle of Ω at x. It holds

$$\left(\frac{2\delta}{m\,\theta}\right)^2 < \left(\frac{4}{m\,\theta}\arctan\delta\right)^2 \le \lambda^2 \quad , \tag{2.6.11}$$

where $m = 1$ if the boundary conditions do not change in x and $m = 2$ if they do. The bound $\left(\frac{4}{m\,\theta}\arctan\delta\right)^2 \le \lambda^2$ is sharp with respect to δ and θ.

PROOF. Dividing k by a δ^{-1} we may assume $\delta^2 \le k_i \le 1$. If x is an interior singular point we conclude as in the proof of Theorems 2.8 that there are two points $\{\varphi_{\min}, \varphi_{\max}\}$ where $s(\varphi)$ achieves an extremum and two zero points $\{\varphi_{zero,1}, \varphi_{zero,2}\}$. It is easy to see that we can order these points like $\varphi_{\min} < \varphi_{zero,1} < \varphi_{\max} < \varphi_{zero,2}$. If x is an interior heterogeneous singular point we are free in the choice of $\varphi_{ex} \in \{\varphi_{\min}, \varphi_{\max}\}$, $\varphi_{zero} \in \{\varphi_{zero,1}, \varphi_{zero,2}\}$ and we may additionally assume $|\varphi_{ex} - \varphi_{zero}| \le \pi/2$.
If x is a boundary heterogeneous singular point we may assume there is an extremum and a zero point of $s(\varphi)$ such that $|\varphi_{ex} - \varphi_{zero}| \le \theta/2$, if the boundary conditions do not change in x and $|\varphi_{ex} - \varphi_{zero}| \le \theta$, if they do. For the sake of brevity we continue the proof in the case of an interior singular point.
We choose j such that $\varphi_{ex} \in [\varphi_j, \varphi_{j+1})$ and show as in the proof of Theorem 2.8 that $c_j = 0$. Further we choose the maximum n such that $\varphi_{j+1} < ... < \varphi_n \le \varphi_{zero}$. Changing the coordinate system we may set $\varphi_{ex} = 0 < \varphi_{zero} < \pi/2$.
We introduce the homogeneous scaling $F : [0, \varphi_{zero}] \to [0, \widehat{\varphi}_{zero}]$ with $F(\varphi) = \widehat{\varphi} = \lambda\varphi$. Define $s_F(F(\varphi)) := s(\varphi), \varphi \in [\varphi_{ex}, \varphi_{zero}]$. Then it holds $\widehat{\varphi}_{zero} \le \lambda\,\pi/2$.
Observe that $s_F(\widehat{\varphi})$ fulfills the assumption of Lemma 2.15 with $k_{\min} := \delta^2$ as $k_i/k_0 \ge k_i \ge k_{\min}$ for all $i = 0, .., n-1$. Since s_F vanishes in $\widehat{\varphi}_{zero}$ we conclude from Lemma 2.15 that

$$2\ \arctan\delta = 2\ \arctan(k_{\min}^{1/2}) \le \widehat{\varphi}_{zero} \le \lambda\,\pi/2 \quad . \tag{2.6.12}$$

The inequality $c < 2\arctan(c)$ for any $0 < c < 1$ is checked easily. This shows assertion (2.6.10). Sharpness follows from remark 2.7.
The case $x \in \partial\Omega$ is proved similarly. To show sharpness of the bound (2.6.11) modify the singular function defined in example 2.2 by restricting the domain of definition and

applying an suitable affine transformation. ∎

According to Theorem 2.16 we are now able to give a regularity result which will depend on the bounds of k.

Theorem 2.17 *Let $\delta < k(x) < \delta^{-1}, \forall x \in \Omega$ for a number $0 < \delta < 1$.*
Then the solution of problem (2.2.1) has regularity

$$u \in H^{1+\delta/(2\pi)}(\Omega) \quad .$$

Let a heterogeneous singular point $x \in \bar{\Omega}$ be given and let $c\,\nu \le k_i \le c\,\nu^{-1}, i = 0, .., n-1$ for some constants $c > 0, \nu > 0$. Denote by U a neighbourhood containing no other singular points and with θ the interior angle of Ω at x (if $x \in \ \Omega$, set $\theta = 2\pi$).
Then

$$u \in H^{1+(4\ \arctan \nu)/(m\theta)-\varepsilon}(U \cap \Omega_i) \subset H^{1+2\nu/(m\theta)}(U \cap \Omega_i) \ \ , \ \ i = 0, .., n-1 \ \ ,$$

where

$$m = \begin{cases} 1/2 & \text{if } x \in \Omega \\ 1 & \text{if } x \in \partial\Omega \ \text{ and the boundary conditions do not change in } x \\ 2 & \text{if } x \in \partial\Omega \ \text{ and the boundary conditions change in } x \end{cases}$$

and $\varepsilon > 0$ is arbitrary.
This is the maximum regularity with respect to δ and ν independent of the number and interior angles of the subdomains.

PROOF. The assertion follows with corollary 2.1 from Theorem 2.16. Sharpness follows from remark 2.7. ∎

An easy consequence of the above theorem is

Corollary 2.4 *Let $\delta > 0$ be given. The singular function ψ_2 defined in example 2.2 with $\lambda = 4\,\pi^{-1}\arctan(\delta)$ is the function with* lowest regularity among all singular functions *for interior heterogeneous singular points under the restriction $\delta \le k \le \delta^{-1}$ and with no other restrictions on the geometry (that means there are no restrictions imposed on the number and interior angles of the subdomains).*

2.6.3 Regularity between the quasi-monotone and the "worst case"

We may say that Theorem 2.17 states regularity results in the "worst case". On the other hand results from section 2.5.4 show regularity for the "regular" - the quasi-monotone - case. The question arises about regularity for diffusion coefficients that are not quasi-monotonically distributed but which have also no checkerboard-like pattern as in the "worst case". In this context it seems naturally to expect that a slight perturbation of quasi-monotonically distributed diffusion coefficients will not result in large changes of the regularity. These questions will be answered in the next theorem. Although a sharp regularity result is given in Theorem 2.17 it can be partly improved introducing additional parameters depending on the diffusion coefficient.

Theorem 2.18 *Let a heterogeneous singular point $x \in \Omega$ be given. We assume that $k_x(\varphi)$ has more than one local maximum. Denote by $k_{\max,1}, k_{\max,2}$ the two largest local maxima and let $k_{\max,1} \geq k_{\max,2}$. Let $k_{\min,1}, k_{\min,2}$ be the two smallest local minima where $k_{\min,1} \leq k_{\min,2}$. Denote by U a neighbourhood containing no other singular points and define $\delta' := \sqrt{k_{\min,2}/k_{\max,2}}$.*
Then the solution of problem (2.2.1) has regularity

$$u \in H^{1+(\arctan \delta')/\pi}(U \cap \Omega_i) \subset H^{1+\delta'/(2\pi)}(U \cap \Omega_i) \ , \ i = 0, .., n-1 \quad .$$

Note that in the limiting case, when the second maximum $k_{\max,2}$ and the second local minimum $k_{\min,2}$ vanishes, that means in the case $k_{\max,2}/k_{\min,2} \to 1$, we reach $H^{1+1/4}$-regularity as in the quasi-monotone case. In the "worst case" example 2.2 it holds $k_{\max,1} = k_{\max,2}$ and $k_{\min,1} = k_{\min,2}$. Accordingly $\delta = \delta'$ and we note that regularity implied by Theorem 2.17 differs only by the constant 4 from the "worst case" result from Theorem 2.17. Hence, we can interpret Theorem 2.18 as a link between the theory of robust regularity results for quasi-monotically distributed diffusion coefficients and results for the "worst case", where no assumptions on the structure of the diffusion coefficients is made.
The case of boundary singular points for pure boundary conditions can be treated a similar fashion. In case of pure Dirichlet conditions Theorem 2.18 is valid with $\delta' = \sqrt{k_{\min,2}/k_{\max,1}}$ and in case of pure Neumann conditions with $\delta' = \sqrt{k_{\min,1}/k_{\max,2}}$.
PROOF. The proof is not self contained but it runs similar to the proof of Theorem 2.17. There Theorem 2.16 is used and we have to prove an analogon of inequality (2.6.10) to hold namely

$$\left(\frac{1}{\pi} \arctan \delta'\right)^2 < \lambda^2 \ , \tag{2.6.13}$$

where λ^2 is the smallest positive eigenvalue of the eigenfunction $s(\varphi)$ for the respective Sturm-Liouville eigenvalue problem. Additionally we will use elements of the eigenvalue bounds for the quasi-monotone case as in Theorem 2.8. We now proceed as in the proof of Theorem 2.17 and therefore give only the differences. The most important point is to observe that we are free in the choice of the local extrema of $k_x(\varphi)$ and we may use the local extrema $k_{\max,2}, k_{\min,2}$ in place of the global ones $k_{\max,1}, k_{\min,1}$.
Let the maxima $k_{\max,1}, k_{\max,2}$ of $k_x(\varphi)$ be attained on the intervals $[\varphi_{i_{\max,1}}, \varphi_{i_{\max,1}+1})$ and $[\varphi_{i_{\max,2}}, \varphi_{i_{\max,2}+1})$ respectively. The minima are attained on $[\varphi_{i_{\min,1}}, \varphi_{i_{\min,1}+1})$ and $[\varphi_{i_{\min,2}}, \varphi_{i_{\min,2}+1})$. Without loss of the generality we may assume that $\varphi_{i_{\max,1}} < \varphi_{i_{\min,1}} < \varphi_{i_{\max,2}} < \varphi_{i_{\min,2}}$ (Figure 2.10). We define the intervals $I := [\varphi_{i_{\max,1}}, \varphi_{i_{\min,1}})$ and $J := [\varphi_{i_{\min,1}}, \varphi_{i_{\max,1}})$ covering the sphere $[0, 2\pi)$.
It is easy to see that there is an extremum φ_{ex} and a root φ_{zero} of the eigenfunction $s(\varphi)$ such that $[\varphi_{ex}, \varphi_{zero}) \subset J$ or $[\varphi_{zero}, \varphi_{ex}) \subset I$. Without loss of generality we may suppose that $[\varphi_{ex}, \varphi_{zero}) \subset J$.
By rotating the coordinate system we obtain

$$\varphi_{ex} = 0 < \varphi_{zero} < 2\pi \quad . \tag{2.6.14}$$

Choose the index j such that $\varphi_{ex} \in [\varphi_j, \varphi_{j+1})$ and note $c_j = \varphi_{ex} = 0$. Note that $k_x(\varphi_{ex}) = k_j$. In the following we regard separately the cases that $k_j \geq k_{\max,2}$, $k_j < k_{\min,2}$ and $k_j \in [k_{\min,2}, k_{\max,2})$.

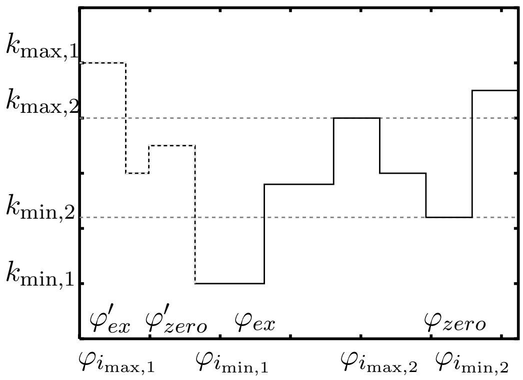

Figure 2.10: local diffusion coefficient $k_x(\varphi)$ has several maxima and minima; possible location of points $\varphi_{ex}, \varphi_{zero}, \varphi'_{ex}, \varphi'_{zero}$; function $k_x(\varphi)$ on interval J is depicted with a black line; grey dashed lines indicate the values $k_{\max,2}, k_{\min,2}$

If $k_j \geq k_{\max,2}$, then $k_x(\varphi)$ is increasing on $[\varphi_{ex}, \varphi_{i_{\max,1}})$, since otherwise there would a local maximum of $k_x(\varphi)$ on $[\varphi_{ex}, \varphi_{i_{\max,1}})$ which is larger than $k_{\max,2}$. We conclude that $k_x(\varphi)$ is increasing on $[\varphi_{ex}, \varphi_{zero}) \subset [\varphi_{ex}, \varphi_{i_{\max,1}})$ and proceed as in the quasi-monotone case to bound λ from below by $1/4$ (Theorem 2.8).

If $k_j < k_{\min,2}$ it is easy to check that $k_i \geq k_j, i = j, .., i_{\max,1}$. Otherwise there would be a local minimum of $k_x(\varphi)$ on $[\varphi_{ex}, \varphi_{i_{\max,1}})$ that is smaller than $k_{\min,2}$. As before we apply a homogeneous scaling F, use inequality (2.6.14) and Lemma 2.15 for the scaled function $s_F(\cdot)$. Renumbering the subdomains we can assure $j = 0$. Now the assumptions of Lemma 2.15 are fulfilled for $k_{\min} := 1$ since it holds $k_i \geq k_{\min}\, k_j, i = j, .., i_{\max,1}$. This shows the eigenvalue bound (2.6.13).

It remains the case that $k_j \in [k_{\min,2}, k_{\max,2})$. Then it holds $k_i \geq k_{\min,2}, i = j, .., i_{\max,1}$. In the opposite case there would be a local minimum of $k_x(\varphi)$ smaller than $k_{\min,2}$. Defining in terms of Lemma 2.15 $k_{\min} := (\delta')^2$ we see $k_i \geq k_{\min,2} \geq k_j\, k_{\min,2}/k_{\max,2} = k_j\, k_{\min}$. Now we apply homogeneous scaling F and Lemma 2.15 to show the eigenvalue bound (2.6.13). This finishes the proof. ■

2.6.4 $W^{2,p}$-regularity

Using the bounds of the eigenvalues in Theorems 2.8, 2.9 and 2.16 it is straightforward to formulate regularity results in Sobolev spaces $W^{2,p}$ for $p \in (1, 2)$. A decomposition theorem which relates the eigenvalue bounds to $W^{2,p}$-regularity can be found in [43, Thm 2.27] resp. [43, Cor 2.28].

Calculation shows that the singular function $r^\lambda s_\lambda(\varphi)$ belongs piecewise to $W^{2,p}$ for $p < 2/(2-\lambda)$. Accordingly, if for a positive number γ holds, $\gamma^2 < \lambda^2$ for all positive eigenvalues λ^2 of the respective Sturm-Liouville eigenvalue problem, each singular solution $r^\lambda s_\lambda(\varphi)$ and the solution u will have piecewise regularity $W^{2,p}$ for $p = 2/(2-\gamma)$.

Corollary 2.5 *Denote by u the solution of problem (2.2.1). Let a singular point $x \in \bar{\Omega}$ with neighbourhood U containing no other singular point be given.*
If the distribution of coefficients $k_i, i = 0, .., n-1$ is quasi-monotone with respect to x, then u has regularity

$$u \in W^{2,1+1/7}(\Omega_i \cap U) \ \ , \ \ i = 0, .., n-1 \quad .$$

If for a $\delta > 0$ holds $\delta \leq k_i \leq \delta^{-1}, i = 0, .., n-1$, then u has regularity

$$u \in W^{2,1+\delta/(4\pi)}(\Omega_i \cap U) \ \ , \ \ i = 0, .., n-1 \quad .$$

PROOF. The result for the quasi-monotone case follows from Theorems 2.8, 2.9 and corollary [43, Cor 2.28]. We check $2/(2-1/4) = 1 + 1/7$. Regularity for the general case follows from Theorem 2.16 and corollary [43, Cor 2.28]. Here we use the inequality

$$\frac{2}{2-\delta/(2\pi)} = \frac{1}{1-\delta/(4\pi)} > 1 + \delta/(4\pi)$$

together with the embedding $W^{2,p} \subset W^{2,q}$ for $1 \leq q < p$. If all eigenvalues λ^2 are different from 1 instead [43, Cor 2.28] one use Theorem 2.2 and the embedding $H^2 \subseteq W^{2,p}$ for any $p \leq 2$. ■

2.7 Regularity in 3D

2.7.1 Overview

In 3D the polyhedral Lipschitz domain Ω is decomposed into polyhedral Lipschitz subdomains $\Omega_i, i = 1, .., n$ and the interface consists of plane faces, with edges and vertices. Singularities occur along edges and at vertices of the interface. A decomposition theorem [17] [45] states that the solution of problem (2.2.1) can be decomposed into a regular part with piecewise H^2-regularity and into a singular part. The singular part can be decomposed into singularities at vertices and on edges. We will shortly review vertex and edge singularities and give regularity results based on those derived in section 2.4.
In the following we rely on results from [17] which were presented for the case of pure Dirichlet or pure Neumann boundary conditions. As in the 2D-case the decomposition into vertex and edge singular functions is more complicated if an exponent λ equals 1. Let us assume in this section that this is not the case.

2.7.2 Vertex singularities

We fix a vertex x of the interface as depicted in Figure 2.7.2 and introduce spherical coordinates (r, φ, ψ) with respect to the vertex. As in the 2D case the vertex singular solutions come from an positive definite eigenvalue problem.
The Laplace operator can be written in spherical coordinates as

$$\frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\Delta' v = 0 \ \ , \tag{2.7.1}$$

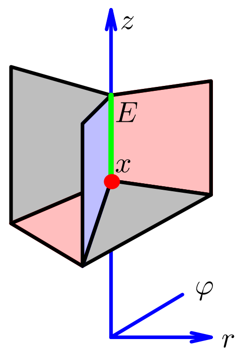

Figure 2.11: Vertex x and edge E of the interface, cylindrical coordinates (r, φ, z) with respect to the edge E

where Δ' denotes the Laplace-Beltrami operator, that is the Laplace operator acting on the sphere.
Then the vertex singular functions are solutions of the homogeneous problem (2.2.1), that means they are piecewise harmonic functions satisfying the interface conditions and having the form [17]

$$v_x(r, \varphi, \psi) = r^\lambda s_x(\varphi, \psi) \quad ,$$

where the piecewise identity

$$-\Delta' s_x(\varphi, \psi) = \mu \; s_x(\varphi, \psi)$$

holds for the eigenvalue $\mu > 0$ on each subdomain Ω_i intersected with the unit sphere. Here for better readability the dependence of λ and μ on the vertex x is not denoted. Insertion of v_x into (2.7.1) yields a relation between λ and μ

$$\lambda(\lambda - 1) + 2\lambda - \mu = 0 \text{ resp. } \lambda = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \mu} \quad .$$

Here admissible values of λ are $\lambda = -1/2 + \sqrt{1/4 + \mu} \geq 0$ and we conclude from [17]:

Remark 2.8 *For the vertex x the vertex singularity has the form of the function v_x and has regularity $v_x \in H^{3/2+\lambda_x-\varepsilon}(\Omega_i), i = 1, .., n$ independent of the diffusion coefficient k, where $\varepsilon > 0$ is an arbitrary number. Therefore, vertex singularities are not critical with respect to H^{1+s}-regularity with $0 < s \leq 1/2$ and in particular with respect to deteriorating regularity with $0 < s \ll 1$. They are important for regularity $H^{1+s}(\Omega_i)$ where $s > 1/2$. Multiplying v_x by a smooth cut-off function it can be assured that the singular function v_x vanishes outside a neighbourhood of the vertex x.*

2.7.3 Edge singularities

The considerations concerning the regularity of 3D edge singularities can be reduced to 2D vertex singularities. Given an edge E let us introduce cylindrical coordinates (r, φ, z) with respect to the edge. Here r denotes the distance from the edge E (see

Figure 2.7.2). Denote by $\Omega_0, .., \Omega_{n(E)-1}$ the subdomains which share the edge E. They introduce in a natural way a partition of a plane H perpendicular to the edge E by lines $\varphi = \varphi_0, .., \varphi = \varphi_{n(E)-1}$ which coincide with a part of the boundary of Ω_i and meet at edge E. In the same manner let the diffusion coefficients $k_i, i = 0, .., n(E)-1$ be defined. We identify the plane H with R^2.
On R^2 the same Sturm-Liouville eigenvalue problem as in section 2.4.1 is defined. If $E \subset \partial\Omega$, set appropriate homogeneous boundary conditions for the Sturm-Liouville eigenvalue problem. Denote by λ^2 the lowest non-vanishing eigenvalue and by $w_E = r^\lambda s_\lambda(\varphi)$ the associated singular solution which is piecewise harmonic and satisfies the interface conditions. Again the dependence of λ on E is not explicitly denoted.
Following [17] with each 2D singular function $w_E, 0 < \lambda < 1$, there is associated a 3D edge singular function through a function $b_E(r,z)$ of the form

$$v_E(r,\varphi,z) = b_E(r,z)\; w_E(r,\varphi) \quad .$$

2.7.4 Regularity results in 3D

The following theorem is shown in [17]:

Theorem 2.19 *Let $\gamma < \min\{\lambda_x + 1/2, \lambda_E, 1\}$ where the minimum is taken over all vertices x and all edges E of the interface and the boundary. Then the solution of (3.2.2) has regularity*

$$u \in H^{1+\gamma}(\Omega_i) \;\; , \;\; i = 1, .., n \quad .$$

We conclude that in case of regularity below $H^{1+1/2}$-regularity it suffices to investigate for each edge the singular exponents for the according 2D problems since the vertex singularities have regularity at least $H^{1+1/2}$. For this we use the lower bounds of eigenvalues of 2D Sturm Liouville eigenvalue problems derived in section 2.4.

Theorem 2.20 *If $\delta \le k_i \le \delta^{-1}, i = 1, .., n$, for a certain $\delta > 0$, then the solution of problem (2.2.1 fulfills $u \in H^{1+\delta/(2\,\pi)}(\Omega)$.*
If for each edge E of the interface the diffusion coefficients $k_0, .., k_{n(E)-1}$ of the according 2D problem are distributed quasi-monotonically, then $\lambda_E > 1/4$ and $u \in H^{1+1/4}(\Omega)$.
For each edge quasi-monotonicity of the diffusion coefficients $k_0, .., k_{n(E)-1}$ is necessary for regularity of the solution of problem (2.2.1) $u \in H^{1+1/4}(\Omega_i)$ independent of the global bounds of k.

PROOF. The proof is based on the regularity result in Theorem 2.19. The first assertion follows from Theorem 2.16. The second assumption is stated in Theorem 2.12. To see that quasi-monotonicity is necessary for $H^{1+1/4}$-regularity which is independent of the bounds of k see remark 2.2 and observe that 2D singular functions can be trivially extended to be constant in the direction of the edge to yield 3D edge singular functions. Global regularity follows from local regularity by Lemma 2.1. ∎