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References

- [Ant95] S.S. Antman. *Nonlinear problems of Elasticity*, volume 107. Springer, 1995.
- [ARS99] M. Ast, H. Rösle, and R. Schenk. FE analysis of frictionally resistant shaft-hub connections. In *Welle-Nabe-Verbindungen — Systemkomponenten im Wandel*, VDI conference proceedings, 1999.
- [Bas96] P. Bastian. *Parallele adaptive Mehrgitterverfahren*. Teubner Skripten zur Numerik. Teubner-Verlag, 1996.
- [BBJ⁺97] P. Bastian, K. Birken, K. Johannsen, S. Lang, N. Neuß, H. Rentz-Reichert, and C. Wieners. UG – a flexible software toolbox for solving partial differential equations. *Computing and Visualization in Science*, 1:27–40, 1997.
- [BC83] A. Brandt and C.W. Cryer. Multigrid algorithms for the solution of linear complementarity problems arising from free boundary problems. *SIAM J. Sci. Stat. Comput.*, 4:655–684, 1983.
- [Bel93] V. Belsky. A multi-grid method for variational inequalities in contact problems. *Computing*, 51:293–311, 1993.
- [Ber82] D.P. Bertsekas. *Constrained optimization and Lagrange multiplier methods*. Academic Press, New York, 1982.
- [BGK87] P. Boieri, F. Gastaldi, and D. Kinderlehrer. Existence, uniqueness and regularity results for the two-body contact problem. *Applied Mathematics and Optimization*, 15:251–227, 1987.
- [BH93] H. Bader and R.H.W. Hoppe. Multigrid solution of Signorini type problems in contact elastostatics. Technical Report TUM-M9304, TU München, 1993.
- [BHL97] F. Ben Belgacem, P. Hild, and P. Laborde. Approximation of the unilateral contact problem by the mortar finite element method. *C. R. Acad. Sci., Paris, Ser. I*, 324:123–127, 1997.
- [BHL99] F. Ben Belgacem, P. Hild, and P. Laborde. Extension of the mortar finite element method to a variational inequality modeling unilateral contact. *Math. Models Methods Appl. Sci.*, 9:287–303, 1999.
- [BL76] J. Bergh and J. Löfström. *Interpolation spaces*. Springer Verlag, Berlin, 1976.
- [BMP93] C. Bernardi, Y. Maday, and A.T. Patera. Domain decomposition by the mortar element method. In H. Kaper et al., editor, *In: Asymptotic and numerical methods for partial differential equations with critical parameters*, pages 269–286. Reidel, Dordrecht, 1993.
- [BMP94] C. Bernardi, Y. Maday, and A.T. Patera. A new nonconforming approach to domain decomposition: The mortar element method. In *Nonlinear Partial Differential Equations and Their Applications*. Pitman, 1994.

- [Bra93] J.H. Bramble. *Multigrid Methods*, volume 294 of *Pitman Research Notes in Mathematical Sciences*. Longman Scientific & Technical, Essex, England, 1993.
- [BS68] P.L. Butzer and K. Scherer. *Approximationsprozesse und Interpolationsmethoden*. Bibliographisches Institut, Mannheim, 1968.
- [Cau23] A.-L. Cauchy. Recherches sur l'équilibre et le mouvement intérieur des corps solides ou fluides, élastiques ou non élastiques. *Bulletin de la Société Philomatique*, pages 9–13, 1823.
- [Cau27] A.-L. Cauchy. De la pression ou tension dans un corps solide. *Exercices de Matheématiques*, 2, 1827.
- [CHP00] P. Coorevits, P. Hild, and J.-P. Pelle. A posteriori error estimation for unilateral contact with matching and non-matching meshes. *Comput. Methods Appl. Mech. Eng.*, 186:65–83, 2000.
- [Cia88] P.G. Ciarlet. *Mathematical Elasticity; Volume 1: Three-Dimensional Elasticity*, volume 20 of *Studies in Mathematics and its Applications*. North-Holland, Amsterdam, 1988.
- [Cla83] F.H. Clarke. *Optimization and Nonsmooth Analysis*. John Wiley & Sons, 1983. A Wiley-Interscience Publication.
- [CSW99] C. Carstensen, O. Scherf, and P. Wriggers. Adaptive finite elements for elastic bodies in contact. *SIAM J. Sci. Comp.*, 20(5):1605–1626, 1999.
- [DL72] G. Duvaut and J.L. Lions. *Les Inéquations en Mécanique et en Physique*. Dunaud, Paris, 1972.
- [Dos97] Z. Dostál. Box constrained quadratic programming with proportioning and projections. *SIAM J. Optim.*, 7:871–887, 1997.
- [Dry99] M. Dryja. A Dirichlet–Neumann algorithm for elliptic mortar finite element problems. In W. Hackbusch and S. Sauter, editors, *Numerical Techniques for Composite Materials*, Notes on Numerical Fluid Mechanics. Vieweg, Braunschweig, Submitted to 15th GAMM–Seminar 1999.
- [Dry01] M. Dryja. The Dirichlet–Neumann algorithm for mortar saddle point problems. *BIT*, 41, to appear 2001.
- [dV79] B.M.F. de Veubeke. *A Course in Elasticity*, volume 29 of *Applied Mathematical Sciences*. Springer, 1979.
- [DV97] Z. Dostál and V. Vondrák. Duality based solution of contact problem with coulomb friction. *Arch. Mech.*, 49(3):453–460, 1997.
- [Eck96] C. Eck. *Existenz und Regularität der Lösungen für Kontaktprobleme mit Reibung*. PhD thesis, Universität Stuttgart, 1996.

- [EJ98] C. Eck and J. Jarušek. Existence results for the static contact problem with Coulomb friction. *Math. Models Methods Appl. Sci.*, 8(3):445–468, 1998.
- [ESW99] C. Eck, O. Steinbach, and W.L. Wendland. A symmetric boundary element method for contact problems with friction. *Mathematics and Computers in Simulation*, 50:43–61, 1999.
- [Eul57] L. Euler. Continuation des recherches sur la théorie du mouvement des fluides. *Hist. Acad. Berlin*, pages 316–361, 1757.
- [Eul71] L. Euler. Sectio tertia de motu fluidorum linearis potissimum aquae. *Novi Comm. Petrop.*, 15:219–360, 1771.
- [GL89] R. Glowinski and P. Le Tallec. *Augmented Lagrangian and operator-splitting methods in nonlinear mechanics*. SIAM, 1989.
- [Glo84] R. Glowinski. *Numerical Methods for Nonlinear Variational Problems*. Series in Computational Physics. Springer, New York, 1984.
- [GLT81] R. Glowinski, J.L. Lions, and J.L. Trémolières. *Numerical analysis of variational inequalities*. North-Holland, Amsterdam, 1981.
- [Gur81] M.E. Gurtin. *An Introduction to Continuum Mechanics*. Academic Press, New York, 1981.
- [Hac85] W. Hackbusch. *Multigrid Methods and Applications*, volume 4 of *Computational Mathematics*. Springer-Verlag, Berlin, 1985.
- [Has83] J. Haslinger. Approximation of the signorini problem with friction, obeying the coulomb law. *Math. Meth. in the Appl. Sci.*, 5:422–437, 1983.
- [Has92] J. Haslinger. Signorini problem with Coulomb’s law of friction. Shape optimization in contact problems. *International J. for numerical methods in engineering*, 34:223–231, 1992.
- [HD97] W. Hackbusch and M. Dryja. On the nonlinear domain decomposition method. *BIT*, 37:296–311, 1997.
- [Her82] H. Hertz. Über die Berührung fester elastischer Körper. *J.f. Math.*, 92, 1882.
- [HH80] J. Haslinger and I. Hlaváček. Contact between elastic bodies. I. continuous problems. *Apl. Mat.*, 25:324–327, 1980.
- [HHNL88] I. Hlaváček, J. Haslinger, J. Nečas, and J. Lovíšek. *Solution of variational inequalities in mechanics*. Springer, Berlin, 1988.
- [Hil00] P. Hild. Numerical implementation of two nonconforming finite element methods for unilateral contact. *Comput. Methods Appl. Mech. Eng.*, 184:99–123, 2000.

- [HK94] R.H.W. Hoppe and R. Kornhuber. Adaptive multilevel–methods for obstacle problems. *SIAM J. Numer. Anal.*, 31:301–323, 1994.
- [HM83] W. Hackbusch and H.D. Mittelmann. On multi–grid methods for variational inequalities. *Numer. Math.*, 42:65–76, 1983.
- [Hop87] R.H.W. Hoppe. Multigrid algorithms for variational inequalities. *SIAM J. Numer. Anal.*, 24:1046–1065, 1987.
- [Hop90] R.H.W. Hoppe. Une méthode multigrille pour la solution des problèmes d’obstacle. *M2 AN*, 24:711–736, 1990.
- [IS93] I.R. Ionescu and M. Sofonea. *Functional and Numerical Methods in Viscoplasticity*. Oxford Science Publications. Oxford University Press, Oxford New York Tokio, 1993.
- [IW92] A. Ibrahimov and E.L. Wilson. Unified computational model for static and dynamic frictional contact. *International J. for numerical methods in engineering*, 34:233–247, 1992.
- [Jar83] J. Jarušek. Contact problems with bounded friction, corecive case. *Czechosl. Math. J.*, 33(108):237–261, 1983.
- [KB92] A. Klarbring and G. Björkman. Solution of large displacement contact problems with friction using Newton’s method for generalized equations. *International J. for numerical methods in engineering*, 34:249–269, 1992.
- [Kin82a] D. Kinderlehrer. Estimates for the solution and its stability in Signorini’s problem. *Appl. Math. Optimization*, 8:159–188, 1982.
- [Kin82b] D. Kinderlehrer. Remarks about signorini’s problem. In *Nonlinear partial differential equations and their applications*, Res. Notes Math. 70, pages 234–251. Coll. de France Semin., 1982.
- [KK00] R. Kornhuber and R.H. Krause. Adaptive multigrid methods for Signorini’s problem in linear elasticity. to appear, 2000.
- [KO88] N. Kikuchi and J.T. Oden. *Contact Problems in elasticity*. SIAM, Philadelphia, 1988.
- [Kor94] R. Kornhuber. Monotone multigrid methods for elliptic variational inequalities I. *Numer. Math.*, 69:167–184, 1994.
- [Kor96] R. Kornhuber. A posteriori error estimates for elliptic variational inequalities. *Computers Math. Applic.*, 31:49–60, 1996.
- [Kor97a] R. Kornhuber. *Adaptive monotone multigrid methods for nonlinear variational problems*. Teubner–Verlag, Stuttgart, 1997.

- [Kor97b] R. Kornhuber. Adaptive monotone multigrid methods for some non-smooth optimization problems. In *Domain Decomposition Methods in Sciences and Engineering*, 8th International Conference, Beijing, P. R. China, pages 177–191. John Wiley & Sons, Chichester, New York, Weinheim, Brisbane, Singapore, Toronto, 1997.
- [Kor01] R. Kornhuber. On constrained Newton linearization and multigrid for variational inequalities. to appear in Numerische Mathematik, 2001.
- [KW00] R.H. Krause and B.I. Wohlmuth. Nonconforming domain decomposition methods techniques for linear elasticity. *East-West Journal of Numerical Mathematics*, 8(3):177–206, 2000.
- [KW01] R.H. Krause and B. Wohlmuth. A Dirichlet–Neumann type algorithm for contact problems with friction. Technical Report A 01-09, FU Berlin, 2001.
- [LPR91] C. Licht, E. Pratt, and M. Raous. Remarks on a numerical method for unilateral contact including friction. In F. Maceri G. del Piero, editor, *Unilateral Problems in Structural Analysis IV*, International Series of Numerical Mathematics, pages 129–144, Basel, 1991. Birkhäuser.
- [Man84] J. Mandel. A multi-level iterative method for symmetric, positive definite linear complementarity problems. *Appl. Math. Optim.*, 11:77–95, 1984.
- [MH94] J.E. Marsden and T.J.R. Hughes. *Mathematical Foundations of Elasticity*. Dover, 1994. Originally published by Prentice Hall, 1983.
- [MO87] J.A.C Martins and J.T. Oden. Existence and uniqueness results for dynamic contact problems with nonlinear normal and friction interface laws. *Nonlinear Anal., Theory Methods Appl.*, 11:407–428, 1987.
- [NJH80] J. Nečas, J. Jarušek, and J. Haslinger. On the solution of the variational inequality to the signorini problem with small friction. *Boll. Unione Math. Ital.*, 5(796–811), 1980.
- [Osw90] P. Oswald. On function spaces related to finite element approximation theory. *Zeitschrift für Analysis und ihre Anwendungen*, 9:43–64, 1990.
- [PC99] G. Pietrzak and A. Curnier. Large deformation frictional contact mechanics: continuum formulation and augmented Lagrangian treatment. *Computer Methods in Applied Mechanics and Engeneering*, 177(3–4):351–381, 1999.
- [RCL88] M. Raous, P. Chabrand, and F. Lebon. Numerical methods for frictional contact problems and applications. *J. Mec. Theor. Appl.*, 7:111–128, 1988. Suppl. 1.
- [RMOC86] P. Rabier, J.A.C. Martins, J.T. Oden, and L. Campos. Existence and local uniqueness of solutions to contact problems in elasticity with nonlinear friction laws. *Int. J. Engng. Sci.*, 24(11):1755–1768, 1986.

-
- [Rod87] J.F. Rodrigues. *Obstacle Problems in Mathematical Physics*. Number 134 in Mathematical Studies. North-Holland, Amsterdam, 1987.
- [Sch88] R. Schumann. *A new method for gaining strong regularity statements for the Signorini problem in linear elasticity*. PhD thesis, arl-Marx-Universität, Leipzig, 1988.
- [Sch98a] J. Schöberl. Efficient contact solvers based on domain decomposition techniques. Technical Report Institutsbericht Nr. 545, Universität Linz, 1998.
- [Sch98b] J. Schöberl. Solving the Signorini problem on the basis of domain decomposition techniques. *Computing*, 60:323–344, 1998.
- [Sig33] A. Signorini. Sopra alcune questioni di elastostatica. *Atti della Società Italiana per il Progresso delle Scienze*, 1933.
- [TX] X.-C. Tai and J. Xu. Global convergence of subspace correction methods for convex optimization problems. Submitted to Mathematics of Computation.
- [Wag98] C. Wagner. *Introduction to Algebraic Multigrid*. IWR, Universität Heidelberg, 1998.
- [WCS98] W.L. Wan, T. Chan, and B. Smith. An energy minimizing interpolation for robust multigrid. Technical Report 98-6, Dept. of Mathematics, UCLA, 1998.
- [Wlo82] J. Wloka. *Partielle Differentialgleichungen*. Teubner Verlag, Stuttgart, 1982.
- [Woh00] B.I. Wohlmuth. A mortar finite element method using dual spaces for the Lagrange multiplier. *SIAM J. Numer. Anal.*, 38:989–1012, 2000.
- [Woh01] B.I. Wohlmuth. *Discretization Methods and Iterative Solvers Based on Domain Decomposition*, volume 17 of *Lecture Notes in Computational Science and Engineering*. Springer Heidelberg, 2001.
- [Wri95] P. Wriggers. Finite element algorithms for contact problems. *Arch. Comp. Meth. Engrg.*, 2:1–49, 1995.
- [Wri00] P. Wriggers. private communication. , 2000.
- [Xu89] J. Xu. *Theory of Multilevel Methods*. PhD thesis, Cornell University, 1989.
- [Xu92] J. Xu. New class of iterative methods for nonselfadjoint or indefinite problems. *SIAM J. Numer. Anal.*, 29:303–319, 1992.

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