K3-surfaces with special symmetry

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Contents

Introduction

K3-surfaces are special two-dimensional holomorphic symplectic manifolds. They come equipped with a symplectic form ω, which is unique up to a scalar factor, and their symmetries are naturally partitioned into symplectic and nonsymplectic transformations. An important class of K3-surfaces consists of those possessing an *antisymplectic involution*, i.e., a holomorphic involution σ such that $\sigma^*\omega = -\omega$.

K3-surfaces with antisymplectic involution occur classically as branched double covers of the projective plane, or more generally of Del Pezzo surfaces. This construction is a prominent source of examples and plays a significant role in the classification of log Del Pezzo surfaces of index two (see the works of Alexeev and Nikulin e.g. in [AN06] and the classification by Nakayama [Nak07]). Moduli spaces of K3-surfaces with antisymplectic involution are studied by Yoshikawa in [Yos04], [Yos07], and lead to new developments in the area of automorphic forms.

In this monograph we study K3-surfaces with antisymplectic involution from the point of view of symmetry. On a K3-surface X with antisymplectic involution it is natural the consider those holomorphic symmetries of X compatible with the given structure (X, ω, σ). These are symplectic automorphisms of X commuting with σ.

Given a finite group G one wishes to understand if it can act in the above fashion on a K3-surface X with antisymplectic involution σ. If this is the case, i.e., if there exists a holomorphic action of G on X such that $g^*\omega = \omega$ and $g \circ \sigma = \sigma \circ g$ for all $g \in G$, then the structure of G can yield strong constraints on the geometry of X. More precisely, if the group G has rich structure or large order, it is possible to obtain a precise description of X. This can be considered the guiding classification problem of this monograph.

In Chapter 3 we derive a classification of K3-surfaces with antisymplectic involution centralized by a group of symplectic automorphisms of order greater than or equal to 96. We prove (cf. Theorem 3.25):

Theorem 1. *Let X be a K3-surface with a symplectic action of G centralized by an antisymplectic involution σ such that* $\mathrm{Fix}(\sigma) \neq \emptyset$*. If $|G| > 96$, then X/σ is a Del Pezzo surface and* $\mathrm{Fix}(\sigma)$ *is a smooth connected curve C with $g(C) \geq 3$.*

By a theorem due to Mukai [Muk88] finite groups of symplectic transformations on K3-surfaces are characterized by the existence of a certain embedding into a particular Mathieu group and are subgroups of eleven specified finite groups of maximal symplectic symmetry. This result naturally limits our considerations and has led us to consider the above classification problem for a group G from this list of eleven *Mukai groups*.

Theorem 1 above can be refined to obtain a complete classification of K3-surfaces with a symplectic action of a Mukai group centralized by an antisymplectic involution with fixed points (cf. Theorem 4.1).

Theorem 2. *Let G be a Mukai group acting on a K3-surface X by symplectic transformations. Let σ be an antisymplectic involution on X centralizing G with $\mathrm{Fix}_X(\sigma) \neq \emptyset$. Then the pair (X, G) can be found in Table 4.1.*

In addition to a number of examples presented by Mukai we find new examples of K3-surfaces with maximal symplectic symmetry as equivariant double covers of Del Pezzo surfaces.

It should be emphasized that the description of K3-surfaces with given symmetry does however not necessary rely on the size of the group or its maximality and a classification can also be obtained for rather small subgroups of the Mukai groups. In order to illustrate that the approach does rather depend on the structure of the group, we prove a classification of K3-surfaces with a symplectic action of the group $C_3 \ltimes C_7$ centralized by an antisymplectic involution in Chapter 5. The surfaces with this given symmetry are characterized as double covers of $\mathbb{P}_2$ branched along invariant sextics in a precisely described one-dimensional family $\mathcal{M}$ (Theorem 5.4).

Theorem 3. *The K3-surfaces with a symplectic action of $G = C_3 \ltimes C_7$ centralized by an antisymplectic involution σ are parametrized by the space $\mathcal{M}$ of equivalence classes of sextic branch curves in $\mathbb{P}_2$.*

The group $C_3 \ltimes C_7$ is a subgroup of the simple group $L_2(7)$ of order 168 which is among the Mukai groups. The actions of $L_2(7)$ on K3-surfaces have been studied by Oguiso and Zhang [OZ02] in an a priori more general setup. Namely, they consider finite groups containing $L_2(7)$ as a proper subgroup and obtain lattice theoretic classification results using the Torelli theorem. Since a finite group containing $L_2(7)$ as a proper subgroup posseses, in the cases considered, an antisymplectic involution centralizing $L_2(7)$, we can apply Theorem 4.1 and improve the existing result (cf. Theorem 6.1).

All classification results summarized above are proved by applying the following general strategy.

The quotient of a K3-surface by an antisymplectic involution σ with fixed points centralized by a finite group G is a rational G-surface Y. We apply an equivariant version of the minimal model program respecting finite symmetry groups to the surface Y. Chapter 2 is dedicated to a detailed derivation of this method, a brief outline of which can also be found in the book of Kollár and Mori ([KM98] Example 2.18, see also Section 2.3 in [Mor82]). In the setup of rational surfaces it leads to the well-known classification of G-minimal rational surfaces ([Man67], [Isk80]).

Equivariant Mori reduction and the theory of G-minimal models have applications in various different context and can also be generalized to higher dimensions. Initiated by Bayle and Beauville in [BB00], the methods have been employed in the classification of subgroups of the Cremona group $\mathrm{Bir}(\mathbb{P}_2)$ of the plane for example by Beauville and Blanc ([Bea07], [BB04], [Bla06]), [Bla07], etc.), de Fernex [dF04], Dolgachev and Iskovskikh [DI06], and Zhang [Zha01].

The equivariant minimal model $Y_{\min}$ of Y is obtained from Y by a finite number of blow-downs of (-1)-curves. Since individual (-1)-curves are not necessarily invariant, each reduction step blows down a number of disjoint (-1)-curves. The surface $Y_{\min}$ is, in all cases considered, a Del Pezzo surface.

Using detailed knowledge of the equivariant reduction map $Y \to Y_{\min}$, the shape of the invariant set $\mathrm{Fix}_X(\sigma)$, and the equivariant geometry of Del Pezzo surfaces, we classify Y, $Y_{\min}$ and $\mathrm{Fix}_X(\sigma)$ and can describe X as an equivariant double cover of a possibly blown-up Del Pezzo surface. Besides the book of Manin, [Man74], our analysis relies, to a certain extend, on Dolgachev's discussion of automorphism groups of Del Pezzo surfaces in [Dol08], Chapter 10.

In addition to classification, this method yields a multitude of new examples of K3-surfaces with given symmetry and a more geometric understanding of existing examples. It should be remarked that a number of these arise when the reduction $Y \to Y_{\min}$ is nontrivial.

In the last two chapters we present two different generalizations of our classification strategy for K3-surfaces with antisymplectic involution.

One of our starting points has been the study of K3-surfaces with $L_2(7)$-symmetry by Oguiso and Zhang mentioned above. Apart from a classification result for K3-surfaces with an action the group $L_2(7) \times C_4$, they also show that there does not exist a K3-surface with an action of a the group $L_2(7) \times C_3$. We give an independent proof of this result in Chapter 6. Assuming the existence of such a surface and following the strategy above, we consider the quotient by the nonsymplectic action of C_3 and apply the equivariant minimal model program to its desingularization. Combining this with additional geometric consideration we reach a contradiction.

In the last chapter we consider K3-surfaces X with an action of a finite group $\tilde{G}$ which contains an antisymplectic involution σ but is not of the form $\tilde{G}_{\mathrm{symp}} \times \langle \sigma \rangle$. Since the action of $\tilde{G}_{\mathrm{symp}}$ does not descend to the quotient X/σ we need to restrict our considerations to the centralizer of σ inside $\tilde{G}$. This strategy is exemplified for a finite group $\tilde{A}_6$ characterized by the short exact sequence $\{\mathrm{id}\} \to A_6 \to \tilde{A}_6 \to C_4 \to \{\mathrm{id}\}$. In analogy to the $L_2(7)$-case, the action of $\tilde{A}_6$ on K3-surfaces has been studied by Keum, Oguiso, and Zhang ([KOZ05], [KOZ07]), and a characterization of X using lattice theory and the Torelli theorem has been derived. Since the existing realization of X does however not reveal its equivariant geometry, we reconsider the problem and, though lacking the ultimate classification, find families of K3-surfaces with D_{16}-symmetry, in which the $\tilde{A}_6$-surface is to be found, as branched double covers. These families are of independent interest and should be studied further. In particular, it remains to find criteria to identify the $\tilde{A}_6$-surface inside these families. Possible approaches are outlined at the end of Chapter 7.

Since none of our results depends on the Torelli theorem, our approach to the classification problem allows generalization to fields of appropriate positive characteristic. This possible direction of further research was proposed to the author by Prof. Keiji Oguiso. Another potential further development would be the adaptation of the methods involved in the present work to related questions in higher dimensions.

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1 Finite group actions on K3-surfaces

This chapter is devoted to a brief introduction to finite groups actions on K3-surfaces and presents a number of basic, well-known results: We consider quotients of K3-surfaces by finite groups of symplectic or nonsymplectic automorphisms. It is shown that the quotient of a K3-surface by a finite group of symplectic automorphisms is a K3-surface, whereas the quotient by a finite group containing nonsymplectic transformations is either rational or an Enriques surface. Our attention concerning nonsymplectic automorphisms is then focussed on antisymplectic involutions and the description of their fixed point set. The chapter concludes with Mukai's classification of finite groups of symplectic automorphisms on K3-surfaces and a discussion of basic examples.

1.1 Basic notation and definitions

Let X be a n-dimensional compact complex manifold. We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X and by $\mathcal{K}_X$ its canonical line bundle. The *i^{th} Betti number of X* is the rank of the free part of $H_i(X)$ and denoted by $b_i(X)$.

A *surface* is a compact connected complex manifold of complex dimension two. A *curve* on a surface X is an irreducible 1-dimensional closed subspace of X. The (arithmetic) genus of a curve C is denoted by $g(C)$.

Definition 1.1. A *K3-surface* is a surface X with trivial canonical bundle $\mathcal{K}_X$ and $b_1(X) = 0$.

Note that a K3-surface is equivalently characterized if the condition $b_1(X) = 0$ is replaced by $q(X) = \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X) = 0$ or $\pi_1(X) = \{\mathrm{id}\}$, i.e., X is simply-connected. Examples of K3-surfaces arise as Kummer surfaces, quartic surfaces in $\mathbb{P}_3$ or double coverings of $\mathbb{P}_2$ branched along smooth curves of degree six.

Let X be a K3-surface. Triviality of $\mathcal{K}_X$ is equivalent to the existence of a nowhere vanishing holomorphic 2-form ω on X. Any 2-form on X can be expressed as a complex multiple of ω. We will therefore mostly refer to ω (or ω_X) as "the" holomorphic 2-form on X . We denote by $\mathrm{Aut}_{\mathcal{O}}(X) = \mathrm{Aut}(X)$ the group of holomorphic automorphisms of X and consider a (finite) subgroup $G \hookrightarrow \mathrm{Aut}(X)$. If the context is clear, the abstract finite group G is identified with its image in $\mathrm{Aut}(X)$. The group G is referred to as a transformation group, symmetry group or automorphism group of X. Note that our considerations are independent of the question whether the group $\mathrm{Aut}(X)$ is finite or not. The order of G is denoted by $|G|$.

Definition 1.2. The action of G on X is called *symplectic* if ω is G-invariant, i.e., $g^*\omega = \omega$ for all $g \in G$.

For a finite group $G < \mathrm{Aut}(X)$ we denote by G_{symp} the subgroup of symplectic transformations in G . This group is the kernel of the homomorphism $\chi : G \to \mathbb{C}^*$ defined by the action of G on the space of holomorphic 2-forms $\Omega^2(X) \cong \mathbb{C}\omega$. It follows that G fits into the short exact sequence

$$\{\mathrm{id}\} \to G_{\mathrm{symp}} \to G \to C_n \to \{\mathrm{id}\}$$

for some cyclic group C_n. If both G_{symp} and $C_n \cong G/G_{\mathrm{symp}}$ are nontrivial, then G is called a symmetry group of *mixed type* .

1.2 Quotients of K3-surfaces

Let X be a surface and let $G < \mathrm{Aut}(X)$ be a finite subgroup of the group of holomorphic automorphisms of X. The orbit space X/G carries the structure of a reduced, irreducible, normal complex space of dimension 2 where the sheaf of holomorphic functions is given by the sheaf G-invariant functions on X. In many cases, the quotient is a singular space. The map $X \to X/G$ is referred to as a quotient map or a covering (map).

For reduced, irreducible complex spaces X, Y of dimension 2 a proper holomorphic map $f : X \to Y$ is called *bimeromorphic* if there exist proper analytic subsets $A \subset X$ and $B \subset Y$ such that $f : X \backslash A \to Y \backslash B$ is biholomorphic. A holomorphic, bimeromorphic map $f : X \to Y$ with X smooth is a *resolution of singularities of Y* .

Definition 1.3. A resolution of singularities $f : X \to Y$ is called *minimal* if it does not contract any (-1)-curves, i.e., there is no curve $E \subset X$ with $E \cong \mathbb{P}_1$ and $E^2 = -1$ such that $f(E) = \{\mathrm{point}\}$.

Every normal surface Y admits a minimal resolution of singularities $f : X \to Y$ which is uniquely determined by Y. In particular, this resolution is equivariant.

1.2.1 Quotients by finite groups of symplectic transformations

In the study and classification of finite groups of symplectic transformations on K3-surfaces, the following well-known result has proved to be very useful (see e.g. [Nik80])

Theorem 1.4. *Let X be a K3-surface, G be a finite group of automorphisms of X and $f : Y \to X/G$ be the minimal resolution of X/G. Then Y is a K3-surface if and only if G acts by symplectic transformations.*

For the reader's convenience we give a detailed proof of this theorem. We begin with the following lemma.

Lemma 1.5. *Let X be a simply-connected surface, G be a finite group of automorphisms and $f : Y \to X/G$ be an arbitrary resolution of singularities of X/G. Then $b_1(Y) = 0$.*

Proof. We denote by $\pi_1(Y)$ the fundamental group of Y and by $[\gamma] \in \pi_1(Y)$ the homotopy equivalence class of a closed continuous path γ. The first Betti number is the rank of the free part of

$$H_1(Y) = \pi_1(Y)/[\pi_1(Y), \pi_1(Y)].$$

We show that for each $[\gamma] \in \pi_1(Y)$ there exists $N \in \mathbb{N}$ such that $[\gamma]^N = 0$, i.e., γ^N is homotopic to zero for some $N \in \mathbb{N}$. It then follows that $H_1(Y)$ is a torsion group and $b_1(Y) = 0$.

Let $C \subset X/G$ be the union of branch curves of the covering $q : X \to X/G$, let $P \subset X/G$ be the set of isolated singularities of X/G, and $E \subset Y$ be the exceptional locus of f. Let $\gamma : [0,1] \to Y$ be a closed path in Y. By choosing a path homotopic to γ which does not intersect $E \cup f^{-1}(C)$ we may assume without loss of generality that $\gamma \cap (E \cup f^{-1}(C)) = \emptyset$.

The path γ is mapped to a closed path in $(X/G)\backslash(C \cup P)$ which we denote also by γ. The quotient $q : X \to X/G$ is unbranched outside $C \cup P$ and we can lift γ to a path $\widetilde{\gamma}$ in X. Let $\widetilde{\gamma}(0) = x \in X$, then $\widetilde{\gamma}(1) = g.x$ for some $g \in G$. Since G is a finite group, it follows that $\widetilde{\gamma^N}$ is closed for some $N \in \mathbb{N}$.

As X is simply-connected, we know that also $X\backslash q^{-1}(P)$ is simply-connected. So $\widetilde{\gamma^N}$ is homotopic to zero in $X\backslash q^{-1}(P)$. We can map the corresponding homotopy to $(X/G)\backslash P$ and conclude that γ^N is homotopic to zero in $(X/G)\backslash P$. It follows that γ^N is homotopic to zero in $Y\backslash E$ and therefore in Y. $\square$

Proof of Theorem 1.4. We let $E \subset Y$ denote the exceptional locus of the map $f : Y \to X/G$. If Y is a K3-surface, let ω_Y denote the nowhere vanishing holomorphic 2-form on Y. Let $(X/G)_{\mathrm{reg}}$ denote the regular part of X/G. Since $f|_{Y\backslash E} : Y\backslash E \to (X/G)_{\mathrm{reg}}$ is biholomorphic, this defines a holomorphic 2-form $\omega_{(X/G)_{\mathrm{reg}}}$ on $(X/G)_{\mathrm{reg}}$. Pulling this form back to X, we obtain a G-invariant holomorphic 2-form on $\pi^{-1}((X/G)_{\mathrm{reg}}) = X\backslash\{p_1, \dots p_k\}$. This extends to a nonzero, i.e., not identically zero, G-invariant holomorphic 2-form on X. In particular, any holomorphic 2-form on X is G-invariant and the action of G is by symplectic transformations.

Conversely, if G acts by symplectic transformations on X, then ω_X defines a nowhere vanishing holomorphic 2-form on $(X/G)_{\mathrm{reg}}$ and on $Y\backslash E$. Our aim is to show that it extends to a nowhere vanishing holomorphic 2-form on Y. In combination with Lemma 1.5 this yields that Y is a K3-surface.

Locally at $p \in X$ the action of G_p can be linearized. I.e., there exist a neighbourhood of p in X which is G_p-equivariantly isomorphic to a neighbourhood of $0 \in \mathbb{C}^2$ with a linear action of G_p. A neighbourhood of $\pi(p) \in X/G$ is isomorphic to a neighbourhood of the origin in $\mathbb{C}^2/\Gamma$ for some finite subgroup $\Gamma < \mathrm{SL}(2, \mathbb{C})$. In particular, the points with nontrivial isotropy are isolated. The singularities of X/G are called simple singularities, Kleinian singularities, Du Val singularities or rational double points. Following [Sha94] IV.4.3, we sketch an argument which yields the desired extension result.

Let $X \times_{(X/G)} Y = \{(x,y) \in X \times Y \mid \pi(x) = f(y)\}$ and let N be its normalization. Consider the diagram

$$\begin{tikzcd} X & N \arrow[l, "p_X"'] \arrow[d, "p_Y"] \\ X/G \arrow[u, leftarrow, "\pi"] & Y. \arrow[l, "f"] \end{tikzcd}$$

We let ω_X denote the nowhere vanishing holomorphic 2-form on X. Its pullback $p_X^*\omega_X$ defines a nowhere vanishing holomorphic 2-form on N_{reg}. Simultaneously, we consider the meromorphic

2-form ω_Y on Y obtained by pulling back the 2-form on X/G induced by the G-invariant 2-form ω_X. By contruction, the pullback $p_Y^*\omega_Y$ coincides with the pullback $p_X^*\omega_X$ on N_{reg}.

Consider the finite holomorphic map $p_Y|_{N_{\text{reg}}} : N_{\text{reg}} \to p_Y(N_{\text{reg}}) \subset Y$. Since $p_Y^*\omega_Y$ is holomorphic on N_{reg}, one checks (by a calculation in local coordinates) that ω_Y is holomorphic on $p_Y(N_{\text{reg}}) = Y\backslash\{y_1, \dots y_k\}$ and consequently extends to a holomorphic 2-form on Y. Since $p_X^*\omega_X = p_Y^*\omega_Y$ is nowhere vanishing on N_{reg}, it follows that ω_Y defines a global, nowhere vanishing holomorphic 2-form on Y. □

Remark 1.6. Let g be a symplectic automorphism of finite order on a K3-surface X. Since K3-surfaces are simply-connected, the covering $X \to X/\langle g \rangle$ can never be unbranched. It follows that g must have fixed points.

Using Theorem 1.4 we give an outline of Nikulin's classification of finite Abelian groups of symplectic transformations on a K3-surface [Nik80]. Let C_p be a cyclic group of prime order acting on a K3-surface X by symplectic transformations and Y be the minimal desingularization of the quotient X/C_p.

Notice that by adjunction the self-intersection number of a curve D of genus $g(D)$ on a K3-surface is given by $D^2 = 2g(D) - 2$. In particular, if D is smooth, then $D^2 = -e(D)$.

The exceptional locus of the map $Y \to X/G$ is a union of (-2)-curves and one can calculate their contribution to the topological Euler characteristic $e(Y)$ in relation to $e(X/C_p)$. Let n_p denote the number of fixed point of C_p on X. Then

$$\begin{aligned} 24 = e(X) &= p \cdot e(X/G) - n_p \\ 24 = e(Y) &= e(X/G) + n_p \cdot p. \end{aligned}$$

Combining these formulas gives $n_p = 24/(p+1)$. For a general finite Abelian group G acting symplectically on a K3-surface X, one needs to consider all possible isotropy groups G_x for $x \in X$. By linearization, $G_x < \mathrm{SL}_2(\mathbb{C})$. Since G is Abelian, it follows that G_x is cyclic and an analoguous formula relating the Euler characteristic of X, X/G, and Y can be derived. A case by case study then yields Nikulin's classification. In particular, we emphasize the following remark.

Remark 1.7. If $g \in \mathrm{Aut}(X)$ is a symplectic automorphism of finite order $n(g)$ on a K3-surface X, then $n(g)$ is bounded by eight and the number of fixed points of g is given by the following table:

$n(g)$	2	3	4	5	6	7	8
$\lvert \mathrm{Fix}_X(g) \rvert$	8	6	4	4	2	3	2

Table 1.1: Fixed points of symplectic automorphisms on K3-surfaces

1.2.2 Quotients by finite groups of nonsymplectic transformations

In this subsection we consider the quotient of a K3-surface X by a finite group G such that $G \neq G_{\text{symp}}$, i.e., there exists $g \in G$ such that $g^*\omega \neq \omega$. We prove

Theorem 1.8. *Let X be a K3-surface and let $G < \mathrm{Aut}(X)$ be a finite group such that $g^*\omega \neq \omega$ for some $g \in G$. Then either*

- *X/G is rational, i.e., bimeromorphically equivalent to $\mathbb{P}_2$, or*
- *the minimal desingularisation of X/G is a minimal Enriques surface and*

$$G/G_{symp} \cong C_2.$$

In this case, $\pi : X \to X/G$ is unbranched if and only if $G_{symp} = \{\mathrm{id}\}$.

Before giving the proof, we establish the necessary notation and state two useful lemmata. We denote by $\pi : X \to X/G$ the quotient map. This map can be ramified at isolated points and along curves. Let $P = \{p_1, \ldots, p_n\}$ denote the set of singularities of X/G. For simplicity, the denote the correspondig subset $\pi^{-1}(P)$ of X also by P. Outside P, the map π is ramified along curves C_i of ramification order $c_i + 1$. We write $C = \sum c_i C_i$.

Let $r : Y \to X/G$ denote a minimal resolution of singularities of X/G. The exceptional locus of r in Y is denoted by D. As Y is not necessarily a minimal surface, we denote by $p : Y \to Y_{\min}$ the sucessive blow-down of (-1)-curves. The union of exceptional curves of p is denoted by E.

$$\begin{tikzcd}
C \subset \mathbf{X} \supset P \arrow[d, "\pi"] & \\
\pi(C) \subset \mathbf{X/G} \supset P & D \subset \mathbf{Y} \supset E \arrow[l, "r"'] \arrow[d, "p"] \\
& \mathbf{Y}_{\min}
\end{tikzcd}$$

The following two lemmata (cf. e.g. [BHPVdV04] I.16 and Thm. I.9.1) will be useful in order to relate the canonical bundles of the spaces X, $(X/G)_{\mathrm{reg}}$, Y and $Y_{\min}$. For a divisor D on a manifold X we denote by $\mathcal{O}_X(D)$ the line bundle associated to D .

Lemma 1.9. *Let X, Y be surfaces and let $\varphi : X \to Y$ be a surjective finite proper holomorphic map ramified along a curve C in X of ramification order k. Then*

$$\mathcal{K}_X = \varphi^*(\mathcal{K}_Y) \otimes \mathcal{O}_X(C)^{\otimes(k-1)}.$$

More generally, if π is ramified along a ramification divisor $R = \sum_i r_i R_i$, where R_i is an irreducible curve and $r_i + 1$ is the ramification order of π along R_i, then

$$\mathcal{K}_X = \pi^*(\mathcal{K}_Y) \otimes \mathcal{O}_X(R).$$

Lemma 1.10. *Let X be a surface and let $b : X \to Y$ be the blow-down of a (-1)-curve $E \subset X$. Then*

$$\mathcal{K}_X = b^*(\mathcal{K}_Y) \otimes \mathcal{O}_X(E).$$

We present a proof of Theorem 1.8 using the Enriques Kodaira classification of surfaces.

Proof of Theorem 1.8. The Kodaira dimension of the K3-surface X is $\mathrm{kod}(X) = 0$. The Kodaira dimension of X/G, which is defined as the Kodaira dimension of some resolution of X/G, is less than or equal to the Kodaira dimension of X. (c.f. Theorem 6.10 in [Uen75]),

$$0 = \mathrm{kod}(X) \geq \mathrm{kod}(X/G) = \mathrm{kod}(Y) = \mathrm{kod}(Y_{\min}) \in \{0, -\infty\}.$$

By Lemma 1.5, the first Betti number of Y and $Y_{\min}$ is zero. If $\mathrm{kod}(Y) = -\infty$, then Y is a smooth rational surface. If $\mathrm{kod}(Y) = \mathrm{kod}(Y_{\min}) = 0$, then, since Y is not a K3-surface by Theorem 1.4, it follows that $Y_{\min}$ is an Enriques surface.

If $Y_{\min}$ is an Enriques surface, then $\mathcal{K}_{Y_{\min}}^{\otimes 2}$ is trivial. Let $s \in \Gamma(Y_{\min}, \mathcal{K}_{Y_{\min}}^{\otimes 2})$ be a nowhere vanishing section. Consecutive application of Lemma 1.10 yields the following formula

$$\mathcal{K}_Y^{\otimes 2} = (p^*\mathcal{K}_{Y_{\min}})^{\otimes 2} \otimes \mathcal{O}_Y(E)^{\otimes 2} = p^*(\mathcal{K}_{Y_{\min}}^{\otimes 2}) \otimes \mathcal{O}_Y(E)^{\otimes 2}.$$

Let $e \in \Gamma(Y, \mathcal{O}_Y(E)^{\otimes 2})$ and write $\tilde{s} = p^*(s) \cdot e$. This global section of $\mathcal{K}_Y^{\otimes 2}$ vanishes along E and is nowhere vanishing outside E. By restricting $\tilde{s}$ to $Y \backslash D$ we obtain a section of $\mathcal{K}_{Y\backslash D}^{\otimes 2}$. Since π is biholomorphic outside D, we can map the restricted section to $(X/G)\backslash P = (X/G)_{\mathrm{reg}}$ and obtain a section $\hat{s}$ of $\mathcal{K}_{(X/G)_{\mathrm{reg}}}^{\otimes 2}$. Note that $\hat{s}$ is not the zero-section. If $E \neq \emptyset$, i.e., Y is not minimal, let $E_1 \subset E$ be a (-1)-curve. The minimality of the resolution $r : Y \to X/G$ implies $E_1 \not\subseteq D$. It follows that $\hat{s}$ vanishes along the image of E_1 in $(X/G)_{\mathrm{reg}}$

We may now apply Lemma 1.9 to the map $\pi|_{X\backslash P}$ to see

$$\begin{aligned}\mathcal{K}_{X\backslash P}^{\otimes 2} &= (\pi^*\mathcal{K}_{(X/G)_{\mathrm{reg}}})^{\otimes 2} \otimes \mathcal{O}_{X\backslash P}(C)^{\otimes 2} \\ &= \pi^*(\mathcal{K}_{(X/G)_{\mathrm{reg}}}^{\otimes 2}) \otimes \mathcal{O}_{X\backslash P}(C)^{\otimes 2}.\end{aligned}$$

Let $c \in \Gamma(X\backslash P, \mathcal{O}_{X\backslash P}(C))^{\otimes 2}$. Then $t := \pi^*\hat{s} \cdot c \in \Gamma(X\backslash P, \mathcal{K}_{X\backslash P}^{\otimes 2})$ is not the zero-section but vanishes along C and along the preimage of the zeroes of $\hat{s}$.

Now t extends to a holomorphic section $\tilde{t} \in \Gamma(X, \mathcal{K}_X^{\otimes 2})$. Since X is K3, it follows that both $\mathcal{K}_X$ and $\mathcal{K}_X^{\otimes 2}$ are trivial and $\tilde{t}$ must be nowhere vanishing. Consequently, both E and C must be empty. It follows that the map π is at worst branched at points P (not along curves) and the minimal resolution Y of X/G is a minimal surface.

$$\begin{tikzcd} P \subset \mathbf{X} \arrow[d, "\pi"] & \\ P \subset \mathbf{X/G} & \mathbf{Y} \supset D \arrow[l, "r"'] \end{tikzcd}$$

The section $\tilde{t}$ on X is G-invariant by construction. Let ω be a nonzero section of the trivial bundle $\mathcal{K}_X$ such that $\tilde{t} = \omega^2$. The action of G on X is nonsymplectic, therefore ω is not invariant but $\tilde{t}$ is. Hence G acts on ω by multiplication with $\{1, -1\}$ and $G/G_{\mathrm{symp}} \cong C_2$.

If $\pi : X \to X/G$ is unbranched, it follows that $\mathrm{Fix}_X(g) = \emptyset$ for all $g \in G\backslash\{\mathrm{id}\}$. Since symplectic automorphisms of finite order necessarily have fixed points, this implies $G_{\mathrm{symp}} = \{\mathrm{id}\}$.

Conversely, if G is isomorphic to C_2, it remains to show that the set $P = \{p_1, \ldots, p_n\}$ is empty. Our argument uses the Euler characteristic e of X, X/G, and Y. By chosing a triangulation of X/G such that all points p_i lie on vertices we calculate $24 = e(X) = 2e(X/G) - n$. Blowing up the C_2-quotient singularities of X/G we obtain $12 = e(Y) = e(X/G) + n$. This implies $e(X/G) = 12$ and $n = 0$ and completes the proof of the theorem. □

1.3 Antisymplectic involutions on K3-surfaces

As a special case of the theorem above we consider the quotient of a K3-surface X by an involution $\sigma \in \mathrm{Aut}(X)$ which acts on the 2-form ω by multiplication by -1 and is therefore called *antisymplectic involution*.

Proposition 1.11. *Let $\pi : X \to X/\sigma$ be the quotient of a K3-surface by an antisymplectic involution σ. If $\mathrm{Fix}_X(\sigma) \neq \emptyset$, then $\mathrm{Fix}_X(\sigma)$ is a disjoint union of smooth curves and X/σ is a smooth rational surface. Furthermore, $\mathrm{Fix}_X(\sigma) = \emptyset$ if and only if X/σ is an Enriques surfaces.*

Proof. If $\mathrm{Fix}_X(\sigma) \neq \emptyset$, then Theorem 1.8 and linearization of the σ-action at its fixed points yields the proposition. If $\mathrm{Fix}_X(\sigma) = \emptyset$, then $X \to X/\sigma$ is unbranched and $\mathrm{kod}(X) = \mathrm{kod}(X/G)$. It follows that X/G is an Enriques surface. □

In order to sketch Nikulin's description of the fixed point set of an antisymplectic involution we summarize some information about the Picard lattice of a K3-surface.

1.3.1 Picard lattices of K3-surfaces

Let X be a complex manifold. The *Picard group of X* is the group of isomorphism classes of line bundles on X and denoted by $\mathrm{Pic}(X)$. It is isomorphic to $H^1(X, \mathcal{O}_X^*)$. Let $\mathbb{Z}_X$ denote the constant sheaf on X corresponding to $\mathbb{Z}$, then the exponential sequence $0 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$ induces a map

$$\delta : H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}).$$

Its kernel is the identity component $\mathrm{Pic}^0(X)$ of the Picard group. The quotient $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is isomorphic to a subgroup of $H^2(X, \mathbb{Z})$ and referred to as the *Néron-Severi group $NS(X)$ of X* . On the space $H^2(X, \mathbb{Z})$ there is the natural intersection or cupproduct pairing. The rank of the Néron-Severi group of X is denoted by $\rho(X)$ and referred to as the *Picard number of X*

If X is a K3-surface, then $H^1(X, \mathcal{O}_X) = \{0\}$ and $\mathrm{Pic}(X)$ is isomorphic to $NS(X)$. In particular, the Picard group carries the structure of a lattice, the *Picard lattice* of X, and is regarded as a sublattice of $H^2(X, \mathbb{Z})$, which is known to have signature $(3, 19)$ (cf. VIII.3 in [BHPVdV04]).

If X is an algebraic K3-surface, i.e., the transcendence degree of the field of meromorphic functions on X equals 2, then $\mathrm{Pic}(X)$ is a nondegenerate lattice of signature $(1, \rho - 1)$ (cf. §3.2 in [Nik80]).

1.3.2 The fixed point set of an antisymplectic involution

We can now present Nikulin's classification of the fixed point set of an antisymplectic involution on a K3-surface [Nik83].

Theorem 1.12. *The fixed point set of an antisymplectic involution σ on a K3-surface X is one of the following three types:*

$$1.)\ \mathrm{Fix}(\sigma) = D_g \cup \bigcup_{i=1}^{n} R_i, \qquad 2.)\ \mathrm{Fix}(\sigma) = D_1 \cup D_1', \qquad 3.)\ \mathrm{Fix}(\sigma) = \emptyset,$$

where D_g denotes a smooth curve of genus $g \geq 0$ and $\bigcup_{i=1}^{n} R_i$ is a possibly empty union of smooth disjoint rational curves. In case 2.), D_1 and D_1' denote disjoint elliptic curves.

Proof. Assume there exists a curve D_g of genus $g \geq 2$ in $\mathrm{Fix}(\sigma)$. By adjunction, this curve has positive self-intersection. We claim that each curve D in $\mathrm{Fix}(\sigma)$ disjoint from D_g is rational.

First note that the existence of an antisymplectic automorphism on X implies that X is algebraic (cf. Thm. 3.1 in [Nik80]) and therefore $\mathrm{Pic}(X)$ is a nondegenerate lattice of signature $(1, \rho - 1)$.

If D is elliptic, then $D^2 = 0$, $D_g^2 > 0$ and $D \cdot D_g = 0$ is contrary to the fact that $\mathrm{Pic}(X)$ has signature $(1, \rho - 1)$. If D is of genus ≥ 2, then $D^2 > 0$ and we obtain the same contradiction.

Now assume that there exists an elliptic curve D_1 in $\mathrm{Fix}(\sigma)$. By the considerations above, there may not be curves of genus ≥ 2 in $\mathrm{Fix}(\sigma)$. If there are no further elliptic curves in $\mathrm{Fix}(\sigma)$, we are in case 1) of the classification. If there is another elliptic curve D_1' in $\mathrm{Fix}(\sigma)$, this has to be linearly equivalent to D_1, as otherwise the intersection form of $\mathrm{Pic}(X)$ would degenerate on the span of D_1 and D_1'. The linear system of D_1 defines an elliptic fibration $X \to \mathbb{P}_1$. The induced action of σ on the base may not be trivial since this would force σ to act trivially in a neighbourhood of D_1 in X. It follows that the induced action of σ on $\mathbb{P}_1$ has precisely two fixed points and that $\mathrm{Fix}(\sigma)$ contains no other curves than D_1 and D_1'. This completes the proof of the theorem. $\square$

1.4 Finite groups of symplectic automorphisms

In preparation for stating Mukai's classification of finite groups of symplectic automorphisms on K3-surfaces we present his list [Muk88] of symplectic actions of finite groups G on K3-surfaces X. It is an important source of examples, many of these will occur in our later discussion. For the sake of brevity, at this point we do not introduce the notation of groups used in this table.

	G	$\lvert G\rvert$	**K3-surface** X
1	$L_2(7)$	168	$\{x_1^3x_2 + x_2^3x_3 + x_3^3x_1 + x_4^4 = 0\} \subset \mathbb{P}_3$
2	A_6	360	$\{\sum_{i=1}^6 x_i = \sum_{i=1}^6 x_1^2 = \sum_{i=1}^6 x_i^3 = 0\} \subset \mathbb{P}_5$
3	S_5	120	$\{\sum_{i=1}^5 x_i = \sum_{i=1}^6 x_1^2 = \sum_{i=1}^5 x_i^3 = 0\} \subset \mathbb{P}_5$
4	M_{20}	960	$\{x_1^4 + x_2^4 + x_3^4 + x_4^4 + 12x_1x_2x_3x_4 = 0\} \subset \mathbb{P}_3$
5	F_{384}	384	$\{x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0\} \subset \mathbb{P}_3$
6	$A_{4,4}$	288	$\{x_1^2 + x_2^2 + x_3^2 = \sqrt{3}x_4^2\} \cap$ $\{x_1^2 + \omega x_2^2 + \omega^2 x_3^2 = \sqrt{3}x_5^2\} \cap$ $\{x_1^2 + \omega^2 x_2^2 + \omega x_3^2 = \sqrt{3}x_6^2\} \subset \mathbb{P}_5$
7	T_{192}	192	$\{x_1^4 + x_2^4 + x_3^4 + x_4^4 - 2\sqrt{-3}(x_1^2x_2^2 + x_3^2x_4^2 = 0\} \subset \mathbb{P}_3$
8	H_{192}	192	$\{x_1^2 + x_3^2 + x_5^2 = x_2^2 + x_4^2 + x_6^2\} \cap$ $\{x_1^2 + x_4^2 = x_2^2 + x_5^2 = x_3^2 + x_6^2\} \subset \mathbb{P}_5$
9	N_{72}	72	$\{x_1^3 + x_2^3 + x_3^3 + x_4^3 = x_1x_2 + x_3x_4 + x_5^2 = 0\} \subset \mathbb{P}_4$
10	M_9	72	Double cover of $\mathbb{P}_2$ branched along $\{x_1^6 + y_2^6 + x_3^6 - 10(x_1^3x_2^3 + x_2^3x_3^3 + x_3^3x_1^3) = 0\}$
11	T_{48}	48	Double cover of $\mathbb{P}_2$ branched along $\{x_1x_2(x_1^4 - x_2^4) + x_3^6 = 0\}$

Table 1.2: Finite groups of symplectic automorphisms on K3-surfaces

The following theorem (Theorem 0.6 in [Muk88]) characterizes finite groups of symplectic automorphisms on K3-surfaces.

Theorem 1.13. *A finite group G has an effective sympletic actions on a K3-surface if and only if it is isomorphic to a subgroup of one of the eleven groups in Table 1.2.*

The "only if"-implication of this theorem follows from the list of eleven examples summarized in Table 1.2. This list of examples is, however, far from being exhaustive. It is therefore desirable to find further examples of K3-surfaces where the groups from this list occur and describe or classify these surfaces with maximal symplectic symmetry..

Definition 1.14. By Proposition 8.8 in [Muk88] there are no subgroup relations among the eleven groups in Mukai's list. Therefore, the groups are *maximal finite groups of symplectic transformations*. We refer to the groups in this list also as *Mukai groups*.

1.4.1 Examples of K3-surfaces with symplectic symmetry

We conclude this chapter by presenting two typical examples of K3-surface with symplectic symmetry.

Example 1.15. The group $L_2(7) = \mathrm{PSL}(2, \mathbb{F}_7) = \mathrm{GL}_3(\mathbb{F}_2)$ is a simple group of order 168. It is generated by the three projective transformations α, β, γ of $\mathbb{P}_1(\mathbb{F}_7)$ given by

$$\alpha(x) = x + 1; \quad \beta(x) = 2x; \quad \gamma(x) = -x^{-1}.$$

In terms of these generators, we define a three-dimensional representation of $L_2(7)$ by

$$\alpha \mapsto \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi^2 & 0 \\ 0 & 0 & \xi^4 \end{pmatrix}; \quad \beta \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \gamma \mapsto \frac{-1}{\sqrt{-7}} \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$$

where $\xi = e^{\frac{2\pi i}{7}}$, $a = \xi^2 - \xi^5$, $b = \xi - \xi^6$, $c = \xi^4 - \xi^3$, and $\sqrt{-7} = \xi + \xi^2 + \xi^4 - \xi^3 - \xi^5 - \xi^6$. Klein's quartic curve

$$C_{\mathrm{Klein}} = \{x_1 x_2^3 + x_2 x_3^3 + x_3 x_1^3 = 0\} \subset \mathbb{P}_2$$

is invariant with respect to induced action of $L_2(7)$ on $\mathbb{P}_2$. Mukai's example of a K3-surface with symplectic $L_2(7)$-symmetry is the smooth quartic hypersurface in $\mathbb{P}_3$ defined by

$$X_{\mathrm{KM}} = \{x_1 x_2^3 + x_2 x_3^3 + x_3 x_1^3 + x_4^4 = 0\} \subset \mathbb{P}_3,$$

where the action of $L_2(7)$ is defined to be trivial on the coordinate x_4 and defined as above on x_1, x_2, x_3. Since $L_2(7)$ is a simple group, it follows that the action is effective and symplectic. The surface X_{KM} is called the *Klein-Mukai surface*. By construction, it is a cyclic degree four cover of $\mathbb{P}_2$ branched along Klein's quartic curve. In fact, there is an action of the group $L_2(7) \times C_4$ on X_{KM}, where the action of C_4 is by nonsymplectic transformations. The Klein-Mukai surface will play an important role in Sections 5.4 and 5.5.

Cyclic coverings

Since many examples of K3-surfaces are constructed as double covers we discuss the construction of branched cyclic covers with emphasis on group actions induced on the covering space.

Let Y be a surface such that Picard group of Y has no torsion, i.e., there does not exist a nontrivial line bundle E on Y such that $E^{\otimes n}$ is trivial for some $n \in \mathbb{N}$.

Let B be an effective and reduced divisor on Y and suppose there exists a line bundle L on Y such that $\mathcal{O}_Y(B) = L^{\otimes n}$ and a section $s \in \Gamma(Y, L^{\otimes n})$ whose zero-divisor is B. Let $p : L \to L^{\otimes n}$ denote the bundle homomorphism mapping each element $(y, z) \in L$ for $y \in Y$ to $(y, z^n) \in L^{\otimes n}$. The preimage $X = p^{-1}(\mathrm{Im}(s))$ of the image of s is an analytic subspace of L. The bundle projection $L \to Y$ restricted to X defines surjective holomorphic map $X \to Y$ of degree n.

$$\begin{tikzcd} X \subset L \arrow[r, "p"] \arrow[d] & L^{\otimes n} \supset \mathrm{Im}(s) \arrow[d] \\ Y \arrow[r, "\mathrm{id}"'] & Y \arrow[u, bend right, "s"'] \end{tikzcd}$$

Since $\mathrm{Pic}(Y)$ is torsion free, the line bundle L is uniquely determined by B. It follows than X is uniquely determined and we refer to X as *the* cyclic degree n covering of Y branched along B. We note that X is normal and irreducible. It is smooth if the divisor B is smooth. (cf. I.17 in [BHPVdV04])

Let G be a finite group in $\mathrm{Aut}(Y)$ and assume that the divisor B is invariant, i.e., $gB = B$ for all $g \in G$. Then the pull-back bundle $g^*L^{\otimes n}$ is isomorphic to $L^{\otimes n}$. We consider the group $\mathrm{BAut}(L^{\otimes n})$ of bundle maps of $L^{\otimes n}$ and the homomorphism $\mathrm{BAut}(L^{\otimes n}) \to \mathrm{Aut}(Y)$ mapping each bundle map to the corresponding automorphism of the base. Its kernel is isomorphic to $\mathbb{C}^*$. The observation $g^*L^{\otimes n} \cong L^{\otimes n}$ implies that the group G is contained in the image of $\mathrm{BAut}(L^{\otimes n})$ in $\mathrm{Aut}(Y)$.

By assumption, the zero set of the section s is G-invariant. The bundle map induced by g^* maps the section s to a multiple $\chi(g)s$ of s for some character $\chi : G \to \mathbb{C}^*$. It follows that the bundle map $\tilde{g}$ induced by $\chi(g)^{-1}g^*$ stabilizes the section. The group $\tilde{G} = \{\tilde{g} \,|\, g \in G\} \subset \mathrm{BAut}(L^{\otimes n})$ is isomorphic to G and stabilizes $\mathrm{Im}(s) \subset L^{\otimes n}$.

In order to define a corresponding action on X, first observe that $g^*L \cong L$ for all $g \in G$. This follows from the observation that $g^*L \otimes L^{-1}$ is a torsion bundle and the assumption that $\mathrm{Pic}(Y)$ has no torsion. As above, we deduce that the group G is contained in the image of $\mathrm{BAut}(L)$ in $\mathrm{Aut}(Y)$. Let $\overline{G}$ be the preimage of G in $\mathrm{BAut}(L)$. Then $\overline{G}$ is a central $\mathbb{C}^*$-extension of G,

$$\{\mathrm{id}\} \to \mathbb{C}^* \to \overline{G} \to G \to \{\mathrm{id}\}.$$

The map $p : L \to L^{\otimes n}$ induces a homomorphism $p_* : \mathrm{BAut}(L) \to \mathrm{BAut}(L^{\otimes n})$. Its kernel is isomorphic to $C_n < \mathbb{C}^*$ and we consider the preimage $H = p_*^{-1}(\tilde{G})$ in $\mathrm{BAut}(L)$. The group $H < \overline{G}$ is a central C_n-extension of $\tilde{G} \cong G$,

$$\{\mathrm{id}\} \to C_n \to H \to G \to \{\mathrm{id}\}.$$

By construction, the subset $X \subset L$ is invariant with respect to H. This discussion proves the following proposition.

Proposition 1.16. *Let Y by a surface such that $\mathrm{Pic}(Y)$ is torsion free and $G < \mathrm{Aut}(Y)$ be a finite group. If $B \subset Y$ is an effective, reduced, G-invariant divisor defined by a section $s \in \Gamma(Y, L^{\otimes n})$ for some line bundle L, then the cyclic degree n covering X of Y branched along B carries the induced action of a central C_n-extension H of G such that the covering map $\pi : X \to Y$ is equivariant.*

Example 1.17 (Double covers)**.** For any finite subgroup $G < \mathrm{PSL}(3, \mathbb{C})$ and any G-invariant smooth curve $C \subset \mathbb{P}_2$ of degree six, the double cover X of $\mathbb{P}_2$ branched along C is a K3-surface with an induced action of a degree two central extension of the group G. Many interesting examples (no. 10 and 11 in Mukai's table) can be contructed this way. For example, the Hessian of Klein's curve $\mathrm{Hess}(C_{\mathrm{Klein}})$ is an $L_2(7)$-invariant sextic curve and the double cover of $\mathbb{P}_2$ branched along $\mathrm{Hess}(C_{\mathrm{Klein}})$ is a K3-surface with a symplectic action of $L_2(7)$ centralized by the antisymplectic covering involution (cf. Section 5.5).

2 Equivariant Mori reduction

This chapter deals with a detailed discussion of Example 2.18 in [KM98] (see also Section 2.3 in [Mor82]) and introduces a minimal model program for surfaces respecting finite groups of symmetries. Given a projective algebraic surface X with G-action, in analogy to the usual minimal model program, one obtains from X a G-minimal model $X_{G\text{-min}}$ by a finite number of G-equivariant blow-downs, each contracting a finite number of disjoint (-1)-curves. The surface $X_{G\text{-min}}$ is either a conic bundle over a smooth curve, a Del Pezzo surface or has nef canonical bundle. The case $G \cong C_2$ is also discussed in [BB00], the case $G \cong C_p$ for p prime in [dF04]. As indicated in the introduction, applications can be found throughout the literature.

2.1 The cone of curves and the cone theorem

Throughout this chapter we let X be a smooth projective algebraic surface and let $\mathrm{Pic}(X)$ denote the group of isomorphism classes of line bundles on X.

Definition 2.1. A *divisor* on X is a formal linear combination of irreducible curves $D = \sum a_i C_i$ with $a_i \in \mathbb{Z}$. A *1-cycle* on X is a formal linear combination of irreducible curves $C = \sum b_i C_i$ with $b_i \in \mathbb{R}$. A 1-cycle is *effective* if $b_i \geq 0$ for all i.

We define a pairing $\mathrm{Pic}(X) \times \{\text{divisors}\} \to \mathbb{Z}$ by $(L, D) \mapsto L \cdot D = \deg(L|_D)$. Extending by linearity, this defines a pairing $\mathrm{Pic}(X) \times \{\text{1-cycles}\} \to \mathbb{R}$. We use this notation for the intersection number also for pairs of divisors C and D and write $C \cdot D = \deg(\mathcal{O}_X(D)|_C)$. Two 1-cycles C, C' are called *numerically equivalent* if $L \cdot C = L \cdot C'$ for all $L \in \mathrm{Pic}(X)$. We write $C \equiv C'$. The numerical equivalence class of a 1-cycle C is denoted by $[C]$. The space of all 1-cycles with real coefficients modulo numerical equivalence is a real vector space denoted by $N_1(X)$. Note that $N_1(X)$ is finite-dimensional.

Remark 2.2. Let L be a line bundle on X and let L^{-1} denote its dual bundle. Then $L^{-1} \cdot C = -L \cdot C$ for all $[C] \in N_1(X)$. We therefore write $L^{-1} = -L$ in the following.

Definition 2.3. A line bundle L is called *nef* if $L \cdot C \geq 0$ for all irreducible curves C.

We set

$$NE(X) = \{\sum a_i [C_i] \mid C_i \subset X \text{ irreducible curve}, 0 \leq a_i \in \mathbb{R}\} \subset N_1(X).$$

The closure $\overline{NE}(X)$ of $NE(X)$ in $N_1(X)$ is called *Kleiman-Mori cone* or *cone of curves* on X.

For a line bundle L, we write $\overline{NE}(X)_{L\geq 0} = \{[C] \in N_1(X) \mid L \cdot C \geq 0\} \cap \overline{NE}(X)$. Analogously, we define $\overline{NE}(X)_{L\leq 0}$, $\overline{NE}(X)_{L>0}$, and $\overline{NE}(X)_{L<0}$.

Using this notation we phrase Kleiman's ampleness criterion (cf. Theorem 1.18 in [KM98])

Theorem 2.4. *A line bundle L on X is ample if and only if $\overline{NE}(X)_{L>0} = \overline{NE}(X)\backslash\{0\}$.*

Definition 2.5. Let V be a finite-dimensional real vector space . A subset $N \subset V$ is called *cone* if $0 \in N$ and N is closed under multiplication by positive real numbers. A subcone $M \subset N$ is called *extremal* if $u, v \in N$ satisfy $u, v \in M$ whenever $u + v \in M$. An extremal subcone is also referred to as an *extremal face*. A 1-dimensional extremal face is called *extremal ray*. For subsets $A, B \subset V$ we define $A + B := \{a + b \,|\, a \in A, b \in B\}$.

The cone of curves $\overline{NE}(X)$ is a convex cone in $N_1(X)$ and the following cone theorem, which is stated here only for surfaces, describes its geometry (cf. Theorem 1.24 in [KM98]).

Theorem 2.6. *Let X be a smooth projective surface and let $\mathcal{K}_X$ denote the canonical line bundle on X. There are countably many rational curves $C_i \in X$ such that $0 < -\mathcal{K}_X \cdot C_i \leq \dim(X) + 1$ and*

$$\overline{NE}(X) = \overline{NE}(X)_{\mathcal{K}_X \geq 0} + \sum_i \mathbb{R}_{\geq 0}[C_i].$$

For any $\varepsilon > 0$ and any ample line bundle L

$$\overline{NE}(X) = \overline{NE}(X)_{(\mathcal{K}_X + \varepsilon L)\geq 0} + \sum_{\textit{finite}} \mathbb{R}_{\geq 0}[C_i].$$

2.2 Surfaces with group action and the cone of invariant curves

Let X be a smooth projective surface and let $G < \mathrm{Aut}_{\mathcal{O}}(X)$ be a group of holomorphic transformations of X. We consider the induced action on the space of 1-cycles on X. For $g \in G$ and an irreducible curve C_i we denote by gC_i the image of C_i under g. For a 1-cycle $C = \sum a_i C_i$ we define $gC = \sum a_i(gC_i)$. This defines a G-action on the space of 1-cycles.

Lemma 2.7. *Let C_1, C_2 be 1-cycles and $C_1 \equiv C_2$. Then $gC_1 \equiv gC_2$ for any $g \in G$.*

Proof. The 1-cycle gC_1 is numerically equivalent to gC_2 if and only if $L \cdot (gC_1) = L \cdot (gC_2)$ for all $L \in \mathrm{Pic}(X)$. For $g \in G$ and $L \in \mathrm{Pic}(X)$ let g^*L denote the pullback of L by g. The claim above is equivalent to $((g^{-1})^*L) \cdot (gC_1) = ((g^{-1})^*L) \cdot (gC_2)$ for all $L \in \mathrm{Pic}(X)$. Now

$$((g^{-1})^*L) \cdot (gC_1) = \deg((g^{-1})^*L|_{gC_1}) = \deg(L|_{C_1}) = L \cdot C_1 = L \cdot C_2 = (g^{-1})^*L(gC_2)$$

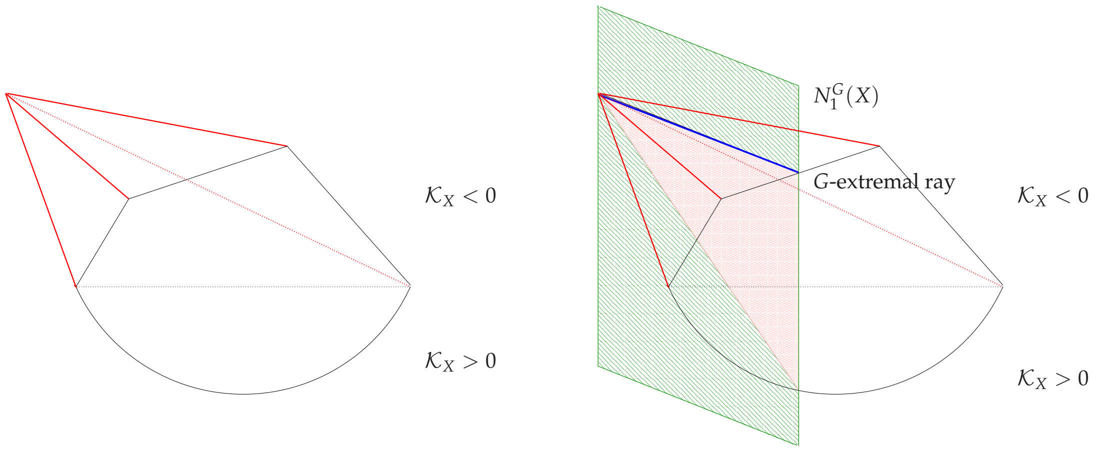
for all $L \in \mathrm{Pic}(X)$. □

This lemma allows us to define a G-action on $N_1(X)$ by setting $g[C] := [gC]$ and extending by linearity. We write $N_1(X)^G = \{[C] \in N_1(X) \mid [C] = [gC] \text{ for all } g \in G\}$, the set of invariant 1-cycles modulo numerical equivalence. This space is a linear subspace of $N_1(X)$.

Since the cone $NE(X)$ is a G-invariant set it follows that its closure $\overline{NE}(X)$ is G-invariant. The subset of invariant elements in $\overline{NE}(X)$ is denoted by $\overline{NE}(X)^G$.

Remark 2.8. $\overline{NE}(X)^G = \overline{NE(X) \cap N_1(X)^G} = \overline{NE}(X) \cap N_1(X)^G$.

The subcone $\overline{NE}(X)^G$ of $\overline{NE}(X)$ is seen to inherit the geometric properties of $\overline{NE}(X)$ established by the cone theorem. Note however that the extremal rays of $\overline{NE}(X)^G$ are in general neither extremal in $\overline{NE}(X)$ (cf. Figure 2.1) nor generated by classes of curves but by classes of 1-cycles.



(a) The cone of curves and its extremal rays

(b) The cone of curves and the invariant subspace $N_1(X)^G$. Their intersection $\overline{NE}(X)^G$ has a new extremal ray.

Figure 2.1: The extremal rays of $\overline{NE}(X)^G$ are not extremal in $\overline{NE}(X)$

Definition 2.9. The extremal rays of $\overline{NE}(X)^G$ are called *G-extremal rays* .

Lemma 2.10. *Let G be a finite group and let R be a G-extremal ray with $\mathcal{K}_X \cdot R < 0$. Then there exists a rational curve C_0 such that R is generated by the class of the 1-cycle $C = \sum_{g \in G} gC_0$.*

Proof. Consider an G-extremal ray $R = \mathbb{R}_{\geq 0}[E]$ where $[E] \in \overline{NE}(X)^G \subset \overline{NE}(X)$. By the cone theorem (Theorem 2.6) it can be written as $[E] = [\sum_i a_i C_i] + [F]$, where $\mathcal{K}_X \cdot F \geq 0$, $a_i \geq 0$ and C_i are rational curves. Let $|G|$ denote the order of G and let $[GF] = G[F] = \sum_{g \in G} g[F]$. Since $g[E] = [E]$ for all $g \in G$ we can write

$$|G|[E] = \sum_{g \in G} g[E] = \sum_{g \in G} ([\sum_i a_i g C_i] + g[F]) = \sum_i a_i G[C_i] + G[F].$$

The element $[\sum a_i(GC_i)] + [GF]$ of the extremal ray $\mathbb{R}_{\geq 0}[E]$ is decomposed as the sum of two elements in $\overline{NE}(X)^G$. Since R is extremal in $\overline{NE}(X)^G$ both must lie in $R = \mathbb{R}_{\geq 0}[E]$.

Consider $[GF] \in R$. Since $g^* \mathcal{K}_X \equiv \mathcal{K}_X$ for all $g \in G$, we obtain

$$\mathcal{K}_X \cdot (GF) = \sum_{g \in G} \mathcal{K}_X \cdot (gF) = \sum_{g \in G} (g^* \mathcal{K}_X) \cdot F = |G| \mathcal{K}_X \cdot F \geq 0.$$

Since $\mathcal{K}_X \cdot R < 0$ by assumption this implies $[F] = 0$ and $\mathbb{R}_{\geq 0}[E] = \mathbb{R}_{\geq 0}[\sum a_i(GCi)]$. Again using the fact that R is extremal in $\overline{NE}(X)^G$, we conclude that each summand of $[\sum a_i(GC_i)]$ must be contained in $R = \mathbb{R}_{\geq 0}[E]$ and the extremal ray $\mathbb{R}_{\geq 0}[E]$ is therefore generated by $[GC_i]$ for some C_i chosen such that $[GC_i] \neq 0$. This completes the proof of the lemma. $\square$

2.3 The contraction theorem and minimal models of surfaces

In this section, we state the contraction theorem for smooth projective surfaces. The proof of this theorem can be found e.g. in [KM98] and needs to be modified slightly in order to give an equivariant contraction theorem in the next section.

Definition 2.11. Let X be a smooth projective surface and let $F \subset \overline{NE}(X)$ be an extremal face. A morphism $\mathrm{cont}_F : X \to Z$ is called the *contraction of F* if

- $(\mathrm{cont}_F)_* \mathcal{O}_X = \mathcal{O}_Z$ and
- $\mathrm{cont}_F(C) = \{\text{point}\}$ for an irreducible curve $C \subset X$ if and only if $[C] \in F$.

The following result is known as the contraction theorem (cf. Theorem 1.28 in [KM98]).

Theorem 2.12. *Let X be a smooth projective surface and $R \subset \overline{NE}(X)$ an extremal ray such that $\mathcal{K}_X \cdot R < 0$. Then the contraction morphism $\mathrm{cont}_R : X \to Z$ exists and is one of the following types:*

1. *Z is a smooth surface and X is obtained from Z by blowing up a point.*
2. *Z is a smooth curve and $\mathrm{cont}_R : X \to Z$ is a minimal ruled surface over Z.*
3. *Z is a point and $-\mathcal{K}_X$ is ample.*

The contraction theorem leads to the minimal model program for surfaces: Starting from X, if $\mathcal{K}_X$ is not nef, i.e, there exists an irreducible curve C such that $\mathcal{K}_X C < 0$, then $\overline{NE}(X)_{\mathcal{K}_X<0}$ is nonempty and there exists an extremal ray R which can be contracted. The contraction morphisms either gives a new surface Z (in case 1) or provides a structure theorem for X which is then either a minimal ruled surface over a smooth curve (case 2) or isomorphic to $\mathbb{P}^2$ (case 3). Note that the contraction theorem as stated above only implies $-\mathcal{K}_X$ ample in case 3. It can be shown that X is in fact $\mathbb{P}^2$. This is omitted here since the statement does not transfer to the equivariant setup. In case 1, we can repeat the procedure if K_Z is not nef. Since the Picard number drops with each blow down, this process terminates after a finite number of steps. The surface obtained from X at the end of this program is called a *minimal model* of X.

Remark 2.13. Let E be a (-1)-curve on X. If C is any irreducible curve on X, then $E \cdot C < 0$ if and only if $C = E$. It follows that $\overline{NE}(X) = \mathrm{span}(\mathbb{R}_{\geq 0}[E], \overline{NE}(X)_{E\geq 0})$. Now $E^2 = -1$ implies $E \notin \overline{NE}(X)_{E\geq 0}$ and E is seen to generate an extremal ray in $\overline{NE}(X)$. By adjunction, $\mathcal{K}_X \cdot E < 0$. The contraction of the extremal ray $R = \mathbb{R}_{\geq 0}[E]$ is precisely the contraction of the (-1)-curve E. Conversely, each extremal contraction of type 1 above is the contraction of a (-1)-curve generating the extremal ray R.

2.4 Equivariant contraction theorem and G-minimal models

We state and prove an equivariant contraction theorem for smooth projective surfaces with finite groups of symmetries. Most steps in the proof are carried out in analogy to the proof of the standard contraction theorem.

Definition 2.14. Let G be a finite group, let X be a smooth projective surface with G-action and let $R \subset \overline{NE}(X)^G$ be G-extremal ray. A morphism $\mathrm{cont}_R^G : X \to Z$ is called the *G-equivariant contraction of R* if

- cont_R^G is equivariant with respect to G
- $(\mathrm{cont}_R^G)_* \mathcal{O}_X = \mathcal{O}_Z$ and
- $\mathrm{cont}_R(C) = \{\text{point}\}$ for an irreducible curve $C \subset X$ if and only if $[GC] \in R$.

Theorem 2.15. *Let G be a finite group, let X be a smooth projective surface with G-action and let R be a G-extremal ray with $\mathcal{K}_X \cdot R < 0$. Then R can be spanned by the class of $C = \sum_{g \in G} gC_0$ for a rational curve C_0, the equivariant contraction morphism $\mathrm{cont}_R^G : X \to Z$ exists and is one of the following three types:*

1. *$C^2 < 0$ and gC_0 are smooth disjoint (-1)-curves. The map $\mathrm{cont}_R^G : X \to Z$ is the equivariant blow down of the disjoint union $\bigcup_{g \in G} gC_0$.*
2. *$C^2 = 0$ and any connected component of C is either irreducible or the union of two (-1)-curves intersecting transversally at a single point. The map $\mathrm{cont}_R^G : X \to Z$ defines an equivariant conic bundle over a smooth curve .*
3. *$C^2 > 0$, $N_1(X)^G = \mathbb{R}$ and $\mathcal{K}_X^{-1}$ is ample, i.e., X is a Del Pezzo surface. The map $\mathrm{cont}_R^G : X \to Z$ is constant, Z is a point.*

Proof. Let R be a G-extremal ray with $\mathcal{K}_X \cdot R < 0$. It follows from Lemma 2.10 that the ray R can be spanned by a 1-cycle of the form $C = GC_0$ for a rational curve C_0. Let $n = |GC_0|$ and write $C = \sum_{i=1}^n C_i$ where the C_i correspond to gC_0 for some $g \in G$. We distinguish three cases according to the sign of the self-intersection of C.

The case $C^2 < 0$

We write $0 > C^2 = \sum_i C_i^2 + \sum_{i \neq j} C_i \cdot C_j$. Since C_i are effective curves we know $C_i \cdot C_j \geq 0$ for all $i \neq j$. Since all curves C_i have the same negative self-intersection and by assumption, $\mathcal{K}_X \cdot C = \sum_i \mathcal{K}_X \cdot C_i = n(\mathcal{K}_X \cdot C_i) < 0$ the adjunction formula reads $2g(C_i) - 2 = -2 = \mathcal{K}_X \cdot C_i + C_i^2$. Consequently, $\mathcal{K}_X \cdot C_i = -1$ and $C_i^2 = -1$. It remains to show that all C_i are disjoint. We assume the contrary and without loss of generality $C_1 \cap C_2 \neq \emptyset$. Now $gC_1 \cap gC_2 \neq \emptyset$ for all $g \in G$ and $\sum_{i \neq j} C_i \cdot C_j \geq n$. This is however contrary to $0 > C^2 = \sum_i C_i^2 + \sum_{i \neq j} C_i \cdot C_j = -n + \sum_{i \neq j} C_i \cdot C_j$.

We let $\mathrm{cont}_R^G : X \to Z$ be the blow-down of $\bigcup_{g \in G} gC_0$ which is equivariant with respect to the induced action on Z and fulfills $(\mathrm{cont}_R^G)_* \mathcal{O}_X = \mathcal{O}_Z$. If D is an irreducible curve such that $\mathrm{cont}_R^G(D) = \{\text{point}\}$, then $D = gC_0$ for some $g \in G$. In particular, $GD = GC_0 = C$ and $[GD] \in R$. Conversely, if $[GD] \in R$ for some irreducible curve D, then $[GD] = \lambda[C]$ for some $\lambda \in \mathbb{R}_{\geq 0}$. Now $(GD) \cdot C = \lambda C^2 < 0$. It follows that D is an irreducible component of C.

The case $C^2 > 0$

This case is treated in precisely the same way as the corresponding case in the standard contraction theorem. Our aim is to show that $[C]$ is in the interior of $\overline{NE}(X)^G$. This is a consequence of the following lemma.

Lemma 2.16. *Let X be a projective surface and let L be an ample line bundle on X. Then the set $Q = \{[E] \in N_1(X) \mid E^2 > 0\}$ has two connected components $Q^+ = \{[E] \in Q \mid L \cdot E > 0\}$ and $Q^- = \{[E] \in Q \mid L \cdot E < 0\}$. Moreover, $Q^+ \subset \overline{NE}(X)$.*

This result follows from the Hodge Index Theorem (cf. Theorem IV.2.14 in [BHPVdV04]) and the fact, that $E^2 > 0$ implies that either E or $-E$ is effective. For a proof of this lemma, we refer the reader to Corollary 1.21 in [KM98].

We consider an effective cycle $C = \sum C_i$ with $C^2 > 0$. By the above lemma, $[C]$ is contained in Q^+ which is an open subset of $N_1(X)$ contained in $\overline{NE}(X)$. It follows that $[C]$ lies in the interior of $\overline{NE}(X)$. The G-extremal ray $R = \mathbb{R}_{\geq 0}[C]$ can only lie in the interior if $\overline{NE}(X)^G = R$. By assumption $\mathcal{K}_X \cdot R < 0$, so that $\mathcal{K}_X$ is negative on $\overline{NE}(X)^G \backslash \{0\}$ and therefore on $\overline{NE}(X) \backslash \{0\}$. The anticanonical bundle $\mathcal{K}_X^{-1}$ is ample by Kleiman ampleness criterion and X is a Del Pezzo surface.

We can define a constant map cont_R^G mapping X to a point Z which is the equivariant contraction of $R = \overline{NE}(X)$ in the sense of Definition 2.14.

The case $C^2 = 0$

Our aim is to show that for some $m > 0$ the linear system $|mC|$ defines a conic bundle structure on X. The argument is seperated into a number of lemmata. For the convenience of the reader, we include also the proofs of well-known preparatory lemmata which do not involve group actions. Recall that $\mathcal{O}(D)$ denotes the line bundle associated to the divisor D on X.

Lemma 2.17. *$H^2(X, \mathcal{O}(mC)) = 0$ for $m \gg 0$.*

Proof. By Serre's duality (cf. Theorem I.5.3 in [BHPVdV04])

$$h^2(X, \mathcal{O}(mC)) = h^0(\mathcal{O}(-mC) \otimes \mathcal{K}_X).$$

Since C is an effective divisor on X, it follows that $h^0(\mathcal{O}(-mC) \otimes \mathcal{K}_X) = 0$ for $m \gg 0$. □

Lemma 2.18. *For $m \gg 0$ the dimension $h^0(X, \mathcal{O}(mC))$ of $H^0(X, \mathcal{O}(mC))$ is at least two.*

Proof. Let m be such that $h^2(X, \mathcal{O}(mC)) = 0$. For a line bundle L on X we denote by $\chi(L) = \sum_i (-1)^i h^i(X, L)$ the Euler characteristic of L. Using the theorem of Riemann-Roch (cf. Theorem V.1.6 in [Har77]),

$$\begin{aligned} h^0(X, \mathcal{O}(mC)) &\geq h^0(X, \mathcal{O}(mC)) - h^1(X, \mathcal{O}(mC)) \\ &= h^0(X, \mathcal{O}(mC)) - h^1(X, \mathcal{O}(mC)) + h^2(X, \mathcal{O}(mC)) \\ &= \chi(\mathcal{O}(mC)) \\ &= \chi(\mathcal{O}) + \frac{1}{2}(\mathcal{O}(mC) \otimes \mathcal{K}_X^{-1}) \cdot (mC) \\ &\overset{C^2=0}{=} \chi(\mathcal{O}) - \frac{m}{2}\mathcal{K}_X \cdot C. \end{aligned}$$

Now $\mathcal{K}_X C < 0$ implies the desired behaviour of $h^0(X, \mathcal{O}(mC))$. □

For a divisor D on X we denote by $|D|$ the complete *linear system of* D, i.e., the set of all effective divisors linearly equivalent to D. A point $p \in X$ is called a *base point* of $|D|$ if $p \in \mathrm{support}(C)$ for all $C \in |D|$.

Lemma 2.19. *There exists $m' > 0$ such that the linear system $|m'C|$ is base point free.*

Proof. We first exclude a positive dimensional set of base points. Let m be chosen such that $h^0(X, \mathcal{O}(mC)) \geq 2$. We denote by B the *fixed part* of the linear system $|mC|$, i.e., the biggest divisor B such that each $D \in |mC|$ can be decomposed as $D = B + E_D$ for some effective divisor E_D. The support of B is the union of all positive dimensional components of the set of base points of $|mC|$. We assume that B is nonempty. The choice of m guarantees that $|mC|$ is not fixed, i.e., there exists $D \in |mC|$ with $D \neq B$. Since $\mathrm{supp}(B) \subset \{s = 0\}$ for all $s \in \Gamma(X, \mathcal{O}(mC))$, each irreducible component of $\mathrm{supp}(B)$ is an irreducible component of C and G-invariance of C implies G-invariance of the fixed part of $|mC|$. It follows that $B = m_0 C$ for some $m_0 < m$. Decomposing $|mC|$ into the fixed part $B = m_0 C$ and the remaining *free part* $|(m - m_0)C|$ shows that some multiple $|m'C|$ for $m' > 0$ has no fixed components. The linear system $|m'C|$ has no isolated base points since these would correspond to isolated points of intersection of divisors linearly equivalent to $m'C$. Such intersections are excluded by $C^2 = 0$. □

We consider the base point free linear system $|m'C|$ and the induced morphism

$$\begin{aligned} \varphi = \varphi_{|m'C|} : & X \to \varphi(X) \subset \mathbb{P}(\Gamma(X, \mathcal{O}(m'C))^*) \\ & x \mapsto \{s \in \Gamma(X, \mathcal{O}(m'C)) \mid s(x) = 0\} \end{aligned}$$

Since C is G-invariant, it follows that φ is an equivariant map with respect to action of G on $\mathbb{P}(\Gamma(X, \mathcal{O}(m'C))^*)$ induced by pullback of sections.

Let us study the fibers of φ. Let z be a linear hyperplane in $\Gamma(X, \mathcal{O}(m'C))$. By definition, $\varphi^{-1}(z) = \bigcap_{s \in z} (s)_0$ where $(s)_0$ denotes the zero set of the section s. Since $(s)_0$ is linearly equivalent to $m'C$ and $C^2 = 0$, the intersection $\bigcap_{s \in z} (s)_0$ does not consist of isolated points but all $(s)_0$ with $s \in z$ have a common component. In particular, each fiber is one-dimensional.

Let $f : X \to Z$ be the Stein factorization of $\varphi : X \to \varphi(X)$. The space Z is normal and 1-dimensional, i.e., Z is a smooth curve. Note that there is a G-action on the smooth curve Z such that f is equivariant.

Lemma 2.20. *The map $f : X \to Z$ defines an equivariant conic bundle, i.e., an equivariant fibration with general fiber isomorphic to $\mathbb{P}_1$.*

Proof. Let F be a smooth fiber of f. By construction, F is a component of $(s)_0$ for some $s \in \Gamma(X, \mathcal{O}(m'C))$. We can find an effective 1-cycle D such that $(s)_0 = F + D$. Averaging over the group G we obtain $\sum_{g \in G} gF + \sum_{g \in G} gD = \sum_{g \in G} g(s)_0$. Recalling $(s)_0 \sim m'C$ and $[C] \in \overline{NE}(X)^G$ we deduce

$$[\sum_{g \in G} gF + \sum_{g \in G} gD] = [\sum_{g \in G} g(s)_0] = m'[\sum_{g \in G} gC] = m|G|[C]$$

showing that $[\sum_{g \in G} gF + \sum_{g \in G} gD]$ in contained in the G-extremal ray generated by $[C]$. Now by the definition of extremality $[\sum_{g \in G} gF] = \lambda[C] \in \mathbb{R}^{>0}[C]$ and therefore $\mathcal{K}_X \cdot (\sum_{g \in G} gF) < 0$. This implies $\mathcal{K}_X F < 0$.

In order to determine the self-intersection of F, we first observe $(\sum_{g \in G} gF)^2 = \lambda^2 C^2 = 0$. Since F is a fiber of a G-equivariant fibration, we know that $\sum_{g \in G} gF = kF + kF_1 + \cdots + kF_l$ where $F, F_1, \ldots F_l$ are distinct fibers of f and $k \in \mathbb{N}^{>0}$. Now $0 = (\sum_{g \in G} gF)^2 = (l + 1)k^2 F^2$ shows $F^2 = 0$. The adjunction formula then implies $g(F) = 0$ and F is isomorphic to $\mathbb{P}_1$. □

The map $\mathrm{cont}_R^G := f$ is equivariant and fulfills $f_* \mathcal{O}_X = \mathcal{O}_Z$ by Stein's factorization theorem. Let D be an irreducible curve in X such that f maps D to a point, i.e., D is contained in a fiber of f. Going through the same arguments as above one checks that $[GD] \in R$. Conversely, if D is an irreducible curve in X such that $[GD] \in R$ it follows that $(GD) \cdot C = 0$. If D is not contracted by f, then $f(D) = Z$ and D meets every fiber of f. In particular, $D \cdot C > 0$, a contradiction. It follows that D must be contracted by f.

This completes the proof of the equivariant contraction theorem. □

The singular fibers of the conic bundle in case 2 of the theorem above are characterized by the following lemma.

Lemma 2.21. *Let $R = \mathbb{R}^{>0}[C]$ be a $\mathcal{K}_X$-negative G-extremal ray with $C^2 = 0$. Let $\mathrm{cont}_R^G := f : X \to Z$ be the equivariant contraction of R defining a conic bundle structure on X. Then every singular fiber of f is a union of two (-1)-curves intersecting transversally.*

Proof. Let F be a singular fiber of f. The same argument as in the previous lemma yields that $\mathcal{K}_X \cdot F < 0$ and $F^2 = 0$. Since F is connected, adjunction implies that the arithmetic genus of F is zero and $\mathcal{K}_X \cdot F = -2$. It follows from the assumption on F being singular that F must be reducible. Let $F = \sum F_i$ be the decomposition into irreducible components. Now $g(F) = 0$ implies $g(F_i) = 0$ for all i.

We apply the same argument as above to the component F_i of F: after averaging over G we deduce that GF_i is in the G-extremal ray R and $\mathcal{K}_X \cdot F_i < 0$. Since $-2 = \mathcal{K}_X \cdot F = \sum \mathcal{K}_X \cdot F_i$, we may conclude that $F = F_1 + F_2$ and $F_i^2 = -1$. The desired result follows. □

G-minimal models of surfaces

Let X be a surface with G-action such that $\mathcal{K}_X$ is not nef, i.e., $\overline{NE}(X)_{\mathcal{K}_X<0}$ is nonempty.

Lemma 2.22. *There exists a G-extremal ray R such that $\mathcal{K}_X \cdot R < 0$.*

Proof. Let $[C] \in \overline{NE}(X)_{\mathcal{K}_X<0} \neq \emptyset$ and consider $[GC] \in \overline{NE}(X)^G$. The G-orbit or G-average of a $\mathcal{K}_X$-negative effective curve is again $\mathcal{K}_X$-negative. It follows that $\overline{NE}(X)^G_{\mathcal{K}_X<0}$ is nonempty. Let L be a G-invariant ample line bundle on X. By the cone theorem, for any $\varepsilon > 0$

$$\overline{NE}(X)^G = \overline{NE}(X)^G_{(\mathcal{K}_X+\varepsilon L)\geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0} G[C_i].$$

where $\mathcal{K}_X \cdot C_i < 0$ for all i. Since $\overline{NE}(X)^G_{\mathcal{K}_X<0}$ is nonempty, we may choose $\varepsilon > 0$ such that $\overline{NE}(X)^G \neq \overline{NE}(X)^G_{(\mathcal{K}_X+\varepsilon L)\geq 0}$. If the ray $R_1 = \mathbb{R}_{\geq 0} G[C_1]$ is not extremal in $\overline{NE}(X)^G$, then its generator $G[C_1]$ can be decomposed as a sum of elements of $\overline{NE}(X)^G$ not contained in R_1. It follows that

$$\overline{NE}(X)^G = \overline{NE}(X)^G_{(\mathcal{K}_X+\varepsilon L)\geq 0} + \sum_{\substack{i\neq 1\\ \text{finite}}} \mathbb{R}_{\geq 0} G[C_i],$$

i.e., the ray R_1 is superfluous in the formula. By assumption $\overline{NE}(X)^G \neq \overline{NE}(X)^G_{(\mathcal{K}_X+\varepsilon L)\geq 0}$ and we may therefore not remove all rays R_i from the formula and at least one ray $R_i = \mathbb{R}_{\geq 0} G[C_i]$ is G-extremal. □

We apply the equivariant contraction theorem to X: In case 1 we obtain from X a new surface Z by blowing down a G-orbit of disjoint (-1)-curves. There is a canonically defined holomorphic G-action on Z such that the blow-down is equivariant. If K_Z is not nef, we repeat the procedure which will stop after a finite number of steps. In case 2 we obtain an equivariant conic bundle structure on X. In case 3 we conclude that X is a Del Pezzo surface with G-action. We call the G-surface obtained from X at the end of this procedure a *G-minimal model of X* .

As a special case, we consider a rational surface X with G-action. Since the canonical bundle $\mathcal{K}_X$ of a rational surface X is never nef (cf. Theorem VI.2.1 in [BHPVdV04]), a G-minimal model of X is an equivariant conic bundle over Z or a Del Pezzo surface with G-action. Note that the base curve Z must be rational: if Z is not rational, one finds nonzero holomorphic one-forms on Z. Pulling these back to X gives rise to nonzero holomorphic one-forms on the rational surface X, a contradiction.

This proves the well-known classification of G-minimal models of rational surfaces (cf. [Man67], [Isk80]). Although this classification does classically not rely on Mori theory, the proof given above is based on Mori's approach. We therefore refer to an equivariant reduction $Y \to Y_{\mathrm{min}}$ as an *equivariant Mori reduction*.

In the following chapters we will apply the equivariant minimal model program to quotients of K3-surfaces by nonsymplectic automorphisms.

3 Centralizers of antisymplectic involutions

This chapter is dedicated to a rough classification of K3-surfaces with antisymplectic involutions centralized by large groups of symplectic transformations (Theorem 3.25).

We consider a K3-surface X with an action of a finite group $G \times C_2 < \mathrm{Aut}(X)$ and assume that the action of G is by symplectic transformations whereas C_2 is generated by an antisymplectic involution σ centralizing G. Furthermore, we assume that $\mathrm{Fix}_X(\sigma) \neq \emptyset$. Let $\pi : X \to X/\sigma = Y$ denote the quotient map. The quotient surface Y is a smooth rational G-surface to which we apply the equivariant minimal model program developed in the previous chapter. A G-minimal model of Y can either be a Del Pezzo surface or an equivariant conic bundle over $\mathbb{P}_1$. In the later case, the possibilities for G are limited by the classification of finite groups with an effective action on $\mathbb{P}_1$

Remark 3.1. The classification of finite subgroups of $\mathrm{SU}(2,\mathbb{C})$ (or $\mathrm{SO}(3,\mathbb{R})$) yields the following list of finite groups with an effective action on $\mathbb{P}_1$:

- cyclic groups C_n
- dihedral groups D_{2n}
- the tetrahedral group $T_{12} \cong A_4$
- the octahedral group $O_{24} \cong S_4$
- the icosahedral group $I_{60} \cong A_5$

If G is any finite group acting on a space X, we refer to the number of elements in an orbit $G.x = \{g.x \,|\, g \in G\}$ as the *length of the G-orbit $G.x$*. Note that the length of a T_{12}-orbit in $\mathbb{P}_1$ is at least four, the length of an O_{24}-orbit in $\mathbb{P}_1$ is at least six, and the length of an I_{60}-orbit in $\mathbb{P}_1$ is at least twelve.

Lemma 3.2. *If a G-minimal model $Y_{\min}$ of Y is an equivariant conic bundle, then $|G| \leq 96$.*

Proof. Let $\varphi : Y_{\min} \to \mathbb{P}_1$ be an equivariant conic bundle structure on $Y_{\min}$. By definition, the general fiber of φ is isomorphic to $\mathbb{P}_1$. We consider the induced action of G on the base $\mathbb{P}_1$. If this action is effective, then G is among the groups specified in the remark above. Since the maximal order of an element in G is eight (cf. Remark 1.7), it follows that the order G is bounded by 60.

If the action of G on the base $\mathbb{P}_1$ is not effective, every element n of the ineffectivity $N < G$ has two fixed points in the general fiber. This gives rise to a positive-dimensional n-fixed point set in $Y_{\min}$ and Y. A symplectic automorphism however has only isolated fixed points. It follows that the action of n on X coincides with the action of σ on $\pi^{-1}(\mathrm{Fix}_Y(N))$. In particular, the order of n is two. Since N acts effectively on the general fiber, it follows that N is isomorphic to either C_2 or $C_2 \times C_2$.

If G/N is isomorphic to the icosahedral group $I_{60} = A_5$, then G fits into the exact sequence $1 \to N \to G \to A_5 \to 1$ for $N = C_2$ or $C_2 \times C_2$. Let η be an element of order five inside A_5. One can find an element ξ of order five in G which is mapped to η. Since neither C_2 nor $C_2 \times C_2$ has automorphisms of order five it follows that ξ centralizes the normal subgroup N. In particular, there is a subgroup $C_2 \times C_5 \cong C_{10}$ in G which is contrary to the assumption that G is a group of symplectic transformations and therefore its elements have order at most eight.

If G/N is cyclic or dihedral, we again use the fact that the order of elements in G is bounded by 8 and conclude $|G/N| \leq 16$. It follows that the maximal possible order of G/N is 24. Using $|N| \leq 4$ we obtain $|G| \leq 96$. □

If $|G| > 96$, the lemma above allows us to restrict our classification to the case where a G-minimal model $Y_{\min}$ of Y is a Del Pezzo surface. The next section is devoted to a brief introduction to Del Pezzo surfaces and their automorphisms groups.

3.1 Del Pezzo surfaces

A *Del Pezzo surface* is a smooth surface Z such that the anticanonical bundle $\mathcal{K}_Z^{-1} = \mathcal{O}_Z(-K_Z)$ is ample. The self-intersection number of the canonical divisor $d := K_Z^2$ is referred to as the *degree* of the Del Pezzo surface and $1 \leq d \leq 9$ (cf. Theorem 24.3 in [Man74]).

Example 3.3. Let $Z = \{f_3 = 0\} \subset \mathbb{P}_3$ be a smooth cubic surface. The anticanonical bundle $\mathcal{K}_Z^{-1}$ of Z is given by the restriction of the hyperplane bundle $\mathcal{O}_{\mathbb{P}_3}(1)$ to Z and therefore ample.

As a consequence of the adjunction formula, an irreducible curve with negative self-intersection on a Del Pezzo surface is a (-1)-curve. The following theorem (cf. Theorem 24.4 in [Man74]) gives a classification of Del Pezzo surfaces according to their degree.

Theorem 3.4. *Let Z be a Del Pezzo surface of degree d.*

- *If $d = 9$, then Z is isomorphic to $\mathbb{P}_2$.*
- *If $d = 8$, then Z is isomorphic to either $\mathbb{P}_1 \times \mathbb{P}_1$ or the blow-up of $\mathbb{P}_2$ in one point.*
- *If $1 \leq d \leq 7$, then Z is isomorphic to the blow-up of $\mathbb{P}_2$ in $9 - d$ points in general position, i.e., no three points lie on one line and no six points lie on one conic.*

In our later considerations of Del Pezzo surfaces Table 3.1 below (cf. Theorem 26.2 in [Man74]) specifying the number of (-1)-curves on a Del Pezzo surface of degree d will be very useful.

degree d	1	2	3	4	5	6	7
number of (-1)-curves	240	56	27	16	10	6	3

Table 3.1: (-1)-curves on Del Pezzo surfaces

Example 3.5. Let Z be a Del Pezzo surface of degree 5. It follows from the theorem above that Z is isomorphic to the blow-up of $\mathbb{P}_2$ in four points $p_1, \dots, p_4$ in general position. We denote by E_i the preimage of p_i is Z. Let L_{ij} denote the line in $\mathbb{P}_2$ joining p_i and p_j and note that there are precisely six lines of this type. The proper transform of L_{ij} is a (-1)-curve in Z. We have thereby specified all ten (-1)-curves in Z. Their incidence graph is known as the *Petersen graph* .

The following theorem summarizes properties of the anticanonical map, i.e., the map associated to the linear system $|-K_Z|$ of the anticanonical divisor (Theorem 24.5 in [Man74] and Theorem 8.3.2 in [Dol08])

Theorem 3.6. *Let Z be a Del Pezzo surface of degree d. If $d \geq 3$, then $\mathcal{K}_Z^{-1}$ is very ample and the anticanonical map is a holomorphic embedding of Z into $\mathbb{P}_d$ such that the image of Z in $\mathbb{P}_d$ is of degree d.*

If $d = 2$, then the anticanonical map is a holomorphic degree two cover $\varphi : Z \to \mathbb{P}_2$ branched along a smooth quartic curve.

If $d = 1$, then the linear system $|-K_Z|$ has exactly one base point p. Let $Z' \to Z$ be the blow-up of p. Then the pull-back of $-K_Z$ to Z' defines an elliptic fibration $f : Z' \to \mathbb{P}_1$. The linear system $|-2K_Z|$ defines a finite map of degree two onto a quadric cone Q in $\mathbb{P}_3$. Its branch locus is given by the intersection of Q with a cubic surface.

Our understanding of Del Pezzo surfaces as surfaces obtained by blowing-up points in $\mathbb{P}_2$ in general position or as degree d subvarieties of $\mathbb{P}_d$ enables us the decide whether certain finite groups G can occur as subgroups of the automorphisms group $\mathrm{Aut}(Z)$ of a Del Pezzo surface Z.

Example 3.7. Consider the semi-direct product $G = C_3 \ltimes C_7$ where the action of C_3 on C_7 is defined by the embedding of C_3 into $\mathrm{Aut}(C_7) \cong C_6$. The group G is a maximal subgroup of the simple group $L_2(7)$ which is discussed below. Let Z be a Del Pezzo surface of degree d with an effective action of G. Since G does not admit a two-dimensional representation, it follows that G does not have fixed points in Z. In particular, $d \neq 1$. For the same reason, Z is not the blow-up of $\mathbb{P}_2$ in one or two points. Since there is no nontrivial homomorphisms $G \to C_2$ and no injective homomorphism $G \to \mathrm{PSL}(2, \mathbb{C})$ it follows that $G \not\hookrightarrow \mathrm{Aut}(\mathbb{P}_1 \times \mathbb{P}_1) = (\mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})) \rtimes C_2$.

In many cases it can be useful to consider possible actions of a finite group G on the union of (-1)-curves on a Del Pezzo surfaces.

Example 3.8. We consider $G = L_2(7)$, the simple group of order 168. Its maximal subgroups are $C_3 \ltimes C_7$ and S_4. Assume G acts effectively on a Del Pezzo surface Z of degree d. Since $L_2(7)$ does not stabilize any smooth rational curve, the G-orbit of a (-1)-curve $E \subset Z$ consists of 7, 8, 14, 24 or more curves. It now follows from Table 3.1 that $d \neq 3, 5, 6$.

If $d = 4$, then the union of (-1)-curves on Z would consist of two G-orbits of length 8. In particular, $\mathrm{Stab}_G(E) \cong C_3 \ltimes C_7$ for any (-1)-curve $E \subset Z$. Blowing down E to a point $p \in Z'$ induces an action of $C_3 \ltimes C_7$ on Z' fixing p. Since $C_3 \ltimes C_7$ does not admit a two-dimensional representation, it follows that the normal subgroup C_7 acts trivially on Z' and therefore on Z. This is a contradiction.

Using the result of the previous example, it follows that Z is either a Del Pezzo surface of degree 2 or isomorphic to $\mathbb{P}_2$. Both cases will play a role in our discussion of K3-surfaces with an action of $L_2(7)$.

Example 3.9. Let be the Del Pezzo surface obtained by blowing up one point p in $\mathbb{P}_2$. Then its automorphims group is the subgroup of $\mathrm{Aut}(\mathbb{P}_2)$ fixing the point p. Similarly, if Z is the Del Pezzo surface obtained by blowing up two points p, q in $\mathbb{P}_2$, then $\mathrm{Aut}(Z) = G \rtimes C_2$ where G is the subgroup of $\mathrm{Aut}(\mathbb{P}_2)$ fixing the two points p, q and C_2 acts by switching the exceptional curves E_p, E_q.

In the previous chapter we have shown that Del Pezzo surfaces can occur as equivariant minimal models. It should be remarked that the blow-up of $\mathbb{P}_2$ in one or two points is never equivariantly minimal: Let Z be the surface obtained by blowing up one or two points in $\mathbb{P}_2$. Then Z contains an $\mathrm{Aut}(Z)$-invariant (-1)-curve, namely the curve E_p in the first case and the proper transform of the line joining p and q in the second case. This curve can always be blown down equivariantly. Using the language of equivariant Mori theory introduced in the previous chapter, the $\mathrm{Aut}(Z)$-invariant (-1)-curve spans a $\mathrm{Aut}(Z)$-extremal ray R of the cone of invariant curves $\overline{NE}(X)^{\mathrm{Aut}(Z)}$ with $\mathcal{K}_Z \cdot R < 0$. Its contraction defines an $\mathrm{Aut}(Z)$-equivariant map to $\mathbb{P}_2$. In particular, Z is not equivariantly minimal.

Remark 3.10. A complete classification of automorphisms groups of Del Pezzo surfaces can be found in [Dol08].

3.2 Branch curves and Mori fibers

We return to the initial setup where X is a K3-surface with an action of $G \times \langle \sigma \rangle$ and $\pi : X \to X/\sigma = Y$ denotes the quotient map, and fix an equivariant Mori reduction $M : Y \to Y_{\mathrm{min}}$.

A rational curve $E \subset Y$ is called a *Mori fiber* if it is contracted in some step of the equivariant Mori reduction $Y \to Y_{\mathrm{min}}$. The set of all Mori fibers is denoted by $\mathcal{E}$. Its cardinality $|\mathcal{E}|$ is denoted by m. We let n denote the total number of rational curves in $\mathrm{Fix}_X(\sigma)$.

Lemma 3.11. *The total number m of Mori fibers in Y is bounded by $m \leq n + 12 - e(Y_{\mathrm{min}}) \leq n + 9$.*

Proof. Recall that $\mathrm{Fix}_X(\sigma)$ is a disjoint union of smooth curves. We choose a triangulation of $\mathrm{Fix}_X(\sigma)$ and extend it to a triangulation of the surface X. The topological Euler characteristic of the double cover is

$$
\begin{aligned}
e(X) = 24 &= 2e(Y) - \sum_{C \subset \mathrm{Fix}_X(\sigma)} e(C) \\
&= 2e(Y) - \sum_{C \subset \mathrm{Fix}_X(\sigma)} (2 - 2g(C)) \\
&= 2e(Y) - 2n + \sum_{\substack{C \subset \mathrm{Fix}_X(\sigma) \\ g(C) \geq 1}} (2g(C) - 2) \\
&\geq 2e(Y) - 2n \\
&= 2(e(Y_{\mathrm{min}}) + m) - 2n
\end{aligned}
$$

This yields $m \leq n + 12 - e(Y_{\mathrm{min}})$, and $e(Y_{min}) \geq 3$ completes the proof of the lemma. □

Let $R := \mathrm{Fix}_X(\sigma) \subset X$ denote the ramification locus of π and let $B := \pi(R) \subset Y$ be its branch locus. In the following, we repeatedly use the fact that for a finite proper surjective holomorphic map of complex manifolds (spaces) $\pi : X \to Y$ of degree d, the intersection number of pullback divisors fulfills $(\pi^* D_1 \cdot \pi^* D_2) = d(D_1 \cdot D_2)$.

Lemma 3.12. *Let $E \in \mathcal{E}$ be a Mori fiber such that $E \not\subset B$ and $|E \cap B| \geq 2$ or $E \cdot B \geq 3$. Then $E^2 = -1$ and $\pi^{-1}(E)$ is a smooth rational curve in X. Furthermore, $E \cdot B = |E \cap B| = 2$.*

Proof. Let $k < 0$ denote self-intersection number of E. By the remark above, the divisor $\pi^{-1}(E) = \pi^* E$ has self-intersection $2k$. Assume that $\pi^{-1}(E)$ is reducible and let $\tilde{E}_1, \tilde{E}_2$ denote its irreducible components. They are rational and therefore, by adjunction on the K3-surface X, have self-intersection number -2. Write

$$0 > 2k = (\pi^{-1}(E))^2 = \tilde{E}_1^2 + \tilde{E}_2^2 + 2(\tilde{E}_1 \cdot \tilde{E}_2) = -4 + 2(\tilde{E}_1 \cdot \tilde{E}_2).$$

Since $\tilde{E}_1$ and $\tilde{E}_2$ intersect at points in the preimage of $E \cap B$, we obtain $\tilde{E}_1 \cdot \tilde{E}_2 \geq 2$, a contradiction. It follows that $\pi^{-1}(E)$ is irreducible. Consequently, $k = -1$ and $\pi^{-1}(E)$ is a smooth rational curve with two σ-fixed points . □

Remark 3.13. Let $E \in \mathcal{E}$ be a Mori fiber.

- If $E \subset B$, then E is the image of a rational curve in X and $E^2 = -4$. (cf. Corollary 3.16 below).
- If $E \not\subset B$ and $\pi^{-1}(E)$ is irreducible, then $2E^2 = (\pi^{-1}(E))^2 < 0$. Adjunction on X implies that $(\pi^{-1}(E))^2 = -2$ and that $\pi^{-1}(E)$ is a smooth rational curve in X. The action of σ has two fixed points on $\pi^{-1}(E)$ and the restricted degree two map $\pi|_{\pi^{-1}(E)} : \pi^{-1}(E) \to E$ is necessarily branched, i.e., $E \cap B \neq \emptyset$.
- If $E \not\subset B$ and $\pi^{-1}(E) = \tilde{E}_1 + \tilde{E}_2$ is reducible, then

  $$2E^2 = \underbrace{\tilde{E}_1^2}_{\geq -2} + \underbrace{2(\tilde{E}_1 \cdot \tilde{E}_2)}_{\geq 0} + \underbrace{\tilde{E}_2^2}_{\geq -2} \geq -4.$$

  In particular, $E^2 \in \{-1, -2\}$.

  – If $E^2 = -1$, then $\tilde{E}_1 \cdot \tilde{E}_2 = 1$ and $E \cap B \neq \emptyset$.
  – If $E^2 = -2$, then $\tilde{E}_1 \cdot \tilde{E}_2 = 0$ and $E \cap B = \emptyset$.

In summary, a Mori fiber $E \not\subset B$ has self-intersection -1 if and only if $E \cap B \neq \emptyset$ and self-intersection -2 if and only if $E \cap B = \emptyset$. A Mori fiber E has self-intersection -4 if and only if $E \subset B$.

More generally, any (-1)-curve E on Y meets B in either one or two points. If $|E \cap B| = 1$, then $\pi^{-1}(E) = E_1 \cup E_2$ is reducible. If $|E \cap B| = 2$, then $\pi^{-1}(E)$ is irreducible and meets $\mathrm{Fix}_X(\sigma) = R = \pi^{-1}(B)$ in two points.

Proposition 3.14. *Every Mori fiber $E \in \mathcal{E}$, $E \not\subset B$ meets the branch locus B in at most two points. If E and B are tangent at p, then $E \cap B = \{p\}$ and $(E \cdot B)_p = 2$.*

Proof. Let $E \in \mathcal{E}$, $E \not\subset B$ and assume $|E \cap B| \geq 2$ or $E \cdot B \geq 3$. Then by the lemma above, $\tilde{E} = \pi^{-1}(E)$ is a smooth rational curve in X. Since $\tilde{E} \not\subset \mathrm{Fix}_X(\sigma)$, the involution σ has exactly two fixed points on $\tilde{E}$ showing $|E \cap B| = 2$. It remains to show that the intersection is transversal.

To see this, let $N_{\tilde{E}}$ denote the normal bundle of $\tilde{E}$ in X. We consider the induced action of σ on $N_{\tilde{E}}$ by a bundle automorphism. Using an equivariant tubular neighbourhood theorem we may equivariantly identify a neighbourhood of $\tilde{E}$ in X with $N_{\tilde{E}}$ via a C^∞-diffeomorphism. The σ-fixed point curves intersecting $\tilde{E}$ map to curves of σ-fixed points in $N_{\tilde{E}}$ intersecting the zero-section and vice versa. Let D be a curve of σ-fixed point in $N_{\tilde{E}}$. If D is not a fiber of $N_{\tilde{E}}$, it follows that σ stabilizes all fibers intersecting D and the induced action of σ on the base must be trivial, a contradiction. It follows that the σ-fixed point curves correspond to fibers of $N_{\tilde{E}}$, and E and B meet transversally.

By negation of the implication above, if E and B are tangent at p, then $|E \cap B| = 1$ and $E \cdot B = 2$. □

3.2.1 Rational branch curves

In this section we find conditions on G, in particular conditions on the order of G, guaranteeing the absence of rational curves in $\mathrm{Fix}_X(\sigma)$.

Lemma 3.15. *Let $\pi : X \to Y$ be a cyclic degree two cover of surfaces and let $C \subset X$ be a smooth curve contained in the ramification locus of π. Then the image of C in Y has self-intersection $(\pi(C))^2 = 2C^2$.*

Proof. We recall that the intersection of pullback divisors fulfills $\pi^*D_1 \cdot \pi^*D_2 = 2(D_1 \cdot D_2)$. In the setup of the lemma, $(\pi^*\pi(C))^2 = 2(\pi(C))^2$. Now $\pi^*\pi(C) \sim 2C$ implies the desired result. □

Note that the lemma above can also be proved by considering the normal bundle N_C of C and the induced action of σ on it. The normal bundle $N_{\pi(C)}$ is isomorphic to N_C^2. Since the self-intersection of a curve is the degree of the normal bundle restricted to the curve, the formula follows.

Corollary 3.16. *Let X be a K3-surface and let $\pi : X \to Y$ be a cyclic degree two cover. Then a rational branch curve of π has self-intersection -4.*

Proof. Let C be a rational curve on the K3-surface X. Then by adjunction $C^2 = -2$ and the image $\pi(C)$ in Y is a (-4)-curve by Lemma 3.15 above. □

On a Del Pezzo surface a curve with negative self-intersection necessarily has self-intersection -1. So if $Y_{\min}$ is a Del Pezzo surface, all rational branch curves of π, which have self-intersection -4 by Corollary 3.16, need to be modified by the Mori reduction when passing to $Y_{\min}$ and therefore have nonempty intersection with the union of Mori fibers.

An important tool in the study of rational branch curves is provided by the following lemma which describes the behaviour of self-intersection numbers under monoidal transformations.

Lemma 3.17. *Let $\tilde{X}$ and X be smooth projective surfaces and let $b : \tilde{X} \to X$ be the blow-down of a (-1)-curve $E \subset \tilde{X}$. For a curve $B \subset \tilde{X}$ having no common component with E the self-intersection of its image in X is given by*

$$b(B)^2 = B^2 + (E \cdot B)^2.$$

Proof. We may choose an ample divisor H in X with $p \notin \mathrm{supp}(H)$ and D linearly equivalent to $b(B) + H$ such that $p \notin \mathrm{supp}(D)$. Since b is biholomorphic away from p, we know

$$(b(B) + H)^2 = D^2 = (b^*D)^2 = (b^*((b(B) + H))^2.$$

Using $(b^*H)^2 = H^2$ and $b^*(b(B)) \cdot b^*H = b(B) \cdot H$ we find $b(B)^2 = (b^*B)^2$. Now $b^*B = B + \mu E$ where μ denotes the multiplicity of the point $p \in b(B)$. This multiplicity equals the intersection multiplicity $E \cdot B$. Therefore,

$$b(B)^2 = (b^*B)^2 = (B + \mu E)^2 = B^2 + 2\mu^2 - \mu^2 = B^2 + \mu^2.$$

and the lemma follows. □

We denote by $\mathcal{C}$ the set of rational branch curves of π. The total number $|\mathcal{C}|$ of these curves is denoted by n. The union of all Mori fibers not contained in the branch locus B is denoted by $\bigcup E_i$.

Let $\mathcal{C}_{\geq k} = \{C \in \mathcal{C} \mid |C \cap \bigcup E_i| \geq k\}$ be the set of those rational branch curves C which meet $\bigcup E_i$ in at least k distinct points and let $|\mathcal{C}_{\geq k}| = r_k$. We let $\mathcal{E}_{\geq k}$ denote the set of Mori fibers $E \not\subset B$ which intersect some C in $\mathcal{C}_{\geq k}$ and define

$$P_k = \{(p, E) \mid p \in C \cap E,\ E \in \mathcal{E}_{\geq k},\ C \in \mathcal{C}_{\geq k}\} \subseteq Y \times \mathcal{E}_{\geq k}$$

and the projection map $\mathrm{pr}_k : P_k \to \mathcal{E}_{\geq k}$ mapping (p, E) to E. This map is surjective by definition of $\mathcal{E}_{\geq k}$ and its fibers consist of ≤ 2 points by Proposition 3.14. Using $|P_k| \geq k r_k$ we see

$$|\mathcal{E}_{\geq k}| \geq \frac{k}{2} r_k. \tag{3.1}$$

Let N be the largest positive integer such that $\mathcal{C}_{\geq N} = \mathcal{C}$, i.e., each rational ramification curve is intersected at least N times by Mori fibers. A curve $C \in \mathcal{C}$ which is intersected precisely N times by Mori fibers is referred to as a *minimizing curve*. In the following, let C be a minimizing curve and let $H = \mathrm{Stab}_G(C) < G$ be the stabilizer of C in G.

Remark 3.18. The index of H in G is bounded by $n = r_N$.

Bounds for n

A smooth rational curve on a K3-surface has self-intersection -2 and all curves in $\mathrm{Fix}_X(\sigma)$ are disjoint. Therefore, the rational curves in $\mathrm{Fix}_X(\sigma)$ generate a sublattice of $\mathrm{Pic}(X)$ of signature $(0, n)$. It follows immediately that $n \leq 19$.

A sharper bound $n \leq 16$ for the number of disjoint (-2)-curves on a K3-surface has been obtained by Nikulin [Nik76] and the following optimal bound in our setup is due to Zhang [Zha98], Theorem 3.

Proposition 3.19. *The total number of connected curves in the fixed point set of an antisymplectic involution on a K3-surface is bounded by 10.*

Corollary 3.20. *The number n of rational curves in $\mathrm{Fix}_X(\sigma)$ is at most 10. If $n = 10$, then $\mathrm{Fix}_X(\sigma)$ is a union of rational curves.*

In the following, we use Zhang's bound $n \leq 10$. Note, however, that all results can likewise be obtained by using the weakest bound $n \leq 19$.

For $N \geq 4$ Zhang's bound can be sharpened using the notion of Mori fibers and minimizing curves.

Lemma 3.21. $\frac{N}{2} n \leq n + 12 - e(Y_{\min}) \leq n + 9$.

Proof. Using Lemma 3.2 and inequality (3.1) $\frac{N}{2} n = \frac{N}{2} r_N \leq |\mathcal{E}_{\geq N}| \leq |\mathcal{E}| \leq n + 12 - e(Y_{\min}) \leq n + 9$. □

In the following we consider the stabilizer H of a minimizing curve C and using the above bounds for n, we obtain bounds for $|G|$.

A bound for $|G|$

Proposition 3.22. *Let X be a K3-surface with an action of a finite group $G \times \langle \sigma \rangle$ such that $G < \mathrm{Aut}_{\mathrm{symp}}(X)$ and σ is an antisymplectic involution with fixed points. If $|G| > 108$, then $\mathrm{Fix}_X(\sigma)$ contains no rational curves.*

Proof. Assume that $\mathrm{Fix}_X(\sigma)$ contains rational curves and consider a minimizing curve $C \subset B$ and its stabilizer $\mathrm{Stab}_G(C) =: H$. Since a symplectic automorphism on X does not admit a one-dimensional set of fixed points, it follows that the action of H on C is effective and H is among the groups discussed in Remark 3.1. We recall the possible lengths of H-orbits in C: the length of an orbit of a dihedral group is at least two, the length of a T_{12}-orbit in $\mathbb{P}_1$ is at least four, the length of an O_{24}-orbit in $\mathbb{P}_1$ is at least six, and the length of an I_{60}-orbit in $\mathbb{P}_1$ is at least twelve.

Let $Y_{\min}$ be a G-minimal model of $X/\sigma = Y$. Recall that by Lemma 3.2 $Y_{\min}$ is a Del Pezzo surface. Each rational branch curve is a (-4)-curve in Y. Since its image in $Y_{\min}$ has self-intersection ≥ -1, it must intersect Mori fibers.

- If $N = 1$, i.e., the rational curve C meets the union of Mori fibers in exactly one point p, then p is a fixed point of the H-action on C. In particular, H is a cyclic group C_k. By Remark 1.7 $k \leq 8$. Since the index of H in G is bounded by $n \leq 10$, it follows that $|G| \leq 80$.

- If $N = 2$, then H is either a cyclic or a dihedral group. By Proposition 3.10 in [Muk88] the maximal order of a dihedral group of symplectic automorphisms on a K3-surface is 12. We first assume $H \cong D_{2m}$ and that the G-orbit $G.C$ of the rational branch curve C has the maximal length $n = |G.C| = 10$, i.e., $B = G \cdot C$. Each curve in $G \cdot C$ meets the union of Mori fibers in precisely two points forming an D_{2m}-orbit. If a Mori fiber E_C meets the curve C twice, then it follows from Proposition 3.14 that E meets no other curve in B. The contraction of E transforms C into a singular curve of self-intersection zero. The Del Pezzo surface $Y_{\min}$ does however not admit a curve of this type. It follows, that E meets a Mori fiber E' which is contracted in a later step of the Mori reduction and meets no other Mori fiber than E'. The described configuration $GE \cup GE'$ requires a total number of at least 20 Mori fibers and therefore contradicts Lemma 3.2. If C meets two distinct Mori fibers E_1, E_2, each of these two can meet at most one further curve in B. The contraction of E_1 and E_2 transforms C into a (-2)-curve. As above, the existence of further Mori fibers meeting E_i follows. Again, by invariance, the total number of Mori fibers exceeds 20, a contradiction. It follows that either H is cyclic or $|G.C| \leq 9$. Both imply $|G| \leq 108$.

- If $N = 3$, let $S = \{p_1, p_2, p_3\}$ be the points of intersection of C with the union $\bigcup E_i$ of Mori fibers. The set S is H-invariant. It follows that H is either trivial or isomorphic to C_2, C_3 or D_6 and that $|G| \leq 60$

- If $N = 4$, it follows from Lemma 3.21 that $n \leq 9$. Now $|H| \leq 12$ implies $|G| \leq 108$. The bound for the order of H is attained by the tetrahedral group T_{12}. If the group G does not contain a tetrahedral group, then $|H| \leq 8$ and $|G| \leq 72$.

- If $N = 5$, the largest possible group acting on C such that there is an invariant subset of cardinality 5 is the dihedral group D_{10}. Since 3.21 implies $n \leq 6$, we conclude $|G| \leq 60$.

- If $N = 6$, then $n \leq 4$ and $|H| \leq 24$ implies $|G| \leq 96$. This bound is attained if and only if $H \cong O_{24}$. If there is no octahedral group in G, then $|H| \leq 12$ and $|G| \leq 48$.

- If $N \geq 12$, then $n = 1$ and $H = G$. The maximal order 60 is attained by the icosahedral group.
- If $6 < N < 12$, we combine $n \leq 4$ and $|H| \leq 24$ to obtain $|G| \leq 96$. If H is not the octahedral group, then $|H| \leq 16$ and $|G| \leq 64$.

The case by case discussion shows that the existence of a rational curve in B implies $|G| \leq 108$ and the proposition follows. □

Remark 3.23. If the group G under consideration does not contain certain subgroups (such as large dihedral groups or T_{12}, O_{24} or I_{60}) then the condition $|G| > 108$ in the proposition above can be improved and non-existence of rational ramification curves also follows for smaller G.

3.2.2 Elliptic branch curves

The aim of this section is to find conditions on the order of G which allow us to exclude elliptic curves in $\mathrm{Fix}_X(\sigma)$. We prove:

Proposition 3.24. *Let X be a K3-surface with an action of a finite group $G \times \langle \sigma \rangle$ such that $G < \mathrm{Aut}_{\mathrm{symp}}(X)$ and σ is an antisymplectic involution with fixed points. If $|G| > 108$, then $\mathrm{Fix}_X(\sigma)$ contains neither rational nor elliptic ramification curves.*

Proof. By the previous proposition $\mathrm{Fix}_X(\sigma)$ contains no rational curves. It follows from Nikulin's description of $\mathrm{Fix}_X(\sigma)$ (cf. Theorem 1.12) that it is either a single curve of genus $g \geq 1$ or the disjoint union of two elliptic curves.

Let $T \subset B$ be an elliptic branch curve and let $H := \mathrm{Stab}_G(T)$. If $H \neq G$, then H has index two in G. The action of H on T is effective. The automorphism group $\mathrm{Aut}(T)$ of T is a semidirect product $L \ltimes T$, where L is a linear cyclic group of order at most 6. We consider the projection $\mathrm{pr}_L : \mathrm{Aut}(T) \to L$ and let $\lambda \in \mathrm{Pr}_L(H)$ be a generating root of unity. We consider T as a quotient $\mathbb{C}/\Gamma$ and choose $h \in H$ with $h(z) = \lambda z + \omega$ and $t \in T$ such that $\omega + (1-\lambda)t = 0$. After conjugation with the translation $z \mapsto z + t$ the group $H < \mathrm{Aut}(T)$ inherits the semidirect product structure of $\mathrm{Aut}(T)$, i.e.,

$$H = (H \cap L) \ltimes (H \cap T).$$

We refer to this decomposition as the *normal form* of H. By Lemma 3.2 a G-minimal model of Y is a Del Pezzo surface and therefore does not admit elliptic curves with self-intersection zero. It follows that T meets the union $\bigcup E_i$ of Mori fibers. Let E be a Mori fiber meeting T. By Proposition 3.14 $|T \cap E| \in \{1, 2\}$. The stabilzer of E in H is denoted by $\mathrm{Stab}_H(E)$. Since the total number of Mori fibers is bounded by 9 (cf. Lemma 3.2), the index of $\mathrm{Stab}_H(E)$ in H is bounded by 9.

If $T \cap E = \{p\}$, then $\mathrm{Stab}_H(E)$ is a cyclic group of order less than or equal to six. It follows that $|G| \leq 6 \cdot 9 \cdot 2 = 108$.

If $T \cap E = \{p_1, p_2\}$, then $B \cap E = T \cap E$ and the stabilizer $\mathrm{Stab}_G(E)$ of E in G is contained in H. If both points p_1, p_2 are fixed by $\mathrm{Stab}_G(E)$, then $|\mathrm{Stab}_G(E)| \leq 6$. If p_1, p_2 form a $\mathrm{Stab}_G(E)$-orbit, then in the normal form $|\mathrm{Stab}_G(E) \cap T| = 2$. It follows that $\mathrm{Stab}_G(E)$ is either C_2 or $D_4 = C_2 \times C_2$. The index of $\mathrm{Stab}_G(E)$ in G is bounded by 9 and $|G| \leq 54$.

In summary, the existence of an elliptic curve in B implies $|G| \leq 108$ and the proposition follows. □

3.3 Rough classification

With the preparations of the previous sections we may now turn to a classification result for K3-surfaces with antisymplectic involution centralized by a large group.

Theorem 3.25. *Let X be a K3-surface with a symplectic action of G centralized by an antisymplectic involution σ such that* $\mathrm{Fix}(\sigma) \neq \emptyset$. *If $|G| > 96$, then Y is a G-minimal Del Pezzo surface and there are no rational or elliptic curves in* $\mathrm{Fix}(\sigma)$. *In particular,* $\mathrm{Fix}(\sigma)$ *is a single smooth curve C with $g(C) \geq 3$ and $\pi(C) \sim -2K_Y$, where K_Y denotes the canonical divisor on Y.*

Proof. The group G is a subgroup of one of the eleven groups on Mukai's list [Muk88] (cf. Theorem 1.13 and Table 1.2). The orders of these Mukai groups are 48, 72, 120, 168, 192, 288, 360, 384, 960. None of these groups can have a subgroup G with $96 < |G| < 120$. In particular, the order of G is at least 120.

We may therefore apply the results of the previous two sections and conclude that $\pi : X \to Y$ is branched along a single smooth curve C of general type. Its genus $g(C)$ must be ≥ 3 by Hurwitz' formula. It remains to show that Y is G-minimal.

Assume the contrary and let $E \subset Y$ be a Mori fiber with $E^2 = -1$. As before we let $B \subset Y$ denote the branch locus of $\pi : X \to Y$. By Remark 3.13 $E \cap B \neq \emptyset$. It follows that $|E \cap B| \in \{1, 2\}$. Let $\mathrm{Stab}_G(E)$ denote the stabilizer of E in G.

If $\pi^{-1}(E)$ is reducible its two irreducible components meet transversally in one point corresponding to $\{p\} = E \cap B$. The curve E is tangent to B at p and we consider the linearization of the action of $\mathrm{Stab}_G(E)$ at p. If the action of $\mathrm{Stab}_G(E)$ on E is not effective, the linearization of the ineffectivity $I < \mathrm{Stab}_G(E)$ yields a trivial action of I on the tangent line of B at p. It follows that the action of I is trivial in a neighbourhood of $\pi^{-1}(p) \in R = \pi^{-1}(B)$. This is contrary to the assumption that G acts symplectically on X. Consequently, the action of $\mathrm{Stab}_G(E)$ on E is effective and in particular, $\mathrm{Stab}_G(E)$ is a cyclic group.

If $\pi^{-1}(E)$ is irreducible, then it is a smooth rational curve with an effective action of $\mathrm{Stab}_G(E)$. It follows that $\mathrm{Stab}_G(E)$ is either cyclic or dihedral. The largest dihedral group with a symplectic action on a K3-surface is D_{12} (Proposition 3.10 in [Muk88]).

We conclude that the order of $\mathrm{Stab}_G(E)$ is bounded 12 and the index of G_E in G is > 9. By Lemma 3.2 the total number m of Mori fibers however satifies $m \leq 9$. This contradiction shows that Y is G-minimal and, in particular, a Del Pezzo surface. □

Remark 3.26. Let X be a K3-surface with a symplectic action of G centralized by an antisymplectic involution σ with $\mathrm{Fix}_X(\sigma) \neq \emptyset$ and let E be a (-1)-curve on $Y = X/\sigma$. Then the argument above can be applied to see that the stabilizer of E in G is cyclic or dihedral and therefore has order at most 12.

In the following chapter, the classification above is applied and extended to the case where G is a maximal group of symplectic transformations on a K3-surface.

4 Mukai groups centralized by antisymplectic involutions

In this chapter we consider K3-surfaces with a symplectic action of one of the eleven groups from Mukai's list (Table 1.2) and assume that it is centralized by an antisymplectic involution. We prove the following classification result.

Theorem 4.1. *Let G be a Mukai group acting on a K3-surface X by symplectic transformations and σ be an antisymplectic involution on X centralizing G with $\mathrm{Fix}_X(\sigma) \neq \emptyset$. Then the pair (X, G) is in Table 4.1 below. In particular, for groups G numbered 4-8 on Mukai's list, there does not exist a K3-surface with an action of $G \times C_2$ with the properties above.*

	G	$\lvert G \rvert$	**K3-surface** X
1a	$L_2(7)$	168	$\{x_1^3x_2 + x_2^3x_3 + x_3^3x_1 + x_4^4 = 0\} \subset \mathbb{P}_3$
1b	$L_2(7)$	168	Double cover of $\mathbb{P}_2$ branched along $\{x_1^5x_2 + x_3^5x_1 + x_2^5x_3 - 5x_1^2x_2^2x_3^2 = 0\}$
2	A_6	360	Double cover of $\mathbb{P}_2$ branched along $\{10x_1^3x_2^3 + 9x_1^5x_3 + 9x_2^3x_3^3 - 45x_1^2x_2^2x_3^2 - 135x_1x_2x_3^4 + 27x_3^6 = 0\}$
3a	S_5	120	$\{\sum_{i=1}^5 x_i = \sum_{i=1}^6 x_1^2 = \sum_{i=1}^5 x_i^3 = 0\} \subset \mathbb{P}_5$
3b	S_5	120	Double cover of $\mathbb{P}_2$ branched along $\{F_{S_5} = 0\}$
9	N_{72}	72	$\{x_1^3 + x_2^3 + x_3^3 + x_4^3 = x_1x_2 + x_3x_4 + x_5^2 = 0\} \subset \mathbb{P}_4$
10	M_9	72	Double cover of $\mathbb{P}_2$ branched along $\{x_1^6 + y_2^6 + x_3^6 - 10(x_1^3x_2^3 + x_2^3x_3^3 + x_3^3x_1^3) = 0\}$
11a	T_{48}	48	Double cover of $\mathbb{P}_2$ branched along $\{x_1x_2(x_1^4 - x_2^4) + x_3^6 = 0\}$
11b	T_{48}	48	Double cover of $\{x_0x_1(x_0^4 - x_1^4) + x_2^3 + x_3^2 = 0\} \subset \mathbb{P}(1,1,2,3)$ branched along $\{x_2 = 0\}$

Table 4.1: K3-surfaces with $G \times C_2$-symmetry

The polynomial F_{S_5} in case 3b) is given by

$$2(x^4yz + xy^4z + xyz^4) - 2(x^4y^2 + x^4z^2 + x^2y^4 + x^2z^4 + y^4z^2 + y^2z^4)$$
$$+2(x^3y^3 + x^3z^3 + y^3z^3) + x^3y^2z + x^3yz^2 + x^2y^3z + x^2yz^3 + xy^3z^2 + xy^2z^3 - 6x^2y^2z^2.$$

Remark 4.2. The examples 1a, 3a, 9, 10, and 11a appaer in Mukai's list, the remaining cases 1b, 2, 3b, and 11b provide additional examples of K3-surfaces with maximal symplectic symmetry.

For the proof of this theorem we consider each group separately and apply the following general strategy.

For a K3-surface X with $G \times C_2$-symmetry we consider the quotient $Y = X/C_2$ and a G-minimal model $Y_{\min}$ of the rational surface Y. We show that $Y_{\min}$ is a Del Pezzo surface and investigate which Del Pezzo surfaces admit an action of the group G.

It is then essential to study the branch locus B of the covering $X \to Y$. As a first step, we exclude rational and elliptic curves in B. In order to exclude rational branch curves, we study their images in $Y_{\min}$ and their intersection with the union of Mori fibers.

We then deduce that B consists of a single curve of genus ≥ 2 with an effective action of the group G. The possible genera of B are restricted by the nature of the group G and the Riemann-Hurwitz formula for the quotient of B by an appropriate normal subgroup N of G. The equations of B or X given in Table 4.1 are derived using invariant theory.

Throughout the remainder of this chapter, the Euler characteristic formula

$$24 = e(X) = 2e(Y_{\min}) + 2m - 2n + \underbrace{(2g-2)}_{\substack{\text{if non-rational} \\ \text{branch curve exists}}}$$

is exploited various times. Here m denotes the total number of Mori contractions of the reduction $Y \to Y_{\min}$, the total number of rational branch curves is denoted by n and g is the genus of a non-rational branch curve.

All classification results are up to *equivariant equivalence*:

Definition 4.3. Let (X_1, σ_1) and (X_2, σ_2) be K3-surfaces with antisymplectic involution and let G be a finite group acting on X_1 and X_2 by

$$\alpha_i : G \to \mathrm{Aut}_{\mathrm{symp}}(X_i),$$

such that $\alpha_i(g) \circ \sigma_i = \sigma_i \circ \alpha_i(g)$ for $i = 1, 2$ and all $g \in G$. Then the surfaces (X_1, σ_1) and (X_2, σ_2) are considered *equivariantly equivalent* if there exist a biholomorphic map $\varphi : X_1 \to X_2$ and a group automorphism $\psi \in \mathrm{Aut}(G)$ such that

$$\alpha_2(g)\varphi(x) = \varphi(\alpha_1(\psi(g))x) \quad \text{and} \quad \sigma_2(\varphi(x)) = \varphi(\sigma_1(x)).$$

for all $x \in X_1$ and all $g \in G$.

More generally, two surfaces Y_1 and Y_2, without additional structure such as a symplectic form or an involution, with actions of a finite group G

$$\alpha_i : G \to \mathrm{Aut}(Y_i)$$

are considered equivariantly equivalent if there exist a biholomorphic map $\varphi : Y_1 \to Y_2$ and a group automorphism $\psi \in \mathrm{Aut}(G)$ such that

$$\alpha_2(g)\varphi(y) = \varphi(\alpha_1(\psi(g))y)$$

for all $y \in Y_1$ and all $g \in G$.

This notion differs from the notion of equivalence in representation theory. Two non-equivalent linear represenations of a group G can induce equivalent actions on the projective plane if they differ by an outer automorphism of the group.

Remark 4.4. If two K3-surfaces (X_1, σ_1) and (X_2, σ_2) are G-equivariantly equivalent, then the quotient surfaces X_i/σ_i are equivariantly equivalent with respect to the induced action of G.

Conversely, let Y be a rational surface with two action of a finite group G which are equivalent in the above sense and let $\varphi \in \mathrm{Aut}(Y)$ be the isomorphims identifying these two actions. We consider a smooth G-invariant curve B linearly equivalent to $-2K_Y$ and the K3-surfaces X_B and $X_{\varphi(B)}$ obtained as double covers branched along B and $\varphi(B)$ equipped with their respective anti-symplectic covering involution.

Note that X_B and $X_{\varphi(B)}$ are constructed as subsets of the anticanonical line bundle where the involution σ is canonically defined. The induced biholomorphic map $\varphi_X : X_B \to X_{\varphi(B)}$ fulfills $\sigma \circ \varphi_X = \varphi_X \circ \sigma$ by construction.

If all elements of the group G can be lifted to symplectic transformations on X_B and $X_{\varphi(B)}$, then the central degree two extensions E of G acting on X_B, $X_{\varphi(B)}$, respectively, split as $E = E_{\mathrm{symp}} \times C_2$ with $E_{\mathrm{symp}} = G$. In this case the group G acts by symplectic transformations on X_B and $X_{\varphi(B)}$ and these are G-equivariantly equivalent in strong sense introduced above. This follows from the assumption that the corresponding G-actions on the base Y are equivalent and the fact that for each $g \in G \subset \mathrm{Aut}(Y)$ there is only one choice of symplectic lifting $\tilde{g} \in \mathrm{Aut}(X_B)$ and $\mathrm{Aut}(X_{\varphi(B)})$.

In the following sections we will go through the lists of Mukai groups and for each group we prove the classification claimed in Theorem 4.1.

4.1 The group $L_2(7)$

Let $G \cong L_2(7)$ be the finite simple group of order 168. If G acts on a K3-surface X, then the kernels of the homomorphism $G \to \mathrm{Aut}(X)$ and the homomorphism $G \to \Omega^2(X)$ are trivial and the action is effective and symplectic. Let σ be an antisymplectic involution on X centralizing G. Since G has an element of order seven which is known to have exactly three fixed points p_1, p_2, p_3 and σ acts on this set of three points, we know that $\mathrm{Fix}_X(\sigma) \neq \emptyset$. By Theorem 3.25, the K3-surface X is a double cover of a Del Pezzo surface Y. Our study of Del Pezzo surfaces with an action of $L_2(7)$ in Example 3.8 has revealed that Y is either $\mathbb{P}_2$ or a Del Pezzo surface of degree 2. In the first case, $\pi : X \to Y$ is branched along a curve of genus 10, in the second case π is branched along a curve of genus 3. Section 5.5 in the next chapter is devoted to an inspection of K3-surfaces with an action of $L_2(7) \times C_2$ and a precise classification result in the setup above will be obtained. The pair (X, G) is equivariantly isomorphic to either the surface 1a) or 1b).

4.2 The group A_6

Let $G \cong A_6$ be the alternating group degree 6. Ii is a simple group and if it acts on a K3-surface X, then this action effective and symplectic. Let σ be an antisymplectic involution on X centralizing G and assume that $\mathrm{Fix}_X(\sigma) \neq \emptyset$. By Theorem 3.25, the K3-surface X is a double cover of a Del Pezzo surface Y with an effective action of A_6.

Lemma 4.5. *The Del Pezzo surface Y is isomorphic to $\mathbb{P}_2$ with a uniquely determined action of A_6 given by a nontrivial central extension $V = 3.A_6$ of degree three known as* Valentiner's group.

Proof. We go through the list of Del Pezzo surfaces.

- If Y has degree one, then $|-K_Y|$ has precisely one base point which would have to be an A_6-fixed point. This is contrary to the fact that A_6 has no faithful two-dimensional representation.

- We recall that the stabilizer of a (-1)-curve E in Y is either cyclic or dihedral (Remark 3.26). In particular, its order is at most 12 and therefore its index in A_6 is at least 30. Using Table 3.1 we see that Y can not be a Del Pezzo surface of degree 2,3,4,5,6.

- Since the blow-up of $\mathbb{P}_2$ in one point is never G-minimal, it remains to exclude $Y \cong \mathbb{P}_1 \times \mathbb{P}_1$. Assume there is an action of A_6 on $\mathbb{P}_1 \times \mathbb{P}_1$. Since A_6 has no subgroups of index two, it follows that $A_6 < \mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C})$ and both canonical projections are A_6-equivariant. Since A_6 has neither an effective action on $\mathbb{P}_1$ nor nontrivial normal subgroups of ineffectivity, it follows that A_6 acts trvially on Y.

It follows that $Y \cong \mathbb{P}_2$. The action of A_6 on $\mathbb{P}_2$ is given by a degree three central extension of A_6. Since A_6 has no faithful three-dimensional representation, this extension is nontrivial and isomorphic the unique nontrivial degree three extension $V = 3.A_6$ known as Valentiner's group. Up to equivariant equivalence, there is a unique action of A_6 on $\mathbb{P}_2$. This follows from the classification of finite subgroup of $\mathrm{SL}_3(\mathbb{C})$ (cf. [MBD16], [Bli17], and [YY93]) and can also be derived as follows: An action of A_6 on $\mathbb{P}_2$ is given by a threedimensional projective representation. We wish to show that any two actions induced by ρ_1, ρ_2 are equivalent. We restrict the projective representations ρ_1 and ρ_2 to the subgroup A_5. The restricted representations are linear and after a change of coordinates $\rho_1(A_5) = \rho_2(A_5) \subset \mathrm{SL}_3(\mathbb{C})$. We fix a subgroup A_4 in A_5 and consider its normalizer N in A_6. The groups N and A_4 generate the full group A_6 and it suffices to prove that $\rho_1(N) = \rho_2(N)$. This is shown by considering an explicit three-dimensional representation of $A_4 < A_5$ and the normalizer $\mathcal{N}$ of A_4 inside $\mathrm{PSL}_3(\mathbb{C})$. The group A_4 has index two in $\mathcal{N}$ and therefore $\mathcal{N} = \rho_1(N) = \rho_2(N)$.. □

The covering $X \to Y$ is branched along an invariant curve C of degree six. This curve is defined by an invariant polynomial F_{A_6} of degree six, which is unique by Molien's formula. Its explicit equation is derived in [Cra99]. In appropriately chosen coordinates,

$$F_{A_6}(x_1, x_2, x_3) = 10x_1^3x_2^3 + 9x_1^5x_3 + 9x_2^3x_3^3 - 45x_1^2x_2^2x_3^2 - 135x_1x_2x_3^4 + 27x_3^6.$$

If a K3-surface with $A_6 \times C_2$-symmetry exists, then it must be the double cover of $\mathbb{P}_2$ branched along $\{F_{A_6} = 0\}$.

The action of A_6 on $\mathbb{P}_2$ induces an action of a central degree two extension of E on the double cover branched along $\{F_{A_6} = 0\}$,

$$\{\mathrm{id}\} \to C_2 \to E \to A_6 \to \{\mathrm{id}\}.$$

Let $E_{\mathrm{symp}} \neq E$ be the normal subgroup of symplectic automorphisms in E. Since A_6 is simple, it follows that E_{symp} is mapped surjectively to A_6 and $E_{\mathrm{symp}} \cong A_6$. In particular, the group E splits as $E_{\mathrm{symp}} \times C_2$ where C_2 is generated by the antisymplectic covering involution. This proves the existence of a unique K3-surface with $A_6 \times C_2$-symmetry. We refer to this K3-surface as the *Valentiner surface*.

4.3 The group S_5

In this section we study K3-surfaces with an symplectic action of the symmetric group S_5 centralized by an antisymplectic involution.

Let X be a K3-surface with a symplectic action of $G = S_5$ and let σ denote an antisymplectic involution centralizing G. We assume that $\mathrm{Fix}_X(\sigma) \neq \emptyset$. We may apply Theorem 3.25 which yields that $X/\sigma = Y$ is a G-minimal Del Pezzo surface and $\pi : X \to Y$ is branched along a smooth connected curve B of genus

$$g(B) = 13 - e(Y).$$

We will see in the following that only very few Del Pezzo surfaces admit an effective action of S_5 or a smooth S_5-invariant curve of appropriate genus.

Lemma 4.6. *The degree $d(Y)$ of the Del Pezzo surface Y is either three or five.*

Proof. We prove the statement by excluding Del Pezzo surfaces of degree $\neq 3, 5$.

- Assume $Y \cong \mathbb{P}_2$. Then $G = S_5$ is acting effectively on $\mathbb{P}_2$, i.e., $S_5 \hookrightarrow \mathrm{PSL}_3(\mathbb{C})$. Let $\tilde{G}$ denote the preimage of G in $\mathrm{SL}_3(\mathbb{C})$. Since A_5 has no nontrivial central extension of degree three, it follows that the preimage of $A_5 < S_5$ in $\tilde{G}$ splits as $\tilde{A}_5 = A_5 \times C_3$. It has index two in $\tilde{G}$ and therefore is a normal subgroup of $\tilde{G}$. Let $g \in S_5$ be any transposition and pick $\tilde{g}$ in its preimage with $\tilde{g}^2 = \mathrm{id}$. Now $\tilde{g}$ and A_5 generate a copy of S_5 in $\mathrm{SL}_3(\mathbb{C})$. The action of S_5 is given by a three-dimensional representation. The irreducible representations of S_5 have dimensions $1, 4, 5$ or 6 and it follows that there is no faithful three-dimensional represenation of S_5 and therefore no effective S_5-action on $\mathbb{P}_2$.

- Assume that Y is isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. We investigate the action of $S_5 = A_5 \rtimes C_2$ and note that A_5 is a simple group. The automorphism group $\mathrm{Aut}(Y)$ is given by

  $$(\mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})) \rtimes C_2.$$

  It follows that $A_5 < \mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$, and the action of A_5 respects the product structure, i.e, the canonical projections onto the factors are A_5-equivariant. If A_5 acts trivially on one of the factors, then the generator τ of the outer C_2 stabilizes this factor because A_5 must act nontrivially on the second factor. It follows that S_5 stablizes the second factor which is impossible since there is no effective action of S_5 on $\mathbb{P}_1$. It follows that A_5 acts effectively on both factors and τ exchanges them. We consider an element λ of order five in A_5 and chose coordinates on $\mathbb{P}_1 \times \mathbb{P}_1$ such that λ acts by

  $$([z_1 : z_2], [w_1 : w_2]) \mapsto ([\xi z_1 : z_2], [\xi^a w_1 : w_2])$$

  for some $a \in \{1, 2, 3, 4\}$ and $\xi^5 = 1$. The automorphism λ has four fixed points

  $$p_1 = ([1:0],[1:0]),\ p_2 = ([1:0],[0:1]),\ p_3 = ([0:1],[1:0]),\ p_4 = ([0:1],[0:1]).$$

  Since it lifts to a symplectic automorphism on the K3-surface X with four fixed points, all fixed points must lie on the branch curve. The branch curve $B \subset Y$ is a smooth invariant curve linearly equivalent to $-2K_Y$ and is therefore given by an S_5-semi-invariant polynomial f of bidegree $(4, 4)$. Since f must be invariant with respect to the subgroup A_5, it is a linear combination of λ-invariant monomials of bidegree $(4, 4)$. For each choice of a one lists all λ-invariant monomials of bidegree $(4, 4)$. For $a = 1$ these are

  $$z_1 z_2^3 w_1^4,\ z_1^2 z_2^2 w_1^3 w_2,\ z_1^3 z_2 w_1^2 w_2^2,\ z_1^4 w_1 w_2^3,\ z_2^4 w_2^4.$$

Since f must vanish at $p_1 \dots p_4$, one sees that f may not contain $z_2^4 w_2^4$. The remaining monomials have a common component $z_1 w_1$ such that f factorizes and C must be reducible, a contradiction. The same argument can be carried out for each choice of a. It follows that the action of S_5 on $\mathbb{P}_1 \times \mathbb{P}_1$ does not admit irreducble curves of bidegree $(4,4)$. This eliminates the case $Y \cong \mathbb{P}_1 \times \mathbb{P}_1$.

- Again using the fact that the largest subgroup of S_5 which can stabilize a (-1)-curve in Y is the group D_{12} of index 10, it follows that the number of (-1)-curves in a G-orbit is at least 10. A Del Pezzo surface of degree six has six (-1)-curves and therefore $d(Y) \neq 6$. A Del Pezzo surface of degree four contains sixteen (-1)-curves. Since 16 does not divide the the order of S_5, the set of these curves is not a single G-orbit. As it cannot be the union of G-orbits either, we conclude $d(Y) \neq 4$.
- If $d(Y) = 2$, then the anticanonical map defines an $\mathrm{Aut}(Y)$-equivariant double cover of $\mathbb{P}_2$. The induced action of S_5 on $\mathbb{P}_2$ would have to be effective and therefore we obtain a contradiction as in the case $Y \cong \mathbb{P}_2$.
- If $d(Y) = 1$ then the anticanonical system $|-K_Y|$ is known to have precisely one base point which has to be fixed point of the action of S_5. Since S_5 has no faithful two-dimensional representation, this is a contradiction.

Since we have considered all possible G-minimal Del Pezzo surfaces the proof of the lemma is completed. □

4.3.1 Double covers of Del Pezzo surfaces of degree three

The following example of a K3-surface X with an action of $S_5 \times C_2$ such that X/σ is a Del Pezzo surface of degree three can be found in Mukai's list [Muk88] (cf. also Table 1.2).

Example 4.7. Let X be the K3-surface in $\mathbb{P}_5$ given by

$$\sum_{i=1}^{5} x_i = \sum_{i=1}^{6} x_1^2 = \sum_{i=1}^{5} x_i^3 = 0$$

and let S_5 act on $\mathbb{P}_5$ by permuting the first five variables and by the character sgn on the last variable. This induces an action on X.

The commutator subgroup $S_5' = A_5 < S_5$ acts by symplectic transformations. In order to show that the full group acts symplectically, consider the transposition $\tau = (12) \in S_5$ acting on $\mathbb{P}_5$ by $[x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \mapsto [x_2 : x_1 : x_3 : x_4 : x_5 : -x_6]$. One checks that the induced involution on X has isolated fixed points and is therefore symplectic. It follows that $S_5 < \mathrm{Aut}_{\mathrm{symp}}(X)$.

Let $\sigma : \mathbb{P}_5 \to \mathbb{P}_5$ be the involution $[x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \mapsto [x_1 : x_2 : x_3 : x_4 : x_5 : -x_6]$. This defines an involution on X with a positive-dimensional set of fixed point $\{x_6 = 0\} \cap X$. Therefore σ is an antisymplectic involution on X which centralizes the action of S_5.

The quotient Y of X by σ is given by restricting then rational map $[x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \mapsto [x_1 : x_2 : x_3 : x_4 : x_5]$ to X. The surface Y is given by

$$\{\sum_{i=1}^{5} y_i = \sum_{i=1}^{5} y_i^3 = 0\} \subset \mathbb{P}_4.$$

and is isomorphic to the Clebsch diagonal surface $\{z_1^2 z_2 + z_1 z_3^2 + z_3 z_4^2 + z_4 z_2^2 = 0\} \subset \mathbb{P}_3$ (cf. Theorem 10.3.10 in [Dol08]), a Del Pezzo surface of degree three. The branch set B is given by $\{\sum_{i=1}^{5} y_1^2 = 0\} \cap Y \subset \mathbb{P}_4$.

By the following proposition, the example above is the unique K3-surface with $S_5 \times C_2$-symmetry such that X/σ is a Del Pezzo surface of degree three.

Proposition 4.8. *Let X be a K3-surface with a symplectic action of the group S_5 centralized by an antisymplectic involution σ. If $Y = X/\sigma$ is a Del Pezzo surface of degree three, then X is equivariantly isomorphic to Mukai's S_5-example $\{\sum_{i=1}^5 x_i = \sum_{i=1}^6 x_1^2 = \sum_{i=1}^5 x_i^3 = 0\} \subset \mathbb{P}_5$.*

Proof. We consider the $\mathrm{Aut}(Y)$-equivariant embedding of the Del Pezzo surface Y into $\mathbb{P}_3$ given by the anticanonical map. Any automorphism of Y induced by an automorphism of the ambient projective space.

It follows from the representation and invariant theory of the group S_5 that a Del Pezzo surface of degree three with an effective action of the group S_5 is equivariantly isomorphic the Clebsch cubic $\{z_1^2 z_2 + z_1 z_3^2 + z_3 z_4^2 + z_4 z_2^2 = 0\} \subset \mathbb{P}_3$ (cf. Theorems 10.3.9 and 10.3.10, Table 10.3 in [Dol08]).

The ramification curve $B \subset Y$ is linearly equivalent to $-2K_Y$. We show that B is given by intersecting Y with a quadric in $\mathbb{P}_3$.

Applying the formula

$$h^0(Y, \mathcal{O}(-rK_Y)) = 1 + \frac{1}{2}r(r+1)d(Y)$$

(cf. e.g. Lemma 8.3.1 in [Dol08]) to $d = d(Y) = 3$ and $r = 2$ we obtain $h^0(Y, \mathcal{O}(-2K_Y)) = 10$. This is also the dimension of the space of sections of $\mathcal{O}_{\mathbb{P}_3}(2)$ in $\mathbb{P}_3$ (homogeneous polynomials of degree two in four variables). It follows that the restriction map

$$H^0(\mathbb{P}_3, \mathcal{O}(2)) \to H^0(Y, \mathcal{O}(-2K_Y))$$

is surjective and $B = Y \cap Q$ for some quadric $Q = \{f = 0\}$ in $\mathbb{P}_3$.

Since B is an S_5-invariant curve in Y, it follows that for each $g \in S_5$ the intersection of $gQ = \{f \circ g^{-1} = 0\}$ with Y coincides with B. It follows that $f|_Y$ is a multiple of $(f \circ g^{-1})|_Y$, i.e., there exists a constant $c \in \mathbb{C}^*$ such that $(f \circ g^{-1}) - cf$ vanishes identically on Y. Since Y is irreducible, this implies $f \circ g^{-1} = cf$. It follows that the polynomial f is an S_5- semi-invariant and therefore invariant with respect to the commutator subgroup A_5.

We have previously noted that after a suitable linear change of coordinates the surface Y is given by $\{\sum_{i=1}^5 y_i = \sum_{i=1}^5 y_i^3 = 0\} \subset \mathbb{P}_4$ where S_5 acts by permutation. The action of any transposition on an S_5-semi-invariant polynomial is given by multiplication by ± 1. It follows that in the coordinates $[y_1 : \cdots : y_5]$ the semi-invariant polynomial f is given by

$$a \sum_{i=1}^5 y_i^2 + b(\sum_{i=1}^5 y_i)^2 = 0$$

for some $a, b \in \mathbb{C}$. Using the fact $Y \subset \{\sum_{i=1}^5 y_i = 0\}$ it follows that B is given by intersecting Y with $\{\sum_{i=1}^5 y_i^2 = 0\}$ and X is Mukai's S_5-example discussed in Example 4.7. □

4.3.2 Double covers of Del Pezzo surfaces of degree five

A second class of candidates of K3-surfaces with $S_5 \times C_2$-symmetry is given by double covers of Del Pezzo surfaces of degree five.

Any two Del Pezzo surfaces of degree five are isomorphic and the automorphisms group of a Del Pezzo surface Y of degree five is S_5. The ten (-1)-curves on Y form a graph known as the *Petersen* *graph.* The Petersen graph has S_5-symmetry and every symmetry of the abstract graph is induced by a unique automorphism of the surface Y.

The following proposition classifies K3-surfaces with $S_5 \times C_2$-symmetry which are double covers of Del Pezzo surfaces of degree five.

Proposition 4.9. *Let X be a K3-surface with a symplectic action of the group S_5 centralized by an antisymplectic involution σ. If $Y = X/\sigma$ is a Del Pezzo surface of degree five, then X is equivariantly isomorphic to the minimal desingularization of the double cover of $\mathbb{P}_2$ branched along the sextic*

$$\{2(x^4yz + xy^4z + xyz^4) - 2(x^4y^2 + x^4z^2 + x^2y^4 + x^2z^4 + y^4z^2 + y^2z^4) + 2(x^3y^3 + x^3z^3 + y^3z^3) \\ + x^3y^2z + x^3yz^2 + x^2y^3z + x^2yz^3 + xy^3z^2 + xy^2z^3 - 6x^2y^2z^2 = 0\}$$

Proof. Let $B \subset Y$ denote the branch locus of the covering $X \to Y$. The curve B is smooth, connected, invariant with respect to the full automorphism group of Y and linearly equivalent to $-2K_Y$.

The Del Pezzo surface Y is the blow-up of $\mathbb{P}_2$ in four points p_1, p_2, p_3, p_4 in general position. We may choose coordinates $[x : y : z]$ on $\mathbb{P}_2$ such that

$$p_1 = [1:0:0], \quad p_2 = [0:1:0], \quad p_3 = [0:0:1], \quad p_4 = [1:1:1].$$

Let $m : Y \to \mathbb{P}_2$ be the blow-down map and let $E_i = m^{-1}(p_i)$. Consider the S_4-action on $\mathbb{P}_2$ permuting the points $\{p_i\}$. The isotropy at the point p_1 is isomorphic to S_3 and induces an effective S_3-action on E_1.

Let E be any (-1)-curve on Y. By adjunction $E \cdot B = 2$. Since Y contains precisely ten (-1)-curves forming an S_5-orbit, the group $H = \mathrm{Stab}_{S_5}(E)$ has order 12 and all stabilizer groups of (-1)-curves in Y are conjugate. It follows that the group H contains S_3, which is acting effectively on E, and therefore H is isomorphic to the dihedral group of order 12. The points of intersection $B \cap E$ form an H-invariant subset of E. Since H has no fixed points in E and precisely one orbit $H.p = \{p, q\}$ consisting of two elements, it follows that B meets E transversally in p and q.

In particular, each curve E_i meets B in two points and the image curve $C = m(B)$ has nodes at the four points p_i. By Lemma 3.17, the self-intersection number of C is $20 + 4 \cdot 4 = 36$, so C is a sextic curve. It is invariant with respect to the action of S_4 given by permutation on $p_1, \ldots p_4$. For simplicity, we first only consider the action of S_3 permuting p_1, p_2, p_3 and conclude that C is given by $\{f = \sum a_i f_i = 0\}$ as a linear combination of the following degree six polynomials

$$\begin{aligned}
f_1 &= x^6 + y^6 + z^6 \\
f_2 &= x^5y + x^5z + xy^5 + xz^5 + y^5z + yz^6 \\
f_3 &= x^4yz + xy^4z + xyz^4 \\
f_4 &= x^4y^2 + x^4z^2 + x^2y^4 + x^2z^4 + y^4z^2 + y^2z^4 \\
f_5 &= x^3y^3 + x^3z^3 + y^3z^3 \\
f_6 &= x^3y^2z + x^3yz^2 + x^2y^3z + x^2yz^3 + xy^3z^2 + xy^2z^3 \\
f_7 &= x^2y^2z^2
\end{aligned}$$

The fact that C passes through p_i and is singular at p_i yields $a_1 = a_2 = 0$ and

$$3a_3 + 6a_4 + 3a_5 + 6a_6 + a_7 = 0.$$

The two tangent lines of C at the node p_i correspond to the unique $\mathrm{Stab}(E_i)$-orbit of length two in E_i. We consider the point p_3 and the subgroup $S_3 < S_4$ stabilizing p_3. The action of S_3 on E_3 is given by the linearized S_3-action on the set of lines through p_3. One checks that in local affine coordinates (x, y) the unique orbit of length two corresponds to the line pair $x^2 - xy + y^2 = 0$. Dehomogenizing f at p_3, i.e., setting $z = 1$, we obtain the local equation f_{dehom} of C at p_3. The polynomial f_{dehom} modulo terms of order three or higher must be a multiple of $x^2 - xy + y^2$. Therefore $a_3 = -a_4$.

Next we consider the intersection of C with the line $L_{34} = \{x = y\}$ joining p_3 and p_4. We know that $f|_{L_{34}}$ vanishes of order two at p_3 and p_4 and at one or two further points on L_{34}.

Let $\widetilde{L}_{34}$ denote the proper transform of L_{34} inside the Del Pezzo surface Y. The curve $\widetilde{L}_{34}$ is a (-1)-curve, hence its stabilizer $\mathrm{Stab}_G(\widetilde{L}_{34})$ is isomorphic to $D_{12} = S_3 \times C_2$. The factor C_2 acts trivially on $\widetilde{L}_{34}$. Since the intersection of $\widetilde{L}_{34}$ with B is $\mathrm{Stab}_G(\widetilde{L}_{34})$ invariant, it follows that $\widetilde{L}_{34} \cap B$ is the unique S_3-orbit a length two in $\widetilde{L}_{34}$.

We wish to transfer our determination of the unique S_3-orbit of length two in E_3 above to the curve $\widetilde{L}_{34}$ using an automorphism of Y mapping E_3 to $\widetilde{L}_{34}$. Consider the automorphism φ of Y induced by the birational map of $\mathbb{P}_2$ given by

$$[x : y : z] \mapsto [x(z - y) : z(x - y) : xz]$$

(cf. Theorem 10.2.2 in [Dol08]) and let ψ be the automorphism of Y induced by the permutation of the points p_2 and p_3 in $\mathbb{P}_2$. Then $\psi \circ \varphi$ is an automorphism of Y mapping E_3 to $\widetilde{L}_{34}$. If $[X : Y]$ denote homogeneous coordinates on E_3 induced by the affine coordinates (x, y) in a neighbourhood of p_3, then a point $[X : Y] \in E_3$ is mapped to the point corresponding to $[X : X : X - Y] \in L_{34} \subset \mathbb{P}_2$. It was derived above that the unique S_3-orbit of length two in E_3 is given by $X^2 - XY + Y^2$ and it follows that the unique S_3-orbit of length two in $\widetilde{L}_{34}$ corresponds to the points $[x : x : z] \in \mathbb{P}_2$ fulfilling $x^2 - xz + z^2 = 0$.

Therefore, $f|_{L_{34}}$ is a multiple of polynomial given by $x^2(x - z)^2(x^2 - xz + z^2)$. Comparing coefficients with $f(x : x : z)$ yields

$$\begin{aligned}
2a_3 + 2a_6 &= 2a_5 + 2a_6 \\
2a_4 + a_5 &= 2a_4 + a_3 \\
8a_4 + 4a_5 &= 2a_4 + 2a_6 + a_7 \\
-6a_4 - 3a_5 &= 2a_5 + 2a_6.
\end{aligned}$$

We conclude $a_3 = a_5 = 2 = -a_4$, $a_6 = 1$, and $a_7 = -6$. So if X as in the lemma exists, it is the double cover of Y branched along the proper transform of $\{f = 0\}$ in Y, where

$$\begin{aligned}
f(x, y, z) =& 2(x^4yz + xy^4z + xyz^4) \\
&- 2(x^4y^2 + x^4z^2 + x^2y^4 + x^2z^4 + y^4z^2 + y^2z^4) \\
&+ 2(x^3y^3 + x^3z^3 + y^3z^3) \\
&+ x^3y^2z + x^3yz^2 + x^2y^3z + x^2yz^3 + xy^3z^2 + xy^2z^3 \\
&- 6x^2y^2z^2.
\end{aligned}$$

In order to prove existence, let X be the minimal desingularisation of the double cover of $\mathbb{P}_2$ branched along $\{f = 0\}$. Then X is the double cover of the Del Pezzo surface Y of degree five branched along the proper transform D of $\{f = 0\}$ in Y. Since all automorphisms of Y are induced by explicit biholomorphic or birational transformation of $\mathbb{P}_2$ one can check by direct computations that D is in fact invariant with respect to the action of $\mathrm{Aut}(Y) = S_5$. The covering involution σ is antisymplectic.

On X there is an action of a central extension E of S_5 by C_2. Let E_{symp} be the subgroup of symplectic automorphisms in E. Since E contains the antisymplectic covering involution $E_{\mathrm{symp}} \neq E$. The image N of E_{symp} in S_5 is normal and therefore either $N \cong S_5$ or $N \cong A_5$.

If $N \cong A_5$ and $|E_{\mathrm{symp}}| = 60$, then $E_{\mathrm{symp}} \cong A_5$. Lifting any transposition from S_5 to an element g of order two in E, the group generated by g and E_{symp} inside E is isomorphic to S_5. It follows that E splits as $S_5 \times C_2$ and $E/E_{\mathrm{symp}} \cong C_2 \times C_2$. This is a contradiction.

If $N \cong A_5$ and $|E_{\mathrm{symp}}| = 120$, then $E = E_{\mathrm{symp}} \times C_2$, where the outer C_2 is generated by the antisymplectic covering involution σ, and $E/C_2 = S_5$ implies that $E_{\mathrm{symp}} \cong S_5$. This is contradictory to the assumption $N \cong A_5$.

In the last remaining case $N \cong S_5$. Since $E_{\mathrm{symp}} \neq E$, also $E_{\mathrm{symp}} \cong S_5$ and E splits as $E_{\mathrm{symp}} \times C_2$. It follows that the action of S_5 on Y induces an symplectic action of S_5 on the double cover X centralized by the antisymplectic covering involution. This completes the proof of the proposition. □

4.3.3 Conclusion

We summarize our results of the previous subsections in the following theorem.

Theorem 4.10. *Let X be a K3-surface with a symplectic action of the group S_5 centralized by an antisymplectic involution σ with* $\mathrm{Fix}_X(\sigma) \neq \emptyset$. *Then X is equivariantly isomorphic to either Mukai's S_5-example or the minimal desingularization of the double cover of* $\mathbb{P}_2$ *branched along the sextic*

$$\begin{aligned}\{F_{S_5}(x_1, x_2, x_3) = \\ 2(x^4yz + xy^4z + xyz^4) - 2(x^4y^2 + x^4z^2 + x^2y^4 + x^2z^4 + y^4z^2 + y^2z^4) + 2(x^3y^3 + x^3z^3 + y^3z^3) \\ + x^3y^2z + x^3yz^2 + x^2y^3z + x^2yz^3 + xy^3z^2 + xy^2z^3 - 6x^2y^2z^2 = 0\}.\end{aligned}$$

4.4 The group $M_{20} = C_2^4 \rtimes A_5$

Proposition 4.11. *There does not exist a K3-surface with a symplectic action of M_{20} centralized by an antisymplectic involution σ with* $\mathrm{Fix}_X(\sigma) \neq \emptyset$.

Proof. Assume that a K3-surface X with these properties exists. Applying Theorem 3.25 we see that $X \to Y$ is branched along a single M_{20}-invariant smooth curve C on the Del Pezzo surface Y. The curve C is neither rational nor elliptic. By Hurtwitz' formula,

$$|\mathrm{Aut}(C)| \leq 84(g(C) - 1),$$

the genus of C must be at least twelve. Since C is linearly equivalent to $-2K_Y$, the adjunction formula

$$2g(C) - 2 = (K_Y, C) + C^2 = 2K_Y^2$$

implies $\deg(Y) = K_Y^2 \geq 11$. This is a contradiction since the degree of a Del Pezzo surface is at most nine. □

4.5 The group $F_{384} = C_2^4 \rtimes S_4$

Before we prove non-existence of K3-surfaces with $F_{384} \times C_2$-symmetry, we note the following useful fact about S_4-actions on Riemann surfaces.

Lemma 4.12. *The group S_4 does not admit an effective action on a Riemann surface of genus one or two.*

Proof. The automorphism group of a Riemann surface T of genus one is of the form $\mathrm{Aut}(T) = L \ltimes T$ for $L \in \{C_2, C_4, C_6\}$. We have seen before (cf. Proof of Proposition 3.24) that any subgroup H of $\mathrm{Aut}(T)$ can be put into the form $H = (H \cap L) \ltimes (H \cap T)$. The nontrivial normal subgroups of S_4 are A_4 and $C_2 \times C_2$. Since A_4 is not Abelian and the quotient of S_4 by $S_4 \cap T = C_2 \times C_2$ is not cyclic, we conclude that S_4 is not a subgroup of $\mathrm{Aut}(T)$.

Assume that S_4 acts effectively on a Riemann surface H of genus two. Note that H is hyperelliptic and the quotient of H by the hyperelliptic involution is branched at six points. Since S_4 has no normal subgroup of order two, the induced action of S_4 on the quotient $\mathbb{P}_1$ is effective and therefore has precisely one orbit consisting of six points. The isotropy subgroup at these points is isomorphic to C_4. The isotropy group at the corresponding points in H must be isomorphic to $C_4 \times C_2$. Since this group is not cyclic, it cannot act effectively with fixed points on a Riemann surface and we obtain a contradiction. □

Proposition 4.13. *There does not exists a K3-surface with a symplectic action of F_{384} centralized by an antisymplectic involution σ with $\mathrm{Fix}_X(\sigma) \neq \emptyset$.*

Proof. As above, assume that a K3-surface X with these properties exists and apply Theorem 3.25 to see that $X \to Y$ is branched along a single F_{384}-invariant smooth curve C on the Del Pezzo surface Y. It follows from Hurwitz' formula that the genus of C is at least 6.

We use the realization of F_{384} as a semi-direct product $C_4^2 \rtimes S_4$ (cf. [Muk88]) and consider the quotient Q of the branch curve C by the normal subgroup $N = C_4^2$. On Q there is the induced action of S_4. It follows from the lemma above that Q is either rational or $g(Q) > 2$. In the second case, if we apply the Riemann-Hurwitz formula to the covering $C \to Q$, then

$$e(C) = 16e(Q) - \text{branch point contributions} \leq -64$$

and $g(C) \geq 33$. This contradicts the adjunction formula on the Del Pezzo surface Y and implies that Q is a rational curve.

It follows from adjunction that $K_Y^2 = g(C) - 1$. Therefore, the degree of the Del Pezzo surface Y is at least five. We consider the action of F_{384} on the configuration of (-1)-curves on Y and recall that the order of a stabilizer of a (-1)-curve in Y is at most twelve (cf. Remark 3.26) and therefore has index greater than or equal to 32 in G. It follows that Y is either $\mathbb{P}_1 \times \mathbb{P}_1$ or $\mathbb{P}_2$. In the first case, the canonical projections of $\mathbb{P}_1 \times \mathbb{P}_1$ are equivariant with respect to a subgroup of index two in F_{384} and thereby contradict Lemma 3.2. Consequently, $Y \cong \mathbb{P}_2$. In particular, $g(C) = 10$ and $e(C) = -18$. It follows that the branch point contribution of the covering $C \to Q$ must be 50. Since isotropy groups must be cyclic, the only possible isotropy subgroups of $N = C_4^2$ at a point in C are C_2 and C_4 and have index four or eight. The full branch point contribution must therefore be a multiple of four. This contradiction yields the non-existence claimed. □

4.6 The group $A_{4,4} = C_2^4 \rtimes A_{3,3}$

By $S_{p,q}$ for $p+q=n$ we denote a subgroup $S_p \times S_q$ of S_n preserving a partition of the set $\{1,\ldots,n\}$ into two subsets of cardinality p and q. The intersection of A_n with $S_{p.q}$ is denoted by $A_{p,q}$.

Proposition 4.14. *There does not exists a K3-surface with a symplectic action of $A_{4,4}$ centralized by an antisymplectic involution σ with* $\mathrm{Fix}_X(\sigma) \neq \emptyset$.

Proof. We again assume that a K3-surface with these properties exists. Applying Theorem 3.25 we see that $X \to Y$ is branched along a single $A_{4,4}$-invariant smooth curve C on the Del Pezzo surface Y. The group $A_{4,4}$ is a semi-direct product $C_2^4 \rtimes A_{3,3}$ (see e.g. [Muk88]). We consider the quotient Q of C by the normal subgroup $N \cong C_2^4$. On Q there is an action of $A_{3,3}$. Since $A_{3,3}$ contains the subgroup $C_3 \times C_3$, which does not act on a rational curve, it follows that Q not rational. We apply the Riemann-Hurwitz formula to the covering $C \to Q$.

If Q is elliptic, then $2g(C)-2$ equals the branch point contribution of the covering $C \to Q$. As above, isotropy groups must be cyclic and the maximal possible isotropy group of the C_2^4-action on C is C_2 and has index eight in C_2^4. Consequently, the branch point contribution at each branch point is eight. Recall that any group H acting on the torus Q can be put into the form $H = (H \cap L) \ltimes (H \cap Q)$ for $L \in \{C_2, C_4, C_6\}$. Since Q acts freely, the action of $C_3 \times C_3 < A_{3,3}$ on the elliptic curve Q has orbits of length greater than or equal to three. Therefore, the total branch point contribution must be greater than or equal to 24. In particular, $g(C) = \deg(Y) + 1 \geq 13$ contrary to $\deg(Y) \leq 9$.

If $g(Q) \geq 2$, then $g(C) \geq 17$ which is also contrary to $\deg(Y) \leq 9$ □

4.7 The groups $T_{192} = (Q_8 * Q_8) \rtimes S_3$ and $H_{192} = C_2^4 \rtimes D_{12}$

By Q_8 we denote the quaternion group $\{+1,-1,+I,-I,+J,-J,+K,-K\}$ where $I^2 = J^2 = K^2 = IJK = -1$. The central product $Q_8 * Q_8$ is defined as the quotient of $Q_8 \times Q_8$ by the central involution $(-1,-1)$, i.e., $Q_8 * Q_8 = (Q_8 \times Q_8)/(-1,-1)$.

Note that both groups T_{192} and H_{192} are semi-direct products $C_2^3 \rtimes S_4$ (cf. [Muk88]).

Proposition 4.15. *For $G = T_{192}$ or $G = H_{192}$ there does not exists a K3-surface with a symplectic action of G centralized by an antisymplectic involution σ with* $\mathrm{Fix}_X(\sigma) \neq \emptyset$.

Proof. Assume that a K3-surface with these properties exists. Applying Theorem 3.25 we see that $X \to Y$ is branched along a single G-invariant smooth curve C on the Del Pezzo surface Y. The genus of C is at least four by Hurwitz' formula and therefore $\deg(Y) \geq 3$. We consider the quotient Q of C by the normal subgroup $N = C_2^3$. By Lemma 4.12 the quotient Q is either rational or $g(Q) > 2$. In the second case $g(C) \geq 19$ and we obtain a contradiction to $\deg(Y) = g(C) - 1 \leq 9$. It follows that Q is a rational curve.

We consider the action of G on the Del Pezzo surface Y of degree ≥ 3, in particular the induced action on its configuration of (-1)-curves. By Remark 3.26 the stabilizer of a (-1)-curve in Y has index ≥ 16 in G and we may immediately exclude the cases $\deg(Y) = 3,5,6,7$. The automorphism group of a Del Pezzo surface of degree four is $C_2^4 \rtimes \Gamma$ for $\Gamma \in \{C_2, C_4, S_3, D_{10}\}$ (cf. [Dol08]). In particular, the maximal possible order is 160 and therefore $\deg(Y) \neq 4$.

Assume that $Y \cong \mathbb{P}_1 \times \mathbb{P}_1$. The canonical projection $\pi_{1,2} : Y \to \mathbb{P}_1$ is equivariant with respect to a subgroup H of G of index at most two. It follows that H fits into the exact sequences

$$\{\mathrm{id}\} \to I_1 \to H \overset{(\pi_1)_*}{\to} H_1 \to \{\mathrm{id}\}$$
$$\{\mathrm{id}\} \to I_2 \to H \overset{(\pi_2)_*}{\to} H_2 \to \{\mathrm{id}\}$$

where $I_i \cong C_2 \times C_2$ is the ineffectivity of the induced H-action on the base and $H_i \cong S_4$ (cf. proof of Lemma 3.2). Since the action of G on $\mathbb{P}_1 \times \mathbb{P}_1$ is effective by assumption, it follows that I_2 acts effectively on $\pi_1(\mathbb{P}_1 \times \mathbb{P}_1)$. We find a set of four points in $\pi_1(\mathbb{P}_1 \times \mathbb{P}_1)$ with nontrivial isotropy with respect to $I_2 \cong C_2 \times C_2$. Since I_2 is a normal subgroup of H, this set is H-invariant. The action of $H_1 \cong S_4$ on $\pi_1(\mathbb{P}_1 \times \mathbb{P}_1)$ does however not admit invariant sets of cardinality four since the minimal S_4-orbit in $\mathbb{P}_1$ has length six.

We conclude that Y must be isomorphic to $\mathbb{P}_2$. It follows that $g(C) = 10$. Return to the covering $C \to Q$,

$$-18 = e(C) = 8 \cdot e(Q) - \text{branch point contributions}.$$

Since Q is rational, the branch point contribution must 34. The possible isotropy of $N = C_2^3$ at a point in C is C_2 and the full branch point contribution must be divisible by four. This contradiction yields the desired non-existence. □

4.8 The group $N_{72} = C_3^2 \rtimes D_8$

We let X be a K3-surface with a symplectic action of $G = N_{72}$ centralized by an antisymplectic involution σ with $\mathrm{Fix}_X(\sigma) \neq \emptyset$. Note that in this case we may not apply Theorem 3.25 and therefore begin by excluding that a G-minimal model of $Y = X/\sigma$ is an equivariant conic bundle.

Lemma 4.16. *A G-minimal model of Y is a Del Pezzo surface.*

Proof. Assume the contrary and let $Y_{\min}$ be an equivariant conic bundle and a G-minimal model of Y. We consider the induced action of G on the base $B = \mathbb{P}_1$ and denote by $I \lhd G$ the ineffectivity of the G-action on B. Arguing as in the proof of Lemma 3.2, we see that I is trivial or isomorphic to either C_2 or $C_2 \times C_2$. In all cases the quotient G/I contains the subgroup $C_3 \times C_3$, which has no effective action on the rational curve B. □

As we will see, only very few Del Pezzo surfaces admit an effective action of the group N_{72}. We will explicitly use the group structure of $N_{72} = C_3^2 \rtimes D_8$: the action of $D_8 = C_2 \ltimes (C_2 \times C_2) = \langle \alpha \rangle \ltimes (\langle \beta \rangle \times \langle \gamma \rangle) = \mathrm{Aut}(C_3 \times C_3)$ on $C_3 \times C_3$ is given by

$$\alpha(a,b) = (b,a), \quad \beta(a,b) = (a^2,b), \quad \gamma(a,b) = (a,b^2).$$

As a first step we show:

Lemma 4.17. *The degree of a Del Pezzo surface $Y_{\min}$ is at most four.*

Proof. We exclude Del Pezzo surface of degree ≥ 5.

- A Del Pezzo surface of degree five has automorphims group S_5 and $N_{72} \not< S_5$.

- The automorphism group of a Del Pezzo surface of degree six is $(\mathbb{C}^*)^2 \rtimes (S_3 \times C_2)$ (cf. Theorem 10.2.1 in [Dol08]). Assume that $N_{72} = C_3^2 \rtimes D_8$ is contained in this group and consider the intersection $A = N_{72} \cap (\mathbb{C}^*)^2$. The quotient of N_{72} by A has order at most 12 and may not contain a copy of C_3^2. Therefore, the order of A is at least six and A contains a copy of C_3. If $|A| = 6$, then $A = C_6 = C_3 \times C_2$ and C_2 is central in N_{72}. Using the group structure of N_{72} specified above one finds that there is no copy of C_2 in N_{72} centralizing $C_3 \times C_3$ and therefore C_2 cannot be contained in the centre of N_{72}. For every choice of C_3 inside $C_3 \times C_3$ there is precisely one element in $\{\alpha, \beta, \gamma\}$ acting trivially on it and the centralizer of C_3 inside D_8 is isomorphic to C_2. If $|A| > 6$, then the centralizer of C_3 in D_8 has order greater then 2, a contradiction.

- A Del Pezzo surface of degree seven is obtained by blowing-up to points p, q in $\mathbb{P}_2$. As was mentioned before, such a surface is never G-minimal.

- If G acts on $\mathbb{P}_1 \times \mathbb{P}_1$, then the canonical projections are equivariant with respect to a subgroup H of index two in G. We consider one of these projections. The action of H induces an effective action of H/I on the base $\mathbb{P}_1$. The group I is either trivial or isomorphic to C_2 or $C_2 \times C_2$. In all case we find an effective action of C_3^2 on the base, a contradiction.

- It remains to exclude $\mathbb{P}_2$. If N_{72} acts on $\mathbb{P}_2$ we consider its embedding into $\mathrm{PSL}_3(\mathbb{C})$, in particular the realization of the subgroup $C_3^2 = \langle a \rangle \times \langle b \rangle$ and its lifting to $\mathrm{SL}_3(\mathbb{C})$.

  We fix a preimage $\tilde{a}$ of a inside $\mathrm{SL}_3(\mathbb{C})$ and may assume that $\tilde{a}$ is diagonal. Since the action of a on $\mathbb{P}_2$ is induced by a symplectic action on X, it follows that a does not have a positive-dimensional set of fixed point. In appropiately chosen coordinates

$$\tilde{a} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi^2 \end{pmatrix},$$

  where ξ is third root of unity. As a next step, we want to specify a preimage $\tilde{b}$ of b inside $\mathrm{SL}_3(\mathbb{C})$. Since a and b commute in $\mathrm{PSL}_3(\mathbb{C})$, we know that

$$\tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1} = \xi^k \mathrm{id}_{\mathbb{C}^3}$$

  for $k \in \{0, 1, 2\}$. Note that $\tilde{b}$ is not diagonal in the coordinates chosen above since this would give rise to C_3^2-fixed points in $\mathbb{P}_2$. As these correspond to C_3^2-fixed points on the double cover $X \to Y$ and a symplectic action of $C_3^2 \not< \mathrm{SL}_2(\mathbb{C})$ on a K3-surface does not admit fixed points, this is a contradiction. An explicit calculation yields

$$\tilde{b} = \tilde{b}_1 = \begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix} \quad \text{or} \quad \tilde{b} = \tilde{b}_2 = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix}.$$

  We can introduce a change of coordinates commuting with $\tilde{a}$ such that

$$\tilde{b} = \tilde{b}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad \tilde{b} = \tilde{b}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

  Since $\tilde{b}_1^2 = \tilde{b}_2$, the two choices above correspond to the two choices of generators b and b^2 of $\langle b \rangle$. We pick $\tilde{b} = \tilde{b}_1$.

  The action of D_8 on C_3^2 is specified above and the element $\beta \in D_8$ acts on C_3^2 by $a \to a^2$ and $b \to b$. There is no element $T \in \mathrm{SL}_3(\mathbb{C})$ such that (projectively) $T\tilde{a}T^{-1} = \tilde{a}^2$ and $T\tilde{b}T^{-1} = \tilde{b}$. It follows that there is no action of N_{72} on $\mathbb{P}_2$.

This completes the proof of the lemma. $\square$

As a next step, we study the possibility of rational curves in $\mathrm{Fix}_X(\sigma)$.

Lemma 4.18. *There are no rational curves in* $\mathrm{Fix}_X(\sigma)$.

Proof. Let n denote the total number of rational curves in $\mathrm{Fix}_X(\sigma)$ and recall $n \leq 10$. If $n \neq 0$, let C be a rational curve in the image of $\mathrm{Fix}_X(\sigma)$ in Y and let $H = \mathrm{Stab}_G(C)$ be its stabilzer. The index of H in G is at most nine, therefore the order of H is at least eight. The action of H on C is effective.

First note that G does not contain $S_4 = O_{24}$ as a subgroup. If this were the case, consider the intersection $S_4 \cap C_3^2$ and the quotient $S_4 \to S_4/(S_4 \cap C_3^2) < D_8$. Since the only nontrivial normal subgroups of S_4 are A_4 and $C_2 \times C_2$, this leads to a contradiction.

Consequently, the order of H is at most twelve. In particular, $n \geq 6$. Since $C_8 \not< G$, the group H is not cyclic and any H-orbit on C consists of at least two points.

It follows from $C^2 = -4$ that C must meet the union of Mori fibers and the union of Mori fibers meets the curve C in at least two points. Recalling that each Mori fibers meets the branch locus B in at most two points we see that at least n Mori fibers meeting B are required. However, no configuration of n Mori fibers is sufficient to transform the curve C into a curve on a Del Pezzo surface and further Mori fibers are required. By invariance, the total number m of Mori fibers must be at least $2n$.

Combining the Euler-characteristic formula

$$24 = 2e(Y_{\min}) + 2m - 2n + \underbrace{2g - 2}_{\substack{\text{if non-rational} \\ \text{branch curve exists}}}$$

with our observation $\deg(Y_{\min}) \leq 4$, i.e., $e(Y_{\min}) \geq 8$ we see that $n \leq 4$. However, it was shown above, that if $n \neq 0$, then $n \geq 6$. It follows that $n = 0$. $\square$

Proposition 4.19. *The quotient surface Y is G-minimal and isomorphic to the Fermat cubic $\{x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0\} \subset \mathbb{P}_3$. Up to equivalence, there is a unique action of G on Y and the branch locus of $X \to Y$ is given by $\{x_1x_2 + x_3x_4 = 0\}$. In particular, X is equivariantly isomorphic to Mukai's N_{72}-example.*

Proof. We first show that the total number m of Mori fibers equals zero. By the Euler-characteristic formula above, the number m is bounded by four. Using the fact that the maximal order of a stabilizer group of a Mori fiber is twelve (cf. proof of Theorem 3.25) we see that Y must be G-minimal.

In order to conclude that Y is the Fermat cubic we consult Dolgachev's lists of automorphisms groups of Del Pezzo surfaces of degree less than or equal to four ([Dol08] Section 10.2.2; Tables 10.3; 10,4; and 10.5): It follows immediately from the order of G that Y is not of degree two or four. If G were a subgroup of an automorphism group of a Del Pezzo surface of degree one, it would contain a central copy of C_3. The group structure of N_{72} does however not allow this. After excluding the cases $\deg(Y) \in \{1, 2, 4\}$ the result now follows from the uniqueness of the cubic surface in $\mathbb{P}_3$ with an action of N_{72} (cf. Appendix A.1). The action of G on Y is induced by a four-dimensional (projective) representation of G and the branch curve $C \subset Y$ is the intersection of Y with an invariant quadric (compare proof of Proposition 4.8).

In the Appendix A.1 it is shown that there is a uniquely determined action of N_{72} on $\mathbb{P}_3$ and a unique invariant quadric hypersurface $\{x_1x_2 + x_3x_4 = 0\}$. In particular, the branch curve in Y is defined by $\{x_1x_2 + x_3x_4 = 0\} \cap Y$.

Mukai's N_{72}-example is defined by $\{x_1^3 + x_2^3 + x_3^3 + x_4^3 = x_1x_2 + x_3x_4 + x_5^2 = 0\} \subset \mathbb{P}_4$. An antisymplectic involution centralizing the action of N_{72} is given by the map $x_5 \mapsto -x_5$. The quotient of Mukai's example by this involution is the Fermat cubic and the fixed point set of the involution is given by $\{x_1x_2 + x_3x_4 = 0\}$. □

4.9 The group $M_9 = C_3^2 \rtimes Q_8$

Let $G = M_9$ and let X be a K3-surface with a symplectic G-action centralized by the antisymplectic involution σ such that $\mathrm{Fix}_X(\sigma) \neq \emptyset$. We proceed in analogy to the case $G = N_{72}$ above. Arguing precisely as in the proof of Lemma 4.16 one shows.

Lemma 4.20. *A G-minimal model of Y is a Del Pezzo surface.*

We may exclude rational branch curves without studying configurations of Mori fibers.

Lemma 4.21. *There are no rational curves in* $\mathrm{Fix}_X(\sigma)$.

Proof. Let n be the total number of rational curves in $\mathrm{Fix}_X(\sigma)$. Assume $n \neq 0$, let C be a rational curve in the image of $\mathrm{Fix}_X(\sigma)$ in Y and let $H < G$ be its stabilizer. The action of H on C is effective. We go through the list of finite groups with an effective action on a rational curve.

Since M_9 is a group of symplectic transformations on a K3-surface, its element have order at most eight. Clearly, $A_6 \not< M_9$ and D_{10}, D_{14}, $D_{16} \not< M_9$. If $S_4 < M_9 = C_3^2 \rtimes Q_8$, then $S_4 \cap C_3^2$ is a normal subgroup of S_4 and it is therefore trivial. Now $S_4 = S_4/(S_4 \cap C_3^2) < M_9/C_3^2 = Q_8$ yields a contradiction. The same argument can be carried out for A_4, D_8 and C_8. If $D_{12} < M_9 = C_3^2 \rtimes Q_8$, then either $D_{12} \cap C_3^2 = C_3$ and $C_2 \times C_2 = D_{12}/C_3 < M_9/C_3^2 = Q_8$ or $D_{12} \cap C_3^2 = \{\mathrm{id}\}$ and $D_{12} < Q_8$, both are impossible.

It follows that the subgroups of M_9 admitting an effective action on a rational curve have index greater than or equal to twelve. Therefore $n \geq 12$, contrary to the bound $n \leq 10$ obtained in Corollary 3.20. □

Proposition 4.22. *The quotient surface Y is G-minimal and isomorphic to $\mathbb{P}_2$. Up to equivalence, there is a unique action of G on Y and the branch locus of $X \to Y$ is given by $\{x_1^6 + x_2^6 + x_3^6 - 10(x_1^3x_2^3 + x_2^3x_3^3 + x_3^3x_1^3) = 0\}$. In particular, X is equivariantly isomorphic to Mukai's M_9-example.*

Proof. We first check that Y is G-minimal. Again, we proceed as in the proof of Theorem 3.25 and Lemma 4.21 above to see that the largest possible stablizer group of a Mori fiber is $D_6 < G$. If Y is not G-minimal, this implies that the total number of Mori fibers is ≥ 12, contradicting $m \leq 9$.

Note that $X \to Y$ is not branched along one or two elliptic curves as this would imply $e(Y) = 12$ and contradict the fact that Y is a Del Pezzo surface.

Let D be the branch curve of $X \to Y$ and consider the quotient Q of D by the normal subgroup $N = C_3^2$ in G. On Q there is an action of Q_8 implying that Q is not rational. We show that Q_8 does not act on an elliptic curve Q. If this were the case, consider the decomposition $Q_8 = (Q_8 \cap Q) \rtimes (Q_8 \cap L)$ where $(Q_8 \cap L)$ is a nontrivial cyclic group. For any choice of generator of $(Q_8 \cap L)$ the center $\{+1,-1\}$ of Q_8 is contained in $(Q_8 \cap L)$. Let $q : Q_8 \to Q_8/(Q_8 \cap Q) \cong Q_8 \cap L$ denote the quotient homomorphism. The commutator subgroup $Q_8' = \{+1,-1\}$ must be contained in the kernel of q. This contradiction yields that Q_8 does not act on an elliptic curve. It follows that the genus of Q is at least two and the genus of D is at least ten. Adjunction on the Del Pezzo surface Y now implies $g = 10$ and $Y \cong \mathbb{P}_2$.

It is shown in Appendix A.2 that, up to natural equivalence, there is a unique action of M_9 on the projective plane. In suitably chosen coordinated the generators a, b of C_3^2 are represented as

$$\tilde{a} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi^2 \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and $I, J \in Q_8$ are represented as

$$\tilde{I} = \frac{1}{\xi - \xi^2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & \xi \end{pmatrix}, \quad \tilde{J} = \frac{1}{\xi - \xi^2} \begin{pmatrix} 1 & \xi & \xi \\ \xi^2 & \xi & \xi^2 \\ \xi^2 & \xi^2 & \xi \end{pmatrix}.$$

We study the action of M_9 on then space of sextic curves. By restricting our consideration to the subgroup C_3^2 first, we see that a polynomial defining an invariant curve must be a linear combination of the following polynomials:

$$\begin{aligned} f_1 &= x_1^6 + x_2^6 + x_3^6; \\ f_2 &= x_1^2 x_2^2 x_3^2; \\ f_3 &= x_1^3 x_2^3 + x_1^3 x_3^3 + x_2^3 x_3^3; \\ f_4 &= x_1^4 x_2 x_3 + x_1 x_2^4 x_3 + x_1 x_2 x_3^4. \end{aligned}$$

Taking now the additional symmetries into account, we find three M_9-invariant sextic curves, namely

$$\{f_1 - 10 f_3 = x_1^6 + x_2^6 + x_3^6 - 10(x_1^3 x_2^3 + x_2^3 x_3^3 + x_3^3 x_1^3) = 0\},$$

which is the example found by Mukai, and additionally

$$\{f_a = f_1 + (18 - 3a) f_2 + 2 f_3 + a f_4 = 0\},$$

where a is a solution of the quadratic equation $a^2 - 6a + 36$, i.e. $a = -6\xi$ or $a = -6\xi^2$. The polynomial f_a is invariant with respect to the action of M_9 for $a = -6\xi^2$ and semi-invariant if $a = -6\xi$.

We wish to show that X is not the double cover of $\mathbb{P}_2$ branched along $\{f_a = 0\}$. If this were the case, consider the fixed point $p = [0 : 1 : -1]$ of the automorphism I and note that $f_a(p) = 0$. So the $\pi^{-1}(p)$ consists of one point $x \in X$ and we linearize the $\langle I \rangle \times \langle \sigma \rangle$ at x. In suitably chosen coordintes the action of the symplectic automorphism I of order four is of the form $(z, w) \mapsto (iz, -iw)$. Since the action of σ commutes with I, the σ-quotient of X is locally given by

$$(z, w) \mapsto (z^2, w) \quad \text{or} \quad (z, w) \mapsto (z, w^2).$$

It follows that the action of I on Y is locally given by either

$$\begin{pmatrix} -1 & 0 \\ 0 & -i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix}.$$

In particular, the local linearization of I at p has determinant $\neq 1$. By a direct computation using the explicit form of $\tilde{I}$ given above, in particular the facts that $\det(\tilde{I}) = 1$ and $\tilde{I}v = v$ for $[v] = p$, we obtain a contradiction.

This completes the proof of the proposition. □

Remark 4.23. In the proof of the propostion above we have observed that an element of $SL_3(\mathbb{C})$ does not necessarily lift to a symplectic transformation on the double cover of $\mathbb{P}_2$ branched along a sextic given by an invariant polynomial. Mukai's M_9-example X is a double cover of $\mathbb{P}_2$ branched along the sextic curve $\{x_1^6 + x_2^6 + x_3^6 - 10(x_1^3x_2^3 + x_2^3x_3^3 + x_3^3x_1^3) = 0\}$ and for this particular example, the action of M_9 does lift to a group of symplectic transformation as claimed by Mukai.

To see this consider the set $\{a, b, I, J\}$ of generators of M_9. Since a and b are commutators in M_9, they can be lifted to symplectic transformation $\bar{a}, \bar{b}$ on X. For I, J consider the linearization at the fixed point $[0 : 1 : -1]$ and check that it has determinant one. Since $[0 : 1 : -1]$ is *not* contained in the branch set of the covering, its preimage in X consists of two points p_1, p_2. We can lift I (J, respectively) to a transformation of X fixing both p_1, p_2 and a neighbourhood of p_1 is I-equivariantly isomorphic to a neighbourhood of $[0 : 1 : -1] \in \mathbb{P}_2$. In particular, the action of the lifted element $\bar{I}$ ($\bar{J}$, respectively) is symplectic. On X there is the action of a degree two central extension E of M_9,

$$\{\mathrm{id}\} \to C_2 \to E \to M_9 \to \{\mathrm{id}\}.$$

The elements $\bar{a}, \bar{b}, \bar{I}, \bar{J}$ generate a subgroup $\tilde{M}_9$ of E_{symp} mapping onto M_9. Since $E_{\mathrm{symp}} \neq E$, the order of $\tilde{M}_9$ is 72 and it follows that $\tilde{M}_9$ is isomorphic to M_9. In particular E splits as $E_{\mathrm{symp}} \times C_2$ with $E_{\mathrm{symp}} = M_9$.

4.10 The group $T_{48} = Q_8 \rtimes S_3$

We let X be a K3-surface with an action of $T_{48} \times C_2$ where the action of $G = T_{48}$ is symplectic and the generator σ of C_2 is antisymplectic and has fixed points. The action of S_3 on Q_8 is given as follows: The element c of order three in S_3 acts on Q_8 by permuting I, J, K and an element d of order two acts by exchanging I and J and mapping K to $-K$.

Lemma 4.24. *A G-minimal model $Y_{\min}$ of Y is either $\mathbb{P}_2$, a Hirzebruch surface Σ_n with $n > 2$, or $e(Y_{\min}) \geq 9$.*

Proof. Let us first consider the case where $Y_{\min}$ is a Del Pezzo surface and go through the list of possibilities.

- Let $Y_{\min} \cong \mathbb{P}_1 \times \mathbb{P}_1$. Since T_{48} acts on $Y_{\min}$, both canonical projections are equivariant with respect to the index two subgroup $G' = Q_8 \rtimes C_3$. Since Q_8 has no effective action on $\mathbb{P}_1$, it follows that the subgroup $Z = \{+1, -1\} < Q_8$ acts trivially on the base. Since this holds with respect to both projections, the subgroup Z acts trivially on $Y_{\min}$, a contradiction.

- Using the group structure of T_{48} one checks that the only nontrivial normal subgroup N of T_{48} such that $N \cap Q_8 \neq Q_8$ is the center $Z = \{+1, -1\}$ of T_{48}. It follows that T_{48} is neither a subgroup of $(\mathbb{C}^*)^2 \rtimes (S_3 \times C_2)$ nor a subgroup of any of the automorphism groups $C_2^4 \rtimes \Gamma$ for $\Gamma \in \{C_2, C_4, S_3, D_{10}\}$ of a Del Pezzo surface of degree four. Furthermore, $T_{48} \not< S_5$. Thus it follows that $d(Y_{\min}) \neq 4, 5, 6$.

So if $Y_{\min}$ is a Del Pezzo surface, then $Y_{\min} \cong \mathbb{P}_2$ or $e(Y_{\min}) \geq 9$.

Let us now turn to the case where $Y_{\min}$ is an equivariant conic bundle. We first show that $Y_{\min}$ is not a conic bundle with singular fibers. We assume the contrary and let $p : Y_{\min} \to \mathbb{P}_1$ be an equivariant conic bundle with singular fibers. The center $Z = \{+1, -1\}$ of $G = T_{48}$ acts trivially on the base an has two fixed points in the generic fiber. Let C_1 and C_2 denote the two curves of Z-fixed points in $Y_{\min}$. By Lemma 2.21 any singular fiber F is the union of two (-1)-curves F_1, F_2 meeting transversally in one point. We consider the action of Z on this union of curves. The group Z does not act trivially on either component of F since linearization at a smooth point of F would yield a trivial action of Z on $Y_{\min}$. Consequently, it has either one or three fixed points on F. The first is impossible since C_1 and C_2 intersect F in two points. It follows that Z stabilizes each curve F_i. We linearize the action of Z at the point of intersection $F_1 \cap F_2$. The intersection is transversal and the action of Z is by $-\mathrm{Id}$ on $T_{F_1} \oplus T_{F_2}$ contradicting the fact the Z acts trivially on the base. Thus Y_{min} is not a conic bundle with singular fibers.

If $Y_{\min} \to \mathbb{P}_1$ is a Hirzebruch surface Σ_n, then the action of T_{48} induces an effective action of S_4 on the base $\mathbb{P}_1$.

The action of T_{48} on Σ_n stabilizes two disjoint sections E_∞ and E_0, the curves of Z-fixed points. This is only possible if $E_0^2 = -E_\infty^2 = n$. Removing the exceptional section E_∞ from Σ_n, we obtain the hyperplane bundle H^n of $\mathbb{P}_1$. Since T_{48} stabilizes the section E_0, we chose this section to be the zero section and conclude that the action of T_{48} on H^n is by bundle automorphisms.

If $n = 2$, then H^n is the anticanonical line bundle of $\mathbb{P}_1$ and the action of S_4 on the base induces an action of S_4 on H^2 by bundle automorphisms. It follows that T_{48} splits as $S_4 \times C_2$, a contradiction. Thus, if $Y_{\min}$ is a Hirzebruch surface Σ_n, then $n \neq 2$. $\square$

Lemma 4.25. *There are no rational curves in* $\mathrm{Fix}_X(\sigma)$.

Proof. We let n denote the total number of rational curves in $\mathrm{Fix}_X(\sigma)$ and assume $n > 0$. Recall $n \leq 10$, let C be a rational curve in $B = \pi(\mathrm{Fix}_X(\sigma)) \subset Y$ and let $H = \mathrm{Stab}_G(C) < G$ be its stabilizer group. The action of H on C is effective, the index of H in G is at most 8. Using the quotient homomorphism $T_{48} \to T_{48}/Q_8 = S_3$ one checks that T_{48} does not contain $O_{24} = S_4$ or $T_{12} = A_4$ as a subgroup. It follows that H is a cyclic or a dihedral group.

If $H \in \{C_6, C_8, D_8\}$, then H and all conjugates of H in G contain the center $Z = \{+1, -1\}$ of G. It follows that Z has two fixed point on each curve gC for $g \in G$. Since there are six (or eight) distinct curves gC in Y, it follows that Z has at least 12 fixed points in Y and in X. This contradicts to assumption that $Z < G$ acts symplectically on X and therefore has eight fixed points in the K3-surface X.

It remains to study the cases $H = D_{12}$ and $H = D_6$ where $n = 8$ or $n = 4$.

We note that a Hirzebruch surface has precisely one curve with negative self-intersection and only fibers have self-intersection zero. A Del Pezzo surface does not contains curves of self-intersection less than -1. The rational branch curves must therefore meet the union of Mori fibers in Y.

The total number of Mori fibers is bounded by $n + 9$. We study the possible stabilizer subgroups $\mathrm{Stab}_G(E) < G$ of Mori fibers. A Mori fiber E with self-intersection (-1) meets the branch locus B in one or two points and its stabilizer is either cyclic or dihedral. If $\mathrm{Stab}_G(E) \in \{C_4, D_8\}$, then the points of intersection of E and B are fixed points of the center Z of G and we find too many Z-fixed points on X.

Assume $n = 4$ and let $R_1, \ldots R_4$ be the rational curves in B. We denote by $\tilde{R}_i$ their images in $Y_{\min}$. The total number m of Mori fibers is bounded by 12. We go through the list of possible configurations:

- If $m = 4$, there is no invariant configuration of Mori fibers such that the contraction maps the four rational branch curves to a configuration on the Hirzebruch or Del Pezzo surface Y_{min}.
- If $m = 6$, then $\text{Stab}_G(E) = C_8$ and the points of intersection of E and B are Z-fixed. Since Z has at most eight fixed points on B, it follows that each curve E meets B only once. The images $\tilde{R}_i$ of the R_i contradict our observations about curves in Del Pezzo and Hirzebruch surfaces.
- If $m = 8$ and all Mori fibers have self-intersection -1, then each Mori fiber meets $\bigcup R_i$ in a Z-fixed point. Since there at at most eight such points, it follows that each Mori fibers meets $\bigcup R_i$ only once and their contractions does not transform the curves R_i sufficiently.
- If $m = 8$ and only four Mori fibers have self-intersection -1, we consider the four Mori fibers of the second reduction step. Each of these meets a Mori fiber E of the first step in precisely one point. By invariance, this would have to be a fixed points of the stabilizer $\text{Stab}_G(E) = D_{12}$, a contradiction.
- If $m = 12$, then either $e(Y_{\text{min}}) = 3$ and there exist a branch curve $D_{g=2}$ of genus two or $e(Y_{\text{min}}) = 4$ and $B = \bigcup R_i$. In the first case, $Y_{\text{min}} \cong \mathbb{P}_2$ and twelve Mori fibers are not sufficient to transform $B = D_{g=2} \cup \bigcup R_i$ into a configuration of curves in the projective plane. So $Y_{\text{min}} = \Sigma_n$ for $n > 2$.

  Since Z has two fixed points in each fiber of $p : \Sigma_n \to \mathbb{P}_1$ the Z-action on Σ_n has two disjoint curves of fixed points. As was remarked above, these curves are the exceptional section E_∞ of self-intersection -n and a section $E_0 \sim E_\infty + nF$ of self-intersection n. Here F denotes a fiber of $p : \Sigma_n \to \mathbb{P}_1$. There is no automorphisms of Σ_n mapping E_∞ to E_0.

  Each rational branch curve $\tilde{R}_i$ has two Z-fixed points. These are exchanged by an element of $\text{Stab}_G(R_i)$ and therefore both lie on either E_∞ or E_0, i.e., $\tilde{R}_i$ cannot have nontrivial intersection with both E_0 and E_∞. By invariance all curves $\tilde{R}_i$ either meet E_0 or E_∞ and not both.

  Using the fact that $\sum \tilde{R}_i$ is linearly equivalent to $-2K_{\Sigma_n} \sim 4E_\infty + (2n+4)F$ we find that $\tilde{R}_i \cdot E_\infty = 0$ and $n = 2$, a contradiction to Lemma 4.24.

We have shown that all possible configurations in the case $n \neq 4$ lead to a contradiction. We now turn to the case $n = 8$ and let $R_1, \dots R_8$ be the rational ramification curves. The total number of Mori fibers is bounded by 16. Note that by invariance, the orbit of a Mori fiber meets $\bigcup R_i$ in at least 16 points or not at all. In particular, Mori fibers meeting R_i come in orbits of length ≥ 8. As above, we go through the list of possible configurations.

- If $m = 16$, then the set of all Mori fibers consists of two orbits of length eight. If all 16 Mori fibers meet B, then each meets B in one point and R_i is mapped to a (-2)-curve in Y_{min}. If only eight Mori fibers meet B, then each of the eight Mori fibers of the second reduction step meets one Mori fiber E of the first reduction step in one point. This point would have to be a $\text{Stab}_G(E)$-fixed point. But if $\text{Stab}_G(E)$ is cyclic, its fixed points coincide with the points $E \cap B$.
- If $m = 12$, then the set of all Mori fibers consists of a single G-orbit and each curve R_i meets three distinct Mori fibers. Their contraction transforms R_i into a (-1)-curve on Y_{min}. It follows that Y_{min} contains at least eight (-1)-curves and is a Del Pezzo surface of degree ≤ 5. We have seen above that $d(Y_{\text{min}}) \neq 4, 5$ and therefore $e(Y_{\text{min}}) \geq 9$. With $m = 12$ and $n = 8$, this contradicts the Euler characteristic formula $24 = 2e(Y_{\text{min}}) + 2m - 2n + (2g - 2)$.

- If $m = 8$ there is no invariant configuration of Mori fibers such that the contraction maps the eight rational branch curves to a configuration on the Hirzebruch or Del Pezzo surface $Y_{\min}$

This completes the proof of the lemma. □

Since there is an effective action of T_{48} on $\mathrm{Fix}_X(\sigma)$, it is neither an elliptic curve nor the union of two elliptic curves. It follows that $X \to Y$ is branched along a single T_{48}-invariant curve B with $g(B) \geq 2$.

Lemma 4.26. *The genus of B is neither three nor four.*

Proof. We consider the quotient $Q = B/Z$ of the curve B by the center Z of G and apply the Euler characteristic formula, $e(B) = 2e(Q) - |\mathrm{Fix}_B(Z)|$. On Q there is an effective action of the group $G/Z = (C_2 \times C_2) \rtimes C_3 = S_4$. Using Lemma 4.12 we see that $e(Q) \in \{2, -4, -6, -8 \dots\}$.

If $g(B) = 3$, then $e(B) = -4$ and the only possibility is $Q \cong \mathbb{P}_1$ and $|\mathrm{Fix}_B(Z)| = 8$. In particular, all Z-fixed points on X are contained in the curve B. Let $A < G$ be the group generated by $I \in Q_8 = \{\pm 1, \pm I, \pm J, \pm K\}$. The four fixed points of A in X are contained in $\mathrm{Fix}_X(Z) = \mathrm{Fix}_B(Z)$ and the quotient group $A/Z \cong C_2$ has four fixed points in Q. This is a contradiction.

If $g(B) = 4$, then $e(B) = -6$ and the only possibility is $Q \cong \mathbb{P}_1$ and $|\mathrm{Fix}_B(Z)| = 10$. This contradicts the fact that Z has at most eight fixed points in B since it has precisely eight fixed points in X. □

In Lemma 4.24 we have reduced the classification to the cases $e(Y_{\min}) \in \{3, 4, 9, 10, 11\}$. In the following, we will exclude the cases $e(Y_{\min}) \in \{4, 9, 10, \}$ and describe the remaining cases more precisely. Recall that the maximal possible stabilizer subgroup of a Mori fiber is D_{12}, in particular, $m = 0$ or $m \geq 4$.

Lemma 4.27. *If $e(Y_{\min}) = 3$, then $Y_{\min} = Y = \mathbb{P}_2$ and $X \to Y$ is branched along the curve $\{x_1 x_2 (x_1^4 - x_2^4) + x_3^6 = 0\}$. In particular, Y is equivariantly isomorphic to Mukai's T_{48}-example.*

Proof. Let $M : Y \to \mathbb{P}_2$ denote a Mori reduction of Y and let $B \subset Y$ be the branch curve of the covering $X \to Y$. If $Y = Y_{\min}$, then $B = M(B)$ is a smooth sextic curve. If $Y \neq Y_{\min}$, then the Euler characteristic formula with $m \in \{4, 6, 8\}$ shows that $g(B) \in \{2, 4, 6\}$. The case $m = 6$, $g(B) = 4$ has been excluded by the previous lemma.

If $m = 4$, then the stabilizer group of each Mori fiber is D_{12} and each Mori fiber meets B in two points. Furthermore, since in this case $g(B) = 6$, the self-intersection of $\mathrm{Fix}_X(\sigma)$ in X equals ten and therefore $B^2 = 20$. The image $M(B)$ of B in $Y_{\min}$ has self-intersection $20 + 4 \cdot 4 = 36$ and follows to be an irreducible singular sextic.

If $m = 8$, then $g(B) = 2$ and $B^2 = 4$. Since the self-intersection number $M(B)^2$ must be a square, one checks that all possible invariant configurations of Mori fibers yield $M(B)^2 = 36$ and involve Mori fibers meeting B is two points. In particular, $M(B)$ is a singular sextic.

We study the action of T_{48} on the projective plane. As a first step, we may choose coordinates on $\mathbb{P}_2$ such that the automorphism $-1 \in Q_8 < T_{48}$ is represented as

$$\widetilde{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We denote by V to the -1-eigenspace of this operator. For each element I, J, K there is a unique choice $\widetilde{I}, \widetilde{J}, \widetilde{K}$ in $\mathrm{SL}_3(\mathbb{C})$ such that $\widetilde{I}^2 = \widetilde{J}^2 = \widetilde{K}^2 = \widetilde{-1}$. One checks $\widetilde{I}\widetilde{J}\widetilde{K} = \widetilde{-1}$. Therefore $\widetilde{I}, \widetilde{J}, \widetilde{K}$ generate a subgroup of $\mathrm{SL}_3(\mathbb{C})$ isomorphic to Q_8. By construction $\widetilde{I}, \widetilde{J}, \widetilde{K}$ stabilze the vector space V. Up to isomorphisms, there is a unique faithful 2-dimensional representation of Q_8 and it follows that I, J, K are represented as

$$\widetilde{I} = \begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{K} = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We recall that the action of S_3 on Q_8 is given as follows: The element c of order three in S_3 acts on Q_8 by permuting I, J, K and an element d of order two acts by exchanging I and J and mapping K to $-K$. With $\mu = \sqrt{\frac{i}{2}}$ and $\nu = \frac{i}{\sqrt{2}}$ it follows that the elements c and d are represented as

$$\widetilde{c} = \begin{pmatrix} -i\mu & i\mu & 0 \\ \mu & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{d} = \begin{pmatrix} -i\nu & -\nu & 0 \\ \nu & i\nu & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In particular, there is a unique action of T_{48} on $\mathbb{P}_2$. In the following, we denote by $[x_1 : x_2 : x_3]$ homogeneous coordintes such that the action of T_{48} is as above. Using the explicit form of the T_{48}-action and the fact that the commutator subgroup of T_{48} is $Q_8 \rtimes C_3$ one can check that any invariant curve of degree six is of the form

$$C_\lambda = \{x_1 x_2 (x_1^4 - x_2^4) + \lambda x_3^6 = 0\}$$

In order to avoid this calculation, one can also argue that the polynomial $x_1 x_2 (x_1^4 - x_2^4)$ is the lowest order invariant of the octahedral group $S_4 \cong T_{48}/Z$.

The curve C_λ is smooth and it follows that $Y = Y_{\mathrm{min}}$. We may adjust the coordinates equivariantly such that $\lambda = 1$ and find that our surface X is precisely Mukai's T_{48}-example. □

Remark 4.28. As claimed by Mukai, the action of T_{48} on $\mathbb{P}_2$ does indeed lift to a symplectic action of T_{48} on the double cover of $\mathbb{P}_2$ branched along the invariant curve $\{x_1 x_2 (x_1^4 - x_2^4) + x_3^6 = 0\}$. The elements of the commutator subgroup can be lifted to symplectic transformation on the double cover X.

The remaining generator d is an involution fixing the point $[0 : 0 : 1]$. Any involution τ with a fixed point p outside the branch locus can be lifted to a symplectic involution on the double cover X as follows:

The linearized action of τ at p has determinant ± 1. We consider the lifting $\tilde{\tau}$ of τ fixing both points in the preimage of p. Its linearization coincides with the linearization on the base and therefore also has determinant ± 1. In particular, $\tilde{\tau}$ is an involution. It follows that either $\tilde{\tau}$ or the second choice of a lifting $\sigma\tilde{\tau}$ acts symplectically on X.

The group generated by all lifted automorphisms is either isomorphic to T_{48} or to the full central extension E

$$\{\mathrm{id}\} \to C_2 \to E \to T_{48} \to \{\mathrm{id}\}$$

acting on the double cover. Since $E_{\mathrm{symp}} \neq E$ the later is impossible it follows that E splits as $E_{\mathrm{symp}} \times C_2$ with $E_{\mathrm{symp}} = T_{48}$.

Finally, we return to the remaining possibilities $e(Y_{\mathrm{min}}) \in \{4, 9, 10, 11\}$.

Lemma 4.29. $e(Y_{\min}) \notin \{4, 9, 10\}$.

Proof. Recalling that the genus of the branch curve B is neither three nor four and that m is either zero or ≥ 4, we may exclude $e(Y_{\min}) = 9, 10$ using the Euler characteristic formula $12 = e(Y_{\min}) + m + g - 1$. It remains to consider the case $Y_{\min} = \Sigma_n$ with $n > 2$ and we claim that this is impossible.

Let $M = Y \to Y_{\min} = \Sigma_n$ denote a (possibly trivial) Mori reduction of Y. The image $M(B)$ of B in Σ_n is linearly equivalent to $-2K_{\Sigma_n}$. Now $M(B) \cdot E_\infty = 2(2-n) < 0$ and it follows that $M(B)$ contains the rational curve E_∞. This is a contradiction since B does not contain any rational curves by Lemma 4.25. □

In the last remaining case, i.e., $e(Y_{\min}) = 11$, the quotient surface Y is a G-minimal Del Pezzo surface of degree 1. Consulting [Dol08], Table 10.5, we find that Y is a hypersurface in weighted projective space $\mathbb{P}(1,1,2,3)$ defined by the degree six equation

$$x_0 x_1 (x_0^4 - x_1^4) + x_2^3 + x_3^2.$$

This follows from the invariant theory of the group $S_4 \cong T_{48}/Z$ and fact that Y is a double cover of a quadric cone Q in $\mathbb{P}_3$ branched along the intersection of Q with a cubic hypersurface (cf. Theorem 3.6).

The linear system of the anticanonical divisor K_Y has precisely one base point p. In coordinates $[x_0 : x_1 : x_2 : x_3]$ this point is given as $[0:0:1:i]$. It is fixed by the action of T_{48}. The linearization of T_{48} at p is given by the unique faithful 2-dimensional represention of T_{48}. This represention has implicitly been discussed above as a subrepresentation V of the three-dimensional representation of T_{48}. It follows that there is a unique action of T_{48} on Y. The branch curve B is linearly equivalent to $-2K_Y$, i.e., $B = \{s = 0\}$ for a section $s \in \Gamma(Y, \mathcal{O}(-2K_Y))$ which is either invariant or semi-invariant.

By an adjunction formula for hypersurfaces in weighted projective space $\mathcal{O}(-2K_Y)) = \mathcal{O}_Y(2)$. The four-dimensional space of sections $\Gamma(Y, \mathcal{O}(-2K_Y))$ is generated by the weighted homogeneous polynomials $x_0^2, x_1^2, x_0x_1, x_2$. We consider the map $Y \to \mathbb{P}(\Gamma(Y, \mathcal{O}(-2K_Y))^*)$ associated to $|-2K_Y|$. Since this map is equivariant with respect to $\mathrm{Aut}(Y)$, the fixed point p is mapped to a fixed point in $\mathbb{P}(\Gamma(Y, \mathcal{O}(-2K_Y))^*)$. It follows that the section corresponding to the homogeneous polynomial x_2 is invariant or semi-invariant with respect to T_{48}. It is the only section of $\mathcal{O}(-2K_Y)$ with this property since the representation of T_{48} on the span of x_0^2, x_1^2, x_0x_1 is irreducible.

The curve $B \subset Y$ defined by $s = 0$ is connected and has arithmetic genus 2. Since T_{48} acts effectively on B and does not act on $\mathbb{P}_1$ or a torus, it follows that B is nonsingular.

It remains to check that the action of T_{48} on Y lifts to a group of symplectic transformation on the double cover X branched along B. First note that B does not contain the base point p. For $I, J, K, c \in T_{48}$ we we choose liftings $\overline{I}, \overline{J}, \overline{K}, \overline{c} \in \mathrm{Aut}(X)$ fixing both points in $\pi^{-1}(p) = \{p_1, p_2\}$. The linearization of $\overline{I}, \overline{J}, \overline{K}, \overline{c}$ at p_1 is the same as the linearization at p and in particular has determinant one. By the general considerations in Remark 4.28 the involution d can be lifted to a symplectic involution on X. The symplectic liftings of I, J, K, c, d generate a subgroup $\tilde{G}$ of $\mathrm{Aut}(X)$ which is isomorphic to either T_{48} or to the central degree two extension of T_{48} acting on X. In analogy to Remarks 4.23 and 4.28 we conclude that $\tilde{G} \cong T_{48}$ and the action of T_{48} on Y induces a symplectic action of T_{48} on the double cover X.

This completes the classification of K3-surfaces with $T_{48} \times C_2$-symmetry. We have shown:

Theorem 4.30. *Let X be a K3-surface with a symplectic action of the group T_{48} centralized by an antisymplectic involution σ with $\mathrm{Fix}_X(\sigma) \neq \emptyset$. Then X is equivariantly isomorphic either to Mukai's T_{48}-example or to the double cover of*

$$\{x_0 x_1(x_0^4 - x_1^4) + x_2^3 + x_3^2 = 0\} \subset \mathbb{P}(1,1,2,3)$$

branched along $\{x_2 = 0\}$

Remark 4.31. The automorphism group of the Del Pezzo surface $Y = \{x_0 x_1(x_0^4 - x_1^4) + x_2^3 + x_3^2 = 0\} \subset \mathbb{P}(1,1,2,3)$ is the trivial central extension $C_3 \times T_{48}$. By contruction, the curve $B = \{s = 0\}$ is invariant with respect to the full automorphism group. The double cover X of Y branched along B carries the action of a finite group $\tilde{G}$ of order $2 \cdot 3 \cdot 48 = 288$ containing $T_{48} < \tilde{G}_{\mathrm{symp}}$. Since T_{48} is a maximal group of symplectic transformations, we find $T_{48} = \tilde{G}_{\mathrm{symp}}$ and therefore

$$\{\mathrm{id}\} \to T_{48} \to \tilde{G} \to C_6 \to \{\mathrm{id}\}.$$

In analogy to the proof of Claim 2.1 in [OZ02], one can check that 288 is the maximal order of a finite group H acting on a K3-surface with $T_{48} < H_{\mathrm{symp}}$. It follows that $\tilde{G}$ is maximal finite subgroup of $\mathrm{Aut}(X)$. For an arbitrary finite group H acting on a K3-surface with $\{\mathrm{id}\} \to T_{48} \to H \to C_6 \to \{\mathrm{id}\}$, there need however not exist an involution in H centralizing T_{48}.

5 K3-surfaces with an antisymplectic involution centralizing $C_3 \ltimes C_7$

In this chapter it is illustrated that a classification of K3-surfaces with antisymplectic involution σ can be carried out even even if the centralizer G of σ inside the group of symplectic transformations is relatively small, i.e., well below the bound 96 obtianed in Theorem 3.25, and not among the maximal groups of symplectic transformations. We consider the group $G = C_3 \ltimes C_7$, which is a subgroup of $L_2(7)$. The principles presented in Chapter 3 can be transferred to this group G and yield a description of K3-surfaces with $G \times \langle \sigma \rangle$-symmetry. Using this, we deduce the classification K3-surfaces with an action of $L_2(7) \times C_2$ announced in Section 4.1. The results presented in this chapter have appeared in [FH08].

To begin with, we present a family of K3-surfaces with $G \times \langle \sigma \rangle$-symmetry.

Example 5.1. We consider the action of G on $\mathbb{P}_2$ given by one of its three-dimensional representations. After a suitable change of coordinates, the action of the commutator subgroup $G' = C_7 < G$ is given by

$$[z_0 : z_1 : z_2] \mapsto [\lambda z_0 : \lambda^2 z_1 : \lambda^4 z_2]$$

for $\lambda = \exp(\frac{2\pi i}{7})$ and C_3 is generated by the permutation

$$[z_0 : z_1 : z_2] \mapsto [z_2 : z_0 : z_1].$$

The vector space of G-invariant homogeneous polynomials of degree six is the span of $P_1 = z_0^2 z_1^2 z_2^2$ and $P_2 = z_0^5 z_1 + z_2^5 z_0 + z_1^5 z_2$.

The family $\mathbb{P}(V)$ of curves defined by polynomials in V contains exactly four singular curves, namely the curve defined by $z_0^2 z_1^2 z_2^2$ and those defined by $3z_0^2 z_1^2 z_2^2 - \zeta^k (z_0^5 z_1 + z_2^5 z_0 + z_1^5 z_2)$, where ζ is a nontrivial third root of unity, $k = 1, 2, 3$. We let $\Sigma = \mathbb{P}(V) \backslash \{z_0^2 z_1^2 z_2^2 = 0\}$.

The double cover of $\mathbb{P}_2$ branched along a curve $C \in \Sigma$ is a K3-surface (singular K3-surface if C is singular) with an action of $G \times C_2$ where C_2 acts nonsymplectically. It follows that Σ parametrizes a family of K3-surface with $G \times C_2$-symmetry.

Remark 5.2. Let us consider the cyclic group Γ of order three generated by the transformation $[z_0 : z_1 : z_2] \mapsto [z_0 : \zeta z_1 : \zeta^2 z_2]$ and its induced action on the space Σ. One finds that the three irreducible singular G-invariant curves form a Γ-orbit. Furthermore, if two curves $C_1, C_2 \in \Sigma$ are equivalent with respect to the action of Γ, then the corresponding K3-surfaces are equivariantly isomorphic (see Section 5.3.2 for a detailed discussion).

Remark 5.3. The singular curve $C_{\text{sing}} \subset \mathbb{P}_2$ defined by $3z_0^2z_1^2z_2^2 - (z_0^5z_1 + z_2^5z_0 + z_1^5z_2)$ has exactly seven singular points $p_1, \ldots p_7$ forming an G-orbit. Since they are in general position (cf. Proposition 5.15), the blow up of $\mathbb{P}_2$ in these points defines a Del Pezzo surface Y_{Klein} of degree two with an action of G. It is seen to be the double cover of $\mathbb{P}_2$ branched along Klein's quartic curve

$$C_{\text{Klein}} := \{z_0z_1^3 + z_1z_2^3 + z_2z_0^3 = 0\}.$$

The proper transform B of C_{sing} in Y_{Klein} is a smooth G-invariant curve. It is a normalization of C_{sing} and has genus three by the genus formula. The curve B coincides with the preimage of C_{Klein} in Y_{Klein}. The minimal resolution $\tilde{X}_{\text{sing}}$ of the singular surface X_{sing} defined as the double cover of $\mathbb{P}_2$ branched along C_{sing} is a K3-surface with an action of G. By construction, it is the double cover of Y_{Klein} branched along B. In particular, $\tilde{X}_{\text{sing}}$ is the degree four cyclic cover of $\mathbb{P}_2$ branched along C_{Klein} and known as the Klein-Mukai-surface X_{KM} (cf. Example 1.15).

Notation. In the following, the notion of "$G \times C_2$-symmetry" abbreviates a symplectic action of G centralized an antisymplectic action of C_2.

In this chapter we will show that the space $\mathcal{M} = \Sigma/\Gamma$ parametrizes K3-surfaces with $G \times C_2$-symmetry up to equivariant equivalence. More precisely, we prove:

Theorem 5.4. *The K3-surfaces with a symplectic action of $G = C_3 \ltimes C_7$ centralized by an antisymplectic involution σ are parametrized by the space $\mathcal{M} = \Sigma/\Gamma$ of equivalence classes of sextic branch curves in $\mathbb{P}_2$. The Klein-Mukai-surface occurs as the minimal desingularization of the double cover branched along the unique singular curve in $\mathcal{M}$.*

Inside the family $\mathcal{M}$ one finds two K3-surfaces with a symplectic action of the larger group $L_2(7)$ centralized by an antisymplectic involution.

Theorem 5.5. *There are exactly two K3-surfaces with an action of the group $L_2(7)$ centralized by an antisymplectic involution. These are the Klein-Mukai-surface X_{KM} and the double cover of $\mathbb{P}_2$ branched along the curve* $\mathrm{Hess}(C_{Klein}) = \{z_0^5z_1 + z_2^5z_0 + z_1^5z_2 - 5z_0^2z_1^2z_2^2 = 0\}$.

5.1 Branch curves and Mori fibers

Let X be a K3 surface with an symplectic action of $G = C_3 \ltimes C_7$ centralized by the antisymplectic involution σ. We consider the quotient $\pi : X \to X/\sigma = Y$. Since the action of G' has precisely three fixed points in X and σ acts on this point set, we know that $\mathrm{Fix}_X(\sigma)$ is not empty. It follows that Y is a smooth rational surface with an effective action of the group G to which we apply the equivariant minimal model program. The following lemma excludes the possibility that a G-minimal model is a conic bundle. The argument resembles that in the proof of Lemma 3.2.

Lemma 5.6. *A G-minimal model of Y is a Del Pezzo surface.*

Proof. Assume the contrary and let $Y_{\text{min}} \to \mathbb{P}_1$ be a G-equivariant conic bundle. Since G has no effective action on the base, there must be a nontrivial normal subgroup acting trivially on the base. This subgroup must be G'. The action of G' on the generic fiber has two fixed points and gives rise to a positive-dimensional G'-fixed point set in Y_{min} and Y. Since the action of G' on Y is induced by a symplectic action of G' on X, this is a contradiction. □

Remark 5.7. Since G has no subgroup of index two, the above proof also shows that $Y_{\text{min}} \not\cong \mathbb{P}_1 \times \mathbb{P}_1$.

In analogy to the procedure of the previous chapter we exclude rational and elliptic ramification curves and show that π is branched along a single curve of genus greater than or equal to three.

Proposition 5.8. *The set* $\mathrm{Fix}_X(\sigma)$ *consists of a single curve* C *and* $g(C) \geq 3$.

Proof. We let $\{x_1, x_2, x_3\} = \mathrm{Fix}_X(G')$. Since G has no faithful two-dimensional representation, it has no fixed points in X an therefore acts transitively on $\{x_1, x_2, x_3\}$. It follows that the central involution σ, which fixes at at least one point x_i, fixes all three points by invariance. Now $\{x_1, x_2, x_3\} \subset \mathrm{Fix}_X(\sigma)$ implies that G' has precisely three fixed points in Y. Let C_i denote the connected component of $\mathrm{Fix}_X(\sigma)$ containing x_i. Since G acts on the set $\{C_1, C_2, C_3\}$, it follows that either $C_1 = C_2 = C_3$ or no two of them coincide.

In the later case, it follows from Theorem 1.12 that at least two curves C_1, C_2 are rational. The action of G' on a rational curves C_i has two fixed points. We therefore find at least five G'-fixed points in X contradicting $|\mathrm{Fix}_X(G')| = 3$.

It follows that all three points x_1, x_2, x_3 lie on one G-invariant connected component C of $\mathrm{Fix}_X(\sigma)$. The action of G on C is effective and it follows that C is not rational.

If $g(C) = 1$, then an effective action of G on C would force G' to act by translations on C, in particular freely, a contradiction.

If $g(C) = 2$, then C is hyperelliptic. The quotient $C \to \mathbb{P}_1$ by the hyperellitic involution is $\mathrm{Aut}(C)$-equivariant and would induce an effective action of G on $\mathbb{P}_1$, a contradiction.

It follows that $g(C) \geq 3$ and it remains to check that there are no rational ramification curves.

We let n denote the total number of rational curves in $\mathrm{Fix}_X(\sigma)$. Since G' acts freely on the complement of C in X, it follows that the number n must be a multiple of seven. Combining this observation with the bound $n \leq 9$ from Corollary 3.20 we conclude that n is either 0 or 7.

We suppose $n = 7$ and let m denote the total number of Mori contractions of a reduction $Y \to Y_{\mathrm{min}}$. The Euler characeristic formula

$$13 - g(C) = e(Y_{\mathrm{min}}) + m - n$$

with $n = 7$, $g(C) \geq 3$ and $e(Y_{\mathrm{min}}) \geq 3$ implies $m \leq 14$.

Let us first check that no Mori fiber E coincides with a rational branch curve B. If this was the case, then all seven rational branch curves coincide with Mori fibers. Rational branch curves have self-intersection -4 by Corollary 3.16. Before they may by contracted, they need to be transformed into (-1)-curves by earlier reduction steps. The remaining seven or less Mori contraction are not sufficient to achieve this transformation. It follows that each rational branch curve is mapped to a curve in Y_{min} and not to a point.

We now first consider the case $m = 14$. The Euler characteristic formula implies $Y_{\mathrm{min}} \cong \mathbb{P}_2$ and $g(C) = 3$. Using our study of Mori fibers and branch curves in Section 3.2, in particular Remark 3.13 and Proposition 3.14, we see that no configuration of 14 Mori fibers is such that the images in $Y_{\mathrm{min}} \cong \mathbb{P}_2$ of any two rational branch curves have nonempty intersection. It follows that $m \leq 13$.

Let $R_1, \ldots, R_7 \subset Y$ denote the rational branch curves. Each curve R_i has self-intersection -4 and therefore has nontrivial intersection with at least one Mori fiber. Let E_1 be a Mori fiber meeting R_1, let $H \cong C_3$ be the stabilizer of R_1 in G and let I be the stabilizer of E_1 in G. Since $m \leq 13$ the group I is nontrivial. If I does not stabilize R_1, then E_1 meets the branch locus in at least three points. This is contrary to Proposition 3.14. It follows that $I = H$. If E_1 meets any other rational branch curve R_2, then it meets all curves in the H-orbit through R_2. Since H acts freely on the set $\{R_2, \ldots, R_7\}$, it follows that E_1 meets three more branch curves. This is again contradictory to Proposition 3.14.

Since $m \leq 13$ it follows that each rational branch curve meets exactly one Mori fiber. Their intersection can be one of the following three types:

1. $E_i \cap R_i = \{p_1, p_2\}$ or
2. $E_i \cap R_i = \{p\}$ and $(E_i, R_i)_p = 2$ or
3. $E_i \cap R_i = \{p\}$ and $(E_i, R_i)_p = 1$.

In all three cases the contraction of E_i alone does not transform the curve R_i into a curve on a Del Pezzo surface. So further reduction steps are needed and require the existence of Mori fibers F_i disjoint from $\bigcup R_i$. Each F_i is a (-2)-curve meeting $\bigcup E_i$ transversally in one point and the total number of Mori fibers exceeds our bound 13.

This contradiction yields $n = 0$ and the proof of the proposition is completed. □

5.2 Classification of the quotient surface Y

We now turn to a classification of the quotient surface Y.

Proposition 5.9. *The surface Y is either G-minimal or the blow up of $\mathbb{P}_2$ in seven singularities of an irreducible G-invariant sextic..*

Proof. Since $n = 0$, the Euler characteristic formula yields $m \leq 7$. The fact that G acts on the set of Mori fibers implies that $m \in \{0, 3, 6, 7\}$. If $m \in \{3, 6\}$, then G' stabilizes every Mori fiber, and consequently it has more then three fixed points, a contradiction. Thus we must only consider the case $m = 7$.

In this case the set of Mori fibers is a G-orbit and it follows that every Mori fiber has self-intersection -1 and therefore has nonempty intersection with $\pi(C)$ by Remark 3.13.

As before, the Euler characteristic formula implies that $g(C) = 3$ and $Y_{\min} = \mathbb{P}_2$ and adjunction in X shows that $(\pi(C))^2 = 8$ in Y. The fact that $\pi(C)$ has nonempty intersection with seven different Mori fibers implies that its image D in $Y_{\min}$ has self-intersection either $15 = 8 + 7$ or $36 = 8 + 4 \cdot 7$. Since the first is impossible it follows that $E \cdot \pi(C) = 2$ for all Mori fibers E and the G-invariant irreducible sextic D has seven singular points corresponding to the images of E in $\mathbb{P}_2$. □

Corollary 5.10. *If Y is not G-minimal, then X is the minimal desingularization of a double cover of $\mathbb{P}_2$ branched along an irreducible G-invariant sextic with seven singular points.*

We conclude this section with a classification of possible G-minimal models of Y.

Proposition 5.11. *The surface $Y_{\min}$ is either a Del Pezzo surface of degree two or $\mathbb{P}_2$.*

Proof. The case $Y_{\min} = \mathbb{P}_1 \times \mathbb{P}_1$ is excluded by Example 3.7 and also by Remark 5.7.

Thus $Y_{\min} = Y_d$ is a Del Pezzo surface of degree $d = 1, \ldots, 9$ which is a blowup of $\mathbb{P}_2$ in $9 - d$ points.

If $Y_{\min} = Y_1$ the anticanonical map has exactly one base point. This point has to be G-fixed and since G has no faithful two-dimensional representations, this case does not occur.

It remains to eliminate $d = 8, \ldots, 3$. In these cases the sets $\mathcal{S}$ of (-1)-curves consist of 1, 2, 6, 10, 16 or 27 elements, respectively (cf. Table 3.1). The G-orbits in $\mathcal{S}$ consist of 1, 3, 7 or 21 curves and there must be orbits of length three or one. If G stablizes a curve in $\mathcal{S}$, then its contraction gives rise to a two-dimensional representation of G which does not exist. If G has an orbit consisting of three curves, then G' stabilizes each of the curves in this orbit. Thus G' has at least six fixed points in $Y_{\min}$ and in Y. This contradicts the fact that $|\mathrm{Fix}_Y(G')| = 3$. □

5.3 Fine classification - Computation of invariants

We have reduced the classification of K3-surfaces with $G \times C_2$-symmetry to the study of equivariant double covers of rational surfaces Y branched along a single invariant curve of genus $g \geq 3$. Here Y is either $\mathbb{P}_2$, the blow-up of $\mathbb{P}_2$ in seven singular points of an irreducible G-invariant sextic, or a Del Pezzo surface of degree two.

5.3.1 The case $Y = Y_{\min} = \mathbb{P}_2$

An effective action of G on $\mathbb{P}_2$ is given by an injective homomorphisms $G \to \mathrm{PSL}_3(\mathbb{C})$. There are two central degree three extension of G, the trivial extension and $C_9 \ltimes C_7$. A study of their three-dimensional representation reveals that in both cases the action of G on $\mathbb{P}_2$ is given by an irreducible representation $G \hookrightarrow \mathrm{SL}_3(\mathbb{C})$. There are two isomorphism classes of irreducible 3-dimensional representations. Since these differ by a group automorphism and the corresponding actions on $\mathbb{P}_2$ are therefore equivalent, we may assume that in appropriately chosen coordinates a generator of G' acts by

$$[z_0 : z_1 : z_2] \mapsto [\lambda z_0, \lambda^2, z_1, \lambda^4 z_2], \tag{5.1}$$

where $\lambda = \exp\frac{2\pi i}{7}$ and a generator of C_3 acts by the cyclic permutation τ which is defined by

$$[z_0 : z_1 : z_2] \mapsto [z_2 : z_0 : z_1]. \tag{5.2}$$

A homogeneous polynomial defining an invariant curve must be a G-semi-invariant with G' acting with eigenvalue one. The G'-invariant monomials of degree six are

$$\mathbb{C}[z_0, z_1, z_2]^{G'}_{(6)} = \mathrm{Span}\{z_0^2 z_1^2 z_2^2, z_0^5 z_1, z_2^5 z_0, z_1^5 z_2\}\,.$$

Letting $P_1 = z_0^2 z_1^2 z_2^2$ and $P_2 = z_0^5 z_1 + z_2^5 z_0 + z_1^5 z_2$, it follows that

$$\mathbb{C}[z_0, z_1, z_2]^{G}_{(6)} = \mathrm{Span}\{P_1, P_2\} =: V\,.$$

There are two G-semi-invariants which are not invariant, namely $z_0^5 z_1 + \zeta z_2^5 z_0 + \zeta^2 z_1^5 z_2$ for $\zeta^3 = 1$ but $\zeta \neq 1$. By direct computation one checks that the curves defined by these polynomials are smooth and that in both cases all τ-fixed points in $\mathbb{P}_2$ lie on them. Thus, τ has only three fixed points on the K3-surface X obtained as a double cover and therefore does not act symplectically (cf. Table 1.1). Consequently, G does not lift to an action by symplectic transformations on the K3-surfaces defined by these two curves. Hence it is enough to consider ramified covers $X \to Y = \mathbb{P}_2$, where the branch curves are defined by invariant polynomials $f \in V$.

We wish to determine which polynomials $P_{\alpha,\beta} = \alpha P_1 + \beta P_2$ define singular curves. Since $\mathrm{Fix}(\tau) = \{[1 : \zeta : \zeta^2] \,|\, \zeta^3 = 1\}$, the curves which contain τ-fixed points are defined by condition $\alpha + 3\zeta\beta = 0$. Let $C_{P_1} = \{P_1 = 0\}$ and let C_ζ be the curve defined by $P_{\alpha,\beta}$ for $\alpha + 3\zeta\beta = 0$. A direct computation shows that C_ζ is singular at the point $[1 : \zeta : \zeta^2]$. We let Σ_{reg} be the complement of this set of four curves, $\Sigma_{\mathrm{reg}} = \mathbb{P}(V) \backslash \{C_{P_1}; C_\zeta \,|\, \zeta^3 = 1\}$.

Lemma 5.12. *A curve $C \in \mathbb{P}(V)$ is smooth if and only if $C \in \Sigma_{\mathrm{reg}}$.*

Proof. Let $C \in \Sigma_{\mathrm{reg}}$. Since τ has no fixed points in C by definition and every subgroup of order three in G is conjugate to $\langle \tau \rangle$, it follows that any G-orbit $G.p$ through a point $p \in C$ has length three or 21.

The only subgroup of order seven in G is the commutator group G'. So the G-orbits of length three are the orbits of the G'-fixed points $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$. One checks by direct computation that every $C \in \Sigma_{\mathrm{reg}}$ is smooth at these three points.

An irreducible curve of degree six has at most ten singular points by the genus formula. Suppose that C is singular at some point q. Then it is singular at each of the 21 points in $G.q$ and C must be reducible. Considering the G-action on the space of irreducible components of C yields a contradiction and it follows that C is smooth. □

For any curve $C \in \Sigma_{\mathrm{reg}}$ the double cover of $\mathbb{P}_2$ branched along C is a K3-surface X_C with an action of a degree two central extension of G. By the following lemma, this action is always of the desired type.

Lemma 5.13. *For every $C \in \Sigma_{\mathrm{reg}}$ the K3-surface X_C carries an action of the group $G \times \langle \sigma \rangle$. The group G acts by symplectic transformations on X_C and σ denotes the covering involution.*

Proof. It follows from the group structure of G that the central degree two extension of G acting on X_C splits as $G \times C_2$. The factor C_2 is by construction generated by the covering involution σ. It remains to check that G acts symplectically. As the commutator subgroup G' acts symplectically it is sufficient to check whether τ lifts to a symplectic automorphism. Consider the τ-fixed point $p = [1 : 1 : 1]$ and check that the linearization of τ at p is in $\mathrm{SL}(2, \mathbb{C})$. Since p is not contained in C, it follows that the linearization of τ at a corresponding fixed point in X_C is also in $\mathrm{SL}(2, \mathbb{C})$. Consequently, the group G acts by symplectic transformations on X_C. □

5.3.2 Equivariant equivalence

We wish to describe the space of K3-surfaces with $G \times C_2$-symmetry modulo equivariant equivalence. For this, we study the family of K3-surfaces parametrized by the family of branch curves Σ_{reg}. Consider the cyclic group Γ of order three in $\mathrm{PGL}(3, \mathbb{C})$ generated by

$$[z_0 : z_1 : z_2] \mapsto [z_0 : \zeta z_1 : \zeta^2 z_2]$$

for $\zeta = \exp(\frac{2\pi i}{3})$. The group Γ acts on Σ_{reg} and by the following proposition the induced equivalence relation is precisely equivariant equivalence formulated in Definition 4.3.

Proposition 5.14. *Two K3-surfaces X_{C_1} and X_{C_2} for $C_1, C_2 \in \Sigma_{\mathrm{reg}}$ are equivariantly equivalent if and only if $C_1 = \gamma C_2$ for some $\gamma \in \Gamma$, i.e., the quotient $\Sigma_{\mathrm{reg}}/\Gamma$ parametrizes equivariant equivalence classes of K3-surfaces X_C for $C \in \Sigma_{\mathrm{reg}}$.*

Proof. If two K3-surfaces X_{C_1} and X_{C_2} for $C_1, C_2 \in \Sigma_{\text{reg}}$ are equivariantly equivalent, then the isomorphism $X_{C_1} \to X_{C_2}$ induces an automorphism of $\mathbb{P}_2$ mapping C_1 to C_2.

Let $C \in \Sigma_{\text{reg}}$ and for $T \in \mathrm{SL}_3(\mathbb{C})$ assume that $T(C) \in \Sigma_{\text{reg}}$. We consider the group span S of TGT^{-1} and G. By Lemma 5.13, the group G acts by symplectic transformations on X_C and $X_{T(C)}$. We argue precisely as in the proof of this lemma to see that TGT^{-1} also acts symplectically on the K3-surface $X_{T(C)}$. It follows that S is acting as a group of symplectic transformations on this K3-surface.

If $S = G$, then T normalizes G. The normalizer N of G in $\mathrm{PGL}_3(\mathbb{C})$ is the product $\Gamma \times G$ and it follows that gT is contained in Γ for some $g \in G$ and $T(C) = gT(C) = \gamma C$.

Note that $L_2(7)$ is the only group in Mukai's list which contains G. Therefore, S is a subgroup of $L_2(7)$. The group G is a maximal subgroup of $L_2(7)$ and if $S \neq G$, then it follows that $S = L_2(7)$. Any two subgroups of order 21 in $L_2(7)$ are conjugate. This implies the existence of $s \in S = L_2(7)$ such that $sTGT^{-1}s^{-1} = G$. Now $sT \in N = \Gamma \times G$ can be written as $sT = \gamma g$ for $(\gamma, g) \in \Gamma \times G$. By assumption, s stabilizes $T(C)$ and $T(C) = sT(C) = \gamma g(C) = \gamma C$. This completes the proof of the proposition. □

5.3.3 The case $Y \neq Y_{\text{min}}$

Let us now consider the three singular irreducible curves in our family $\mathbb{P}(V)$. They are identified by the action of Γ. Using Corollary 5.10 we see that if $Y = X/\sigma$ is not G-minimal, then, up to equivariant equivalence, the K3-surface X is the minimal desingularization of the double cover of $\mathbb{P}_2$ branched along $C_{\zeta=1} = C_{\text{sing}}$ and Y is the blow-up of $\mathbb{P}_2$ in the seven singular points of C_{sing}. These points are the G'-orbit of $[1:1:1]$. In the following propostion we prove that these are in general position and therefore Y is a Del Pezzo surface.

Proposition 5.15. *If Y is not minimal, then it is the Del Pezzo surface of degree two which arises by blowing up the seven singular points $p_1, \dots, p_7$ on the curve C_{sing} in $\mathbb{P}_2$. The corresponding map $Y \to \mathbb{P}_2$ is G-equivariant and therefore a Mori reduction of Y.*

Proof. We show that the points $\{p_1, \dots, p_7\} = G'.[1:1:1]$ are in general position, i.e., no three lie on one line and no six lie on one conic. It follows from direct computation that no three points in $G'.[1:1:1]$ lie on one line. If $p_1, \dots p_6$ lie on a conic Q, then $g.p_1, \dots, g.p_6$ lie on $g.Q$ for every $g \in G$. Since $\{p_1, \dots, p_7\}$ is a G-invariant set, the conics Q and $g.Q$ intersect in at least five points and therefore coincide. It follows that Q is an invariant conic meeting C_{sing} at its seven singularities and $(Q, C_{\text{sing}}) \geq 14$ implies $Q \subset C_{\text{sing}}$, a contradiction. □

5.4 Klein's quartic and the Klein-Mukai surface

In this section we show that the Del Pezzo surface discussed in Proposition 5.15 above can be realized as the double cover of $\mathbb{P}_2$ branched along Klein's quartic curve.

Proposition 5.16. *A Del Pezzo surface of degree two with an action of G is equivariantly isomorphic to the double cover Y_{Klein} of $\mathbb{P}_2$ branched along Klein's quartic curve.*

Proof. Recall that the anticanonical map of a Del Pezzo surface Y of degree two defines a 2:1 map to $\mathbb{P}_2$. This map is branched along a smooth curve of degree four and equivariant with respect to

$\mathrm{Aut}(Y)$. We obtain an action of G on $\mathbb{P}_2$ stabilizing a smooth quartic. As before, we may choose coordinates such that G is acting as in equations (5.1) and (5.2). Then

$$\mathbb{C}[z_0 : z_1 : z_2]^{G'}_{(4)} = \mathrm{Span}\{z_0^3 z_2, z_1^3 z_0, z_2^3 z_1\}.$$

is a direct sum of G-eigenspaces. The eigenspace of the eigenvalue ζ is spanned by the polynomial $Q_\zeta := z_0^3 z_2 + \zeta z_2^3 z_1 + \zeta^2 z_1^3 z_0$ with ζ being a third root of unity.

In order to take into account equivariant equivalence we consider the cyclic group $\Gamma \subset \mathrm{SL}_3(\mathbb{C})$ which is generated by the transformation γ, $[z_0 : z_1 : z_2] \mapsto [z_0 : \zeta z_1 : \zeta^2 z_2]$. The induced action on $\mathbb{C}[z_0 : z_1 : z_2]^{G'}_{(4)}$ is transitive on the G-eigenspaces spanned by the Q_ζ. Consequently, up to equivariant equivalence, we may assume that $Y \to \mathbb{P}_2$ is branched along Klein's curve C_{Klein} which is defined by Q_1. □

Corollary 5.17. *A Del Pezzo surface of degree two with an action of G is never G-minimal. Its Mori reduction $Y_{\mathrm{Klein}} \to \mathbb{P}_2$ is precisely the map discussed in Proposition 5.15.*

We summarize our observartions in the following proposition.

Proposition 5.18. *If X is a K3-surface with a symplectic G-action centralized by an antisymplectic involution σ, then $Y_{min} = \mathbb{P}_2$. In all but one case $X/\sigma = Y = Y_{min}$. In the exceptional case $Y = Y_{\mathrm{Klein}}$, the Mori reduction $Y \to Y_{min}$ is the contraction of seven (-1)-curves to the singular points of C_{sing} and the branch set B of $X \to Y$ is the proper transform of C_{Klein} in Y.*

Proof. It remains to prove that B is the proper transform of C_{Klein} in Y. Suppose that the branch curve of $X \to Y$ is some other curve $\widetilde{B}$ linearly equivalent to $-2K_Y$. Let $I := \widetilde{B} \cap B$ and note that $|I| \leq B \cdot \widetilde{B} = 4K_Y^2 = 8$. Since G has no fixed points in B, it follows that $|I| = 3$ and that I is a G-orbit. Thus the intersection multiplicities at the three points in $\widetilde{B} \cap B$ are the same. Since 3 does not divide 8, this is a contradiction. □

In order to complete the proof of Theorem 5.4 it remains to show that the action of G on Y_{Klein} lifts to a group of symplectic transformation on the K3-surface $X = X_{KM}$ defined as a double cover of Y_{Klein} branched along the proper transform of C_{sing}.

Since G stabilizes C_{Klein} and does not admit nontrivial central extensions of degree two, it lifts to a subgroup of $\mathrm{Aut}(Y_{\mathrm{Klein}})$ and subsequently to a subgroup of $\mathrm{Aut}(X)$.

The covering involution $Y_{\mathrm{Klein}} \to \mathbb{P}_2$, lifts to a holomorphic transformation of X where we also find the involution defining $X \to Y_{\mathrm{Klein}}$. These two transformations generate a group F of order four. The elements of F all have a positive-dimensional fixed point set. It follows that F acts solely by nonsymplectic transformations and is therefore isomorphic to C_4. The full preimage of G in $\mathrm{Aut}(X)$ splits as $G \times C_4$.

Since the commutator group G' automatically acts by symplectic transformations, we must only check that the lift of the cyclic permutation τ, $[z_0 : z_1 : z_2] \mapsto [z_2 : z_0 : z_1]$, acts symplectically. As above, this follows from a linearization argument at a τ-fixed point not in C_{Klein}.

In conclusion, up to equivalence there is a unique action of G by symplectic transformations on the K3-surface X_{KM}. It is centralized by a cyclic group of order four which acts faithfully on the symplectic form.

The Klein-Mukai-surface is the only surface with $G \times C_2$-symmetry for which $Y \ncong \mathbb{P}_2$. As in the introduction of this chapter, we define Σ as the complement of C_{P_1} in $\mathbb{P}(V)$. Then $\Sigma = \Sigma_{\mathrm{reg}} \cup \{C_\zeta \mid \zeta^3 = 1\}$. Using this notation the space

$$\mathcal{M} = \Sigma/\Gamma$$

parametrizes the space of K3-surfaces with $G \times C_2$-symmetry up to equivariant equivalence. This completes the proof of Theorem 5.4.

5.5 The group $L_2(7)$ centralized by an antisymplectic involution

We consider the simple group of order 168. This group is $\mathrm{PSL}(2, \mathbb{F}_7)$ and usually denoted by $L_2(7)$. It contains our group $G = C_3 \ltimes C_7$ as a subgroup. Since $L_2(7)$ is a simple group, if it acts on a K3-surface, it automatically acts by symplectic transformations.

We wish to prove Theorem 5.5 stating that there are exactly two K3-surfaces with an action of the group $L_2(7)$ centralized by an antisymplectic involution. These are the Klein-Mukai-surface X_{KM} and the double cover of $\mathbb{P}_2$ branched along the curve $\mathrm{Hess}(C_{\mathrm{Klein}}) = \{z_0^5 z_1 + z_2^5 z_0 + z_1^5 z_2 - 5z_0^2 z_1^2 z_2^2 = 0\}$.

We have to check which elements of $\mathcal{M}$ have the symmetry of the larger group. The Klein-Mukai-surface is known to have $L_2(7) \times C_4$-symmetry (cf. Example 1.15). If $X \neq X_{\mathrm{KM}}$ has $L_2(7)$-symmetry, then it follows from the considerations of the previous sections that X is an $L_2(7)$-equivariant double cover of $\mathbb{P}_2$ branched along a smooth $L_2(7)$-invariant sextic curve. I.e., it remains to identify the surfaces with $L_2(7)$-symmetry in the family parametrized by $\Sigma_{\mathrm{reg}}/\Gamma$.

Lemma 5.19. *The action of $L_2(7)$ on $\mathbb{P}_2$ is necessarily given by a three-dimensional represention.*

Proof. The lemma follows from the fact that the group $L_2(7)$ does not admit nontrivial degree three central extensions. This can be derived from the cohomology group $H^2(L_2(7), \mathbb{C}^*) \cong C_2$ known as the Schur Multiplier. □

There are two isomorphism classes of three-dimensional representations and these differ by an outer automorphism. We may therefore consider the particular representation given in Example 1.15. One checks that the curve $\mathrm{Hess}(C_{\mathrm{Klein}})$ is $L_2(7)$-invariant. The maximal possible isotropy group is C_7 and each $L_2(7)$-orbit in $\mathrm{Hess}(C_{\mathrm{Klein}})$ consists of at least 21 elements. If there was another $L_2(7)$-invariant curve C in Σ_{reg}, then the invariant set $C \cap \mathrm{Hess}(C_{\mathrm{Klein}})$ consists of at most 36 points. This is a contradiction and it follows that $\mathrm{Hess}(C_{\mathrm{Klein}})$ is the only $L_2(7)$-invariant curve in Σ_{reg}.

It remains to check that $L_2(7)$ lifts to a subgroup of $\mathrm{Aut}(X_{\mathrm{Hess}(C_{\mathrm{Klein}})})$: On $X_{\mathrm{Hess}(C_{\mathrm{Klein}})}$ we find an action of a central degree two extension E of $L_2(7)$. Since $E \neq E_{\mathrm{symp}}$ and $L_2(7)$ is simple, the subgroup of symplectic transformations inside E must be isomorphic to $L_2(7)$.

It follows that X_{KM} and the double cover of $\mathbb{P}_2$ branched along $\mathrm{Hess}(C_{\mathrm{Klein}})$ are the only examples of K3-surfaces with $L_2(7) \times C_2$ symmetry. This completes the proof of Theorem 5.5.

Remark 5.20. If we consider the quotient Y_{Klein} of X_{KM} by the antisymplectic involution $\sigma \in C_4$, this surface was seen not to be minimal with respect to the action of $C_3 \ltimes C_7$. It is however $L_2(7)$-minimal as we cannot find a equivariant contraction morphism blowing down an orbit of disjoint (-1)-curves in Y_{Klein} . Such an orbit would have to consists of seven Mori fibers. The only subgroup of index seven is S_4. A Mori fiber of self-intersection (-1) does however not admit an action of the group S_4 (cf. Proof of Theorem 3.25).

6 The simple group of order 168

In this chapter we consider finite groups containing $L_2(7)$, the simple group of order 168, and their actions on K3-surfaces. Based on our considerations about $L_2(7) \times C_2$-actions on K3-surfaces in Section 5.5 we derive a classification result (Theorem 6.1). This gives a refinement of a lattice-theoretic result due to Oguiso and Zhang [OZ02]. The main part of this chapter is dedicated to proving the non-existence of K3-surfaces with an action of the group $L_2(7) \times C_3$ (Theorem 6.3) using equivariant Mori reduction.

6.1 Finite groups containing $L_2(7)$

If H is a finite group acting on a K3-surface and $L_2(7) \subsetneq H$, then it follows from Mukai's theorem and the fact that $L_2(7)$ is simple, that H fits into the short exact sequence

$$1 \to L_2(7) = H_{\mathrm{symp}} \to H \to C_m \to 1$$

for some $m \in \mathbb{N}$. As it is noted by Oguiso and Zhang, Claim 2.1 in [OZ02], it follows from Proposition 3.4 in [Muk88] that $m \in \{1, 2, 3, 4, 6\}$.

The action of H on $L_2(7)$ by conjugation defines a homomorphism $H \to \mathrm{Aut}(L_2(7))$. Factorizing by the group of inner automorphism of $L_2(7)$ we obtain a homomorphism

$$C_m \cong H/L_2(7) \to \mathrm{Out}(L_2(7)) \cong C_2.$$

If H is not the nontrivial semidirect product $L_2(7) \rtimes C_2$, this homomorphism has a nontrivial kernel. In particular, we find a cyclic group $C_k < C_m$ centralizing $L_2(7)$. If k is even, we may apply our results on K3-surfaces with $L_2(7) \times C_2$-symmetry from the previous chapter.

If $m = 3, 6$, then $k = 3$ or $k = 6$. These cases may be excluded as is shown in [OZ02], Added in proof, Proposition 1. An independent proof of this fact, i.e., the non-existence of K3-surfaces with $L_2(7) \times C_3$ symmetry, using equivariant Mori theory, in particular the classification of $L_2(7)$-minimal models, is given below (Theorem 6.3).

We summarize our observations about K3-surfaces with $L_2(7)$-symmetry in the following theorem, which improves the classification result due to Oguiso and Zhang.

Theorem 6.1. *Let H be finite group acting on a K3-surface X with $L_2(7) \subsetneq H$. Then*

- $|H/L_2(7)| \in \{2, 4\}$.
- *If* $|H/L_2(7)| = 4$, *then* $H = L_2(7) \times C_4$ *and* $X \cong X_{\mathrm{KM}}$.
- *If* $|H/L_2(7)| = 2$ *and* $H = L_2(7) \times C_2$, *then either* $X \cong X_{\mathrm{KM}}$ *or* $X \cong X_{\mathrm{Hess}(C_{\mathrm{Klein}})}$

The first statement follows from the non-existence of K3-surfaces with $L_2(7) \times C_3$-symmetry (Theorem 6.3 below) and the third statement follows from Theorem 5.5. The remaining part ist covered in the following lemma (cf. Main Theorem in [OZ02]).

Lemma 6.2. *If X is a K3-surface with an action of a finite group containing the $L_2(7)$ as a subgroup of index four, then X is the Klein-Mukai-surface.*

Proof. We let X be a K3-surface and H be a finite subgroup of $\mathrm{Aut}(X)$ with $L_2(7) < H$ and $|H/L_2(7)| = 4$.

Since $L_2(7)$ is simple and a maximal group of symplectic transformations, it coincides with the group of symplectic transformations in H. In particular, $H/L_2(7) = C_4$ and a group $\langle\sigma\rangle$ of order two is contained in the kernel of the homomorphism $H \to \mathrm{Aut}(L_2(7))$. It follows that we are in the setting of Theorem 5.5 where $\Lambda := H/\langle\sigma\rangle$ acts on $Y = X/\sigma$. If $X \neq X_{KM}$, then $Y = \mathbb{P}_2$. This possibility needs to be eliminated.

Let τ be any element of Λ which is not in $L_2(7)$ and let $\Gamma = C_3 \ltimes C_7 < L_2(7)$. Since any two subgroups of order 21 in $L_2(7)$ are conjugate by an element of $L_2(7)$, it follows that there exists $h \in L_2(7)$ with $(h\tau)\Gamma(h\tau)^{-1} = \Gamma$. Thus, the normalizer $N(\Gamma)$ of Γ in Λ is a group of order 42 which also normalizes the commutator subgroup Γ' and therefore stabilizes its set F of fixed points.

Using coordinates $[z_0 : z_1 : z_2]$ of $\mathbb{P}_2$ as in Theorem 5.5 one checks by direct computation that the only transformations in $\mathrm{Stab}(F)$ which stabilize the branch curve $\mathrm{Hess}(C_{\mathrm{Klein}})$ are those in Γ itself. This contradiction shows that $Y \neq \mathbb{P}_2$ and therefore $X = X_{KM}$. □

6.2 Non-existence of K3-surfaces with an action of $L_2(7) \times C_3$

The method of equivariant Mori reduction can be applied to obtain both classification and non-existence results. In the following, we exemplify a general approach to prove non-existence of K3-surfaces with specified symmetry by considering the group $L_2(7) \times C_3$ and give an independent proof of the following observation of Oguiso and Zhang [OZ02]:

Theorem 6.3. *There does not exist a K3-surface with an action of $L_2(7) \times C_3$.*

The remainder of this chapter is dedicated to the proof of this theorem.

6.2.1 Global structure

Let $G \cong L_2(7)$, let $D \cong C_3$, and assume there exists a K3-surface X with a holomorphic action of $G \times D$. Since G is a simple group and a maximal group of symplectic transformations on a

K3-surface, it follows that G acts symplectically whereas the action of D is nonsymplectic. We obtain the following commuting diagram.

$$\begin{tikzcd} X \arrow[d, "\pi"] & \hat{X} \arrow[l, "b_X"'] \arrow[d, "\hat{\pi}"] \\ X/D = Y & \hat{Y} \arrow[l, "b_Y"'] \arrow[d, "M_{\mathrm{red}}"] \\ & \hat{Y}_{\min} = Z \end{tikzcd}$$

Here b_X is the blow-up of the isolated D-fixed points in X. The singularities of X/D correspond to isolated D-fixed points. Since the linearization of the D-action at an isolated fixed point is locally of the form $(z, w) \mapsto (\chi z, \chi w)$ for some nontrivial character $\chi : D \to \mathbb{C}^*$, each singularity of X/D is resolved by a single blow-up. We let b_Y denote the simultanious blow-up of all singularities of Y. We fix a G-Mori reduction $M_{\mathrm{red}} : \hat{Y} \to \hat{Y}_{\min} = Z$. All maps in the diagram are G-equivariant. By Theorem 1.8, the surface $\hat{Y}$ is rational. As conic bundles do not admit an action of G (cf. Lemma 5.6), we know that $\hat{Y}_{\min}$ is a Del Pezzo surface . The following lemma specifies Z.

Lemma 6.4. *The Del Pezzo surface Z is either $\mathbb{P}_2$ or a surface obtained from $\mathbb{P}_2$ by blowing up 7 points in general position. In the later case, Z is a G-equivariant double cover of $\mathbb{P}_2$ branched along Klein's quartic curve. The action of G on $\mathbb{P}_2$ is given by a three-dimensional representation.*

Proof. The first part of the lemma follows from our observations in Example 3.8, the last part has been discussed in Lemma 5.19. If Z is a Del Pezzo surface of degree two, then the anticanonical map realizes it as an equivariant double cover of $\mathbb{P}_2$ branched along a smooth quartic curve C. We choose coordinates on $\mathbb{P}_2$ such that the action of G is given by the representation ρ of Example 1.15 (or its dual represenation ρ^*) and have already seen that Klein's quartic curve

$$C_{\mathrm{Klein}} = \{x_1 x_2^3 + x_2 x_3^3 + x_3 x_1^3 = 0\} \subset \mathbb{P}_2$$

is G-invariant. If $C \neq C_{\mathrm{Klein}}$, then $C \cap C_{\mathrm{Klein}}$ is a G-invariant subset of $\mathbb{P}_2$. Since the maximal cyclic subgroup of G is of order seven, it follows that a G-orbit $G.p$ for a point $p \in C \cap C_{\mathrm{Klein}}$ consists of at least 24 elements. Since $C \cap C_{\mathrm{Klein}}$ however consists of at most 16 points, this is a contradiction. Therefore, $C = C_{\mathrm{Klein}}$ and the lemma follows. □

D-fixed points

The map π is in general ramified both at points and along curves. Let x be an isolated D-fixed point in X. As was noted above, the isotropy representation of the nonsymplectic D-action at x in local coordinates (z, w) is given by $(z, w) \mapsto (\chi z, \chi w)$ for some nontrivial character $\chi : D \to \mathbb{C}^*$. The action of D on the rational curve $\hat{E}$ obtained by blowing up x is trivial and therefore $\hat{E}$ is contained in the ramification set $\mathrm{Fix}_{\hat{X}}(D)$. Let $\{\hat{E}_i\}$ denote the set of (-1)-curves in $\hat{X}$ obtained from blowing up isolated D-fixed points in X and define $E_i = \hat{\pi}(\hat{E}_i)$.

If C is a curve of D-fixed points in X, it follows that $\hat{\pi}$ is ramified along $b_X^{-1}(C)$. Let $\{\hat{F}_j\}$ denote the set of all ramification curves of type $b_X^{-1}(C)$ and define $F_j = \hat{\pi}(\hat{F}_j)$. The map $\hat{\pi}$ is a D-quotient and ramified along curves

$$\mathrm{Fix}_{\hat{X}}(D) = \bigcup \hat{E}_i \cup \bigcup \hat{F}_j.$$

6.2.2 Mori contractions and C_7-fixed points

Many aspects of the group theory of G can be well understood in term of its generators α, β, γ of order 7,3,2, respectively. Since the action of G on $\mathbb{P}_2$ is given by a three-dimensional irreducible representation, the action of G on Z is given explicitly in terms of α, β, γ. We let $S = \langle \alpha \rangle \cong C_7 < G$ be a cyclic subgroup of order seven in G.

The symplectic action of a cyclic group of order seven on an K3-surface has exactly three fixed points. Since $p_1 = [1:0:0]$, $p_2 = [0:1:0]$ and $p_3 = [0:0:1]$ all lie on $C_{\mathrm{Klein}} \subset \mathbb{P}_2$, the action of S on Z has exactly three fixed points.

Let $\mathrm{Fix}_{\hat{Y}}(S) =: \{y_1, \dots, y_k\}$ and let $\mathrm{Fix}_{\hat{X}}(S) =: \{x_1, \dots, x_l\}$. Since blowing-up an S-fixed point in X replaces the fixed point by a rational curve with two S-fixed points in $\hat{X}$, we find $3 \leq k \leq l \leq 6$.

Lemma 6.5. *The fixed points of S in $\hat{X}$ are contained in the D-ramification set, i.e.,* $\mathrm{Fix}_{\hat{X}}(S) \subset \mathrm{Fix}_{\hat{X}}(D)$.

Proof. Since D centralizes S, the action of D stabilizes the S-fixed point set. We first show that $\mathrm{Fix}_X(S) \subset \mathrm{Fix}_X(D)$. Assume the contrary and let $\mathrm{Fix}_X(S) = \{s_1, s_2, s_3\}$ be a D-orbit and $\pi(s_i) = y$. Then y is a smooth point and fixed by the action of S on Y. There exists a neighbourhood of y in Y which is biholomorphic to a neighbourhood of $b_Y^{-1}(y) = \tilde{y}$ in $\hat{Y}$. By construction, $\tilde{y} \in \mathrm{Fix}_{\hat{Y}}(S)$. Since $\mathrm{Fix}_{\hat{Y}}(S)$ consists of at least three points, we let $\tilde{\tilde{y}} \neq \tilde{y}$ be an additional S-fixed point on $\hat{Y}$. The fibre $\pi^{-1}(b_Y(\tilde{\tilde{y}}))$ consists of one or three points and is disjoint from $\{s_1, s_2, s_3\}$. Since the point $\tilde{\tilde{y}}$ is a fixed point of S, we know that $S \cong C_7$ acts on the fiber $\pi^{-1}(b_Y(\tilde{\tilde{y}}))$ and is seen to fix it pointwise. This is contrary to the fact that $\mathrm{Fix}_X(S) = \{s_1, s_2, s_3\}$. It follows that $\mathrm{Fix}_X(S) \subset \mathrm{Fix}_X(D)$.

It remains to show the corresponding inclusion on $\hat{X}$. If the points s_i do not coincide with isolated D-fixed points, the statement follows since b_X is equivariant and biholomorphic outside the isolated D-fixed points. If s_i is an isolated D-fixed point, we have seen above that the action of D on the blow-up of s_i is trivial. In particular, $\mathrm{Fix}_{\hat{X}}(S) \subset \mathrm{Fix}_{\hat{X}}(D)$. □

Excluding the case $|\mathrm{Fix}_{\hat{Y}}(S)| = 3$

Lemma 6.6. *If $|\mathrm{Fix}_{\hat{Y}}(S)| = 3$, then $\mathrm{Fix}_{\hat{Y}}(S) \cap \bigcup E_i = \emptyset$.*

Proof. Fixed points of S on a curve $\hat{E}_i$ always come in pairs: If the curve $\hat{E}_i$ contains a fixed point of S, then the isotropy representation of S at the fixed point $b_X(\hat{E}_i)$ in X defines an action of the cyclic group S on the rational curve $\hat{E}_i$ with exactly two fixed points. If $|\mathrm{Fix}_{\hat{Y}}(S)| = |\mathrm{Fix}_{\hat{X}}(S)| = 3$ and $\mathrm{Fix}_{\hat{Y}}(S) \cap \bigcup E_i \neq \emptyset$, then two of the S-fixed point lie on the same curve $\hat{E}_i$ and $|\mathrm{Fix}_X(S)| \leq 2$, a contradiction. □

Lemma 6.7. *If $|\mathrm{Fix}_{\hat{Y}}(S)| = 3$, then the set $\mathrm{Fix}_{\hat{Y}}(S)$ has empty intersection with the exceptional locus of the full equivariant Mori reduction $M_{\mathrm{red}} : \hat{Y} \to Z$.*

Proof. Let C be any exceptional curve of the Mori reduction and assume there is a fixed point of S on C. As the point p obtained from blowing down C has to be a fixed point of S, it follows that the curve C is S-invariant. In particular, we know that the action of S on C has exactly two fixed points. Now blowing down C reduces the number of S-fixed point by 1. This contradicts the fact that $|\mathrm{Fix}_Z(S)| = 3$. □

Lemma 6.8. *Let $|\mathrm{Fix}_{\hat{Y}}(S)| = 3$ and let $p \in \mathrm{Fix}_Z(S)$. Then there exist local coordinates (u,v) at p and a nontrivial character $\mu : S \to \mathbb{C}^*$ such that the action of S at p is locally given by either*

$$(u,v) \mapsto (\mu^3 u, \mu^{-1} v) \quad \textit{or} \quad (u,v) \mapsto (\mu u, \mu^{-3} v).$$

Proof. On the K3-surface X the action of S at a fixed point is in local coordinates (z,w) given by $(z,w) \mapsto (\mu z, \mu^{-1} w)$ for some nontrivial character $\mu : S \to \mathbb{C}^*$. Since $\mathrm{Fix}_{\hat{Y}}(S) \cap \bigcup E_i = \emptyset$, the map b_X is biholomorphic in a neighbourhood of the fixed point. Recalling that $\mathrm{Fix}_{\hat{X}}(S)$ is contained in the ramification locus of $\hat{\pi}$ (i.e., $p \in \mathrm{Fix}_{\hat{X}}(D)$) the action of D may be linearized at p. Since S and D commute, the action of D is diagonal in the chosen local coordinates (z,w). We conclude that $\hat{\pi}$ is locally of the form $(z,w) \mapsto (z^3, w)$ or (z, w^3). The action of S at a fixed point in $\hat{Y}$ is defined by (μ^3, μ^{-1}) or (μ, μ^{-3}), respectively. By the lemma above, the fixpoints of S are not affected by the Mori reduction. The map M_{red} is S-equivariant and locally biholomorphic in a neighbourhood of a fixed point of S. The lemma follows. □

Using our explicit knowledge of the G-action on Z we will show in the following that the linearization of the action of $S < G$ at a fixed point in the Del Pezzo surface Z is not of the type described by the lemma above. We distinguish two cases when studying Z.

Let $Z \cong \mathbb{P}_2$ and $[x_0 : x_1 : x_2]$ denote homogeneous coordinates on $\mathbb{P}_2$ such that the action of $S < G$ on $\mathbb{P}_2$ is given by $[x_0 : x_1 : x_2] \mapsto [\zeta x_0, \zeta^2 x_1, \zeta^4 x_2]$ where ζ is a 7^{th} root of unity. Using affine coordinates $z = \frac{x_1}{x_0}, w = \frac{x_2}{x_0}$ we check that the action of S at $p_1 = [1 : 0 : 0]$ is locally given by $(z,w) \mapsto (\zeta z, \zeta^3 w)$. This contradicts Lemma 6.8.

Let $Z \xrightarrow{q} \mathbb{P}_2$ be the double cover of $\mathbb{P}_2$ branched along Klein's quartic curve and let $[x_0 : x_1 : x_2]$ denote homogeneous coordinates on $\mathbb{P}_2$. As above, using affine coordinates $u = \frac{x_1}{x_0}, v = \frac{x_2}{x_0}$ we check that the action of S in a neighbourhood of $[1 : 0 : 0]$ is locally given by $(u,v) \mapsto (\zeta u, \zeta^3 v)$. The branch curve $C_{\mathrm{Klein}} \subset \mathbb{P}_2$ is defined by the equation $u^3 + uv^3 + v$. In new coordinates $(\tilde{u}(u,v), \tilde{v}(u,v)) = (u, u^3 + uv^3 + v)$ the branch curve is defined by $\tilde{v} = 0$ and the action of S is given by $(\tilde{u}, \tilde{v}) \mapsto (\zeta \tilde{u}, \zeta^3 \tilde{v})$. Consider the fixed point $[1 : 0 : 0] \in \mathbb{P}_2$ and its preimage $p \in Z$. At p, coordinates (z,w) can be chosen such that the covering map is locally given by $(z,w) \mapsto (z, w^2) = (\tilde{u}, \tilde{v})$. It follows that the action of S at $p \in Z$ is locally given by $(z,w) \mapsto (\zeta z, \zeta^5 w)$. This is again contrary to Lemma 6.8.

In summary, if $|\mathrm{Fix}_{\hat{Y}}(S)| = 3$, the action of $S < G$ on the Del Pezzo surface Z cannot be induced by a symplectic C_7-action on the K3-surface X. This proves the following lemma.

Lemma 6.9. $|\mathrm{Fix}_{\hat{Y}}(S)| \geq 4$.

6.2.3 Lifting Klein's quartic

The discussion of the previous section shows that there must be a step in the Mori reduction where the blow-down of a (-1)-curve identifies two S-fixed points. Let $z \in Z$ be a fixed point of S. Then, by equivariance, all points in the G-orbit of z are obtained by blowing down (-1)-curves in the process of Mori reduction. If $Z \cong \mathbb{P}_2$, we denote by $C_{\mathrm{Klein}} \subset Z$ Klein's quartic curve. If Z is the double cover of $\mathbb{P}_2$ branched along Klein's curve, we abuse notation and denote by C_{Klein} the ramification curve in Z. In the later case C_{Klein} is a G-invariant curve of genus 3 and self-intersection 8 by Lemma 3.15.

Let $z \in \mathrm{Fix}_Z(S) \subset C_{\mathrm{Klein}}$ and consider the G-orbit $G \cdot z$. By invariance, $G \cdot z \subset C_{\mathrm{Klein}}$. The isotropy group G_z must be cyclic and $G_z = S$ implies $|G \cdot z| = 24$. Let B denote the strict transform of

C_{Klein} in $\hat{Y}$. The curve B is a smooth G-invariant curve of genus 3 and meets at least 24 Mori fibers. Applying Lemma 3.17 to $M_{\text{red}}(B) = C_{\text{Klein}}$ we obtain

$$B^2 \leq C_{\text{Klein}}^2 - 24 \leq -8.$$

Lemma 6.10. *The curve B does not coincide with any of the curves of type E or F. Its preimage $\hat{B} := \hat{\pi}^{-1}(B) \subset \hat{X}$ is a cyclic degree three cover of B branched at $B \cap (\bigcup E_i \cup \bigcup F_j)$.*

Proof. The curves $E_i \subset \hat{Y}$ are (-3)-curves whereas B has self-intersection less than or equal to -8. Assume $B = F_j$ for some j. Then $\hat{B}$ is a curve of self-intersection less than or equal to -4 by Lemma 3.15 which is mapped biholomorphically to the K3-surface X. We obtain a contradiction since K3-surfaces do not admit curves of self-intersection less than -2. □

Since $\text{Fix}_Z(S) \subset C_{\text{Klein}}$ there are three fixed points of S on $\hat{B}$. From $\text{Fix}_{\hat{X}}(S) \subset \text{Fix}_{\hat{X}}(D)$ it follows that $\hat{\pi}|_{\hat{B}} : \hat{B} \to B$ is branched at three or more points. In particular, the curve $\hat{B}$ is connected. In the following, we will distinguish two cases: the curve $\hat{B}$ being reducible or irreducible.

Case 1: The curve $\hat{B}$ is reducible

The three irreducible components $\hat{B}_i$, $i = 1, 2, 3$ of $\hat{B}$ are smooth curves which are mapped biholomorphically onto B. Since B is exceptional, the configuration of curves $\hat{B}$ is also exceptional. It follows that the intersection matrix $(\hat{B}_i \cdot \hat{B}_j)_{ij}$ is negative definite. In the following we study the intersection matrix of $\hat{B}$ and will obtain a contradiction.

The restricted map $b_X : \hat{B}_i \to b_X(\hat{B}_i)$ is the normalization of $b_X(\hat{B}_i)$ and consequently the arithmetic genus of $b_X(\hat{B}_i)$ is given by the formula (cf. II.11 in [BHPVdV04])

$$g(b_X(\hat{B}_i)) = g(\hat{B}_i) + \delta(b_X(\hat{B}_i)),$$

where the number δ is computed as $\delta(b_X(\hat{B}_i)) = \sum_{p \in b_X(\hat{B}_i)} \dim_{\mathbb{C}}(b_{X*}\mathcal{O}_{\hat{B}_i}/\mathcal{O}_{b_X(\hat{B}_i)})_p$. Note that the sum can also be taken over the singular points $p \in b_X(\hat{B}_i)$ only, since smooth points do not contribute to the sum. Since X is a K3-surface, the adjunction formula for $b_X(\hat{B}_i)$ reads

$$(b_X(\hat{B}_i))^2 = 2g(b_X(\hat{B}_i)) - 2 = 2g(\hat{B}_i) + 2\delta(b_X(\hat{B}_i)) - 2.$$

By Lemma 3.17, the self-intersection number $(b_X(\hat{B}_i))^2$ can be expressed in terms of the self-intersection $\hat{B}_i^2$ and intersection multiplicities $E_j \cdot \hat{B}_i$:

$$(b_X(\hat{B}_i))^2 = \hat{B}_i^2 + \sum_j (\hat{E}_j \cdot \hat{B}_i)^2.$$

It follows that the self-intersection number of $\hat{B}_i$ can be expressed as

$$\hat{B}_i^2 = 2g(\hat{B}_i) + 2\delta(b_X(\hat{B}_i)) - 2 - \sum_j (\hat{E}_j \cdot \hat{B}_i)^2. \tag{6.1}$$

For simplicity, we first consider the case where $\hat{B}_i$ has nontrivial intersection with only one curve of type $\hat{E}$. We refer to this curve as $\hat{E}$. The general case then follows by addition over all curves $\hat{E}_j$, the number δ for the full contraction b_X is the sum of all numbers δ obtained when blowing down disjoint curves $\hat{E}_j$ stepwise.

Estimating the number δ

Example 6.11. Let $C = C_1 \cup C_2$ be a connected curve consisting of two irreducible components. Then the arithmetic genus of C is calculated as $g(C) = g(C_1) + g(C_2) + C_1 \cdot C_2 - 1$. The normalization $\tilde{C}$ of C is given by the disjoint union of the normalizations $\tilde{C}_i$ of C_1 and C_2. In particular, $g(\tilde{C}) = g(\tilde{C}_1) + g(\tilde{C}_2) - 1$, so that $\delta(C) = \delta(C_1) + \delta(C_2) + C_1 \cdot C_2$ (cf. II.11 in [BHPVdV04]).

Since the number δ is a sum of contributions δ_p at singular points p, we can calculate the number δ_p locally at each singularity where we decompose the germ of the curve as the union of irreducible components and use a formula generalizing the example above. We refer to an irreducible component of a curve germ realized in a open neighbourhood of the surface as a *curve segment*.

In order to study the singularities of $b_X(\hat{B}_i)$ one needs to consider the points of intersection $\hat{E} \cap \hat{B}_i$. These points of intersection can be of different quality:

- **Type $m = 1$:** The intersection at $b \in \hat{B}_i$ is transversal and the local intersection multiplicity at b is equal to 1. A neighbourhood of b in $\hat{B}_i$ is mapped to a smooth curve segment in $b_X(\hat{B}_i)$.
- **Type $m > 1$:** The intersection at $b \in \hat{B}_i$ is of higher multiplicity $m(b)$, i.e., $\hat{E}$ is tangent to $\hat{B}_i$ and in local coordinates (z, w) we may write $\hat{E} = \{z = 0\}$ and $\hat{B}_i = \{z - w^m\}$. Blowing down $\hat{E}$ transforms a neighbourhood of b into a a curve segment isomorphic to $\{x^{m+1} - y^m = 0\}$. For the singularity $(0,0)$ of this curve we calculate

$$\delta_{(0,0)} = \frac{1}{2} m(m-1).$$

Let b_m denote the number of points in $\hat{E} \cap \hat{B}_i$ with local intersection multiplicity m. For each point of intersection of $\hat{E}$ and $\hat{B}_i$ we obtain an irreducible component of the germ of $b_X(\hat{B}_i)$ at $p = b_X(\hat{E})$. We compute δ_p by decomposing this germ and need to determine local intersection multiplicities of all combinations of irreducible components.

Lemma 6.12. *Two irreducible components of the germ of $b_X(\hat{B}_i)$ at p corresponding to points in $\hat{E} \cap \hat{B}_i$ of type m and n meet with local intersection multiplicity greater than or equal to mn.*

Proof. In order to determine the intersection multiplicity of two irreducible components corresponding to points of type m and n, we write one curve as $\{x^{m+1} - y^m = 0\}$. The second curve can be expressed as $\{h_1(x,y)^{n+1} - h_2(x,y)^n = 0\}$ where $(x,y) \mapsto (h_1(x,y), h_2(x,y))$ is a holomorphic change of coordinates. Now normalizing the first curve by $\xi \mapsto (\xi^m, \xi^{m+1})$ and pulling back the equation of the second curve to the normalization $\mathbb{C}$, we obtain the equation $h_1(\xi^m, \xi^{m+1})^{n+1} - h_2(\xi^m, \xi^{m+1})^n = 0$ which has degree at least mn in ξ. It follows that the local intersection multiplicity is greater than or equal to mn. $\square$

Counting different types of intersections of irreducible components we obtain the following estimate for δ_p

$$\begin{aligned} \delta_p &= \sum \delta_p(C_i) + \sum_{i \neq j} (C_i \cdot C_j)_p \\ &\geq \sum_{m \in \mathbb{N}} \frac{b_m}{2} m(m-1) + \frac{1}{2} \sum_{m \in \mathbb{N}} b_m(b_m - 1)m^2 + \sum_{m > n} b_m b_n mn \end{aligned}$$

where $\sum_{i\neq j}(C_i \cdot C_j)_p$ decomposes into intersections $(C_i \cdot C_j)_p$ of type mm and intersections of type mn for $m \neq n$. The formula above applies to each curve $\hat{E}_j$ having nontrivial intersection with $\hat{B}_i$. Let p_j be the point on X obtained by blowing down $\hat{E}_j$ and let b^j_m denote the number of points of type m in $\hat{B}_i \cap \hat{E}_j$. Then

$$\begin{aligned}
\delta(b_X(\hat{B}_i)) &= \sum_j \delta_{p_j}(b_X(\hat{B}_i)) \\
&\geq \sum_j \Big(\sum_{m\in\mathbb{N}} \frac{b^j_m}{2} m(m-1) + \frac{1}{2} \sum_{m\in\mathbb{N}} b^j_m(b^j_m - 1)m^2 + \sum_{m>n} b^j_m b^j_n mn \Big).
\end{aligned}$$

Returning to the formula (6.1) for $\hat{B}_i^2$ we obtain

$$\begin{aligned}
\hat{B}_i^2 &= 2g(\hat{B}_i) + 2\delta(b_X(\hat{B}_i)) - 2 - \sum_j (\hat{E}_j \cdot \hat{B}_i)^2 \\
&\geq \sum_j \Big(\sum_{m\in\mathbb{N}} b^j_m m(m-1) + \sum_{m\in\mathbb{N}} b^j_m(b^j_m - 1)m^2 + 2\sum_{m>n} b^j_m b^j_n mn \Big) \\
&\quad - 2 - \sum_j \Big(\sum_m b^j_m m\Big)^2 \\
&\geq -2 - \sum_j \sum_m b^j_m m.
\end{aligned}$$

As a next step, we will find a bound for $(\hat{B}_i \cdot \hat{B}_k)$ in the case $i \neq k$. If a curve $\hat{B}_i$ intersects a ramification curve of type $\hat{E}$ or $\hat{F}$ in a point x, then $(\hat{B}_i \cdot \hat{B}_k)_x \geq 1$. If $(\hat{B}_i \cdot \hat{E}_j)_x = m$, then for $k \neq i$

$$(\hat{B}_k \cdot \hat{E}_j)_x = (\varphi_D(\hat{B}_i) \cdot \hat{E}_j)_x = (\varphi_D(\hat{B}_i) \cdot \varphi_D(\hat{E}_j))_x = (\hat{B}_i \cdot \hat{E}_j)_x = m$$

where $\varphi_D \in D$ is a biholomorphic transformation and E_j is in the fixed locus of D.

Lemma 6.13. *Assume $\hat{B}_i$ meets a curve of type $\hat{E}$ or $\hat{F}$ in x with local intersection multiplicity m. Then $(\hat{B}_i \cdot \hat{B}_k)_x \geq m$.*

Proof. Let $\hat{E}$, $\hat{F}$ respectively, be locally given by $\{z = 0\}$. Then $\hat{B}_i$ is locally given by $\{z - w^m = 0\}$ and $\hat{B}_k$ by $\{h_1(z,w) - h_2(z,w)^m = 0\}$ where $(z,w) \mapsto (h_1(z,w), h_2(z,w))$ is, as in the proof of Lemma 6.12, a holomorphic change of coordinates. Note that it stabilizes $\{z = 0\}$, i.e., $h_1(0,w) = 0$ for all w and we can write $h_1(z,w) = z\tilde{h}_1(z,w)$. The intersection of $\hat{B}_i$ and $\hat{B}_j$ corresponds to the equation $w^m\tilde{h}_1(w^m,w) - h_2(w^m,w)$ which is of degree greater than or equal to m. The lemma follows. ☐

Summing over all points of intersection of $\hat{B}_i$ and $\hat{B}_k$ one finds $\hat{B}_i \cdot \hat{B}_k \geq \sum_j \sum_m b^j_m m$. Recall that by Lemma 6.5 $\mathrm{Fix}_{\hat{X}}(S)$ is contained in $\mathrm{Fix}_{\hat{X}}(D)$ and that the curve B contains three S-fixed points. Therefore, it intersects the ramification locus of $\hat{\pi}$ in at least three points. At these points the three irreducible components of $\hat{B}$ must meet. In particular, $(\hat{B}_i, \hat{B}_k) \geq 3$. This yields

$$\begin{aligned}
&(1,1,1)\begin{pmatrix} \hat{B}_1^2 & \hat{B}_1 \cdot \hat{B}_2 & \hat{B}_1 \cdot \hat{B}_3 \\ \hat{B}_2 \cdot \hat{B}_1 & \hat{B}_2^2 & \hat{B}_2 \cdot \hat{B}_3 \\ \hat{B}_3 \cdot \hat{B}_1 & \hat{B}_3 \cdot \hat{B}_2 & \hat{B}_3^2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \hat{B}_1^2 + \hat{B}_2^2 + \hat{B}_3^2 + 2(\hat{B}_1 \cdot \hat{B}_2 + \hat{B}_2 \cdot \hat{B}_3 + \hat{B}_1 \cdot \hat{B}_3) \\
&\geq -6 - 3\sum_j \sum_m b^j_m + 3\sum_j \sum_m b^j_m m + (\hat{B}_1 \cdot \hat{B}_2 + \hat{B}_2 \cdot \hat{B}_3 + \hat{B}_1 \cdot \hat{B}_3) \\
&= -6 + (\hat{B}_1 \cdot \hat{B}_2 + \hat{B}_2 \cdot \hat{B}_3 + \hat{B}_1 \cdot \hat{B}_3) \\
&\geq 3.
\end{aligned}$$

Hence, the intersection matrix $(\hat{B}_i \cdot \hat{B}_j)_{ij}$ is not negative-definite contradicting the fact that $\hat{B}$ is exceptional. It follows that the curve $\hat{B}$ must be irreducible.

Case 2: The curve $\hat{B}$ is irreducible

Let $n : N \to \hat{B}$ be the normalization of $\hat{B}$. Since b_X is a blow-up, $b_X \circ n : N \to b_X(\hat{B})$ is the normalization of the curve $b_X(\hat{B}) \subset X$. It follows that $g(b_X(\hat{B})) = g(N) + \delta(b_X(\hat{B}))$. By adjunction, the self-intersection of $b_X(\hat{B})$ is given by

$$(b_X(\hat{B}))^2 = 2g(b_X(\hat{B})) - 2 = 2g(N) + 2\delta(b_X(\hat{B})) - 2.$$

As above, by Lemma 3.17, $(b_X(\hat{B}))^2 = \hat{B}^2 + \sum_j (\hat{E}_j \cdot \hat{B})^2$. Thus, the self-intersection of $\hat{B}$ can be expressed as

$$\hat{B}^2 = 2g(N) + 2\delta(b_X(\hat{B})) - 2 - \sum_j (\hat{E}_j \cdot \hat{B})^2.$$

Since the curve $\hat{B}$ is exceptional, this self-intersection number must be negative. By finding a lower bound for $\hat{B}^2$ we will obtain a contradiction.

Let us first examine the points of intersection $\hat{B} \cap \hat{E}$ for one curve $\hat{E}$ among the exceptional curves of the blow-down b_X. We consider the corresponding points of intersection of B and E in $\hat{Y}$ and we choose coordinates (ξ, η) such that E is locally defined by $\{\xi = 0\}$, the map $\hat{\pi}$ is locally given by $(z, w) \mapsto (z^3, w) = (\xi, \eta)$ and $B = \{f(\xi, \eta) = 0\}$. It follows that $\hat{B}$ is locally defined by $\{h = f \circ \hat{\pi} = 0\}$.

If E and B meet transversally, we know that the function $f(\xi, \eta)$ fulfills $\frac{\partial f}{\partial \eta}|_{(0,0)} \neq 0$. It follows that $\frac{\partial h}{\partial w}|_{(0,0)} \neq 0$ and after a suitable change of coordinates $h(z, w) = z^m - w$.

If E and B meet tangentially, we know that the function $f(\xi, \eta)$ fulfills $\frac{\partial f}{\partial \eta}|_{(0,0)} = 0$. Since B is smooth, we know $\frac{\partial f}{\partial \xi}|_{(0,0)} \neq 0$. After a suitable change of coordinates $h(z, w) = z^3 - w^n$ with $n > 0$. Note that in both cases the coordinate change on $\hat{X}$ is such that $\hat{E}$ is still defined by $\{z = 0\}$. This will be important when describing the blow-down b_X of $\hat{E}$.

Consider a curve segment $\{h = 0\}$ in $\hat{X}$ and its image under the map b_X. If $h(z, w) = z^m - w$ then the corresponding smooth segment of $b_X(\hat{B})$ is defined by $x^{m+1} - y = 0$. If $h(z, w) = z^3 - w^n$ then the corresponding piece of $b_X(\hat{B})$ is defined by $x^{n+3} - y^n = 0$ and has a singular point if $n > 1$.

Let $p = b_X(\hat{E})$. We will determine δ_p by decomposing the germ of $b_X(\hat{B})$ at p into its irreducible components. There are three different types of such components:

1. smooth components locally defined by $x^{m+1} - y = 0$,
2. singular components locally defined by $x^{n+3} - y^n = 0$ for $n > 1$ not divisible by 3,
3. triplets of smooth components locally defined by $x^6 - y^3 = 0$,
4. triplets of singular components locally defined by $x^{n+3} - y^n = 0$ for $n = 3k$ and $k > 1$.

The singularity in case 2) gives $\delta = \frac{n^2+n-2}{2}$. In case 4), each component is defined by an equation of type $x^{k+1} - y^k = 0$ and the singularity of each component gives $\delta = \frac{k^2-k}{2}$.
In order to determine δ_p we need to specify intersection multiplicities for all combinations of irreducible components.

Lemma 6.14. *The local intersection multiplicities of pairs of irreducible components of the germ of $b_X(\hat{B})$ at p in general position are given by the following table.*

local equation	$x^{m_1+1}-y$	$x^{n_1+3}-y^{n_1}$	x^2-y	$x^{k_1+1}-y^{k_1}$
$x^{m_2+1}-y$	1	n_1	1	k_1
$x^{n_2+3}-y^{n_2}$	n_2	n_1n_2	n_2	n_2k_1
x^2-y	1	n_1	1 or (2)	k_1
$x^{k_2+1}-y^{k_2}$	k_2	k_2n_1	k_2	k_1k_2 or (k^2+k)

Note that the local equations in the first row and column, although all written as functions of (x,y), describe the curve segments in different choices of local coordinates.

Sketch of proof. As above, we rewrite one equation as $f(h_1(x,y),h_2(x,y))$ where (h_1,h_2) is a holomorphic change of local coordinates. The intersection multiplicities can then be calculated by the method introduced in the proof of Lemma 6.12. Two irreducible components in a triplet of type 3) meet with intersection multiplicity 2. Two irreducible components in a triplet of type 4) meet with intersection multiplicity k^2+k. These quantities are indicated in brackets as they differ from the intersection multiplicities of two irreducible components from different triplets. □

Remark 6.15. If two irreducible components of the germ of $b_X(\hat{B})$ at p are in special position, their local intersection multiplicity is greater than the value specified in the above table. In particular, the table gives lower bounds for the respective intersection numbers.

Let a denote the number of irreducible components of type 1), let b_n the number of irreducible components of type 2) where $n \notin 3\mathbb{N}$, let $c \in 3\mathbb{N}$ denote the number of irreducible components of type 3) and let $d_k \in 3\mathbb{N}$ denote the number of irreducible components of type 4). We summarize $e = a + c$.

A lower bound for δ_p is given by

$$\begin{aligned}\delta_p \geq & \sum_n b_n \frac{n^2+n-2}{2} + \sum_k d_k \frac{k^2-k}{2} \\ & + \frac{1}{2}e(e-1) + c + \sum_n e b_n n + \sum_k e d_k k \\ & + \frac{1}{2}\sum_n b_n(b_n-1)n^2 + \sum_{n_1>n_2} b_{n_1}b_{n_2}n_1n_2 + \sum_{n,k} b_n d_k nk \\ & + \frac{1}{2}\sum_k d_k(d_k-1)k^2 + \sum_k d_k k + \sum_{k_1>k_2} d_{k_1}d_{k_2}k_1k_2.\end{aligned}$$

For simplicity, we first consider only one curve $\hat{E}$ intersecting $\hat{B}$. The formula for $\hat{B}^2$ becomes

$$\begin{aligned}\hat{B}^2 &= 2g(N) + 2\delta(b_X(\hat{B})) - 2 - (\hat{E}\cdot\hat{B})^2 \\ &= 2g(N) + 2\delta(b_X(\hat{B})) - 2 - \underbrace{(e + \sum_n b_n n + \sum_k d_k k)^2}_{(\hat{E}\cdot\hat{B})^2} \\ &= 2g(N) - 2 - e + 2c + \sum_k d_k k + \sum_n b_n(n-2) \\ &\geq 2g(N) - 2 - a. \end{aligned} \tag{6.2}$$

The same formula also holds if we consider the general case of curves $\bigcup_i \hat{E}_i$ intersecting $\hat{B}$ since both the calculation of δ and the intersection number $\sum_i (\hat{B}, \hat{E}_i)^2$ can be obtained from the above by addition. The number a now represents the number of points of type 1) in the union of curves $\hat{E}_i$.

The map $n \circ \hat{\pi} : N \to \hat{B} \to B$ is a degree three cover of the smooth curve B branched at $V \subset B$. The genus of B is three, the topological Euler characteristic is $e(B) = -4$. Let $\tilde{V} := B \cap (\bigcup E_i \cup \bigcup F_j)$ denote the branch locus of $\hat{\pi} : \hat{B} \to B$. Then $V \subset \tilde{V}$ and V must contain those points in $\tilde{V}$ which correspond to smooth points on $\hat{B}$. In partcular, $|V| \geq a$.

The Euler characteristic of N is given by $e(N) = 3e(B) - 2|V| = -12 - 2|V| = 2 - 2g(N)$. and inequality (6.2) becomes

$$\hat{B}^2 \geq 12 + 2|V| - a \geq 12 + |V| \geq 0$$

contradicting the fact that $\hat{B}$ is exceptional.

Conclusion

The above contradiction shows the non-existence of a K3-surface with an action of $G \times C_3$. This completes the prove of Theorem 6.3.

7 The alternating group of degree six

In the previous chapters we have considered symplectic automorphisms groups of K3-surfaces centralized by an antisymplectic involution, i.e., the groups under consideration were of the form $\tilde{G} = G \times \langle \sigma \rangle$ where $\tilde{G}_{\text{symp}} = G$. In this chapter we wish to discuss more general automorphims groups $\tilde{G}$ of mixed type: if $\tilde{G}$ contains an antisymplectic involution σ with fixed points we consider the quotient by σ. In general, if σ does not centralize the group $\tilde{G}_{\text{symp}}$ inside $\tilde{G}$, the action of $\tilde{G}_{\text{symp}}$ does *not* descend to the quotient surface. We therefore restrict our consideration to the centralizer $Z_{\tilde{G}}(\sigma)$ of σ inside $\tilde{G}$ (or $\tilde{G}_{\text{symp}}$) and study its action on the quotient surface.

If we are able to describe the family of K3-surfaces with $Z_{\tilde{G}}(\sigma)$-symmetry, it remains to detect the surfaces with $\tilde{G}$-symmetry inside this family. This chapter is devoted to a situation where the group $\tilde{G}$ contains the alternating group of degree six. Although, a precise classification cannot be obtained at present, we achieve an improved understanding of the equivariant geometry of K3-surfaces with $\tilde{G}$-symmetry and classify families of K3-surfaces with $Z_{\tilde{G}}(\sigma)$-symmetry (cf. Theorem 7.31). In this sense, this closing chapter serves as an outlook on how the method of equivariant Mori reduction allows generalization to more advanced classification problems.

7.1 The group $\tilde{A}_6$

We let $\tilde{G}$ be any finite group which fits into the exact sequence

$$\{\text{id}\} \to A_6 \to \tilde{G} \xrightarrow{\alpha} C_n \to \{\text{id}\}.$$

and in the following consider a K3-surface X with an effective action of $\tilde{G}$. The group of symplectic automorphisms $(\tilde{G})_{\text{symp}}$ in $\tilde{G}$ coincides with A_6.

This particular situation is considered by Keum, Oguiso, and Zhang in [KOZ05] and [KOZ07]. They lay special emphasis on the maximal possible choice of $\tilde{G}$ and therefore consider a group $\tilde{G} = \tilde{A}_6$ characterized by the exact sequence

$$\{\text{id}\} \to A_6 \to \tilde{A}_6 \xrightarrow{\alpha} C_4 \to \{\text{id}\}. \tag{7.1}$$

Let $N := \text{Inn}(\tilde{A}_6) \subset \text{Aut}(A_6)$ denote the group of inner automorphisms of $\tilde{A}_6$ and let int : $\tilde{A}_6 \to N$ be the homomorphisms mapping an element $g \in \tilde{A}_6$ to conjugation with g. It can be shown that the group $\tilde{A}_6$ is a semidirect product $A_6 \rtimes C_4$ embedded in $N \times C_4$ by the map (int, α) (Theorem 2.3 in [KOZ07]). By Theorem 4.1 in [KOZ07] the group N is isomorphic to M_{10} and the isomorphism class of $\tilde{A}_6$ is uniquely determined by (7.1) and the condition that it acts on a K3-surface.

In [KOZ05] a lattice-theoretic proof of the following classification result (Theorem 5.1, Theorem 3.1, Proposition 3.5) is given.

Theorem 7.1. *A K3 surface X with an effective action of $\tilde{A}_6$ is isomorphic to the minimal desingularization of the surface in $\mathbb{P}_1 \times \mathbb{P}_2$ given by*

$$S^2(X^3 + Y^3 + Z^3) - 3(S^2 + T^2)XYZ = 0.$$

Although this realization is very concrete, the action of $\tilde{A}_6$ on this surface is hidden. The existence of an isomorphism from a K3-surface with $\tilde{A}_6$-symmetry to the surface defined by the equation above follows abstractly since both surfaces are shown to have the same transcendental lattice. It is therefore desirable to achieve a more geometric understanding of K3-surfaces with $\tilde{A}_6$-symmetry in general and in particular to obtain an explicit realization of X where the action of $\tilde{A}_6$ is visible.

We let the generator of the factor C_4 in the semidirect product $\tilde{A}_6 = A_6 \rtimes C_4$ be denoted by τ. The order four automorphism τ is nonsymplectic and has fixed points. It follows that the antisymplectic involution $\sigma := \tau^2$ fulfils

$$\mathrm{Fix}_X(\sigma) \neq \emptyset.$$

Since σ is mapped to the trivial automorphism in $\mathrm{Out}(A_6) = \mathrm{Aut}(A_6)/\mathrm{int}(A_6) \cong C_2 \times C_2$ there exists $h \in A_6$ such that $\mathrm{int}(h) = \mathrm{int}(\sigma) \in \mathrm{Aut}(A_6)$. The antisymplectic involution $h\sigma$ centralizes A_6 in $\tilde{A}_6$.

Remark 7.2. If $\mathrm{Fix}_X(h\sigma) \neq \emptyset$, we are in the situation dealt with in Section 4.2, i.e., the K3-surface X is an A_6-equivariant double cover of $\mathbb{P}_2$ where A_6 acts as Valentiner's group and the branch locus is given by $F_{A_6}(x_1, x_2, x_3) = 10x_1^3x_2^3 + 9x_1^5x_3 + 9x_2^3x_3^3 - 45x_1^2x_2^2x_3^2 - 135x_1x_2x_3^4 + 27x_3^6$. By construction, there is an evident action of $A_6 \times C_2$ on the Valentiner surface, it is however not clear whether this surface admits the larger symmetry group $\tilde{A}_6$.

In the following we assume that $h\sigma$ acts without fixed points on X as otherwise the remark above yields an A_6-equivariant classification of X.

7.1.1 The centralizer G of σ in $\tilde{A}_6$

We study the quotient $\pi : X \to X/\sigma = Y$. As mentioned above, the action of the centralizer of σ descends to an action on Y. We therefore start by identifying the centralizer $G := Z_{\tilde{A}_6}(\sigma)$ of σ in $\tilde{A}_6$.

Lemma 7.3. *The group G equals $Z_{A_6}(\sigma) \rtimes C_4$ and $Z_{A_6}(\sigma) = Z_{A_6}(h)$*

Proof. The lemma follows from direct computations: we write an element of $\tilde{A}_6$ as $a\tau^k$ with $a \in A_6$. Then $a\tau^k$ is in $Z_{\tilde{A}_6}(\sigma)$ if and only if $a\tau^k\tau^2 = \tau^2 a\tau^k$. This is the case if and only if $a\tau^2 = \tau^2 a$, i.e., if $a \in Z_{A_6}(\sigma)$. Now $\langle\tau\rangle < Z_{\tilde{A}_6}(\sigma)$ implies the first part of the lemma. The second part follows from the equality $\mathrm{int}(\sigma) = \mathrm{int}(h)$. □

Lemma 7.4. $Z_{A_6}(h) = D_8$

Proof. Since $\operatorname{int}(\sigma) = \operatorname{int}(h)$ and $\sigma^2 = \mathrm{id}$, it follows that h^2 commutes with any element in A_6. As $Z(A_6) = \{\mathrm{id}\}$, it follows that h is of order two. There is only one conjugacy class of elements of order two in A_6. We calculate $Z_{A_6}(h) = D_8$ for one particular choice of h. Let

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 5 & 6 \end{pmatrix}.$$

Any element in the centralizer of h must be of the form

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ * & * & * & * & 5 & 6 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ * & * & * & * & 6 & 5 \end{pmatrix}$$

It is therefore sufficient to perform all calculations in S_4. If an element of S_4 is a composition of an even (odd) number of transpositions, the corresponding element of $Z_{A_6}(h)$ is given by completing it with the identity map (transposition map) on the fifth and sixth letter.

Let

$$g_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, g_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}, g_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}.$$

and check that $g_1, g_2, g_3 \in Z_{A_6}(h)$. Define $g_1 g_2 =: c$ and check

$$c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad c^2 = h.$$

Now $g_3 c g_3 = c^3$ and the subgroup of S_4 (A_6, respectively) generated by c and g_3 is seen to be a dihedral group of order eight; $\langle g_3 \rangle \ltimes \langle c \rangle = D_8 < Z_{A_6}(h)$. In order to show equality, assume that $Z_{A_6}(h)$ is bigger. It then follows that the centralizer of h in S_4 is a subgroup of order 12, in particular, it has a subgroup of order three. Going through the list of elements of order three in S_4 one checks that none commutes with h and obtains a contradiction. □

Let $D_8 = C_2 \ltimes C_4$ where C_2 is generated by $g = g_3$ and C_4 by c and note that $c^2 = h$. We study the action of τ on D_8 by conjugation. Since C_4 is the only cyclic subgroup of order four in D_8, it is τ-invariant. If c is τ-fixed, i.e. $\tau c = c\tau$, then

$$(\tau c)^2 = c\tau\tau c = c\sigma c \overset{c \in Z(\sigma)}{=} \sigma c^2 = \sigma h.$$

In this case τc generates a cyclic group of order four acting freely on X, a contradiction. So τ acts on $\langle c \rangle$ by $c \mapsto c^3$ and $c^2 \mapsto c^2$. Now $\tau g \tau^{-1} = c^k g$ for some $k \in \{0, 1, 2, 3\}$. If $k = 2$, then

$$(\tau g)^2 = \tau g \tau g = \tau g \tau^{-1} \tau^2 g = c^2 g \sigma g \overset{g \in Z(\sigma)}{=} c^2 \sigma = \sigma h$$

and we obtain the same contradiction as above. So $k \in \{1, 3\}$ and by choosing the appropriate generator of $\langle c \rangle$ we may assume that $k = 3$. The action of τ on $Z_{A_6}(h) = D_8$ given by $g \mapsto c^3 g$ and $c \mapsto c^3$.

Lemma 7.5. $G' = \langle c \rangle$.

Proof. The commutator subgroup G' is the smallest normal subgroup N of G such that G/N is Abelian. We use the above considerations about the action of τ on D_8 by conjugation. The subgroup $\langle c \rangle$ is normal in $G = D_8 \rtimes \langle \tau \rangle$ and $G/\langle c \rangle$ is seen to be Abelian. Since $G/\langle c^2 \rangle$ is not Abelian, $G' \neq \langle c^2 \rangle$ and the lemma follows. □

7.1.2 The group $H = G/\langle\sigma\rangle$

We consider the quotient $Y = X/\sigma$ equipped with the action of $G/\sigma =: H = Z_{\tilde{A}_6}(\sigma)/\langle\sigma\rangle = D_8 \rtimes C_2$. The group C_2 is generated by $[\tau]_\sigma$. For simplicity, we transfer the above notation from G to H by writing e.g. τ for $[\tau]_\sigma$. etc.. Since $\tau g \tau^{-1} = c^3 g = gc$, it follows as above that $H' = \langle c \rangle$.

Let $K < G$ be the cyclic group of order eight generated by $g\tau$.

$$K = \{\mathrm{id}, g\tau, c\sigma, g\tau^3 c, c^2, g\tau c^2, \sigma c^3, gc\tau^3\}.$$

We denote the image of K in G/σ by the same symbol. Since $[\sigma c]_\sigma = [c]_\sigma \in K$ it contains $H' = \langle c \rangle$ and we can write $H = \langle \tau \rangle \ltimes K = D_{16}$.

Lemma 7.6. *There is no nontrivial normal subgroup N in H with $N \cap H' = \{\mathrm{id}\}$.*

Proof. If such a group exists, first consider the case $N \cap K = \{\mathrm{id}\}$. Then $N \cong C_2$ and $H = K \times N$ would be Abelian, a contradiction. If $N \cap K \neq \{\mathrm{id}\}$ then $N \cap K = \langle (g\tau)^k \rangle$ for some $k \in \{1, 2, 4\}$. This implies $(g\tau)^4 = c^2 \in N$ and contradicts $N \cap H' = N \cap \langle c \rangle = \emptyset$. □

The following observations strongly rely the assumption that σh acts freely on X.

Lemma 7.7. *The subgroup H' acts freely on the branch set $B = \pi(\mathrm{Fix}_X(\sigma))$ in Y.*

Proof. If for some $b \in B$ the isotropy group H'_b is nontrivial, then $c^2(b) = h(b) = b$ and σh fixes the corresponding point $\tilde{b} \in X$. □

Corollary 7.8. *The subgroup H' acts freely on the set $\mathcal{R}$ of rational branch curves. In particular, the number of rational branch curves n is a multiple of four.*

Corollary 7.9. *The subgroup H' acts freely on the set of τ-fixed points in Y.*

Proof. We show $\mathrm{Fix}_Y(\tau) \subset B$. Since $\sigma = \tau^2$ on X, a $\langle\tau\rangle$-orbit $\{x, \tau x, \sigma x, \tau^3 x\}$ in X gives rise to a τ-fixed point y in the quotient $Y = X/\sigma$ if and only if $\sigma x = \tau x$. Therefore, τ-fixed points in Y correspond to τ-fixed points in X. By definition $\mathrm{Fix}_X(\tau) \subset \mathrm{Fix}_X(\sigma)$ and the claim follows. □

7.2 H-minimal models of Y

Since $\mathrm{Fix}_X(\sigma) \neq \emptyset$, the quotient surface Y is a smooth rational H-surface to which we apply the equivariant minimal model program. We denote by $Y_{\min}$ an H-minimal model of Y. It is known that $Y_{\min}$ is either a Del Pezzo surface or an H-equivariant conic bundle over $\mathbb{P}_1$.

Theorem 7.10. *An H-minimal model $Y_{\min}$ does not admit an H-equivariant $\mathbb{P}_1$-fibration. In particular, $Y_{\min}$ is a Del Pezzo surface.*

In order to prove this statement we begin with the following general fact (cf. Proof of Lemma 6.7).

Lemma 7.11. *If $Y \to Y_{\min}$ is an H-equivariant Mori reduction and A a cyclic subgroup of H, then*

$$|\mathrm{Fix}_Y(A)| \geq |\mathrm{Fix}_{Y_{\min}}(A)|.$$

Proof. Each step of a Mori reduction is known to contract a disjoint union of (-1)-curves. It is sufficient to prove the statement for one step in a Mori reduction. If such a step changes the number of fixed points, then some Mori fiber E of the reduction is contracted to an A-fixed point. The rational curve E is A-invariant and therefore contains two A-fixed points. The number of fixed points drops. □

Suppose that some $Y_{\min}$ is an H-equivariant conic bundle, i.e., there is an H-equivariant fibration $p : Y_{\min} \to \mathbb{P}_1$ with generic fiber $\mathbb{P}_1$. We let $p_* : H \to \mathrm{Aut}(\mathbb{P}_1)$ be the associated homomorphism.

Lemma 7.12. $\mathrm{Ker}(p_*) \cap H' = \{\mathrm{id}\}$.

Proof. The elements of $\mathrm{Ker}(p_*)$ fix two points in every generic p-fiber. If $h = c^2 \in H' = \langle c \rangle$ fixes points in every generic p-fiber, then h acts trivially on a one-dimensional subset $C \subset Y$. Since $h = c^2$ acts symplectically on X it has only isolated fixed points in X. Therefore, on the preimage $\tilde{C} = \pi^{-1}(C) \subset X$, the action of h coincides with the action of σ. But then $\sigma h|_{\tilde{C}} = \mathrm{id}|_{\tilde{C}}$ contradicts the assumption that σh acts freely on X. □

Proof of Theorem 7.10. Since there are no nontrivial normal subgroups in H which have trivial intersection with H' (Lemma 7.6), it follows from Lemma 7.12 that $\mathrm{Ker}(p_*) = \{\mathrm{id}\}$, i.e., the group H acts effectively on the base.

We regard H as the semidirect product $H = \langle \tau \rangle \ltimes K$, where $K = C_8$ is described above. The group H acts on the base as a dihedral group and therefore τ exchanges the K-fixed points. We will obtain a contraction by analyzing the K-actions on the fibers over its two fixed points. Since τ exchanges these fibers, it is enough to study the K-action on one of them which we denote by F.

By Lemma 2.21 there are two situations which we must consider:

1. F is a regular fiber of $Y_{\min} \to \mathbb{P}_1$.
2. $F = F_1 \cup F_2$ is the union of two (-1)-curves intersecting transversally in one point.

We study the fixed points of c, $h = c^2$ and $g\tau$ in $Y_{\min}$. Recall that in X the symplectic transformation c has precisely four fixed points and h has precisely eight fixed points. This set of eight points is stabilized by the full centralizer of h, in particular by $K = \langle g\tau \rangle \cong C_8$.

Since $h\sigma$ acts by assumption freely on X, it follows that σ acts freely on the set of h-fixed points in X. If $hy = y$ for some $y \in Y$, then the preimage of y in X consists of two elements $x_1, \sigma x_1 = x_2$. If these form an $\langle h \rangle$-orbit, then both are σh-fixed, a contradiction. It follows that $\{x_1, x_2\} \subset \mathrm{Fix}_X(h)$ and the number of h-fixed points in Y is precisely four. In particular, h acts effectively on any curve in Y.

Let us first consider Case 2 where $F = F_1 \cup F_2$ is reducible. Since $\langle c \rangle$ is the only subgroup of index two in K, it follows that $\langle c \rangle$ stabilizes F_i and both c and h have three fixed points in F (two on each irreducible component, one is the point of intersection $F_1 \cap F_2$), i.e., six fixed points on $F \cup \tau F \subset Y_{\min}$. This is contrary to Lemma 7.11 because h has at most four fixed points in $Y_{\min}$.

If F is regular (Case 1), then the cyclic group K has two fixed points on the rational curve F. Since $h \in K$, the four K-fixed points on $F \cup \tau F$ are contained in the set of h-fixed points on $Y_{\min}$. As $|\mathrm{Fix}_{Y_{\min}}(h)| \leq 4$, the K-fixed points coincide with the four h-fixed points in $Y_{\min}$;

$$\mathrm{Fix}_{Y_{\min}}(h) = \mathrm{Fix}_{Y_{\min}}(K).$$

In particular, the Mori reduction does not affect the four h-fixed points $\{y_1, \dots y_4\}$ in Y. By equivariance of the reduction, the group K acts trivially on this set of four points. Passing to the double cover X, we conclude that the action of $g\tau \in K$ on a preimage $\{x_i, \sigma x_i\}$ of y_i is either trivial or coincides with the action of σ. In both cases it follows that $(g\tau)^2 = c\sigma$ acts trivially on the set of h-fixed points in X. As $\mathrm{Fix}_X(c) \subset \mathrm{Fix}_X(h)$, this is contrary to the fact that σ acts freely on $\mathrm{Fix}_X(h)$. □

In the following we wish to identify the Del Pezzo surface Y_{min}. For thus, we use the Euler characteristic formulas,

$$24 = e(X) = 2e(Y) - 2n + \underbrace{2g-2}_{\text{if } D_g \text{ is present}},$$

where $D_g \subset B$ is of general type, $g = g(D_g) \geq 2$, and

$$e(Y) = e(Y_{\mathrm{min}}) + m,$$

where $m = |\mathcal{E}|$ denotes the total number of Mori fibers. For convenience we introduce the difference $\delta = m - n$. If a branch curve D_g of general type is present, then $13 - g - \delta = e(Y_{\mathrm{min}})$ and if it is not present $12 - \delta = e(Y_{\mathrm{min}})$.

Proposition 7.13. *For every Mori fiber E the orbit $H.E$ consists of at least four Mori fibers.*

Proof. We need to distinguish three cases:

$$1.)\ E \cap B \neq \emptyset \text{ and } E \not\subset B; \qquad 2.)\ E \subset B; \qquad 3.)\ E \cap B = \emptyset$$

Case 1 Since H' acts freely on the branch curves and E meets B in at most two points, we know $|H'.E| \geq 2$. If $|H.E| = 2$, then the isotropy group H_E is a normal subgroup of index two which necessarily contains the commutator group H', a contradiction.

Case 2 We show that the H'-orbit of E consists of four Mori fibers. If it consisted of less than four Mori fibers, the stabilizer $H'_E \neq \{\mathrm{id}\}$ of E in H' would fix two points in $E \subset B$. This contradicts Lemma 7.7.

Case 3 All Mori fibers disjoint from B have self-intersection (-2) and meet exactly one Mori fiber of the previous steps of the reduction in exactly one point. If $E \cap B = \emptyset$ there is a chain of Mori fibers $E_1, \dots, E_k = E$ connecting E and B with the following properties: The Mori fiber E_1 is the only one to have nonempty intersection with B and is the first curve of this configuration to be blown down in the reduction process. The curves fulfil $(E_i, E_{i+1}) = 1$ for all $i \in \{1, \dots, k-1\}$ and $(E_i, E_j) = 0$ for all $j \neq i+1$. The curves are blown down subsequently and meet no Mori fibers outside this chain.

The H-orbit of this union of Mori fibers consists of at least four copies of this chain. This is due to that fact that the H-orbit of E_1 consists of at least four Mori fibers by Case 1. In particular, the H-orbit of E consists of at least four copies of E. □

Corollary 7.14. *The difference δ is a non-negative multiple $4k$ of four. If $\delta = 0$, then X is a double cover of $Y = Y_{\mathrm{min}} = \mathbb{P}_1 \times \mathbb{P}_1$ branched along a curve of genus nine.*

Proof. Above we have shown that m and and n are multiples of four. Therefore $\delta = 4k$.

If δ was negative, i.e., $m < n$, there is no configuration of Mori fibers meeting the rational branch curves such that the corresponding contractions transform the (-4)-curves in Y to curves on a Del Pezzo surface Y_{min}. It follows that δ is non-negative.

If $\delta = 0$, then $n = m = 0$ and Y is H-minimal. The commutator subgroup $H' \cong C_4$ acts freely on the branch locus B implying $e(B) \in \{0, -8, -16, \dots\}$. Since the Euler characteristic of the Del Pezzo surface Y is at least 3 and at most 11,

$$6 \leq 2e(Y) = 24 + e(B) \leq 22,$$

we only need to consider the case $e(Y) \in \{4, 8\}$ and $B = D_g$ for $g \in \{9, 5\}$.

The automorphism group of a Del Pezzo surface of degree 4 is $C_2^4 \rtimes \Gamma$ for $\Gamma \in \{C_2, C_4, S_3, D_{10}\}$. If $D_{16} < C_2^4 \rtimes \Gamma$ then $A := D_{16} \cap C_2^4 \lhd D_{16}$ and A is either trivial or isomorphic to C_2. In both case D_{16}/A is not a subgroup of Γ in any of the cases listed above. Therefore, $e(Y) \neq 8$.

A Del Pezzo surface of degree 8 is either the blow-up of $\mathbb{P}_2$ in one point or $\mathbb{P}_1 \times \mathbb{P}_1$. Since the first is never equivariantly minimal, it follows that $Y \cong \mathbb{P}_1 \times \mathbb{P}_1$ and $g(B) = 9$. □

Theorem 7.15. *Any H-minimal model $Y_{\min}$ of Y is $\mathbb{P}_1 \times \mathbb{P}_1$.*

Proof. Suppose $\delta \neq 0$. Since $\delta \geq 4$, it follows that $e(Y_{\min}) = 13 - g - \delta \leq 7$ if a branch curve D_g of general type is present, and $e(Y_{\min}) = 12 - \delta \leq 8$ if not. We go through the list of of Del Pezzo surfaces with $e(Y_{\min}) \leq 8$.

- If $e(Y_{\min}) = 8$, i.e., $\deg(Y_{\min}) = 4$, then the possible automorphism groups are very limited and we have alredy noted above that D_{16} does not occur.
- If $e(Y_{\min}) = 7$, then $\mathrm{Aut}(Y_{\min}) = S_5$. Since 120 is not divisible by 16, we see that a Del Pezzo surface of degree five does not admit an effective action of the group H.
- If $e(Y_{\min}) = 6$, then $A := \mathrm{Aut}(Y_{\min}) = (\mathbb{C}^*)^2 \rtimes (S_3 \times C_2)$. We denote by $A^\circ \cong (\mathbb{C}^*)^2$ the connected component of A. If $q : A \to A/A^\circ$ is the canonical quotient homomorphism then $q(H') < q(A)' \cong C_3$. Consequently $H' = C_4 < A^\circ$. We may realize $Y_{\min}$ as $\mathbb{P}_2$ blown up at the three corner points and A° as the space of diagonal matrices in $\mathrm{SL}_3(\mathbb{C})$. Every possible representation of C_4 in this group has ineffectivity along one of the lines joining corner points. But, as we have seen before, the elements of H', in particular $c^2 = h$, have only isolated fixed points in $Y_{\min}$.
- A Del Pezzo surface obtained by blowing up one or two points in $\mathbb{P}_2$ is never H-minimal and therefore does not occur
- Finally, $Y_{\min} \neq \mathbb{P}_2$: If $e(Y_{\min}) = 3$ then either $\delta = 9$ (if D_g is not present), a contradiction to $\delta = 4k$, or $g + \delta = 10$. In the later case, $\delta = 4, 8$ forces $g = 6, 2$. In both cases, the Euler characteristic $2 - 2g$ of D_g is not divisible by 4. This contradicts the fact that H' acts freely on D_g.

We have hereby excluded all possible Del Pezzo surfaces except $\mathbb{P}_1 \times \mathbb{P}_1$ and the proposition follows. □

7.3 Branch curves and Mori fibers

We let $M : Y \to Y_{\min} = \mathbb{P}_1 \times \mathbb{P}_1$ denote an H-equivariant Mori reduction of Y.

Lemma 7.16. *The length of an orbit of Mori fibers is at least eight.*

Proof. Consider the action of H on $\mathbb{P}_1 \times \mathbb{P}_1$. Both canonical projections are equivariant with respect to the commutator subgroup $H' = \langle c \rangle \cong C_4$. Since $c^2 \in H'$ does not act trivially on any curve in Y or $Y_{\min}$, it follows that H' has precisely four fixed points in $Y_{\min} = \mathbb{P}_1 \times \mathbb{P}_1$. Since $h = c^2$ has precisely four fixed points in Y and $\mathrm{Fix}_Y(H') = \mathrm{Fix}_Y(c) \subset \mathrm{Fix}_Y(c^2)$, we conclude that H' has precisely four fixed points in Y and it follows that the Mori fibers do not pass through H'-fixed points. Note that the H'-fixed points in Y coincide with the h-fixed points.

Suppose there is an H-orbit $H.E$ of Mori fibers of length strictly less then eight and let $p = M(E)$. We obtain an H-orbit $H.p$ in $\mathbb{P}_1 \times \mathbb{P}_1$ with $|H.p| \leq 4$. Now $|K.p| \leq 4$ implies that $K_p \neq \{\mathrm{id}\}$, in particular, $h = c^2 \in K_p$. It follows that p is a h-fixed point. This contradicts the fact that the Mori fibers do not pass through fixed points of h. □

Corollary 7.17. *The total number m of Mori fibers equals 0, 8, or 16..*

Proof. A total number of 24 or more Mori fibers would require 16 rational curves in B. This contradicts the bound for the number of connected components of the fixed point set of an anti-symplectic involution on a K3-surface (cf. Corollary 3.20) □

Recalling that the number of rational branch curves is a multiple of four, i.e., $n \in \{0, 4, 8\}$ and using the fact $m \in \{0, 8, 16\}$ along with $m \leq n + 9$, we conclude that the surface Y is of one of the following types.

1. $m = 0$
   The quotient surface Y is H-minimal. The map $X \to Y \cong \mathbb{P}_1 \times \mathbb{P}_1$ is branched along a single curve B. This curve B is a smooth H-invariant curve of bidegree $(4, 4)$.
2. $m = 8$ and $e(Y) = 12$
   The surface Y is the blow-up of $\mathbb{P}_1 \times \mathbb{P}_1$ in an H-orbit consisting of eight points.
   (a) If the branch locus B of $X \to Y$ contains no rational curves, then $e(B) = 0$ and B is either an elliptic curve or the union of two elliptic curves defining an elliptic fibration on X.
   (b) If the branch locus B of $X \to Y$ contains rational curves, their number is exactly four (Observe that eight or more rational branch curves of self-intersection (-4) cannot be modified sufficiently and mapped to curves on a Del Pezzo surface by contracting eight Mori fibers). It follows that the branch locus is the disjoint union of an invariant curve of higher genus and four rational curves.
3. $m = 16$ and $e(Y) = 20$
   The map $X \to Y$ is branched along eight disjoint rational curves.

We can simplify the above situation by studying rational curves in B, their intersection with Mori fibers and their images in $\mathbb{P}_1 \times \mathbb{P}_1$.

Proposition 7.18. *If $e(Y) = 12$, then $n = 0$.*

Proof. Suppose $n \neq 0$ and let $C_i \subset Y$ be a rational branch curve. Since $C_i^2 = -4$ and $M(C_i) \subset \mathbb{P}_1 \times \mathbb{P}_1$ has self-intersection ≥ 0 it must meet the union of Mori fibers $\bigcup E_j$. All possible configurations of Mori fibers yield image curves $M(C_i)$ of self-intersection ≤ 4. If $M(C_i)$ is a curve a bidegree (a, b), then, by adjunction.

$$2g(M(C_i)) - 2 = (M(C_i))^2 + (M(C_i) \cdot K_{\mathbb{P}_1 \times \mathbb{P}_1}) = 2ab - 2a - 2b,$$

and $(M(C_i))^2 = 2ab \leq 4$ implies that $g(M(C_i)) = 0$. In particular, $M(C_i)$ must be nonsingular. Hence each Mori fiber meets C_i in at most one point. It follows that C_i meets four Mori fibers, each in one point, and $(M(C_i))^2 = 0$. Curves of self-intersection zero in $\mathbb{P}_1 \times \mathbb{P}_1$ are fibers of the canonical projections $\mathbb{P}_1 \times \mathbb{P}_1 \to \mathbb{P}_1$. The curve C_1 meets four Mori fibers $E_1, \ldots E_4$ and each of these Mori fibers meets some C_i for $i \neq 1$. After renumbering, we may assume that E_1 and E_2 meet C_2 and therefore $M(C_1)$ and $M(C_2)$ meet in more than one point, a contradiction. It follows that $e(Y) = 12$ implies $n = 0$ □

Proposition 7.19. *If $e(Y) = 20$, then Y is the blow-up of $\mathbb{P}_1 \times \mathbb{P}_1$ in sixteen points*

$$\{p_1, \ldots p_{16}\} = (F_1 \cup F_2 \cup F_3 \cup F_4) \cap (F_5 \cup F_6 \cup F_7 \cup F_8),$$

where $F_1, \ldots F_4$ are fibers of the canonical projection π_1 and $F_5, \ldots F_8$ are fibers of π_2. The branch locus is given by the proper transform of $\bigcup F_i$ in Y.

Proof. We denote the eight rational branch curves by $C_1, \ldots C_8$. The Mori reduction can have two steps. A slightly more involved study of possible configurations of Mori fibers shows that $0 \leq (M(C_i))^2 \leq 4$. As above $M(C_i)$ is seen to be nonsingular and each Mori fiber can meet C_i in at most one point. Any configuration of curves with this property yields $(M(C_i))^2 = 0$ and $F_i = M(C_i)$ is a fiber of a canonical projection $\mathbb{P}_1 \times \mathbb{P}_1 \to \mathbb{P}_1$.

If there are Mori fibers disjoint from B these are blown down in the second step of the Mori reduction. Let $E_1, \ldots, E_8$ denote the Mori fibers of the first step and $\tilde{E}_1, \ldots, \tilde{E}_8$ those of the second step. We label them such that $\tilde{E}_i$ meets E_i. Each curve E_i meets two rational branch curves C_i and C_{i+4} and their images $F_i = M(C_i)$ and $F_{i+4} = M(C_{i+4})$ meet with multiplicity ≥ 2. This is contrary to the fact that they are fibers of the canonical projections. It follows that there are no Mori fibers disjoint from B and all 16 Mori fibers are contrancted simultaniously. There is precisely one possible configuration of Mori fibers on Y such that all rational brach curves are mapped to fibers of the canonical projections of $\mathbb{P}_1 \times \mathbb{P}_1$: The curves $C_1, \ldots C_4$ are mapped to fibers of π_1 and $C_5, \ldots, C_8$ are mapped to fibers of π_2. The Mori reduction contracts 16 curves to the 16 points of intersection $\{p_1, \ldots p_{16}\} = (\bigcup_{i=1}^4 F_i) \cap (\bigcup_{i=5}^8 F_i) \subset \mathbb{P}_1 \times \mathbb{P}_1$. □

Let us now restrict our attention to the case where the branch locus B is the union of two linearly equivalent elliptic curves and exclude this case.

7.3.1 Two elliptic branch curves

In this section we prove:

Theorem 7.20. $\mathrm{Fix}_X(\sigma)$ *is not the union of two elliptic curves.*

We assume the contrary, let $\mathrm{Fix}_X(\sigma) = D_1 \cup D_2$ with D_i elliptic and let $f : X \to \mathbb{P}_1$ denote the elliptic fibration defined by the curves D_1 and D_2. Recall that σ acts effectively on the base $\mathbb{P}_1$ as otherwise σ would act trivially in a neighbourhood of D_i by a linearization argument (cf. Theorem 1.12). It follows that the group of order four generated by τ acts effectively on $\mathbb{P}_1$.

Let I be the ineffectivity of the induced G-action on the base $\mathbb{P}_1$. We regard $G = C_4 \ltimes D_8$ where $C_4 = \langle \tau \rangle$ and D_8 is the centralizer of σ in A_6 (cf. Section 7.1.1) and define $J := I \cap D_8$. First, note that I is nontrivial:

Lemma 7.21. *The group G does not act effectively on $\mathbb{P}_1$, i.e., $I \neq \{\mathrm{id}\}$.*

Proof. If G acts effectively on $\mathbb{P}_1$, then G is among the groups specified in Remark 3.1. In our special case $|G| = 32$ and G would have to be cyclic or dihedral. Since the group G does not contain a cyclic group of order 16, this is a contradiction. □

Lemma 7.22. *The intersection $J = I \cap D_8$ is nontrivial.*

Proof. Assume the contrary and let $J = I \cap D_8 = \{e\}$. We consider the quotient $G \to G/D_8 \cong C_4$ and see that either $I \cong C_2$ or $I \cong C_4$.

- If $I \cap D_8 = \{e\}$ and $I \cong C_2$, we write $I = \langle \sigma\xi \rangle$ with $\xi \in D_8$ an element of order two. Now I is normal if and only if $\xi = h$, i.e., $I = \langle \sigma h \rangle$. In this case, since $\sigma h \notin K$, the image of K in G/I is a normal subgroup of index two and one checks that $G/I \cong D_{16}$. The group K is mapped injectively into G/I. The equivalence relation defining this quotient identifies σ and h and both are in the image of K. So h-fixed points in X must lie in the fibers over the σ-fixed points in $\mathbb{P}_1$, i.e., the σ-fixed points sets D_1, D_2. Since σh acts freely on X, this is a contradiction.
- If $I \cap D_8 = \{e\}$ and $I \cong C_4$ we write $I = \langle \tau\xi \rangle$ and show that for no choice of ξ the group $I = \langle \tau\xi \rangle$ is normal in G: If $\xi = c^k g$, then $\langle \tau\xi \rangle = K$ is of order eight. If $\xi = c^k$, then $\langle \tau\xi \rangle$ is of order four and has trivial intersection with D_8. It is however not normalized by g.

As we obtain contradictions in all cases, we see that the intersection $J = I \cap D_8$ is nontrivial. □

In the following, we consider the different possibilities for the order of J and show that in fact none of these occur.

If $|J| = 8$ then $D_8 \subset I$. Recall that any automorphism group of an elliptic curve splits into an Abelian part acting freely and a cyclic part fixing a point. Since D_8 is not Abelian, any D_8-action on the fibers of f must have points with nontrivial isotropy and gives rise to a positive-dimensional fixed point set of some subgroup of D_8 on X contradicting the fact that D_8 acts symplectically on X. It follows that the maximal possible order of J is four.

Lemma 7.23. *The ineffectivity I does not contain $\langle c \rangle$.*

Proof. Assume the contrary and consider the fixed points of c^2. If a c^2-fixed point lies at a smooth point of a fiber of f, then the linearization of the c^2-action at this fixed point gives rise to a positive-dimensional fixed point set in X and yields a contradiction. It follows that the fixed points of c^2 are contained in the singular f-fibers. Since $\langle \tau \rangle$ normalizes $\langle c \rangle$ and the $\langle \tau \rangle$-orbit of a singular fiber consists of four such fibers, we must only consider two cases:

1. The eight c^2-fixed points are contained in four singular fibers (one $\langle \tau \rangle$-orbit of fibers), each of these fibers contains two c^2-fixed points.
2. The eight c^2-fixed points are contained in eight singular fibers (two $\langle \tau \rangle$-orbits).

Note that $\langle c^2 \rangle$ is normal in I and therefore I acts on the set of $\langle c^2 \rangle$-fixed points. In the second case, all eight c^2-fixed points are also c-fixed. This is contrary to c having only four fixed points and therefore the second case does not occur.

The first case does not occur for similar reasons: If c^2 has exactly two fixed points x_1 and x_2 in some fiber F, then $\langle c \rangle$ either acts transitively on $\{x_1, x_2\}$ or fixes both points. Since $\mathrm{Fix}_X(c) \subset \mathrm{Fix}_X(c^2)$ and $\langle c \rangle$ must have exactly one fixed point on F, this is impossible. □

Corollary 7.24. $|J| \neq 4$.

Proof. Assume $|J| = 4$. Using τ we check that no subgroup of D_8 isomorphic to $C_2 \times C_2$ is normal in G. It follows that the group $\langle c \rangle$ is the only order four subgroup of D_8 which is normal in G and therefore $J = \langle c \rangle$. By the lemma above this is however impossible. $\square$

It remains to consider the case where $|J| = 2$. The only normal subgroup of order two in D_8 is $J = \langle h \rangle$.

Lemma 7.25. *If $|J| = 2$, then $I = \langle \sigma c \rangle$.*

Proof. We first show that $|J| = 2$ implies $|I| = 4$: If $|I| = 2$, then $I = \langle h \rangle$ and $G/I = C_4 \ltimes (C_2 \times C_2)$. Since this group does not act effectively on $\mathbb{P}_1$, this is a contradiction. If $|I| \geq 8$, then G/I is Abelian and therefore I contains the commutator subgroup $G' = \langle c \rangle$. This contradicts Lemma 7.23. It follows that $|I| = 4$ and either $I \cong C_4$ or $I \cong C_2 \times C_2$. In the later case, the only possible choice is $I = \langle \sigma \rangle \times \langle h \rangle$ which contradicts the fact that σ acts effectively on the base. It follows that $I = \langle \sigma \xi \rangle$, where $\xi^2 = h$ and therefore $\xi = c$. $\square$

Let us now consider the action of G on X with $I = \langle \sigma c \rangle$. Recall that the cyclic group $\langle \tau \rangle$ acts effectively on the base and has two fixed points there. Since $\sigma = \tau^2$, these are precisely the two σ-fixed points. In particular, $\langle \tau \rangle$ stabilizes both σ-fixed point curves D_1 and D_2 in X. Furthermore, the transformations σc and c stabilize D_i for $i = 1, 2$. Since the only fixed points of c in $\mathbb{P}_1$ are the images of D_1 and D_2,

$$\mathrm{Fix}_X(c) \subset D_1 \cup D_2 = \mathrm{Fix}_X(\sigma).$$

On the other hand, we know that $\mathrm{Fix}_X(c) \cap \mathrm{Fix}_X(\sigma) = \emptyset$. Thus $I = \langle \sigma c \rangle$ is not possible and the case $|J| = 2$ does not occur.

We have hereby eleminated all possibilities for $|J|$ and completed the proof of Theorem 7.20.

7.4 Rough classification of X

We summerize the observations of the previous section in the following classification result.

Theorem 7.26. *Let X be a K3-surface with an effective action of the group G such that $\mathrm{Fix}_X(h\sigma) = \emptyset$. Then X is one of the following types:*

1. *a double cover of $\mathbb{P}_1 \times \mathbb{P}_1$ branched along a smooth H-invariant curve of bidegree (4,4).*
2. *a double cover of a blow-up of $\mathbb{P}_1 \times \mathbb{P}_1$ in eight points and branched along a smooth elliptic curve B. The image of B in $\mathbb{P}_1 \times \mathbb{P}_1$ has bidegree (4,4) and eight singular points.*
3. *a double cover of a blow-up Y of $\mathbb{P}_1 \times \mathbb{P}_1$ in sixteen points $\{p_1, \dots p_{16}\} = (\bigcup_{i=1}^4 F_i) \cap (\bigcup_{i=5}^8 F_i)$, where $F_1, \dots F_4$ are fibers of the canonical projection π_1 and $F_5, \dots F_8$ are fibers of π_2. The branch locus ist given by the proper transform of $\bigcup F_i$ in Y. The set $\bigcup F_i$ is an invariant reducible subvariety of bidegree (4,4).*

Proof. It remains to consider case 2. and show that the image of B in $\mathbb{P}_1 \times \mathbb{P}_1$ has bidegree (4,4) and eight singular points. We prove that each Mori fiber E meets the branch locus B either in two points or once with multiplicity two, i.e., we need to check that E may not meet B transversally in exactly one point. If this was the case, the image $M(B)$ of the branch curve is a smooth H-invariant curve of bidegree $(2,2)$. The double cover X' of $\mathbb{P}_1 \times \mathbb{P}_1$ branched along the smooth curve $M(B) = C_{(2,2)}$ is a smooth surface. Since X is K3 and therefore minimal the induced birational map $X \to X'$ is an isomorphism. This is a contradiction since X' is not a K3-surface.

As each Mori fiber meets B with multiplicity two, the self-intersection number of $M(B)$ is 32 and $M(B)$ is a curve of bidegree (4,4) with eight singular points. These singularities are either nodes or cusps depending on the kind of intersection of E and B. We obtain a diagram

$$\begin{tikzcd} X_{\mathrm{sing}} \arrow[d, "2:1"] & X \arrow[l, "\text{desing.}"'] \arrow[d, "2:1"] \\ C_{(4,4)} \subset \mathbb{P}_1 \times \mathbb{P}_1 & Y \supset B \arrow[l, "M"'] \end{tikzcd}$$

□

In order to obtain a description of possible branch curves, we study the action of H on $\mathbb{P}_1 \times \mathbb{P}_1$ and its invariants.

7.4.1 The action of H on $\mathbb{P}_1 \times \mathbb{P}_1$

Recall that we consider the dihedral group $H \cong D_{16}$ generated by τg of order eight and τ. For convenience, we recall the group structure of H:

$$\begin{aligned} c = (g\tau)^2, &\quad \tau g \tau = gc, \\ g^2 = \mathrm{id}, &\quad \tau c \tau = c^3, \\ c^4 = \mathrm{id}, &\quad \tau^2 = \mathrm{id}. \end{aligned}$$

In this section, we prove:

Proposition 7.27. *In appropriately chosen coordinates the action of H on $\mathbb{P}_1 \times \mathbb{P}_1$ given by*

- $c([z_0 : z_1], [w_0 : w_1]) = ([iz_0 : z_1], [-iw_0 : w_1])$
- $\tau([z_0 : z_1], [w_0 : w_1]) = ([z_1 : z_0], [iw_1 : w_0])$
- $g([z_0 : z_1], [w_0 : w_1]) = ([w_0 : w_1], [z_0 : z_1])\,.$

Sketch of proof. First note that the index two subgroup H_1 of H preserving the canonical projections is generated by τ and c, i.e, $H_1 = \langle \tau \rangle \ltimes \langle c \rangle \cong D_8$. We begin by choosing coordinates such that

$$c([z_0 : z_1], [w_0 : w_1]) = ([\chi_1(c) z_0 : z_1], [\chi_2(c) w_0 : w_1])$$

where $\chi_i : H' \to S^1$ are faithful characters. Since τ acts transitively on the set of H'-fixed points, we conclude that after an appropriate change of coordinates not affecting the H'-action

$$\tau([z_0 : z_1], [w_0 : w_1]) = ([z_1 : z_0], [w_1 : w_0]).$$

The automorphism g permutes the factors of $\mathbb{P}_1 \times \mathbb{P}_1$, stabilizes the fixed point set of H' and fulfills $gcg^{-1} = c^3$ and $g\tau g^{-1} = c\tau$. Therefore, one finds

- $c([z_0 : z_1], [w_0 : w_1]) = ([iz_0 : z_1], [-iw_0 : w_1])$
- $\tau([z_0 : z_1], [w_0 : w_1]) = ([z_1 : z_0], [w_1 : w_0])$
- $g([z_0 : z_1], [w_0 : w_1]) = ([\lambda w_0 : w_1], [\lambda^{-1} z_0 : z_1])$, where $\lambda^2 = i$.

We introduce a change of coordinates such that g is of the simple form

$$g([z_0 : z_1], [w_0 : w_1]) = ([w_0 : w_1], [z_0 : z_1]).$$

This does affect the shape of the τ-action and yields the action of H described in the propostion. □

7.4.2 Invariant curves of bidegree $(4, 4)$

Given the action of H on $\mathbb{P}_1 \times \mathbb{P}_1$ discussed above, we wish to study the invariants and semi-invariants of bidegree $(4, 4)$. The space of (a, b)- bihomogeneous polynomials in $[z_0 : z_1][w_0 : w_1]$ is denoted by $\mathbb{C}_{(a,b)}([z_0 : z_1][w_0 : w_1])$.

An invariant curve C is given by a D_{16}-eigenvector $f \in \mathbb{C}_{(4,4)}([z_0 : z_1][w_0 : w_1])$. The kernel of the D_{16}-representation on the line $\mathbb{C}f$ spanned f contains the commutator subgroup $H' = \langle c \rangle$. It follows that f is a linear combination of c-invariant monomials of bidegree $(4, 4)$. These are

$$z_0^4 w_0^4,\ z_0^4 w_1^4,\ z_1^4 w_0^4,\ z_1^4 w_1^4,\ z_0^2 z_1^2 w_0^2 w_1^2,\ z_0^3 z_1 w_0^3 w_1,\ z_0 z_1^3 w_0 w_1^3.$$

The polynomials

$$f_1 = z_0^4 w_0^4 + z_1^4 w_1^4, \quad f_2 = z_0^4 w_1^4 + z_1^4 w_0^4, \quad f_3 = z_0^3 z_1 w_0^3 w_1 - i z_0 z_1^3 w_0 w_1^3$$

span the space of D_{16}-invariants. Semi-invariants are appropiate linear combinations of

$$g_1 = z_0^4 w_0^4 - z_1^4 w_1^4, \quad g_2 = z_0^4 w_1^4 - z_1^4 w_0^4, \quad g_3 = z_0^3 z_1 w_0^3 w_1 + i z_0 z_1^3 w_0 w_1^3, \quad g_4 = z_0^2 z_1^2 w_0^2 w_1^2.$$

Note

$$\begin{array}{llll} \tau(g_1) = -g_1, & \tau(g_2) = -g_2, & \tau(g_3) = -g_3, & \tau(g_4) = -g_4, \\ g(g_1) = g_1, & g(g_2) = -g_2, & g(g_3) = g_3, & g(g_4) = g_4. \end{array}$$

It follows that a D_{16}-invariant curve of bidegree $(4, 4)$ in $\mathbb{P}_1 \times \mathbb{P}_1$ is of the following three types

$$\begin{aligned} C_a &= \{a_1 f_1 + a_2 f_2 + a_3 f_3 = 0\}, \\ C_b &= \{b_1 g_1 + b_3 g_3 + b_4 g_4 = 0\}, \\ C_0 &= \{g_2 = 0\}. \end{aligned}$$

7.4.3 Refining the classification of X

Using the above description of invariant curves of bidegree (4,4) we may refine Theorem 7.26.

Reducible curves of bidegree $(4,4)$

Theorem 7.28. *Let X be a K3-surface with an effective action of the group G such that $\mathrm{Fix}_X(h\sigma)=\emptyset$. If $e(X/\sigma)=20$, then X/σ is equivariantly isomorphic to the blow up of $\mathbb{P}_1\times\mathbb{P}_1$ in the singular points of the curve $C=\{f_1-f_2=0\}$ and $X\to Y$ is branched along the proper transform of C in Y.*

Proof. It follows from Theorem 7.26 that X is the double cover of $\mathbb{P}_1\times\mathbb{P}_1$ blown up in sixteen points. These sixteen points are the points of intersection of eight fibers of $\mathbb{P}_1\times\mathbb{P}_1$, four for each of fibration.

By invariance these fibers lie over the base points $[1:1],[1:-1],[1:i],[1:-1]$ and the configurations of eight fibres is defined by the invariant polynomial f_1-f_2.

The double cover $X\to Y$ is branched along the proper transform of this configuration of eight rational curves. This proper transform is a disjoint union of eight rational curves in Y, each with self-intersection (-4). □

Smooth curves of bidegree $(4,4)$

Theorem 7.29. *Let X be a K3-surface with an effective action of the group G such that $\mathrm{Fix}_X(h\sigma)=\emptyset$. If $X/\sigma\cong\mathbb{P}_1\times\mathbb{P}_1$, then after a change of coordinates the branch locus is C_a for some $a_1,a_2,a_3\in\mathbb{C}$.*

Proof. The surface X is a double cover of $\mathbb{P}_1\times\mathbb{P}_1$ branched along a smooth H-invariant curve of bidegree (4,4). The invariant (4,4)-curves C_b and C_0 discussed above are seen to be singular at $([1:0],[1:0])$ or $([1:0],[0:1])$. □

Note that the general curve C_a is smooth. We obtain a 2-dimensional family $\{C_a\}$ of smooth branch curves and a corresponding family of K3-surfaces $\{X_{C_a}\}$.

Curves of bidegree $(4,4)$ with eight singular points

It remains to consider the case 2. of the classification. Our aim is to find an example of a K3-surface X such that $X/\sigma=Y$ has a nontrivial Mori reduction $M:Y\to\mathbb{P}_1\times\mathbb{P}_1=Z$ contracting a single H-orbit of Mori fibers consisting of precisely 8 curves. In this case the branch locus $B\subset Y$ is mapped to a singular $(4,4)$-curve $C=M(B)$ in Z. The curve C is irreducible and has precisely 8 singular points along a single H-orbit in Z.

As we have noted above, many of the curves C_a,C_b,C_0 are seen to be singular at $([1:0],[1:0])$ or $([1:0],[0:1])$. Since both points lie in H-orbits of length two, these curves are not candidates for our construction. This argument excludes the curves C_b,C_0 and C_a if $a_1=0$ or $a_2=0$.

For C_a with $a_3=0$ one checks that C_a has singular points if and only if $a_1=-a_2$, i.e., if C_a is reducible. It therefore remains to consider curves C_a where all coefficients $a_i\neq 0$. We choose $a_3=1$.

Lemma 7.30. *If $a_i\neq 0$ for $i=1,2,3$, then C_a is irreducible.*

Sketch of proof. First note that C_a does not pass through $([1:0],[1:0])$ or $([1:0],[0:1])$. Therefore, possible singularities or points of intersection of irreducible components come in orbits of length eight. Assume that C_a is reducible, consider the decomposition into irreducible components and the H-action on it. A curve of type $(n, 0)$ is always reducible for $n > 1$ and therefore does not occur in the decomposition.

If C_a contains a (2,2)-curve $C_a^{(2,2)}$, then the H-orbit of $C_a^{(2,2)}$ has length ≤ 2 and $C_a^{(2,2)}$ is stable with respect to the subgroup $H' = \langle c \rangle$ of H. All c-semi-invariants of bidegree (2,2) are, however, reducible. Similary, all c-semi-invariants of bidegree (1,2) or (2,1) are reducible an therefore C does not have a curve of this type as an irreducible component.

The curve C_a is not the union of a (1,3)- and a (3,1)-curve, since their intersection number is 10 and contradicts invariance. Similarly one excludes the union of a (1,1) and a (3,3)-curve.

If C_a is a union of (1,1) or (1,0) and (0,1)-curves, one checks by direct computation that the requirement that C_a is H-invariant gives strong restrictions and finds that in all cases at least one coefficient a_i has to be zero. $\square$

One possible choice of an orbit of length eight is given by the orbit through a τ-fixed point $p_\tau = ([1:1], [\pm\sqrt{i}:1])$. One checks that $p_\tau \in C_a$ for any choice of a_i. However, if we want C_a to be singular in p_τ, then $a_2 = 0$. It then follows that C_a is singular at points outside $H.p_\tau$. It has more than eight singular points and is therefore reducible.

All other orbits of length eight are given by orbits through g-fixed points $p_x = ([1:x], [1:x])$ for $x \neq 0$. One can choose coefficients $a_i(x)$ such that $C_{a(x)}$ is singular at p_x if and only if $x^8 \neq 1$. If the curve $C_{a(x)}$ is irreducible, then it has precisely eight singular points $H.p_x$ of multiplicity 2 (cusps or nodes) and the double cover of $\mathbb{P}_1 \times \mathbb{P}_1$ branched along $C_{a(x)}$ is a singular K3-surface with precisely eight singular points. We obtain a diagram

$$\begin{tikzcd} X_{\mathrm{sing}} \arrow[d, "2:1"] & & X \arrow[ll, "\text{desing.}"'] \arrow[d, "2:1"] \\ C_{(4,4)} \subset \mathbb{P}_1 \times \mathbb{P}_1 & & Y \supset B \arrow[ll, "M"] \end{tikzcd}$$

If p_x is a node in $C_{a(x)}$, then the corresponding singularity of X_{sing} is resolved by a single blow-up. The (-2)-curve in X obtained from this desingularization is a double cover of a (-1)-curve in Y meeting B in two points.

If p_x is a cusp in $C_{a(x)}$, then the corresponding singularity of X_{sing} is resolved by two blow-ups. The union of the two intersecting (-2)-curves in X obtained from this desingularization is a double cover of a (-1)-curve in Y tangent to B is one point.

The information determining whether p_x is a cusp or a node is encoded in the rank of the Hessian of the equation of $C_{a(x)}$ at p_x. The condition that this rank is one gives a nontrivial polynomial condition. For a general irreducible member of the family $\{C_{a(x)} \mid x \neq 0,\, x^8 \neq 1\}$ the singularities of $C_{a(x)}$ are nodes.

We let q be the polynomial in x that vanishes if and only if the rank of the Hessian of $C_{a(x)}$ at p_x is one. It has degree 24, but 16 of its solutions give rise to reducible curves $C_{a(x)}$. The remaining eight solution give rise to four different irreducible curves. These are identified by the action of the normalizer of H in $\mathrm{Aut}(\mathbb{P}_1 \times \mathbb{P}_1)$ and therefore define equivalent K3-surfaces.

We summarize the discussion in the following main classification theorem.

Theorem 7.31. *Let X be a K3-surface with an effective action of the group G such that* $\mathrm{Fix}_X(h\sigma) = \emptyset$. *Then X is an element of one the following families of K3-surfaces:*

1. *the two-dimensional family $\{X_{C_a}\}$ for C_a smooth,*
2. *the one-dimensional family of minimal desingularization of double covers of $\mathbb{P}_1 \times \mathbb{P}_1$ branched along curves in $\{C_{a(x)} \,|\, x \neq 0,\, x^8 \neq 1\}$. The general curve $C_{a(x)}$ has precisely eight nodes along an H-orbit. Up to natural equivalence there is a unique curve $C_{a(x)}$ with eight cusps along an H-orbit.*
3. *the trivial family consisting only of the minimal desingularization of the double cover of $\mathbb{P}_1 \times \mathbb{P}_1$ branched along the curve $C_a = \{f_1 - f_2 = 0\}$ where $a_1 = 1, a_2 = -1, a_3 = 0$.*

Corollary 7.32. *Let X be a K3-surface with an effective action of the group $\tilde{A}_6$. If* $\mathrm{Fix}_X(h\sigma) = \emptyset$, *then X is an element of one the families 1. -3. above. If* $\mathrm{Fix}_X(h\sigma) \neq \emptyset$, *then X is A_6-equivariantly isomorphic to the Valentiner surface.*

7.5 Summary and outlook

Recall that our starting point was the description of K3-surfaces with $\tilde{A}_6$-symmetry. Using the group structure of $\tilde{A}_6$ we have divided the problem into two possible cases corresponding to the question whether $\mathrm{Fix}_X(h\sigma)$ is empty or not. If it is nonempty, the K3-surface with $\tilde{A}_6$-symmetry is the Valentiner surface discussed in Section 4.2. If is is empty, our discussion in the previous sections has reduced the problem to finding the $\tilde{A}_6$-surface in the families of surfaces X_{C_a} with D_{16}-symmetry.

It is known that a K3-surface with $\tilde{A}_6$-symmetry has maximal Picard rank 20. This follows from a criterion due to Mukai (cf. [Muk88]) and is explicitely shown in [KOZ05].

All surfaces X_{C_a} for $C_a \subset \mathbb{P}_1 \times \mathbb{P}_1$ a (4,4)-curve are elliptic since the natural fibration of $\mathbb{P}_1 \times \mathbb{P}_1$ induces an elliptic fibration on the double cover (or is desingularization).

A possible approach for finding the $\tilde{A}_6$-example inside our families is to find those surfaces with maximal Picard number by studying the elliptic fibration. It would be desirable to apply the following formula for the Picard rank of an elliptic surface $f : X \to \mathbb{P}_1$ with a section (cf. [SI77]):

$$\rho(X) = 2 + \mathrm{rank}(MW_f) + \sum_i (m_i - 1)$$

where the sum is taken over all singular fibers, m_i denotes the number of irreducible components of the singular fiber and $\mathrm{rank}(MW_f)$ is the rank of the Mordell-Weil group of sections of f. The number two in the formula is the dimension of the hyperbolic lattice spanned by a general fiber and the section.

First, one has to ensure that the fibration under consideration has a section. One approach to find sections is to consider the quotient $q : \mathbb{P}_1 \times \mathbb{P}_1 \to \mathbb{P}_2$ and the image of the curve C_a inside $\mathbb{P}_2$. If we find an appropiate bitangent to $q(C_a)$ such that its preimage in $\mathbb{P}_1 \times \mathbb{P}_1$ is everywhere tangent to C_a, then its preimage in the double cover of $\mathbb{P}_1 \times \mathbb{P}_1$ is reducible and both its components define sections of the elliptic fibration. For C_a the curve with eight nodes the existence of a section (two sections) follows from an application of the Plücker formula to the curve $q(C_a)$ with 3 cusps and its dual curve.

As a next step, one wishes to understand the singular fibers of the elliptic fibrations. Singular fibers occur whenever the branch curve C_a intersects a fiber F of the $\mathbb{P}_1 \times \mathbb{P}_1$ in less than four points. Depending on the nature of intersection $F \cap C_a$ one can describe the corresponding singular fiber of the elliptic fibration. For C_a the curve with eight cusps one finds precisely eight singular fibres of type I_3, i.e., three rational curves forming a closed cycle. In particular, the contribution of all singular fibres $\sum_i (m_i - 1)$ in the formula above is 16. In the case where C_a is smooth or has eight nodes, this contribution is less.

In order to determine the number $\rho(X_{C_a})$ it is neccesary to either understand the Mordell-Weil group and its $\mathrm{rank}(MW_f)$ or to find curves which give additional contribution to $\mathrm{Pic}(X_{C_a}$ not included in $2 + \sum_i (m_i - 1)$.

In conclusion, the method of equivariant Mori reduction applied to quotients X/σ yields an explicit description of a families of K3-surfaces with $D_{16} \times \langle \sigma \rangle$-symmetry and by construction, the K3-surface with $\tilde{A}_6$-symmetry is contained in one of these families. It remains to find criteria to characterize this particular surface inside this family. The possible approach by understanding the function

$$a \mapsto \rho(X_{C_a})$$

using the elliptic structure of X_{C_a} requires a detailed analysis of the Mordell-Weil group.

A Actions of certain Mukai groups on projective space

In this appendix, we derive the unique action of the group N_{72} on $\mathbb{P}_3$ and the unique action of M_9 on $\mathbb{P}_2$ in the context of Sections 4.8 and 4.9. We consider the homomorphism $\mathrm{SL}_n(\mathbb{C}) \to \mathrm{PSL}_n(\mathbb{C})$ and determine preimages $\tilde{g} \in \mathrm{SL}_n(\mathbb{C})$ of the generators $g \in G \subset \mathrm{PSL}_n(\mathbb{C})$. Our considerations benefit from fact that both actions are induced by symplectic actions of the corresponding group on a K3-surface X.

A.1 The action of N_{72} on $\mathbb{P}_3$

One can calculate explicitly the realization of the N_{72}-action on $\mathbb{P}_3$ by using the decomposition $C_3^2 \rtimes D_8$ where $D_8 = C_2 \ltimes (C_2 \times C_2) = \mathrm{Aut}(C_3^2)$. For each generator of N_{72} we will specify the corresponding element in $\mathrm{SL}_4(\mathbb{C})$. We denote the center of $\mathrm{SL}_4(\mathbb{C})$ by Z. Recall that the action of $D_8 = C_2 \ltimes (C_2 \times C_2) = \langle \alpha \rangle \ltimes (\langle \beta \rangle \times \langle \gamma \rangle)$ on $C_3 \times C_3$ is given by

$$\alpha(a,b) = (b,a), \quad \beta(a,b) = (a^2,b), \quad \gamma(a,b) = (a,b^2).$$

In suitably chosen coordinates the generator a of C_3^2 can be represented as

$$\tilde{a} = \begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & \xi^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where ξ is a third root of unity. Next we wish to specify γ in $\mathrm{SL}_4(\mathbb{C})$. We know that $a\gamma = \gamma a$, i.e., $\tilde{a}\tilde{\gamma}\tilde{a}^{-1}\tilde{\gamma}^{-1} \in Z$, and γ is seen to be of the form

$$\tilde{\gamma} = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

where $*$ denotes a nonzero matrix entry. Since a and b commute in N_{72}, we know that $\tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1} \in Z$ and

$$\tilde{b} = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

Since γ acts on b by $\gamma b \gamma = b^{-1} = b^2$, it follows that

$$\tilde{b} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

We apply a change of coordinates affecting only the lower (2×2)-block of b and therefore not affecting the shape of a auch that

$$\tilde{b} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & \xi^2 \end{pmatrix}.$$

It follows that α interchanges the two (2×2)-blocks of the matrices a and b and

$$\tilde{\alpha} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Finally, γ and β can be put into the form

$$\tilde{\gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \tilde{\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A.1.1 Invariant quadrics and cubics

Let $f \in \mathbb{C}_2[x_1 : x_2 : x_3 : x_4]$ be a semi-invariant homogeneous polynomial of degree two,

$$f = \sum_i a_i x_1^2 + \sum_{i \neq j} b_{ij} x_i x_j.$$

If $a_i \neq 0$ for some $1 \leq i \leq 4$, then semi-invariance with respect to the transformations α, β, γ yields $a_1 = a_2 = a_3 = a_4$. It follows that f is not semi-invariant with respect to a.

If $b_{13} \neq 0$, then semi-invariance with respect to the transformations α, β, γ yields $b_{13} = b_{23} = b_{24} = b_{14}$. As above, the polynomial f is not semi-invariant with respect to a.

Therefore, if f is semi-invariant, then $a_i = b_{13} = b_{23} = b_{24} = b_{14} = 0$ and $b_{12} = b_{34}$. In particular, all degree two semi-invariants are in fact invariant. There is a unique N_{72}-invariant quadric hypersurface in $\mathbb{P}_3$ given by the equation $x_1x_2 + x_3x_4$.

Analogous considerations show that a semi-invariant polynomial of degree three is a multiple of $f_{\text{Fermat}} = x_1^3 + x_2^3 + x_3^3 + x_4^3$ and the Fermat cubic $\{f_{\text{Fermat}} = 0\}$ is seen to be the unique N_{72}-invariant cubic hypersurface in $\mathbb{P}_3$.

A.2 The action of M_9 on $\mathbb{P}_2$

We consider the decompostion of $M_9 = (C_3 \times C_3) \rtimes Q_8$. The generators of $C_3 \times C_3$ are denoted a and b and the generators of Q_8 are denoted by I, J, K. Recall $I^2 = J^2 = K^2 = IJK = -1$. We choose the factorization of $C_3 \times C_3$ such that -1 acts as

$$(-1)a(-1) = a^2, \quad (-1)b(-1) = b^2.$$

Furthermore, $Ia(-I) = b$ and $Ja(-J) = b^2a$.

We repeatedly use the fact that the action of M_9 is induced by a symplectic action of M_9 on a K3-surface X which is a double cover of $\mathbb{P}_2$.

We begin by fixing a representation of a. Since a may not have a positive dimensional set of fixed points in $\mathbb{P}_2$, it follows that in appropriately chosen coordinates

$$\tilde{a} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi^2 \end{pmatrix},$$

where ξ is third root of unity.

As a next step, we want to specify a representation of b inside $\mathrm{SL}_3(\mathbb{C})$. Since a and b commute in $\mathrm{PSL}_3(\mathbb{C})$, we know that

$$\tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1} = \xi^k \mathrm{id}_{\mathbb{C}^3}$$

for $k \in \{0, 1, 2\}$. Note that $\tilde{b}$ is not diagonal in the coordinates chosen above since this would give rise to C_3^2-fixed points in $\mathbb{P}_2$. As these correspond to C_3^2-fixed points on the double cover $X \to Y$ and a symplectic action of $C_3^2 \not< \mathrm{SL}_2(\mathbb{C})$ on a K3-surface does not admit fixed points, this is a contradiction. An explicit calculation yields

$$\tilde{b} = \tilde{b}_1 = \begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix} \quad \text{or} \quad \tilde{b} = \tilde{b}_2 = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix}.$$

We can introduce a change of coordinates commuting with $\tilde{a}$ such that

$$\tilde{b} = \tilde{b}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad \tilde{b} = \tilde{b}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since $\tilde{b}_1 = \tilde{b}_2^2$, the two choices above correspond to choices of generators b and b^2 of $\langle b \rangle$ and are therefore equivalent. In the following we fix the second choice of b. A direct computation yields that the element -1 must be represented in the form

$$\begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}.$$

After reordering the coordinates, we can assume that

$$\widetilde{-1} = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix}.$$

The relation $(-1)b(-1) = b^2$ yields

$$\widetilde{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \eta \\ 0 & \eta^2 & 0 \end{pmatrix}.$$

for some third root of unity η. The element I fulfills $Ia(-I) = b$ and, using the representation of a and b given above, we conclude

$$\widetilde{I} = \frac{1}{\xi - \xi^2} \begin{pmatrix} 1 & 1 & 1 \\ \zeta^2 & \zeta^2\xi & \zeta^2\xi^2 \\ \zeta & \zeta\xi^2 & \zeta\xi \end{pmatrix}$$

for some third root of unity ζ. Now $I^2 = -1$ implies $\zeta = 1$ and $\eta = 1$. Analogous considerations yield the following shape of J:

$$\widetilde{J} = \frac{1}{\xi - \xi^2} \begin{pmatrix} 1 & \xi & \xi \\ \xi^2 & \xi & \xi^2 \\ \xi^2 & \xi^2 & \xi \end{pmatrix}.$$

In appropiately chosen coordinates the action on M_9 is precisely of the type claimed in Section 4.9.

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Index of Notation

$\mathcal{K}_X$ the canonical line bundle of X

K_X the canonical divisor of X

$\mathcal{O}_X$ the sheaf of holomorphic functions on X

$\mathcal{O}_X(D)$ the line bundle associated to the divisor D

$\mathrm{Aut}(X)$ the group of holomorphic automorphisms of X

ω_X the holomorphic 2-form on a K3-surface X

$NS(X)$ the Néron-Severi group of X

$\mathrm{Pic}(X)$ the Picard group of X

$\rho(X)$ the Picard number of X

$L \cdot C$ the intersection number of a line bundle L and a 1-cycle C

$\overline{NE}(X)$ the cone of curves on X

$\overline{NE}(X)^G$ the intersection of $\overline{NE}(X)$ with the space of invariant numerical equivalence classes of 1-cycles

cont_F the contraction of an extremal face F

$\pi_1(X)$ the fundamental group of X

$b_i(X)$ the i^{th} Betti number of X

$e(X)$ the topological Euler characteristic of X

$g(C)$ the (arithmetic) genus of a curve C

G_{symp} the subgroup of symplectic transformations in G

C_n the cyclic group of order n

D_{2n} the dihedral group of order $2n$

Q_8 the quaternion group

T_{12} the tetrahedral group

O_{24} the octahedral group

I_{60} the icosahedral group

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