On the local well-posedness of the Kadomtsev-Petviashvili II equation

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Chapter 1

Introduction

In this thesis we are concerned with the local well-posedness theory of the initial value problem for the Kadomtsev-Petviashvili II equation in two space dimensions

$$(u_t + u_{xxx} + (u^2)_x)_x + u_{yy} = 0 \quad \text{in } (-T,T) \times \mathbb{R}^2, \quad u(0) = u_0$$

as well as in three space dimensions

$$(u_t + u_{xxx} + (u^2)_x)_x + \Delta_{\vec{y}} u = 0 \quad \text{in } (-T,T) \times \mathbb{R}^3, \quad u(0) = u_0$$

and dispersive generalisations thereof.

The Kadomtsev-Petviashvili II equations are universal models for the propagation of long weakly dispersive waves which are essentially one dimensional with weak transverse effects.[1] They can be seen as multidimensional generalisations of the Korteweg-de Vries equation[2]

$$u_t + u_{xxx} + (u^2)_x = 0 \quad \text{in } (-T,T) \times \mathbb{R}, \quad u(0) = u_0$$

We consider initial values u_0 in non-isotropic Sobolev spaces $H^{s_1,s_2}(\mathbb{R}^d)$ and our goal is to show the local well-posedness for low regularity data, i. e. data in $H^{s_1,s_2}(\mathbb{R}^d)$ with s_1 and s_2 as small as possible. Our notion of well-posedness comprises, for given regularities s_1 and s_2, the *existence* and *uniqueness* of solutions in a suitable space of space-time functions (or more generally distributions) X_T, the *persistence of regularity*, i. e. the solution u is a continuous function in t with values in the Banach space $H^{s_1,s_2}(\mathbb{R}^d)$,

[1] See [16].

[2] For an explanation how the Kadomtsev-Petviashvili equations are (formally) obtained from the one dimensional models (also for more general dispersion terms), see also the introduction of [22].

and the *continuous dependence* of the solutions on the initial data, i. e. the *flow map*, which assigns the solution u to the initial value u_0, is a continuous mapping from $H^{s_1,s_2}(\mathbb{R}^d)$ to X_T. In fact, all flow maps turn out to be analytic mappings. This stems from the fact that we use a *Picard iteration method* on the Duhamel formulation[3] of the Kadomtsev-Petviashvili II equation to construct the solution and from the fact that the nonlinearity is polynomial.

The spaces X_T where the solutions are constructed are modifications of the spaces first used by BOURGAIN [5] in the context of the Kadomtsev-Petviashvili II equation (on $\mathbb{T}^2$ rather than on $\mathbb{R}^2$).[4] BOURGAIN'S idea was to include the symbol of the linear part of the equation into the definition of the spaces, which makes it possible to easily exploit dispersive properties of the linear equation in the context of these spaces and which also allows to exploit certain algebraic properties of the symbol in order to overcome the loss of derivatives in the nonlinearity. The proof of local well-posedness then reduces to showing a suitable estimate for the nonlinearity in these spaces.[5]

By using the Picard iteration method in the modified Bourgain spaces, we show the local well-posedness of the Kadomtsev-Petviashvili II equation in two space dimensions for $s_1 > -\frac{1}{2}$ and $s_2 \geq 0$. On the scale of spaces $H^{s_1,0}(\mathbb{R}^2)$ this includes the full subcritical range because the homogeneous space $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$ is scale invariant for this problem. Since it is not possible to obtain the crucial bilinear estimate in the standard Bourgain spaces for $-\frac{1}{2} < s_1 < -\frac{1}{3}$ which can be seen by the counterexamples in [31], we include a low frequency condition into the definition of the spaces.[6] The drawback of this low frequency condition is that the resulting spaces do not contain the (time localized) solutions of the linearized equation unless the initial value obeys the same low frequency condition. Therefore, we choose the space X_T to be the sum of the low-frequency modified space and a standard space. This sum structure is the crucial ingredient to be able to lower the x-regularity without imposing a low frequency condition on the initial values.[7]

By the same method, we show the local well-posedness of the Kadomtsev-

[3]More precisely, because the product in the nonlinearity does not make sense a priori for very rough initial values, we consider an operator equation which coincides with the Duhamel formulation for smooth functions. However, we will show in Theorem 3.3 that the solutions thus constructed are, in fact, distributional solutions of the original equation.

[4]These spaces have already been used by BOURGAIN [3, 4] in the context of the Korteweg-de Vries and the nonlinear Schrödinger equation.

[5]For a good overview of the general scheme how to prove local well-posedness of the equation from the multilinear estimates see [6] or the first part of [7].

[6]A similar condition was already used by TAKAOKA [30] to get local well-posedness in the range $-\frac{1}{2} < s_1 < -\frac{1}{3}$ but only if the initial value also satisfies a low frequency condition, i. e. for initial data in $H^{s_1,0}(\mathbb{R}^2) \cap \dot{H}^{-\frac{1}{2}+\varepsilon,0}(\mathbb{R}^2)$ with suitably chosen ε.

[7]Cf. also Remark 4.9.

Petviashvili II equation in three space dimensions for $s_1 > \frac{1}{2}$ and $s_2 > 0$. In this case, $\dot{H}^{\frac{1}{2},0}(\mathbb{R}^3)$ is scale invariant.

More generally, we prove, by the method described above, the local well-posedness of the *dispersion generalised Kadomtsev-Petviashvili II equation*

$$(u_t - |D_x|^\alpha u_x + (u^2)_x)_x + \Delta_{\vec{y}} u = 0 \quad \text{in } (-T,T) \times \mathbb{R}^d, \quad u(0) = u_0 \tag{1.1}$$

for $\frac{4}{3} < \alpha \leq 6$, if $d = 2$, and $2 \leq \alpha \leq 6$, if $d = 3$. These equations are multidimensional generalisations of the one dimensional models

$$u_t - |D_x|^\alpha u_x + (u^2)_x = 0 \quad \text{in } (-T,T) \times \mathbb{R}, \quad u(0) = u_0 \tag{1.2}$$

For $d = 2$, we obtain local well-posedness of (1.1) for

$$s_1 > \max\left(1 - \frac{3}{4}\alpha, \frac{1}{4} - \frac{3}{8}\alpha\right)$$

and $s_2 \geq 0$. Because the L^2-norm of real valued solutions of (1.1) is conserved, this immediately implies global well-posedness for real-valued initial data in $H^{s_1,0}(\mathbb{R}^2)$ for $s_1 \geq 0$. We note that if $\frac{4}{3} < \alpha < 2$, we still get the full subcritical range on the scale $H^{s_1,0}(\mathbb{R}^2)$. It is interesting that for these α the two dimensional models "behave better" than the one dimensional equation (1.2) in the sense that the flow map of the one dimensional model cannot be C^2-differentiable at the origin in any Sobolev space $H^s(\mathbb{R})$. This also means that it is not possible to solve (1.2) in $H^s(\mathbb{R})$ with a Picard iteration scheme.[8]

The case $\alpha = 4$ is also known as *fifth order Kadomtsev-Petviashvili II equation*

$$(u_t - u_{xxxxx} + (u^2)_x)_x + u_{yy} = 0 \quad \text{in } (-T,T) \times \mathbb{R}^2, \quad u(0) = u_0$$

Our result in this case shows local well-posedness for $s_1 > -\frac{5}{4}$ and $s_2 \geq 0$.[9]

For $d = 3$ and $2 < \alpha \leq 6$, we obtain local well-posedness of (1.1) for

$$s_1 > \max\left(\frac{3}{2} - \frac{\alpha}{2}, \frac{1}{4} - \frac{5}{24}\alpha\right)$$

and $s_2 \geq 0$. As in the two dimensional case, the global well-posedness for real-valued initial data in $H^{s_1,0}(\mathbb{R}^3)$ for $s_1 \geq 0$ and $\alpha > 3$ follows.

[8] This has been proven by MOLINET, SAUT AND TZVETKOV [21]. Note, however, that a Picard iteration has been applied by HERR (cf. [8], Chapter 4) to prove well-posedness for initial values in Sobolev spaces which include a low frequency condition.

[9] Note that well-posedness for the same class of initial data has recently been obtained by ISAZA, LÓPEZ AND MEJÍA [12].

In the case $\alpha = 4$ of the *fifth order Kadomtsev-Petviashvili II equation in three space dimensions*

$$(u_t - u_{xxxxx} + (u^2)_x)_x + \Delta_{\vec{y}} u = 0 \quad \text{in } (-T,T) \times \mathbb{R}^3, \quad u(0) = u_0$$

our result shows the local well-posedness for $s_1 > -\frac{1}{2}$ and $s_2 \geq 0$.

We now give an overview of the organization of this thesis:

In Chapter 2, after fixing some notation, we introduce the Bourgain spaces and show a general well-posedness result which reduces the question of local well-posedness for equation (1.1) to a bilinear estimate in the Bourgain spaces.

In Chapter 3, we first give local smoothing as well as Strichartz estimates for solutions of the linear equation

$$(u_t - |D_x|^\alpha u_x)_x + \Delta_{\vec{y}} u = 0 \quad \text{in } (-T,T) \times \mathbb{R}^d, \quad u(0) = u_0$$

It is shown that the local smoothing estimate implies that solutions of the Duhamel formulation of (1.1) are actually solutions in the distributional sense. In Section 3.3 we give an overview of the techniques used to derive bilinear Strichartz type estimates and discuss some of their properties. Finally, in Section 3.4 and Section 3.5 we derive bilinear Strichartz type estimates in the two dimensional, respectively three dimensional case. These estimates are the building blocks used to derive the bilinear estimate which is needed to apply the general well-posedness result of Section 2.4.

In Chapter 4, the main results for the two dimensional case are proven. The main bilinear estimate for the two dimensional case is announced in Section 4.2 and proven in Section 4.3 and Section 4.4. This is done by first splitting the nonlinearity into various pieces and then using for each piece a pointwise estimate to reduce the case to an appropriate bilinear Strichartz type estimate of Section 3.4.

In Chapter 5, the main results for the three dimensional case are proven. This is done analogously to the two dimensional case.

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Chapter 2

Bourgain spaces and well-posedness

2.1 Preliminaries

Let us first recall some known facts about standard function spaces and fix some notation that will be used throughout this thesis:

- For $x \in \mathbb{R}^n$ let $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$.

- Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space, i. e. the space of all $u \in C^\infty(\mathbb{R}^n)$ such that for all $j \in \mathbb{N}$

$$q_j(u) := \max_{|\gamma| \le j} \max_{x \in \mathbb{R}^n} \langle x \rangle^j |\partial^\gamma u(x)| < \infty \tag{2.1}$$

  It is well known that, endowed with this family of seminorms, $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space, i. e. a completely metrizable topological vector space. The dual space $\mathcal{S}'(\mathbb{R}^n)$ is called the space of tempered distributions on $\mathbb{R}^n$.

- d always denotes the number of space variables in the equation, i. e. $d = 2$ when we consider the two dimensional case and $d = 3$ when we consider the three dimensional case. The space variable will always be denoted by $(x, \vec{y})$ where $x \in \mathbb{R}$ and $\vec{y} \in \mathbb{R}^{d-1}$. If we consider the case $d = 2$, we will often write y instead of $\vec{y}$. If we consider the case $d = 3$, we will write $\vec{y} = (y, \tilde{y})$.

- $n := d + 1$ always denotes the number of total variables (including the time variable t) in the equation.

- For $u \in L^1(\mathbb{R}^n)$ the *Fourier transform* $\mathcal{F}u$ of u is defined as

$$(\mathcal{F}u)(\tau, \xi, \vec{\eta}) := \int_{\mathbb{R}^n} e^{-i(t\tau + x\xi + \vec{y}\cdot\vec{\eta})} u(t, x, \vec{y}) dt dx d\vec{y}, (\tau, \xi, \vec{\eta}) \in \mathbb{R}^n \quad (2.2)$$

  It is well known that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a topological and linear isomorphism with

$$(\mathcal{F}^{-1}v)(t, x, \vec{y}) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(t\tau + x\xi + \vec{y}\cdot\vec{\eta})} v(\tau, \xi, \vec{\eta}) d\tau d\xi d\vec{\eta} \quad (2.3)$$

  for $v \in \mathcal{S}(\mathbb{R}^n)$. Furthermore, $\mathcal{F}$ can be extended to a linear and continuous isomorphism on $\mathcal{S}'(\mathbb{R}^n)$. If we only consider a partial Fourier transform in some of the variables, we will denote this by $\mathcal{F}_1$ for the Fourier transform in the first variable, etc.

- For $s \in \mathbb{R}$ we define the operators J_x^s, $J_{\vec{y}}^s$, and $|D_x|^s$ as *Fourier multiplier operators* with multiplier $\langle \xi \rangle^s$, $\langle \vec{y} \rangle^s$, and $|\xi|^s$, respectively. This means, for example, that $(\mathcal{F}_2 J_x^s u)(t, \xi, \vec{y}) = \langle \xi \rangle^s \mathcal{F}_2 u(t, \xi, \vec{y})$, $\xi \in \mathbb{R}$.

- The (non-isotropic) Sobolev space $H^{s_1,s_2}(\mathbb{R}^d)$ is the space of $u_0 \in \mathcal{S}'(\mathbb{R}^d)$ such that the norm

$$\|u_0\|_{H^{s_1,s_2}} := \|\langle \xi \rangle^{s_1} \langle \vec{\eta} \rangle^{s_2} \mathcal{F}u_0\|_{L^2_{\xi\vec{\eta}}} \quad (2.4)$$

  is finite.

- $\mu = (\tau, \xi, \vec{\eta}) \in \mathbb{R}^3$ always denotes the Fourier variable dual to $(t, x, \vec{y})$. In the case $d = 2$ we will again write η instead of $\vec{\eta}$. In the case $d = 3$ we will write $\vec{\eta} = (\eta, \tilde{\eta})$.

- For $\mu = (\tau, \xi, \vec{\eta})$ let

$$\lambda := \lambda(\mu) := \tau - \xi|\xi|^\alpha + \frac{\vec{\eta}^2}{\xi} \quad (2.5)$$

  where $\vec{\eta}^2 := \vec{\eta} \cdot \vec{\eta}$ is the scalar product. If there are two frequency variables μ and μ_1, we will write $\mu_2 := \mu - \mu_1$, $\lambda_1 := \lambda(\mu_1)$, $\lambda_2 := \lambda(\mu_2)$ for short. The elements of μ_2 are also denoted by $(\tau_2, \xi_2, \vec{\eta}_2)$. Furthermore, let $|\lambda_{\max}| := \max(|\lambda|, |\lambda_1|, |\lambda_2|)$, $|\xi_{\max}| := \max(|\xi|, |\xi_1|, |\xi_2|)$, and $|\xi_{\min}| := \min(|\xi|, |\xi_1|, |\xi_2|)$.

- $A \lesssim B$ means that there is a (harmless) constant C such that $A \leq CB$. $A \sim B$ is equivalent to $A \lesssim B$ and $B \lesssim A$.

- For a Banach space X and a Hausdorff topological vector space $\mathcal{A}$, the notation $X \hookrightarrow \mathcal{A}$ means that there is a continuous embedding from X into $\mathcal{A}$. Let $C_b(\mathbb{R};X)$ denote the Banach space of all continuous and bounded functions $f:\mathbb{R}\to X$ with the sup-norm. If X_1 and X_2 are Banach spaces with $X_i \hookrightarrow \mathcal{A}$, where $\mathcal{A}$ is a Hausdorff topological vector space, we will often consider the two Banach spaces $X_1 \cap X_2$, endowed with the norm

$$\|x\|_{X_1\cap X_2} := \|x\|_{X_1} + \|x\|_{X_2}, \quad x \in X_1 \cap X_2 \tag{2.6}$$

and $X_1 + X_2 := \{x \in \mathcal{A} \mid x = x_1 + x_2, x_i \in X_i (i = 1,2)\}$, endowed with the norm

$$\|x\|_{X_1+X_2} := \inf\{\|x_1\|_{X_1} + \|x_2\|_{X_2} \mid x = x_1 + x_2, x_i \in X_i (i = 1,2)\} \tag{2.7}$$

2.2 Bourgain spaces

In this section we define the function spaces which are adapted to the linear part of equation (1.1). As the symbol of the linear operator has a singularity along $\xi = 0$ and as we want to be able to deal with a low frequency condition in ξ, we will consider the following space of test functions.

Definition 2.1.

$$\mathcal{S}_{-\infty} := \{\phi \in \mathcal{S}(\mathbb{R}^n) | \partial_\xi^k \mathcal{F}\phi(\tau, 0, \vec{\eta}) = 0 \ \forall k \in \mathbb{N}_0 \ \forall (\tau,\vec{\eta}) \in \mathbb{R}^{n-1}\} \tag{2.8}$$

Remark 2.2. $\mathcal{S}_{-\infty}$ is a closed subspace of the Fréchet space $\mathcal{S}(\mathbb{R}^n)$. The functions in $\mathcal{S}_{-\infty}$ have the property that for $k \in \mathbb{N}_0$ and for $(\tau,\xi,\vec{\eta}) \in \mathbb{R}^n$, we have $|\mathcal{F}\phi(\tau,\xi,\vec{\eta})| \le q_k(\phi)|\xi|^k$.

Therefore, for $s_1, s_2, b, \sigma \in \mathbb{R}$, the following definition makes sense.

Definition 2.3. Let $s_1, s_2, b, \sigma \in \mathbb{R}$. For $\phi \in \mathcal{S}_{-\infty}$ let

$$\|\phi\|_{X_\sigma^{b,s_1,s_2}} := \| |\xi|^{-\sigma} \langle\xi\rangle^{s_1+\sigma} \langle\vec{\eta}\rangle^{s_2} \langle\lambda\rangle^b \mathcal{F}\phi\|_{L^2_\mu} \tag{2.9}$$

with λ as defined in (2.5). We define the space X_σ^{b,s_1,s_2} as the completion of $\mathcal{S}_{-\infty}$ with respect to the norm (2.9).

Remark 2.4. If $s_2 = 0$, we simply write X_σ^{b,s_1} instead of $X_\sigma^{b,s_1,0}$.

We can identify X_σ^{b,s_1,s_2} with a subspace of tempered distributions on $\mathbb{R}^n$, at least for $\sigma > -\frac{1}{2}$ and $b > -\frac{1}{2} - \sigma$. In order to prove this, we shall need the following lemma.

Lemma 2.5. *Let $s_1, s_2, b, \sigma \in \mathbb{R}$ with $\sigma > -\frac{1}{2}$ and $b > -\frac{1}{2} - \sigma$. Then there exists a $j = j(s_1, s_2, b, \sigma, \alpha)$ such that for all $\phi \in \mathcal{S}_{-\infty}$ and all $\psi \in \mathcal{S}(\mathbb{R}^n)$*

$$|\langle \phi, \psi \rangle_{\mathcal{S}',\mathcal{S}}| \lesssim \|\phi\|_{X_\sigma^{b,s_1,s_2}} q_j(\psi) \tag{2.10}$$

Proof. Let

$$k(\mu) := |\xi|^{-\sigma} \langle \xi \rangle^{s_1+\sigma} \langle \vec{\eta} \rangle^{s_2} \langle \lambda \rangle^b \tag{2.11}$$

By Plancherel's theorem and the Cauchy-Schwarz inequality, it holds that

$$|\langle \phi, \psi \rangle_{\mathcal{S}',\mathcal{S}}| = c \left| \int_{\mathbb{R}^n} k(\mu) \mathcal{F}\phi(\mu) k(\mu)^{-1} \overline{\mathcal{F}\overline{\psi}(\mu)} d\mu \right| \lesssim \|\phi\|_{X_\sigma^{b,s_1,s_2}} \|k^{-1} \mathcal{F}\overline{\psi}\|_{L^2_\mu} \tag{2.12}$$

Now, there exists a $j' = j'(s_1, s_2, b, \sigma, \alpha) \in \mathbb{N}$ such that

$$k(\mu)^{-1} \lesssim (|\xi|^\sigma + |\xi|^{\sigma+\min(0,b)}) \langle \tau \rangle^{-1} \langle \xi \rangle^{-1-\sigma} \langle \vec{\eta} \rangle^{-2} \langle \mu \rangle^{j'}$$

Since $\sigma > -\frac{1}{2}$ and $b > -\frac{1}{2} - \sigma$, this implies that $\langle \cdot \rangle^{-j'} k^{-1} \in L^2_\mu$. Therefore, we have for every $\tilde{\psi} \in \mathcal{S}(\mathbb{R}^n)$ that

$$\|k^{-1} \tilde{\psi}\|_{L^2_\mu} \lesssim q_{j'}(\tilde{\psi}) \tag{2.13}$$

In particular, we get for $\tilde{\psi} = \mathcal{F}\overline{\psi}$ that

$$\|k^{-1} \mathcal{F}\overline{\psi}\|_{L^2_\mu} \lesssim q_{j'}(\mathcal{F}\overline{\psi}) \lesssim q_j(\psi) \tag{2.14}$$

where the last inequality follows for some $j = j(j') \in \mathbb{N}$ because of the continuity of the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$. Now, (2.10) follows from (2.12) and (2.14). □

Proposition 2.6. *For $\sigma > -\frac{1}{2}$ and $b > -\frac{1}{2} - \sigma$, it holds that*

$$X_\sigma^{b,s_1,s_2} = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid |\xi|^{-\sigma} \langle \xi \rangle^{s_1+\sigma} \langle \vec{\eta} \rangle^{s_2} \langle \lambda \rangle^b \mathcal{F}u \in L^2(\mathbb{R}^n)\} \tag{2.15}$$

Moreover, we have for all $u \in X_\sigma^{b,s_1,s_2}$

$$\|u\|_{X_\sigma^{b,s_1,s_2}} = \||\xi|^{-\sigma} \langle \xi \rangle^{s_1+\sigma} \langle \vec{\eta} \rangle^{s_2} \langle \lambda \rangle^b \mathcal{F}u\|_{L^2_\mu} \tag{2.16}$$

Proof. Let us first suppose that $u \in X_\sigma^{b,s_1,s_2}$. We show that we can identify u with a tempered distribution such that (2.16) holds. Let $(\phi_l)_{l\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{S}_{-\infty}$ with respect to the norm (2.9) in the equivalence class defined by u in X_σ^{b,s_1,s_2}. By (2.10), we have for all $l, m \in \mathbb{N}$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ that

$$|\langle \phi_l - \phi_m, \psi \rangle_{\mathcal{S}',\mathcal{S}}| \lesssim \|\phi_l - \phi_m\|_{X_\sigma^{b,s_1,s_2}} q_j(\psi)$$

This implies that $(\langle\phi_l,\psi\rangle_{\mathcal{S}',\mathcal{S}})_{l\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for all $\psi\in\mathcal{S}(\mathbb{R}^n)$. We can therefore define a linear functional u on $\mathcal{S}(\mathbb{R}^n)$ by letting

$$\langle u,\psi\rangle_{\mathcal{S}',\mathcal{S}} := \lim_{l\to\infty}\langle\phi_l,\psi\rangle_{\mathcal{S}',\mathcal{S}},\quad \psi\in\mathcal{S}(\mathbb{R}^n) \tag{2.17}$$

u does not depend on the special choice of the Cauchy sequence (ϕ_l) but only on its equivalence class with respect to the norm (2.9). This follows from the standard argument of mixing two Cauchy sequences from the same equivalence class. Let us show that u is in fact a tempered distribution. (2.10) implies $|\langle\phi_l,\psi\rangle_{\mathcal{S}',\mathcal{S}}| \lesssim \|\phi_l\|_{X_\sigma^{b,s_1,s_2}}q_j(\psi)$ for $l\in\mathbb{N}$ and $\psi\in\mathcal{S}(\mathbb{R}^n)$. Letting $l\to\infty$ in the last inequality, we obtain $|\langle u,\psi\rangle_{\mathcal{S}',\mathcal{S}}| \lesssim \|u\|_{X_\sigma^{b,s_1,s_2}}q_j(\psi)$ which implies $u\in\mathcal{S}'(\mathbb{R}^n)$. We will now show (2.16). Let $f_l := k\mathcal{F}\phi_l$ for $l\in\mathbb{N}$ with k as defined in (2.11). Due to Definition 2.3, we have that f_l is a Cauchy sequence in $L^2(\mathbb{R}^n)$. Therefore, there is an $f\in L^2(\mathbb{R}^n)$ with $\|f_l-f\|_{L^2_\mu}\to 0$. For $\psi\in\mathcal{S}(\mathbb{R}^n)$, we have

$$\langle\mathcal{F}u,\psi\rangle_{\mathcal{S}',\mathcal{S}} = \lim_{l\to\infty}\langle\mathcal{F}\phi_l,\psi\rangle_{\mathcal{S}',\mathcal{S}} = \lim_{l\to\infty}\int_{\mathbb{R}^n} f_l(\mu)k(\mu)^{-1}\psi(\mu)d\mu$$

However, it follows by (2.13) that $k^{-1}\psi\in L^2(\mathbb{R}^n)$. Therefore, letting $l\to\infty$, we obtain

$$\langle\mathcal{F}u,\psi\rangle_{\mathcal{S}',\mathcal{S}} = \int_{\mathbb{R}^n} f(\mu)k(\mu)^{-1}\psi(\mu)d\mu$$

i. e. $\mathcal{F}u = fk^{-1}$ is a regular distribution and $k\mathcal{F}u = f\in L^2(\mathbb{R}^n)$. Furthermore,

$$\|u\|_{X_\sigma^{b,s_1,s_2}} = \lim_{l\to\infty}\|\phi_l\|_{X_\sigma^{b,s_1,s_2}} = \lim_{l\to\infty}\|f_l\|_{L^2} = \|f\|_{L^2}$$

which proves (2.16). From (2.16), it follows immediately that the identification operator, which maps the equivalence class $u\in X_\sigma^{b,s_1,s_2}$ to the distribution u, is injective.

Let us now show that this identification operator is onto $\tilde{X}$, where

$$\tilde{X} := \{u\in\mathcal{S}'(\mathbb{R}^n) \mid |\xi|^{-\sigma}\langle\xi\rangle^{s_1+\sigma}\langle\vec{\eta}\rangle^{s_2}\langle\lambda\rangle^b\mathcal{F}u\in L^2(\mathbb{R}^n)\} \tag{2.18}$$

We suppose that $u\in\mathcal{S}'(\mathbb{R}^n)$ such that $f := k\mathcal{F}u\in L^2(\mathbb{R}^n)$. Let

$$\tilde{\mathcal{S}} := \{\phi\in\mathcal{S}(\mathbb{R}^n)|\exists\varepsilon>0\forall(\tau,\xi,\vec{\eta})\in\mathbb{R}^n\ :\ |\xi|<\varepsilon \implies \mathcal{F}\phi(\tau,\xi,\vec{\eta})=0\}$$

We obviously have that $\tilde{\mathcal{S}}\subset\mathcal{S}_{-\infty}$ and $\mathcal{F}\tilde{\mathcal{S}}$ is dense in $L^2(\mathbb{R}^n)$. Therefore, there exists a sequence $(f_l)_{l\in\mathbb{N}}$ in $\mathcal{F}\tilde{\mathcal{S}}$ such that $\|f_l-f\|_{L^2_\mu}\to 0$ for $l\to\infty$. Let $\phi_l := \mathcal{F}^{-1}(k^{-1}f_l)$ for $l\in\mathbb{N}$. Then $\phi_l\in\tilde{\mathcal{S}}\subset\mathcal{S}_{-\infty}$ and

$$\|\phi_l-\phi_m\|_{X_\sigma^{b,s_1,s_2}} = \|f_l-f_m\|_{L^2}$$

Therefore, $(\phi_l)_{l\in\mathbb{N}}$ is a Cauchy sequence with respect to the norm (2.9) and defines an element of X_σ^{b,s_1,s_2}. It remains to prove that

$$\langle u,\psi\rangle_{\mathcal{S}',\mathcal{S}} = \lim_{l\to\infty}\langle \phi_l,\psi\rangle_{\mathcal{S}',\mathcal{S}}$$

This follows from

$$\begin{aligned} |\langle u-\phi_l,\psi\rangle_{\mathcal{S}',\mathcal{S}}| &= c|\langle \mathcal{F}(u-\phi_l),\mathcal{F}^{-1}\psi\rangle_{\mathcal{S}',\mathcal{S}}| \\ &= c\left|\int_{\mathbb{R}^n}(f-f_l)(\mu)k(\mu)^{-1}(\mathcal{F}^{-1}\psi)(\mu)d\mu\right| \\ &\lesssim \|f_l-f\|_{L^2}q_{j'}(\mathcal{F}^{-1}\psi)\to 0 \quad (l\to\infty) \end{aligned}$$

where we used (2.13) for the last inequality. □

In the following, we always assume that $\sigma > -\frac{1}{2}$ and $b > -\frac{1}{2}-\sigma$ hold, so that we can always compute the norm of elements $u\in X_\sigma^{b,s_1,s_2}$ by (2.16).

The following embedding property of the X_σ^{b,s_1,s_2}-spaces is obvious.

Lemma 2.7. *Let $s_1,s_2,\sigma,b,s_1',s_2',\sigma',b'\in\mathbb{R}$ with $s_1'\geq s_1$,$s_2'\geq s_2$,$\sigma'\geq\sigma$ and $b'\geq b$. Then $X_{\sigma'}^{b',s_1',s_2'}\hookrightarrow X_\sigma^{b,s_1,s_2}$ and*

$$\|u\|_{X_\sigma^{b,s_1,s_2}} \leq \|u\|_{X_{\sigma'}^{b',s_1',s_2'}} \tag{2.19}$$

for every $u\in X_{\sigma'}^{b',s_1',s_2'}$.

For $b>\frac{1}{2}$ we also have the following embedding of the X_σ^{b,s_1,s_2}-spaces into spaces of bounded and continuous vector-valued functions in t.

Proposition 2.8. *Let $s_1,s_2\in\mathbb{R}$, $\sigma\geq 0$ and $b>\frac{1}{2}$. Then*

$$X_\sigma^{b,s_1,s_2}\hookrightarrow C_b(\mathbb{R};H^{s_1,s_2}(\mathbb{R}^d))$$

Proof. For $\sigma=0$ see, for example, [7], Lemma 1.5. For $\sigma>0$ we combine this with Lemma 2.7 to get

$$X_\sigma^{b,s_1,s_2}\hookrightarrow X_0^{b,s_1,s_2}\hookrightarrow C_b(\mathbb{R};H^{s_1,s_2}(\mathbb{R}^d))$$

□

Definition 2.9. Let $X\hookrightarrow\mathcal{D}'(\mathbb{R}^n)$. For $T>0$ we define the *restriction operator* $R_T: X\to\mathcal{D}'((-T,T)\times\mathbb{R}^d)$, $u\mapsto u\big|_{(-T,T)\times\mathbb{R}^d}$. Furthermore, we define the space X_T to be the quotient space $X/\mathcal{N}_T$, where

$$\mathcal{N}_T=\{u\in X\mid R_Tu=0\}$$

Remark 2.10. The norm in X_T is given by

$$\|u\|_{X_T} = \inf\{\|\tilde{u}\|_X \mid \tilde{u} \in X,\ \tilde{u}\big|_{(-T,T)\times\mathbb{R}^d} = u\}$$

We note that, under the assumptions of Proposition 2.8, we have that $(X_\sigma^{b,s_1,s_2})_T \hookrightarrow C([-T,T];H^{s_1,s_2}(\mathbb{R}^d))$. For $u \in X$, we often simply write u instead of $R_T u$ to denote the corresponding element of X_T. It will always be clear from the context what is meant.

2.3 The linear equation

In this section we consider the linear equation

$$(u_t - |D_x|^\alpha u_x)_x + \Delta_{\vec{y}} u = 0 \quad \text{in } (-T,T)\times\mathbb{R}^d, \quad u(0) = u_0 \tag{2.20}$$

and derive some of its properties.

Definition 2.11. Let $s_1, s_2 \in \mathbb{R}$ and $\alpha > 0$. We define a unitary group $(U_\alpha(t))_{t\in\mathbb{R}}$ on $H^{s_1,s_2}(\mathbb{R}^d)$ by

$$\mathcal{F}(U_\alpha(t)u_0)(\xi,\vec{\eta}) = e^{itp_\alpha(\xi,\vec{\eta})}(\mathcal{F}u_0)(\xi,\vec{\eta}), \quad u_0 \in H^{s_1,s_2}(\mathbb{R}^d) \tag{2.21}$$

where the phase function p_α is defined by

$$p_\alpha(\xi,\vec{\eta}) := \xi|\xi|^\alpha - \frac{\vec{\eta}^2}{\xi} \tag{2.22}$$

Remark 2.12. Note that $u(t) := U_\alpha(t)u_0$ is only formally a solution of (2.20) for $u_0 \in H^{s_1,s_2}(\mathbb{R}^d)$ because, in general, u is not differentiable as a function with values in any space $H^{s_1',s_2'}(\mathbb{R}^d)$ due to the singularity of p_α along $\xi = 0$. We can deal with this problem in two ways:

a) We can restrict to initial values u_0 in the space

$$H^{s_1,s_2}_{-1}(\mathbb{R}^d) := \{u_0 \in H^{s_1,s_2} \mid \partial_x^{-1}u_0 := \mathcal{F}^{-1}(-i\xi^{-1}\mathcal{F}u_0) \in H^{s_1,s_2}\}$$

endowed with the norm $\|u_0\|_{H^{s_1,s_2}_{-1}} := \|u_0\|_{H^{s_1,s_2}} + \|\partial_x^{-1}u_0\|_{H^{s_1,s_2}}$. If, for $u_0 \in H^{s_1,s_2}_{-1}(\mathbb{R}^d)$, we let $u(t) := U_\alpha(t)u_0$, $t \in (-T,T)$, we can easily check that $u \in C^1((-T,T);H^{s_1-(\alpha+1),s_2-2})$ and that (2.20) holds in $H^{s_1-(\alpha+2),s_2-2}$ for all $t \in (-T,T)$. We will not pursue this approach because we do not want to put any low frequency condition on the initial values.

b) We can change the order of differentiation in the first term in (2.20). We easily verify that if $u_0 \in H^{s_1,s_2}(\mathbb{R}^d)$ and $u(t) := U_\alpha(t)u_0$, $t \in (-T,T)$, then $\partial_x u \in C^1((-T,T); H^{s_1-(\alpha+2),s_2-2})$ and for all $t \in (-T,T)$ we have

$$\partial_t \partial_x u(t) = |D_x|^\alpha u_{xx}(t) - \Delta_{\vec{y}} u(t)$$

This also implies that (2.20) is fulfilled in the sense of distributions, which, for $s_1, s_2 \geq 0$, means that for all $\varphi \in C_c^\infty((-T,T) \times \mathbb{R}^d)$

$$\int_{\mathbb{R}^n} u(\varphi_{tx} - |D_x|^\alpha \varphi_{xx} + \Delta_{\vec{y}}\varphi) dt dx d\vec{y} = 0 \tag{2.23}$$

We now show how the action of the linear group U_α on initial values whose ξ-frequency is localized in an annulus $|\xi| \sim 2^k$ can be reduced to the special case $k = 0$ by scaling. This result will be used in Section 3.2 to derive Strichartz estimates for the solution of the linear equation. We first introduce the Littlewood-Paley and scaling operators.

Definition 2.13. Let $\varphi \in C_c^\infty(\mathbb{R})$ be such that $0 \leq \varphi \leq 1$, $\varphi(-\xi) = \varphi(\xi)$, and

$$\varphi(\xi) = \begin{cases} 1, & |\xi| \leq 1 \\ 0, & |\xi| \geq 2 \end{cases} \tag{2.24}$$

If we let $\psi_k(\xi) := \varphi(2^{-k}\xi) - \varphi(2^{1-k}\xi)$ for $k \in \mathbb{Z}$, $\xi \in \mathbb{R}$, then $0 \leq \psi_k \leq 1$, $\psi_k(-\xi) = \psi_k(\xi)$, $\psi_k(\xi) = 0$ for $|\xi| \notin (2^{k-1}, 2^{k+1})$, and

$$\sum_{k \in \mathbb{Z}} \psi_k(\xi) = 1 \quad \text{for } \xi \in \mathbb{R} \setminus \{0\} \tag{2.25}$$

Furthermore, if $\varphi_0 := \varphi$ and $\varphi_k := \psi_k$ for $k \in \mathbb{N}$, then

$$\sum_{k \in \mathbb{N}_0} \varphi_k(\xi) = 1 \quad \text{for } \xi \in \mathbb{R} \tag{2.26}$$

If we also let $\tilde{\psi}_k := \psi_{k-1} + \psi_k + \psi_{k+1}$, then $\tilde{\psi}_k \psi_k = \psi_k$ for $k \in \mathbb{Z}$.

Definition 2.14. For $k \in \mathbb{Z}$ and $\delta > 0$ define Δ_k, $\tilde{\Delta}_k$ and S_δ by

$$\begin{aligned} (S_\delta u)(t, x, \vec{y}) &= u(t, \delta x, \delta^{\frac{\alpha}{2}+1}\vec{y}) \\ (\mathcal{F}_2 \Delta_k u)(t, \xi, \vec{y}) &= \psi_k(\xi) \mathcal{F}_2 u(t, \xi, \vec{y}) \\ \tilde{\Delta}_k &= \Delta_{k-1} + \Delta_k + \Delta_{k+1} \end{aligned}$$

Proposition 2.15. *For every $k \in \mathbb{Z}$, it holds that*

$$U_\alpha(t)\tilde{\Delta}_k = S_{2^k} U_\alpha(2^{(\alpha+1)k} t) \tilde{\Delta}_0 S_{2^{-k}}$$

Proof. For $u_0 \in \mathcal{S}(\mathbb{R}^d)$, it holds that

$$(U_\alpha(t)\tilde{\Delta}_k u_0)(x,\vec{y}) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\xi + i\vec{y}\cdot\vec{\eta}} e^{itp_\alpha(\xi,\vec{\eta})} \tilde{\psi}_k(\xi)(\mathcal{F}u_0)(\xi,\vec{\eta}) d\xi d\vec{\eta}$$

Using the change of variables $\xi = 2^k \xi'$, $\vec{\eta} = 2^{\left(\frac{\alpha}{2}+1\right)k}\vec{\eta}'$, we see that the last integral is equal to

$$\begin{aligned}(2\pi)^{-d} \int_{\mathbb{R}^d} & e^{i2^k x\xi' + i2^{\left(\frac{\alpha}{2}+1\right)k}\vec{y}\cdot\vec{\eta}'} e^{i2^{(\alpha+1)k}tp_\alpha(\xi',\vec{\eta}')} \tilde{\psi}_0(\xi')(\mathcal{F}S_{2^{-k}}u_0)(\xi',\vec{\eta}')d\xi' d\vec{\eta}' \\ &= (S_{2^k}U_\alpha(2^{(\alpha+1)k}t)\tilde{\Delta}_0 S_{2^{-k}}u_0)(x,\vec{y})\end{aligned}$$

The claim follows by the density of $\mathcal{S}(\mathbb{R}^d)$ in $H^{s_1,s_2}(\mathbb{R}^d)$. □

Proposition 2.16. *For* $1 \le q, r \le \infty$ *let*

$$\bar{\gamma}(d,\alpha,q,r) = \left(\left(\frac{\alpha}{2}+1\right)d - \frac{\alpha}{2}\right)\left(\frac{1}{2}-\frac{1}{r}\right) - (\alpha+1)\frac{1}{q} \tag{2.27}$$

and suppose that

$$\|U_\alpha(t)\tilde{\Delta}_0 u_0\|_{L_t^q L_{x\vec{y}}^r} \lesssim \|u_0\|_{L^2} \tag{2.28}$$

Then it follows that

$$\|U_\alpha(t)\tilde{\Delta}_k u_0\|_{L_t^q L_{x\vec{y}}^r} \lesssim 2^{k\bar{\gamma}(d,\alpha,q,r)}\|u_0\|_{L^2} \tag{2.29}$$

for every $k \in \mathbb{Z}$.

Proof. (2.29) follows from (2.28), Proposition 2.15 and the facts that

$$\begin{aligned}\|S_\delta v_0\|_{L_{x\vec{y}}^r} &= \delta^{-\left(\left(\frac{\alpha}{2}+1\right)d - \frac{\alpha}{2}\right)\frac{1}{r}}\|v_0\|_{L_{x\vec{y}}^r} \\ \|U_\alpha(2^{(\alpha+1)k}t)v_0\|_{L_t^q L_{x\vec{y}}^r} &= 2^{-k(\alpha+1)\frac{1}{q}}\|U_\alpha(t)v_0\|_{L_t^q L_{x\vec{y}}^r}\end{aligned}$$

□

Definition 2.17. Let $s_1, s_2 \in \mathbb{R}$. Let $L : H^{s_1,s_2}(\mathbb{R}^d) \to C_b(\mathbb{R}, H^{s_1,s_2}(\mathbb{R}^d))$ be defined by

$$(Lu_0)(t) := U_\alpha(t)u_0 \tag{2.30}$$

Also, for $T \in (0,1]$ let $L_T : H^{s_1,s_2}(\mathbb{R}^d) \to C_b(\mathbb{R}, H^{s_1,s_2}(\mathbb{R}^d))$ be defined by

$$(L_T u_0)(t) := \zeta_T(t)U_\alpha(t)u_0 \tag{2.31}$$

where $\zeta \in C_c^\infty((-2,2))$ with $\zeta_{|[-1,1]} = 1$ and $\zeta_T := \zeta(\cdot/T)$.

Definition 2.18. Let $(s_1, s_2) \in \mathbb{R}^2$. For $f \in L^2(\mathbb{R}, H^{s_1,s_2}(\mathbb{R}^d)) = X_0^{0,s_1,s_2}$ let

$$(\Gamma f)(t) := \int_0^t U_\alpha(t-t')f(t')dt', \quad t \in \mathbb{R} \tag{2.32}$$

Furthermore, for $T \in (0,1]$ let

$$(\Gamma_T f)(t) := \zeta_T(t) \int_0^t U_\alpha(t-t')f(t')dt', \quad t \in \mathbb{R} \tag{2.33}$$

where ζ_T is as in Definition 2.17.

Remark 2.19. If $s_1, s_2, s_1', s_2' \in \mathbb{R}$ and $f \in X_0^{0,s_1,s_2} \cap X_0^{0,s_1',s_2'}$, then Γf and $\Gamma_T f$ do not depend on whether we compute the integral in $H^{s_1,s_2}(\mathbb{R}^d)$ or in $H^{s_1',s_2'}(\mathbb{R}^d)$.

We have the following well-known linear estimates.

Proposition 2.20. *For $b \geq 0$ and $s_1, s_2 \in \mathbb{R}$, it holds that*

$$\|L_1 u_0\|_{X_0^{b,s_1,s_2}} \lesssim \|u_0\|_{H^{s_1,s_2}(\mathbb{R}^2)} \tag{2.34}$$

Proof. See for example [6]. □

Proposition 2.21. *For $-\frac{1}{2} < b' \leq 0 \leq b \leq b'+1$, $T \leq 1$ and $s_1, s_2 \in \mathbb{R}$, the operator Γ_T can be continuously extended to a linear operator from X_σ^{b',s_1,s_2} to X_σ^{b,s_1,s_2} and*

$$\|\Gamma_T f\|_{X_\sigma^{b,s_1,s_2}} \lesssim T^{1-(b-b')}\|f\|_{X_\sigma^{b',s_1,s_2}} \tag{2.35}$$

for $f \in X_\sigma^{b',s_1,s_2}$.

Proof. For $\sigma = 0$ see [6]. For $\sigma \neq 0$ consider the operator I_σ defined for $u \in \mathcal{S}_{-\infty}$ by $(\mathcal{F}_2 I_\sigma u)(t,\xi,\vec{y}) = (\frac{\langle\xi\rangle}{|\xi|})^\sigma \mathcal{F}_2 u(t,\xi,\vec{y})$ (i. e. $I_\sigma = J_x^\sigma |D_x|^{-\sigma}$). Then $I_\sigma : X_\sigma^{b,s_1,s_2} \to X_0^{b,s_1,s_2}$ is an isometric isomorphism. It follows for $f \in \mathcal{S}_{-\infty}$ that

$$\begin{aligned}
\|\Gamma_T f\|_{X_\sigma^{b,s_1,s_2}} &= \|\zeta_T \int_0^t U_\alpha(t-t')f(t')dt'\|_{X_\sigma^{b',s_1,s_2}} \\
&= \|I_\sigma \zeta_T \int_0^t U_\alpha(t-t')f(t')dt'\|_{X_0^{b',s_1,s_2}} \\
&= \|\zeta_T \int_0^t U_\alpha(t-t')I_\sigma f(t')dt'\|_{X_0^{b',s_1,s_2}} \\
&\lesssim T^{1-(b-b')}\|I_\sigma f\|_{X_0^{b,s_1,s_2}} = T^{1-(b-b')}\|f\|_{X_\sigma^{b,s_1,s_2}}
\end{aligned}$$

The claim then follows by the density of $\mathcal{S}_{-\infty}$ in X_σ^{b',s_1,s_2}. □

2.4 The general well-posedness result

Theorem 2.22. *Suppose that there exist parameters $s_1, s_2 \in \mathbb{R}$, $b > \frac{1}{2}$, $b' \in (b-1, 0]$, $b_1 \in [0, -b']$, and $\sigma \in [0,1]$ such that for the Banach spaces X_1, X_2, and Y defined by*

$$X_1 := X_0^{b-b',s_1,s_2}, \quad X_2 := X_\sigma^{b,s_1,s_2} \cap X_\sigma^{b+b_1,s_1-(\alpha+1)b_1,s_2} \tag{2.36}$$

$$Y := X_\sigma^{b',s_1,s_2} \cap X_\sigma^{b'+b_1,s_1-(\alpha+1)b_1,s_2} \tag{2.37}$$

we have the following bilinear estimate for all $u_1, u_2 \in \mathcal{S}_{-\infty}$ and $k, l \in \{1,2\}$

$$\|B(u_1,u_2)\|_Y \lesssim \|u_1\|_{X_k}\|u_2\|_{X_l} \tag{2.38}$$

where $B(u_1,u_2) := \partial_x(u_1u_2)$.

Then there exists a non increasing function $T : (0,\infty) \to (0,\infty)$ such that the following holds true:

a) For every $r > 0$ and $u_0 \in \mathcal{B}_r := \{u_0 \in H^{s_1,s_2}(\mathbb{R}^d) \mid \|u_0\|_{H^{s_1,s_2}(\mathbb{R}^d)} < r\}$ there is a unique element $u \in X_{T(r)}$ where $X := X_1 + X_2$ such that we have

$$u(t) = U_\alpha(t)u_0 + \Gamma(B(u,u))(t), \quad t \in [-T(r), T(r)] \tag{2.39}$$

Furthermore, u is of the form

$$u(t) = U_\alpha(t)u_0 + w(t), \quad t \in [-T(r), T(r)] \tag{2.40}$$

with $w \in X_{2,T(r)}$.

b) For every $r > 0$ the flow map $F_r : \mathcal{B}_r \to X_{T(r)}, u_0 \mapsto u$ defined by a) is analytic.

c) If $r_2 > r_1 > 0$ and $u_0 \in B_{r_1}$, then $R_{T(r_2)}F_{r_1}(u_0) = F_{r_2}(u_0)$.

Remark 2.23. Note that in the proof of Theorem 2.22 we construct the solution u of (2.39) in such a way that it is an element of $X_{1,T} + X_{2,T}$ with $T := T(r)$. But it is easy to see that $X_{1,T} + X_{2,T} = (X_1 + X_2)_T$.

Remark 2.24. The spaces X_2 and Y defined in Theorem 2.22 are built by taking intersections of the Bourgain type spaces of Section 2.2. Therefore, it is easy to see that they also satisfy the linear estimate of Proposition 2.21, i. e.

$$\|\Gamma_T f\|_{X_2} \lesssim T^{1-(b-b')}\|f\|_Y \tag{2.41}$$

Furthermore, by Proposition 2.20, we have for $u_0 \in H^{s_1,s_2}(\mathbb{R}^d)$ that

$$\|L_1 u_0\|_{X_1} \lesssim \|u_0\|_{H^{s_1,s_2}} \tag{2.42}$$

We first prove two lemmata used in the proof of Theorem 2.22. We assume that the assumptions of Theorem 2.22 hold throughout the rest of the section.

Lemma 2.25. *Under the assumptions of Theorem 2.22, we have that* ΓB *is a well defined continuous bilinear operator from* $X_T \times X_T$ *to* $X_{2,T}$ *for every* $T \in (0,1]$ *and we have that*

$$\|\Gamma(B(u_1,u_2))\|_{X_{2,T}} \lesssim T^\kappa \|u_1\|_{X_T} \|u_2\|_{X_T} \tag{2.43}$$

where $\kappa := 1 - (b - b') > 0$. *In particular, we see that the right hand side of* (2.39) *is well defined. Furthermore,* L *is a well defined continuous linear operator from* $H^{s_1,s_2}(\mathbb{R}^d)$ *to* $X_{1,T}$*, i. e.*

$$\|Lu_0\|_{X_{1,T}} \lesssim \|u_0\|_{H^{s_1,s_2}(\mathbb{R}^d)} \tag{2.44}$$

Proof. First, (2.38) shows that B extends continuously to a bilinear operator $B : X_k \times X_l \to Y$ for all $k, l \in \{1,2\}$ and (2.38) holds for all $u_1 \in X_k$ and $u_2 \in X_l$. Also, (2.41) shows that Γ_T extends continuously to a linear operator $\Gamma_T : Y \to X_2$ for every $T \in (0,1]$. Altogether, we see that for every $T \in (0,1]$ and $k, l \in \{1,2\}$ we have that $\Gamma_T(B(u_1,u_2))$ is well defined for $u_1 \in X_k$ and $u_2 \in X_l$ and

$$\|\Gamma_T(B(u_1,u_2))\|_{X_2} \lesssim T^\kappa \|u_1\|_{X_k} \|u_2\|_{X_l} \tag{2.45}$$

As $R_T \Gamma_T B(u_1,u_2)$ only depends on $R_T u_1$ and $R_T u_2$ and

$$R_T \Gamma B(u_1,u_2) = R_T \Gamma_T B(u_1,u_2),$$

ΓB is a well defined continuous bilinear operator from $X_{k,T} \times X_{l,T}$ to $X_{2,T}$ for $k, l \in \{1,2\}$ and

$$\|\Gamma(B(u_1,u_2))\|_{X_{2,T}} \lesssim T^\kappa \|u_1\|_{X_{k,T}} \|u_2\|_{X_{l,T}} \tag{2.46}$$

holds. For $u_1, u_2 \in X_T$ and $u_i = v_i + w_i$ with $v_i \in X_{1,T}$ and $w_i \in X_{2,T}$ $(i = 1,2)$, it follows from (2.46) that

$$\|\Gamma(B(u_1,u_2))\|_{X_{2,T}} \lesssim T^\kappa (\|v_1\|_{X_{1,T}} + \|w_1\|_{X_{2,T}})(\|v_2\|_{X_{1,T}} + \|w_2\|_{X_{2,T}})$$

Now, (2.43) follows from this and (2.7) (using that $X_{1,T} + X_{2,T} = (X_1 + X_2)_T$, cf. Remark 2.23) Furthermore, it is easy to see that (2.44) follows from Proposition 2.20. □

Lemma 2.26. *Under the assumptions of Theorem 2.22, we have for* $T \in (0,1]$, $T_0 \in (-T,T)$ *and* $\delta \in (0, T - |T_0|)$ *that* τ_{T_0}*, which is defined for* $u \in \mathcal{S}_{-\infty}$ *by*

$$(\tau_{T_0} u)(t) := u(T_0 + t), \quad t \in \mathbb{R} \tag{2.47}$$

is a well defined continuous linear operator from $X_{k,T}$ to $X_{k,\delta}$, $k = 1, 2$, and we have for every $u \in X_{k,T}$ that

$$\|\tau_{T_0} u\|_{X_{k,\delta}} \leq \|u\|_{X_{k,T}} \tag{2.48}$$

Furthermore, we have for $u_1, u_2 \in X_T$ that

$$\begin{aligned}(\tau_{T_0}\Gamma B(u_1, u_2))(t)& \\ = U_\alpha(t)(\Gamma B(u_1, u_2)(T_0)) &+ \Gamma B(\tau_{T_0} u_1, \tau_{T_0} u_2)(t), \quad t \in [-\delta, \delta]\end{aligned} \tag{2.49}$$

Proof. By definition (2.9) of the X_σ^{b,s_1,s_2}-norm and definition (2.6), we have for $u \in \mathcal{S}_{-\infty}$ and $k \in \{1, 2\}$ that

$$\|\tau_{T_0} u\|_{X_k} = \|u\|_{X_k} \tag{2.50}$$

Because $\mathcal{S}_{-\infty}$ is dense in X_k for $k = 1, 2$, we find that (2.50) actually holds for all $u \in X_k$, i. e. τ_{T_0} is an isometry on X_k. Since $X_k \hookrightarrow C_b(\mathbb{R}, H^{s_1,s_2}(\mathbb{R}^d))$, we have for every $u \in X_k$ that $(\tau_{T_0} u)(t) = u(t + T_0)$, $t \in \mathbb{R}$. Therefore, it is obvious that $R_\delta \tau_{T_0} u$ only depends on $R_T u$. However, this implies that τ_{T_0} is a well defined continuous linear operator from $X_{k,T}$ to $X_{k,\delta}$ and that (2.48) holds.

Let us prove (2.49). We note that by the bilinearity of B it suffices to prove (2.49) for $u_1 \in X_{k,T}$, $u_2 \in X_{l,T}$, $k, l \in \{1, 2\}$. For $u_1, u_2 \in \mathcal{S}_{-\infty}$ we have

$$\begin{aligned}(\tau_{T_0}\Gamma B(u_1, u_2))(t) &= \int_0^{t+T_0} U_\alpha(t + T_0 - t')\partial_x(u_1(t')u_2(t'))dt' \\ &= U_\alpha(t) \int_0^{T_0} U_\alpha(T_0 - t')\partial_x(u_1(t')u_2(t'))dt' \\ &+ \int_0^t U_\alpha(t - t'')\partial_x(u_1(t'' + T_0)u_2(t'' + T_0))dt''\end{aligned}$$

where we made the change of variables $t' = t'' + T_0$ in the second integral. This proves (2.49) for $u_1, u_2 \in \mathcal{S}_{-\infty}$. Since all terms in (2.49) depend continuously on $u_1 \in X_{k,T}$ and $u_2 \in X_{l,T}$, (2.49) follows. ☐

Now, we can prove Theorem 2.22. Note that we restrict ourselves to $0 < T \leq 1$ but that the same arguments apply to any compact time interval.

Proof (of Theorem 2.22). Existence of a solution: Let $r > 0$ and suppose $T := T(r) \in (0, 1]$ has already been chosen. For $u_0 \in \mathcal{B}_r$ we search for a solution $u \in X_T$ of the operator equation

$$u = Lu_0 + \Gamma B(u, u) \tag{2.51}$$

where the right hand side of this equation is well defined due to Lemma 2.25. Let us define $w := u - L_1 u_0$. Then $u \in X_T$ is a solution of (2.51) if and only if $w \in X_T$ is a solution of

$$w = \Gamma B(w + Lu_0, w + Lu_0) \tag{2.52}$$

Note that by the mapping properties of ΓB this implies that $w \in X_{2,T}$. We will show that for T small enough (2.52) has indeed a solution $w \in X_{2,T}$. For fixed u_0 we define the operator Φ_T on $X_{2,T}$ by

$$\Phi_T(w) := \Gamma B(w + Lu_0, w + Lu_0) \tag{2.53}$$

so that w is a solution of (2.52) if and only if w is a fixed point of Φ_T. Let us show that for T small enough Φ_T has a fixed point in $X_{2,T}$. For $R > 0$ set $\mathcal{A}_R := \{u \in X_{2,T} \mid \|u\|_{X_{2,T}} \leq R\}$. If $w, w_1, w_2 \in \mathcal{A}_R$, we get by the bilinearity and symmetry of ΓB, (2.43), (2.44), and (2.7) that there is a constant $A > 0$ such that

$$\|\Phi_T(w)\|_{X_{2,T}} \leq AT^\kappa(\|w\|_{X_{2,T}} + \|u_0\|_{H^{s_1,s_2}})^2 \leq AT^\kappa(R + r)^2 \tag{2.54}$$

and

$$\begin{aligned}
&\|\Phi_T(w_1) - \Phi_T(w_2)\|_{X_{2,T}} \\
&\qquad \leq AT^\kappa(\|w_1\|_{X_{2,T}} + \|w_2\|_{X_{2,T}} + 2\|u_0\|_{H^{s_1,s_2}})\|w_1 - w_2\|_{X_{2,T}} \\
&\qquad \leq 2AT^\kappa(R + r)\|w_1 - w_2\|_{X_{2,T}}
\end{aligned} \tag{2.55}$$

For given $r > 0$, let $R := r$ and $T = T(r) := \min(1, (8Ar)^{-\frac{1}{\kappa}})$. Then it follows from (2.54) and (2.55) that Φ_T is a contraction mapping from the complete metric space $\mathcal{A}_R$ into itself. By Banach's fixed point theorem, there exists a fixed point $w \in \mathcal{A}_R$ of Φ_T. But then $F_r(u_0) := u := Lu_0 + w \in X_T$ is a solution of (2.51). For every $t \in [-T, T]$, we have that $(Lu_0)(t) = U_\alpha(t)u_0$. Therefore, (2.51) implies (2.39). Furthermore, we obviously have (2.40).

Uniqueness of the solution: Let us suppose that there are two solutions $u_1, u_2 \in X_T$ of (2.39). We have to show that $v := u_1 - u_2 = 0$. Let us suppose that $v \neq 0$. Then there exists $t \in (-T, T)$ such that $v(t) \neq 0$. We can restrict to the case $t \in (0, T)$ because the proof for the case $t \in (-T, 0)$ is analogous. Let

$$T_0 := \inf\{t \in (0, T] \mid v(t) \neq 0\}$$

Then we have $T_0 \in (0, T)$ and $v(t) = 0$ for every $t \in [0, T_0]$. Since u_1 and u_2 are solutions of (2.39), we have by the bilinearity and symmetry of ΓB that

$$v(t) = \Gamma B(u_1 + u_2, v)(t), \quad t \in [-T, T] \tag{2.56}$$

i. e. $v = \Gamma B(u_1 + u_2, v) \in X_{2,T}$. Now, $v(t) = 0$ for $t \in [0, T_0]$ implies that $\Gamma B(u_1 + u_2, v)(T_0) = 0$, so that by (2.49) we have for every $\delta \in (0, T - T_0)$ that

$$\tau_{T_0} v(t) = \Gamma B(\tau_{T_0}(u_1 + u_2), \tau_{T_0} v)(t), \quad t \in [-\delta, \delta] \tag{2.57}$$

Combining (2.57) with (2.43) and (2.48), we see that there is a $D > 0$ such that for every $\delta \in (0, T - T_0)$

$$\|\tau_{T_0} v\|_{X_{2,\delta}} \le D\delta^\kappa \|\tau_{T_0} v\|_{X_{2,\delta}}$$

If we choose $\delta \le (2D)^{-\frac{1}{\kappa}}$, we find that $\|\tau_{T_0} v\|_{X_{2,\delta}} = 0$, i. e. $v(T_0 + t) = 0$ for all $t \in [-\delta, \delta]$. However, this contradicts the choice of T_0.

Consistency of the flow map: If $r_2 > r_1 > 0$ and $u_0 \in B_{r_1}$, then it is obvious that both $R_{T(r_2)} F_{r_1}(u_0)$ and $F_{r_2}(u_0)$ are solutions of (2.39) in $X_{T(r_2)}$. So, by the uniqueness of the solution, it follows that $R_{T(r_2)} F_{r_1}(u_0) = F_{r_2}(u_0)$.

Analyticity of the flow map: For $r > 0$ define $\Lambda_r : B_r \times X_{T(r)} \to X_{T(r)}$ by

$$\Lambda_r(u_0, u) := u - (L_1 u_0 + \Gamma B(u, u)) \tag{2.58}$$

so that for $u_0 \in B_r$ and $u \in X_{T(r)}$ we have that $\Lambda_r(u_0, u) = 0$ if and only if $u = F_r(u_0)$. The mapping Λ_r is obviously analytic. Therefore, we deduce from the implicit function theorem that F_r is analytic. □

By Theorem 2.22, the (local in time) well-posedness of equation (1.1) follows from a bilinear estimate of the form (2.38). While deriving the bilinear estimate (2.38), we can, in most of the cases, take u_1 and u_2 in the simpler space X_0^{b,s_1,s_2} instead of X_l for the calculations and then use the following simple embedding property.

Proposition 2.27. *For X defined as in Theorem 2.22 we have $X \hookrightarrow X_0^{b,s_1,s_2}$. More precisely, we have the estimate*

$$\|u\|_{X_0^{b,s_1,s_2}} \le \|u\|_X \le \|u\|_{X_k} \tag{2.59}$$

for $k \in \{1, 2\}$

Proof. Since $\sigma \ge 0$, we have by Lemma 2.7 and by definition (2.6) that $\|u\|_{X_0^{b,s_1,s_2}} \le \|u\|_{X_\sigma^{b,s_1,s_2}} \le \|u\|_{X_2}$. As $b - b' \ge b$, it follows by Lemma 2.7 that $\|u\|_{X_0^{b,s_1,s_2}} \le \|u\|_{X_0^{b-b',s_1,s_2}} = \|u\|_{X_1}$. Therefore, if $u \in X$ and $u = v + w$ with $v \in X_1$ and $w \in X_2$, then $\|u\|_{X_0^{b,s_1,s_2}} \le \|v\|_{X_1} + \|w\|_{X_2}$. If we now take the infimum on the right hand side of this inequality over all possible decompositions of u of the form $u = v + w$ with $v \in X_1$ and $w \in X_2$, we get the left inequality of (2.59). The right inequality of (2.59) follows directly from the definition of the norm of X in (2.7). □

If $s_1 \geq 0$ and $s_2 = 0$, then we can use the the conservation of the L^2-norm, which holds for real valued solutions of (2.39), to obtain the following global result, where $H^{s_1,0}(\mathbb{R}^d;\mathbb{R})$ denotes the subspace of all real valued functions in $H^{s_1,0}(\mathbb{R}^d)$.

Theorem 2.28. *Suppose that there are parameters $b > \frac{1}{2}$, $b' \in (b-1, 0]$, $b_1 \in [0, -b']$ and $\sigma \in [0,1]$ such that for the Banach spaces $X_1^{(s)}$, $X_2^{(s)}$, and $Y^{(s)}$ defined by*

$$X_1^{(s)} := X_0^{b-b',s}, \quad X_2^{(s)} := X_\sigma^{b,s} \cap X_\sigma^{b+b_1,s-(\alpha+1)b_1} \tag{2.60}$$

$$Y^{(s)} := X_\sigma^{b',s} \cap X_\sigma^{b'+b_1,s-(\alpha+1)b_1} \tag{2.61}$$

we have the following bilinear estimate for all $u_1, u_2 \in \mathcal{S}_{-\infty}$ and $k, l \in \{1,2\}$

$$\|B(u_1,u_2)\|_{Y^{(0)}} \lesssim \|u_1\|_{X_k^{(0)}} \|u_2\|_{X_l^{(0)}} \tag{2.62}$$

where $B(u_1,u_2) := \partial_x(u_1 u_2)$.

Then, for every $s \geq 0$, $r > 0$ and $T > 0$, there is an analytic map $F_r : B_r \to X_T^{(s)}$, where $\mathcal{B}_r := \{u_0 \in H^{s,0}(\mathbb{R}^d;\mathbb{R}) \mid \|u_0\|_{H^{s,0}} < r\}$ and $X^{(s)} := X_1^{(s)} + X_2^{(s)}$, such that for every $u_0 \in \mathcal{B}_r$ the function $u := F_r(u_0)$ is the unique solution of

$$u(t) = U_\alpha(t)u_0 + \Gamma(B(u,u))(t), \quad t \in [-T,T] \tag{2.63}$$

in $X_T^{(s)}$. Furthermore, u is of the form

$$u(t) = U_\alpha(t)u_0 + w(t), \quad t \in [-T,T] \tag{2.64}$$

with $w \in X_{2,T}^{(s)}$.

We need the following lemma in the proof of Theorem 2.28.

Lemma 2.29. *Under the assumptions of Theorem 2.28, we have for every $s \geq 0$ and every $T \in (0,1]$ that the operator ΓB is a well defined continuous bilinear operator from $X_T^{(s)} \times X_T^{(s)}$ to $X_{2,T}^{(s)}$ and for all $u_1, u_2 \in X_T^{(s)}$ we have*

$$\|\Gamma B(u_1,u_2)\|_{X_{2,T}^{(s)}} \lesssim \|u_1\|_{X_T^{(s)}} \|u_2\|_{X_T^{(0)}} + \|u_1\|_{X_T^{(0)}} \|u_2\|_{X_T^{(s)}} \tag{2.65}$$

Proof. We define the bilinear operator P_1 for $u_1, u_2 \in \mathcal{S}_{-\infty}$ by

$$\mathcal{F}P_1(u_1,u_2)(\mu) = \int_{\mathbb{R}^n} \chi_{|\xi_1| \leq |\xi_2|}(\mu_1,\mu)(\mathcal{F}u_1)(\mu_1)(\mathcal{F}u_2)(\mu_2) d\mu_1 \tag{2.66}$$

Then it obviously holds that $\Gamma B(u_1,u_2) = \Gamma\partial_x P_1(u_1,u_2) + \Gamma\partial_x P_1(u_2,u_1)$. Hence, the claim follows if we show that $\Gamma\partial_x P_1$ is a well defined continuous bilinear operator from $X_T^{(0)} \times X_T^{(s)}$ to $X_{2,T}^{(s)}$ and for all $u_1 \in X_T^{(0)}$ and $u_2 \in X_T^{(s)}$ we have

$$\|\Gamma \partial_x P_1(u_1, u_2)\|_{X_{2,T}^{(s)}} \lesssim \|u_1\|_{X_T^{(0)}} \|u_2\|_{X_T^{(s)}}$$

This follows exactly as in the proof of Lemma 2.25 if we show that for all $u_1, u_2 \in \mathcal{S}_{-\infty}$ and $k, l \in \{1, 2\}$ we have

$$\|\partial_x P_1(u_1, u_2)\|_{Y^{(s)}} \lesssim \|u_1\|_{X_k^{(0)}} \|u_2\|_{X_l^{(s)}} \tag{2.67}$$

By the definitions of $X_1^{(s)}$, $X_2^{(s)}$, and $Y^{(s)}$ and the definition of the $X_\sigma^{b,s}$-norm (2.9), we see that $\|u\|_{X_k^{(s)}} = \|J_x^s u\|_{X_k^{(0)}}$ for $k = 1, 2$ and $\|u\|_{Y^{(s)}} = \|J_x^s u\|_{Y^{(0)}}$. Therefore, we have to show that

$$\|J_x^s \partial_x P_1(u_1, u_2)\|_{Y^{(0)}} \lesssim \|u_1\|_{X_k^{(0)}} \|J_x^s u_2\|_{X_l^{(0)}} \tag{2.68}$$

We easily see that the norms on the right hand side of (2.68) only depend on the modulus of $\mathcal{F}u_1$ and $\mathcal{F}u_2$. Furthermore, we have $\|u\|_{Y^{(0)}} \leq \|v\|_{Y^{(0)}}$ if $|\mathcal{F}u| \leq |\mathcal{F}v|$. Since $|\xi_1| \leq |\xi_2|$ implies $|\xi| \leq 2|\xi_2|$, we deduce that

$$\begin{aligned}
|\mathcal{F}J_x^s \partial_x P_1(u_1, u_2)(\mu)| &= \left| \langle \xi \rangle^s i\xi \int_{\mathbb{R}^n} \chi_{|\xi_1| \leq |\xi_2|}(\mu_1, \mu)(\mathcal{F}u_1)(\mu_1)(\mathcal{F}u_2)(\mu_2) d\mu_1 \right| \\
&\lesssim |\xi| \int_{\mathbb{R}^n} \langle \xi_2 \rangle^s |\mathcal{F}u_1(\mu_1)||\mathcal{F}u_2(\mu_2)| d\mu_1 \\
&= |\mathcal{F}B(\mathcal{F}^{-1}(|\mathcal{F}u_1|), \mathcal{F}^{-1}(|\mathcal{F}J_x^s u_2|))|
\end{aligned}$$

Using (2.62), we obtain

$$\begin{aligned}
\|J_x^s \partial_x P_1(u_1, u_2)\|_{Y^{(0)}} &\lesssim \|B(\mathcal{F}^{-1}(|\mathcal{F}u_1|), \mathcal{F}^{-1}(|\mathcal{F}J_x^s u_2|))\|_{Y^{(0)}} \\
&\lesssim \|\mathcal{F}^{-1}(|\mathcal{F}u_1|)\|_{X_k^{(0)}} \|\mathcal{F}^{-1}(|\mathcal{F}J_x^s u_2|)\|_{X_l^{(0)}} \\
&= \|u_1\|_{X_k^{(0)}} \|J_x^s u_2\|_{X_l^{(0)}}
\end{aligned}$$

This proves (2.67). □

Proof of Theorem 2.28. We restrict ourselves to $0 < T \leq 1$ but the same arguments apply to any compact time interval. First of all, we note that if $u_0 \in H^{s,0}(\mathbb{R}^d; \mathbb{R})$ and $u \in X_T^{(s)}$ is a solution of (2.63) (which is unique by the proof of Theorem 2.22), then u is also real-valued. This follows from the uniqueness of the solution in $X_T^{(s)}$ and the fact that $\bar{u}$ is also a solution of (2.63) in $X_T^{(s)}$.

For $r_0 \in [0, r]$ let

$$\mathcal{B}_{r,r_0} := \{u_0 \in H^{s,0}(\mathbb{R}^d; \mathbb{R}) \mid \|u_0\|_{H^{s,0}} < r,\ \|u_0\|_{L^2} < r_0\} \tag{2.69}$$

Note that for $r_0 = r$ we have $\mathcal{B}_{r,r} = \mathcal{B}_r$. Now, let T^* be the supremum of all $T \in (0, 1]$ such that the following holds true: There is an analytic map $F : \mathcal{B}_{r,r_0} \to X_T$ such that for every $u_0 \in \mathcal{B}_{r,r_0}$ the function $u := F(u_0)$ is the unique solution of (2.63) in $X_T^{(s)}$. We are going to show that $T^* = 1$. First of all, $T^* > 0$ by Theorem 2.22. By (2.63), (2.44), and (2.65), we have that there is a constant $A \geq 1$ such that for all $T \in (0, T^*)$ we get

$$\|u\|_{X_T^{(s)}} \leq \|Lu_0\|_{X_{1,T}^{(s)}} + \|\Gamma B(u,u)\|_{X_{2,T}^{(s)}} \leq Ar + 2AT^\kappa \|u\|_{X_T^{(0)}} \|u\|_{X_T^{(s)}} \quad (2.70)$$

and

$$\|u\|_{X_T^{(0)}} \leq \|Lu_0\|_{X_{1,T}^{(0)}} + \|w\|_{X_{2,T}^{(0)}} \leq Ar_0 + \|w\|_{X_{2,T}^{(0)}} \quad (2.71)$$

where $w := u - Lu_0$. If $T^* < \min(1, (16A^2 r_0)^{-\frac{1}{\kappa}})$, we have by (2.54) and (2.55) with $R := Ar_0$ and $r := r_0$ that $\|w\|_{X_{2,T}^{(0)}} \leq Ar_0$ and therefore, by (2.71), that $\|u\|_{X_T^{(0)}} \leq 2Ar_0$. Combining this with (2.70), we find that

$$\|u\|_{X_T^{(s)}} \leq Ar + \frac{1}{4}\|u\|_{X_T^{(s)}}$$

It follows that $\sup_{|t| \leq T} \|u(t)\|_{H^{s,0}} \leq C\|u\|_{X_T^{(s)}} < 2ACr$. This upper bound does only depend on r and not on T. Now, by Theorem 2.22 there is a $T' = T'(r) > 0$ and an analytic flow map $\tilde{F} := F_{2ACr} : \mathcal{B}_{2ACr} \to X_{T'}$. We choose $T \in (0, T^*)$ such that $T^* < T + \frac{T'}{2} \leq 1$ and define a map on $\mathcal{B}_{r,r_0}$ with values in $X_{T+\frac{T'}{2}}$ by

$$H(u_0) := \zeta_0 F(u_0) + \zeta_+ \tau_{-T} \tilde{F}(u(T)) + \zeta_- \tau_T \tilde{F}(u(-T)) \quad (2.72)$$

where $\zeta_0, \zeta_+, \zeta_-$ is a smooth partition of unity on $I := [-T - \frac{T'}{2}, T + \frac{T'}{2}]$, which is adapted to the covering $I \subset I_0 \cup I_+ \cup I_-$ with $I_0 := (-T, T)$, $I_+ := (T - T', T + T')$ and $I_- := (-T - T', -T + T')$. We can verify that H is well-defined, analytic, and that for every $u_0 \in \mathcal{B}_{r,r_0}$ the function $H(u_0)$ is the unique solution of (2.63) in $X_{T+\frac{T'}{2}}$. As $T + \frac{T'}{2} > T^*$, this contradicts the choice of T^*. Hence, we deduce that

$$T^* \geq \min(1, (16A^2 r_0)^{-\frac{1}{\kappa}}) \quad (2.73)$$

i. e. T^* can be bounded from below by a bound only dependent on the L^2-norm of the initial values. But the L^2-norm of real valued solutions u of (2.63) is conserved, i. e. $\|u(\pm T)\|_{L^2} = \|u_0\|_{L^2}$ for all $T \in (0, T^*)$. So, if we had $T^* < 1$, then we could extend the flow map beyond the time interval $[-T^*, T^*]$ by a similar argument as above, which would again contradict the choice of T^*. Therefore, we have $T^* = 1$. $\square$

2.5 Notes and references

All concepts introduced in Section 2.1 are standard and described in many textbooks. For a detailed survey of distributions and the Fourier transform and their use in the theory of partial differential equations, see, for example, the textbooks by HÖRMANDER [9], YOSIDA [35], and KABALLO [15]. For general results on the non-isotropic Sobolev spaces, see SCHMEISSER AND TRIEBEL [27], Chapter 2. Note that the spaces called "non-isotropic" in this thesis are called "spaces with dominating mixed smoothness properties" in [27] whereas the "anisotropic" spaces considered in [27] are different.

The spaces of Section 2.2 were first used in the context of the Kadomtsev-Petviashvili II equation by BOURGAIN [5]. These kind of spaces, which are adapted to the symbol of the linear part of the equation, had already been used by BOURGAIN [3,4] to prove well-posedness results for the Korteweg-de Vries and a nonlinear Schrödinger equation.

For an introduction to the theory of semigroups and their use in well-posedness problems in partial differential equations, see PAZY [23].

For the methods used in the proof of the general well-posedness result of Section 2.4 we refer the reader to the survey article of GINIBRE [6] and the first part of the thesis of GRÜNROCK [7].

Chapter 3

Dispersive inequalities for KP-type equations

3.1 Local smoothing estimates

In this section we give a local smoothing estimate for the solution of the linear equation (2.20). We then use this estimate to show that the solutions of the operator equation (2.39) in X_T (cf. Theorem 2.22) are actually locally integrable functions (in all variables) and satisfy (1.1) in the sense of distributions (at least in the range of parameters s_1, s_2 that we consider in Chapters 4 and 5).

Similar to the case $\alpha = 2$ and $d = 2$ (cf. [20], Lemma 3.2), one can prove the following local smoothing estimate.

Theorem 3.1. *For $u_0 \in L^2(\mathbb{R}^d)$ we have*

$$\| |D_x|^{\frac{\alpha}{2}} U_\alpha(t) u_0 \|_{L_x^\infty L_{t\vec{y}}^2} \lesssim \|u_0\|_{L_{x\vec{y}}^2} \tag{3.1}$$

Proof. The proof is analogous to the one given in [20], Lemma 3.2. □

This local smoothing estimate can be restated as an embedding of a Bourgain space into $L_x^\infty L_{t\vec{y}}^2$.

Corollary 3.2. *For $s_1 \geq -\frac{\alpha}{2}$, $s_2 \geq 0$ and $b > \frac{1}{2}$ we have*

$$\|u\|_{L_x^\infty L_{t\vec{y}}^2} \lesssim \|u\|_{X_0^{b,s_1,s_2}} \tag{3.2}$$

i. e. $X_0^{b,s_1,s_2} \hookrightarrow L_x^\infty L_{T\vec{y}}^2$.

Proof. By well-known methods (see, for example, [6], Lemme 3.3), (3.1) implies for $b > \frac{1}{2}$

$$\||D_x|^{\frac{\alpha}{2}} u\|_{L^\infty_x L^2_{t\vec{y}}} \lesssim \|u\|_{X_0^{b,0}} \tag{3.3}$$

Let φ be the function defined in Definition 2.13. We decompose $u = u_l + u_h$, where $u_l := \mathcal{F}^{-1}(\phi \mathcal{F} u)$ and $u_h := \mathcal{F}^{-1}((1-\phi)\mathcal{F} u)$. For u_h it follows from (3.3) that

$$\|u_h\|_{L^\infty_x L^2_{t\vec{y}}} \lesssim \||D_x|^{-\frac{\alpha}{2}} u_h\|_{X_0^{b,0}} = \||D_x|^{-\frac{\alpha}{2}} J_x^{-s_1} J_{\vec{y}}^{-s_2} u_h\|_{X_0^{b,s_1,s_2}} \lesssim \|u\|_{X_0^{b,s_1,s_2}} \tag{3.4}$$

where for the last inequality we used $-s_2 \leq 0$, $-\frac{\alpha}{2} - s_1 \leq 0$ and that $|\xi| \sim \langle \xi \rangle$ on the support of $\mathcal{F} u_h$. For u_l we have by Minkowski's and Sobolev's inequality that

$$\|u_l\|_{L^\infty_x L^2_{t\vec{y}}} \lesssim \|J_x u_l\|_{L^2_{tx\vec{y}}} \lesssim \|u\|_{X_0^{b,s_1,s_2}} \tag{3.5}$$

where the last inequality follows because of $-s_2 \leq 0$, $-b \leq 0$ and $\langle \xi \rangle \sim 1$ on the support of $\mathcal{F} u_l$. Now, (3.4) and (3.5) together imply (3.2). $\square$

Theorem 3.3. *Suppose that the conditions of Theorem 2.22 are fulfilled with $s_1 \geq -\frac{\alpha}{2}$ and $s_2 \geq 0$. Let $u_0 \in H^{s_1,s_2}(\mathbb{R}^d)$ and $u \in X_T$ be the unique solution of (2.39). Then u is a solution of (1.1) in the sense of distributions, i. e. $u \in L^\infty_x L^2_{T\vec{y}} := L^\infty_x(\mathbb{R}; L^2_{t\vec{y}}((-T,T)\times\mathbb{R}^{d-1}))$ and for all $\varphi \in C_c^\infty((-T,T)\times\mathbb{R}^d)$ we have that*

$$\int_{\mathbb{R}^n} u(\varphi_{tx} - |D_x|^\alpha \varphi_{xx} + \Delta_{\vec{y}}\varphi) + u^2 \varphi_{xx} dt dx d\vec{y} = 0 \tag{3.6}$$

Proof. By the density of $\mathcal{S}_{-\infty}$ in X we have a sequence $(\tilde{u}_j)_{j\in\mathbb{N}}$ in $\mathcal{S}_{-\infty}$ such that $u_j := R_T(\tilde{u}_j) \to u$ in X_T for $j \to \infty$. As $X_T \hookrightarrow C([-T,T]; H^{s_1,s_2}(\mathbb{R}^d))$, this especially implies that $u_j(0) \to u_0$ in $H^{s_1,s_2}(\mathbb{R}^d)$ for $j \to \infty$. Let

$$v_j := L u_j(0) + \Gamma B(u_j, u_j) \in X_T$$

Then, $v_j \to L u_0 + \Gamma B(u,u) = u$ in X_T by Lemma 2.25. It is easy to compute that

$$\partial_t \partial_x v_j(t) = (|D_x|^\alpha \partial_x^2 - \Delta_{\vec{y}}) v_j(t) - \partial_x^2(u_j(t)^2), \quad t \in (-T,T)$$

Multiplying this last equation with $\varphi \in C_c^\infty((-T,T)\times\mathbb{R}^d)$, integrating by parts, and using Plancherel's theorem, we obtain

$$\int_{\mathbb{R}^n} v_j(\varphi_{tx} - |D_x|^\alpha \varphi_{xx} + \Delta_{\vec{y}}\varphi) + u_j^2 \varphi_{xx} dt dx d\vec{y} = 0 \tag{3.7}$$

By Proposition 2.27 and Corollary 3.2, we have the embedding

$$X \hookrightarrow X_0^{b,s_1,s_2} \hookrightarrow L_x^\infty L_{t\vec{y}}^2$$

and therefore also $X_T \hookrightarrow L_x^\infty L_{T\vec{y}}^2$. This implies that $u \in L_x^\infty L_{T\vec{y}}^2$, $v_j \to u$ in $L_x^\infty L_{T\vec{y}}^2$, and $u_j^2 \to u^2$ in $L_x^\infty L_{T\vec{y}}^1$ for $j \to \infty$. Furthermore, we can show that $|D_x|^\alpha \varphi_{xx} \in L_x^1 L_{T\vec{y}}^2$. Therefore, letting $j \to \infty$ in (3.7), we get (3.6). □

3.2 Linear Strichartz estimates

In this section, we will use the abstract results of KEEL and TAO [17] to derive Strichartz estimates for the solutions of the linear equation (2.20). In order to be able to apply the results of [17], we need the following decay estimates, which are proved exactly as in the case $\alpha = 2$, $d = 2$ (cf. SAUT [24]). For the case $\alpha \in 2\mathbb{N}$ in dimensions $d = 2, 3$, see also BEN-ARTZI AND SAUT [1]. For the convenience of the reader, we will give the full proof here.

Theorem 3.4. *We have for $\alpha > d - 2$ and $u_0 \in L^1(\mathbb{R}^d)$*

$$\||D_x|^{\frac{\alpha}{2}-\frac{d}{2}} U_\alpha(t) u_0\|_{L^\infty_{x\vec{y}}(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{2}} \|u_0\|_{L^1_{x\vec{y}}(\mathbb{R}^d)} \tag{3.8}$$

Proof. Let $\theta := \frac{d}{2} - \frac{\alpha}{2}$. Because $\theta < 1$, we have that

$$m_t(\xi, \eta) := |\xi|^{-\theta} e^{it(\xi|\xi|^\alpha - \frac{\vec{\eta}^2}{\xi})} \in \mathcal{S}'(\mathbb{R}^d)$$

for every $t \in \mathbb{R}$. Therefore, we have for $u_0 \in \mathcal{S}(\mathbb{R}^d)$ and $t \in \mathbb{R}$ that

$$|D_x|^{-\theta} U_\alpha(t) u_0 = \mathcal{F}^{-1}(m_t \mathcal{F} u_0) = \mathcal{F}^{-1}(m_t) * u_0 \in \mathcal{S}'(\mathbb{R}^d)$$

For $\delta_1, \delta_2 > 0$ let us define $m_t^{\delta_1,\delta_2}(\xi, \vec{\eta}) := e^{-\delta_1 \xi^2 - \delta_2 \vec{\eta}^2} m_t(\xi, \vec{\eta})$ Then, by the theorem of dominated convergence, we have that $\lim_{\delta_1,\delta_2 \to 0+} m_t^{\delta_1,\delta_2} = m_t$ in $\mathcal{S}'(\mathbb{R}^d)$. Therefore, we have that

$$|D_x|^{-\theta} U_\alpha(t) u_0 = \lim_{\delta_1,\delta_2 \to 0+} \mathcal{F}^{-1}(m_t^{\delta_1,\delta_2}) * u_0$$

Furthermore, we see that

$$\begin{aligned}
\mathcal{F}^{-1}(m_t^{\delta_1,\delta_2})(x, \vec{y}) &= c \int_{\mathbb{R}^d} e^{i(x\xi + \vec{y}\cdot\vec{\eta})} m_t^{\delta_1,\delta_2}(\xi, \vec{\eta}) d\xi d\vec{\eta} \\
&= c \int_{\mathbb{R}} |\xi|^{-\theta} e^{-\delta_1 \xi^2 + i(x\xi + t\xi|\xi|^\alpha)} \left(\int_{\mathbb{R}^{d-1}} e^{i\vec{y}\cdot\vec{\eta}} e^{-(\delta_2 + \frac{it}{\xi})\vec{\eta}^2} d\vec{\eta} \right) d\xi \\
&= c \int_{\mathbb{R}} |\xi|^{-\theta} e^{-\delta_1 \xi^2 + ix\xi + it\xi|\xi|^\alpha} (\delta_2 + \frac{it}{\xi})^{-\frac{d-1}{2}} e^{-\frac{1}{4}(\delta_2 + \frac{it}{\xi})^{-1} \vec{y}^2} d\xi
\end{aligned}$$

By the theorem of dominated convergence, we can take the limit $\delta_2 \to 0+$ in the last expression and get that

$$\mathcal{F}^{-1}(m_t) = c|t|^{\frac{1}{2}-\frac{d}{2}} \lim_{\delta_1 \to 0+} \int_{\mathbb{R}} |\xi|^{\frac{\alpha}{2}-\frac{1}{2}} e^{-\delta_1 \xi^2 + i(x\xi + t\xi|\xi|^\alpha - (d-1)\operatorname{sign}(\frac{\xi}{t})\frac{\pi}{4})} e^{\frac{i\xi}{4t}\bar{y}^2} d\xi$$

Let $\psi(\xi) := e^{-\delta_1\xi^2 - i(d-1)\operatorname{sign}(\frac{\xi}{t})\frac{\pi}{4}}$ and $\phi(\xi) := \xi|\xi|^\alpha$. Then $|\phi''(\xi)|^{\frac{1}{2}} \sim |\xi|^{\frac{\alpha}{2}-\frac{1}{2}}$ and we can use Corollary 2.9 of [18] to see that

$$\left| \int_{\mathbb{R}} |\xi|^{\frac{\alpha}{2}-\frac{1}{2}} e^{-\delta_1 \xi^2 + i(x\xi + t\xi|\xi|^\alpha - (d-1)\operatorname{sign}(\frac{\xi}{t})\frac{\pi}{4})} e^{\frac{i\xi}{4t}\bar{y}^2} d\xi \right| \lesssim |t|^{-\frac{1}{2}}$$

where the implicit constant does not depend on $\delta_1 > 0$. Therefore, we get that $\mathcal{F}^{-1}(m_t) \in L^\infty(\mathbb{R}^d)$ and $\|\mathcal{F}^{-1}(m_t)\|_{L^\infty(\mathbb{R}^d)} \leq C|t|^{-\frac{d}{2}}$. It follows that we have the decay estimate (3.8) for all $u_0 \in \mathcal{S}(\mathbb{R}^d)$ and then, by continuity, also for all $u_0 \in L^1(\mathbb{R}^d)$. $\square$

Definition 3.5. Let

$$\gamma(d, \alpha, r) := \left(\frac{d}{2} - \frac{\alpha}{2}\right)\left(\frac{1}{2} - \frac{1}{r}\right) \tag{3.9}$$

Theorem 3.6. *For $d = 2$ let $2 < q \leq \infty$ and $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$. We then have that*

$$\||D_x|^{-\gamma(2,\alpha,r)} U_\alpha(t) u_0\|_{L_t^q L_{xy}^r} \lesssim \|u_0\|_{L_{xy}^2} \tag{3.10}$$

and for $b > \frac{1}{2}$

$$\||D_x|^{-\gamma(2,\alpha,r)} u\|_{L_t^q L_{xy}^r} \lesssim \|u\|_{X_0^{b,0}} \tag{3.11}$$

Proof. Let Δ_k and $\tilde{\Delta}_k$ be defined as in Definition 2.14 and $\tilde{U}(t) := U_\alpha(t)\tilde{\Delta}_0$. Then, $\tilde{U}(t)$ is a linear and continuous operator on the Hilbert space $L^2(\mathbb{R}^2)$ with $\tilde{U}(t)^* = U_\alpha(-t)\tilde{\Delta}_0$. By Theorem 3.4, it follows that

$$\begin{aligned}
\|\tilde{U}(s)\tilde{U}(t)^* u_0\|_{L_{xy}^\infty(\mathbb{R}^2)} &= \|\tilde{\Delta}_0^2 U_\alpha(s-t) u_0\|_{L_{xy}^\infty(\mathbb{R}^2)} \\
&\lesssim \||D_x|^{\frac{\alpha}{2}-\frac{d}{2}} U_\alpha(s-t) u_0\|_{L_{xy}^\infty(\mathbb{R}^2)} \\
&\lesssim |s-t|^{-1} \|u_0\|_{L_{xy}^1(\mathbb{R}^2)}.
\end{aligned}$$

Therefore, we can use Theorem 1.2 of [17] to get

$$\|U_\alpha(t)\tilde{\Delta}_0 u_0\|_{L_t^q L_{xy}^r} \lesssim \|u_0\|_{L_{xy}^2} \tag{3.12}$$

We can check that $\frac{1}{q} = \frac{1}{2} - \frac{1}{r}$ implies that $\gamma(2, \alpha, r) = \bar{\gamma}(2, \alpha, q, r)$ where γ is defined as in (3.9) and $\bar{\gamma}$ is defined as in (2.27). By Proposition 2.16, it follows that

$$\|U_\alpha(t)\tilde{\Delta}_k u_0\|_{L_t^q L_{xy}^r} \lesssim 2^{k\gamma(2,\alpha,r)} \|u_0\|_{L_{xy}^2} \tag{3.13}$$

Because $2 < q$ and $2 \leq r < \infty$, we can use Littlewood-Paley theory and Minkowski's inequality to see that

$$\|U_\alpha(t)u_0\|_{L_t^q L_{xy}^r} \lesssim \|(\sum_{k\in\mathbb{Z}} |\Delta_k U_\alpha(t)u_0|^2)^{\frac{1}{2}}\|_{L_t^q L_{xy}^r} \lesssim (\sum_{k\in\mathbb{Z}} \|\Delta_k U_\alpha(t)u_0\|_{L_t^q L_{xy}^r}^2)^{\frac{1}{2}} \tag{3.14}$$

By (3.13) and because of $\Delta_k = \tilde{\Delta}_k \Delta_k$, we can estimate

$$\|\Delta_k U_\alpha(t)u_0\|_{L_t^q L_{xy}^r} = \|U_\alpha(t)\tilde{\Delta}_k \Delta_k u_0\|_{L_t^q L_{xy}^r} \lesssim 2^{k\gamma(2,\alpha,r)}\|\Delta_k u_0\|_{L_{xy}^2}$$

If we substitute this into (3.14) and use Plancherel's theorem and the fact that $|\xi| \sim 2^k$ on the support of ψ_k, we get that

$$\|U_\alpha(t)u_0\|_{L_t^q L_{xy}^r} \lesssim (\sum_{k\in\mathbb{Z}} (2^{k\gamma(2,\alpha,r)}\|\Delta_k u_0\|_{L_{xy}^2})^2)^{\frac{1}{2}} \lesssim \||D_x|^{\gamma(2,\alpha,r)}u_0\|_{L_{xy}^2}$$

If we now substitute $|D_x|^{-\gamma(2,\alpha,r)}u_0$ for u_0, we see that (3.10) holds. Now, (3.11) follows from (3.10) by standard methods (cf. [6], Lemme 3.3). □

Theorem 3.7. *For $d = 3$ and $\alpha > 1$ let $2 \leq q \leq \infty$ and $\frac{2}{q} + \frac{3}{r} = \frac{3}{2}$. We then have that*

$$\||D_x|^{-\gamma(3,\alpha,r)}U_\alpha(t)u_0\|_{L_t^q L_{x\vec{y}}^r} \lesssim \|u_0\|_{L_{x\vec{y}}^2} \tag{3.15}$$

and for $b > \frac{1}{2}$

$$\||D_x|^{-\gamma(3,\alpha,r)}u\|_{L_t^q L_{x\vec{y}}^r} \lesssim \|u\|_{X_0^{b,0}} \tag{3.16}$$

Furthermore, we have that

$$\||D_x|^{-\frac{1}{4}+\frac{\alpha}{12}}u\|_{L_t^2 L_{x\vec{y}}^3} \lesssim \|u\|_{X_0^{\frac{1}{4},0}} \tag{3.17}$$

Proof. As in the proof of Theorem 3.6, we can deduce from Theorem 3.4, Theorem 1.2 of [17] and Proposition 2.16 that for $k \in \mathbb{Z}$

$$\|U_\alpha(t)\tilde{\Delta}_k u_0\|_{L_t^q L_{x\vec{y}}^r} \lesssim 2^{k\gamma(3,\alpha,r)}\|u_0\|_{L_{x\vec{y}}^2} \tag{3.18}$$

Now, (3.15) and (3.16) follow as in the proof of Theorem 3.6.

It remains to show (3.17). Let φ_j be defined as in Definition 2.13. Let operators Q_j, for $j \in \mathbb{N}_0$, be defined by

$$(\mathcal{F}Q_j u)(\mu) := \varphi_j(\lambda)\mathcal{F}u(\mu) \tag{3.19}$$

Using these operators, we can define the following Besov-type refinements of the spaces $X_0^{b,0}$. Let $XB^{b,p}$ for $b \in \mathbb{R}$ and $1 \leq p < \infty$ be the space defined as the completion of $\mathcal{S}_{-\infty}$ with respect to the norm

$$\|u\|_{XB^{b,p}} := \left(\sum_{j \in \mathbb{N}_0} (2^{jb} \|Q_j u\|_{L^2(\mathbb{R}^4)})^p \right)^{\frac{1}{p}} \tag{3.20}$$

By using Plancherel's theorem, we easily see that for $b \in \mathbb{R}$

$$\|u\|_{X_0^{b,0}} \sim \|u\|_{XB^{b,2}}$$

so that $X_0^{b,0} = XB^{b,2}$. Now, (3.18) for $q = 2$ and $r = 6$ reads

$$\|U_\alpha(t) \tilde{\Delta}_k u_0\|_{L_t^2 L_{x\vec{y}}^6} \lesssim 2^{k\gamma} \|u_0\|_{L_{x\vec{y}}^2} \tag{3.21}$$

where $\gamma := \gamma(3, \alpha, 6) = \frac{3-\alpha}{6}$. For $u \in \mathcal{S}_{-\infty}$ let $g(\tau) := \mathcal{F}_1(U(-\cdot)u)(\tau)$. By standard methods (see [6], Lemme 3.3), it follows from (3.21) that

$$\|\tilde{\Delta}_k u\|_{L_t^2 L_{x\vec{y}}^6} \lesssim 2^{k\gamma} \|\tilde{\Delta}_k g\|_{L_\tau^1 L_{x\vec{y}}^2} \lesssim 2^{k\gamma} \|g\|_{L_\tau^1 L_{x\vec{y}}^2}$$

where for the last inequality we used that the operators $\tilde{\Delta}_k$ are uniformly bounded on L_x^2. We have that $(\mathcal{F}_{23} g)(\mu) = (\mathcal{F}u)(\tau + p_\alpha(\xi, \vec{\eta}), \xi, \vec{\eta})$, so by Plancherel's theorem, (2.26), and Minkowski's inequality, we find that

$$\begin{aligned} \|g\|_{L_\tau^1 L_{x\vec{y}}^2} &= c \int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} |(\mathcal{F}u)(\tau' + p_\alpha(\xi, \vec{\eta}), \xi, \vec{\eta})|^2 d\xi d\vec{\eta} \right)^{\frac{1}{2}} d\tau' \\ &\leq c \sum_{j \in \mathbb{N}_0} \int_{|\tau'| \sim 2^j} \left(\int_{\mathbb{R}^3} |\varphi_j(\tau')(\mathcal{F}u)(\tau' + p_\alpha(\xi, \vec{\eta}), \xi, \vec{\eta})|^2 d\xi d\vec{\eta} \right)^{\frac{1}{2}} d\tau' \end{aligned}$$

Using the Cauchy-Schwarz inequality in τ' and making the change of variables $\tau' = \lambda(\mu) = \tau - p_\alpha(\xi, \vec{\eta})$, we see that the last expression is bounded above by

$$c \sum_{j \in \mathbb{N}_0} 2^{\frac{j}{2}} \left(\int_{\mathbb{R}^4} |\varphi_j(\lambda) \mathcal{F}u(\mu)|^2 d\mu \right)^{\frac{1}{2}} = c \sum_{l \in \mathbb{N}_0} 2^{\frac{j}{2}} \|Q_j u\|_{L^2} = c \|u\|_{XB^{\frac{1}{2},1}}$$

Altogether, we have for every $k \in \mathbb{Z}$ that

$$\|\tilde{\Delta}_k u\|_{L_t^2 L_{x\vec{y}}^6} \lesssim 2^{k\gamma} \|u\|_{XB^{\frac{1}{2},1}} \tag{3.22}$$

We also have the trivial bound

$$\|\tilde{\Delta}_k u\|_{L_t^2 L_{x\vec{y}}^2} \lesssim \|u\|_{L_t^2 L_{x\vec{y}}^2} \lesssim \|u\|_{XB^{0,2}} \tag{3.23}$$

Now, we interpolate between (3.22) and (3.23). Note that the space $XB^{b,p}$ is a retract of the space $l_p^b(L^2(\mathbb{R}^4))$ of all sequences $(f_j)_{j\in\mathbb{N}_0}$ with $f_j \in L^2(\mathbb{R}^4)$, endowed with the norm

$$\|(f_j)\|_{l_p^b(L^2(\mathbb{R}^4))} = \left(\sum_{j\in\mathbb{N}_0} (2^{jb}\|f_j\|_{L^2(\mathbb{R}^4)})^p\right)^{\frac{1}{p}} \tag{3.24}$$

(For the definition of retract, see [2], Definition 6.4.1.) Therefore, it follows from [2], Theorem 5.6.1 that

$$(XB^{0,2}, XB^{\frac{1}{2},1})_{\frac{1}{2},2} = XB^{\frac{1}{4},2} = X_0^{\frac{1}{4},0} \tag{3.25}$$

where $(\cdot,\cdot)_{\frac{1}{2},2}$ denotes the real interpolation method. From the Lions-Peetre interpolation theorem (see [2], 5.8.6), it follows that

$$(L_t^2 L_{x\vec{y}}^6, L_t^2 L_{x\vec{y}}^2)_{\frac{1}{2},2} = L_t^2 L_{x\vec{y}}^{3,2} \hookrightarrow L_t^2 L_{x\vec{y}}^3 \tag{3.26}$$

where $L_{x\vec{y}}^{3,2}$ denotes the Lorentz space. Altogether, we can now deduce from (3.22) and (3.23) that

$$\|\tilde{\Delta}_k u\|_{L_t^2 L_{x\vec{y}}^3} \lesssim 2^{k\frac{\gamma}{2}} \|u\|_{X_0^{\frac{1}{4},0}} \tag{3.27}$$

Finally, this implies (3.17) by using standard Littlewood-Paley theory (as in the proof of Theorem 3.6). □

3.3 Bilinear Strichartz-type estimates: Generalities

In this section we are concerned with bilinear estimates, which express dispersive properties of the equation, just as the linear Strichartz estimates of the last section. In fact, the linear Strichartz estimates from the last section imply certain bilinear estimates. As an example, suppose $d = 2$ and $\alpha = 2$. Then, by Theorem 3.6, we have that $\|u\|_{L^4_{txy}} \lesssim \|u\|_{X_0^{b,0}}$ for $b > \frac{1}{2}$. Combining this with Hölder's inequality, we get that

$$\|u_1 u_2\|_{L^2} \le \|u_1\|_{L^4}\|u_2\|_{L^4} \lesssim \|u_1\|_{X_0^{b,0}}\|u_2\|_{X_0^{b,0}}$$

If we use Plancherel's theorem and the definition (2.9) of the $X_0^{b,0}$-norm, we see that this is equivalent to

$$\left\| \int_{\mathbb{R}^n} \frac{f_1(\mu_1) f_2(\mu_2) d\mu_1}{\langle\lambda_1\rangle^{b_1}\langle\lambda_2\rangle^{b_2}} \right\|_{L^2_\mu} \lesssim \|f_1\|_{L^2}\|f_2\|_{L^2}$$

where we let $f_i(\mu) := \langle\lambda\rangle^b \mathcal{F}u_i(\mu)$, $i = 1, 2$. By duality, the last estimate is equivalent to

$$\left|\int_{\mathbb{R}^{2n}} \frac{f_1(\mu_1)f_2(\mu_2)f_3(\mu)}{\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b} d\mu d\mu_1\right| \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2}$$

Because the L^2-norm of a function f only depends on the modulus of f, we finally see that this is equivalent to

$$\int_{\mathbb{R}^{2n}} \frac{f_1(\mu_1)f_2(\mu_2)f_3(\mu)}{\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b} d\mu d\mu_1 \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2}$$

for all $f_i \geq 0$. If we define $K(\mu_1, \mu) := \frac{1}{\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b}$, then we can write this last estimate as

$$\int_{\mathbb{R}^{2n}} K(\mu_1, \mu) f_1 f_2 f_3 d\mu_1 d\mu \leq A \prod_{i=1}^{3} \|f_i\|_{L^2(\mathbb{R}^n)} \tag{3.28}$$

where we used the convention that in an integral over μ_1 and μ, $f_1 f_2 f_3$ always means $f_1(\mu_1)f_2(\mu_2)f_3(\mu)$, where $\mu_2 = \mu - \mu_1$. We will derive other bilinear estimates of the type (3.28), which will not follow from the linear estimates in Section 3.2. Our main tool to derive these estimates is the use of the Cauchy-Schwarz inequality as shown in the next proposition.

Proposition 3.8. *If $K \geq 0$ such that*

$$A := \sup_{\mu \in \mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(\mu_1, \mu)^2 d\mu_1\right)^{\frac{1}{2}} < \infty \tag{3.29}$$

then we have (3.28) *for all $f_i \geq 0$.*

Proof. By using the Cauchy-Schwarz inequality in μ_1, we get that

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} K(\mu_1, \mu) f_1(\mu_1) f_2(\mu_2) f_3(\mu) d\mu_1 d\mu \\
&\quad \leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(\mu_1, \mu)^2 d\mu_1\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} f_1(\mu_1)^2 f_2(\mu_2)^2 d\mu_1\right)^{\frac{1}{2}} f_3(\mu)\, d\mu \\
&\quad \leq A \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f_1(\mu_1)^2 f_2(\mu_2)^2 d\mu_1\right)^{\frac{1}{2}} f_3(\mu)\, d\mu
\end{aligned}$$

If we now use the Cauchy-Schwarz inequality in μ, we finally obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} K(\mu_1, \mu) f_1(\mu_1) f_2(\mu_2) f_3(\mu) d\mu_1 d\mu \\
&\quad \leq A \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f_1(\mu_1)^2 f_2(\mu_2)^2 d\mu_1 d\mu\right)^{\frac{1}{2}} \|f_3\|_{L^2(\mathbb{R}^n)} = A \prod_{i=1}^{3} \|f_i\|_{L^2(\mathbb{R}^n)}
\end{aligned}$$

□

Proposition 3.9. *Let us suppose that we have $K_1, K_2 \geq 0$ such that for $j = 1, 2$*

$$\int_{\mathbb{R}^{2n}} K_j(\mu_1, \mu) f_1 f_2 f_3 d\mu_1 d\mu \leq A_j \prod_{i=1}^{3} \|f_i\|_{L^2(\mathbb{R}^n)}$$

for $f_i \geq 0$. Then, for every $\theta \in [0, 1]$, we have that

$$\int_{\mathbb{R}^{2n}} K_1(\mu_1, \mu)^{\theta} K_2(\mu_1, \mu)^{1-\theta} f_1 f_2 f_3 \; d\mu_1 d\mu \leq A_1^{\theta} A_2^{1-\theta} \prod_{i=1}^{3} \|f_i\|_{L^2(\mathbb{R}^n)} \tag{3.30}$$

for all $f_i \geq 0$.

Proof. For $\theta = 0$ and $\theta = 1$, there is nothing to prove. Suppose $\theta \in (0, 1)$. Let $p := 1/\theta$ and $p' := 1/(1 - \theta)$. Then $p, p' \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Therefore, using Hölder's inequality, we get that

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} K_1(\mu_1, \mu)^{\theta} K_2(\mu_1, \mu)^{1-\theta} f_1 f_2 f_3 \; d\mu_1 d\mu \\
&\quad = \int_{\mathbb{R}^{2n}} (K_1(\mu_1, \mu) f_1 f_2 f_3)^{\frac{1}{p}} (K_2(\mu_1, \mu) f_1 f_2 f_3)^{\frac{1}{p'}} d\mu_1 d\mu \\
&\quad \leq \left(\int_{\mathbb{R}^{2n}} K_1(\mu_1, \mu) f_1 f_2 f_3 d\mu_1 d\mu \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^{2n}} K_2(\mu_1, \mu) f_1 f_2 f_3 d\mu_1 d\mu \right)^{\frac{1}{p'}} \\
&\quad \leq A_1^{\theta} A_2^{1-\theta} \prod_{i=1}^{3} \|f_i\|_{L^2(\mathbb{R}^n)}
\end{aligned}$$

□

Proposition 3.10. *Let $K \geq 0$. Suppose that for all $f_i \geq 0$ we have*

$$\int_{\mathbb{R}^{2n}} K(\mu_1, \mu) f_1 f_2 f_3 d\mu_1 d\mu \leq A \prod_{i=1}^{3} \|f_i\|_{L^2(\mathbb{R}^n)} \tag{3.31}$$

We then also have that

$$\int_{\mathbb{R}^{2n}} K(\mu, \mu_1) f_1 f_2 f_3 d\mu_1 d\mu \leq A \prod_{i=1}^{3} \|f_i\|_{L^2(\mathbb{R}^n)} \tag{3.32}$$

$$\int_{\mathbb{R}^{2n}} K(\mu_1, -\mu_2) f_1 f_2 f_3 d\mu_1 d\mu \leq A \prod_{i=1}^{3} \|f_i\|_{L^2(\mathbb{R}^n)} \tag{3.33}$$

$$\int_{\mathbb{R}^{2n}} K(\mu_2, \mu) f_1 f_2 f_3 d\mu_1 d\mu \leq A \prod_{i=1}^{3} \|f_i\|_{L^2(\mathbb{R}^n)} \tag{3.34}$$

for all $f_i \geq 0$.

Proof. Let us prove (3.33). The other proofs are similar. We use the change of variables $T : (\mu_1, \mu) \mapsto (\mu_1', \mu')$ with $\mu_1' = \mu_1$ and $\mu' = \mu_1 - \mu = -\mu_2$ in the integral on the left hand side of (3.33) to see that this integral is equal to

$$\int_{\mathbb{R}^{2n}} K(\mu_1', \mu') f_1(\mu_1') f_2(-\mu') f_3(-\mu_2')\ d\mu_1' d\mu'$$

If we put $g_1 := f_1$, $g_2 := f_3(-\cdot)$ and $g_3 := f_2(-\cdot)$, we can use (3.31) to see that the last integral is less or equal to

$$A \prod_{i=1}^{3} \|g_i\|_{L^2(\mathbb{R}^n)} = A \prod_{i=1}^{3} \|f_i\|_{L^2(\mathbb{R}^n)}$$

□

3.4 Bilinear Strichartz type estimates in two dimensions

In this section we will derive bilinear estimates of type (3.28) for $d = 2$. These estimates will then be used in Chapter 4 to deduce local well-posedness results for the two dimensional generalised Kadomtsev-Petviashvili II equation. Let us assume $d = 2$ throughout this section. Recall the convention that in an integral over μ_1 and μ, $f_1 f_2 f_3$ always means $f_1(\mu_1) f_2(\mu_2) f_3(\mu)$, where $\mu_2 = \mu - \mu_1$.

Corollary 3.11. *For $b > \frac{1}{2}$ it holds that*

$$\|u_1 u_2\|_{L^2} \lesssim \||D_x|^{\frac{1}{4} - \frac{\alpha}{8}} u_1\|_{X_0^{b,0}} \||D_x|^{\frac{1}{4} - \frac{\alpha}{8}} u_2\|_{X_0^{b,0}} \tag{3.35}$$

Furthermore, we have

$$\int_{\mathbb{R}^6} \frac{|\xi_1|^{-\frac{1}{4} + \frac{\alpha}{8}} |\xi_2|^{-\frac{1}{4} + \frac{\alpha}{8}}}{\langle \lambda_1 \rangle^b \langle \lambda_2 \rangle^b} f_1 f_2 f_3\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{3.36}$$

$$\int_{\mathbb{R}^6} \frac{|\xi_1|^{-\frac{1}{4} + \frac{\alpha}{8}} |\xi|^{-\frac{1}{4} + \frac{\alpha}{8}}}{\langle \lambda_1 \rangle^b \langle \lambda \rangle^b} f_1 f_2 f_3\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{3.37}$$

$$\int_{\mathbb{R}^6} \frac{|\xi|^{-\frac{1}{4} + \frac{\alpha}{8}} |\xi_2|^{-\frac{1}{4} + \frac{\alpha}{8}}}{\langle \lambda \rangle^b \langle \lambda_2 \rangle^b} f_1 f_2 f_3\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{3.38}$$

Proof. Letting $r = q = 4$ in (3.11), we get that $\||D_x|^{-(\frac{1}{4}-\frac{\alpha}{8})}u\|_{L^4_{txy}} \lesssim \|u\|_{X_0^{b,0}}$ or equivalently $\|u\|_{L^4_{txy}} \lesssim \||D_x|^{\frac{1}{4}-\frac{\alpha}{8}}u\|_{X_0^{b,0}}$. Now, (3.35) follows by combining this estimate with Hölder's inequality. Letting $f_i(\mu) := |\xi|^{\frac{1}{4}-\frac{\alpha}{8}}\langle\lambda\rangle^b\mathcal{F}u_i(\mu)$ for $i = 1, 2$ and using duality, we see that (3.35) is equivalent to (3.36). By Proposition 3.10, we also obtain (3.37) and (3.38). □

For the part of the product u_1u_2 where the ξ-frequency of the first factor is significantly smaller than the ξ-frequency of the second factor, we can improve this bilinear Strichartz estimate. To formulate this improvement, let us define for $c > 0$ the following operator:

$$\mathcal{F}P_c(u_1, u_2)(\mu) := \int_{\mathbb{R}^n} \chi_{|\xi_1|\leq c|\xi_2|}\mathcal{F}u_1(\mu_1)\mathcal{F}u_2(\mu_2)\, d\mu_1 \tag{3.39}$$

We have the following refined bilinear Strichartz estimate, which for the case $\alpha = 2$ was already implicitly used in [13, 29–31, 33, 34].

Theorem 3.12. *For $b > \frac{1}{2}$ it holds that*

$$\|P_{\frac{1}{3}}(u_1, u_2)\|_{L^2} \lesssim \||D_x|^{\frac{1}{2}}u_1\|_{X_0^{b,0}}\||D_x|^{-\frac{\alpha}{4}}u_2\|_{X_0^{b,0}} \tag{3.40}$$

For the proof of the theorem, we need the following lemma.

Lemma 3.13. *For $\alpha > 0$ let $\phi_\alpha(\xi) := \xi|\xi|^\alpha$ and*

$$r_\alpha(\xi, \xi_1) := \phi_\alpha(\xi) - \phi_\alpha(\xi_1) - \phi_\alpha(\xi_2), \quad \xi, \xi_1 \in \mathbb{R} \tag{3.41}$$

We then have for every $\xi, \xi_1 \in \mathbb{R}$ that

$$\frac{\alpha}{2^\alpha}|\xi_{\min}||\xi_{\max}|^\alpha \leq |r_\alpha(\xi, \xi_1)| \leq (\alpha + 1 + \frac{1}{2^\alpha})|\xi_{\min}||\xi_{\max}|^\alpha \tag{3.42}$$

Proof of Lemma 3.13. Suppose first that $|\xi_{\min}| = |\xi_1|$. We then have that

$$|\phi_\alpha(\xi_1)| = |\xi_{\min}|^{\alpha+1} \leq \frac{1}{2^\alpha}|\xi_{\min}||\xi_{\max}|^\alpha$$

because $|\xi_{\min}| \leq \frac{1}{2}|\xi_{\max}|$. Furthermore, there is a $\theta \in [0, 1]$ such that

$$|\phi_\alpha(\xi) - \phi_\alpha(\xi_2)| = |\phi'_\alpha(\xi - \theta\xi_1)||\xi_1| = (\alpha + 1)|\xi - \theta\xi_1|^\alpha|\xi_{\min}|$$

Because $|\xi_1| \leq |\xi|$, it follows that

$$\min_{\theta\in[0,1]} |\xi - \theta\xi_1| = \min\{|\xi|, |\xi_2|\} = |\xi_{\text{med}}| \geq \frac{1}{2}|\xi_{\max}|$$

and

$$\max_{\theta\in[0,1]} |\xi-\theta\xi_1| = \max\{|\xi|,|\xi_2|\} = |\xi_{\max}|$$

Combining these estimates, we get that

$$\begin{aligned}|r_\alpha(\xi,\xi_1)| &\geq |\phi_\alpha(\xi)-\phi_\alpha(\xi_2)| - |\phi_\alpha(\xi_1)| \\ &\geq (\alpha+1)\frac{1}{2^\alpha}|\xi_{\max}|^\alpha|\xi_{\min}| - \frac{1}{2^\alpha}|\xi_{\min}||\xi_{\max}|^\alpha \\ &= \frac{\alpha}{2^\alpha}|\xi_{\min}||\xi_{\max}|^\alpha\end{aligned}$$

and

$$\begin{aligned}|r_\alpha(\xi,\xi_1)| &\leq |\phi_\alpha(\xi)-\phi_\alpha(\xi_2)| + |\phi_\alpha(\xi_1)| \\ &\leq (\alpha+1)|\xi_{\max}|^\alpha|\xi_{\min}| + \frac{1}{2^\alpha}|\xi_{\min}||\xi_{\max}|^\alpha \\ &= (\alpha+1+\frac{1}{2^\alpha})|\xi_{\min}||\xi_{\max}|^\alpha\end{aligned}$$

which proves (3.42) in the case $|\xi_{\min}| = |\xi_1|$. Taking into account that $r_\alpha(\xi,\xi_1) = r_\alpha(\xi,\xi_2) = -r_\alpha(\xi_2,\xi)$, we see that (3.42) also holds in the other cases. □

Proof of Theorem 3.12. Let

$$f_1(\mu) := |\xi|^{\frac{1}{2}}\langle\lambda\rangle^b\mathcal{F}u_1(\mu), \quad f_2(\mu) := |\xi|^{-\frac{\alpha}{4}}\langle\lambda\rangle^b\mathcal{F}u_2(\mu)$$

We have to show that

$$\left\|\int_{\mathbb{R}^3}\chi_{|\xi_1|\leq\frac{1}{3}|\xi_2|}\frac{|\xi_1|^{-\frac{1}{2}}|\xi_2|^{\frac{\alpha}{4}}}{\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b}f_1(\mu_1)f_2(\mu_2)d\mu_1\right\|_{L^2_\mu} \lesssim \|f_1\|_{L^2}\|f_2\|_{L^2}$$

which by duality is equivalent to

$$\int_{\mathbb{R}^6}\chi_{|\xi_1|\leq\frac{1}{3}|\xi_2|}\frac{|\xi_1|^{-\frac{1}{2}}|\xi_2|^{\frac{\alpha}{4}}}{\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b}f_1f_2f_3\,d\mu_1d\mu \lesssim \prod_{i=1}^3\|f_i\|_{L^2} \tag{3.43}$$

for $f_i\geq 0$. By Proposition 3.8, it suffices to show that $\sup_\mu I(\mu)^{\frac{1}{2}} < \infty$ where

$$I(\mu) := \int_{\mathbb{R}^3}\chi_{|\xi_1|\leq\frac{1}{3}|\xi_2|}\frac{|\xi_1|^{-1}|\xi_2|^{\frac{\alpha}{2}}}{\langle\lambda_1\rangle^{2b}\langle\lambda_2\rangle^{2b}}\,d\mu_1$$

For fixed μ, we now use the change of variables $T:\mu_1\mapsto(\nu,\lambda_1,\lambda_2)$, where

$$\nu(\mu_1) := r_\alpha(\xi,\xi_1) = \xi|\xi|^\alpha - \xi_1|\xi_1|^\alpha - \xi_2|\xi_2|^\alpha$$

Let us also recall the definition of λ_1 and λ_2

$$\lambda_1(\mu_1) = \tau_1 - \xi_1|\xi_1|^\alpha + \frac{\eta_1^2}{\xi_1}$$
$$\lambda_2(\mu_1) = \tau_2 - \xi_2|\xi_2|^\alpha + \frac{\eta_2^2}{\xi_2}$$

Observe that

$$\lambda_1 + \lambda_2 - \lambda = \nu + \frac{(\xi\eta_1 - \eta\xi_1)^2}{\xi\xi_1\xi_2} \tag{3.44}$$

Therefore, we have that

$$|\partial_{\eta_1}(\lambda_1+\lambda_2)| = 2|\xi\frac{\xi\eta_1 - \eta\xi_1}{\xi\xi_1\xi_2}| = 2\frac{|\xi|^{\frac{1}{2}}|\lambda_1+\lambda_2-\lambda-\nu|^{\frac{1}{2}}}{|\xi_1|^{\frac{1}{2}}|\xi_2|^{\frac{1}{2}}}$$

Furthermore, we have $\partial_{\xi_1}\nu = (\alpha+1)(|\xi_2|^\alpha - |\xi_1|^\alpha)$. Since we only consider the region where $|\xi_1| \leq \frac{1}{3}|\xi_2|$, which implies $|\xi_1| = |\xi_{\min}|$ and $|\xi_2| \sim |\xi| \sim |\xi_{\max}|$, it follows by (3.42) that $|\nu| \sim |\xi_1||\xi_2|^\alpha$. We also have $|\partial_{\xi_1}\nu| \gtrsim |\xi_2|^\alpha$ in this region. Therefore, we deduce that

$$\begin{aligned}|\det D_{\mu_1}T| &= |\partial_{\xi_1}\nu||\partial_{\eta_1}\lambda_1 + \partial_{\eta_1}\lambda_2| \gtrsim |\xi_1|^{-\frac{1}{2}}|\xi_2|^\alpha|\lambda_1+\lambda_2-\lambda-\nu|^{\frac{1}{2}}\\ &\gtrsim |\xi_1|^{-1}|\xi_2|^{\frac{\alpha}{2}}|\nu|^{\frac{1}{2}}|\lambda_1+\lambda_2-\lambda-\nu|^{\frac{1}{2}}\end{aligned}$$

Let us note that it is possible to divide the region of integration into a finite number of open subsets U_i such that T is an injective C^1-function in U_i with non vanishing Jacobian. Because we are in the KP II case, both terms on the right hand side of (3.44) have the same sign, which implies that $|\nu| \leq |\lambda_1+\lambda_2-\lambda|$. So, performing the change of variables and using the elementary inequality

$$\int_{-K}^{K} \frac{d\nu}{|\nu|^{\frac{1}{2}}|a-\nu|^{\frac{1}{2}}} \lesssim \frac{K^{\frac{1}{2}}}{|a|^{\frac{1}{2}}}, \quad a \neq 0$$

we obtain

$$I(\mu) \lesssim \int_{\mathbb{R}^3} \frac{\chi_{|\nu|\leq|\lambda_1+\lambda_2-\lambda|} d\nu d\lambda_1 d\lambda_2}{\langle\lambda_1\rangle^{2b}\langle\lambda_2\rangle^{2b}|\nu|^{\frac{1}{2}}|\lambda_1+\lambda_2-\lambda-\nu|^{\frac{1}{2}}} \lesssim \int_{\mathbb{R}^2} \frac{d\lambda_1 d\lambda_2}{\langle\lambda_1\rangle^{2b}\langle\lambda_2\rangle^{2b}} \lesssim 1$$

□

In fact, (3.43) also holds without the cut-off function $\chi_{|\xi_1|\leq\frac{1}{3}|\xi_2|}$ and we also get dual versions of (3.43).

Proposition 3.14. *For $b > \frac{1}{2}$ we have that*

$$\int_{\mathbb{R}^6} \frac{|\xi_1|^{-\frac{1}{2}}|\xi_2|^{\frac{\alpha}{4}}}{\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{3.45}$$

$$\int_{\mathbb{R}^6} \frac{|\xi_1|^{-\frac{1}{2}}|\xi|^{\frac{\alpha}{4}}}{\langle\lambda_1\rangle^b\langle\lambda\rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{3.46}$$

$$\int_{\mathbb{R}^6} \frac{|\xi|^{-\frac{1}{2}}|\xi_2|^{\frac{\alpha}{4}}}{\langle\lambda\rangle^b\langle\lambda_2\rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{3.47}$$

$$\int_{\mathbb{R}^6} \frac{|\xi|^{-\frac{1}{2}}|\xi_1|^{\frac{\alpha}{4}}}{\langle\lambda\rangle^b\langle\lambda_1\rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{3.48}$$

Proof. In the region where $|\xi_1| \leq \frac{1}{3}|\xi_2|$ the estimate (3.45) follows from (3.43). In the region where $|\xi_1| > \frac{1}{3}|\xi_2|$ we have that $|\xi_1|^{-\frac{1}{2}}|\xi_2|^{\frac{\alpha}{4}} \lesssim |\xi_1|^{-\frac{1}{4}+\frac{\alpha}{8}}|\xi_2|^{-\frac{1}{4}+\frac{\alpha}{8}}$, so that the estimate in this region follows from the bilinear Strichartz estimate (3.36). We then get (3.46), (3.47) and (3.48) by Proposition 3.10. $\square$

3.5 Bilinear Strichartz type estimates in three dimensions

In this section we will derive bilinear estimates of type (3.28) for $d = 3$. These estimates will then be used in Chapter 5 to deduce local well-posedness results for the three dimensional generalised Kadomtsev-Petviashvili II equation. Let us assume $d = 3$ and $\alpha > 1$ throughout this section. Recall the convention that in an integral over μ_1 and μ, $f_1 f_2 f_3$ always means $f_1(\mu_1)f_2(\mu_2)f_3(\mu)$, where $\mu_2 = \mu - \mu_1$.

Theorem 3.15. *We have for $b > \frac{1}{2}$ and $f_i \geq 0$ that*

$$\int_{\mathbb{R}^8} \frac{|\xi\xi_1\xi_2|^{-\frac{1}{4}+\frac{\alpha}{12}}}{\langle\lambda\rangle^{\frac{1}{4}}\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{3.49}$$

as well as

$$\int_{\mathbb{R}^8} \frac{|\xi\xi_1\xi_2|^{-\frac{1}{4}+\frac{\alpha}{12}}}{\langle\lambda\rangle^b\langle\lambda_1\rangle^{\frac{1}{4}}\langle\lambda_2\rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{3.50}$$

Proof. By Theorem 3.7, it follows that

$$\||D_x|^{-\frac{1}{4}+\frac{\alpha}{12}}u\|_{L_t^4 L_{xy}^3} \lesssim \|u\|_{X_0^{b,0}}$$

and

$$\||D_x|^{-\frac{1}{4}+\frac{\alpha}{12}}u\|_{L_t^2L_{x\vec{y}}^3} \lesssim \|u\|_{X_0^{\frac{1}{4},0}}$$

By the definition of the $X_0^{b,0}$-norm, this can be rewritten as

$$\|\mathcal{F}^{-1}(|\xi|^{-\frac{1}{4}+\frac{\alpha}{12}}\langle\lambda\rangle^{-b}f)\|_{L_t^4L_{x\vec{y}}^3} \lesssim \|f\|_{L^2} \tag{3.51}$$

and

$$\|\mathcal{F}^{-1}(|\xi|^{-\frac{1}{4}+\frac{\alpha}{12}}\langle\lambda\rangle^{-\frac{1}{4}}f)\|_{L_t^2L_{x\vec{y}}^3} \lesssim \|f\|_{L^2} \tag{3.52}$$

Using Plancherel's theorem and Hölder's inequality, we see that the left-hand side of (3.49) is bounded by

$$c\prod_{i=1}^{2}\|\mathcal{F}^{-1}(|\xi|^{-\frac{1}{4}+\frac{\alpha}{12}}\langle\lambda\rangle^{-b}f_i)\|_{L_t^4L_{x\vec{y}}^3}\|\mathcal{F}^{-1}(|\xi|^{-\frac{1}{4}+\frac{\alpha}{12}}\langle\lambda\rangle^{-\frac{1}{4}}f_3)\|_{L_t^2L_{x\vec{y}}^3}$$

Combining this with (3.51) and (3.52), we obtain (3.49). Furthermore, (3.50) follows from (3.49) by Proposition 3.10. □

Theorem 3.16. *For $b > \frac{1}{2}$, $\delta > 0$ and $f_i \geq 0$, we have that*

$$\int_{\mathbb{R}^8} \frac{|\xi|^{\frac{1}{2}}|\xi_1|^{-\frac{1}{2}}|\xi_2|^{-\frac{1}{2}}}{\langle\xi_1\rangle^{\frac{1}{2}+\delta}\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b} f_1f_2f_3\,d\mu_1d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2} \tag{3.53}$$

Proof. By Proposition 3.8, it suffices to show that $\sup_\mu I(\mu)^{\frac{1}{2}} < \infty$, where

$$I(\mu) := |\xi|\int_{\mathbb{R}^4} \frac{|\xi_1|^{-1}|\xi_2|^{-1}\,d\mu_1}{\langle\xi_1\rangle^{1+2\delta}\langle\lambda_1\rangle^{2b}\langle\lambda_2\rangle^{2b}}$$

For fixed μ and ξ_1, we use the change of variables $S : (\tau_1, \eta_1, \tilde{\eta}_1) \mapsto (\theta, \lambda_1, \lambda_2)$, where

$$\theta(\tau_1, \eta_1, \tilde{\eta}_1) := \frac{(\xi\eta_1 - \eta\xi_1)^2}{\xi\xi_1\xi_2}$$

Let us recall the definition of λ_1 and λ_2

$$\lambda_1(\mu_1) = \tau_1 - \xi_1|\xi_1|^\alpha + \frac{\eta_1^2}{\xi_1} + \frac{\tilde{\eta}_1^2}{\xi_1}$$

$$\lambda_2(\mu_1) = \tau_2 - \xi_2|\xi_2|^\alpha + \frac{\eta_2^2}{\xi_2} + \frac{\tilde{\eta}_2^2}{\xi_2}$$

Observe that

$$\lambda_1 + \lambda_2 - \lambda - \nu = \theta + \frac{(\xi\tilde{\eta}_1 - \tilde{\eta}\xi_1)^2}{\xi\xi_1\xi_2} \tag{3.54}$$

where ν only depends on ξ_1. Therefore, we have that

$$|\partial_{\eta_1}\theta| = 2|\xi\frac{\xi\eta_1 - \eta\xi_1}{\xi\xi_1\xi_2}| = 2\frac{|\xi|^{\frac{1}{2}}|\theta|^{\frac{1}{2}}}{|\xi_1|^{\frac{1}{2}}|\xi_2|^{\frac{1}{2}}}$$

and

$$|\partial_{\tilde{\eta}_1}(\lambda_1+\lambda_2)| = 2\frac{|\xi|^{\frac{1}{2}}|\lambda_1+\lambda_2-\lambda-\nu-\theta|^{\frac{1}{2}}}{|\xi_1|^{\frac{1}{2}}|\xi_2|^{\frac{1}{2}}}$$

Altogether, we get

$$\begin{aligned} |\det D_{(\tau_1,\eta_1,\tilde{\eta}_1)}S| &= |\partial_{\eta_1}\theta||\partial_{\tilde{\eta}_1}(\lambda_1+\lambda_2)| \\ &= 4|\xi||\xi_1|^{-1}|\xi_2|^{-1}|\theta|^{\frac{1}{2}}|\lambda_1+\lambda_2-\lambda-\nu-\theta|^{\frac{1}{2}} \end{aligned}$$

Let us note that it is possible to divide the region of integration into a finite number of open subsets U_i such that S is an injective C^1-function in U_i with non vanishing Jacobian. Both terms on the right hand side of (3.54) have the same sign, which implies that $|\theta| \le |\lambda_1+\lambda_2-\lambda-\nu|$. So, performing the change of variables, we obtain

$$I(\mu) \lesssim \int_{\mathbb{R}} \frac{1}{\langle\xi_1\rangle^{1+2\delta}} \left(\int_{\mathbb{R}^3} \frac{\chi_{|\theta|\le|\lambda_1+\lambda_2-\lambda-\nu|}\, d\theta d\lambda_1 d\lambda_2}{\langle\lambda_1\rangle^{2b}\langle\lambda_2\rangle^{2b}|\theta|^{\frac{1}{2}}|\lambda_1+\lambda_2-\lambda-\nu-\theta|^{\frac{1}{2}}} \right) d\xi_1$$

Using the elementary inequality

$$\int_{-K}^{K} \frac{dx}{|x|^{\frac{1}{2}}|a-x|^{\frac{1}{2}}} \lesssim \frac{K^{\frac{1}{2}}}{|a|^{\frac{1}{2}}}, \quad a \neq 0 \tag{3.55}$$

we find that

$$\int_{\mathbb{R}^3} \frac{\chi_{|\theta|\le|\lambda_1+\lambda_2-\lambda-\nu|}\, d\theta d\lambda_1 d\lambda_2}{\langle\lambda_1\rangle^{2b}\langle\lambda_2\rangle^{2b}|\theta|^{\frac{1}{2}}|\lambda_1+\lambda_2-\lambda-\nu-\theta|^{\frac{1}{2}}} \lesssim 1$$

Altogether, we deduce that

$$I(\mu) \lesssim \int_{\mathbb{R}} \frac{d\xi_1}{\langle\xi_1\rangle^{1+2\delta}} \lesssim 1$$

□

Remark 3.17. By Proposition 3.10, (3.53) implies that

$$\int_{\mathbb{R}^8} \frac{|\xi|^{-\frac{1}{2}}|\xi_1|^{-\frac{1}{2}}|\xi_2|^{\frac{1}{2}}}{\langle\xi_1\rangle^{\frac{1}{2}+\delta}\langle\lambda\rangle^b\langle\lambda_1\rangle^b} f_1 f_2 f_3\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2} \tag{3.56}$$

$$\int_{\mathbb{R}^8} \frac{|\xi|^{-\frac{1}{2}}|\xi_1|^{\frac{1}{2}}|\xi_2|^{-\frac{1}{2}}}{\langle\xi\rangle^{\frac{1}{2}+\delta}\langle\lambda\rangle^b\langle\lambda_2\rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{3.57}$$

$$\int_{\mathbb{R}^8} \frac{|\xi|^{-\frac{1}{2}}|\xi_1|^{-\frac{1}{2}}|\xi_2|^{\frac{1}{2}}}{\langle\xi\rangle^{\frac{1}{2}+\delta}\langle\lambda\rangle^b\langle\lambda_1\rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{3.58}$$

Let

$$\Xi_1 = \{(\mu_1,\mu) \in \mathbb{R}^8 \mid |\xi_1| \le \frac{1}{3}|\xi_2|, |\xi_2| \ge 1\}$$
$$\Xi_2 = \{(\mu_1,\mu) \in \mathbb{R}^8 \mid \frac{1}{3}|\xi_2| \le |\xi_1| \le |\xi_2|, |\xi_2| \ge 1\}$$

If we now interpolate between (3.49) and (3.53) restricted to Ξ_2, we get

Corollary 3.18. *We have for $\delta > 0$, $\theta \in [0,1)$ and $f_i \ge 0$ that*

$$\int_{\Xi_2} \frac{|\xi|^{\frac{1}{2}-\theta(\frac{3}{4}-\frac{\alpha}{12})}|\xi_1|^{-\frac{3}{2}+\theta(1+\frac{\alpha}{6})-\delta}}{\langle\lambda\rangle^{\frac{\theta}{4}}\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{3.59}$$

Proof. In Ξ_2, we have that $\langle\xi_1\rangle \sim |\xi_1| \sim |\xi_2|$. Therefore, (3.49) implies

$$\int_{\Xi_2} \frac{|\xi|^{-\frac{1}{4}+\frac{\alpha}{12}}|\xi_1|^{-\frac{1}{2}+\frac{\alpha}{6}}}{\langle\lambda\rangle^{\frac{1}{4}}\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{3.60}$$

Similarly, (3.53) implies

$$\int_{\Xi_2} \frac{|\xi|^{\frac{1}{2}}|\xi_1|^{-\frac{3}{2}-\frac{\delta}{1-\theta}}}{\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{3.61}$$

By Proposition 3.9, (3.59) now follows from (3.60) and (3.61). □

Theorem 3.19. *For $b > \frac{1}{2}$ and $s > \frac{1}{2}$ we have that*

$$\int_{\Xi_1} \frac{|\xi|^{\frac{\alpha}{4}}|\xi_1|^{-\frac{1}{2}}\langle\vec{\eta}\rangle^s f_1 f_2 f_3}{\langle\vec{\eta}_1\rangle^s\langle\vec{\eta}_2\rangle^s\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b} \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{3.62}$$

Proof. By Proposition 3.8, it suffices to show that $\sup_\mu I(\mu)^{\frac{1}{2}} < \infty$, where

$$I(\mu) := \int_{\mathbb{R}^4} \frac{\chi_{\Xi_1}|\xi|^{\frac{\alpha}{2}}|\xi_1|^{-1}\langle\vec{\eta}\rangle^{2s} d\mu_1}{\langle\vec{\eta}_1\rangle^{2s}\langle\vec{\eta}_2\rangle^{2s}\langle\lambda_1\rangle^{2b}\langle\lambda_2\rangle^{2b}}$$

We write $I(\mu) = I_1(\mu) + I_2(\mu)$, where $I_1(\mu)$ and $I_2(\mu)$ are the parts of the integral where we have $|\xi\eta_1 - \eta\xi_1| \geq |\xi\tilde{\eta}_1 - \tilde{\eta}\xi_1|$ and $|\xi\eta_1 - \eta\xi_1| \leq |\xi\tilde{\eta}_1 - \tilde{\eta}\xi_1|$, respectively. Let us note that by an exchange of the variables η_1 and $\tilde{\eta}_1$ in the integral $I_2(\mu)$, it is easy to see that $I_2(\tau, \xi, \eta, \tilde{\eta}) = I_1(\tau, \xi, \tilde{\eta}, \eta)$. Hence, it suffices to consider $I_1(\mu)$. We have that

$$\frac{\langle \vec{\eta} \rangle^{2s}}{\langle \vec{\eta}_1 \rangle^{2s} \langle \vec{\eta}_2 \rangle^{2s}} \lesssim \frac{1}{\langle \tilde{\eta}_1 \rangle^{2s}} + \frac{1}{\langle \tilde{\eta}_2 \rangle^{2s}}$$

and therefore

$$\begin{aligned} I_1(\mu) \lesssim \int_{\mathbb{R}} &\left(\frac{1}{\langle \tilde{\eta}_1 \rangle^{2s}} + \frac{1}{\langle \tilde{\eta}_2 \rangle^{2s}} \right) \times \\ &\left(\int_{\mathbb{R}^3} \chi_{|\xi_1| \leq \frac{1}{3}|\xi_2|} \chi_{|\xi\eta_1 - \eta\xi_1| \geq |\xi\tilde{\eta}_1 - \tilde{\eta}\xi_1|} \frac{|\xi|^{\frac{\alpha}{2}} |\xi_1|^{-1}}{\langle \lambda_1 \rangle^{2b} \langle \lambda_2 \rangle^{2b}} \, d\tau_1 d\xi_1 d\eta_1 \right) \, d\tilde{\eta}_1 \end{aligned}$$

For fixed μ and $\tilde{\eta}_1$, we use the change of variables $S : (\tau_1, \xi_1, \eta_1) \mapsto (\nu, \lambda_1, \lambda_2)$, where $\nu(\xi_1) := \xi|\xi|^\alpha - \xi_1|\xi_1|^\alpha - \xi_2|\xi_2|^\alpha$. Observe that

$$\lambda_1 + \lambda_2 - \lambda = \nu(\xi_1) + \frac{(\xi\eta_1 - \eta\xi_1)^2}{\xi\xi_1\xi_2} + \frac{(\xi\tilde{\eta}_1 - \tilde{\eta}\xi_1)^2}{\xi\xi_1\xi_2} \tag{3.63}$$

Therefore, it follows that

$$\begin{aligned} |\partial_{\eta_1}(\lambda_1 + \lambda_2)| &= 2|\xi \frac{\xi\eta_1 - \eta\xi_1}{\xi\xi_1\xi_2}| \\ &\geq \frac{|\xi|^{\frac{1}{2}}}{|\xi_1|^{\frac{1}{2}} |\xi_2|^{\frac{1}{2}}} \left(\frac{|\xi\eta_1 - \eta\xi_1|}{|\xi|^{\frac{1}{2}} |\xi_1|^{\frac{1}{2}} |\xi_2|^{\frac{1}{2}}} + \frac{|\xi\tilde{\eta}_1 - \tilde{\eta}\xi_1|}{|\xi|^{\frac{1}{2}} |\xi_1|^{\frac{1}{2}} |\xi_2|^{\frac{1}{2}}} \right) \end{aligned}$$

where for the last inequality we used that $|\xi\eta_1 - \eta\xi_1| \geq |\xi\tilde{\eta}_1 - \tilde{\eta}\xi_1|$. Using (3.63), we get

$$|\partial_{\eta_1}(\lambda_1 + \lambda_2)| \gtrsim \frac{|\xi|^{\frac{1}{2}} |\lambda_1 + \lambda_2 - \lambda - \nu|^{\frac{1}{2}}}{|\xi_1|^{\frac{1}{2}} |\xi_2|^{\frac{1}{2}}}$$

Furthermore, taking into account that we are in the region where $|\xi_1| \leq \frac{1}{3}|\xi_2|$, we see that

$$|\partial_{\xi_1}\nu| \gtrsim |\xi_2|^\alpha \sim |\xi_2|^{\frac{\alpha}{2}} |\xi_1|^{-\frac{1}{2}} |\nu|^{\frac{1}{2}}$$

Using also that $|\xi| \sim |\xi_2|$ in this region, we finally obtain

$$\begin{aligned} |\det D_{(\tau_1, \xi_1, \eta_1)} S| &= |\partial_{\xi_1}\nu| |\partial_{\eta_1}(\lambda_1 + \lambda_2)| \\ &\gtrsim |\xi|^{\frac{\alpha}{2}} |\xi_1|^{-1} |\nu|^{\frac{1}{2}} |\lambda_1 + \lambda_2 - \lambda - \nu|^{\frac{1}{2}} \end{aligned}$$

Let us note that it is possible to divide the region of integration into a finite number of open subsets U_i such that S is an injective C^1-function in U_i with non vanishing Jacobian. Because we are in the KP II case, all terms on the right hand side of (3.63) have the same sign, which implies that $|\nu| \leq |\lambda_1 + \lambda_2 - \lambda|$. So, performing the change of variables, we obtain

$$I_1(\mu) \lesssim \int_{\mathbb{R}} \left(\frac{1}{\langle \tilde{\eta}_1 \rangle^{2s}} + \frac{1}{\langle \tilde{\eta}_2 \rangle^{2s}} \right) \left(\int_{\mathbb{R}^3} \frac{\chi_{|\nu| \leq |\lambda_1+\lambda_2-\lambda|} \; d\nu d\lambda_1 d\lambda_2}{\langle \lambda_1 \rangle^{2b} \langle \lambda_2 \rangle^{2b} |\nu|^{\frac{1}{2}} |\lambda_1 + \lambda_2 - \lambda - \nu|^{\frac{1}{2}}} \right) d\tilde{\eta}_1$$

Using the elementary inequality (3.55), we deduce that

$$\int_{\mathbb{R}^3} \frac{\chi_{|\nu| \leq |\lambda_1+\lambda_2-\lambda|} \; d\nu d\lambda_1 d\lambda_2}{\langle \lambda_1 \rangle^{2b} \langle \lambda_2 \rangle^{2b} |\nu|^{\frac{1}{2}} |\lambda_1 + \lambda_2 - \lambda - \nu|^{\frac{1}{2}}} \lesssim 1$$

Altogether, it follows that

$$I_1(\mu) \lesssim \int_{\mathbb{R}} \frac{1}{\langle \tilde{\eta}_1 \rangle^{2s}} d\tilde{\eta}_1 + \int_{\mathbb{R}} \frac{1}{\langle \tilde{\eta}_2 \rangle^{2s}} d\tilde{\eta}_1 \lesssim 1$$

where for the last inequality we used that $2s > 1$. □

Corollary 3.20. *For $b, \bar{b} > \frac{1}{2}$, $\delta > 0$, $s_2 > 0$, $0 < \theta < \min(1, 2s_2)$ and $f_i \geq 0$ we have that*

$$\int_{\Xi_1} \frac{|\xi|^{\theta \frac{\alpha}{4}} |\xi_1|^{-\frac{1}{2}} \langle \xi_1 \rangle^{-\frac{1-\theta}{2} - \delta} \langle \vec{\eta} \rangle^{s_2}}{\langle \vec{\eta}_1 \rangle^{s_2} \langle \vec{\eta}_2 \rangle^{s_2} \langle \lambda_1 \rangle^{b} \langle \lambda_2 \rangle^{b}} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{3.64}$$

and

$$\int_{\Xi_1} \frac{|\xi|^{\theta \frac{\alpha}{4}} |\xi_1|^{-\frac{1}{2}} \langle \xi_1 \rangle^{-\frac{1-\theta}{2} - \delta} \langle \vec{\eta} \rangle^{s_2}}{\langle \vec{\eta}_1 \rangle^{s_2} \langle \vec{\eta}_2 \rangle^{s_2} \langle \lambda_1 \rangle^{b} \langle \lambda_2 \rangle^{\theta \bar{b}} \langle \lambda \rangle^{(1-\theta)\bar{b}}} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{3.65}$$

Proof. In Ξ_1, we have that $|\xi| \sim |\xi_2|$, i. e. $|\xi|^{\frac{1}{2}} |\xi_2|^{-\frac{1}{2}} \sim 1$. Now, (3.53) with δ replaced by $\delta/(1-\theta)$ implies

$$\int_{\Xi_1} \frac{|\xi_1|^{-\frac{1}{2}} \langle \xi_1 \rangle^{-\frac{1}{2} - \frac{\delta}{1-\theta}}}{\langle \lambda_1 \rangle^{b} \langle \lambda_2 \rangle^{b}} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2}$$

By Proposition 3.9, the last inequality and (3.62) with $s = s_2/\theta > 1/2$ imply that (3.64) holds.

Similarly, (3.56) with δ replaced by $\delta/(1-\theta)$ and $\langle \lambda \rangle^{b}$ replaced by $\langle \lambda \rangle^{\bar{b}}$ implies that

$$\int_{\Xi_1} \frac{|\xi_1|^{-\frac{1}{2}} \langle \xi_1 \rangle^{-\frac{1}{2} - \frac{\delta}{1-\theta}}}{\langle \lambda \rangle^{\bar{b}} \langle \lambda_1 \rangle^{b}} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2}$$

Now again, by Proposition 3.9, the last inequality and (3.62) with $s = s_2/\theta$ and $\langle \lambda_2 \rangle^{b}$ replaced by $\langle \lambda_2 \rangle^{\bar{b}}$ imply (3.65). □

3.6 Notes and references

The local smoothing estimates of Section 3.1 were used in the case of the (modified) two dimensional Kadomtsev-Petviashvili II equation by Kenig and Ziesler in [20]. For the three dimensional Kadomtsev-Petviashvili II equation, these estimates were used by Isaza, López and Mejía in [11] in order to prove local well-posedness results for this equation. Note that, in contrast to [11], we do not use the local smoothing estimate in the proof of local well-posedness of the dispersion generalised Kadomtsev-Petviashvili II equations but only need them to establish that the solutions of the operator equation (2.39) are solutions in the sense of distributions. (Cf. Theorem 3.3.)

The name "Strichartz estimates" for estimates of the solution of a linear dispersive equation in mixed Lebesgue spaces $L_t^q L_{x\vec{y}}^r$ goes back to the work of Strichartz [28] on the Schrödinger, Klein-Gordon and wave equation. For the general scheme of how to deduce the Strichartz estimates from decay estimates of the form (3.8), we again refer to Keel and Tao [17]. The decay estimates and the linear Strichartz estimates proven in Section 3.2 are well-known in the case $\alpha = 2$ of the Kadomtsev-Petviashvili II equation in two or three space dimensions. See, for example, Saut [24] for the two dimensional case and Tzvetkov [33] for the three dimensional case. For the case $\alpha \in 2\mathbb{N}$ in dimensions $d = 2, 3$, the decay and linear Strichartz estimates are proven in Ben-Artzi and Saut [1]. See also Saut and Tzvetkov [25] for the case $\alpha = 4$.

The method of writing the bilinear Bourgain space estimates as integral inequalities and reducing them by the Cauchy-Schwarz inequality to an estimate of the form (3.29) was first used by Kenig, Ponce and Vega in [19]. For a much more detailed account of bilinear (and, more generally, multilinear) estimates and methods to prove them, cf. Tao [32].

The refined bilinear Strichartz estimate of Theorem 3.12 was, in the case $\alpha = 2$ of the Kadomtsev-Petviashvili II equation, already implicitly used in [13, 29–31, 33, 34], although it was not formulated explicitly. In the case $\alpha = 4$ of the fifth order Kadomtsev-Petviashvili II equation, it was already implicitly used by Isaza, López and Mejía in [12].

The bilinear Strichartz estimate of Theorem 3.16 in the case $\alpha = 2$ of the Kadomtsev-Petviashvili II equation in three space dimensions has been used implicitly by Isaza, López and Mejía in [11].

Chapter 4

The two dimensional case

In this chapter we consider the *(two dimensional) Kadomtsev-Petviashvili II equation*

$$(u_t + u_{xxx} + (u^2)_x)_x + u_{yy} = 0 \quad \text{in } (-T,T) \times \mathbb{R}^2, \quad u(0) = u_0 \tag{4.1}$$

and, more generally, the *dispersion generalised Kadomtsev-Petviashvili II type equation*

$$(u_t - |D_x|^\alpha u_x + (u^2)_x)_x + u_{yy} = 0 \quad \text{in } (-T,T) \times \mathbb{R}^2, \quad u(0) = u_0 \tag{4.2}$$

with $\frac{4}{3} < \alpha \leq 6$. Note that (4.1) is just (4.2) for $\alpha = 2$. For $\alpha = 4$, (4.2) is the *fifth order Kadomtsev-Petviashvili II equation*

$$(u_t - u_{xxxxx} + (u^2)_x)_x + u_{yy} = 0 \quad \text{in } (-T,T) \times \mathbb{R}^2, \quad u(0) = u_0 \tag{4.3}$$

Our goal is to solve (4.2) for low regularity initial data, i. e. for $u_0 \in H^{s_1,s_2}(\mathbb{R}^2)$ with s_1 and s_2 as small as possible.

4.1 Main results

Our main result for the Kadomtsev-Petviashvili II equation (4.1) is the following.

Theorem 4.1. *Let $s_1 > -\frac{1}{2}$ and $s_2 \geq 0$. Then there exist a Banach space $X \hookrightarrow C_b(\mathbb{R}; H^{s_1,s_2}(\mathbb{R}^2))$ and a non increasing function $T : (0,\infty) \to (0,\infty)$ such that the following holds true:*

a) For every $r > 0$ and $u_0 \in \mathcal{B}_r := \{u_0 \in H^{s_1,s_2}(\mathbb{R}^2) \mid \|u_0\|_{H^{s_1,s_2}(\mathbb{R}^2)} < r\}$ there is a unique solution $u \in X_{T(r)} \hookrightarrow C([-T(r),T(r)]; H^{s_1,s_2}(\mathbb{R}^2))$ of (4.1).

b) For every $r > 0$ the flow map $F_r : \mathcal{B}_r \to X_{T(r)}, u_0 \mapsto u$ defined by a) is analytic.

c) If $r_2 > r_1 > 0$ and $u_0 \in \mathcal{B}_{r_1}$, then $R_{T(r_2)}F_{r_1}(u_0) = F_{r_2}(u_0)$.

Theorem 4.1 is just the special case $\alpha = 2$ of the following more general theorem.

Theorem 4.2. *For $\frac{4}{3} < \alpha \leq 6$ let $s_1 > \max(1 - \frac{3}{4}\alpha, \frac{1}{4} - \frac{3}{8}\alpha)$ and $s_2 \geq 0$. Then there exist a Banach space $X \hookrightarrow C_b(\mathbb{R}; H^{s_1,s_2}(\mathbb{R}^2))$ and a non increasing function $T : (0,\infty) \to (0,\infty)$ such that the following holds true:*

a) For every $r > 0$ and $u_0 \in \mathcal{B}_r := \{u_0 \in H^{s_1,s_2}(\mathbb{R}^2) \mid \|u_0\|_{H^{s_1,s_2}(\mathbb{R}^2)} < r\}$ there is a unique solution $u \in X_{T(r)} \hookrightarrow C([-T(r),T(r)]; H^{s_1,s_2}(\mathbb{R}^2))$ of (4.2).

b) For every $r > 0$ the flow map $F_r : \mathcal{B}_r \to X_{T(r)}, u_0 \mapsto u$ defined by a) is analytic.

c) If $r_2 > r_1 > 0$ and $u_0 \in \mathcal{B}_{r_1}$, then $R_{T(r_2)}F_{r_1}(u_0) = F_{r_2}(u_0)$.

Theorem 4.2 follows from Theorem 2.22 and the bilinear estimate which is proven in Theorem 4.7 (cf. Section 4.2).

Remark 4.3. By a solution of (4.2) in $X_{T(r)}$ we always mean a solution of the corresponding operator equation (2.39). Note, however, that because of $\max(1-\frac{3}{4}\alpha, \frac{1}{4}-\frac{3}{8}\alpha) > -\frac{\alpha}{2}$, Theorem 3.3 shows that these solutions also solve (4.2) in the sense of distributions, i. e. (3.6) holds.

Remark 4.4. In the particular case $\alpha = 4$ of the fifth order Kadomtsev-Petviashvili II equation, Theorem 4.2 shows the local well-posedness of (4.3) for $s_1 > -\frac{5}{4}$ and $s_2 \geq 0$. We therefore get a local well-posedness result for the same class of initial data as Isaza, López and Mejía in [12]. Note, though, that the spaces X_T, where the local well-posedness result of Theorem 4.2 holds, are different from those used in [12] (cf. Remark 4.10).

Remark 4.5. Let us note that if u is a solution of (4.2), then so is

$$u_\delta(t,x,y) = \delta^\alpha u(\delta^{\alpha+1}t, \delta x, \delta^{\frac{\alpha}{2}+1}y)$$

Considering the homogeneous Sobolev norm

$$\|u_0\|_{\dot{H}^{s_1,s_2}} := \||\xi|^{s_1}|\eta|^{s_2}\mathcal{F}u_0\|_{L^2_{\xi,\eta}}$$

we get $\|u_\delta(0,\cdot,\cdot)\|_{\dot{H}^{s_1,s_2}} = \delta^{\frac{3}{4}\alpha-1+s_1+(\frac{\alpha}{2}+1)s_2}\|u(0,\cdot,\cdot)\|_{\dot{H}^{s_1,s_2}}$. This scaling argument suggests that we get ill-posedness for $s_1 + (1+\frac{\alpha}{2})s_2 < 1 - \frac{3}{4}\alpha$. Note that for $\frac{4}{3} < \alpha \leq 2$ and $s_2 = 0$ we reach the critical value $1 - \frac{3}{4}\alpha$ of s_1, except for the endpoint. For $\alpha = 2$ it is proven in [20], Theorem 4.2 that the flow map cannot be C^3 at the origin from $H^{s_1,0}(\mathbb{R}^2)$ to $H^{s_1,0}(\mathbb{R}^2)$ for $s_1 < -\frac{1}{2}$, so that our result is sharp (except at the endpoint) for the scale $H^{s_1,0}(\mathbb{R}^2)$ in the sense of C^3-wellposedness. (Note that while in general the space $H^{s_1,s_2}(\mathbb{R}^2)$ defined in [20] differs from the one defined by the norm (2.4), they coincide for $s_2 = 0$.) For $\alpha > 2$, though, we have that $\frac{1}{4} - \frac{3}{8}\alpha > 1 - \frac{3}{4}\alpha$, so that we do not reach the scaling limit in this case.

By combining the local well-posedness result of Theorem 4.2 with the conservation of the L^2-norm, which holds for real valued solutions of (4.2), we obtain the following global result, where $H^{s_1,0}(\mathbb{R}^2;\mathbb{R})$ denotes the subspace of all real valued functions in $H^{s_1,0}(\mathbb{R}^2)$.

Theorem 4.6. *For $\frac{4}{3} < \alpha \leq 6$ let $s_1 \geq 0$. Then there exists a Banach space $X \hookrightarrow C_b(\mathbb{R}; H^{s_1,0}(\mathbb{R}^2;\mathbb{R}))$ such that for every $u_0 \in H^{s_1,0}(\mathbb{R}^2;\mathbb{R})$ and every $T > 0$, there is exactly one solution u of equation (4.2) in X_T.*

Theorem 4.6 follows from Theorem 2.28 and the bilinear estimate which is proven in Theorem 4.7 (cf. Section 4.2).

4.2 The main bilinear estimate

In the following formulation and proof of the crucial bilinear estimate needed to prove Theorem 4.2, we will only consider the case $s_2 = 0$ (and write s for s_1) to simplify the presentation. Note that the case $s_2 > 0$ follows from this special case because in the general case we only get an extra term $\frac{\langle \eta \rangle^{s_2}}{\langle \eta_1 \rangle^{s_2} \langle \eta_2 \rangle^{s_2}}$ in the integral inequalities we have to prove (see (4.12)). However, this term is always bounded above for $s_2 \geq 0$.

Theorem 4.7. *For $\frac{4}{3} < \alpha \leq 6$ and*

$$s > \max\left(1 - \frac{3}{4}\alpha, \frac{1}{4} - \frac{3}{8}\alpha\right) \tag{4.4}$$

there exist $b > \frac{1}{2}$, $b' \in (b-1, 0]$, $b_1 \in [0, -b']$, and $\sigma \in [0,1]$ such that for the spaces X_1, X_2, and Y defined by

$$X_1 := X_0^{b-b',s}, \quad X_2 := X_\sigma^{b,s} \cap X_\sigma^{b+b_1, s-(\alpha+1)b_1} \tag{4.5}$$

$$Y := X_\sigma^{b',s} \cap X_\sigma^{b'+b_1, s-(\alpha+1)b_1} \tag{4.6}$$

we have that

$$\|\partial_x(u_1 u_2)\|_Y \lesssim \|u_1\|_{X_k} \|u_2\|_{X_l} \tag{4.7}$$

for $u_1, u_2 \in \mathcal{S}_{-\infty}$ and $k, l \in \{1, 2\}$.

Remark 4.8. If we let $X := X_1 + X_2$, then by the bilinearity of

$$B(u_1, u_2) := \partial_x(u_1 u_2)$$

and by the definition (2.7) of the norm in X, it follows from (4.7) that

$$\|B(u_1, u_2)\|_Y \lesssim \|u_1\|_X \|u_2\|_X \quad (u_1, u_2 \in \mathcal{S}_{-\infty}) \tag{4.8}$$

This implies that B can be extended to a continuous and bilinear operator from $X \times X$ to Y.

Remark 4.9. In the case $\alpha = 2$ of the Kadomtsev-Petviashvili II equation, the bilinear estimate (4.8) allows us to show the local well-posedness for initial data in $H^{s,0}(\mathbb{R}^2)$ for all $s > -\frac{1}{2}$. Let us explain how this relates to the counterexamples in [31]. The counterexamples presented in [31] show that it is not possible to get the bilinear estimate for $-\frac{1}{2} < s < -\frac{1}{3}$ without including the low frequency condition (i. e. the term $(\frac{\langle\xi\rangle}{|\xi|})^\sigma$) into the definition of X_2. The drawback of this, however, is that the space X_2 does not contain the (localized) solutions $L_1 u_0$ of the linear equation (2.20) anymore, unless we impose the same low frequency condition on u_0. Therefore, we introduce an auxiliary space X_1 which does not include the low frequency condition and contains $L_1 u_0$ for every $u_0 \in H^{s,0}(\mathbb{R}^2)$. Although there is no low frequency weight in the definition of the space X_1, we can show the bilinear estimate (4.7) also for $u_i \in X_1$ because the term $\langle\lambda\rangle^{b-b'}$ is included in the definition of the norm of X_1 and $b - b'$ is close to 1 (and significantly greater than $b + b_1$). Then we can construct the solution in the sum space $X := X_1 + X_2$, which obeys the bilinear estimate (4.8) and contains $L_1 u_0$ for every $u_0 \in H^{s,0}(\mathbb{R}^2)$.

Remark 4.10. In the case $\alpha = 4$ of the fifth order Kadomtsev-Petviashvili II equation, it is possible to get the bilinear estimate (4.7) in the spaces $X_2 = X_0^{b,s}$ and $Y = X_0^{b',s}$, i. e. choosing $b_1 = 0$ and $\sigma = 0$, as can be seen from the fact that for $\alpha \geq 4$ we use Lemma 4.15 instead of Lemma 4.13 and Lemma 4.14. More generally, this is true for all $\alpha > \frac{5}{2}$ which can be seen by refining the estimate of Lemma 4.15 by an additional dyadic decomposition and interpolation argument as used in [31], pp. 89-92.

In order to prove Theorem 4.7, we will split the nonlinear term $\partial_x(u_1 u_2)$ into various pieces and give estimates in appropriate $X_\sigma^{b,s}$-spaces for each of these pieces (cf. Section 4.3). We will then combine these estimates to give the proof of Theorem 4.7 in Section 4.4. Because we will use exactly the same splitting of the nonlinear term $\partial_x(u_1 u_2)$ in the three dimensional case (cf. Chapter 5), we will describe it here for general dimension d.

First of all, with P_c defined as in (3.39), we can write

$$\partial_x(u_1 u_2) = \partial_x P_1(u_1, u_2) + \partial_x P_1(u_2, u_1) \tag{4.9}$$

Since the main bilinear estimate (4.7) is symmetric in u_1 and u_2 (at least if we consider all combinations of $k, l \in \{1, 2\}$), it suffices to prove it only for $\partial_x P_1(u_1, u_2)$. This expression can be decomposed further into

$$\partial_x P_1(u_1 u_2) = Q_{00}(u_1, u_2) + \sum_{i=1}^{2} \sum_{j=0}^{2} Q_{ij}(u_1, u_2) \tag{4.10}$$

The operators Q_{ij} are defined by

$$\mathcal{F}Q_{ij}(u_1, u_2)(\mu) = i\xi \int_{\mathbb{R}^n} \chi_{A_{ij}}(\mu_1, \mu)\mathcal{F}u_1(\mu_1)\mathcal{F}u_2(\mu_2)d\mu_1$$

where $A_{00} := \{(\mu_1, \mu) \in \mathbb{R}^{2n} \mid |\xi_1| \leq |\xi_2| \leq 1\}$ and $A_{ij} := \Xi_i \cap \Lambda_j$ for $1 \leq i \leq 2$, $0 \leq j \leq 2$ with

$$\begin{aligned}
\Xi_1 &= \{(\mu_1, \mu) \in \mathbb{R}^{2n} \mid |\xi_1| \leq \frac{1}{3}|\xi_2|, |\xi_2| \geq 1\} \\
\Xi_2 &= \{(\mu_1, \mu) \in \mathbb{R}^{2n} \mid \frac{1}{3}|\xi_2| \leq |\xi_1| \leq |\xi_2|, |\xi_2| \geq 1\} \\
\Lambda_0 &= \{(\mu_1, \mu) \in \mathbb{R}^{2n} \mid |\lambda| = |\lambda_{\max}|\} \\
\Lambda_j &= \{(\mu_1, \mu) \in \mathbb{R}^{2n} \mid |\lambda_j| = |\lambda_{\max}|\} \quad (j = 1, 2)
\end{aligned}$$

Let us explain what the meaning of the regions Ξ_1 and Ξ_2 is. In Ξ_1 we have that $2 \leq 2|\xi_2| \leq 3|\xi| \leq 4|\xi_2|$, i. e. ξ and ξ_2 are comparable in size and are both bounded away from zero, whereas ξ_1 is the smallest of the frequencies dual to the x-variable, i. e. $|\xi_1| = |\xi_{\min}|$. In Ξ_2 we have that ξ_1 and ξ_2 are comparable in size and are both bounded away from zero, whereas ξ may be small here and we have $|\xi| \sim |\xi_{\min}|$. For each of the operators Q_{ij}, we will show estimates of the form

$$\|Q_{ij}(u_1, u_2)\|_{X_{\sigma}^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_{\bar{\sigma}}^{\bar{b},\bar{s}}} \|u_2\|_{X_0^{b,s}} \tag{4.11}$$

for appropriately chosen $\tilde{s}, \tilde{b}, \sigma, \bar{s}, \bar{b}, \bar{\sigma}$. By definition (2.9) of the $X_{\sigma}^{b,s}$-norm (and (2.16)) and by setting

$$\begin{aligned}
f_1(\mu_1) &:= |\xi_1|^{-\bar{\sigma}} \langle \xi_1 \rangle^{\bar{s}+\bar{\sigma}} \langle \lambda_1 \rangle^{\bar{b}} \mathcal{F}u_1(\mu_1) \\
f_2(\mu_2) &:= \langle \xi_2 \rangle^{s} \langle \lambda_2 \rangle^{b} \mathcal{F}u_2(\mu_2)
\end{aligned}$$

we see that this is equivalent to

$$\left\| \frac{|\xi| \langle \xi \rangle^{\tilde{s}+\sigma}}{|\xi|^{\sigma} \langle \lambda \rangle^{-\tilde{b}}} \int_{\mathbb{R}^n} \chi_{A_{ij}}(\mu_1, \mu) \frac{|\xi_1|^{\bar{\sigma}} f_1(\mu_1) f_2(\mu_2) d\mu_1}{\langle \xi_1 \rangle^{\bar{s}+\bar{\sigma}} \langle \lambda_1 \rangle^{\bar{b}} \langle \xi_2 \rangle^{s} \langle \lambda_2 \rangle^{b}} \right\|_{L^2_{\mu}} \lesssim \|f_1\|_{L^2} \|f_2\|_{L^2}$$

Using duality, this estimate is equivalent to

$$\int_{A_{ij}} \frac{|\xi|\langle\xi\rangle^{\tilde{s}+\sigma}|\xi_1|^{\bar{\sigma}} f_1 f_2 f_3 \, d\mu_1 d\mu}{|\xi|^{\sigma}\langle\xi_1\rangle^{\bar{s}+\bar{\sigma}}\langle\xi_2\rangle^{s}\langle\lambda\rangle^{-\tilde{b}}\langle\lambda_1\rangle^{\bar{b}}\langle\lambda_2\rangle^{b}} \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{4.12}$$

for all $f_i \geq 0$, where we again used the convention that in an integral over μ_1 and μ, $f_1 f_2 f_3$ always means $f_1(\mu_1)f_2(\mu_2)f_3(\mu)$. The main ingredients we use in the proof of these estimates are the bilinear Strichartz type estimates of Section 3.4 and the "resonance identity" (3.44). We already noted that the two terms on the right hand side of (3.44) have the same sign. Therefore, we have

$$|\lambda_{\max}| \geq \frac{1}{3}|\lambda_1 + \lambda_2 - \lambda| \geq \frac{1}{3}|\nu| \geq \frac{\alpha}{3 \cdot 2^{\alpha}}|\xi_{\min}||\xi_{\max}|^{\alpha} \tag{4.13}$$

where for the last inequality we used Lemma 3.13.

4.3 Estimates for the Q_{ij}

In this section we will derive suitable estimates for the "pieces" $Q_{ij}(u_1, u_2)$ of the bilinear term $\partial_x(u_1 u_2)$ (see (4.10)). In most cases it will be possible to derive an estimate of the form (4.11) with $(\bar{b}, \bar{s}, \bar{\sigma}) = (b, s, 0)$, i. e. with $u_1, u_2 \in X_0^{b,s}$. We then use that by the embedding property of $X = X_1 + X_2$ (cf. Proposition 2.27), we have $\|u\|_{X_0^{b,s}} \leq \|u\|_{X_1}$ and $\|u\|_{X_0^{b,s}} \leq \|u\|_{X_2}$. In these cases the integral inequality we have to prove reads

$$\int_{A_{ij}} \frac{|\xi|^{1-\sigma}\langle\xi\rangle^{\tilde{s}+\sigma}\langle\lambda\rangle^{\tilde{b}}}{\langle\xi_1\rangle^{s}\langle\xi_2\rangle^{s}\langle\lambda_1\rangle^{b}\langle\lambda_2\rangle^{b}} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{4.14}$$

Now, this is proven in most cases under various conditions on the parameters $b, s, \tilde{b}, \tilde{s}$ and σ. In the critical case, i. e. $(i,j) = (1,1)$ and $\alpha < 4$, we cannot choose $u_1 \in X_0^{b,s}$ but have to prove separate estimates suitable for the two cases $u_1 \in X_1$ and $u_1 \in X_2$ (cf. Lemma 4.13 and Lemma 4.14).

The following two conditions will be assumed in all cases:

$$s, \tilde{s} \in \mathbb{R}, 0 \leq \sigma \leq 1, b > \frac{1}{2}, -\frac{1}{2} < \tilde{b} \leq 0 \tag{4.15}$$

and

$$\tilde{b} \leq \frac{1}{\alpha+1}\left(\frac{\alpha}{4} - \frac{3}{2} + 2s - \tilde{s}\right) \tag{4.16}$$

Let us now consider the different cases.

Lemma 4.11. *We have that*

$$\|Q_{00}(u_1,u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_0^{b,s}}\|u_2\|_{X_0^{b,s}} \tag{4.17}$$

provided that (4.15) *and*

$$\sigma \leq \frac{3}{2} - \frac{\alpha}{4} \tag{4.18}$$

hold.

Proof. We have to prove (4.14) for $(i,j)=(0,0)$. Because in A_{00} we have $|\xi| \leq 2|\xi_2| \leq 2$ and $|\xi_1| \leq |\xi_2| \leq 1$, it follows that $\langle\xi\rangle \sim \langle\xi_1\rangle \sim \langle\xi_2\rangle \sim 1$. Hence, it suffices to show

$$\int_{A_{00}} k_{00}(\mu_1,\mu)\frac{|\xi_1|^{-\frac{1}{2}}|\xi_2|^{\frac{\alpha}{4}}}{\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

where $k_{00}(\mu_1,\mu) := \langle\lambda\rangle^{\tilde{b}}|\xi|^{1-\sigma}|\xi_1|^{\frac{1}{2}}|\xi_2|^{-\frac{\alpha}{4}}$. If we show that k_{00} is bounded on A_{00}, then the claim follows from the bilinear estimate (3.45).

In A_{00}, we have $|\xi_1| \leq |\xi_2|$ and $|\xi| \leq 2|\xi_2|$. Furthermore, $\tilde{b} \leq 0$ and $1-\sigma \geq 0$ by (4.15). Altogether, it follows that $k_{00}(\mu_1,\mu) \lesssim |\xi_2|^{\frac{3}{2}-\frac{\alpha}{4}-\sigma} \lesssim 1$, where the last inequality follows from (4.18). □

Lemma 4.12. *We have that*

$$\|Q_{10}(u_1,u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_0^{b,s}}\|u_2\|_{X_0^{b,s}} \tag{4.19}$$

provided that (4.15), (4.16) *and*

$$\tilde{b} \leq \frac{1}{\alpha}\left(\frac{\alpha}{4} - 1 + s - \tilde{s}\right) \tag{4.20}$$

hold.

Proof. We have to prove (4.14) for $(i,j)=(1,0)$. Because in A_{10} we have $\langle\xi_2\rangle \sim \langle\xi\rangle \sim |\xi| \sim |\xi_2|$ and $\langle\lambda\rangle^{\tilde{b}} \lesssim |\xi|^{\alpha\tilde{b}}|\xi_1|^{\tilde{b}}$, it suffices to show

$$\int_{A_{10}} k_{10}(\mu_1,\mu)\frac{|\xi_1|^{-\frac{1}{2}}|\xi_2|^{\frac{\alpha}{4}}}{\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

where

$$k_{10}(\mu_1,\mu) := |\xi|^{1-\frac{\alpha}{4}+\alpha\tilde{b}+\tilde{s}-s}\langle\xi_1\rangle^{-s}|\xi_1|^{\frac{1}{2}+\tilde{b}} \tag{4.21}$$

If we show that k_{10} is bounded on Ξ_1, then it is especially bounded on $A_{10} \subset \Xi_1$ and the lemma follows from the bilinear estimate (3.45).

Let us first consider the case $|\xi_1| \leq 1$. Then $\langle \xi_1 \rangle \sim 1$. We have $\frac{1}{2} + \tilde{b} > 0$ because of (4.15) and $1 - \frac{\alpha}{4} + \alpha\tilde{b} + \tilde{s} - s \leq 0$ because of (4.20). Since $|\xi| \gtrsim 1$ in Ξ_1, we conclude that $k_{10}(\mu_1, \mu) \lesssim 1$ in this case.

Now, we consider the case $|\xi_1| \geq 1$. Then $\langle \xi_1 \rangle \sim |\xi_1|$. Using this and $|\xi| \gtrsim |\xi_1|$ in Ξ_1, it follows from (4.20) that $k_{10}(\mu_1, \mu) \lesssim |\xi_1|^{\frac{3}{2} - \frac{\alpha}{4} + (\alpha+1)\tilde{b} + (\tilde{s} - 2s)}$. Because (4.16) implies $\frac{3}{2} - \frac{\alpha}{4} + (\alpha + 1)\tilde{b} + (\tilde{s} - 2s) \leq 0$, we also obtain $k_{10}(\mu_1, \mu) \lesssim 1$ in this case. □

Lemma 4.13. *We have that*

$$\|Q_{11}(u_1, u_2)\|_{X_\sigma^{0,\tilde{s}}} \lesssim \|u_1\|_{X_0^{b-\tilde{b},s}} \|u_2\|_{X_0^{b,s}} \tag{4.22}$$

provided that (4.15), (4.16) *and* (4.20) *hold.*

Proof. We have to prove that

$$\int_{A_{11}} \frac{|\xi|^{1-\sigma} \langle \xi \rangle^{\tilde{s}+\sigma} f_1 f_2 f_3}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \lambda_1 \rangle^{b-\tilde{b}} \langle \lambda_2 \rangle^b} \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2}$$

In A_{11} we have $\langle \xi_2 \rangle \sim \langle \xi \rangle \sim |\xi| \sim |\xi_2|$ and $\langle \lambda_1 \rangle^{\tilde{b}} \lesssim |\xi|^{\alpha\tilde{b}} |\xi_1|^{\tilde{b}}$, so it suffices to show

$$\int_{A_{11}} k_{10}(\mu_1, \mu) \frac{|\xi_1|^{-\frac{1}{2}} |\xi_2|^{\frac{\alpha}{4}}}{\langle \lambda_1 \rangle^b \langle \lambda_2 \rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2}$$

where k_{10} is defined as in (4.21). In Lemma 4.12, it was shown that under the conditions (4.15), (4.16), and (4.20), k_{10} is bounded on Ξ_1. This implies that it is especially bounded on $A_{11} \subset \Xi_1$. Therefore, (4.22) follows from the bilinear estimate (3.45). □

Lemma 4.14. *We have that*

$$\|Q_{11}(u_1, u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_\sigma^{b+b_1, s-(\alpha+1)b_1}} \|u_2\|_{X_0^{b,s}} \tag{4.23}$$

provided that (4.15), (4.16), $b_1 \geq 0$ *and*

$$\tilde{b} - b_1 \leq \frac{1}{\alpha} \left(\frac{\alpha}{4} - \frac{3}{2} + s - \tilde{s} \right) \tag{4.24}$$

$$\sigma \geq b_1 - \tilde{b} \tag{4.25}$$

hold.

Proof. We have to prove that

$$\int_{A_{11}} \frac{|\xi|^{1-\sigma}\langle\xi\rangle^{\tilde{s}+\sigma}|\xi_1|^{\sigma}\langle\lambda\rangle^{\tilde{b}} f_1 f_2 f_3}{\langle\xi_1\rangle^{s-(\alpha+1)b_1+\sigma}\langle\xi_2\rangle^{s}\langle\lambda_1\rangle^{b+b_1}\langle\lambda_2\rangle^{b}}\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

In A_{11} we have $\langle\xi_2\rangle \sim \langle\xi\rangle \sim |\xi| \sim |\xi_2|$, so it suffices to show

$$\int_{A_{11}} k_{11}(\mu_1,\mu)\frac{|\xi|^{-\frac{1}{4}+\frac{\alpha}{8}}|\xi_2|^{-\frac{1}{4}+\frac{\alpha}{8}}}{\langle\lambda\rangle^{b}\langle\lambda_2\rangle^{b}} f_1 f_2 f_3\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

where $k_{11}(\mu_1,\mu) := \langle\lambda\rangle^{\tilde{b}+b}\langle\lambda_1\rangle^{-b-b_1}|\xi|^{\frac{3}{2}-\frac{\alpha}{4}+\tilde{s}-s}\langle\xi_1\rangle^{-s+(\alpha+1)b_1-\sigma}|\xi_1|^{\sigma}$. Now, the lemma follows from the bilinear Strichartz estimate (3.38) if we show that k_{11} is bounded on A_{11}.

Because of $|\lambda| \le |\lambda_1|$ in A_{11} and $\tilde{b}+b > 0$ by (4.15), we find that $\langle\lambda\rangle^{\tilde{b}+b}\langle\lambda_1\rangle^{-b-b_1} \le \langle\lambda_1\rangle^{\tilde{b}-b_1}$. In A_{11} we have $|\lambda_1| \gtrsim |\xi|^{\alpha}|\xi_1|$. Since $\tilde{b}-b_1 \le 0$, we obtain

$$k_{11}(\mu_1,\mu) \lesssim |\xi|^{\frac{3}{2}-\frac{\alpha}{4}+\alpha(\tilde{b}-b_1)+\tilde{s}-s}\langle\xi_1\rangle^{-s+(\alpha+1)b_1-\sigma}|\xi_1|^{\tilde{b}-b_1+\sigma}$$

Let us first suppose that $|\xi_1| \le 1$. We then have $\langle\xi_1\rangle \sim 1$. Note that $|\xi| \gtrsim 1$ in A_{11}. Because $\frac{3}{2}-\frac{\alpha}{4}+\alpha(\tilde{b}-b_1)+\tilde{s}-s \le 0$ by (4.24) and $\tilde{b}-b_1+\sigma \ge 0$ by (4.25), it follows that $k_{11}(\mu_1,\mu) \lesssim 1$ in this case.

Let us now consider the case $|\xi_1| \ge 1$. We then have $\langle\xi_1\rangle \sim |\xi_1|$. By (4.24) and since $|\xi| \gtrsim |\xi_1|$ in A_{11}, we find that $k_{11}(\mu_1,\mu) \lesssim |\xi_1|^{\frac{3}{2}-\frac{\alpha}{4}+(\alpha+1)\tilde{b}+(\tilde{s}-2s)}$. Now, (4.16) implies $k_{11}(\mu_1,\mu) \lesssim 1$ in this case. □

If $\alpha \ge 4$, we shall rely on the following lemma to estimate Q_{11} instead of Lemmata 4.13 and 4.14.

Lemma 4.15. *We have that*

$$\|Q_{11}(u_1,u_2)\|_{X_{\sigma}^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_0^{b,s}}\|u_2\|_{X_0^{b,s}} \tag{4.26}$$

provided that (4.15), (4.16) *and*

$$\tilde{s} \le s + \frac{\alpha}{4} - 1 \tag{4.27}$$

$$\tilde{b} \le -\frac{1}{2\alpha} \tag{4.28}$$

hold.

Proof. We have to prove (4.14) for $(i,j)=(1,1)$. We split $A_{11} = A_{11}^{\leq} \cup A_{11}^{\geq}$, where

$$A_{11}^{\leq} := \{(\mu_1,\mu) \in A_{11} \mid |\xi_1| \leq 1\}, \quad A_{11}^{\geq} := \{(\mu_1,\mu) \in A_{11} \mid |\xi_1| \geq 1\}$$

Using $\langle \xi_2 \rangle \sim \langle \xi \rangle \sim |\xi|$ in A_{11}, $\langle \xi_1 \rangle \sim 1$ in $A_{11}^{\leq}$, and $\langle \xi_1 \rangle \sim |\xi_1|$ in $A_{11}^{\geq}$, we see that it suffices to show

$$\int_{A_{11}^{\leq}} \frac{|\xi|^{1+\tilde{s}-s} \langle \lambda \rangle^{\tilde{b}}}{\langle \lambda_1 \rangle^b \langle \lambda_2 \rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{4.29}$$

and

$$\int_{A_{11}^{\geq}} \frac{|\xi|^{1+\tilde{s}-s} |\xi_1|^{-s} \langle \lambda \rangle^{\tilde{b}}}{\langle \lambda_1 \rangle^b \langle \lambda_2 \rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2} \tag{4.30}$$

Let us first consider the case $A_{11}^{\leq}$. Because $\tilde{b} \leq 0$ by (4.15), it follows that $\langle \lambda \rangle^{\tilde{b}} \leq 1$. Furthermore, since $|\xi| \sim |\xi_2|$, $|\xi_1| \leq 1$ in $A_{11}^{\leq}$, and $1+\tilde{s}-s-\frac{\alpha}{4} \leq 0$ by (4.27), we obtain

$$\frac{|\xi|^{1+\tilde{s}-s}}{\langle \lambda_1 \rangle^b \langle \lambda_2 \rangle^b} \lesssim \frac{|\xi_1|^{-\frac{1}{2}} |\xi_2|^{\frac{\alpha}{4}}}{\langle \lambda_1 \rangle^b \langle \lambda_2 \rangle^b}$$

Therefore, (4.29) follows from the bilinear estimate (3.45).

Let us now consider the case $A_{11}^{\geq}$. Then since $|\xi| \sim |\xi_2|$ in $A_{11}^{\geq}$, we see that (4.30) follows from

$$\int_{A_{11}^{\geq}} \tilde{k}_{11}(\mu_1,\mu) \frac{|\xi|^{-\frac{1}{4}+\frac{\alpha}{8}} |\xi_2|^{-\frac{1}{4}+\frac{\alpha}{8}}}{\langle \lambda \rangle^b \langle \lambda_2 \rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2}$$

where $\tilde{k}_{11}(\mu_1,\mu) := \langle \lambda \rangle^{\tilde{b}+b} \langle \lambda_1 \rangle^{-b} |\xi|^{\frac{3}{2}-\frac{\alpha}{4}+\tilde{s}-s} |\xi_1|^{-s}$. If we prove that $\tilde{k}_{11}$ is bounded on $A_{11}^{\geq}$, then the claim follows from the bilinear estimate (3.38).

As $|\lambda| \leq |\lambda_1|$ in $A_{11}^{\geq}$ and $\tilde{b}+b > 0$ by (4.15), we have $\langle \lambda \rangle^{\tilde{b}+b} \langle \lambda_1 \rangle^{-b} \leq \langle \lambda_1 \rangle^{\tilde{b}}$. Using $|\lambda_1| \gtrsim |\xi|^{\alpha} |\xi_1|$ in $A_{11}^{\geq}$ and $\tilde{b} \leq 0$ by (4.15), we find that

$$\tilde{k}_{11}(\mu_1,\mu) \lesssim |\xi|^{\frac{3}{2}-\frac{\alpha}{4}+\alpha\tilde{b}+\tilde{s}-s} |\xi_1|^{\tilde{b}-s}$$

Now, (4.27) and (4.28) imply $\frac{3}{2}-\frac{\alpha}{4}+\alpha\tilde{b}+\tilde{s}-s \leq 0$. Because $|\xi| \gtrsim |\xi_1|$ in $A_{11}^{\geq}$, we obtain $\tilde{k}_{11}(\mu_1,\mu) \lesssim |\xi_1|^{\frac{3}{2}-\frac{\alpha}{4}+(\alpha+1)\tilde{b}+(\tilde{s}-2s)} \lesssim 1$, where the last inequality follows from (4.16). □

Lemma 4.16. *We have that*

$$\|Q_{12}(u_1,u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_0^{b,s}} \|u_2\|_{X_0^{b,s}} \tag{4.31}$$

provided that (4.15), (4.16) *and* (4.20) *hold.*

Proof. We have to prove (4.14) with $(i,j)=(1,2)$. Because $\langle\xi_2\rangle\sim\langle\xi\rangle\sim|\xi|$ in A_{12}, it suffices to show

$$\int_{A_{12}} k_{12}(\mu_1,\mu)\frac{|\xi_1|^{-\frac{1}{2}}|\xi|^{\frac{\alpha}{4}}}{\langle\lambda_1\rangle^b\langle\lambda\rangle^b} f_1f_2f_3\,d\mu_1d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

where $k_{12}(\mu_1,\mu):=\langle\lambda\rangle^{\tilde{b}+b}\langle\lambda_2\rangle^{-b}|\xi|^{1-\frac{\alpha}{4}+\tilde{s}-s}\langle\xi_1\rangle^{-s}|\xi_1|^{\frac{1}{2}}$. If we show that k_{12} is bounded on A_{12}, then (4.31) follows from the bilinear estimate (3.46).

Since $|\lambda|\leq|\lambda_2|$ in A_{12} and $\tilde{b}+b>0$ by (4.15), we get $\langle\lambda\rangle^{\tilde{b}+b}\langle\lambda_2\rangle^{-b}\leq\langle\lambda_2\rangle^{\tilde{b}}$. Furthermore, we have $|\lambda_2|=|\lambda_{\max}|\gtrsim|\xi|^\alpha|\xi_1|$ in A_{12}. Because $\tilde{b}\leq 0$ by (4.15), we find that $k_{12}(\mu_1,\mu)\lesssim k_{10}(\mu_1,\mu)$, where k_{10} is defined as in (4.21). In Lemma 4.12, it was shown that k_{10} is bounded on Ξ_1 under the conditions (4.15), (4.16) and (4.20). Therefore, it is especially bounded on $A_{12}\subset\Xi_1$ and the claim follows. □

Lemma 4.17. *We have that*

$$\|Q_{20}(u_1,u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}}\lesssim\|u_1\|_{X_0^{b,s}}\|u_2\|_{X_0^{b,s}} \tag{4.32}$$

provided that (4.15), (4.16) *and*

$$\max(\tilde{b},\sigma-1)\leq\frac{1}{\alpha}\left(\frac{\alpha}{4}-\frac{1}{2}+2s\right) \tag{4.33}$$

hold.

Proof. We have to prove (4.14) for $(i,j)=(2,0)$. As $\langle\xi_2\rangle\sim\langle\xi_1\rangle\sim|\xi_1|\sim|\xi_2|$ and $|\lambda|=|\lambda_{\max}|$ in A_{20}, it suffices to show

$$\int_{A_{20}} k_{20}(\mu_1,\mu)\frac{|\xi_1|^{-\frac{1}{4}+\frac{\alpha}{8}}|\xi_2|^{-\frac{1}{4}+\frac{\alpha}{8}}}{\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b} f_1f_2f_3\,d\mu_1d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

where

$$k_{20}(\mu_1,\mu):=\langle\lambda_{\max}\rangle^{\tilde{b}}|\xi|^{1-\sigma}\langle\xi\rangle^{\tilde{s}+\sigma}|\xi_1|^{\frac{1}{2}-\frac{\alpha}{4}-2s} \tag{4.34}$$

If we show that k_{20} is bounded on Ξ_2, then it is especially bounded on $A_{20}\subset\Xi_2$ and (4.32) follows from the bilinear Strichartz estimate (3.36).

Let us first consider the case $|\xi|\leq 1$. We then have that $\langle\xi\rangle\sim 1$. Let $m:=\max(\tilde{b},\sigma-1)$. By (4.15) we have $m\leq 0$. Using this and $|\lambda_{\max}|\gtrsim|\xi_1|^\alpha|\xi|$ in Ξ_2,we find that $\langle\lambda_{\max}\rangle^{\tilde{b}}\leq\langle\lambda_{\max}\rangle^m\lesssim|\xi_1|^{\alpha m}|\xi|^m$. This implies

$$k_{20}(\mu_1,\mu)\lesssim|\xi|^{1-\sigma+m}|\xi_1|^{\frac{1}{2}-\frac{\alpha}{4}+\alpha m-2s}$$

We have $1-\sigma+m \geq 0$ by the definition of m and $\frac{1}{2}-\frac{\alpha}{4}+\alpha m-2s \leq 0$ by (4.33). Taking into account that $|\xi_1| \gtrsim 1$ in Ξ_2, this implies $k_{20}(\mu_1,\mu) \lesssim 1$ in this case.

Let us now consider the case $|\xi| \geq 1$. We then have $\langle\xi\rangle \sim |\xi|$. Using this and $\langle\lambda_{\max}\rangle^{\tilde{b}} \lesssim |\xi_1|^{\alpha\tilde{b}}|\xi|^{\tilde{b}}$ in Ξ_2, it follows that

$$k_{20}(\mu_1,\mu) \lesssim |\xi|^{1+\tilde{b}+\tilde{s}}|\xi_1|^{\frac{1}{2}-\frac{\alpha}{4}+\alpha\tilde{b}-2s}$$

Because $\frac{1}{2}-\frac{\alpha}{4}+\alpha\tilde{b}-2s \leq 0$ by (4.33) and $|\xi_1| \gtrsim |\xi|$ in Ξ_2, we conclude that $k_{20}(\mu_1,\mu) \lesssim |\xi|^{\frac{3}{2}-\frac{\alpha}{4}+(\alpha+1)\tilde{b}+(\tilde{s}-2s)} \lesssim 1$, where the last inequality follows from (4.16). □

Lemma 4.18. *We have that*

$$\|Q_{21}(u_1,u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_0^{b,s}}\|u_2\|_{X_0^{b,s}} \tag{4.35}$$

provided that (4.15), (4.16) *and* (4.33) *hold.*

Proof. We have to prove (4.14) for $(i,j)=(2,1)$. As $\langle\xi_2\rangle \sim \langle\xi_1\rangle \sim |\xi_1| \sim |\xi_2|$ in A_{21}, it suffices to show

$$\int_{A_{21}} k_{21}(\mu_1,\mu)\frac{|\xi|^{-\frac{1}{2}}|\xi_2|^{\frac{\alpha}{4}}}{\langle\lambda\rangle^b\langle\lambda_2\rangle^b} f_1 f_2 f_3\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

where $k_{21}(\mu_1,\mu) := \langle\lambda\rangle^{\tilde{b}+b}\langle\lambda_1\rangle^{-b}|\xi|^{\frac{3}{2}-\sigma}\langle\xi\rangle^{\tilde{s}+\sigma}|\xi_1|^{-\frac{\alpha}{4}-2s}$. If we show that k_{21} is bounded on A_{21}, then (4.35) follows from the bilinear estimate (3.47).

In A_{21} we have $|\lambda| \leq |\lambda_1| = |\lambda_{\max}|$. Using this and $\tilde{b}+b>0$, which follows from (4.15), we conclude that $k_{21}(\mu_1,\mu) \leq \langle\lambda_{\max}\rangle^{\tilde{b}}|\xi|^{\frac{3}{2}-\sigma}\langle\xi\rangle^{\tilde{s}+\sigma}|\xi_1|^{-\frac{\alpha}{4}-2s}$. Taking into account that $|\xi|^{\frac{1}{2}} \lesssim |\xi_1|^{\frac{1}{2}}$ in A_{21}, we get $k_{21}(\mu_1,\mu) \lesssim k_{20}(\mu_1,\mu)$, where k_{20} is defined as in (4.34). In Lemma 4.17, it was shown that under the conditions (4.15), (4.16), and (4.33), k_{20} is bounded on Ξ_2. Therefore, it is especially bounded on $A_{21} \subset \Xi_2$ and the claim follows. □

Lemma 4.19. *We have that*

$$\|Q_{22}(u_1,u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_0^{b,s}}\|u_2\|_{X_0^{b,s}} \tag{4.36}$$

provided that (4.15), (4.16) *and* (4.33) *hold.*

Proof. We have to prove (4.14) for $(i,j)=(2,2)$. Because $\langle\xi_2\rangle \sim \langle\xi_1\rangle \sim |\xi_1|$ in A_{22}, it suffices to show

$$\int_{A_{22}} k_{22}(\mu_1,\mu)\frac{|\xi|^{-\frac{1}{2}}|\xi_1|^{\frac{\alpha}{4}}}{\langle\lambda\rangle^b\langle\lambda_1\rangle^b} f_1 f_2 f_3\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

where $k_{22}(\mu_1,\mu) := \langle\lambda\rangle^{\tilde{b}+b}\langle\lambda_2\rangle^{-b}|\xi|^{\frac{3}{2}-\sigma}\langle\xi\rangle^{\tilde{s}+\sigma}|\xi_1|^{-\frac{\alpha}{4}-2s}$. If we show that k_{22} is bounded on A_{22}, then (4.36) follows from the bilinear estimate (3.48). However, the boundedness of k_{22} on A_{22} follows exactly in the same way as the boundedness of k_{21} in Lemma 4.18. □

4.4 Proof of the main bilinear estimate

In this section we will use the estimates of the last section to give the proof of Theorem 4.7.

Proof of Theorem 4.7. Let

$$b := \frac{1}{2} + \frac{\varepsilon}{2}, \quad b' := -\frac{1}{2} + \varepsilon \tag{4.37}$$

where $\varepsilon > 0$ will be restricted by various upper bounds given in the course of the proof. By definition (4.37), we obviously have $b' > b - 1$. Let

$$b_1 := \begin{cases} \frac{3}{2\alpha} - \frac{3}{4} + \varepsilon & (\frac{4}{3} < \alpha \le 2) \\ 0 & (2 < \alpha \le 6) \end{cases} \tag{4.38}$$

and

$$\sigma := \begin{cases} -b' + b_1 & (\frac{4}{3} < \alpha < 4) \\ \frac{3}{2} - \frac{\alpha}{4} & (4 \le \alpha \le 6) \end{cases} \tag{4.39}$$

As noted before, because of the symmetry of (4.7) in u_1 and u_2 (at least if we consider all combinations of $k, l \in \{1, 2\}$), it suffices to show (4.7) for $\partial_x P_1(u_1, u_2)$ instead of $\partial_x(u_1 u_2)$, where P_1 is the operator defined in (3.39). We decompose $\partial_x P_1(u_1, u_2)$ further as in (4.10). Therefore, we have to show that for every Q_{ij} and every $k, l \in \{1, 2\}$

$$\|Q_{ij}(u_1, u_2)\|_Y \lesssim \|u_1\|_{X_k}\|u_2\|_{X_l} \tag{4.40}$$

By the embedding property of the space $X = X_1 + X_2$, which was proven in Proposition 2.27, we see that (4.40) follows from

$$\|Q_{ij}(u_1, u_2)\|_Y \lesssim \|u_1\|_{X_0^{b,s}}\|u_2\|_{X_0^{b,s}} \tag{4.41}$$

which we actually prove in all cases, except the case $(i, j) = (1, 1)$ and $\alpha < 4$. By the definition (4.6) of Y, it suffices to prove

$$\|Q_{ij}(u_1, u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_0^{b,s}}\|u_2\|_{X_0^{b,s}} \tag{4.42}$$

in the two cases

$$(\tilde{b}, \tilde{s}) = (b', s) \tag{4.43}$$

and

$$(\tilde{b}, \tilde{s}) = (b' + b_1, s - (\alpha + 1)b_1) \tag{4.44}$$

We will now use the Lemmata 4.11–4.19 of Section 4.3 to prove (4.42) for all $(i, j) \neq (1, 1)$. In order to apply any of these lemmata, we need that the conditions (4.15) and (4.16) are fulfilled. Let us, therefore, first show that there exists an $\varepsilon > 0$ such that, in both cases (4.43) and (4.44), the conditions (4.15) and (4.16) indeed hold. If

$$\varepsilon \leq \frac{1}{16} \tag{4.45}$$

then by the definitions (4.37), (4.38), and (4.39) and taking into account that $\alpha > \frac{4}{3}$, we see that (4.15) holds. Now, (4.16) is fulfilled if

$$\varepsilon \leq \frac{1}{\alpha + 1} \left(\frac{3}{4}\alpha - 1 + s \right) \tag{4.46}$$

where the right hand side of this inequality is positive because of (4.4).

Let us now turn to the proof of (4.42) for all $(i, j) \neq (1, 1)$. We will consider three cases:

1. $(i, j) = (0, 0)$: We will use Lemma 4.11 in order to prove (4.42) in this case. Therefore, we have to check that condition (4.18) holds. Taking into account the definition (4.39) of σ, we easily see that (4.18) holds.

2. $(i, j) = (1, 0)$ and $(i, j) = (1, 2)$: In order to prove (4.42) in these two cases, we will use Lemma 4.12 and Lemma 4.16, respectively. Therefore, we have to check that (4.20) holds for $(\tilde{b}, \tilde{s}) = (b', s)$ and $(\tilde{b}, \tilde{s}) = (b' + b_1, s - (\alpha + 1)b_1)$. Now, (4.20) is fulfilled if

$$\varepsilon \leq \frac{1}{\alpha} \left(\frac{3}{4}\alpha - 1 \right) \tag{4.47}$$

where the right hand side of this inequality is positive because $\alpha > \frac{4}{3}$.

3. $(i, j) = (2, 0)$, $(i, j) = (2, 1)$ and $(i, j) = (2, 2)$: In order to prove (4.42) in these three cases, we will use Lemma 4.17, Lemma 4.18 and Lemma 4.19, respectively. Therefore, we have to check that condition (4.33) holds for $(\tilde{b}, \tilde{s}) = (b', s)$ and $(\tilde{b}, \tilde{s}) = (b' + b_1, s - (\alpha + 1)b_1)$. Since $\tilde{s}$ does not appear in condition (4.33) and $b' \leq b' + b_1$, we see that we only have to verify this condition for $(\tilde{b}, \tilde{s}) = (b' + b_1, s - (\alpha + 1)b_1)$. In order to see that condition (4.33) holds, we have to show that

$$b' + b_1 \leq \frac{1}{\alpha} \left(\frac{\alpha}{4} - \frac{1}{2} + 2s \right) \tag{4.48}$$

and

$$\sigma - 1 \leq \frac{1}{\alpha}\left(\frac{\alpha}{4} - \frac{1}{2} + 2s\right) \tag{4.49}$$

hold.

Let us first show (4.48). This follows from (4.46) if $\frac{4}{3} < \alpha \leq 2$ and from

$$\varepsilon \leq \frac{1}{\alpha}\left(\frac{3}{4}\alpha - \frac{1}{2} + 2s\right) \tag{4.50}$$

if $\alpha > 2$, where the right hand side of the last inequality is positive because of (4.4).

Let us now show (4.49). If $\frac{4}{3} < \alpha < 4$, we find that $\sigma - 1 < b' + b_1$ by the definition (4.39) of σ and because of $b' > -\frac{1}{2}$. On the other hand, if $\alpha \geq 4$, we also have

$$\sigma - 1 \leq -\frac{1}{2} < b' + b_1$$

Therefore, (4.48) implies (4.49).

Let us now consider the case $(i, j) = (1, 1)$. If $\alpha \geq 4$, we can also prove (4.42) in this case. We will use Lemma 4.15. Therefore, we have to show that the conditions (4.27) and (4.28) hold. But if ε fulfills (4.45), it follows that these two conditions indeed hold. This ends the proof in the case $\alpha \geq 4$.

Let us now suppose that $\frac{4}{3} < \alpha < 4$. We show that

$$\|Q_{11}(u_1, u_2)\|_Y \lesssim \|u_1\|_{X_k}\|u_2\|_{X_0^{b,s}} \tag{4.51}$$

holds for $k \in \{1, 2\}$. Then it follows from Proposition 2.27 again that this implies (4.40) for all $k, l \in \{1, 2\}$. By the definition of Y, it suffices to prove

$$\|Q_{11}(u_1, u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_k}\|u_2\|_{X_0^{b,s}} \tag{4.52}$$

for $(\tilde{b}, \tilde{s}) = (b', s)$ and $(\tilde{b}, \tilde{s}) = (b' + b_1, s - (\alpha + 1)b_1)$ and for $k \in \{1, 2\}$.

Let us treat the cases $k = 1$ and $k = 2$ separately.

1. $k = 1$: We will use Lemma 4.13 in this case. By the embedding properties of the Bourgain spaces, we have

$$\|Q_{11}(u_1, u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \leq \|Q_{11}(u_1, u_2)\|_{X_\sigma^{0,s}}$$

for $(\tilde{b}, \tilde{s}) = (b', s)$ and $(\tilde{b}, \tilde{s}) = (b' + b_1, s - (\alpha + 1)b_1)$. Therefore, it suffices to show

$$\|Q_{11}(u_1, u_2)\|_{X_\sigma^{0,s}} \lesssim \|u_1\|_{X_0^{b-b',s}}\|u_2\|_{X_0^{b,s}} \tag{4.53}$$

This follows from Lemma 4.13 with $(\tilde{b}, \tilde{s}) = (b', s)$. We have to check that (4.20) holds for this choice of $(\tilde{b}, \tilde{s})$. However, this was already proven to hold under the assumption (4.47).

2. $k = 2$: By the definition of X_2, we find that

$$\|u_1\|_{X_\sigma^{b+b_1, s-(\alpha+1)b_1}} \leq \|u_1\|_{X_2}$$

so that it suffices to show

$$\|Q_{11}(u_1, u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_\sigma^{b+b_1, s-(\alpha+1)b_1}} \|u_2\|_{X_0^{b,s}} \tag{4.54}$$

for $(\tilde{b}, \tilde{s}) = (b', s)$ and $(\tilde{b}, \tilde{s}) = (b'+b_1, s-(\alpha+1)b_1)$. We use Lemma 4.14 in order to prove this. Therefore, we have to show that the conditions (4.24) and (4.25) hold. If $\frac{4}{3} < \alpha \leq 2$, then the definition (4.38) of b_1 implies that (4.24) holds. If $\alpha > 2$, then (4.24) is fulfilled provided that

$$\varepsilon \leq \frac{1}{\alpha}\left(\frac{3}{4}\alpha - \frac{3}{2}\right) \tag{4.55}$$

where the right hand side of this inequality is positive because of $\alpha > 2$. Since $\alpha < 4$, (4.25) follows from the definition (4.39) of σ.

This ends the proof also in the case that $\frac{4}{3} < \alpha < 4$. □

4.5 Notes and references

The use of the Picard iteration scheme in Bourgain spaces to obtain local well-posedness for the Cauchy problem of the Kadomtsev-Petviashvili II equation goes back to the seminal work [5] of Bourgain. In this work, Bourgain showed the (global) well-posedness of (4.1) (on $\mathbb{T}^2$ rather than on $\mathbb{R}^2$) with initial values in L^2, i. e. for $s_1 = s_2 = 0$. This result has been improved afterwards by Takaoka and Tzvetkov [31] and Isaza and Mejía [13] to the local well-posedness of (4.1) for $s_1 > -\frac{1}{3}$ and $s_2 \geq 0$. (For previous results see also [33], [34], [29].) In [30], Takaoka showed local well-posedness for $s_1 > -\frac{1}{2}$, $s_2 = 0$, but only if the additional low frequency condition $|D_x|^{-\frac{1}{2}+\varepsilon} u_0 \in L^2$, with suitably chosen ε, is imposed on the initial values. Note that global well-posedness for equation (4.1) holds for $s_1 > -\frac{1}{14}$ and $s_2 = 0$. This was shown by Isaza and Mejía in [14].

For the fifth order Kadomtsev-Petviashvili II equation (4.3), Saut and Tzvetkov [25] proved local well-posedness for $s_1 \geq -\frac{1}{4}$ and $s_2 \geq 0$ under the additional low frequency condition $\partial_x^{-1} u_0 \in H^{s_1,s_2}(\mathbb{R}^2)$ (which is removed for

$s_1 = s_2 = 0$ by the same authors in [26]). Note that the equation considered in [25,26] is slightly more general than (4.3) because it also contains the third order term. Very recently, ISAZA, LÓPEZ AND MEJÍA [12] have improved the local well-posedness result to $s_1 > -\frac{5}{4}$ and $s_2 \geq 0$. (These authors also show global well-posedness of (4.3) for $s_1 > -\frac{4}{7}$ and $s_2 = 0$.) In [26], SAUT AND TZVETKOV also showed local well-posedness for the fifth order Kadomtsev-Petviashvili II equation with periodic boundary condition, i. e. posed on $(-T,T) \times \mathbb{T}^2$, for initial values in the subset of the non-isotropic Sobolev space with $s_1 \geq -\frac{1}{8}$ and $s_2 = 0$ with constant mean value in x.

For general $\alpha \in (\frac{4}{3}, 6]$, IÓRIO AND NUNES [10] showed the local well-posedness for initial values u_0 in the isotropic Sobolev space $H^s(\mathbb{R}^2)$, $s > 2$ with the additional low frequency condition $\partial_x^{-1} u_0 \in H^s(\mathbb{R}^2)$ using parabolic regularization. Note that these authors consider much more general equations and do not use the dispersive structure of the equation.

Chapter 5

The three dimensional case

In this chapter we consider the *three dimensional Kadomtsev-Petviashvili II equation*

$$(u_t + u_{xxx} + (u^2)_x)_x + \Delta_{\vec{y}} u = 0 \quad \text{in } (-T,T) \times \mathbb{R}^3, \quad u(0) = u_0 \tag{5.1}$$

and the more general *dispersion generalised Kadomtsev-Petviashvili II type equation*

$$(u_t - |D_x|^\alpha u_x + (u^2)_x)_x + \Delta_{\vec{y}} u = 0 \quad \text{in } (-T,T) \times \mathbb{R}^3, \quad u(0) = u_0 \tag{5.2}$$

with $2 \le \alpha \le 6$. Note that (5.1) is just (5.2) for $\alpha = 2$. For $\alpha = 4$, (5.2) is the *three dimensional fifth order Kadomtsev-Petviashvili II equation*

$$(u_t - u_{xxxxx} + (u^2)_x)_x + \Delta_{\vec{y}} u = 0 \quad \text{in } (-T,T) \times \mathbb{R}^3, \quad u(0) = u_0 \tag{5.3}$$

5.1 Main results

Our main result for the Kadomtsev-Petviashvili II equation in three space dimensions (5.1) is the following.

Theorem 5.1. *Let $s_1 > \frac{1}{2}$ and $s_2 > 0$. Then there exist a Banach space $X \hookrightarrow C_b(\mathbb{R}; H^{s_1,s_2}(\mathbb{R}^3))$ and a non increasing function $T : (0,\infty) \to (0,\infty)$ such that the following holds true:*

a) For every $r > 0$ and $u_0 \in \mathcal{B}_r := \{u_0 \in H^{s_1,s_2}(\mathbb{R}^3) \mid \|u_0\|_{H^{s_1,s_2}(\mathbb{R}^3)} < r\}$ there is a unique solution $u \in X_{T(r)} \hookrightarrow C([-T(r),T(r)]; H^{s_1,s_2}(\mathbb{R}^3))$ of (5.1).

b) For every $r > 0$ the flow map $F_r : \mathcal{B}_r \to X_{T(r)}, u_0 \mapsto u$ defined by a) is analytic.

c) If $r_2 > r_1 > 0$ and $u_0 \in \mathcal{B}_{r_1}$, then $R_{T(r_2)} F_{r_1}(u_0) = F_{r_2}(u_0)$.

We also have the following theorem concerning solutions of the generalised equation (5.2) in three space dimensions.

Theorem 5.2. *For $2 < \alpha \leq 6$ let $s_1 > \max(\frac{3}{2} - \frac{\alpha}{2}, \frac{1}{4} - \frac{5}{24}\alpha)$ and $s_2 \geq 0$. Then there exist a Banach space $X \hookrightarrow C_b(\mathbb{R}; H^{s_1,s_2}(\mathbb{R}^3))$ and a non increasing function $T : (0, \infty) \to (0, \infty)$ such that the following holds true:*

a) For every $r > 0$ and $u_0 \in \mathcal{B}_r := \{u_0 \in H^{s_1,s_2}(\mathbb{R}^3) \mid \|u_0\|_{H^{s_1,s_2}(\mathbb{R}^3)} < r\}$ there is a unique solution $u \in X_{T(r)} \hookrightarrow C([-T(r), T(r)]; H^{s_1,s_2}(\mathbb{R}^3))$ of (5.2).

b) For every $r > 0$ the flow map $F_r : \mathcal{B}_r \to X_{T(r)}, u_0 \mapsto u$ defined by a) is analytic.

c) If $r_2 > r_1 > 0$ and $u_0 \in \mathcal{B}_{r_1}$, then $R_{T(r_2)} F_{r_1}(u_0) = F_{r_2}(u_0)$.

Theorem 5.1 and Theorem 5.2 follow from Theorem 2.22 and the bilinear estimates of Theorem 5.6 and Theorem 5.7, respectively (cf. Section 5.2).

Remark 5.3. In the particular case $\alpha = 4$ of the fifth order Kadomtsev-Petviashvili II equation in three space dimensions, Theorem 5.2 shows the local well-posedness of (5.3) for $s_1 > -\frac{1}{2}$ and $s_2 \geq 0$.

Remark 5.4. Just as in the two dimensional case, we have that if u is a solution of (5.2), so is

$$u_\delta(t, x, \vec{y}) = \delta^\alpha u(\delta^{\alpha+1} t, \delta x, \delta^{\frac{\alpha}{2}+1} \vec{y})$$

Considering the homogeneous Sobolev norm

$$\|u_0\|_{\dot{H}^{s_1,s_2}} := \||\xi|^{s_1} |\vec{\eta}|^{s_2} \mathcal{F} u_0\|_{L^2_{\xi\vec{\eta}}}$$

we get $\|u_\delta(0, \cdot, \cdot)\|_{\dot{H}^{s_1,s_2}} = \delta^{\frac{\alpha}{2} - \frac{3}{2} + s_1 + (\frac{\alpha}{2}+1)s_2} \|u(0, \cdot, \cdot)\|_{\dot{H}^{s_1,s_2}}$. This scaling argument suggests that we have ill-posedness for $s_1 + (1 + \frac{\alpha}{2})s_2 < \frac{3}{2} - \frac{\alpha}{2}$. Note that, for $\alpha = 2$, we again come arbitrarily close to the scale invariant space $\dot{H}^{\frac{1}{2},0}(\mathbb{R}^3)$ but in this case we "loose an ε" in the x- as well as the $\vec{y}$-regularity. For $2 < \alpha \leq \frac{30}{7}$, though, we can let $s_2 = 0$ in Theorem 5.2 and reach the critical value $\frac{3}{2} - \frac{\alpha}{2}$ of s_1, except for the endpoint. Note that this result includes the case of the fifth order Kadomtsev-Petviashvili II equation in three space dimensions (5.3). For $\alpha > \frac{30}{7}$, we have that $\frac{1}{4} - \frac{5}{24}\alpha > \frac{3}{2} - \frac{\alpha}{2}$, so that we do not reach the scaling limit in this case.

For $\alpha > 3$, Theorem 5.2 shows, in particular, the local in time well-posedness of (5.2) for $u_0 \in L^2(\mathbb{R}^3)$. By combining this local well-posedness result with the conservation of the L^2-norm, which holds for real valued solutions of (5.2), we get the following global result, where $H^{s_1,0}(\mathbb{R}^3;\mathbb{R})$ denotes the subspace of all real valued functions in $H^{s_1,0}(\mathbb{R}^3)$.

Theorem 5.5. *For $3 < \alpha \leq 6$ let $s_1 \geq 0$ Then there exists a Banach space $X \hookrightarrow C_b(\mathbb{R}; H^{s_1,0}(\mathbb{R}^3;\mathbb{R}))$ such that for every $u_0 \in H^{s_1,0}(\mathbb{R}^3;\mathbb{R})$ and $T > 0$, there is exactly one solution u of equation (5.2) in X_T.*

Theorem 5.5 follows from Theorem 2.28 and the bilinear estimate which is proven in Theorem 5.7 (cf. Section 5.2).

5.2 The main bilinear estimate

Let us announce the theorem first in the case $\alpha = 2$. We denote the regularity in x by s instead of s_1 as in the case of two space dimensions.

Theorem 5.6. *For $\alpha = 2$, $s > \frac{1}{2}$ and $s_2 > 0$ there exist $b > \frac{1}{2}$, $b' \in (b-1, 0]$, $b_1 \in [0, -b']$ and $\sigma \in [0, 1]$ such that for the spaces X_1, X_2, and Y defined by*

$$X_1 := X_0^{b-b',s,s_2}, \quad X_2 := X_\sigma^{b,s,s_2} \cap X_\sigma^{b+b_1,s-(\alpha+1)b_1,s_2} \tag{5.4}$$

$$Y := X_\sigma^{b',s,s_2} \cap X_\sigma^{b'+b_1,s-(\alpha+1)b_1,s_2} \tag{5.5}$$

we have that

$$\|\partial_x(u_1 u_2)\|_Y \lesssim \|u_1\|_{X_k} \|u_2\|_{X_l} \tag{5.6}$$

for $u_1, u_2 \in \mathcal{S}_{-\infty}$ and $k, l \in \{1, 2\}$.

In the case $\alpha > 2$ we will formulate and prove the main bilinear estimate only for $s_2 = 0$. The general case $s_2 \geq 0$ then follows by the same arguments as in the two dimensional case (see the beginning of Section 4.2).

Theorem 5.7. *Let $2 < \alpha \leq 6$ and*

$$s > \max\left(\frac{3}{2} - \frac{\alpha}{2}, \frac{1}{4} - \frac{5}{24}\alpha\right) \tag{5.7}$$

Then there exist $b > \frac{1}{2}$, $b' \in (b-1, 0]$, $b_1 \in [0, -b']$ and $\sigma \in [0, 1]$ such that for the spaces X_1, X_2, and Y defined by

$$X_1 := X_0^{b-b',s}, \quad X_2 := X_\sigma^{b,s} \cap X_\sigma^{b+b_1,s-(\alpha+1)b_1} \tag{5.8}$$

$$Y := X_\sigma^{b',s} \cap X_\sigma^{b'+b_1,s-(\alpha+1)b_1} \tag{5.9}$$

we have that

$$\|\partial_x(u_1u_2)\|_{\tilde{X}} \lesssim \|u_1\|_{X_k}\|u_2\|_{X_l} \tag{5.10}$$

for $u_1, u_2 \in \mathcal{S}_{-\infty}$ *and* $k, l \in \{1, 2\}$

The proofs of Theorem 5.6 and Theorem 5.7 go along the same lines as the proof of Theorem 4.7 in Chapter 4: We use the decompositions (4.9) and (4.10) of the bilinear term $\partial_x(u_1u_2)$ and then give estimates for the "pieces" $Q_{ij}(u_1, u_2)$ in suitable X_σ^{b,s_1,s_2}-spaces. In the case $\alpha > 2$, i. e. when we have $s_2 = 0$, this means showing estimates of the form (4.11) which are equivalent to the integral estimates (4.12). If $\alpha = 2$, then we need the presence of $\vec{y}$-regularity, i. e. that $s_2 > 0$, in order to be able to prove the estimates for some of the $Q_{ij}(u_1, u_2)$. We then show estimates of the form

$$\|Q_{ij}(u_1, u_2)\|_{X_\sigma^{\tilde{b},\tilde{s},s_2}} \lesssim \|u_1\|_{X_{\bar{\sigma}}^{\bar{b},\bar{s},s_2}}\|u_2\|_{X_0^{b,s,s_2}} \tag{5.11}$$

Note, however, that even if $\alpha = 2$, we do not need the $\vec{y}$-regularity in all of the cases Q_{ij} and that (5.11) for all $s_2 \geq 0$ follows from (4.11) because of $\frac{\langle\vec{\eta}\rangle^{s_2}}{\langle\vec{\eta}_1\rangle^{s_2}\langle\vec{\eta}_2\rangle^{s_2}} \lesssim 1$.

As in the two dimensional case, the main ingredients in the proof of these estimates are the bilinear Strichartz type estimates of Section 3.5 and the "resonance identity" (3.63), which again implies by Lemma 3.13 that

$$|\lambda_{\max}| \gtrsim |\xi_{\min}||\xi_{\max}|^\alpha \tag{5.12}$$

5.3 Estimates for the Q_{ij}

In this section we will derive suitable estimates for the "pieces" $Q_{ij}(u_1, u_2)$ of the bilinear term $\partial_x(u_1u_2)$ (see (4.10)). As in the two dimensional case, it will, in most cases, be possible to derive an estimate of the form (4.11) with $(\bar{b}, \bar{s}, \bar{\sigma}) = (b, s, 0)$, i. e. with $u_1, u_2 \in X_0^{b,s}$. In these cases, the integral inequality we have to prove reads

$$\int_{A_{ij}} \frac{|\xi|^{1-\sigma}\langle\xi\rangle^{\tilde{s}+\sigma}\langle\lambda\rangle^{\tilde{b}}}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b} f_1f_2f_3\,d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2} \tag{5.13}$$

This is proven under various conditions on the parameters $b, s, \tilde{b}, \tilde{s}$ and σ.

In the critical case, i. e. $(i, j) = (1, 1)$, we will not choose $u_1 \in X_0^{b,s}$ but prove separate estimates suitable for the two cases $u_1 \in X_1$ and $u_1 \in X_2$ (cf. Lemma 5.10 and Lemma 5.11). Also, in the case $\alpha = 2$, we need some extra estimates which require the presence of $\vec{y}$-regularity (cf. Lemmata 5.17–5.19).

The following two conditions will be assumed in all cases:

$$s, \tilde{s} \in \mathbb{R},\ 0 \le \sigma \le 1,\ b > \frac{1}{2},\ -\frac{1}{2} < \tilde{b} \le 0 \tag{5.14}$$

and

$$\tilde{b} < \frac{1}{\alpha+1}(-2+2s-\tilde{s}) \tag{5.15}$$

Let us now consider the different cases.

Lemma 5.8. *We have that*

$$\|Q_{00}(u_1,u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_0^{b,s}}\|u_2\|_{X_0^{b,s}} \tag{5.16}$$

provided that (5.14) *and*

$$\sigma \le \frac{1}{2} \tag{5.17}$$

hold.

Proof. We have to prove (5.13) for $(i,j)=(0,0)$. Since in A_{00} we have $|\xi| \le 2|\xi_2| \le 2$ and $|\xi_1| \le |\xi_2| \le 1$, it follows that $\langle\xi\rangle \sim \langle\xi_1\rangle \sim \langle\xi_2\rangle \sim 1$. Therefore, it suffices to show

$$\int_{A_{00}} k_{00}(\mu_1,\mu)\frac{|\xi|^{\frac{1}{2}}|\xi_1|^{-\frac{1}{2}}|\xi_2|^{-\frac{1}{2}}}{\langle\xi_1\rangle^{\frac{1}{2}+\delta}\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b} f_1f_2f_3\,d\mu_1d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

where $k_{00}(\mu_1,\mu) := |\xi|^{\frac{1}{2}-\sigma}|\xi_1|^{\frac{1}{2}}|\xi_2|^{\frac{1}{2}}\langle\lambda\rangle^{\tilde{b}}$ and $\delta > 0$ can be chosen arbitrarily. k_{00} is bounded on A_{00} because of (5.17) and $\tilde{b} \le 0$. Therefore, the claim follows from the bilinear estimate (3.53). □

Lemma 5.9. *We have that*

$$\|Q_{10}(u_1,u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_0^{b,s}}\|u_2\|_{X_0^{b,s}} \tag{5.18}$$

provided that (5.14), (5.15) *and*

$$\tilde{b} \le \frac{1}{\alpha}(-1+s-\tilde{s}) \tag{5.19}$$

hold.

Proof. We have to prove (5.13) for $(i,j)=(1,0)$. Using that in A_{10} we have $\langle\xi\rangle \sim \langle\xi_2\rangle \sim |\xi_2| \sim |\xi|$ and $\langle\lambda\rangle^{\tilde{b}} \lesssim |\xi|^{\alpha\tilde{b}}|\xi_1|^{\tilde{b}}$, we see that it suffices to show

$$\int_{A_{10}} k_{10}(\mu_1,\mu)\frac{|\xi|^{\frac{1}{2}}|\xi_1|^{-\frac{1}{2}}|\xi_2|^{-\frac{1}{2}}}{\langle\xi_1\rangle^{\frac{1}{2}+\delta}\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b} f_1f_2f_3\,d\mu_1d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

where

$$k_{10}(\mu_1,\mu) := |\xi|^{1+\alpha\tilde{b}+\tilde{s}-s}|\xi_1|^{\frac{1}{2}+\tilde{b}}\langle\xi_1\rangle^{-s+\frac{1}{2}+\delta} \tag{5.20}$$

and $\delta > 0$ will be chosen later. We show that k_{10} is bounded on Ξ_1, then it follows that it is especially bounded on $A_{10} \subset \Xi_1$. The lemma then follows from the bilinear estimate (3.53).

Let us first consider the case $|\xi_1| \leq 1$. We then have $\langle\xi_1\rangle \sim 1$. Because of (5.14) and (5.19), we have $\frac{1}{2} + \tilde{b} > 0$ and $1 + \alpha\tilde{b} + \tilde{s} - s \leq 0$. Because we have $|\xi| \gtrsim 1$ in Ξ_1, we find that $k_{10}(\mu_1,\mu) \lesssim 1$ in this case.

Let us now consider the case $|\xi_1| \geq 1$. We then have $\langle\xi_1\rangle \sim |\xi_1|$. Using this and $|\xi| \gtrsim |\xi_1|$ in Ξ_1, we get $k_{10}(\mu_1,\mu) \lesssim |\xi_1|^{2+(\alpha+1)\tilde{b}+(\tilde{s}-2s)+\delta}$. Because of (5.15), there exists a $\delta > 0$ such that $2 + (\alpha+1)\tilde{b} + (\tilde{s}-2s) + \delta \leq 0$. It follows that $k_{10}(\mu_1,\mu) \lesssim 1$. □

Lemma 5.10. *We have that*

$$\|Q_{11}(u_1,u_2)\|_{X_\sigma^{0,\tilde{s}}} \lesssim \|u_1\|_{X_0^{b-\tilde{b},s}}\|u_2\|_{X_0^{b,s}} \tag{5.21}$$

provided that (5.14), (5.15) *and* (5.19) *hold.*

Proof. We have to prove that

$$\int_{A_{11}} \frac{|\xi|^{1-\sigma}\langle\xi\rangle^{\tilde{s}+\sigma} f_1 f_2 f_3}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\lambda_1\rangle^{b-\tilde{b}}\langle\lambda_2\rangle^b}\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

In A_{11} we have $\langle\xi\rangle \sim \langle\xi_2\rangle \sim |\xi_2| \sim |\xi|$ and $\langle\lambda_1\rangle^{\tilde{b}} \lesssim |\xi|^{\alpha\tilde{b}}|\xi_1|^{\tilde{b}}$. Therefore, it suffices to show

$$\int_{A_{11}} k_{10}(\mu_1,\mu)\frac{|\xi|^{\frac{1}{2}}|\xi_1|^{-\frac{1}{2}}|\xi_2|^{-\frac{1}{2}}}{\langle\xi_1\rangle^{\frac{1}{2}+\delta}\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b} f_1 f_2 f_3\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

where k_{10} is defined as in (5.20) and $\delta > 0$ is chosen as in Lemma 5.9. In Lemma 5.9, k_{10} was shown to be bounded on Ξ_1 under the conditions (5.14), (5.15) and (5.19). Hence, it is especially bounded on $A_{11} \subset \Xi_1$ and the lemma follows from the bilinear estimate (3.53). □

Lemma 5.11. *We have that*

$$\|Q_{11}(u_1,u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_\sigma^{b+b_1,s-(\alpha+1)b_1}}\|u_2\|_{X_0^{b,s}} \tag{5.22}$$

provided that (5.14), (5.15) *and*

$$\tilde{b} - b_1 \leq -\frac{1}{4} \tag{5.23}$$

$$-\sigma - \frac{1}{2} + \frac{\alpha}{12} \leq \tilde{b} - b_1 \leq -\frac{3}{2\alpha} - \frac{1}{12} + \frac{s-\tilde{s}}{\alpha} \tag{5.24}$$

hold.

Proof. We have to prove that

$$\int_{A_{11}} \frac{|\xi|^{1-\sigma}\langle\xi\rangle^{\tilde{s}+\sigma}|\xi_1|^{\sigma}\langle\lambda\rangle^{\tilde{b}} f_1 f_2 f_3}{\langle\xi_1\rangle^{s-(\alpha+1)b_1+\sigma}\langle\xi_2\rangle^{s}\langle\lambda_1\rangle^{b+b_1}\langle\lambda_2\rangle^{b}}\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

Because of $\langle\xi\rangle \sim \langle\xi_2\rangle \sim |\xi_2| \sim |\xi|$ in A_{11}, it suffices to show

$$\int_{A_{11}} k_{11}(\mu_1,\mu)\frac{|\xi\xi_1\xi_2|^{-\frac{1}{4}+\frac{\alpha}{12}}}{\langle\lambda_1\rangle^{\frac{1}{4}}\langle\lambda\rangle^{b}\langle\lambda_2\rangle^{b}} f_1 f_2 f_3\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

where

$$k_{11}(\mu_1,\mu) := \langle\lambda\rangle^{\tilde{b}+b}\langle\lambda_1\rangle^{-b-b_1+\frac{1}{4}}|\xi|^{\frac{3}{2}-\frac{\alpha}{6}+\tilde{s}-s}|\xi_1|^{\sigma+\frac{1}{4}-\frac{\alpha}{12}}\langle\xi_1\rangle^{-s+(\alpha+1)b_1-\sigma}$$

Now, (5.22) follows from the bilinear estimate (3.50) if we show that k_{11} is bounded on A_{11}.

Because of (5.14), we have $\tilde{b}+b>0$. Using this and $|\lambda| \le |\lambda_1|$ in A_{11}, we get $\langle\lambda\rangle^{\tilde{b}+b}\langle\lambda_1\rangle^{-b-b_1+\frac{1}{4}} \le \langle\lambda_1\rangle^{\tilde{b}-b_1+\frac{1}{4}}$. Furthermore, $|\lambda_1| \gtrsim |\xi|^{\alpha}|\xi_1|$ in A_{11}, so using that by (5.23) we have $\tilde{b}-b_1+\frac{1}{4} \le 0$, we find that

$$k_{11}(\mu_1,\mu) \lesssim |\xi|^{\frac{3}{2}+\frac{\alpha}{12}+\alpha(\tilde{b}-b_1)+(\tilde{s}-s)}|\xi_1|^{\sigma+\frac{1}{2}-\frac{\alpha}{12}+(\tilde{b}-b_1)}\langle\xi_1\rangle^{-s+(\alpha+1)b_1-\sigma}$$

Let us first consider the case $|\xi_1| \le 1$. We then have $\langle\xi_1\rangle \sim 1$. By (5.24), it follows that $\frac{3}{2}+\frac{\alpha}{12}+\alpha(\tilde{b}-b_1)+(\tilde{s}-s) \le 0$ and $\sigma+\frac{1}{2}-\frac{\alpha}{12}+(\tilde{b}-b_1) \ge 0$. Using that $|\xi| \gtrsim 1$ in A_{11}, we obtain $k_{11}(\mu_1,\mu) \lesssim 1$.

Let us now consider the case $|\xi_1| \ge 1$. We then have $\langle\xi_1\rangle \sim |\xi_1|$. Using this and $|\xi| \gtrsim |\xi_1|$ in A_{11}, we find that $k_{11}(\mu_1,\mu) \lesssim |\xi_1|^{2+(\alpha+1)\tilde{b}+(\tilde{s}-2s)}$. Therefore, $k_{11}(\mu_1,\mu) \lesssim 1$ follows from (5.15). □

Lemma 5.12. *We have that*

$$\|Q_{12}(u_1,u_2)\|_{X_{\sigma}^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_0^{b,s}}\|u_2\|_{X_0^{b,s}} \tag{5.25}$$

provided that (5.14)*,* (5.15) *and* (5.19) *hold.*

Proof. We have to show (5.13) for $(i,j)=(1,2)$. Using $\langle\xi\rangle \sim \langle\xi_2\rangle \sim |\xi_2| \sim |\xi|$ in A_{12}, we see that it suffices to show

$$\int_{A_{12}} k_{12}(\mu_1,\mu)\frac{|\xi|^{-\frac{1}{2}}|\xi_1|^{-\frac{1}{2}}|\xi_2|^{\frac{1}{2}}}{\langle\xi_1\rangle^{\frac{1}{2}+\delta}\langle\lambda\rangle^{b}\langle\lambda_1\rangle^{b}} f_1 f_2 f_3\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

where $k_{12}(\mu_1,\mu) := \langle\lambda_2\rangle^{-b}\langle\lambda\rangle^{\tilde{b}+b}|\xi|^{1+\tilde{s}-s}|\xi_1|^{\frac{1}{2}}\langle\xi_1\rangle^{-s+\frac{1}{2}+\delta}$ and $\delta>0$ is chosen as in Lemma 5.9. If we show that k_{12} is bounded on A_{12}, then (5.25) follows from the bilinear estimate (3.56).

Because of (5.14), we have $\tilde{b}+b>0$. Since $|\lambda|\le|\lambda_2|=|\lambda_{\max}|$ in A_{12}, we get $\langle\lambda_2\rangle^{-b}\langle\lambda\rangle^{\tilde{b}+b}\le\langle\lambda_2\rangle^{\tilde{b}}\lesssim|\xi|^{\alpha\tilde{b}}|\xi_1|^{\tilde{b}}$. Therefore, $k_{12}(\mu_1,\mu)\lesssim k_{10}(\mu_1,\mu)$, where k_{10} is defined as in (5.20). Because, in Lemma 5.9, k_{10} was shown to be bounded on Ξ_1 under the conditions (5.14), (5.15) and (5.19), it is especially bounded on $A_{12}\subset\Xi_1$. □

Lemma 5.13. *We have that*

$$\|Q_{20}(u_1,u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}}\lesssim\|u_1\|_{X_0^{b,s}}\|u_2\|_{X_0^{b,s}} \tag{5.26}$$

provided that (5.14), (5.15) *and*

$$\sigma-\frac{3}{2}+\frac{\alpha}{12}\le\tilde{b}\le\min\left(-\frac{1}{4},-\frac{1}{2\alpha}-\frac{1}{12}+\frac{2s}{\alpha}\right) \tag{5.27}$$

hold.

Proof. We have to prove (5.13) for $(i,j)=(2,0)$. As $\langle\xi_2\rangle\sim\langle\xi_1\rangle\sim|\xi_1|\sim|\xi_2|$ in A_{20}, we see that it suffices to show

$$\int_{A_{20}}k_{20}(\mu_1,\mu)\frac{|\xi\xi_1\xi_2|^{-\frac{1}{4}+\frac{\alpha}{12}}}{\langle\lambda\rangle^{\frac{1}{4}}\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b}f_1f_2f_3\,d\mu_1d\mu\lesssim\prod_{i=1}^3\|f_i\|_{L^2}$$

where $k_{20}(\mu_1,\mu):=\langle\lambda\rangle^{\tilde{b}+\frac{1}{4}}|\xi|^{\frac{5}{4}-\frac{\alpha}{12}-\sigma}\langle\xi\rangle^{\tilde{s}+\sigma}|\xi_1|^{-2s+\frac{1}{2}-\frac{\alpha}{6}}$. If we show that k_{20} is bounded on A_{20}, then (5.26) follows from (3.49).

Because of (5.27), we have $\tilde{b}+\frac{1}{4}\le 0$. Using this and $|\lambda|\gtrsim|\xi_1|^\alpha|\xi|$ in A_{20}, we find that $k_{20}(\mu_1,\mu)\lesssim|\xi|^{\frac{3}{2}-\frac{\alpha}{12}+\tilde{b}-\sigma}\langle\xi\rangle^{\tilde{s}+\sigma}|\xi_1|^{\frac{1}{2}+\frac{\alpha}{12}+\alpha\tilde{b}-2s}$.

Let us first suppose that $|\xi|\le 1$. We then have $\langle\xi\rangle\sim 1$. By (5.27), it follows that $\frac{3}{2}-\frac{\alpha}{12}+\tilde{b}-\sigma\ge 0$ and $\frac{1}{2}+\frac{\alpha}{12}+\alpha\tilde{b}-2s\le 0$. Because $|\xi_1|\gtrsim 1$ in A_{20}, we obtain $k_{20}(\mu_1,\mu)\lesssim 1$.

Now, suppose that $|\xi|\ge 1$. Then we have $\langle\xi\rangle\sim|\xi|$. Because $|\xi_1|\gtrsim|\xi|$ in A_{20}, we find that $k_{20}(\mu_1,\mu)\lesssim|\xi|^{2+(\alpha+1)\tilde{b}+(\tilde{s}-2s)}\lesssim 1$, where the last inequality follows from (5.15). □

Lemma 5.13 can only be used if $\tilde{b}\le -1/4$. Since we also have to deal with values of $\tilde{b}$ greater than $-1/4$ (at least in the case $\alpha\le 9/4$), we also need

Lemma 5.14. *We have* (5.26) *provided that* (5.14), (5.15), $\sigma\le\frac{1}{2}$, $\alpha\le 9$ *and*

$$-\frac{1}{4}<\tilde{b}<\frac{12s-9}{24+4\alpha} \tag{5.28}$$

hold.

Proof. We have to prove (5.13) for $(i,j)=(2,0)$. Let $\theta := -4\tilde{b}$, then $\tilde{b} = -\frac{\theta}{4}$. By (5.14) and (5.28), we have $\theta \in [0,1)$. Because $\langle \xi_2 \rangle \sim \langle \xi_1 \rangle \sim |\xi_1| \sim |\xi_2|$ in A_{20}, we see that it suffices to show

$$\int_{A_{20}} \tilde{k}_{20}(\mu_1,\mu) \frac{|\xi|^{\frac{1}{2}-\theta(\frac{3}{4}-\frac{\alpha}{12})}|\xi_1|^{-\frac{3}{2}+\theta(1+\frac{\alpha}{6})-\delta}}{\langle \lambda \rangle^{\frac{\theta}{4}} \langle \lambda_1 \rangle^b \langle \lambda_2 \rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2}$$

where $\tilde{k}_{20}(\mu_1,\mu) := |\xi|^{\frac{1}{2}+\theta(\frac{3}{4}-\frac{\alpha}{12})-\sigma} \langle \xi \rangle^{\tilde{s}+\sigma} |\xi_1|^{-2s+\frac{3}{2}-\theta(1+\frac{\alpha}{6})+\delta}$ and $\delta > 0$ will be chosen later. If we show that $\tilde{k}_{20}$ is bounded on A_{20}, then (5.26) follows from the bilinear estimate (3.59).

By the definition of θ, we have

$$\tilde{k}_{20}(\mu_1,\mu) = |\xi|^{\frac{1}{2}-\tilde{b}(3-\frac{\alpha}{3})-\sigma} \langle \xi \rangle^{\tilde{s}+\sigma} |\xi_1|^{-2s+\frac{3}{2}+\tilde{b}(4+\frac{2}{3}\alpha)+\delta}$$

Let us first consider the case $|\xi| \le 1$. (5.28) implies $-2s+\frac{3}{2}+\tilde{b}(4+\frac{2}{3}\alpha) < 0$. Therefore, for δ sufficiently small, we have $-2s+\frac{3}{2}+\tilde{b}(4+\frac{2}{3}\alpha)+\delta \le 0$. Because $\sigma \le \frac{1}{2}$, we obtain $\frac{1}{2}-\sigma \ge 0$. Since $\alpha \le 9$ and $\tilde{b} \le 0$, we have $-\tilde{b}(3-\frac{\alpha}{3}) \ge 0$. Altogether, we get $\frac{1}{2}-\sigma-\tilde{b}(3-\frac{\alpha}{3}) \ge 0$. It follows that $\tilde{k}_{20}(\mu_1,\mu) \lesssim 1$ in this case.

Let us now consider the case $|\xi| \ge 1$. We then have $\langle \xi \rangle \sim |\xi|$. Because $|\xi_1| \gtrsim |\xi|$ in A_{20}, we find that $\tilde{k}_{20}(\mu_1,\mu) \lesssim |\xi|^{2+(\alpha+1)\tilde{b}+(\tilde{s}-2s)+\delta} \lesssim 1$, where the last inequality follows from (5.15) if we choose δ sufficiently small. □

Lemma 5.15. *We have that*

$$\|Q_{21}(u_1,u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_0^{b,s}} \|u_2\|_{X_0^{b,s}} \tag{5.29}$$

provided that (5.14), (5.15) *and*

$$\tilde{b} \le \frac{2s}{\alpha} \tag{5.30}$$

hold.

Proof. We have to prove (5.13) for $(i,j)=(2,1)$. As $\langle \xi_1 \rangle \sim \langle \xi_2 \rangle \sim |\xi_2| \sim |\xi_1|$ and $\langle \lambda \rangle^{\tilde{b}} \langle \lambda_1 \rangle^{-b} \le \langle \lambda \rangle^{-b} \langle \lambda_1 \rangle^{\tilde{b}} = \langle \lambda \rangle^{-b} \langle \lambda_{\max} \rangle^{\tilde{b}}$ in A_{21}, it suffices to show

$$\int_{A_{21}} k_{21}(\mu_1,\mu) \frac{|\xi|^{-\frac{1}{2}}|\xi_1|^{\frac{1}{2}}|\xi_2|^{-\frac{1}{2}}}{\langle \xi \rangle^{\frac{1}{2}+\delta} \langle \lambda \rangle^b \langle \lambda_2 \rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2}$$

where

$$k_{21}(\mu_1,\mu) := \langle \lambda_{\max} \rangle^{\tilde{b}} |\xi|^{\frac{3}{2}-\sigma} \langle \xi \rangle^{\frac{1}{2}+\tilde{s}+\sigma+\delta} |\xi_1|^{-2s} \tag{5.31}$$

and $\delta > 0$ will be chosen later. If we show that k_{21} is bounded on Ξ_2, then it is especially bounded on $A_{21} \subset \Xi_2$ and the lemma follows from (3.57).

Because $|\lambda_{\max}| \gtrsim |\xi_1|^\alpha |\xi|$ in Ξ_2, we obtain

$$k_{21}(\mu_1, \mu) \lesssim |\xi|^{\frac{3}{2}+\tilde{b}-\sigma} \langle \xi \rangle^{\frac{1}{2}+\tilde{s}+\sigma+\delta} |\xi_1|^{-2s+\alpha\tilde{b}}$$

Let us first consider the case that $|\xi| \leq 1$. We then have $\langle \xi \rangle \sim 1$. Because of (5.14), we have $\frac{3}{2} + \tilde{b} - \sigma > 0$. By (5.30), it follows that $-2s + \alpha\tilde{b} \leq 0$. Since $|\xi_1| \gtrsim 1$ in Ξ_2, this implies $k_{21}(\mu_1, \mu) \lesssim 1$ in this case.

Let us now consider the case that $|\xi| \geq 1$. This implies $\langle \xi \rangle \sim |\xi|$. Using this and $|\xi_1| \gtrsim |\xi|$ in Ξ_2, we get $k_{21}(\mu_1, \mu) \lesssim |\xi|^{2+(\alpha+1)\tilde{b}+(\tilde{s}-2s)+\delta}$. Because of (5.15), there exists a $\delta > 0$ such that $2 + (\alpha+1)\tilde{b} + (\tilde{s} - 2s) + \delta \leq 0$. We then obtain $k_{21}(\mu_1, \mu) \lesssim 1$. $\square$

Lemma 5.16. *We have that*

$$\|Q_{22}(u_1, u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_0^{b,s}} \|u_2\|_{X_0^{b,s}} \tag{5.32}$$

provided that (5.14), (5.15) *and* (5.30) *hold.*

Proof. We have to prove (5.13) for $(i, j) = (2, 2)$. As $\langle \xi_1 \rangle \sim \langle \xi_2 \rangle \sim |\xi_2| \sim |\xi_1|$ and $\langle \lambda \rangle^{\tilde{b}} \langle \lambda_2 \rangle^{-b} \leq \langle \lambda \rangle^{-b} \langle \lambda_2 \rangle^{\tilde{b}} = \langle \lambda \rangle^{-b} \langle \lambda_{\max} \rangle^{\tilde{b}}$ in A_{22}, it suffices to show

$$\int_{A_{22}} k_{21}(\mu_1, \mu) \frac{|\xi|^{-\frac{1}{2}} |\xi_1|^{-\frac{1}{2}} |\xi_2|^{\frac{1}{2}}}{\langle \xi \rangle^{\frac{1}{2}+\delta} \langle \lambda \rangle^b \langle \lambda_1 \rangle^b} f_1 f_2 f_3 \, d\mu_1 d\mu \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2}$$

where k_{21} is defined as in (5.31) and $\delta > 0$ is chosen as in Lemma 5.15. It was shown in Lemma 5.15 that k_{21} is bounded on Ξ_2 under the conditions (5.14), (5.15) and (5.30). Therefore, it is especially bounded on $A_{22} \subset \Xi_2$ and (5.32) follows from the bilinear estimate (3.58). $\square$

The lemmata proven so far are sufficient to give the proof of well-posedness for $\alpha > 2$. For $\alpha = 2$, however, we see that (5.14) and (5.19) for $\tilde{s} = s$ imply

$$-\frac{1}{2} < \tilde{b} \leq -\frac{1}{\alpha} = -\frac{1}{2}$$

which is a contradiction. Therefore, we need substitutes for Lemma 5.9, Lemma 5.10 and Lemma 5.12 in this case.

Lemma 5.17. *Let $\alpha = 2$. We have that*

$$\|Q_{10}(u_1, u_2)\|_{X_\sigma^{\tilde{b},\tilde{s},s_2}} \lesssim \|u_1\|_{X_0^{b,s,s_2}} \|u_2\|_{X_0^{b,s,s_2}} \tag{5.33}$$

provided that (5.14), (5.15) *and*

$$\tilde{b} \leq \frac{1}{2}\left(-1 + \frac{s_2}{2+2s_2} + (s - \tilde{s})\right) \tag{5.34}$$

hold.

Proof. We have to prove that

$$\int_{A_{10}} \frac{|\xi|^{1-\sigma}\langle\xi\rangle^{\tilde{s}+\sigma}\langle\vec{\eta}\rangle^{s_2}\langle\lambda\rangle^{\tilde{b}} f_1 f_2 f_3}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\vec{\eta}_1\rangle^{s_2}\langle\vec{\eta}_2\rangle^{s_2}\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b}\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

Let $\theta = s_2/(1+s_2)$ and $\delta = \theta/2$. Then $0 < \theta < s_2$ and $(1-\theta)/2 + \delta = 1/2$. Using $\langle\xi_2\rangle \sim \langle\xi\rangle \sim |\xi|$ and $\langle\lambda\rangle^{\tilde{b}} \lesssim |\xi|^{2\tilde{b}}|\xi_1|^{\tilde{b}}$ in A_{10}, we see that it suffices to show

$$\int_{A_{10}} \tilde{k}_{10}(\mu_1,\mu)\frac{|\xi|^{\frac{\theta}{2}}|\xi_1|^{-\frac{1}{2}}\langle\xi_1\rangle^{-\frac{1}{2}}\langle\vec{\eta}\rangle^{s_2} f_1 f_2 f_3}{\langle\vec{\eta}_1\rangle^{s_2}\langle\vec{\eta}_2\rangle^{s_2}\langle\lambda_1\rangle^b\langle\lambda_2\rangle^b}\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

where

$$\tilde{k}_{10}(\mu_1,\mu) := |\xi|^{1+2\tilde{b}-\frac{\theta}{2}+\tilde{s}-s}|\xi_1|^{\frac{1}{2}+\tilde{b}}\langle\xi_1\rangle^{-s+\frac{1}{2}} \tag{5.35}$$

If we show that $\tilde{k}_{10}$ is bounded on Ξ_1, then it is especially bounded on $A_{10} \subset \Xi_1$ and the lemma follows from the bilinear estimate (3.64).

Let us first consider the case that $|\xi_1| \leq 1$. This implies $\langle\xi_1\rangle \sim 1$. Because of (5.14), we have $\frac{1}{2} + \tilde{b} > 0$. Using (5.34) and the definition of θ, it follows that $1 + 2\tilde{b} - \frac{\theta}{2} + \tilde{s} - s \leq 0$. Because $|\xi| \gtrsim 1$ in Ξ_1, we obtain $\tilde{k}_{10}(\mu_1,\mu) \lesssim 1$ in this case.

Let us now consider the case $|\xi_1| \geq 1$. We then have $\langle\xi_1\rangle \sim |\xi_1|$. Using this and $|\xi| \gtrsim |\xi_1|$ in A_{10}, we find that $\tilde{k}_{10}(\mu_1,\mu) \lesssim |\xi_1|^{2+3\tilde{b}+\tilde{s}-2s-\frac{\theta}{2}} \lesssim 1$, where the last inequality follows because of (5.15) and $\theta > 0$. $\square$

Lemma 5.18. *Let* $\alpha = 2$. *We have that*

$$\|Q_{11}(u_1,u_2)\|_{X_\sigma^{0,\tilde{s},s_2}} \lesssim \|u_1\|_{X_0^{b-\tilde{b},s,s_2}}\|u_2\|_{X_0^{b,s,s_2}} \tag{5.36}$$

provided that (5.14), (5.15) *and* (5.34) *hold.*

Proof. We have to prove that

$$\int_{A_{11}} \frac{|\xi|^{1-\sigma}\langle\xi\rangle^{\tilde{s}+\sigma}\langle\vec{\eta}\rangle^{s_2} f_1 f_2 f_3}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\vec{\eta}_1\rangle^{s_2}\langle\vec{\eta}_2\rangle^{s_2}\langle\lambda_1\rangle^{b-\tilde{b}}\langle\lambda_2\rangle^b}\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

Let $\theta = s_2/(1+s_2)$ and $\delta = \theta/2$ as in Lemma 5.17. Because $\langle\xi_2\rangle \sim \langle\xi\rangle \sim |\xi|$ and $\langle\lambda_1\rangle^{\tilde{b}} \lesssim |\xi|^{2\tilde{b}}|\xi_1|^{\tilde{b}}$ in A_{11}, it suffices to show

$$\int_{A_{11}} \tilde{k}_{10}(\mu_1,\mu)\frac{|\xi|^{\frac{\theta}{2}}|\xi_1|^{-\frac{1}{2}}\langle\xi_1\rangle^{-\frac{1}{2}}\langle\vec{\eta}\rangle^{s_2} f_1 f_2 f_3}{\langle\vec{\eta}_1\rangle^{s_2}\langle\vec{\eta}_2\rangle^{s_2}\langle\lambda_1\rangle^{b}\langle\lambda_2\rangle^{b}}\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

where $\tilde{k}_{10}$ is defined as in (5.35). In Lemma 5.17, it was shown that $\tilde{k}_{10}$ is bounded on Ξ_1 under the conditions (5.14), (5.15) and (5.34). Therefore, it is especially bounded on $A_{11} \subset \Xi_1$ and (5.36) follows from (3.64). □

Lemma 5.19. *Let $\alpha = 2$. We have that*

$$\|Q_{12}(u_1,u_2)\|_{X_\sigma^{\tilde{b},\tilde{s},s_2}} \lesssim \|u_1\|_{X_0^{b,s,s_2}}\|u_2\|_{X_0^{b,s,s_2}} \tag{5.37}$$

provided that (5.14), (5.15) *and*

$$-\frac{1}{2}+\frac{2}{3}\left(b-\frac{1}{2}\right) \le \tilde{b} \le -\frac{1}{2}+\frac{2}{3}\left(b-\frac{1}{2}\right)+\frac{s_2}{8}+\frac{s-\tilde{s}}{2} \tag{5.38}$$

$$\tilde{b} \le -\frac{1}{2}+\frac{4}{3}\left(b-\frac{1}{2}\right)+(s-\tilde{s}) \tag{5.39}$$

hold.

Proof. Without loss of generality, we can assume $s_2 \le 1$. We have to show

$$\int_{A_{12}} \frac{|\xi|^{1-\sigma}\langle\xi\rangle^{\tilde{s}+\sigma}\langle\vec{\eta}\rangle^{s_2}\langle\lambda\rangle^{\tilde{b}} f_1 f_2 f_3}{\langle\xi_1\rangle^{s}\langle\xi_2\rangle^{s}\langle\vec{\eta}_1\rangle^{s_2}\langle\vec{\eta}_2\rangle^{s_2}\langle\lambda_1\rangle^{b}\langle\lambda_2\rangle^{b}}\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

Let $\theta = \min(2\tilde{b}+1, s_2/2)$. Due to $s_2 > 0$ and (5.14), we have $0 < \theta < s_2 \le 1$. Furthermore, let $\bar{b} = 1/3 + b/3$. Then,

$$b - \bar{b} = \frac{2}{3}\left(b-\frac{1}{2}\right) \tag{5.40}$$

and $1/2 < \bar{b} < b$. Using that $\langle\xi_2\rangle \sim \langle\xi\rangle \sim |\xi|$ in A_{12}, it suffices to show

$$\int_{A_{12}} \frac{\tilde{k}_{12}(\mu_1,\mu)|\xi|^{\frac{\theta}{2}}|\xi_1|^{-\frac{1}{2}}\langle\xi_1\rangle^{-\frac{1-\theta}{2}-\delta}\langle\vec{\eta}\rangle^{s_2} f_1 f_2 f_3}{\langle\vec{\eta}_1\rangle^{s_2}\langle\vec{\eta}_2\rangle^{s_2}\langle\lambda_1\rangle^{b}\langle\lambda_2\rangle^{\theta\bar{b}}\langle\lambda\rangle^{(1-\theta)\bar{b}}}\, d\mu_1 d\mu \lesssim \prod_{i=1}^{3}\|f_i\|_{L^2}$$

where $\tilde{k}_{12}(\mu_1,\mu) := \langle\lambda\rangle^{\tilde{b}+(1-\theta)\bar{b}}\langle\lambda_2\rangle^{-b+\theta\bar{b}}|\xi|^{1-\frac{\theta}{2}+\tilde{s}-s}|\xi_1|^{\frac{1}{2}}\langle\xi_1\rangle^{-s+\frac{1}{2}+\frac{\theta}{2}}$ and $\delta := \theta$. If we show that k_{12} is bounded on A_{12}, then the lemma follows from (3.65).

Because of $\theta \le 2\tilde{b}+1$, it follows that $\tilde{b}+(1-\theta)\bar{b} \ge (1-2\bar{b})\tilde{b} \ge 0$, where the last inequality follows from $\bar{b} > 1/2$ and $\tilde{b} \le 0$. Since $|\lambda| \le |\lambda_2|$ in A_{12}, we get $\langle\lambda\rangle^{\tilde{b}+(1-\theta)\bar{b}}\langle\lambda_2\rangle^{-b+\theta\bar{b}} \le \langle\lambda_2\rangle^{\tilde{b}-(b-\bar{b})}$. Because $\tilde{b}-(b-\bar{b}) < 0$, it follows that

$$\tilde{k}_{12}(\mu_1,\mu) \lesssim |\xi|^{1+2\tilde{b}-\frac{4}{3}(b-\frac{1}{2})-\frac{\theta}{2}+\tilde{s}-s}|\xi_1|^{\frac{1}{2}+\tilde{b}-\frac{2}{3}(b-\frac{1}{2})}\langle\xi_1\rangle^{-s+\frac{1}{2}+\frac{\theta}{2}}$$

where we used (5.40).

Let us first consider the case $|\xi_1| \le 1$. This implies $\langle\xi_1\rangle \sim 1$. Because of (5.38), (5.39) and the definition of θ, we have $1+2\tilde{b}-\frac{4}{3}(b-\frac{1}{2})-\frac{\theta}{2}+\tilde{s}-s \le 0$. By (5.38), we have $\frac{1}{2}+\tilde{b}-\frac{2}{3}(b-\frac{1}{2}) \ge 0$. Since $|\xi| \gtrsim 1$ in A_{12}, we altogether obtain $\tilde{k}_{12}(\mu_1,\mu) \lesssim 1$ in this case.

Let us now consider the case $|\xi_1| \ge 1$. We then have $\langle\xi_1\rangle \sim |\xi_1|$. Using that $|\xi| \gtrsim |\xi_1|$ in A_{12}, we get $\tilde{k}_{12}(\mu_1,\mu) \lesssim |\xi_1|^{2+3\tilde{b}+\tilde{s}-2s-2(b-\frac{1}{2})}$. By (5.14) and (5.15), it follows that $2+3\tilde{b}+\tilde{s}-2s-2(b-\frac{1}{2}) < 0$. Therefore, we obtain $\tilde{k}_{12}(\mu_1,\mu) \lesssim 1$. □

5.4 Proof of the main bilinear estimates

We now give the proofs of Theorem 5.6 and Theorem 5.7.

Proof (of Theorem 5.6). Let

$$b := \frac{1}{2}+\frac{3}{4}\varepsilon, \quad b' := -\frac{1}{2}+\varepsilon \tag{5.41}$$

where $\varepsilon > 0$ will be restricted by various upper bounds given in the course of the proof. We obviously have that $b' > b-1$. Let

$$b_1 := \frac{1}{3}+\varepsilon \tag{5.42}$$

and

$$\sigma := \frac{1}{2} \tag{5.43}$$

Just as in the proof of the two dimensional case (cf. Section 4.4), we see that it suffices to show for all $(i,j) \ne (1,1)$

$$\|Q_{ij}(u_1,u_2)\|_{X_\sigma^{\tilde{b},\tilde{s},s_2}} \lesssim \|u_1\|_{X_0^{b,s,s_2}}\|u_2\|_{X_0^{b,s,s_2}} \tag{5.44}$$

in the two cases

$$(\tilde{b},\tilde{s}) = (b',s) \tag{5.45}$$

and

$$(\tilde{b},\tilde{s}) = (b'+b_1, s-3b_1) \tag{5.46}$$

as well as

$$\|Q_{11}(u_1, u_2)\|_{X_\sigma^{\tilde{b},\tilde{s},s_2}} \lesssim \|u_1\|_{X_k} \|u_2\|_{X_0^{b,s,s_2}} \tag{5.47}$$

in the cases (5.45) and (5.46) for $k \in \{1, 2\}$.

We will use the Lemmata 5.8–5.19 of Section 5.3 to prove this. Since we need that conditions (5.14) and (5.15) are fulfilled to apply any of these lemmata, we will first show that we can choose $\varepsilon > 0$ in such a way that, in both cases (5.45) and (5.46), the conditions (5.14) and (5.15) indeed hold. If

$$\varepsilon \leq \frac{1}{32} \tag{5.48}$$

then by the definitions (5.41), (5.42) and (5.43) of b, b', b_1, and σ, we certainly have that condition (5.14) holds. Now, (5.15) is fulfilled if

$$\varepsilon < \frac{1}{3}\left(-\frac{1}{2} + s\right) \tag{5.49}$$

where the right hand side of this inequality is positive because of $s > \frac{1}{2}$.

Let us now prove (5.44) for all possible choices of $(i, j) \neq (1, 1)$. We will consider four cases:

1. $(i, j) = (0, 0)$: We will use Lemma 5.8 to prove (5.44) in this case. Therefore, we have to check that condition (5.17) holds. This is obviously the case, as $\sigma = \frac{1}{2}$.

2. $(i, j) = (1, 0)$ and $(i, j) = (1, 2)$: In order to prove (5.44) in these two cases, we will apply Lemma 5.17 and Lemma 5.19, respectively. Therefore, we have to check that conditions (5.34), (5.38) and (5.39) hold for $(\tilde{b}, \tilde{s}) = (b', s)$ and for $(\tilde{b}, \tilde{s}) = (b' + b_1, s - 3b_1)$. If

$$\varepsilon \leq \frac{s_2}{4 + 4s_2} \tag{5.50}$$

where the right hand side of this inequality is positive because of $s_2 > 0$, then (5.34) holds. Now, (5.50) implies that $\varepsilon \leq \frac{s_2}{4}$. From this and the definitions (5.41) of b and b', it follows that also (5.38) and (5.39) hold.

3. $(i, j) = (2, 0)$: In order to prove (5.44) in this case, we use Lemma 5.13 if $(\tilde{b}, \tilde{s}) = (b', s)$ and Lemma 5.14 if $(\tilde{b}, \tilde{s}) = (b' + b_1, s - 3b_1)$.

   Let us first consider the case $(\tilde{b}, \tilde{s}) = (b', s)$. We have to check that condition (5.27) holds. The left inequality of (5.27) holds because of $\sigma = \frac{1}{2}$ and $b' > -\frac{1}{2}$. By $s > \frac{1}{2}$, we see that the right inequality of (5.27) also holds.

Now, let us suppose $(\tilde{b}, \tilde{s}) = (b' + b_1, s - 3b_1)$. We obviously have $\sigma \le \frac{1}{2}$. We have to check that (5.28) holds. We find that $b' + b_1 = -\frac{1}{6} + 2\varepsilon > -\frac{1}{4}$. Now, (5.48) and $s > \frac{1}{2}$ imply that (5.28) holds.

4. $(i, j) = (2, 1)$ and $(i, j) = (2, 2)$: In order to prove (5.44) in these two cases, we apply Lemma 5.15 and Lemma 5.16, respectively. Therefore, we have to check that (5.30) holds for $(\tilde{b}, \tilde{s}) = (b', s)$ and for $(\tilde{b}, \tilde{s}) = (b' + b_1, s - 3b_1)$. This follows from $s > \frac{1}{2}$.

Let us finally consider (5.47). We have to distinguish the two cases $k = 1$ and $k = 2$.

1. $k = 1$: As in the two dimensional case, we see, by the embedding properties of the Bourgain spaces, that it suffices to show

$$\|Q_{11}(u_1, u_2)\|_{X_\sigma^{0,s,s_2}} \lesssim \|u_1\|_{X_0^{b-b',s,s_2}} \|u_2\|_{X_0^{b,s,s_2}} \tag{5.51}$$

This follows from Lemma 5.18 if we show that condition (5.34) holds for $(\tilde{b}, \tilde{s}) = (b', s)$. However, it was already shown above that (5.34) is implied by (5.50).

2. $k = 2$: As in the two dimensional case, we see that it suffices to show

$$\|Q_{11}(u_1, u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_\sigma^{b+b_1,s-(\alpha+1)b_1}} \|u_2\|_{X_0^{b,s}} \tag{5.52}$$

for $(\tilde{b}, \tilde{s}) = (b', s)$ and for $(\tilde{b}, \tilde{s}) = (b' + b_1, s - 3b_1)$. This follows from Lemma 5.11 if we show that conditions (5.23) and (5.24) hold for $(\tilde{b}, \tilde{s}) = (b', s)$ and $(\tilde{b}, \tilde{s}) = (b' + b_1, s - 3b_1)$. By (5.48), we obviously have (5.23). By the definitions (5.41) of b' and (5.42) of b_1, we have $b' - b_1 = -\frac{5}{6}$. Using this and $\sigma = \frac{1}{2}$, we find that (5.24) also holds.

This ends the proof of Theorem 5.6. □

Now, we give the proof of Theorem 5.7.

Proof (of Theorem 5.7). Let

$$b := \frac{1}{2} + \frac{\varepsilon}{2}, \quad b' := -\frac{1}{2} + \varepsilon \tag{5.53}$$

where $\varepsilon > 0$ will be restricted by various upper bounds given in the course of the proof. We have by (5.53) that $b' > b - 1$. Let

$$b_1 := \begin{cases} \frac{3}{2\alpha} - \frac{5}{12} + \varepsilon & (2 < \alpha \le \frac{18}{5}) \\ 0 & (\frac{18}{5} < \alpha \le 6) \end{cases} \tag{5.54}$$

and

$$\sigma := \frac{1}{2} \tag{5.55}$$

If $\alpha > \frac{9}{4}$, we have $\frac{1}{3} - \frac{3}{4\alpha} > 0$. Let us then choose

$$\varepsilon \le \frac{1}{3} - \frac{3}{4\alpha} \tag{5.56}$$

This implies $b' + b_1 \le -\frac{1}{4}$ for $\frac{9}{4} < \alpha \le 6$.

Just as in the proof of the two dimensional case (cf. Section 4.4), we see that it suffices to show for all $(i,j) \neq (1,1)$

$$\|Q_{ij}(u_1, u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_0^{b,s}} \|u_2\|_{X_0^{b,s}} \tag{5.57}$$

in the two cases

$$(\tilde{b}, \tilde{s}) = (b', s) \tag{5.58}$$

and

$$(\tilde{b}, \tilde{s}) = (b' + b_1, s - (\alpha+1)b_1) \tag{5.59}$$

as well as

$$\|Q_{11}(u_1, u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_k} \|u_2\|_{X_0^{b,s}} \tag{5.60}$$

in the cases (5.58) and (5.59) for $k \in \{1, 2\}$.

We will use the Lemmata 5.8–5.16 of Section 5.3 to prove this. Because we need that conditions (5.14) and (5.15) are fulfilled to apply any of these Lemmata, we first prove that there exists an $\varepsilon > 0$ such that, in both cases (5.58) and (5.59), the conditions (5.14) and (5.15) indeed hold. If

$$\varepsilon \le \frac{1}{12} \tag{5.61}$$

then by the definitions (5.53), (5.54) and (5.55) of b, b', b_1, and σ and taking into account that $\alpha > 2$, we see that we have (5.14). Now, (5.15) is fulfilled if

$$\varepsilon < \frac{1}{\alpha+1}\left(\frac{\alpha}{2} - \frac{3}{2} + s\right) \tag{5.62}$$

where the right hand side of this inequality is positive because of (5.7).

Let us now prove (5.57) for all possible choices of $(i,j) \neq (1,1)$. We will consider four cases:

1. $(i,j) = (0,0)$: We will apply Lemma 5.8 in order to prove (5.57) in this case. Therefore, we have to show that condition (5.17) is fulfilled. However, this follows from $\sigma = \frac{1}{2}$.

2. $(i,j) = (1,0)$ and $(i,j) = (1,2)$: In order to prove (5.57) in these two cases, we apply Lemma 5.9 and Lemma 5.12, respectively. Therefore, we have to check that (5.19) holds for $(\tilde{b},\tilde{s}) = (b',s)$ and $(\tilde{b},\tilde{s}) = (b'+b_1, s-(\alpha+1)b_1)$. This follows if

$$\varepsilon \leq \frac{1}{\alpha}\left(\frac{\alpha}{2} - 1\right) \tag{5.63}$$

where the right hand side of this inequality is positive because of $\alpha > 2$.

3. $(i,j) = (2,0)$: Let us first prove (5.57) in the case $(\tilde{b},\tilde{s}) = (b',s)$. We apply Lemma 5.13. Therefore, we have to show that condition (5.27) holds. Now, the left inequality in (5.27) holds because of $\sigma = \frac{1}{2}$, $b' > -\frac{1}{2}$ and $\alpha \leq 6$. By (5.61), we certainly have $b' \leq -\frac{1}{4}$. Furthermore, if

$$\varepsilon \leq \frac{1}{\alpha}\left(\frac{5}{12}\alpha - \frac{1}{2} + 2s\right) \tag{5.64}$$

where the right hand side of this inequality is positive because of (5.7), then we also have the right inequality of (5.27).

Let us now prove (5.57) for $(\tilde{b},\tilde{s}) = (b'+b_1, s-(\alpha+1)b_1)$. Since $b_1 = 0$ for $\alpha > \frac{18}{5}$, we can suppose $2 < \alpha \leq \frac{18}{5}$. We apply Lemma 5.13, if $\frac{9}{4} < \alpha \leq \frac{18}{5}$, and Lemma 5.14, if $2 < \alpha \leq \frac{9}{4}$.
Let us first consider the case $\frac{9}{4} < \alpha \leq \frac{18}{5}$. We have to check that condition (5.27) holds. The left inequality in (5.27) holds because of $\sigma = \frac{1}{2}$, $b'+b_1 > -\frac{1}{2}$ and $\alpha \leq 6$. By (5.56), we have that $b'+b_1 \leq -\frac{1}{4}$. If

$$\varepsilon \leq \frac{1}{\alpha}\left(\frac{5}{12}\alpha - 1 + s\right) \tag{5.65}$$

where the right hand side is positive because of (5.7) and $\alpha \leq \frac{18}{5}$, then the right inequality of (5.27) holds.

Let us now consider the case $2 < \alpha \leq \frac{9}{4}$. We apply Lemma 5.14 to prove (5.57). For this range of α, we find that $b'+b_1 > -\frac{1}{4}$. Let

$$\varepsilon \leq \frac{1}{2}\left(\frac{9-6\alpha}{24+4\alpha} + \frac{11}{12} - \frac{3}{2\alpha}\right) \tag{5.66}$$

where the right hand side of this inequality is positive for $2 < \alpha \leq \frac{9}{4}$. Then (5.66) and (5.7) imply (5.28).

4. $(i,j) = (2,1)$ and $(i,j) = (2,2)$: In order to prove (5.57) in these two cases, we apply Lemma 5.15 and Lemma 5.16, respectively. Therefore,

we have to check that condition (5.30) holds for $(\tilde{b}, \tilde{s}) = (b', s)$ and $(\tilde{b}, \tilde{s}) = (b' + b_1, s - (\alpha + 1)b_1)$. By (5.64), if $\alpha > \frac{18}{5}$, and (5.65), if $\alpha \leq \frac{18}{5}$, we see that (5.30) holds.

Let us finally consider (5.60). We have to distinguish the two cases $k = 1$ and $k = 2$.

1. $k = 1$: As in the two dimensional case, we have, by the embedding properties of the Bourgain spaces, that it suffices to show

$$\|Q_{11}(u_1, u_2)\|_{X_\sigma^{0,s}} \lesssim \|u_1\|_{X_0^{b-b',s}} \|u_2\|_{X_0^{b,s}} \tag{5.67}$$

   This follows from Lemma 5.10 if we show that condition (5.19) holds for $(\tilde{b}, \tilde{s}) = (b', s)$. However, it has already been shown that this follows from (5.63).

2. $k = 2$: As in the two dimensional case, we see that it suffices to show

$$\|Q_{11}(u_1, u_2)\|_{X_\sigma^{\tilde{b},\tilde{s}}} \lesssim \|u_1\|_{X_\sigma^{b+b_1, s-(\alpha+1)b_1}} \|u_2\|_{X_0^{b,s}} \tag{5.68}$$

   for $(\tilde{b}, \tilde{s}) = (b', s)$ and $(\tilde{b}, \tilde{s}) = (b' + b_1, s - (\alpha + 1)b_1)$. This follows from Lemma 5.11 if we show that conditions (5.23) and (5.24) hold. Now, (5.23) follows obviously from (5.61). The left inequality in (5.24) holds by the definition (5.54) of b_1 and because of $\sigma = \frac{1}{2}$ and $2 < \alpha \leq 6$. The right inequality in (5.24) follows from the definition (5.54) of b_1 if $2 < \alpha \leq \frac{18}{5}$. If $\frac{18}{5} < \alpha \leq 6$, let

$$\varepsilon \leq \frac{5}{12} - \frac{3}{2\alpha} \tag{5.69}$$

   where the right hand side of this inequality is positive because of $\alpha > \frac{18}{5}$. Then (5.69) implies the right inequality in (5.24) for $\frac{18}{5} < \alpha \leq 6$.

This ends the proof of Theorem 5.7. □

5.5 Notes and references

Compared to the case of two space dimensions, there are only few well-posedness results concerning the Kadomtsev-Petviashvili II equation in three space dimensions. Local well-posedness for initial values in the isotropic Sobolev space $H^s(\mathbb{R}^3)$, $s > \frac{3}{2}$, that obey the low-frequency condition $\partial_x^{-1} u_0 \in H^s(\mathbb{R}^3)$, was obtained by TZVETKOV [33]. Only recently, this result has been improved by ISAZA, LÓPEZ AND MEJÍA [11] to the local well-posedness in non-isotropic Sobolev spaces which are similar to our spaces $H^{s_1,s_2}(\mathbb{R}^3)$ for $s_1 > 1$ and $s_2 > 0$.

For the fifth order Kadomtsev-Petviashvili II equation in three dimensions, SAUT AND TZVETKOV [25] proved local well-posedness for $s_1 \geq -\frac{1}{8}$ and $s_2 \geq 0$ under the additional low frequency condition $\partial_x^{-1} u_0 \in H^{s_1,s_2}(\mathbb{R}^3)$. Note that the equation considered in [25] is slightly more general than (5.3) because it also contains the third order term. In [26], SAUT AND TZVETKOV showed the local well-posedness of the fifth order Kadomtsev-Petviashvili II equation with periodic boundary conditions, i. e. posed on $(-T, T) \times \mathbb{T}^3$, for initial data in the subset of a non-isotropic Sobolev space with $s_1 > 0$ and $s_2 = 0$ with constant mean value in x.

Bibliography

[1] M. Ben-Artzi and J.-C. Saut. Uniform decay estimates for a class of oscillatory integrals and applications. *Differential Integral Equations*, 12(2):137–145, 1999.

[2] J. Bergh and J. Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[3] J. Bourgain. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations. *Geom. Funct. Anal.*, 3(2):107–156, 1993.

[4] J. Bourgain. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation. *Geom. Funct. Anal.*, 3(3):209–262, 1993.

[5] J. Bourgain. On the Cauchy problem for the Kadomtsev-Petviashvili equation. *Geom. Funct. Anal.*, 3(4):315–341, 1993.

[6] J. Ginibre. Le problème de Cauchy pour des EDP semi-linéaires périodiques en variables d'espace (d'après Bourgain). *Astérisque*, 237:Exp. No. 796, 4, 163–187, 1996. Séminaire Bourbaki, Vol. 1994/95.

[7] A. Grünrock. *New applications of the Fourier restriction norm method to wellposedness problems for nonlinear evolution equations.* PhD thesis, Bergische Universität-Gesamthochschule Wuppertal, 2002.

[8] S. Herr. *Well-posedness results for dispersive equations with derivative nonlinearities.* PhD thesis, Universität Dortmund, 2006. http://hdl.handle.net/2003/22856.

[9] L. Hörmander. *The analysis of linear partial differential operators. I.* Springer Study Edition. Springer-Verlag, Berlin, second edition, 1990. Distribution theory and Fourier analysis.

[10] R. J. Iório, Jr. and W. V. L. Nunes. On equations of KP-type. *Proc. Roy. Soc. Edinburgh Sect. A*, 128(4):725–743, 1998.

[11] P. Isaza, J. López, and J. Mejía. The Cauchy problem for the Kadomtsev-Petviashvili (KPII) equation in three space dimensions. preprint.

[12] P. Isaza, J. López, and J. Mejía. Cauchy problem for the fifth order Kadomtsev-Petviashvili (KPII) equation. *Commun. Pure Appl. Anal.*, 5(4):887–905, 2006.

[13] P. Isaza and J. Mejía. Local and global Cauchy problems for the Kadomtsev-Petviashvili (KP-II) equation in Sobolev spaces of negative indices. *Comm. Partial Differential Equations*, 26(5-6):1027–1054, 2001.

[14] P. Isaza J. and J. Mejía. Global solution for the Kadomtsev-Petviashvili equation (KPII) in anisotropic Sobolev spaces of negative indices. *Electron. J. Differential Equations*, pages No. 68, 12 pp. (electronic), 2003.

[15] W. Kaballo. *Einführung in die Analysis III.* Spektrum Hochschultaschenbuch. Heidelberg: Spektrum Akademischer Verlag, 1999.

[16] B. B. Kadomtsev and V. I. Petviashvili. On the stability of solitary waves in weakly dispersing media. *Sov. Phys., Dokl.*, 15:539–541, 1970.

[17] M. Keel and T. Tao. Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980, 1998.

[18] C. E. Kenig, G. Ponce, and L. Vega. Oscillatory integrals and regularity of dispersive equations. *Indiana Univ. Math. J.*, 40(1):33–69, 1991.

[19] C. E. Kenig, G. Ponce, and L. Vega. A bilinear estimate with applications to the KdV equation. *J. Amer. Math. Soc.*, 9(2):573–603, 1996.

[20] C. E. Kenig and S. N. Ziesler. Local well posedness for modified Kadomstev-Petviashvili equations. *Differential Integral Equations*, 18(10):1111–1146, 2005.

[21] L. Molinet, J. C. Saut, and N. Tzvetkov. Ill-posedness issues for the Benjamin-Ono and related equations. *SIAM J. Math. Anal.*, 33(4):982–988 (electronic), 2001.

[22] L. Molinet, J. C. Saut, and N. Tzvetkov. Remarks on the mass constraint for KP type equations. preprint, arXiv:math.AP/0603303, 2006.

[23] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[24] J. C. Saut. Remarks on the generalized Kadomtsev-Petviashvili equations. *Indiana Univ. Math. J.*, 42(3):1011–1026, 1993.

[25] J. C. Saut and N. Tzvetkov. The Cauchy problem for higher-order KP equations. *J. Differential Equations*, 153(1):196–222, 1999.

[26] J. C. Saut and N. Tzvetkov. The Cauchy problem for the fifth order KP equations. *J. Math. Pures Appl. (9)*, 79(4):307–338, 2000.

[27] H.-J. Schmeisser and H. Triebel. *Topics in Fourier analysis and function spaces*, volume 42 of *Mathematik und ihre Anwendungen in Physik und Technik [Mathematics and its Applications in Physics and Technology]*. Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1987.

[28] R. S. Strichartz. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.*, 44:705–714, 1977.

[29] H. Takaoka. Global well-posedness for the Kadomtsev-Petviashvili II equation. *Discrete Contin. Dynam. Systems*, 6(2):483–499, 2000.

[30] H. Takaoka. Well-posedness for the Kadomtsev-Petviashvili II equation. *Adv. Differential Equations*, 5(10-12):1421–1443, 2000.

[31] H. Takaoka and N. Tzvetkov. On the local regularity of the Kadomtsev-Petviashvili-II equation. *Internat. Math. Res. Notices*, 2001(2):77–114, 2001.

[32] T. Tao. Multilinear weighted convolution of L^2-functions, and applications to nonlinear dispersive equations. *Amer. J. Math.*, 123(5):839–908, 2001.

[33] N. Tzvetkov. On the Cauchy problem for Kadomtsev-Petviashvili equation. *Comm. Partial Differential Equations*, 24(7-8):1367–1397, 1999.

[34] N. Tzvetkov. Global low-regularity solutions for Kadomtsev-Petviashvili equation. *Differential Integral Equations*, 13(10-12):1289–1320, 2000.

[35] K. Yosida. *Functional analysis*, volume 123 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, sixth edition, 1980.