Stochastic Control, Optimal Saving, and Job Search in Continuous Time

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Abstract. Economic uncertainty may affect significantly people's behavior and hence macroeconomic variables. It is thus important to understand how people behave in presence of different kinds of economic risk. The present dissertation focuses therefore on the impact of the uncertainty in capital and labor income on the individual saving behavior. The underlying uncertain variables are here modeled as stochastic processes that each obey a specific stochastic differential equation, where uncertainty stems either from Poisson or Lévy processes. The results on the optimal behavior are derived by maximizing the individual expected lifetime utility. The first chapter is concerned with the necessary mathematical tools, the change-of-variables formula and the Hamilton-Jacobi-Bellman equation under Poisson uncertainty. We extend their possible field of application in order make them appropriate for the analysis of the dynamic stochastic optimization problems occurring in the following chapters and elsewhere. The second chapter considers an optimum-saving problem with labor income, where capital risk stems from asset prices that follow geometric Lévy processes. Chapter 3, finally, studies the optimal saving behavior if agents face not only risk but also uncertain spells of unemployment. To this end, we turn back to Poisson processes, which here are used to model properly the separation and matching process.

Contents

Introduction and Summary

Uncertainty in economic variables is a major concern of individuals and may affect their decisions on, for example, saving or investment considerably. The presence of uncertainty may thus have a significant impact on macroeconomic variables, such as the capital stock or the output of an economy. It is hence for both individuals and policy makers important to understand how economic uncertainty — or risk, as it is also called among economists — affects individual behavior.

In the present thesis we focus on the impact of the uncertainty in capital and labor income. More precisely, we study the optimal saving behavior of an agent facing either uncertain returns from his capital investments or risk of unemployment. The results are derived by maximizing the agent's expected lifetime utility in continuous time. The underlying uncertain variables, such as asset prices, are here, and as typically done in the economic literature, modeled as stochastic processes by assuming they each obey a specific stochastic differential equation. The advantage of this kind of modeling is that mathematical theory provides a bunch of, to a certain extent, well-known tools that can be used in the following analysis. Most prominently, these are the change-of-variables formula (sometimes referred to as "Ito's-Lemma") for deriving the differentials of mappings of stochastic processes and the Hamilton-Jacobi-Bellman equation for tackling dynamic stochastic optimization problems, such as the expected-utility maximization problem occurring subsequently. The thesis proceeds as follows. The first chapter is concerned with the generalization of the aforementioned mathematical methods, i.e., the change-of-variables formula and the Hamilton-Jacobi-Bellman equation. The second chapter studies the optimal saving behavior assuming uncertainty in capital income, while the last chapter is dedicated to the impact of risk of unemployment and the uncertain job search process.

The first chapter "*Controlled Stochastic Differential Equations under Poisson Uncertainty and with Unbounded Utility*" deals with the Hamilton-Jacobi-Bellman equation for solving optimal stochastic control problems as occurring in the subsequent chapters and elsewhere in the economic literature. We assume here that uncertainty stems from

Poisson processes. So far, applying the Hamilton-Jacobi-Bellman equation has required strong assumptions on the model, such as a bounded utility function and bounded coefficients in the controlled differential equation. This chapter relaxes these assumptions. We show that one can still use the Hamilton-Jacobi-Bellman equation as a necessary criterion for optimality if the utility function and the coefficients are linearly bounded. We also derive sufficiency in a verification theorem without imposing any boundedness condition at all. Finally, we show that the result on necessity extends to the case in which Brownian motion as an additional source for uncertainty is taken into account. An accompanying paper, Sennewald and Wälde (2006), illustrates the results derived in this chapters. There we consider an optimum consumption and investment problem with labor income, where uncertainty of the risky-asset price stems from a Poisson process.

In the second chapter "*Keynes-Ramsey Rules in Continuous-Time Setups Under Lévy Uncertainty*" we propose a more general modeling of uncertain asset prices by introducing Lévy processes. Here we describe the optimal consumption behavior by means of a Keynes-Ramsey rule. Observe that so far, Keynes-Ramsey rules for describing the optimal consumption behavior in a continuous time setup under uncertainty have been "incomplete" in the sense that they merely provide the evolution of the marginal utility process, but not the evolution of the optimal consumption process itself. Only recently, "complete" Keynes-Ramsey rules have been derived in a setup with CRRA (constant relative risk aversion) utility functions and uncertainty caused by Brownian motion or Poisson processes. These processes, however, only provide a limited tool for modeling dynamic uncertainty and new results can be achieved by using Lévy processes. This chapter therefore shows how the evolution of the optimal consumption process can be derived if uncertainty stems from a Lévy process and the consumption function is not necessarily of the CRRA type.

In the third and last chapter, titled "*Optimal Saving under Risk of Unemployment*", we consider an optimum consumption problem in which an agent is exposed to both risk and uncertain spells of unemployment. The back and forth in the employment status is properly modeled by a stochastic differential equation with Poisson processes. The resulting stochastic income process gives rise to precautionary saving which decreases in the level of wealth. We find that this excess saving, jointly with the jumps in labor income, lead to consumption paths that are totally different from what we know from deterministic setups. In particular, there can be, dependent on the interest rate, target saving or temporary poverty traps. We further find that uncertainty in the employment status raises the average (though not necessary the actual) consumption growth.

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Part 1

Mathematical Methods

CHAPTER 1

Controlled Stochastic Differential Equations under Poisson Uncertainty and with Unbounded Utility*

Abstract. The present paper is concerned with the optimal control of stochastic differential equations, where uncertainty stems from Poisson processes. Optimal behavior (e.g., optimal consumption) is usually determined by employing the Hamilton-Jacobi-Bellman equation. This requires strong assumptions on the model, such as a bounded utility function and bounded coefficients in the controlled differential equation. The present paper relaxes these assumptions. We show that one can still use the Hamilton-Jacobi-Bellman equation as a necessary criterion for optimality if the utility function and the coefficients are linearly bounded. We also derive sufficiency in a verification theorem without imposing any boundedness condition at all. It is finally shown that, under very mild assumptions, an optimal Markov control is optimal even within the class of general controls.

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1. Introduction

This paper is concerned with the optimal control of stochastic differential equations (*SDEs*) in an infinite time horizon setup where uncertainty stems from one or more Poisson processes. Such controlled SDEs are a standard tool in the economic literature for modeling dynamic behavior of economic variables that are hit by randomly occurring shocks and that can be controlled by an agent. They can be found (in a deterministic disguise) in quality-ladder models of growth (e.g., Grossman and Helpman, 1991, Segerstrom, 1998, Howitt, 1999), in the endogenous cycles and growth literature with uncertainty (e.g., Wälde, 1999, 2005, Steger, 2005), in the Mortensen-Pissaridis type labor market literature (e.g., Mortensen, 1994, Pissaridis, 2000), and in finance (e.g., Merton, 1971 and subsequent work), to name only a few applications. Often, Poisson processes are included as a special case in a framework with jump-diffusion, piecewise deterministic or general Markov processes, see, e.g., Aase (1984), Bellamy (2001), Framstad et al. (2001), and, in a more mathematical context, Davis (1993) or Fleming and Soner (1993).

Usually, the objective consists in finding an optimal control that maximizes (or minimizes) a certain performance criterion. Consider, for example, the following extension of Merton's (1971) optimal consumption and portfolio problem.[1] A household can invest its wealth a either in a safe bank account with interest rate r_2 or in a risky asset whose price grows continuously at the rate r_1 and jumps at random points in time by $\beta \times 100$ percent. The random times are modeled by the jump times of a Poisson processes N and occur with a frequency $\lambda > 0$, the arrival rate of N. At each instant t, the household receives labor income w and consumes the amount c_t. Then its budget constraint obeys the SDE,

$$da_t = [r_1 b_t + r_2 (a_t - b_t) + w - c_t]\, dt + \beta b_{t\_} dN_t, \tag{1}$$

where b_t stands for the amount invested in the risky asset. Given the household's time preference rate $\rho > 0$ and the CRRA (constant relative risk aversion) utility function $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, $\sigma > 0$, $\sigma \neq 1$, the household's objective is to find the optimal consumption and investment stream that maximize the expected lifetime utility $E_s \int_s^\infty e^{-\rho(t-s)} u(c_t)\, dt$ subject to budget constraint (1).

The performance achieved with the optimal control is called the value function. As is well known, under certain assumptions the value function and, if existing, the optimal

[1] For more details see Section 3 of the accompanying paper Sennewald and Wälde (2006).

Markov control satisfy a partial differential equation, generally known as the Hamilton-Jacobi-Bellman (*HJB*) equation. On the other hand, if there is a function and a Markov control solving the HJB equation and satisfying certain terminal conditions, this function is the value function and the Markov control is optimal. Hence, the HJB equation provides both a necessary and sufficient criterion for optimality. In the economic literature, Merton (1971) was one of the first to state a HJB equation for an optimal control problem with Poisson processes. Since then it has found widespread use.

Unfortunately, the required conditions that allow the application of the HJB equation as either necessary or sufficient criterion are rather strong. In particular, besides a sufficiently smooth value function, many authors assume the utility or cost function to be bounded, see, e.g., Gihman and Skorohod (1979) for jump-diffusion processes or Dempster (1991) and Davis (1993) for piecewise deterministic processes.[2] The same applies for the coefficient functions in the controlled SDE, which govern the evolution of the controlled process. Other authors impose, instead of boundedness, other underlying conditions, such as a countable state and action space, cf., e.g., van Dijk (1988) for controlled jump processes. In some cases the required conditions are rather difficult to check without strong mathematical background, see, e.g., Kushner (1967) and Fleming and Soner (1993), who assume the value function to be in the domain of the infinitesimal generator of the controlled Markov process or the Dynkin formula to hold. Kushner (1967) requires furthermore a certain uniform integrability condition. In other cases, precise assumptions on, for example, utility are missing, or the HJB equation is derived at a rather heuristic level, see, e.g., Kushner (1967), Malliaris and Brock (1982), Kushner and Dupuis (1992), Fleming and Soner (1993), and Dixit and Pindyck (1994).[3]

If one thinks, for example, of the frequently used class of CRRA utility functions, such as given in the example above, and considers that the consumption stream c_t may grow toward infinity, the condition of bounded utility is apparently too strong for economic modeling. Also, in light of, for example, budget constraint (1), the assumption of bounded

[2] If the smoothness conditions are not satisfied, the value function can still be a viscosity solution of the HJB equation. This result was first derived by Crandall and Lions (1983) for general HJB equations. An excellent survey on that issue is provided by Crandall, Ishi, and Lions (1992).

[3] In both Kushner (1967) and Fleming and Soner (1993) only the necessity part is derived heuristically, whereas sufficiency is proven rigorously.

coefficients in the controlled SDE is obviously too restrictive.[4] Further, the assumption of countable state or action spaces is not convenient.

The objective of the present paper is therefore to present rigorous proofs for the necessity and sufficiency of the HJB equation under weaker assumptions than before. In particular, to show necessity, we only require linear boundedness of the utility function and of the coefficients, whereas for deriving sufficiency we do not impose — apart from a terminal condition — any boundedness restrictions at all. Additionally, since the HJB equation applies only for Markov controls, and one might feel that considering Markov controls only is too restrictive, it is also shown that the performance of Markov controls is as that good as for any other class of controls. That is, an optimal Markov control is also optimal within the class of general controls. Finally, as a major tool for the derivations in this paper and also because of its relevance in economic modeling, a change-of-variables formula (CVF) is presented which can directly be applied on multidimensional SDEs with many Poisson processes.

For discrete time and in a deterministic environment, Rincón-Zapatero and Rodriguez-Palmero (2003) and Le Van and Morhaim (2002) study a similar problem. They show for unbounded utility that the HJB equation possesses a unique solution and that this solution is the value function. In this paper, the HJB equation is derived via the dynamic programming approach, cf. Kushner and Dupuis (1992) and Fleming and Soner (1993). It is crucial for the necessity property of the HJB equation that the value function belongs to the domain of the infinitesimal generator of the controlled process, what, e.g., Fleming and Soner (1993) simply assumed. Herein lies a major improvement compared to the literature. Whereas this condition was so far almost trivially satisfied due to the boundedness assumption for the utility and coefficient functions, we show that it also holds true for the more general case where these functions are linearly bounded.

The proof of sufficiency draws from the fact that Poisson processes are, unlike Brownian motion and general Markov processes, of bounded variation. This property implicates according to García and Griego (1994) that any stochastic integral with respect to a compensated Poisson process is a martingale if the integrand is an adapted and cádlág process. In turn, the Dynkin formula, which is fundamental for the proof, holds for fairly general

[4]Consider, for example, the "jump coefficient" in (1): As b_t is not bounded, βb_t will be neither.

processes and functions. That finally permits the mild assumptions we impose for the sufficiency part.[5]

Our result on the performance of Markov controls was derived by, e.g., Gihman and Skorohod (1979) and Fleming and Soner (1993), but under stronger assumptions, as mentioned above. For the proof we adopt the arguments given in Øksendal (2003), who arrives at an analogous conclusion for controlled diffusion processes.

Supposing similar boundedness assumptions as in the paper at hand, Krylov (1980) derives necessity of the HJB equation for controlled diffusion processes without jumps. Here we show that the necessity property extends to the Poisson-diffusion setup. Sufficiency for that setup, on the other hand, requires assumptions such as certain integrability conditions, that are more restrictive than those for the pure Poisson setting and which are due to the unbounded variation of Brownian motion. Here, we refer to Øksendal and Sulem (2005, Theorem 3.1) who presents a sufficiency result in case of controlled Lévy type processes.

As an illustration of the proofs and results presented in this paper, the accompanying paper by Sennewald and Wälde (2006) provides various examples, among them the optimum consumption and investment problem stated above. A reader that is not interested in the proofs can directly refer to this paper and use it as a toolbox for own modeling.

The organization of this paper is as follows. The subsequent section gives some general assumptions and definitions concerning the formal background. In Section 3 we establish the control problem with the necessary assumptions. Then, Section 4 provides the CVF and useful properties of the controlled state process and the value function. Section 5 is devoted to the main results of the paper, the HJB equation as optimality criterion. Subsequently, in Section 6, we present the extension to the Poisson-diffusion setup. The proofs are given in Section 7, and the last section, finally, concludes.

2. General definitions and assumptions

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\{\mathcal{F}_t, t \geq 0\}$. Let $\{X_t(\omega), t \geq s\}$ be a n-dimensional adapted stochastic process starting at time $s \geq 0$ and with cádlág paths. Throughout the paper we suppress the stochastic argument ω, and we write

[5] Recall that for general Markov processes Fleming and Soner (1993) assumed the value function to be in the domain of the infinitesimal generator as well as the validity of the Dynkin formula. Only, in case of controlled diffusion processes, they relax these assumptions, but still require the value function to be polynomially bounded.

shortly X for $\{X_t(\omega), t \geq s\}$ whenever there is no risk of confusion. The left limit at time t, $\lim_{\tau \nearrow t} X_\tau$, is denoted by X_{t_-}, where $X_{s_-} := 0$. Trivially, X_{t_-} coincides with X_t if X possesses continuous paths. In the following the expression cádlág is also used for any real-valued function $g(x)$ that is continuous from the right with left limits in its argument x. If $g : \mathbb{R}^n \to \mathbb{R}^m$, $n, m \in \mathbb{N}$, is such a cádlág function, and the process X adapted and cádlág, the process $g(X)$ becomes adapted and cádlág, too, and we denote the left limit in t, $\lim_{\tau \nearrow t} g(X_\tau)$, by $g(X_t)_-$. Then, if g is continuous, $g(X_t)_- = g(X_{t_-})$.

For vectors $x, y \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{(n,m)}$, $x \cdot y$ stands for the standard scalar product and $\|x\|$ and $\|M\|$ for the Euclidean norm. $\mathcal{C}^k$ denotes the space of k-times continuously differentiable functions. If $f \in \mathcal{C}^1 : (0, \infty) \times \mathbb{R}^n$, $y = (t, x) \mapsto f(y) = f(t, x)$, then f_t stands for the partial derivative with respect to t and f_x and f_y for the gradients with respect to x and y, respectively. If $f \in \mathcal{C}^2$, f_{xx} and f_{yy} denote the Hesse matrices with respect to x and y, respectively.

3. The control problem

Let C be a r-dimensional adapted cádlág process and $N^1, ..., N^d$ independent adapted Poisson processes with arrival rates $\lambda^1, \ldots, \lambda^d$. Then the n-dimensional *state process* X controlled by the process C and starting at time s in point $x \in \mathbb{R}^n$ is supposed to obey a SDE of the form

$$X_t = x + \int_s^t \alpha(\tau, X_\tau, C_\tau)\, d\tau + \sum_{k=1}^{d} \int_s^t \beta_k(\tau, X_{\tau_-}, C_{\tau_-}) dN_\tau^k, \text{[6]} \tag{2}$$

with continuous *coefficient functions* $\alpha, \beta_1, \ldots, \beta_d : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$. Note that, due to the continuity, we can write $\beta_k(\tau, X_{\tau_-}, C_{\tau_-})$ for $\beta_k(\tau, X_\tau, C_\tau)_-$, $k = 1, \ldots, d$. The coefficient function α describes the time continuous evolution of the state process X, whereas for each $k = 1, \ldots, d$ the function β_k gives the magnitude of the jump in X whenever Poisson process N^k jumps. Both the time continuous behavior and the jump size are controlled by the choice of the control process C. In the following we always assume that SDE (2) possesses a unique adapted solution $X^{C,s,x}$. A detailed analysis of

[6] The n-dimensional stochastic integral is to be read componentwise. That is, for each $k = 1, \ldots, d$, $\int_s^t \beta_k(\tau, X_{\tau_-}, C_{\tau_-}) dN_\tau^k = \begin{pmatrix} \int_s^t \beta_{1k}(\tau, X_{\tau_-}, C_{\tau_-}) dN_\tau^k \\ \vdots \\ \int_s^t \beta_{nk}(\tau, X_{\tau_-}, C_{\tau_-}) dN_\tau^k \end{pmatrix}$, with β_{ik}, $i = 1, \ldots, n$ being the components of β_k.

SDEs with sufficient conditions for the existence of such a unique solution can be found in, e.g., García and Griego (1994) and Protter (1995).

According to requirements in many economic models, we introduce a state space constraint by assuming that X is allowed to lie only within a certain convex *state space* $\Theta \subset \mathbb{R}^n$. We require furthermore that, if at time t state $z \in \Theta$ is observed, the control at this time, C_t, can take only values in a certain convex *control space* $\Gamma_{t,z} \subset \mathbb{R}^r$. Let $\Gamma := \cup_{(t,z)\in[0,\infty)\times\Theta}\Gamma_{t,z}$ be the union of all possible control spaces. A control C with $C_t \in \Gamma_{t,X_t^{C,s,x}}$ for all $t \geq s$ and of which the corresponding state process $X^{C,s,x}$ remains in Θ is called *admissible control.*

Notice that in the economic literature SDEs appear often in differential notation. In this somewhat shorter notation, SDE (2) reads

$$dX_t = \alpha\left(t, X_t, C_t\right) dt + \sum_{k=1}^{d} \beta_k \left(t, X_{t_-}, C_{t_-}\right) dN_t^k, X_s = x.$$

This expression might appear more intuitive since it seems to show more clearly what the (infinitesimal) change of X at time t is driven by. Nevertheless, the differential notation is only an "abbreviation" of the integral form, and both notations have the same meaning. Throughout this paper, we shall always use the integral notation.

Let $u : [0,\infty) \times \Theta \times \Gamma \to \mathbb{R}$ (the "*instantaneous utility function*") and $\rho : [0,\infty) \to \mathbb{R}_+$ (the "*time preference rate*") be continuous functions. Suppose that for all admissible controls,

$$E_s \int_s^\infty e^{-\int_s^t \rho(\tau)d\tau} \left| u\left(t, X_t^{C,s,x}, C_t\right) \right| dt < \infty, \tag{3}$$

where E_s denotes the expectation conditional on $\mathcal{F}_s$. Then the objective is to find an admissible control that maximizes the *performance criterion* ("*expected lifetime utility*")[7]

$$W^C(s,x) := E_s \int_s^\infty e^{-\int_s^t \rho(\tau)d\tau} u\left(t, X_t^{C,s,x}, C_t\right) dt. \tag{4}$$

Such a control is called *optimal control* for the starting point (s,x). Being a function of the initial point $(s,x) \in [0,\infty) \times \Theta$, W^C is also called *performance function.*

There exist various types of controls that may be considered. Following Øksendal (2003), these are, e.g.,

[7]In some cases one may wish to minimize W^C, for example, if u stands for a cost rate. Then one can equivalently maximize $-W^C$, where u in (4) is replaced with $-u$, and the following still applies.

- *Feedback or closed loop controls*, which are adapted to the Filtration $\{\mathcal{M}_t, t \geq s\}$ where $\mathcal{M}_t$ denotes the σ-algebra generated by $\left\{X_\tau^{s,x,C}, s \leq \tau \leq t\right\}$. That is, the choice of the control value at time t depends on the whole history of $X_t^{s,x,C}$.
- *Deterministic* or *open loop controls*, which do not depend on ω.
- *Markov controls*, whose value at time t is given as a function of current time and state. That is, $C_t(\omega) = \phi(t, X_t^{s,x,C}(\omega))$ for some function $\phi : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^r$. In that case the corresponding state process $X_t^{s,x,C}$ is Markovian.

In applied optimization problems, Markov controls present the most practical class of controls since they "say clearly" what to do if at a certain time a certain state is observed. Moreover, the HJB equation provides a powerful tool to characterize and verify optimal Markov controls, as we shall see in Theorems 3 and 4. It even turns out that, under very mild assumptions, one obtains as good a performance with a Markov control as with any other admissible control, see Theorem 5. Hence, it is justified if we work in our analysis with Markov controls only.[8] The following definitions give the necessary tools to formulate our control problem precisely:

(i) A cádlág function $\phi : [0, \infty) \times \mathbb{R}^n \to \Gamma$, $(t, z) \longmapsto \phi(t, z)$ is called a *policy*. If X is an adapted cádlág process, a Markov control C^ϕ induced by a policy ϕ via $C_t^\phi := \phi(t, X_t)$ is adapted and cádlág, too. Observe that in this case the integrals in the controlled SDE (2) are well-defined. For SDE (2) we write then

$$X_t = x + \int_s^t \alpha^\phi(\tau, X_\tau)\, ds + \sum_{k=1}^{d} \int_s^t \beta_k^\phi(\tau, X_\tau)_- dN_\tau^k, \tag{5}$$

where $\alpha^\phi(t, z) := \alpha(t, z, \phi(t, z))$ and $\beta_k^\phi(t, z) := \beta_k(t, z, \phi(t, z))$. The unique solution is denoted by $X^{\phi,s,x}$. Furthermore, the performance function, defined according to (4), is indicated by the superscript ϕ (instead of C) and reads with $u^\phi(t, z) := u(t, z, \phi(t, z))$ and $\overline{\rho}_s(t) := \frac{1}{t-s} \int_s^t \rho(\tau)\, d\tau$ (the "average time preference rate" from s to t):

$$W^\phi(s, x) = E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} u^\phi\left(t, X_t^{\phi,s,x}\right) dt. \tag{6}$$

(ii) A policy ϕ is called *admissible* if $\phi(t, z) \in \Gamma_{t,z}$ for all $(t, z) \in [0, \infty) \times \Theta$ and if for any starting point $(s, x) \in [0, \infty) \times \Theta$ the controlled process $X^{\phi,s,x}$ never leaves Θ, i.e., $X_t^{\phi,s,x} \in \Theta$ for all $t \geq s$. The space of admissible policies is denoted by Π.

[8] Restricting ourselves only to deterministic controls is clearly not sufficient since in a stochastic environment it is not likely that a deterministic control is optimal.

(iii) If the supremum is finite for all $(s,x) \in [0,\infty) \times \Theta$, we call the function $V : [0,\infty) \times \Theta \to \mathbb{R}$ given by

$$V(s,x) := \sup_{\phi \in \Pi} W^{\phi}(s,x) \tag{7}$$

the *value function.*

(iv) An admissible policy $\phi^* \in \Pi$ is called *optimal policy* if its performance function is equal to the value function (7). That is, $W^{\phi^*}(s,x) = V(s,x)$ for all $(s,x) \in [0,\infty) \times \Theta$. Notice that the function ϕ^* does not depend on the initial point (s,x).

The control problem consists in finding an optimal admissible policy and can be tackled with the HJB equation. As mentioned in the introductory section, we do not limit ourselves to a bounded utility function or to bounded coefficients in order to ensure applicability for more general setups. Nevertheless, to show the necessity of the HJB equation for optimality in Theorem 3 we assume the following conditions to be satisfied. For the sufficiency part in Theorem 4 they are *not* required.

(H1) We say that u satisfies a *linear boundedness condition* if there exists a constant $\mu > 0$ such that for all $(t,z) \in [0,\infty) \times \Theta$ and $c \in \Gamma_{t,z}$,

$$|u(t,z,c)| \le \mu \left[\|z\| + \|c\| + 1\right]. \tag{8}$$

(H2) If existing, an optimal policy ϕ^* is said to be *linearly bounded* if there exists a constant $\gamma \ge 0$ such that for all $(t,z) \in [0,\infty) \times \Theta$

$$\|\phi^*(t,z)\| \le \gamma \left[\|z\| + 1\right]. \tag{9}$$

(H3) We say that a coefficient function $g \in \{\alpha, \beta_1, \ldots, \beta_d\}$ satisfies a *linear growth condition* if for any admissible policy ϕ *boundedness coefficients* $p_g(t) \ge 0$ and $q_g(t) \ge 0$ exist for each $t \ge 0$ such that for all $z \in \Theta$,

$$\left\|g^{\phi}(t,z)\right\| \le p_g(t)\,\|z\| + q_g(t), \tag{10}$$

and the mappings $t \mapsto p_g(t)$ and $t \mapsto q_g(t)$ are cádlág. Notice that this condition holds uniformly over the set of admissible policies.

(H4) Define for any $s \in [0,\infty)$

$$P_s(t) := \frac{1}{t-s} \int_s^t \left(p_\alpha(\tau) + \sum_{k=1}^{d} \lambda_k p_{\beta_k}(\tau) \right) d\tau, \quad t \ge s, \tag{11}$$

and

$$Q_s(t) := \int_s^t e^{-P_s(\tau)(\tau-s)} \left(q_\alpha(\tau) + \sum_{k=1}^{d} \lambda_k q_{\beta_k}(\tau) \right) d\tau, \quad t \geq s. \tag{12}$$

If for some $g \in \{\alpha, \beta_1, \ldots, \beta_d\}$ there exists a $t^* \geq 0$ with $q_g(t^*) > 0$, the right continuity of q_g implies that $Q_0(t) > 0$ for all $t > t^*$, and we say that the *regularity condition* is satisfied if

$$B := \int_0^\infty e^{-[\overline{\rho}_0(t) - P_0(t)]t} Q_0(t)\, dt < \infty. \tag{13}$$

If, in contrast, in the degenerated case, for each $g \in \{\alpha, \beta_1, \ldots, \beta_d\}$ the boundedness coefficient $q_g(t)$ is equal to 0 for all $t \geq 0$, then $Q_0(t) = 0$ and the regularity condition is said to be satisfied if

$$A := \int_0^\infty e^{-[\overline{\rho}_0(t) - P_0(t)]t} dt < \infty. \tag{14}$$

Let us give a quick preview of the results presented in the subsequent sections in order to explain why and where we shall use the conditions stipulated in (H1)-(H4). The linear growth condition (10) gives an upper bound for the growth rate of the controlled process $X^{\phi,s,x}$. It allows to derive a finite upper bound for the expectation of $X_t^{\phi,s,x}$, which can be expressed in terms of the initial state x, see Lemma 1. Regularity conditions (13) and (14), respectively, make sure that the expected present value of the controlled process is finite for any admissible policy ϕ, see Corollary 2. Then, together with the linear boundedness conditions (8) and (9), we can deduce that the value function is linearly bounded with respect to the initial state x, see Theorem 2. This result will be used to show that the value function is in the domain of the infinitesimal generator of the controlled process (see Lemma 3), which in turn is crucial for deriving the HJB equation as a necessary criterion for optimality in Theorem 3.

The linear boundedness condition (8) is a substantial progress compared to the absolute boundedness required in the literature. It is, for example, indispensable for the CRRA utility function given in the example above.[9] Assumption (9) is not very restrictive. In most applications such as, again, our example[9] or models of exploiting renewable resources, linear boundedness of the controls is even naturally implied. The linear growth condition (10) is a common requirement in the theory of SDEs. It ensures that the solution $X^{\phi,s,x}$ does not explode. In addition, observe that (10) follows from another common

[9] For more details see Sennewald and Wälde (2006, Subsection 3.2).

assumption on SDEs, namely that the coefficients satisfy a Lipschitz condition, which ensures the existence of a unique solution $X^{\phi,s,s}$.[10] Regularity conditions (13) and (14) are easily met for sufficiently high time preference rates, cf. also part (iii) of the following remark.

REMARK 1. *(i) Condition (10) can be replaced by the following "easy-to-check" growth condition: There exist cádlág mappings* $p_g(t), \tilde{p}_g(t), q_g(t) \geq 0$ *such that* $\|g(t,z,c)\| \leq p_g(t)\|z\| + \tilde{p}_g(t)\|c\| + q_g(t)$. *Then (10) follows immediately with (9).*
(ii) The following conclusion will be helpful for the proofs in Section 7. In the non-degenerated case, where there exist some $g \in \{\alpha, \beta_1, \ldots, \beta_d\}$ *and* $t^* > 0$ *with* $q_g(t^*) > 0$, *regularity condition (13) implies* $A < \infty$, *where* A *is defined as in (14). This result is derived in appendix A.1. On the other hand, if* $q_g(t) = 0$ *for all* g *and* t, *we obtain immediately* $B = 0$ *and, by assumption,* $A < \infty$. *Thus, in either case we have* $A < \infty$ *and* $B < \infty$.
(iii) If the linear boundedness coefficients q_g *and* p_g *as well as the time preference rate* ρ *are constants, regularity conditions (13) and (14) hold if and only if* $\rho > p_\alpha + \sum_{k=1}^d \lambda_k p_{\beta_k}$.

4. Properties of the state process and the value function

This section serves as preparation for the derivation of the HJB equation as a necessary and sufficient condition for optimality. It provides a CVF, the central tool in this paper, and furthermore some useful properties of the controlled state process and the value function if the assumptions (H1)-(H4) from the preceding section are met. The proofs are given in Section 7. The CVF is presented in the following theorem. It can directly be applied on multidimensional SDEs as given by (5) and allows to describe the evolution of processes induced by a C^1-mapping of the time-state process $\{(t, X_t^{\phi,s,x}) : t \geq s\}$.

THEOREM 1. *Let* $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ *be a* C^1 *- function. Then the process* $\{f(t, X_t^{\phi,s,x}) : t \geq s\}$ *is adapted and cádlág, too, and it obeys*[11]

$$\begin{aligned} f\left(t, X_t^{\phi,s,x}\right) &= f(s,x) + \int_s^t \left[f_t\left(\tau, X_\tau^{\phi,s,x}\right) + \alpha^\phi\left(\tau, X_\tau^{\phi,s,x}\right) \cdot f_x\left(\tau, X_\tau^{\phi,s,x}\right)\right] d\tau \\ &\quad + \sum_{k=1}^d \int_s^t \left[f\left(\tau, X_{\tau_-}^{\phi,s,x} + \beta_k^\phi(\tau, X_{\tau_-}^{\phi,s,x})\right) - f\left(\tau, X_{\tau_-}^{\phi,s,x}\right)\right] dN_\tau^k. \end{aligned}$$

[10] For more details see, e.g., García and Griego (1994) or Gihman and Skorohod (1979, Ch. 3).

[11] Recall that the operator "$\cdot$" denotes the standard scalar product.

The following lemma shows that the expectation of $||X_t^{\phi,s,x}||$ is linearly bounded with respect to the initial value x. This property holds uniformly over all admissible policies $\phi \in \Pi$.

LEMMA 1. *If the coefficients $\alpha, \beta_1, \ldots, \beta_d$ satisfy the linear growth condition (10), then for all admissible policies $\phi \in \Pi$,*

$$E_s \left\| X_t^{\phi,s,x} \right\| \leq e^{P_s(t)(t-s)} \left[\|x\| + Q_s(t) \right],$$

where $P_s(t)$ and $Q_s(t)$ are defined as in (11) and (12), respectively.

From Lemma 1 we deduce the following corollary.

COROLLARY 1. *If the coefficients $\alpha, \beta_1, \ldots, \beta_d$ satisfy the linear growth condition (10), then for all admissible policies $\phi \in \Pi$,*

$$E_s \sup_{s \leq \tau \leq t} \left\| X_\tau^{\phi,s,x} \right\| \leq e^{P_s(t)(t-s)} \left[\|x\| + Q_s(t) \right].$$

The next corollary shows that, for any admissible policy ϕ, the expected present value of the controlled process $X^{\phi,s,x}$ discounted with the time preference rate is finite and linearly bounded with respect to the initial state x.

COROLLARY 2. *If the coefficients $\alpha, \beta_1, \ldots, \beta_d$ satisfy the linear growth condition (10) such that regularity conditions (13) and (14), respectively, hold, then for all admissible policies $\phi \in \Pi$,*

$$E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} \left\| X_t^{\phi,s,x} \right\| dt \leq A(s) \|x\| + B(s) < \infty,$$

where

$$A(s) := \int_s^\infty e^{-[\overline{\rho}_s(t) - P_s(t)](t-s)} dt \tag{15}$$

and

$$B(s) := \int_s^\infty e^{-[\overline{\rho}_s(t) - P_s(t)](t-s)} Q_s(t) \, dt, \tag{16}$$

and $P_s(t)$ and $Q_s(t)$ are defined as in (11) and (12), respectively.

If the utility function u is linearly bounded in the sense of (8), we derive the following theorem from the preceding results. It shows that the value function, as well, is linearly bounded with respect to the initial state x.

THEOREM 2. *Suppose the utility function u satisfies the linear boundedness condition (8) and the coefficients $\alpha, \beta_1, \ldots, \beta_d$ the linear growth condition (10) such that regularity conditions (13) and (14), respectively, hold. In addition, let the optimal policy ϕ^* be linearly bounded according to (9). Then for all $(s, x) \in [0, \infty) \times \Theta$,*

$$|V(s, x)| \leq (1 + \gamma) \mu A(s) \|x\| + K(s) < \infty,$$

where $A(s)$ is defined as in (15), and $K(s)$ is a deterministic value that depends on the boundedness coefficients γ, μ, q_g, and p_g, where $g \in \{\alpha, \beta_1, \ldots, \beta_d\}$.

5. The Hamilton-Jacobi-Bellman equation

This section presents the main results of the paper, the HJB equation as a necessary and sufficient criterion for optimality. In order to achieve a shorter notation, we first define the following differential operator D associated with the controlled SDE (5). For a C^1-function $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ let D be given by

$$D^c f(s, x) := f_t(s, x) + \alpha(s, x, c) \cdot f_x(s, x) + \sum_{k=1}^{d} \lambda_k [f(s, x + \beta_k(s, x, c)) - f(s, x)]. \quad (17)$$

Then the necessity part is presented in the following theorem.

THEOREM 3. *Assume that for any $(s, x) \in [0, \infty) \times \Theta$ and $c \in \Gamma_{s,x}$ there exists an admissible policy ϕ with $\phi(s, x) = c$. Let the utility function u satisfy the linear boundedness condition (8), and the coefficients $\alpha, \beta_1, \ldots, \beta_d$ the linear growth condition (10) such that regularity conditions (13) and (14), respectively, hold. Assume that an optimal policy ϕ^* satisfying (9) exists. If furthermore the value function V is once continuously differentiable with bounded first derivatives, the following equation is satisfied for all $(s, x) \in [0, \infty) \times \Theta$:*

$$\rho(s) V(s, x) = \max_{c \in \Gamma_{s,x}} \{u(s, x, c) + D^c V(s, x)\}, \quad (18)$$

and the maximum is achieved by $\phi^*(s, x)$.

Equation (18) is called the HJB equation. Theorem 3 says that under the stipulated conditions the HJB equation must be necessarily satisfied by the value function and the optimal policy. Based on the fact that the optimal policy maximizes the right-hand side of (18), we derive the following corollary.

COROLLARY 3. *Let the conditions of Theorem 3 be satisfied, and let furthermore u be differentiable with respect to c. Then, for all $(s, x) \in [0, \infty) \times \Theta$ for which $\phi^*(s, x)$ lies* *in the interior of* $\Gamma_{s,x}$*, the following first-order condition holds:*

$$\frac{\partial}{\partial c_i} u(s,x,\phi^*(s,x)) = -\frac{\partial}{\partial c_i} D^{\phi^*(s,x)} V(s,x), \quad i = 1,\dots,r. \tag{19}$$

If the value function and the optimal policy are unknown, equation (19) can be used for further analysis. For example, starting from (19) it is possible to derive a Keynes-Ramsey rule for optimum-consumption problems, see, e.g., Wälde (1999) and the accompanying paper Sennewald and Wälde (2006) or, for the case of Brownian motion, Turnovsky (2000). In some cases, one may even derive explicit expressions for candidates of both the value function and the optimal policy, see also Sennewald and Wälde (2006).

So far, we only know that the HJB equation is necessary. The subsequent theorem shows that it is also a sufficient condition for optimality.

Theorem 4. *Let a* C^1 *- function* $J : [0,\infty) \times \Theta \to \mathbb{R}$ *satisfy*

$$\rho(s) J(s,x) \geq u(s,x,c) + D^c J(s,x), \quad \forall (s,x) \in [0,\infty) \times \Theta, \forall c \in \Gamma_{s,x}, \tag{20}$$

and suppose in addition that there exists an admissible policy ϕ^* *such that*

$$\rho(s) J(s,x) = u^{\phi^*}(s,x) + D^{\phi^*(s,x)} J(s,x), \quad \forall (s,x) \in [0,\infty) \times \Theta. \tag{21}$$

If furthermore for all $(s,x) \in [0,\infty) \times \Theta$ *the limiting condition*

$$\lim_{t\to\infty} E\left[e^{-\overline{\rho}_s(t)t} J(t, X_t^{\phi^*,s,x})\right] = 0 \tag{22}$$

and the limiting inequality

$$\lim_{t\to\infty} E\left[e^{-\overline{\rho}_s(t)t} J(t, X_t^{\phi,s,x})\right] \geq 0, \quad \forall \phi \in \Pi, \tag{23}$$

are satisfied, then J *is the value function* V *and the policy* ϕ^* *is optimal.*

The HJB equation from Theorem 3 is divided here into inequality (20) and equation (21). The theorem tells us that, if there exist a C^1-function and a policy such that this policy maximizes the HJB equation and terminal conditions (22) and (23) are satisfied, then this policy is optimal and the function is the value function. Thus, one can use Theorem 4 to verify whether a given function and a given policy (which were, for example, found by "guessing" or via the first-order conditions in Corollary 3)[12] coincide with the value function and the optimal policy. Such theorems are therefore also called verification

[12]The method of "guessing" the value function and then verifying it has first been applied by Merton (1971). He showed that, if the utility function u is of the HARA class, the value function can easily be guessed as it is of similar form as u.

theorems. Notice that the conditions in Theorem 4 are much milder than for the necessity part in Theorem 3. In particular, one can show that the linear boundedness and growth conditions (8), (9), and (10) together with regularity conditions (13) and (14) are sufficient for both terminal conditions (22) and (23) to be satisfied.

Limiting condition (22) generalizes the boundary condition for finite time horizon settings, see, e.g., Kushner and Dupuis (1992). In a deterministic framework, Michel (1982) and later Kamihigashi (2001) show that such terminal (or transversality) conditions may even be necessary conditions. In many control problems, the utility function u is assumed to be nonnegative, for example, if $u(c) = c^{\sigma}$, $\sigma > 0$. Then limiting inequality (23) holds obviously since only candidates J for the value function with $J(s, x) \geq 0$ for all $(s, x) \in [0, \infty) \times \Theta$ are sensible.

The following corollary shows that, under certain conditions and making use of the fact that a concave function can have only a unique maximum point, the verification can be carried out quite easily.

Corollary 4. *Let the instantaneous utility function u be nonnegative as well as strictly concave and differentiable in the control variable c. Assume furthermore that also the coefficients $\alpha, \beta_1, \dots, \beta_d$ are concave in c.*[13] *Then, if a concave C^1 - function $J : [0, \infty) \times \Theta \to \mathbb{R}$ and an admissible policy ϕ^* satisfy equation (21) and the first-order condition*

$$\frac{\partial}{\partial c_i} u(s, x, \phi^*(s, x)) = -\frac{\partial}{\partial c_i} D^{\phi^*(s,x)} J(s, x), \quad i = 1, \dots, r, \tag{24}$$

and if furthermore limiting condition (22) holds, ϕ^* is an optimal policy and J is the value function V.

The following theorem tells us that an optimal Markov control is even optimal within the class of general admissible controls under very mild assumptions.

Theorem 5. *Suppose that an optimal Markov policy ϕ^* exists and assume the value function V to be once continuously differentiable and to satisfy*

$$\rho(s) V(s, x) \geq u(s, x, c) + D^c V(s, x), \quad \forall (s, x) \in [0, \infty) \times \Theta, \forall c \in \Gamma_{s,x}. \tag{25}$$

Furthermore, let the following limiting inequality hold for all admissible controls C:

$$\lim_{t \to \infty} E_s \left[e^{-\overline{\rho}_s(t) t} V(t, X_t^{C,s,x}) \right] \geq 0. \tag{26}$$

[13] Note that $\alpha, \beta_1, \dots, \beta_d$ can be linear in the control variable as well, cf. budget constraint (1).

Define the supremum of the performance function over all general admissible controls C *by* $V^a(s,x) := \sup_{\{C\ adm.\ control\}} W^C(s,x)$. *Then,* $V(s,x) = V^a(s,x)$ *for all* $(s,x) \in [0,\infty) \times \Theta$.

The result in Theorem 5 is not surprising since the "implicit" Markov nature of the controlled SDE (2) suggests that Markov controls represent, so to speak, the natural class of controls, and no wider class has to be taken into account. Note that the HJB equation is sufficient for inequality (25) to be satisfied. That is, under the conditions of Theorems 3 and 4, inequality (25) holds and only limiting condition (26) has to be checked.

6. The Poisson-diffusion setting

In the following we show that the necessity property of the HJB equation extends to the Poisson-diffusion case under the same mild conditions as before. Let $B^1, \ldots, B^{\tilde{d}}$ be $\tilde{d}$ independent standard Brownian motions that are also independent of the Poisson processes. The controlled process $X_t^{\phi,s,x}$ is now given as the solution of

$$X_t = x + \int_s^t \alpha^\phi(\tau, X_\tau)\, ds + \sum_{l=1}^{\tilde{d}} \int_s^t \sigma_l^\phi(\tau, X_\tau) dB_\tau^l + \sum_{k=1}^{d} \int_s^t \beta_k^\phi(\tau, X_\tau)_- dN_\tau^k, \tag{27}$$

where for each $l = 1, \ldots, \tilde{d}$ the *diffusion coefficient* $\sigma_l^\phi(t,x) := \sigma_l(t,x,\phi(t,x))$ is defined by a continuous vector function $\sigma_l : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$.[14] The differential operator $\tilde{D}$ corresponding to the controlled SDE (27) applies to $\mathcal{C}^{1,2}$ - functions $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ and reads

$$\begin{aligned} \tilde{D}^c f(s,x) \quad : \quad &= f_t(s,x) + \alpha(s,x,c) \cdot f_x(s,x) + \frac{1}{2} \sum_{l=1}^{\tilde{d}} \sigma_l(s,x,c)' f_{xx}(s,x)\, \sigma_l(s,x,c) \\ &+ \sum_{k=1}^{d} \lambda_k [f(s, x + \beta_k(s,x,c)) - f(s,x)]. \end{aligned} \tag{28}$$

Then the necessity of the HJB equation analogous to Theorem 3 is stated in the following theorem.

Theorem 6. *Assume the conditions from Theorem 3 to be satisfied. In addition let the diffusion coefficients* $\sigma_1^\phi, \ldots, \sigma_{\tilde{d}}^\phi$ *be linearly bounded in the sense of (10), i.e.,*

$$\left\| \sigma_l^\phi(t,x) \right\| \le p_{\sigma_l}(t) \|x\| + q_{\sigma_l}(t). \tag{29}$$

[14] As the jump integral, the n-dimensional diffusion integral is to be read componentwise.

If then the value function V is twice continuously differentiable with bounded Hesse matrix V_{yy}, the following equation is satisfied for all $(s,x) \in [0,\infty) \times \Theta$:

$$\rho(s)\,V(s,x) = \max_{c \in \Gamma_{s,x}} \left\{ u(s,x,c) + \tilde{D}^c V(s,x) \right\}, \tag{30}$$

where the maximum is achieved by the optimal policy $\phi^*(s,x)$.

The proof is in analogy to the proof of Theorem 3. Appendix A.2 shows where and how to make the necessary adjustments.

7. Proof of results

This part presents the proofs for the findings from Sections 4 and 5. Before starting, we repeat a crucial property of the stochastic integral with respect to Poisson processes. It is due to the fact that Poisson processes are of bounded variation. We are given a Poisson process N with arrival rate λ and a cádlág process X. Both processes are adapted. Then, according to García and Griego (1994, Section 3), the following relation holds true for any $0 \le \nu \le s < t$:

$$E_\nu \left[\int_s^t X_{\tau_-} dN_\tau \right] = \lambda E_\nu \left[\int_s^t X_\tau d\tau \right]. \tag{31}$$

For the reader's convenience we recall the following result from real analysis. It can be proven using the (ε,δ) - definition of continuity at point t. A proof can be found in many textbooks on real analysis as in, e.g., Browder (1996).

Lemma 2. *Let the function $f : [0,\infty) \to \mathbb{R}$ be integrable and right continuous at point $t \in [0,\infty)$. Then,*

$$\lim_{h \searrow 0} \frac{1}{h} \int_t^{t+h} f(\tau) d\tau = f(t).$$

We now turn to the proofs. Theorem 1 is derived from Garcia and Griego's (1994) CVF on p. 344. The necessary assumptions ($X^{\phi,s,x}$ is cádlág and the stochastic integrals in (5) are in the Lebesgue-Stieltjes sense) are obviously met. Lemma 1, which shows that the expectation of $||X_t^{\phi,s,x}||$ is linearly bounded with respect to the initial state x, is proven as follows.

Proof of Lemma 1. Using a comparison principle as, e.g., Bassan et al. (1993, Corollary 3.5), we deduce from the linear growth condition (10) that $\left\| X_t^{\phi,s,x} \right\| \le Z_t^{s,x}$, where $Z_t^{s,x}$ denotes the unique solution of [15]

$$Z_t = \|x\| + \int_s^t \left[p_\alpha(\tau) Z_\tau + q_\alpha(\tau)\right] d\tau + \sum_{k=1}^{d} \int_s^t \left[p_{\beta_k}(\tau_-) Z_{\tau_-} + q_{\beta_k}(\tau_-)\right] dN_\tau^k. \tag{32}$$

Hence,

$$E_s \left\| X_t^{\phi,s,x} \right\| \leq E_s Z_t^{s,x}. \tag{33}$$

We now compute $E_s Z_t^{s,x}$. Taking expectation on SDE (32) and using (31) yields

$$E_s Z_t^{s,x} = \|x\| + E_s \int_s^t \left[p_\alpha(\tau) Z_\tau + q_\alpha(\tau) + \sum_{k=1}^{d} \lambda_k \left[p_{\beta_k}(\tau) Z_\tau + q_{\beta_k}(\tau)\right]\right] d\tau. \tag{34}$$

Interchanging expectation and integral due to the theorem of bounded convergence leads to[16]

$$E_s Z_t^{s,x} = \|x\| + \int_s^t \left[\left(p_\alpha(\tau) + \sum_{k=1}^{d} \lambda_k p_{\beta_k}(\tau)\right) E_s Z_\tau^{s,x} + q_\alpha(\tau) + \sum_{k=1}^{d} \lambda_k q_{\beta_k}(\tau)\right] d\tau.$$

This deterministic linear differential equation in $E_s Z_t^{s,x}$ has the unique solution

$$E_s Z_t^{s,x} = e^{P_s(t)(t-s)} \left[\|x\| + Q_s(t)\right], \tag{35}$$

where $P_s(t)$ and $Q_s(t)$ are defined as in (11) and (12), respectively. This relation together with (33) finishes the proof. $\square$

The preceding proof immediately implies the subsequent proof of Corollary 1.

Proof of Corollary 1. Since the boundedness coefficients p_g and q_g, $g \in \{\alpha, \beta_1, \ldots, \beta_d\}$, are nonnegative, $Z^{s,x}$ has increasing paths. Remember from the proof of Lemma 1 that $\left\| X_t^{\phi,s,x} \right\| \leq Z_t^{s,x}$ for all $t \geq s$. Thus, $\sup_{s \leq \tau \leq t} \left\| X_\tau^{\phi,s,x} \right\| \leq \sup_{s \leq \tau \leq t} Z_\tau^{s,x} = Z_t^{s,x}$ and hence, $E_s \sup_{s \leq \tau \leq t} \left\| X_\tau^{\phi,s,x} \right\| \leq E_s Z_t^{s,x}$, which together with (35) yields Corollary 1. $\square$

Proof of Corollary 2. From the proof of Lemma 1 we know that $\left\| X_t^{\phi,s,x} \right\| \leq Z_t^{s,x}$. Thus,

$$E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} \left\| X_t^{\phi,s,x} \right\| dt \leq E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt. \tag{36}$$

Using (35) and assuming for the moment that $A(s)$ and $B(s)$ defined as in (15) and (16), respectively, are finite, we can now apply the theorems of bounded and monotone

[15] Using Protter (1990, theorem V.6), one can show easily that (32) possess a unique solution with finite expectation.

[16] See Appendix A.3 to see how to use the theorem of bounded convergence in this case.

convergence in order to interchange expectation and integral on the right-hand side of (36),[17] which yields

$$E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} \left\| X_t^{\phi,s,x} \right\| dt \leq A(s) \|x\| + B(s). \tag{37}$$

It remains to be shown that $A(s)$ and $B(s)$ are finite. For this purpose we use that

$$A(s) \leq e^{[\overline{\rho}_0(s) - P_0(s)]s} A \tag{38}$$

and

$$B(s) \leq e^{[\overline{\rho}_0(s) - P_0(s)]s} B. \tag{39}$$

But since we know from Remark 1 (ii) that due to regularity conditions (13) and (14), respectively, A and B are always finite, the result follows. □

We proceed with the proof of Theorem 2, which shows that the value function is linearly bounded with respect to the initial value x.

Proof of Theorem 2. Using the linear boundedness conditions (8) and (9), we find the following upper bound for the value function:

$$\begin{aligned} |V(s,x)| &= \left| W^{\phi^*}(s,x) \right| \leq E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} \left| u^{\phi^*}\left(t, X_t^{\phi^*,s,x}\right)\right| dt \\ &\leq \mu E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} \left[\left\| X_t^{\phi^*,s,x} \right\| + \left\| \phi^*\left(X_t^{\phi^*,s,x}\right) \right\| + 1 \right] dt \\ &\leq (1+\gamma)\mu \left[E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} \left\| X_t^{\phi^*,s,x} \right\| dt + \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} dt \right]. \end{aligned} \tag{40}$$

Since $A(s)$ is an upper bound for $\int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} dt$ and $A(s)$ is finite according to (38) and Remark 1 (ii), the second term in brackets on the right-hand side is finite, too. The first term is finite according to Corollary 2. Hence, setting $K(s) := (1+\gamma)\mu[\int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} dt + B(s)]$, we finally obtain $K(s) < \infty$ and hence $|V(s,x)| \leq (1+\gamma)\mu A(s)\|x\| + K(s) < \infty$, which is what was to be shown. □

In order to simplify the notation in the following, we drop the explicit time argument by introducing the time-state process

$$\left\{ Y_t^{\phi,y} = \left(s+t, X_{s+t}^{\phi,s,x} \right), t \geq 0 \right\}, \quad Y_0^{\phi,y} := y := (s,x). \tag{41}$$

Then the state space corresponding to this process is $\tilde{\Theta} := [0,\infty) \times \Theta \subset \mathbb{R}^{n+1}$, and $Y^{\phi,y}$ solves the transformed SDE

$$Y_t = y + \int_0^t \tilde{\alpha}^\phi(Y_\tau)\, d\tau + \sum_{k=1}^d \int_0^t \tilde{\beta}_k^\phi(Y_\tau)_-\, d\tilde{N}_\tau^k, \tag{42}$$

[17] See Appendix A.4.

where the coefficients are given by $\tilde{\alpha}^{\phi}(t,z) := \left(1, \alpha^{\phi}(t,z)\right)'$ and $\tilde{\beta}_k^{\phi}(t,z) := \left(0, \beta_k^{\phi}(t,z)\right)'$, and for each $k = 1, \ldots, d$ the process $\tilde{N}^k$ defined by $\tilde{N}_\tau^k := N_{s+\tau}^k - N_s^k$ forms a Poisson process. The corresponding filtration is $\left\{\tilde{\mathcal{F}}_t, t \geq 0\right\}$, where $\tilde{\mathcal{F}}_t := \mathcal{F}_{s+t}$. We rewrite the performance function by time transformation as

$$W^{\phi}(y) = \tilde{E}_0 \int_0^t e^{-\tilde{\rho}_s(t)t} u^{\phi}\left(Y_t^{\phi,y}\right) dt, \tag{43}$$

where $\tilde{\rho}_s(t) := \frac{1}{t}\int_0^t \rho(s+r)\, dr = \overline{\rho}_s(s+t)$, and $\tilde{E}_t$ denotes the conditional expectation with respect to $\tilde{\mathcal{F}}_t$.

Altogether, by deriving (42) and (43), we have transformed the general control problem into a time-autonomous one. The corresponding differential operator D is the same as in (17) and reads, adapted to the time-autonomous setup,

$$D^c f(y) = \tilde{\alpha}(y,c) \cdot f_y(y) + \sum_{k=1}^{d} \lambda_k [f(y + \tilde{\beta}_k(y,c)) - f(y)]. \tag{44}$$

The following lemma shows that the value function V belongs to the domain of the infinitesimal generator of the controlled process $X^{\phi,s,x}$ for any admissible policy ϕ. This result is crucial for deriving the necessity of the HJB equation in Theorem 3. Whereas the proof is almost trivial if utility (or value function)[18] and the coefficients are bounded, it becomes more complex for the more general case with linearly bounded utility and coefficient functions.

LEMMA 3. *Under the conditions of Theorem 3 we obtain for any admissible policy ϕ,*

$$\lim_{h \searrow 0} \frac{1}{h} \tilde{E}_0 \left[e^{-\tilde{\rho}_s(h)h} V(Y_h^{\phi,y}) - V(y)\right] = D^{\phi(y)} V(y) - \rho(s)\, V(y).$$

PROOF. Applying the CVF from Theorem 1 to the C^1 - function $f(v) = f(t,z) = e^{-\tilde{\rho}_s(t)t} V(v)$ yields

$$\begin{aligned}
&e^{-\tilde{\rho}_s(h)(h)} V\left(Y_h^{\phi,y}\right) - V(y) \\
&\quad = \int_0^h \left[\tilde{\alpha}^{\phi}\left(Y_\tau^{\phi,y}\right) \cdot e^{-\tilde{\rho}_s(\tau)\tau} V_y\left(Y_\tau^{\phi,y}\right) - \rho(s+\tau)\, e^{-\tilde{\rho}_s(\tau)\tau} V(Y_\tau^{\phi,y})\right] d\tau \\
&\quad + \sum_{k=1}^{d} \int_0^h \left[e^{-\tilde{\rho}_s(\tau)\tau} V\left(Y_{\tau_-}^{\phi,y} + \tilde{\beta}_k^{\phi}\left(Y_\tau^{\phi,y}\right)_-\right) - e^{-\tilde{\rho}_s(\tau)\tau} V(Y_{\tau_-}^{\phi,y})\right] d\tilde{N}_\tau^k.
\end{aligned}$$

[18] As one can show easily, a bounded utility function implies that the value function is bounded as well.

Taking expectation and dividing by h gives, together with (31),

$$\frac{1}{h}\tilde{E}_0\left[e^{-\tilde{\rho}_s(h)h}V(Y_h^{\phi,y})-V(y)\right]$$
$$=\tilde{E}_0\left\{\frac{1}{h}\int_0^h e^{-\tilde{\rho}_s(\tau)\tau}\left[\tilde{\alpha}^\phi\left(Y_\tau^{\phi,y}\right)\cdot V_y\left(Y_\tau^{\phi,y}\right)-\rho\left(s+\tau\right)V(Y_\tau^{\phi,y})\right]d\tau\right\}$$
$$+\sum_{k=1}^d\lambda_k\tilde{E}_0\left\{\frac{1}{h}\int_0^h e^{-\tilde{\rho}_s(\tau)\tau}\left[V\left(Y_\tau^{\phi,y}+\tilde{\beta}_k^\phi\left(Y_\tau^{\phi,y}\right)\right)-V(Y_\tau^{\phi,y})\right]d\tau\right\}. \quad (45)$$

Now let h tend to 0. We show that the theorem of bounded convergence can be applied to interchange limit and expectation on the right-hand side in (45). For this purpose we have to find an upper bound with finite expectation for each of the $d+1$ random variables inside the expectations. Notice that such a bound must hold uniformly over all h that are small enough. Whereas the bound is obvious if the utility function and the coefficients are bounded, we have to do some more calculation for the more general case with linear boundedness.

We first consider the most-left integral on the right-hand side of (45). Remember from real analysis that for any univariate piecewise continuous function f, $\int_x^y f(z)\,dz\leq(y-x)\max_{x\leq z\leq y}f(z)$. According to this result we derive for $h\leq 1$, using the linear boundedness of α^ϕ, the linear boundedness of V according to Theorem 2, and the boundedness of the first derivative of V:

$$\left|\frac{1}{h}\int_0^h e^{-\tilde{\rho}_s(\tau)\tau}\left[\tilde{\alpha}^\phi\left(Y_\tau^{\phi,y}\right)\cdot V_y\left(Y_\tau^{\phi,y}\right)-\rho\left(s+\tau\right)V(Y_\tau^{\phi,y})\right]d\tau\right|$$
$$\leq\left[\left\|p_\alpha\right\|_1\left\|V_y\right\|+(1+\gamma)\,\mu\left\|\rho\right\|_1\left\|A\right\|_1\right]\sup_{\tau\in[0,1]}\left\|X_{s+\tau}^{\phi,s,x}\right\|+\left(1+\left\|q_\alpha\right\|_1\right)\left\|V_y\right\|+\left\|\rho\right\|_1\left\|K\right\|_1$$

where, by assumption, $\|V_y\|:=\sup_{y\in\tilde{\Theta}}\|V_y(y)\|<\infty$ and, due to their cádlág property (respective continuity), $\|p_\alpha\|_1:=\sup_{\tau\in[0,1]}p_\alpha(s+\tau)<\infty$, $\|A\|_1:=\sup_{\tau\in[0,1]}A(s+\tau)<\infty$, and so forth. According to Corollary 1, $\sup_{\tau\in[0,1]}\left\|X_{s+\tau}^{\phi,s,x}\right\|$ possesses finite expectation. Hence, the right-hand side in the latter inequality is an upper bound with finite expectation for the first integral on the right-hand side in (45). In analogy, for each of the remaining k integrals in (45) an upper bound for all $h\leq 1$ is given by

$$\left|\frac{1}{h}\int_0^h e^{-\tilde{\rho}_s(\tau)\tau}\left[V\left(Y_\tau^{\phi,y}+\tilde{\beta}_k^\phi\left(Y_\tau^{\phi,y}\right)\right)-V(Y_\tau^{\phi,y})\right]d\tau\right|$$
$$\leq(1+\gamma)\,\mu\left\|A\right\|_1\left(2+\left\|p_{\beta_k}\right\|_1\right)\sup_{\tau\in[0,1]}\left\|X_{s+\tau}^{\phi,s,x}\right\|+(1+\gamma)\,\mu\left\|A\right\|_1\left\|q_{\beta_k}\right\|_1+2\left\|K\right\|_1$$

Again with Lemma 1 we deduce that the expectation of this upper bound is finite. The theorem of bounded convergence can hence be applied on (45), and interchanging limit and expectation finally yields jointly with Lemma 2

$$\lim_{h\searrow 0}\frac{1}{h}\tilde{E}_0\left[e^{-\tilde{\rho}_s(h)h}V(Y_h^{\phi,y})-V(y)\right]$$
$$=\tilde{\alpha}^{\phi}\left(y\right)\cdot V_y\left(y\right)-\rho\left(s\right)V(y)+\sum_{k=1}^{d}\lambda_k\left(V\left(y+\tilde{\beta}_k^{\phi}\left(y\right)\right)-V(y)\right)$$
$$=D^{\phi(y)}V(y)-\rho\left(s\right)V(y),$$

which is what was to be shown. □

In the remaining part of this section we finally present the proofs of the main results from Section 5.

Proof of Theorem 3. Let $y\in\tilde{\Theta}$. We first prove that the optimal policy ϕ^* yields equality in the HJB equation (18). For some small $h>0$ we obtain, cf. also Fleming and Soner (1993) or Kushner and Dupuis (1992),

$$\begin{aligned}
0 &= \tilde{E}_0\int_0^\infty e^{-\tilde{\rho}_s(t)t}u^{\phi^*}\left(Y_t^{\phi^*,y}\right)dt-V(y)\\
&= \tilde{E}_0\int_0^h e^{-\tilde{\rho}_s(t)t}u^{\phi^*}\left(Y_t^{\phi^*,y}\right)dt+\tilde{E}_0\int_h^\infty e^{-\tilde{\rho}_s(t)t}u^{\phi^*}\left(Y_t^{\phi^*,y}\right)dt-V(y)\\
&= \tilde{E}_0\int_0^h e^{-\tilde{\rho}_s(t)t}u^{\phi^*}\left(Y_t^{\phi^*,y}\right)dt\\
&\quad+\tilde{E}_0\left\{e^{-\tilde{\rho}_s(h)h}E\left[\int_0^\infty e^{-\tilde{\rho}_{s+h}(t)t}u^{\phi^*}\left(Y_{h+t}^{\phi^*,y}\right)dt\middle|\,Y_h^{\phi^*,y}\right]\right\}-V(y)\\
&= \tilde{E}_0\int_0^h e^{-\tilde{\rho}_s(t)t}u^{\phi^*}\left(Y_t^{\phi^*,y}\right)dt+\tilde{E}_0\left[e^{-\tilde{\rho}_s(h)h}V(Y_h^{\phi^*,y})-V(y)\right]. \qquad (46)
\end{aligned}$$

Dividing by h and applying the limit $h\searrow 0$, this becomes

$$0=\lim_{h\searrow 0}\tilde{E}_0\frac{1}{h}\int_0^h e^{-\tilde{\rho}_s(t)t}u^{\phi^*}\left(Y_t^{\phi^*,y}\right)dt+\lim_{h\searrow 0}\tilde{E}_0\frac{1}{h}\left[e^{-\tilde{\rho}_s(h)h}V(Y_h^{\phi^*,y})-V(y)\right].$$

For the first term we use in analogy to appendix A.3 the theorem of bounded convergence to interchange expectation and integral.[19] Then, we obtain with Lemma 2,

$$\lim_{h\searrow 0}\tilde{E}_0\frac{1}{h}\int_0^h e^{-\tilde{\rho}_s(t)t}u^{\phi^*}\left(Y_t^{\phi^*,y}\right)dt=u^{\phi^*}\left(y\right).$$

For the second term, Lemma 3 gives the limit. Thus, altogether, $0=u^{\phi^*}\left(y\right)+D^{\phi^*(y)}V(y)-\rho\left(s\right)V(y)$, which shows that equality in (18) is satisfied for the optimal policy.

It remains to be shown that for any $c\in\Gamma_y$, $\rho\left(s\right)V(y)\geq u(y,c)+D^cV(y)$. For this purpose we follow an argument applied by Kushner and Dupuis (1992) and Duffie (1992),

[19] An upper bound is $\int_0^\infty e^{-\tilde{\rho}_s(t)t}\left|u^{\phi^*}\left(Y_t^{\phi^*,y}\right)\right|dt$, which possess finite expectation due to assumption (3).

in defining a policy

$$\psi_{y,h}(v) := \begin{cases} \phi(v) & \text{for } s \leq t < s+h \\ \phi^*(v) & \text{for } t \geq s+h \end{cases}, \quad v = (t,z) \in \tilde{\Phi},$$

where ϕ is an arbitrary admissible policy with $\phi(y) = c$.[20] Since from time $s+h$ on the policies $\psi_{y,h}$ and ϕ^* equal each other, we obtain

$$W^{\psi_{y,h}}(Y_t^{\psi_{y,h},y}) = W^{\phi^*}(Y_t^{\psi_{y,h},y}) = V(Y_t^{\psi_{y,h},y}), \quad \forall t \geq h.$$

Then in analogy to (46),

$$0 \geq W^{\psi_{y,h}}(y) - V(y) = \tilde{E}_0 \int_0^h e^{-\tilde{\rho}_s(t)t} u^{\phi}(Y_t^{\phi,y}) dt + \tilde{E}_0 \left[e^{-\tilde{\rho}_s(h)h} V(Y_h^{\phi,y}) - V(y) \right].$$

Now, if we divide by h and let h tend toward 0, we obtain

$$0 \geq \lim_{h \searrow 0} \tilde{E}_0 \frac{1}{h} \int_0^h e^{-\tilde{\rho}_s(t)t} u^{\phi}(Y_t^{\phi,y}) dt + \lim_{h \searrow 0} \frac{1}{h} \tilde{E}_0 \left[e^{-\tilde{\rho}_s(h)h} V(Y_h^{\phi^*,y}) - V(y) \right].$$

Again, the limit of the first term is derived by first interchanging expectation and integral according to the theorem of bounded convergence and then by applying Lemma 2, whereas Lemma 3 gives the second limit. Hence, $0 \geq u(y,c) + D^c V(y) - \rho(s) V(y)$. Since $c \in \Gamma_y$ was chosen arbitrarily, the proof is completed. □

Proof of Corollary 3. Let $y \in \tilde{\Theta}$. Since according to Theorem 3, $u^{\phi^*}(y) + D^{\phi^*(y)} V(y) \geq u(y,c) + D^c V(y)$ for all $c \in \Gamma_y$, (19) must hold as a first order condition if $\phi^*(y)$ lies in the interior of Γ_y. □

Proof of Theorem 4. We have a continuously differentiable function $J : \tilde{\Theta} \to \mathbb{R}$ that satisfies inequality (20) and, with an admissible policy ϕ^*, equation (21). We show (i) $J(y) \geq W^{\phi}(y)$ for any arbitrary admissible policy ϕ and (ii) $J(y) = W^{\phi^*}(y)$, which implies that ϕ^* is an optimal policy and that $J = W^{\phi^*}$ is the value function V.

Step (i): Let $\phi \in \Pi$ be an arbitrary admissible policy. Then inequality (20) gives

$$-\rho(s) J(y) + D^{\phi(y)} J(y) \leq -u^{\phi}(y), \quad \forall y \in \tilde{\Theta}. \tag{47}$$

Applying the change of variables formula from Theorem 1 to the C^1 - function $f(v) = f(t,z) = e^{-\tilde{\rho}_s(t)t} J(v)$ and taking the expectation on both sides yields together with (31) the following version of the Dynkin formula:

$$\tilde{E}_0 e^{-\tilde{\rho}_s(t)t} J(Y_t^{\phi,y}) - J(y) = \tilde{E}_0 \int_0^t e^{-\tilde{\rho}_s(\tau)\tau} \left[-\rho(s+\tau) J(Y_\tau^{\phi,y}) + D^{\phi\left(Y_\tau^{\phi,y}\right)} J(Y_\tau^{\phi,y}) \right] d\tau.$$

[20]By assumption, there exists an admissible policy ϕ with $\phi(y) = c$ for any $c \in \Gamma_y$.

Then, inequality (47) implies $J(y) \geq \tilde{E}_0 \int_0^t e^{-\tilde{\rho}_s(\tau)\tau} u^{\phi}(Y_\tau^{\phi,y}) d\tau + \tilde{E}_0 e^{-\tilde{\rho}_s(t)t} J(Y_t^{\phi,y})$. Letting t approach infinity and applying the theorem of bounded convergence on the first term on the right-hand side gives $J(y) \geq W^{\phi}(y) + \lim_{t\to\infty} \tilde{E}_0 e^{-\tilde{\rho}_s(t)t} J(Y_t^{\phi,y})$.[21] Thus, since by assumption (23) the limit on the right-hand side is equal to or greater as 0, $J(y) \geq W^{\phi}(y)$.

Step (ii): We may rewrite (21) as $-\rho(s) J(y) + D^{\phi^*(y)} J(y) = -u^{\phi^*}(y)$. Then, in exactly the same way as in step (i), only with "=" instead of "$\leq$", we deduce that $J(y) = W^{\phi^*}(y) + \lim_{t\to\infty} \tilde{E}_0 e^{-\tilde{\rho}_s(t)t} J(Y_t^{\phi^*,y})$. Since by limiting condition (22) the right-most term goes to zero, we obtain $J(y) = W^{\phi^*}(y)$, which completes the proof. $\square$

Proof of Corollary 4. We show that the conditions of Theorem 4 are satisfied. Then, by Theorem 4, the result follows. At first we derive from the nonnegativity of u that the value function V is nonnegative, too. Hence, limiting inequality (23) holds. Thus, it remains to be shown that $\phi^*(y)$ is a global maximum point of $u(y,c) + D^c J(y)$, or $u^{\phi^*}(y) + D^{\phi^*(y)} J(y) \geq u(y,c) + D^c J(y)$ for all $c \in \Gamma_y$. The first order condition for $\phi^*(y)$ to be a local maximum point is satisfied by assumption (24). From the strict concavity of u and V and the concavity of $\alpha, \beta_1, \ldots, \beta_d$ we know that $u(y,c) + D^c J(y)$ is strictly concave in c as well. Hence, $\phi^*(y)$ is both a local and a global maximum point. $\square$

Proof of Theorem 5. This proof is similar to the one presented in Øksendal (2003) for controlled diffusion processes. In analogy to part (i) of the proof of Theorem 4, we get for any admissible control C, $V(y) \geq W^C(y) + \lim_{t\to\infty} e^{-\tilde{\rho}_s(t)t} \tilde{E}_0 J(Y_t^{C,y})$. According to limiting inequality (26) the limit on the right-hand side is equal to or greater than 0. Thus, $V(y) \geq W^C(y)$. Since the control C was chosen arbitrarily and the class of Markov controls is included in the class of generalized admissible controls (and thus $V(y) \leq V^a(y)$), the theorem follows. $\square$

8. Conclusion

In a model of optimal control where the state variable is subject to random jumps driven by one or more independent Poisson processes we have presented rigorous proofs for both the necessity and the sufficiency of the HJB equation under milder conditions than before. We especially relax the assumption of bounded utility and coefficient functions. More precisely, it could be shown that the HJB equation is still a necessary condition for optimality if these functions are linearly bounded. On the other hand, apart from a

[21] An upper bound with finite expectation is given by $\int_0^\infty e^{-\tilde{\rho}_s(\tau)\tau} \left| u^{\phi}(Y_\tau^{\phi,y}) \right| d\tau$.

terminal condition, sufficiency could be derived even without requiring any boundedness condition at all.

Nevertheless, we required, at least in the necessity part, other underlying, extrinsic conditions to be satisfied, namely (i) (implicitly) the expected present value of the state process to be finite (see assumption (H4) and Lemma 1) and (ii) the value function to be once continuously differentiable with bounded first derivatives. Relaxing these issues is left for further research.

A. Appendix

A.1. Derivation of Remark 1 (ii). If there exist some g and t^* with $q_g(t^*) > 0$, the cádlág property of the boundedness coefficient q_g yields $Q_0(t) > 0$ for all $t \geq t^*$. Thus, for some $T > t^*$,

$$A \leq \int_0^T e^{-[\overline{\rho}_0(t)-P_0(t)]t}dt + \frac{1}{Q_0(T)}\int_T^\infty e^{-[\overline{\rho}_0(t)-P_0(t)]t}Q_0(t)\,dt \leq \int_0^T e^{-[\overline{\rho}_0(t)-P_0(t)]t}dt + \frac{B}{Q_0(T)},$$

and hence, due to (13), $A < \infty$.

A.2. The Poisson-diffusion setting - Proofs. In the present section we show where and how the proofs from Section 7 have to be adjusted in order to prove Theorem 6. First, we find that the assertions from the preparatory Section 4 carry over to the Poisson-diffusion setup if the following modifications are carried out. The CVF corresponding to the Poisson-diffusion SDE (27) reads as stated in the following theorem. It can be derived by "translating" the generalized Itô formula from Øksendal and Sulem (2005, Theorem 1.16) to the setup at hand.

THEOREM 7. *Let $X_t^{\phi,s,x}$ obey SDE (27). For a $\mathcal{C}^{1,2}$ - function $f : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ the process $\{f(t, X_t^{\phi,s,x}) : t \geq s\}$ is adapted and cádlág and follows*

$$\begin{aligned}
f\left(t, X_t^{\phi,s,x}\right) &= f(s,x) + \int_s^t \left[\begin{array}{c} f_t\left(\tau, X_\tau^{\phi,s,x}\right) + \alpha^\phi\left(\tau, X_\tau^{\phi,s,x}\right)\cdot f_x\left(\tau, X_\tau^{\phi,s,x}\right) \\ +\frac{1}{2}\sum\limits_{k=1}^{\tilde{d}} \sigma_l^\phi(\tau, X_\tau^{\phi,s,x})' f_{xx}\left(\tau, X_\tau^{\phi,s,x}\right)\sigma_l^\phi(\tau, X_\tau^{\phi,s,x}) \end{array}\right] d\tau \\
&\quad + \sum_{l=1}^{\tilde{d}} \int_s^t \sigma_l^\phi(\tau, X_\tau^{\phi,s,x})\cdot f_x\left(\tau, X_\tau^{\phi,s,x}\right) dB_\tau^l \\
&\quad + \sum_{k=1}^{d} \int_s^t \left[f\left(\tau, X_{\tau_-}^{\phi,s,x} + \beta_k^\phi(\tau, X_{\tau_-}^{\phi,s,x})\right) - f\left(\tau, X_{\tau_-}^{\phi,s,x}\right)\right] dN_\tau^k.
\end{aligned}$$

In the proof of Lemma 1, SDE (32), now bounding (27), has to be changed to

$$\begin{aligned} Z_t &= \|x\| + \int_s^t \left[p_\alpha(\tau) Z_\tau + q_\alpha(\tau)\right] d\tau + \sum_{l=1}^{\tilde{d}} \int_s^t \left[p_{\sigma_l}(\tau) Z_\tau + q_{\sigma_l}(\tau)\right] dB_\tau^l \\ &\quad + \sum_{k=1}^{d} \int_s^t \left[p_{\beta_k}(\tau_-) Z_{\tau_-} + q_{\beta_k}(\tau_-)\right] dN_\tau^k. \end{aligned} \tag{48}$$

According to Øksendal and Sulem (2005, Theorem 1.19), this linear SDE possesses a unique solution with finite second moment. Then, after taking expectation and using the martingale property of the Brownian motion, we see that the proof and hence the assertion of Lemma 1 remain unchanged. In the proof of Corollary 1 we find that $Z_t^{s,x}$ following (48) has still increasing paths since the boundedness coefficients of the diffusion components σ_l given by (29) are nonnegative. Thus Corollary 1 still holds true, as do Corollary 2 and Theorem 2 whose proofs do not need to be altered at all.

We now turn to the proof of Theorem 6. The time-state process $Y_t^{\phi,y}$ from (41) obeys here $Y_t^{\phi,y} = y + \int_0^t \tilde{\alpha}^\phi\left(Y_\tau^{\phi,y}\right) d\tau + \sum_{l=1}^{\tilde{d}} \int_s^t \tilde{\sigma}_l^\phi(Y_\tau^{\phi,y}) d\tilde{B}_\tau^l + \sum_{k=1}^{d} \int_0^t \tilde{\beta}_k^\phi\left(Y_\tau^{\phi,y}\right)_- d\tilde{N}_\tau^k$, where for each $l = 1, \ldots, \tilde{d}$, $\tilde{\sigma}_l^\phi(t,x) := \left(0, \sigma_l^\phi(t,x)\right)'$ and $\tilde{B}_\tau^l := B_{s+\tau}^l - B_s^l$ is a Brownian motion starting at s. The corresponding differential operator is given by $\tilde{D}^c$ in (28). If the conditions from Theorem 6 are satisfied, Lemma 3 holds true with $\tilde{D}^c$ instead of D^c. For its proof we now apply CVF from Theorem 7, and equation (45) changes to

$$\begin{aligned} &\frac{1}{h}\tilde{E}_0\left[e^{-\tilde{\rho}_s(h)(h)} V\left(Y_h^{\phi,y}\right) - V(y)\right] \\ =\ & \tilde{E}_0 \frac{1}{h}\int_0^h \left[\begin{array}{c} \alpha^\phi\left(Y_\tau^{\phi,y}\right) \cdot e^{-\tilde{\rho}_s(\tau)\tau} V_y\left(Y_\tau^{\phi,y}\right) + \frac{1}{2}\sum_{l=1}^{\tilde{d}} \tilde{\sigma}_l^\phi(Y_\tau^{\phi,y})' V_{yy}\left(Y_\tau^{\phi,y}\right)\tilde{\sigma}_l^\phi(Y_\tau^{\phi,y}) \\ -\rho(s+\tau) e^{-\tilde{\rho}_s(\tau)\tau} V(Y_\tau^{\phi,y}) \end{array}\right] d\tau \\ &+ \sum_{k=1}^{d} \lambda_k \tilde{E}_0 \frac{1}{h}\int_0^h \left[e^{-\tilde{\rho}_s(\tau)\tau} V\left(Y_{\tau_-}^{\phi,y} + \beta_k^\phi\left(Y_\tau^{\phi,y}\right)_-\right) - e^{-\tilde{\rho}_s(\tau)\tau} V(Y_{\tau_-}^{\phi,y})\right] d\tau. \end{aligned}$$

In order to be allowed to apply the theorem of bounded convergence we have to find an upper bound with finite expectation for the additionally obtained diffusion term. Using the linear boundedness (29) of the coefficient σ, we arrive for each $l = 1, \ldots, \tilde{d}$ at the upper bound

$$\begin{aligned} \frac{1}{h}\int_0^h \left[\frac{1}{2}\sum_{l=1}^{\tilde{d}} \tilde{\sigma}_l^\phi(Y_\tau^{\phi,y})' V_{yy}\left(Y_\tau^{\phi,y}\right)\tilde{\sigma}_l^\phi(Y_\tau^{\phi,y})\right] &\leq \sup_{\tau\in[0,1]} \frac{1}{2}\sum_{l=1}^{\tilde{d}} \left\|\tilde{\sigma}_l^\phi(Y_\tau^{\phi,y})\right\|^2 \|V_{yy}\| \\ \leq \sup_{\tau\in[0,1]} \frac{1}{2}\|V_{yy}\| \sum_{l=1}^{\tilde{d}} &\left[\begin{array}{c} \left\|p_{\sigma_l}^2\right\|_1 \sup_{\tau\in[0,1]}\left\|X_\tau^{\phi,s,x}\right\|^2 \\ +2\|p_{\sigma_l}\|_1 \|q_{\sigma_l}\|_1 \sup_{\tau\in[0,1]}\left\|X_\tau^{\phi,s,x}\right\| + \left\|q_{\sigma_l}^2\right\|_1 \end{array}\right], \end{aligned} \tag{49}$$

where $\|p_{\sigma_l}\|_1 := \sup_{\tau\in[0,1]} p_{\sigma_l}(s+\tau) < \infty$, $\|q_{\sigma_l}\|_1 := \sup_{\tau\in[0,1]} q_{\sigma_l}(s+\tau) < \infty$ and, by assumption, $\|V_{yy}\| := \sup_{(t,z)\in[0,\infty)\times\Theta} \|V_{yy}(t,z)\| < \infty$. From Corollary 1 we know that $E_0 \sup_{\tau\in[0,1]} \left\|X_\tau^{\phi,s,x}\right\| < \infty$. Finiteness of $E_0 \sup_{\tau\in[0,1]} \left\|X_\tau^{\phi,s,x}\right\|^2$ follows from $E_0 \sup_{\tau\in[0,1]} \left\|X_\tau^{\phi,s,x}\right\|^2 \leq E_0 Z_1^2$, which is according to the adapted proof of Corollary 1, and $E_s Z_1^2 < \infty$, as is mentioned above. The right-hand side in (49) is hence of finite expectation and the theorem of bounded convergence can be applied. The remaining part of this proof as well as the proof of Theorem 6 are then exactly as in the pure jump setting. We only have to add, whenever it is necessary, the quadratic diffusion term, which leads finally to $\tilde{D}^c$ instead of D^c.

A.3. Interchanging expectation and integral in (34). If we define the process $H_\tau := p_\alpha(\tau) Z_\tau + q_\alpha(\tau) + \sum_{k=1}^d \lambda_k \left[p_{\beta_k}(\tau) Z_\tau + q_{\beta_k}(\tau)\right]$, (34) reads $E_s Z_t^{s,x} = \|x\| + E_s \int_s^t H_\tau d\tau$. We may express the integral as a limit of Riemann sums by $\int_s^t H_\tau d\tau = \lim_{\Delta\to 0} \Delta \sum_{T=0}^{n_\Delta - 1} H_{s+T}$, where Δ is the length of the subintervals for an equidistant partition of the interval $[s,t]$ and n_Δ the number of these subintervals, i.e., $\Delta \cdot n_\Delta = t-s$. Now the problem of interchanging expectation and integral has been converted into a problem of interchanging expectation and limit. Here the theorem of bounded convergence comes into play. We have to find an upper bound with finite expectation for the absolute value of $\Delta \sum_{T=0}^{n_\Delta - 1} H_{s+T}$ that holds uniformly for all Δ small enough. Since the boundedness coefficients p_g and q_g, $g \in \{\alpha, \beta_1, \ldots, \beta_d\}$ are nonnegative, $Z^{s,x}$ is nonnegative, too, and has increasing paths. Therefore,

$$\begin{aligned}\left\|\Delta \sum_{T=0}^{n_\Delta-1} H_{s+T}\right\| &= \Delta \sum_{T=0}^{n_\Delta-1} H_{s+T} \\ &\leq (t-s)\left[\left(\|p_\alpha\|_{s,t} + \sum_{k=1}^d \lambda_k \left\|p_{\beta_k}\right\|_{s,t}\right) Z_t^{s,x} + \|q_\alpha\|_{s,t} + \sum_{k=1}^d \lambda_k \left\|q_{\beta_k}\right\|_{s,t}\right],\end{aligned}$$

where, for $g = \alpha, \beta_1, \ldots, \beta_d$, $\|p_g\|_{s,t} := \sup_{s\leq\tau\leq t} |p_g(\tau)| < \infty$ and $\|q_g\|_{s,t} := \sup_{s\leq\tau\leq t} |q_g(\tau)| < \infty$. Thus, since the right-hand side has clearly finite expectation, the theorem of bounded convergence allows to interchange expectation and limit, and we obtain

$$E_s Z_t^{s,x} = \|x\| + E_s \lim_{\Delta\to 0} \Delta \sum_{T=0}^{n_\Delta-1} H_{s+T} = \|x\| + \lim_{\Delta\to 0} \Delta \sum_{T=0}^{n_\Delta-1} E_s H_{s+T} = \|x\| + \int_s^t E_s H_\tau d\tau.$$

A.4. Interchanging expectation and integral in (36). Assuming $A(s)$ and $B(s)$ to be finite, we show how the theorems of monotone and bounded convergence can be used to interchange expectation and integral in $E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt$. At first, we consider the expectation of the finite horizon integral $\int_s^T e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt$. Here, in analogy to appendix A.3 and with the upper bound $(T-s)\,Z_T^{s,x}$, the theorem of bounded convergence yields together with (35)

$$E_s \int_s^T e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt = \int_s^T e^{-[\overline{\rho}_s(t)-P_s(t)](t-s)} \left[\|x\| + Q_s(t)\right] dt. \tag{50}$$

In the next step, we write $\int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt = \lim_{T\to\infty} \int_s^T e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt$. Since $\int_s^T e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt$ is increasing in T and since according to (50),

$$\begin{aligned} \sup_{T\geq s} E_s \int_s^T e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt &= \int_s^\infty e^{-[\overline{\rho}_s(t)-P_s(t)](t-s)} \left[\|x\| + Q_s(t)\right] dt \\ &= A(s)\,\|x\| + B(s) < \infty, \end{aligned}$$

the theorem of monotone convergence tells us that $\int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt$ possesses finite expectation and that

$$E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt = \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} E_s Z_t^{s,x} dt = A(s)\,\|x\| + B(s)\,.$$

Part 2

Uncertainty in Capital Income

CHAPTER 2

Keynes-Ramsey Rules in Continuous-Time Setups Under Lévy Uncertainty*

Abstract. Keynes-Ramsey rules for describing the optimal consumption behavior in a continuous time setup under uncertainty are "incomplete" in the sense that they provide only the evolution of the marginal utility process and not the evolution of the optimal consumption process itself. Only recently, "complete" Keynes-Ramsey rules have been derived in a setup with CRRA (constant relative risk aversion) utility functions and uncertainty caused by Brownian motion or Poisson processes. But these processes provide only a limited tool for modeling dynamic uncertainty. New results can be achieved by using Lévy processes. We show how the evolution of the optimal consumption process can be derived if uncertainty stems from a Lévy process and the consumption function is not necessarily of the CRRA type.

JEL classification: C61; D81; D91

Keywords: Optimal consumption; Lévy processes; Keynes-Ramsey rule

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1. Introduction

Understanding optimal consumption and saving behavior of households is central to understand many macroeconomic phenomena. For example, in models of economic growth the aggregate savings of the households in an economy yield the capital stock of that economy, see, e.g., Steger (2005) and Wälde (2005). In a dynamic setup, the most "valuable" result is a closed-form expression for the optimal consumption or saving rule, usually given by a Markov policy, i.e., by a function of current wealth and other state variables, cf., e.g., Merton (1971). But unfortunately, finding such an expression is rather the exception than the rule. Nevertheless, it is in most cases possible to derive a Keynes-Ramsey rule, which describes the evolution of the optimal consumption process over time. This rule is available for deterministic models in both continuous and discrete time (see, e.g., Cass, 1965, and Koopman, 1965) and for stochastic models in discrete time (see, e.g., de Hek, 1999). As will be discussed in turn, it has not yet been completely derived for stochastic models in continuous time. The present paper therefore provides a Keynes-Ramsey rule in a continuous-time setup with general utility and uncertainty stemming from a Lévy process, i.e., from a process with stationary and independent increments, thus including the special cases with Brownian motion and Poisson processes.

Keynes-Ramsey rules for stochastic continuous-time models provided in the economic literature are "incomplete" in the sense that they describe only the evolution of the marginal utility process induced by optimal consumption behavior and not the evolution of the optimal consumption process itself, see, e.g., Turnovsky (2000) and Wälde (1999, 2002). Only recently, Steger (2005) and Sennewald and Wälde (2006) presented "complete" Keynes-Ramsey rules assuming CRRA utility functions and uncertainty stemming from Brownian motion or Poisson processes. But in many cases, these processes represent a poor model for dynamic uncertainty in real data. For example, a frequently used model for asset price dynamics is the geometric Brownian motion, which implies normally distributed logreturns for all time spans in question. But whereas monthly logreturn data are fitted very well by a normal distribution, daily or weekly logreturns exhibit more mass around the origin and in the tails than the normal distribution can provide, see, e.g., Eberlein and Keller (1995). The authors show that the logreturns can be fitted much better by the hyperbolic distribution, and that therefore hyperbolic Lévy processes are more appropriate for modeling uncertainty of stock prices. Other authors as, e.g., Barndorff-Nielsen (1997), propose a similar class of Lévy processes — processes of the normal inverse Gaussian type — for a more realistic stock-price modeling. The contribution of the present paper lies in introducing Lévy processes and in providing a method that allows to deduce the evolution of the optimal consumption process from the evolution of the marginal utility process for arbitrary utility functions.

We consider a model in which the household's objective is to maximize his expected lifetime utility by choosing between consumption and investment in a risky asset. At first glance, this one-asset framework seems to be rather unrealistic. But introducing alternative investment opportunities, risky or risk-free, as in, e.g., Merton (1971) or Turnovsky (2000), and thus taking into account that households may react to changes in riskiness by reallocation among assets, does not change the results substantially. Furthermore, many types of macroeconomic uncertainty affect the returns of all assets, including risk-free assets as, e.g., a bank account or government bonds. Moreover, for an economy as a whole, real capital constitutes the only form of saving. Therefore the model presented here can also be understood as a model of growth under uncertainty. Here, again, Lévy processes may appear more adequate for modeling shocks in, e.g., capital or productivity, even though there are no empirical results on that issue available yet.

The paper is organized as follows. In the subsequent section we introduce the optimum-consumption problem. Section 3 contains the derivation of the Keynes-Ramsey rule. In Section 4 we consider some examples with special Lévy processes (Brownian motion, Poisson processes, and hyperbolic Lévy processes, which were proposed by Eberlein and Keller, 1995), and the results in Steger (2005) for the optimal growth of average consumption in a model of economic growth under Gaussian and Poisson uncertainty are reproduced. Section 5, finally, concludes.

2. The Setup

2.1. The budget constraint. Let the wealth of a household at time t, $a(t)$, be given by the number of stocks $n(t)$ it owns times their price $v(t)$. That is, $a(t) = n(t)\, v(t)$. Assume the price $v(t)$ to be a geometric Lévy process, following the stochastic differential equation

$$dv(t) = \mu v(t)\, dt + v(t_-)\, dx(t), \tag{1}$$

where $\mu \in \mathbb{R}$ and $x(t)$ is a Lévy process, i.e., $x(t)$ is cádlág and has stationary and independent increments.[1,2] Suppose that $x(t)$ has finite expectation, i.e., $E|x(t)| < \infty$, and that its jumps are not smaller than -1. The latter condition ensures that the asset price $v(t)$ always remains positive. As shown in Appendix A.1, the first condition, $E|x(t)| < \infty$, holds if and only if the logreturns have finite expectation. Thus, since asset prices exhibit usually bounded logreturns (see, e.g., Eberlein and Keller, 1995), assuming $E|x(t)| < \infty$ does not limit the scope of the model. The existence of the first moments of $x(t)$ is equivalent to the fact that $x(t)$ has not "too many big jumps", which in turn allows the following representation of $x(t)$ (see, e.g., Sato, 1999):

$$x(t) = E[x(1)]\,t + \tilde{z}(t) + \textstyle\int_{-1}^{\infty} \zeta \tilde{q}(t, d\zeta). \tag{2}$$

Here $\tilde{z}(t)$ denotes a Brownian motion and $\tilde{q}(t,\cdot) = q(t,\cdot) - t\lambda(\cdot)$ an independent compensated Poisson measure, where $q(t,\cdot)$ is the Poisson measure that, roughly speaking, counts the jumps of $x(t)$ and $\lambda(\cdot) = E[q(1,\cdot)]$ can be understood as the corresponding arrival rate of $q(t,\cdot)$. More precisely, given a Borel set $A \subset \mathbb{R}$, $q(t,A)$ counts the jumps of $x(t)$ with size $\zeta \in A$ that occur up to time t. Thus, $\lambda(A)$ represents the average number of jumps with size in A that occur per unit of time, and it is called the Lévy measure of $x(t)$.

Equation (2) shows that $x(t)$ can be decomposed into a sum consisting of a linear deterministic part, a Brownian motion, and a jump part, which is independent of the Brownian motion. The integral $\int_{-1}^{\infty} \zeta \tilde{q}(t,\zeta)$ may be rewritten as $\int_{-1}^{\infty} \zeta \tilde{q}(t,d\zeta) = \int_{-1}^{\infty} \zeta q(t,d\zeta) - t\int_{-1}^{\infty} \zeta \lambda(d\zeta)$, where $\int_{-1}^{\infty} \zeta q(t,d\zeta)$ represents the "sum" of all jumps the process $x(t)$ makes up to time t and $t\int_{-1}^{\infty} \zeta\lambda(d\zeta)$ is the expectation of that sum. Thus, unlike composite Poisson processes, Lévy processes allow for a continuum of jump sizes. Recall that, in order to ensure positive prices $v(t)$, we assumed only jumps of a size greater than -1, whereas in the general case $x(t)$ may also have jumps less than -1. If the jump size can only take a certain value $\alpha > -1$, we obtain the simple Poisson setup, i.e., $\int_{-1}^{\infty} \zeta \tilde{q}(t,d\zeta) = \alpha q(t) - \alpha\lambda t$, where $q(t)$ is a Poisson process with arrival rate λ.

The increments of the Brownian motion $\tilde{z}(t)$ have variance $Var[\tilde{z}(t) - \tilde{z}(s)] = (t-s)\sigma^2$, where the parameter $\sigma > 0$ depends on the underlying Lévy process, $x(t)$.

[1] A process is called cádlág if its paths are continuous from the right with left limits. The expression cádlág is an acronym from the french "continu á droite, limites á gauche".

[2] A detailed introduction into the topic of Lévy processes can be found in, e.g., Jacod and Shiryaev (1987), Protter (1995), or Sato (1999).

Hence, we may express $\tilde{z}(t)$ by $\tilde{z}(t) = \sigma z(t)$, where $z(t)$ denotes a standard Brownian motion with $Var[z(t) - z(s)] = t - s$.

Clearly, both the Brownian motion and the jump part are martingales. Thus, $E[x(t)] = E[x(1)]\,t$ grows linearly in time, and we can say that the paths of a Levy process are given by a linear deterministic trend with both continuous and jump disturbances that are zero in average. The variance of $x(t)$ is given by $Var[x(t)] = \left(\sigma^2 + \int_{-1}^{\infty} \zeta^2 \lambda(d\zeta)\right) t$.

Inserting the Lévy decomposition (2) into the price differential (1) and assuming without loss of generality the expectation of the Lévy process to be zero, i.e., $E[x(1)] = 0$,[3] we obtain

$$dv(t) = \mu v(t)\,dt + \sigma v(t)\,dz(t) + v(t_-)\int_{-1}^{\infty} \zeta \tilde{q}(dt, d\zeta). \tag{3}$$

Thus, the asset price grows in average with the deterministic rate μ and is disturbed by "white" and jump noise.

Let the household earn dividend payments $\pi(t)$ per unit of asset it owns and labor income $w(t)$. Assume furthermore that it spends $c(t)$ on consumption. When buying assets is the only way of saving, the number of stocks held by the household changes in a deterministic way according to

$$dn(t) = \frac{n(t)\pi(t) + w(t) - c(t)}{v(t)} dt. \tag{4}$$

Thus, when savings $n(t)\pi(t) + w(t) - p(t)c(t)$ are positive, the number of stocks held by the household increases by savings divided by the price of one stock. When savings are negative, the number of stocks decreases.

The change of the household's wealth, i.e., the household's budget constraint, is then simply given by applying the change-of-variables formula on $a(t) = n(t)v(t)$ and the differentials (3) and (4)[4]

$$\begin{aligned} da(t) &= v(t)\frac{n(t)\pi(t) + w(t) - c(t)}{v(t)}dt + \mu n(t)v(t)\,dt \\ &\quad + \sigma n(t)v(t)\,dz(t) + n(t_-)v(t_-)\int_{-1}^{\infty}\zeta\tilde{q}(dt,d\zeta) \\ &= [r(t)a(t) + w(t) - c(t)]\,dt + \sigma a(t)\,dz(t) + a(t_-)\int_{-1}^{\infty}\zeta\tilde{q}(dt,d\zeta), \end{aligned} \tag{5}$$

where the interest rate is defined as $r(t) \equiv \mu + \pi(t)/v(t)$. Budget constraint (5) shows that the general framework with Lévy processes includes also the special cases with Brownian motion and Poisson processes as considered in, e.g., Steger (2005) or, in a slightly

[3] If $E[x(1)] \neq 0$, we include $E[x(1)]$ in the drift parameter μ in (1), cf. also Subsection 4.2.

[4] The change-of-variables formula is given in Appendix A.2, Corollary 1.

different setup with an additional riskless asset, in Turnovsky (2000) and Sennewald and Wälde (2006).

2.2. The optimal control problem. Let the household derive utility from consumption with an instantaneous utility function $u(c)$ that is three times continuously differentiable. Starting at time $s \geq 0$ and given a fixed time preference rate $\rho > 0$ the household's objective consists in choosing an optimal consumption path that maximizes the expected lifetime utility,

$$U^{c(t)}(s, a(s)) = E \int_s^\infty e^{-\rho[t-s]} u(c(t)) dt$$

subject to budget constraint (5). Assume that an optimal consumption process $c^*(t)$ that is given by a Markov policy exists. That is, the optimal consumption expenditure at time t depends only on wealth at t and time t itself. Thus, the only way that uncertainty affects $c^*(t)$ is through $a(t)$ and we may express, if we find it convenient, $c^*(t)$ by $c^*(t, a(t))$.

The value function V is defined as the expected lifetime utility derived from the optimal consumption process. That is,

$$V(s, a(s)) \equiv E \int_s^\infty e^{-\rho[t-s]} u(c^*(t)) dt. \tag{6}$$

3. The Keynes-Ramsey rule

In this section we show, how starting with the Hamilton-Jacobi-Bellman (HJB) equation, one can derive a Keynes-Ramsey rule for the optimal control problem stated above. Assume that the value function V is twice continuously differentiable. Then, according to, e.g., Øksendal and Sulem (2005, Ch. 3), it solves the HJB equation[5]

$$\rho V(t, a) = \max_{c \geq 0} \left\{ \begin{array}{c} u(c) + V_t(t, a) + [r(t) a + w(t) - c] V_a(t, a) + \frac{1}{2}\sigma^2 a^2 V_{aa}(t, a) \\ + \int_{-1}^\infty [V(t, (1+\zeta) a) - V(t, a) - \zeta a V_a(t, a)] \lambda(d\zeta) \end{array} \right\}, \tag{7}$$

where V_t denotes the partial derivative of V with respect to the time argument t and V_a and V_{aa} the first and second derivatives, respectively, with respect to a. The maximum

[5]Similar to the problem for Poisson uncertainty described in Sennewald (2007) or Chapter 1 of this thesis, the boundedness condition for the instantaneous utility function and the differential coefficients still holds if one wishes to apply the HJB equation as a necessary condition under Lévy uncertainty. The reference mentioned here, Øksendal and Sulem (2005), actually only covers the sufficiency property of the HJB equation. A heuristic derivation of the HJB equation can be found in Appendix A.3.

in (7) is attained by the optimal Markov control $c^*(t,a)$. Assuming that c^* is an interior maximum, i.e., $c^* > 0$, the first-order condition is then found to be

$$u'(c^*) = V_a(t,a). \tag{8}$$

In the next step, we compute the evolution of the marginal value $V_a(t,a(t))$ evaluated along the optimally controlled wealth process $a(t)$. Since there is no risk of confusion in the following, we shall as from now write c instead of c^* for the optimal consumption expenditure. With budget constraint (5), the change-of-variables formula then yields[6]

$$\begin{aligned} dV_a(t,a(t)) &= V_{ta}(t,a(t))\,dt \\ &+ [r(t)a(t)+w(t)-c(t)]\,V_{aa}(t,a(t))\,dt + \frac{1}{2}\sigma^2a^2(t)\,V_{aaa}(t,a(t))\,dt \\ &+\sigma a(t)\,V_{aa}(t,a(t))\,dz(t) \\ &+ \textstyle\int_{-1}^{\infty}[V_a(t,(1+\zeta)a(t)) - V_a(a(t)) - \zeta a(t)\,V_{aa}(t,a(t))]\,\lambda(d\zeta)\,dt \\ &+ \textstyle\int_{-1}^{\infty}[V_a(t,(1+\zeta)a(t_-)) - V_a(t,a(t_-))]\,\tilde{q}(dt,d\zeta), \end{aligned} \tag{9}$$

where V_{aaa} stands for the third derivative of V with respect to a. On the other hand, the derivation of the maximized HJB equation (7) with respect to a yields according to the envelope theorem

$$\begin{aligned} \rho V_a(t,a) &= V_{ta}(t,a) + r(t)\,V_a(t,a) + [r(t)a + w(t) - c]\,V_{aa}(t,a) \\ &\quad +\sigma^2 aV_{aa}(t,a) + \frac{1}{2}\sigma^2a^2V_{aaa}(t,a) \\ &\quad +\frac{d}{da}\textstyle\int_{-1}^{\infty}[V(t,(1+\zeta)a) - V(t,a) - \zeta aV_a(t,a)]\,\lambda(d\zeta). \end{aligned}$$

Assuming that differentiation and integral on the right-most term on the right-hand hand side are interchangeable, rearranging leads to

$$\begin{aligned} &(\rho - r(t))\,V_a(t,a) - \sigma^2aV_{aa}(t,a) - \textstyle\int_{-1}^{\infty}\zeta\,[V_a(t,(1+\zeta)a) - V_a(t,a)]\,\lambda(d\zeta) \\ &\quad = V_{ta}(t,a) + [r(t)a + w(t) - c]\,V_{aa}(t,a) + \frac{1}{2}\sigma^2a^2V_{aaa}(a) \\ &\qquad + \textstyle\int_{-1}^{\infty}[V_a(t,(1+\zeta)a) - V_a(t,a) - \zeta aV_{aa}(t,a)]\,\lambda(d\zeta). \end{aligned}$$

Inserting this expression evaluated at $a(t)$ into the stochastic differential (9) yields

$$\begin{aligned} dV_a(a(t)) &= \left[\begin{array}{c} (\rho - r(t))\,V_a(t,a(t)) - \sigma^2a(t)\,V_{aa}(t,a(t)) \\ -\int_{-1}^{\infty}\zeta\,[V_a(t,(1+\zeta)a(t)) - V_a(t,a(t))]\,\lambda(d\zeta) \end{array}\right]dt \\ &\quad +\sigma a(t)\,V_{aa}(t,a(t))\,dz(t) + \textstyle\int_{-1}^{\infty}[V_a(t,(1+\zeta)a(t_-)) - V_a(t,a(t_-))]\,\tilde{q}(dt,d\zeta). \end{aligned}$$

[6] For the change-of-variables formula see Corollary 1 in Appendix A.2.

Now replacing V_a with u', according to the first-order condition (8), and V_{aa} with $u''(c)\,c_a$ (which is also due to (8) and where $c_a \equiv \partial c(t,a)/\partial a$ stands for the marginal prospensity to consume), we arrive at the Keynes-Ramsey rule in the "traditional" form as derived in, e.g., Turnovsky (2000) for Gaussian uncertainty (and a slightly different setup in which also a riskless asset is available)

$$\begin{aligned}\frac{du'(c(t))}{u'(c(t_-))} &= \left\{\rho - r(t) - \frac{u''(c(t))}{u'(c(t))}c_a(t)\,a(t)\,\sigma^2 - \int_{-1}^{\infty}\zeta\left[\frac{u'(c_\zeta(t))}{u'(c(t))} - 1\right]\lambda(d\zeta)\right\}dt \\ &\quad + \frac{u''(c(t))}{u'(c(t))}c_a(t)\,a(t)\,\sigma dz(t) + \int_{-1}^{\infty}\left[\frac{u'(c_\zeta(t_-))}{u'(c(t_-))} - 1\right]\tilde{q}(dt,d\zeta). \qquad (10)\end{aligned}$$

Here $c_\zeta(t)$ denotes the optimal consumption expenditure when a jump in wealth by ζ percent has occurred, i.e., $c_\zeta(t) = c(t,(1+\zeta)\,a(t))$. In this version of the Keynes-Ramsey rule, the optimal consumption behavior is given implicitly by the evolution of the marginal utility. In the following we derive from (10) the evolution of the optimal consumption process itself. To this end, we only need to apply the change-of-variables formula from Corollary 1 in Appendix A.2 on the mapping $y \mapsto (u')^{-1}(y)$, where for y we insert the marginal utility $u'(c(t))$ whose evolution is given by (10). That yields

$$\begin{aligned}dc(t) &= -\frac{1}{u''(c(t))}\left\{\begin{array}{c}[r(t)-\rho]\,u'(c(t)) \\ +\left[\frac{1}{2}u'''(c(t))\,c_a(t)\,a(t) + u''(c(t))\right]c_a(t)\,a(t)\,\sigma^2 \\ -\int_{-1}^{\infty}(1+\zeta)\left[u'(c(t)) - u'(c_\zeta(t))\right]\lambda(d\zeta)\end{array}\right\}dt \\ &\quad + \int_{-1}^{\infty}\left[c_\zeta(t_-) - c(t_-)\right]\lambda(d\zeta)\,dt + c_a(t)\,a(t)\,\sigma dz(t) + \int_{-1}^{\infty}\left[c_\zeta(t_-) - c(t_-)\right]\tilde{q}(dt,d\zeta),\end{aligned}$$

where we used that according to the rule $df^{-1}(f(x))/df(x) = 1/f'(x)$,

$$\frac{d(u')^{-1}(u(c))}{du(c)} = \frac{1}{u''(c)} \quad \text{and} \quad \frac{d^2(u')^{-1}(u(c))}{[du(c)]^2} = -\frac{u'''(c)}{[u''(c)]^3}.$$

If we denote the Arrow-Pratt measure for relative risk aversion by

$$\theta(t) \equiv -\frac{u''(c(t))}{u'(c(t))}c(t)$$

and the measure for relative prudence by

$$\eta(t) \equiv -\frac{u'''(c(t))}{u''(c(t))}c(t),$$

rearranging leads to

$$\begin{aligned}
\theta(t_-)\frac{dc(t)}{c(t_-)} &= [r(t)-\rho]\,dt + \left[\frac{1}{2}\eta(t)\frac{c_a(t)\,a(t)}{c(t)} - 1\right]\theta(t)\frac{c_a(t)\,a(t)}{c(t)}\sigma^2 dt \\
&\quad - \textstyle\int_{-1}^{\infty}(1+\zeta)\left[1-\frac{u'(c_\zeta(t))}{u'(c(t))}\right]\lambda(d\zeta)\,dt + \theta(t)\frac{c_a(t)\,a(t)}{c(t)}\sigma dz(t) \\
&\quad + \theta(t_-)\textstyle\int_{-1}^{\infty}\left[\frac{c_\zeta(t_-)}{c(t_-)} - 1\right] q(dt,d\zeta), \qquad (11)
\end{aligned}$$

Unlike (10), this version of the Keynes-Ramsey rule describes explicitly the evolution of the optimal consumption process. In a deterministic setup, the left-hand side must be equal to the first term on the right-hand side, see, e.g., Barro and Sala-i-Martín (1995). The other terms on the right-hand side are thus due to the uncertainty introduced by the Lévy process $x(t)$. More precisely, the second and fourth term appear because of the diffusion part, whereas the third and fifth term stem from the jump part of the Lévy process.

Keynes-Ramsey rule (11) shows that some interpretations from the deterministic setup carry over to the stochastic environment: Assuming that u satisfies the usual conditions, i.e., $u' > 0$ and $u'' < 0$, we can conclude that the higher the interest rate $r(t)$ or the lower the time preference rate ρ the more consumption the household sacrifices today for consumption tomorrow, i.e., $dc(t)/c(t_-)$ goes up.

If the utility function is of the CRRA type, given by

$$u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}, \quad \gamma > 0, \gamma \neq 1,^{7} \qquad (12)$$

the measures for relative risk aversion and prudence are constants and read $\theta = \gamma$ and $\eta = 1+\gamma$, respectively. Then Keynes-Ramsey rule (11) reads

$$\begin{aligned}
\frac{dc(t)}{c(t_-)} &= \left\{\frac{r(t)-\rho}{\gamma} - \left[1-\frac{1}{2}(1+\gamma)\frac{c_a(t)\,a(t)}{c(t)}\right]\frac{c_a(t)\,a(t)}{c(t)}\sigma^2\right\} dt \\
&\quad - \frac{1}{\gamma}\textstyle\int_{-1}^{\infty}(1+\zeta)\left[1-\left(\frac{c_\zeta(t)}{c(t)}\right)^{-\gamma}\right]\lambda(d\zeta)\,dt \\
&\quad + \frac{c_a(t)\,a(t)}{c(t)}\sigma dz(t) + \textstyle\int_{-1}^{\infty}\left[\frac{c_\zeta(t_-)}{c(t_-)} - 1\right] q(dt,d\zeta). \qquad (13)
\end{aligned}$$

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[7] The following also applies to the special case $\gamma \to 1$, i.e. $u(c) = \log c$.

Taking expectation on the latter differential yields the expected growth rate of optimal consumption

$$\begin{aligned}\frac{dEc(t)/dt}{c(t)} &= \frac{r(t)-\rho}{\gamma}-\left[1-\frac{1}{2}(1+\gamma)\frac{c_a(t)a(t)}{c(t)}\right]\frac{c_a(t)a(t)}{c(t)}\sigma^2 \\ &\quad -\frac{1}{\gamma}\int_{-1}^{\infty}\left\{(1+\zeta)\left[1-\left(\frac{c_\zeta(t)}{c(t)}\right)^{-\gamma}\right]-\gamma\left[\frac{c_\zeta(t)}{c(t)}-1\right]\right\}\lambda(d\zeta). \quad (14)\end{aligned}$$

Here we used the martingale property of the Brownian motion $z(t)$ and the compensated Poisson measure $\tilde{q}(t,\cdot)$, cf. Subsection 2.1. If the flow of labor income w is equal to zero, one can show that the optimal consumption expenditure is proportional to current wealth. In this case, using that then

$$\frac{c_\zeta(t)}{c(t)}=\frac{(1+\zeta)a(t)}{a(t)}=1+\zeta,$$

Keynes-Ramsey rule (13) and the expected consumption growth rate (14) read

$$\begin{aligned}\frac{dc(t)}{c(t_-)} &= \left[\frac{r(t)-\rho}{\gamma}-\frac{1}{2}(1-\gamma)\sigma^2\right]dt-\frac{1}{\gamma}\int_{-1}^{\infty}\left[(1+\zeta)-(1+\zeta)^{1-\gamma}-\gamma\zeta\right]\lambda(d\zeta)\,dt \\ &\quad +\sigma dz(t)+\int_{-1}^{\infty}\zeta\tilde{q}(dt,d\zeta)\end{aligned}$$

and

$$\frac{dEc(t)/dt}{c(t)}=\left[\frac{r(t)-\rho}{\gamma}-\frac{1}{2}(1-\gamma)\sigma^2\right]dt-\frac{1}{\gamma}\int_{-1}^{\infty}\left[(1+\zeta)-(1+\zeta)^{1-\gamma}-\gamma\zeta\right]\lambda(d\zeta)\,dt. \quad (15)$$

Equation (15) allows us analyze the effects of risk on the average consumption growth. Recall that the variance of the underlying stochastic process, i.e., our measure of risk, is given by $Var[x(t)]=\left(\sigma^2+\int_{-1}^{\infty}\zeta^2\lambda(d\zeta)\right)t$. Then, looking at Equation (15), it is easy to see that an increased risk due to a rise in the variance of the Brownian-motion part, σ^2, leads to lower (higher) consumption growth if the relative risk aversion γ is less (greater) than 1, whereas consumption growth remains unaffected in the log-utility case with $\gamma=1$. As is shown in Appendix A.4, the same result holds true if risk increases due to a higher variance of the jump part, $\int_{-1}^{\infty}\zeta^2\lambda(d\zeta)$. Notice that while rising the variance of $x(t)$, the expected value remains unchanged. Hence, the well-known results on the effects of capital risk on the average consumption growth obtained for different setups with, e.g., Brownian motion and Poisson processes (cf. Steger, 2005 and Sennewald and Wälde, 2006) or generally distributed increments in discrete time (cf. de Hek, 1999) carry over to the continuous-time case with Lévy uncertainty. The intuitive argumentation behind these findings is standard. On the one hand, higher risk makes the consumer less willing to save since he faces a higher probability of loss. This is the so called substitution effect. On the other hand, increased risk leads to more saving in order to protect oneself against very low levels of future consumption. This is called the income effect. In case of low risk aversion (i.e., $\gamma < 1$) the substitution effect dominates the income effect, whereas the opposite holds true if risk aversion is high (i.e., $\gamma > 1$). Only in case of log-utility ($\gamma = 1$) both effects are balanced.

4. Examples

4.1. Geometric Brownian motion. As mentioned before, the Lévy-process framework covers the special case in which the asset price is given as a geometric Brownian motion, i.e., $dv(t) = \mu v(t)\,dt + \sigma v(t)\,dz(t)$. Under this assumption the budget constraint turns out to be

$$da(t) = [r(t)\,a(t) + w(t) - c(t)]\,dt + \sigma a(t)\,dz(t),$$

and the corresponding Keynes-Ramsey rule reads (cf. Equation (11))

$$\theta(t)\frac{dc(t)}{c(t)} = [r(t) - \rho]\,dt + \left[\frac{1}{2}\eta(t)\frac{c_a(t)\,a(t)}{c(t)} - 1\right]\frac{c_a(t)\,a(t)}{c(t)}\sigma^2 dt + \theta(t)\frac{c_a(t)\,a(t)}{c(t)}\sigma dz(t),$$

or, for the CRRA utility function (12),

$$\frac{dc(t)}{c(t)} = \left\{\frac{r(t)-\rho}{\gamma} - \left[1 - \frac{1}{2}(1+\gamma)\frac{c_a(t)\,a(t)}{c(t)}\right]\frac{c_a(t)\,a(t)}{c(t)}\sigma^2\right\}dt + \frac{c_a(t)\,a(t)}{c(t)}\sigma dz(t).$$

Taking expectation on the latter stochastic differential gives the average consumption growth rate

$$\frac{dEc(t)/dt}{c(t)} = \frac{r(t)-\rho}{\gamma} - \left[1 - \frac{1}{2}(1+\gamma)\frac{c_a(t)\,a(t)}{c(t)}\right]\frac{c_a(t)\,a(t)}{c(t)}\sigma^2.$$

If labor income w is equal to zero, this growth rate simplifies to

$$\frac{dEc(t)/dt}{c(t)} = \frac{r(t)-\rho}{\gamma} - \frac{1}{2}(1-\gamma)\,\sigma^2,$$

which, with a slightly different notation, coincides with Steger's (2005) Equation (5).

4.2. Poisson processes. Assume that the asset price evolves deterministically with jumps at random times according to

$$dv(t) = \mu v(t)\,dt + \alpha v(t_-)\,dq_1(t) - \beta v(t_-)\,dq_2(t), \tag{16}$$

where $\alpha \geq 0$, $0 \leq \beta < 1$, and q_1 and q_2 are Poisson processes with arrival rates λ_1 and λ_2, respectively. Now the price grows continuously with the rate μ, and at random times it jumps upwards by α percent (when $dq_1(t) = 1$) or downwards by β percent (when $dq_2(t) = 1$). To achieve a martingale structure as in differential (3), we rewrite (16) to

$$dv(t) = (\mu + \lambda_1\alpha - \lambda_2\beta) v(t) dt + \alpha v(t_-) d\tilde{q}_1(t) - \beta v(t_-) d\tilde{q}_2(t),$$

where $\tilde{q}_i(t) = q_i(t) - \lambda_i t$, $i = 1, 2$, denote the compensated Poisson processes. Then, the budget constraint reads

$$da(t) = [ra(t) + w(t) - c(t)] dt + a(t_-) [\alpha d\tilde{q}_1(t) - \beta d\tilde{q}_2(t)],$$

with $r = \mu + \lambda_1\alpha - \lambda_2\beta$ and where for sake of simplicity dividend payments are not taken into account. In analogy to (11), optimal consumption growth must then obey Keynes-Ramsey rule

$$\begin{aligned} \theta(t_-)\frac{dc(t)}{c(t_-)} = {} & (\mu - \rho) dt - \lambda_1(1+\alpha)\left[1 - \frac{u'(c_\alpha(t))}{u'(c(t))}\right] dt + \lambda_2(1-\beta)\left[\frac{u'(c_{-\beta}(t))}{u'(c(t))} - 1\right] dt \\ & + \theta(t_-)[c_\alpha(t_-) - c(t_-)] dq_1(t) - \theta(t_-)[c(t_-) - c_{-\beta}(t_-)] dq_2(t), \end{aligned}$$

where $c_\alpha(t)$ and $c_{-\beta}(t)$ denote the optimal consumption expenditures if the asset price at t jumps upwards ($dq_1(t) = 1$) and downwards ($dq_2(t) = 1$), respectively. In case of CRRA utility function (12) the Keynes-Ramsey rule reads

$$\begin{aligned} \frac{dc(t)}{c(t_-)} = {} & \frac{\mu-\rho}{\gamma} dt - \frac{\lambda_1(1+\alpha)}{\gamma}\left[1 - \left(\frac{c_\alpha(t)}{c(t)}\right)^{-\gamma}\right] dt + \frac{\lambda_2(1-\beta)}{\gamma}\left[\left(\frac{c_{-\beta}(t)}{c(t)}\right)^{-\gamma} - 1\right] dt \\ & + \left[\frac{c_\alpha(t_-)}{c(t_-)} - 1\right] dq_1(t) - \left[1 - \frac{c_{-\beta}(t_-)}{c(t_-)}\right] dq_2(t). \end{aligned}$$

Setting $w = 0$, assuming that the jumps are symmetric, i.e., $\lambda_1 = \lambda_2$ and $\alpha = \beta$, and taking expectation, we find further

$$\frac{dEc(t)/dt}{c(t)} = \frac{\mu-\rho}{\gamma} - \lambda_1 \frac{2 - (1+\alpha)^{1-\gamma} - (1-\alpha)^{1-\gamma}}{\gamma}$$

which, only with different notation, is the same result as derived by Steger (2005, Eq. (7)).

4.3. Hyperbolic Lévy motion. Let $x(t)$ be a centered and symmetric hyperbolic Lévy process. That is, the increments of $x(t)$ are hyperbolic distributed symmetrically around zero. Then x is a pure jump martingale with representation $x(t) = \int_0^t \int_{\mathbb{R}\setminus\{0\}} \zeta [q(dt, d\zeta) - \lambda(d\zeta) dt]$, cf. Eberlein and Keller (1995). The Lévy measure λ

has density[8]

$$g(\zeta) = \frac{1}{\pi^2 |\zeta|} \int_0^\infty \frac{e^{-|\zeta|\sqrt{2y+(\phi/\delta)^2}}}{y\left[J_1^2\left(\delta\sqrt{2y}\right) + Y_1^2\left(\delta\sqrt{2y}\right)\right]} dy + \frac{e^{-|\zeta|}}{|\zeta|},$$

where J_1 and Y_1 denote the Bessel functions with index 1 of the first and second kind, respectively. Notice that $x(t)$ may have jumps minor than -1, so that the asset price $v(t)$ may become negative. We therefore stop the decision problem at the time the asset price falls below -1 for the first time, i.e., at $T \equiv \inf\{t > 0 : \Delta x(t) < -1\}$. The household maximizes then $U^{c(t)}(s, a(s)) = E_s \int_s^T e^{-\rho[t-s]} u(c(t)) dt$ and, as long as $t < T$, the Keynes-Ramsey rule reads[9]

$$\begin{aligned} \theta(t_-) \frac{dc(t)}{c(t_-)} &= [r(t) - \rho] dt - \int_{-1}^\infty (1+\zeta) \left[1 - \frac{u'(c_\zeta(t))}{u'(c(t))}\right] g(\zeta) d\zeta dt \\ &\quad + \theta(t_-) \int_{-1}^\infty \left[\frac{c_\zeta(t_-)}{c(t_-)} - 1\right] q(dt, d\zeta). \end{aligned}$$

For CRRA utility function (12) this Keynes-Ramsey rule becomes

$$\begin{aligned} \frac{dc(t)}{c(t_-)} &= \frac{r(t) - \rho}{\gamma} dt - \frac{1}{\gamma} \int_{-1}^\infty (1+\zeta) \left[1 - \left(\frac{c_\zeta(t)}{c(t)}\right)^{-\gamma}\right] g(\zeta) d\zeta dt \\ &\quad + \int_{-1}^\infty \left[\frac{c_\zeta(t_-)}{c(t_-)} - 1\right] q(dt, d\zeta). \end{aligned}$$

Taking expectation yields the average consumption growth rate

$$\frac{dEc(t)/dt}{c(t)} = \frac{r(t) - \rho}{\gamma} - \frac{1}{\gamma} \int_{-1}^\infty \left\{(1+\zeta)\left[1 - \left(\frac{c_\zeta(t)}{c(t)}\right)^{-\gamma}\right] - \gamma \left[\frac{c_\zeta(t)}{c(t)} - 1\right]\right\} g(\zeta) d\zeta.$$

and for $w = 0$ we finally obtain

$$\frac{dEc(t)/dt}{c(t)} = \frac{r(t) - \rho}{\gamma} - \frac{1}{\gamma} \int_{-1}^\infty \left[(1+\zeta) - (1+\zeta)^{1-\gamma} - \gamma\zeta\right] g(\zeta) d\zeta.$$

5. Conclusion

In a continuous-time setup and under Lévy uncertainty we have derived a version of the Keynes-Ramsey rule that describes explicitly the evolution of the optimal consumption process, given any arbitrary sufficiently smooth utility function. Lévy processes allow a much more realistic modelling of stock prices. From the general Lévy framework we have drawn the special cases with Brownian motion, Poisson processes, and processes

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[8] That is, $\lambda(d\eta) = g(\eta) d\eta$.

[9] According to Øksendal and Sulem (2005, Ch. 3), the HJB equation does not change by introducing a stopping time, such as our T. The Keynes-Ramsey rule can therefore be derived as shown in Section 3.

with hyperbolic distributed increments, and the results for the average evolution of optimal consumption in a growth model under uncertainty from Steger (2005) have been reproduced.

The model presented in this paper can be applied easily to the analysis of economic growth under uncertainty. Here also, Lévy processes may be more suitable for modeling dynamic uncertainty as, e.g., shocks in productivity. To find out which class of Lévy processes fits best to real data is left for further (empirical) research.

A. Appendix

A.1. Logreturns and noise with finite expectation. We prove that $E\left|x\left(t\right)\right|<\infty$ iff $E\log v\left(t\right)<\infty$. Recall that the asset price $v\left(t\right)$ obeys the stochastic differential equation (1). If we define the process $y\left(t\right)\equiv\mu t+x\left(t\right)$, which is also a Lévy process, we may rewrite (1) to $dv\left(t\right)=v\left(t_{-}\right)dy\left(t\right)$. Then, since $v\left(t\right)$ is positive by construction, we know from Goll and Kallsen (1999, Lemma 5.8) that $v\left(t\right)$ is an exponential Lévy process, i.e., there exists a Lévy process $\tilde{y}\left(t\right)$ with $v\left(t\right)=v_0e^{\tilde{y}\left(t\right)}$. This in turn allows, again with Goll and Kallsen (1999, Lemma 5.8), to conclude that $\tilde{y}\left(t\right)$ has finite expectation if and only if $x\left(t\right)$ has finite expectation.

A.2. The change-of-variables formula (Itô's formula) for Lévy processes. The following change-of-variables formula is taken from Gihman and Skorohod (1972, p.128).

THEOREM 1. *Let for $k=1,\ldots,d$ the process $y_k\left(t\right)$ exhibits the following decomposition:*

$$y_k\left(t\right)=y_k\left(0\right)+\alpha_k\left(t\right)+\beta_k\left(t\right)+\gamma_k\left(t\right),$$

where $\alpha_k\left(t\right)$ is a deterministic and continuous process, $\beta_k\left(t\right)$ a continuous martingale, and $\gamma_k\left(t\right)$ a stochastic pure-jump process with Lévy measure λ. More precisely, $\gamma_k\left(t\right)$ is given by

$$\gamma_k\left(t\right)=\int_0^t\int_{\mathbb{R}}\phi_k\left(s_{-},\zeta\right)\tilde{q}\left(ds,d\zeta\right)=\int_0^t\int_{\mathbb{R}}\phi_k\left(s_{-},\zeta\right)q\left(ds,d\zeta\right)-\int_0^t\int_{\mathbb{R}}\phi_k\left(s,\zeta\right)\lambda\left(d\zeta\right)ds,$$

where the measure $q\left(t,\cdot\right)$ does not depend on the martingales $\beta_k\left(t\right)$ and $\phi_k\left(s,\zeta\right)$ is a integrable, cádlág process that depends on the jumps size ζ. If $f:[0,\infty)\times\mathbb{R}^d\rightarrow\mathbb{R}$ is a *twice continuously differentiable function, the process* $f(t, y(t))$ *obeys*

$$\begin{aligned} df(t, y(t)) &= f_t(t, y(t))\,dt + \sum_{k=1}^{d} f_{y_k}(t, y(t))\,d\left[\alpha_k(t) + \beta_k(t)\right] \\ &\quad + \frac{1}{2}\sum_{k,l=1}^{d} f_{y_k y_l}(t, y(t))\,\langle \beta_k(t), \beta_l(t)\rangle \\ &\quad + \int_{\mathbb{R}^d} \begin{bmatrix} f(t, y(t) + \phi(t,\zeta)) - f(t, y(t)) \\ -\sum_{k=1}^{d} f_{y_k}(t, y(t))\,\phi_k(t,\zeta) \end{bmatrix} \lambda(d\zeta)\,dt \\ &\quad + \int_{\mathbb{R}^d} \left[f(t, y(t_-) + \phi(t_-,\zeta)) - f(t, y(t_-))\right] \tilde{q}(dt, d\zeta). \end{aligned}$$

Here $\langle \beta_k(t), \beta_l(t)\rangle$ *stands for the quadratic covariation process of* $\beta_k(t)$ *and* $\beta_l(t)$*,* f_t *and* f_{y_k} *for the partial derivatives of* f *with respect to* t *and* y_k*, and* ϕ *is the vector function consisting of the components* $\phi_1, \ldots, \phi_d$*.*

Using that the quadratic covariation is bilinear and that for two independent standard Brownian motions $z_1(t)$ and $z_2(t)$, $\langle z_i(t), z_j(t)\rangle = \delta_{ij} t$, $i, j = 1, 2$, we deduce with a slight change of the notation the following corollary.

Corollary 1. *Let* $y(t)$ *be a* d*-dimensional stochastic process whose components obey*

$$dy_k(t) = \alpha_k(t, y(t))\,dt + \beta_k(t, y(t))\,dz_k(t) + \int_{\mathbb{R}} \gamma_k(t_-, y(t_-), \zeta)\,\tilde{q}(dt, d\zeta).$$

If the function f *is given as above, then*

$$\begin{aligned} df(t, y(t)) &= f_t(t, y(t))\,dt + \sum_{k=1}^{d} f_{y_k}(t, y(t))\,\alpha_k(t, y(t))\,dt \\ &\quad + \frac{1}{2}\sum_{k,l=1}^{d} f_{y_k y_l}(t, y(t))\,\beta_k(t, y(t))\,\beta_l(t, y(t))\,dt \\ &\quad + \sum_{k=1}^{d} f_{y_k}(t, y(t))\,\beta_k(t, y(t))\,dz_k(t) \\ &\quad + \int_{\mathbb{R}^d} \begin{bmatrix} f(t, y(t) + \gamma(t, y(t), \zeta)) - f(t, y(t)) \\ -\sum_{k=1}^{d} f_{y_k}(t, y(t))\,\gamma_k(t, y(t_-), \zeta) \end{bmatrix} \lambda(d\zeta)\,dt \\ &\quad + \int_{\mathbb{R}^d} \left[f(t, y(t_-) + \gamma(t_-, y(t_-), \zeta)) - f(t, y(t_-))\right] \tilde{q}(dt, d\zeta), \end{aligned}$$

where $\gamma = (\gamma_1, \ldots, \gamma_d)^T$*.*

A.3. A heuristic derivation of the HJB equation. This appendix shows how HJB equation (7) can be heuristically derived. As a starting point, we write the HJB equation in the general form as

$$\rho V(t, a(t)) = \max_{c(t)} \left\{ u(c(t)) + \frac{1}{dt} E_t dV(t, a(t)) \right\}, \tag{17}$$

where the maximum is achieved by the optimal consumption choice $c^*(t)$, and V denotes the value function (6). The general HJB equation (17) says that the household chooses consumption in t such that she maximizes her instantaneous return from consumption, which consists of the instantaneous utility flow $u(c(t))$ plus the expected change $\frac{1}{dt} E_t dV(t, a(t))$ in the value of wealth corresponding to the consumption choice in t. It tells furthermore that the intertemporal return $\rho V(t, a(t))$ from holding $a(t)$ is given by the return from the optimal consumption in t, $u(c^*(t)) + \frac{1}{dt} E_t dV(t, a(t))$. We see that, when determining the optimal behavior at t, the household only needs to consider the value function at t and its expected change in order to cover future behavior. This is a direct result of Bellman's principle of optimality, see, e.g., Bellman (1957).

Assume that V is once continuously differentiable. Obtaining the HJB equation for a specific maximization problem then requires (i) application of CVF on $V(t, a(t))$, (ii) computing expectations and (iii) "dividing" by dt. With budget constraint (5), CVF from Corollary 1 in Appendix A.2 yields

$$\begin{aligned} dV(t, a(t)) &= V_t(t, a(t))\, dt \\ &\quad + [r(t)\, a(t) + w(t) - c(t)]\, V_a(t, a(t))\, dt + \frac{1}{2}\sigma^2 a^2(t)\, V_{aa}(t, a(t))\, dt \\ &\quad + \sigma a(t)\, V_a(t, a(t))\, dz(t) \\ &\quad + \textstyle\int_{-1}^{\infty} [V(t, (1+\zeta)\, a(t)) - V(a(t)) - \zeta a(t)\, V_a(t, a(t))]\, \lambda(d\zeta)\, dt \\ &\quad + \textstyle\int_{-1}^{\infty} [V(t, (1+\zeta)\, a(t_-)) - V(t, a(t_-))]\, \tilde{q}(dt, d\zeta), \end{aligned} \tag{18}$$

where V_x and V_{xy} denote the partial derivatives of the value function. Using that, analogously to Poisson processes, the expected value of the compensated jumps of size ζ, $\tilde{q}(t, \zeta) = q(t, \zeta) - t\lambda(\zeta)$, is equal to zero, we find that the expected value of the last term in (18) is zero, too. Also, the $dz(t)$-term becomes zero when taking expectation. The expected value of (18) hence reads

$$\begin{aligned} E_t d(t, a(t)) &= V_t(t, a(t))\, dt \\ &\quad + \{[r(t)\, a(t) + w(t) - c(t)]\, V_a(t, a(t)) + \frac{1}{2}\sigma^2 a^2(t)\, V_{aa}(t, a(t))\} dt \\ &\quad + \textstyle\int_{-1}^{\infty} [V(t, (1+\zeta)\, a(t)) - V(a(t)) - \zeta a(t)\, V_a(t, a(t))]\, \lambda(d\zeta)\, dt. \end{aligned}$$

Dividing by dt gives and inserting into (17) finally yields HJB equation (7).

A.4. The effects of jump risk on the optimal consumption growth. Recall that the variance of the Lévy process $x(t)$ is given by $Var[x(t)] = \left(\sigma^2 + \int_{-1}^{\infty} \zeta^2 \lambda(d\zeta)\right) t$. Thus, an increase of risk in the jump part may be due to a rise in the "arrival rate" λ or due to higher jump sizes ζ. We shall show that in both cases the usual results on the effect of risk hold true, namely that for low risk aversion ($\gamma < 1$) consumption grows more slowly, for high risk aversion ($\gamma > 1$) consumption grows faster, and for $\gamma = 1$ risk has no impact on the consumption growth rate at all.

A.4.1. *Increasing* λ. A rise in $\lambda(\zeta)$ means that jumps with size ζ occur with a higher intensity. To see how this affects the optimal consumption growth, we consider Keynes-Ramsey rule (15). Then we find that the sign of the term $g(\gamma) \equiv -1/\gamma[(1+\zeta) - (1+\zeta)^{1-\gamma} - \gamma\zeta]$ determines whether a higher $\lambda(\zeta)$ has a positive or negative impact on the consumption growth. We show that

$$g(\gamma) \begin{cases} < 0 \text{ for } \gamma < 1; \\ = 0 \text{ for } \gamma = 1; \\ > 0 \text{ for } \gamma < 1. \end{cases} \tag{19}$$

Since for $\gamma = 1$, $g(\gamma)$ obviously becomes zero, this statement is true if the derivative of g,

$$g'(\gamma) = \frac{1}{\gamma^2}\left[(1+\zeta) - (1+\zeta)^{1-\gamma} - \gamma\zeta\right] - \frac{1}{\gamma}\left[(1+\zeta)^{1-\gamma}\ln(1+\zeta) - \zeta\right],$$

is negative. Collecting terms and rearranging shows that this holds if and only if $\ln(1+\zeta)^{\gamma} < (1+\zeta)^{\gamma} - 1$, which in turn is satisfied if and only if $(1+\zeta)^{\gamma} < \exp^{(1+\zeta)^{\gamma}-1}$. Now, using the power series representation of the exponential function, we can conclude that the latter inequality is true if and only if $0 < \sum_{n=2}^{\infty} \frac{[(1+\zeta)^{\gamma}-1]^n}{n!}$, which in turn can be rewritten to

$$0 < \sum_{k=2}^{\infty} \frac{\left\{[(1+\zeta)^{\gamma}-1]^2\right\}^k}{(2k)!}\left[1 + \frac{[(1+\zeta)^{\gamma}-1]}{2k+1}\right].$$

It is easy to see that for any $\zeta > -1$ all summands on the right-hand side are positive. The latter inequality therefore holds true, and (19) follows, which in turn establishes the results on the effects of risk as stated above.

A.4.2. *Increasing the jump size.* Let $\zeta_0 \in (-1, \infty)$ and assume that $\lambda(\{\zeta_0\}) > 0$. If we increase all jumps with size ζ_0 by a factor $\kappa > 0$ (assuming that $(1+\kappa)\zeta_0 > -1$), the underlying Lévy process becomes[10]

$$x_\kappa(t) \equiv x(t) + \int_{\{\zeta_0\}} \kappa\zeta\tilde{q}(t, d\zeta),$$

where the corresponding Lévy measure reads

$$\lambda_\kappa(\zeta) = \begin{cases} 0 \text{ for } \zeta = \zeta_0 \\ \lambda(\zeta) + \lambda(\zeta_0) \text{ for } \zeta = (1+\kappa)\zeta_0 \\ \lambda(\zeta) \text{ otherwise} \end{cases}.$$

Then, due to Keynes-Ramsey rule (15), the optimal average consumption growth according to $x_\kappa(t)$ obeys

$$\begin{aligned} \frac{dEc_\kappa(t)/dt}{c_\kappa(t)} &= \left[\frac{r(t)-\rho}{\gamma} - \frac{1}{2}(1-\gamma)\sigma^2\right]dt \\ &\quad -\frac{1}{\gamma}\int_{-1}^{\infty}\left[(1+\zeta) - (1+\zeta)^{1-\gamma} - \gamma\zeta\right]\lambda_\kappa(d\zeta)\,dt \\ &= \frac{dEc(t)/dt}{c(t)} - \frac{1}{\gamma}\left[g(\kappa) - g(0)\right]\lambda(\{\zeta_0\})\,dt, \end{aligned}$$

where we set $g(\kappa) \equiv [1+(1+\kappa)\zeta_0] - [1+(1+\kappa)\zeta_0]^{1-\gamma} - \gamma(1+\kappa)\zeta_0$. Hence, the sign of the derivative of $g(\kappa)$ determines whether a rise in the jump size increases or decreases consumption growth. The derivative reads $g'(\kappa) = (1-\gamma)\left[1-(1+(1+\kappa)\zeta_0)^{-\gamma}\right]\zeta_0$. Now it is easy to see that, since $(1+\kappa)\zeta_0$ was assumed to be greater than -1, $g'(\kappa) > 0$ for $\gamma < 1$, $g'(\kappa) = 0$ for $\gamma = 1$, and $g'(\kappa) < 0$ for $\gamma > 1$. This shows that $(dEc_\kappa(t)/dt)/c_\kappa(t)$ is less than $(dEc(t)/dt)/c(t)$ if $\gamma < 1$, equal if $\gamma = 1$, and greater if $\gamma > 1$, which finally yields the aforementioned results on the effects of risk.

[10] The following demonstration can be easily extended to the case in which jumps of (infinitely) many sizes increase.

Part 3

Labor Income Uncertainty

CHAPTER 3

Optimal Saving Under Risk of Unemployment*

Abstract. We consider a continuous-time optimum-consumption problem in which an agent is exposed to both risk and uncertain spells of unemployment. The back and forth in the employment status is properly modeled by a stochastic differential equation with Poisson processes. The resulting stochastic income process gives rise to precautionary saving which is decreasing in the level of wealth. We find that this excess saving jointly with the jumps in labor income lead to consumption paths that are totally different from what we know from deterministic setups. In particular, there can be, dependent on the interest rate, target saving or temporary poverty traps. We further find that the uncertainty in the employment status raises the average (though not necessary the actual) consumption growth.

JEL classification: C61; D11; E24

Keywords: Optimal consumption; Risk of unemployment; Labor income risk; Precautionary Saving

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1. Introduction

Uncertainty in labor income is a major concern of individuals. Theoretic and empirical results show that such earnings uncertainty leads to precautionary saving as a kind of personal insurance against bad income draws. This behavior increases not only individual and aggregate wealth but also, at least in average, individual and aggregate consumption growth. The theoretic basis for precautionary saving was established by Leland (1968) and Sandmo (1970); important extensions were provided by, e.g., Kimball (1990a, 1990b). The authors find that precautionary saving is associated with convexity of the marginal utility (i.e., with a positive third derivative of the utility function), such as exhibited by, for example, the widely used class of CRRA[1] utility functions. Empirical studies confirm the theoretic results. Caballero's (1991) simulation, for example, conjectures that on the aggregate level the excess wealth due to precautionary saving may account for as much as 60 percent of U.S. household's net worth. Similarly, Gourinchas and Parker (2001) estimate precautionary wealth to be 65 percent of total liquid wealth. They find further that in average four percent per annum of consumption growth at young ages is due to precautionary motives. Though there are authors presenting less dramatic results (see, e.g., Guiso et al., 1992, or Lusardi, 1998), precautionary saving is a widely accepted and important issue of individuals' behavior.[2]

A major source for the uncertainty in labor income is risk of unemployment and the uncertain outcome of the job search process. In the paper at hand we therefore present a continuous-time optimum-consumption problem in which an agent with CRRA preferences is exposed to both risk and uncertain spells of unemployment. We find that the back and forth between job and unemployment leads through the channel of precautionary saving to consumption trajectories that are totally different from what we know from deterministic setups. In particular, the usual rule that the agent (dis)saves whenever the interest rate is greater (less) than the time preference does not hold anymore. Instead we obtain, depending on the parameter settings, target saving, where the agent's wealth and consumption converge toward a target state, or temporary poverty traps in case the agent's employment history has been unlucky. Furthermore and not surprising in light of the precautionary saving motive, we find that the uncertainty in labor income due to risk of unemployment raises the average (though not necessarily the actual) consumption

[1]Constant relative risk aversion.

[2]The large differences in the estimated contribution of precautionary saving stem mainly from the various variables employed in order to proxy earnings uncertainty.

growth. As a consequence, if the interest rate is equal to the time preference rate — which in absence of uncertainty leads to constant levels for both consumption and wealth —, expected consumption expenditure and wealth always grow with a positive rate.

Beside the references mentioned above, there are many authors who are concerned with the theory of the optimal saving behavior in presence of uncertain labor income, see, e.g., Kimball and Mankiw (1989), Zeldes (1989), Carroll and Kimball (1996), Talmain (1998), and Rendon (2006). Several authors even include a simultaneous portfolio choice, see, e.g., Merton (1971), Svensson and Werner (1993), Duffie et al. (1997), and Lentz and Tranæs (2005), or endogenize individual labor supply as, e.g., Bodie et al. (1992) and Basak (1999). Still other authors, such as Deaton (1991) and Carroll (2004), add liquidity constraints, which on its own lead to precautionary saving.

Most of the models describing earnings uncertainty are, however, not suitable for capturing properly the back and forth between employment and unemployment. Some authors simply assume (in discrete time) the income process to be given by i.i.d. random shocks or to follow a geometric random walk, see, e.g., Zeldes (1989), Deaton (1991), Aiyagari (1994), and Carroll (2001), while others describe (in continuous time) the income process as a geometric Brownian motion, see, e.g., Merton (1971), Bodie et al. (1992), and Duffie et al. (1997). To the best of our knowledge, only Lentz and Tranæs (2005) and Rendon (2006) address and model risk of unemployment explicitly. They, however, consider discrete-time models in which the agent chooses both consumption and, while unemployed, job-search effort which affects the arrival rate of new job offers. In neither of these models the authors derive a closed-form solution.[3] Lentz and Tranæs (2005) provide instead implicit results on the interaction between wealth, saving, unemployment spell, and search effort, while Rendon (2006) determines the optimal policies and other variables numerically.

While finishing the article at hand, a related, but independent work by Toche (2005) was drawn to our attention. Toche (2005) considers an optimum-consumption problem in continuous-time in which unemployment is assumed to be an absorbing state. That

[3]Finding a closed-form solution for optimum-saving problems with uncertain labor income is restricted to special cases. These include models with exponential utility (see, e.g., Kimball and Mankiw, 1989), risk-neutral agents (see, e.g., Aghion and Howitt, 1992), perfect correlation of risky securities and uncertain labor income (see, e.g., Merton, 1971), or a combination of these properties (see, e.g., Svensson and Werner, 1993). Approaches for numerical and analytical approximations to the solution can be found in, e.g., Zeldes (1989) or Talmain (1998).

means, once the agent has been laid off, he remains unemployed throughout the rest of his life. This assumption is obviously a very unrealistic simplification since unemployed persons usually find a new job again.

The present paper provides a more realistic modeling by allowing the agent to jump back and forth between job and unemployment throughout his lifetime. Technically this is achieved, by assuming the job state, or more precisely the associated income process, to obey a stochastic differential equation driven by Poisson processes. This modeling is not only more suitable in light of the random changes in the employment status but also allows the rigorous use of the Hamilton-Jacobi-Bellman equation as a tool for tackling the optimum-consumption problem.

Unfortunately, but not surprisingly, a closed-form solution cannot be derived. However, using previous work on the subject of precautionary saving, which includes many of the aforementioned references, we obtain an analytical characterization of the optimal consumption rule. Jointly with a suitable form of the Keynes-Ramsey rule, which is also derived in this paper, these characterizations lead to interesting insights into the optimal saving behavior. We find that some of our results, such as target saving in case of employment, also hold (at least qualitatively) in the simple setup from Toche (2005). In this context we may also refer to, e.g., Carroll (2001) who, too, proves the existence of a target level of wealth, though for a different setup as discussed above. Other findings, in contrast, such as poverty traps, are new and only explainable with our specific modeling of the labor income process.

The model is kept as simple as possible in order to identify the pure effect of both risk and uncertain spells of unemployment on the saving behavior, cf. also Toche (2005). In detail, that means we do neither consider portfolio and leisure choice nor liquidity constraints, i.e., the agent can borrow freely whenever he thinks it is useful to do so. Furthermore we assume, unlike Lentz and Tranæs (2005) and Rendon (2006), both the separation and the matching process to be exogenously given.

The remainder of the paper is organized as follows. Section 2 describes the model. In Section 3 we provide fundamental characteristics of the optimal consumption rule. Subsequently, in Section 4, we derive the Keynes-Ramsey rule and analyze the resulting differential systems. Section 5 presents and discusses the results on the saving behavior, while in Section 6 we consider the average consumption growth. The last section finally concludes.

2. The model

We consider an agent who is at time t_0 endowed with some initial wealth a_0. At each instant $t \geq t_0$ he is paid a wage rate w while working in a job at t, whereas, while unemployed at t, he receives unemployment benefits b. For sake of simplicity, we assume both b and w to be constants. The agent can spend his wealth at t, $a(t)$, on either consumption $c(t)$ or investment that bears a constant interest rate $r > 0$. Then, wealth changes according to

$$da(t) = [ra(t) + z(t) - c(t)]\, dt, \tag{1}$$

where $z(t) \in \{b, w\}$. Whenever we find it useful in the following, we shall write $a^b(t)$ or $a^w(t)$ instead of $a(t)$ in order to distinguish between the two different laws of motion that wealth can obey with respect to the employment status. We abstract away from liquidity constraint, which means that the agent, whenever necessary, can borrow without limit.

In case of unemployment, job offers arrive Poisson distributed with an exogenous matching rate $\lambda_m > 0$. We assume that the agent accepts any job offer that arrives during unemployment and that there is no on-the-job search.[4] Let q_m denote the Poisson process that counts how often the agent has found a new job after unemployment. The arrival rate of q_m is λ_m in case of unemployment and 0 while the agent is working in a job. That is, q_m "stops" after it has jumped (i.e., a new job has been found) and starts not until the agent has become unemployed again. The average unemployment spell is thus $1/\lambda_m$.

Similarly, while working in a job, the agent is laid off with a separation rate $\lambda_s > 0$, and a Poisson process q_s counts how often he has been separated from a firm. Again, q_s stops after it has jumped and starts again not until a jump of q_m has occurred, i.e., the agent has found a new job. Thus, the arrival rate of q_s is λ_s in periods with job and 0 during unemployment, resulting in an average job duration of $1/\lambda_s$.

Summarizing, the dynamics of $z(t)$ can be described by the stochastic differential equation

$$dz(t) = [w - z(t_-)]\, dq_m(t) + [b - z(t_-)]\, dq_s(t), \tag{2}$$

[4]The assumption that job offers are always accepted is justified if the potential output of a vacancy is sufficiently high, or the unemployment benefits sufficiently low, such that the wage (which is usually achieved by bargaining between agent and firm) exceeds the agent's reservation wage (which is mainly determined by the amount of the unemployment benefits), cf. Pissarides (2000) or the survey of Rogerson et al. (2004) and the references therein.

where the initial income $z(t_0) = z_0 \in \{b, w\}$ also indicates the initial employment status. It turns out that $z(t)$ is a two-state birth-death process with birth rate λ_m and death rate λ_s, see, e.g., Ross (1983, Ch. 5). In the following we shall occasionally make use of the properties of this special class of Markov jump processes.

Notice that with (2) we have expressed $z(t)$ by a stochastic differential equation. As mentioned in the introductory section most authors assume the income process either to be given by i.i.d. distributed shocks or to follow a geometric random walk or Brownian motion. Though such modeling may simplify the analysis, it is not at all suitable to mirror the back and forth in the employment status. In addition, having $z(t)$ described by a stochastic differential equation allows a straightforward use of the dynamic programming approach, i.e., of the Hamilton-Jacobi-Bellman equation, in order to tackle the following maximization problem.[5]

We denote by E_t the expectation operator conditional on information available at time t, i.e., conditional on $z(t)$ and $a(t)$. Let the agent's time preference rate be given by the constant $\rho > 0$ and assume the planning horizon to be infinite. Then, given the CRRA utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}, \quad \sigma > 0, \sigma \neq 1,^{6} \tag{3}$$

the agent's objective consists in maximizing his expected lifetime utility

$$U^{c(t)}(z_0, a_0) = E_{t_0} \int_{t_0}^{\infty} e^{-\rho(t - t_0)} u(c(t))\, dt \tag{4}$$

subject to budget constraint (1) and labor income dynamics (2). Assume that an optimal (Markov) consumption rule $c(z, a)$ maximizing (4) exists. Then, the value function V reads

$$V(z_0, a_0) \equiv U^{c(z,a)}(z_0, a_0) = E_{t_0} \int_{t_0}^{\infty} e^{-\rho(t - t_0)} u(c(z(t), a(t)))\, dt.^{7}$$

3. Properties of the optimal consumption rule

Before studying the stated consumption problem more closely, we provide in the present section some important properties of the optimal consumption rule $c(z, a)$ and the marginal prospensity to consume, $\partial c(z, a)/\partial a$, which are illustrated in Figure 1. We

[5] Naturally, the dynamic programming approach also applies if the income process is a geometric Brownian motion.

[6] The following also applies to the special case $\sigma \to 1$, i.e. $u(c) = \log c$.

[7] Since the dynamics of $a(t)$ and $z(t)$ do not explicitly depend on time t, the state space is completely described be these two variables. Thus, as utility function (3) does not depend on time neither, we conclude that neither U^c, V and c are explicit functions of time.

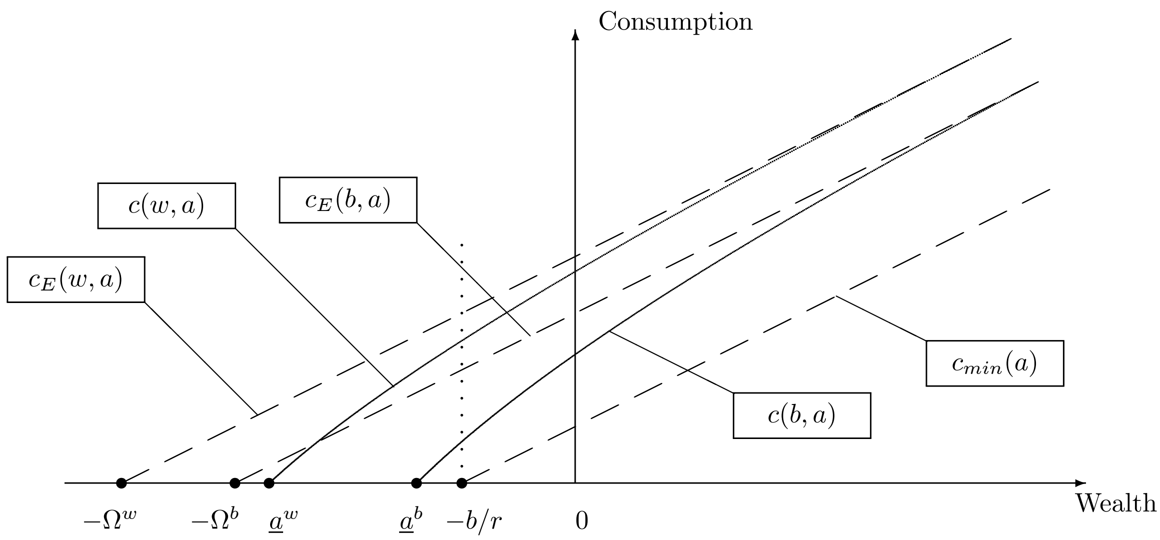

FIGURE 1. The optimal consumption rules $c(b,a)$ and $c(w,a)$.

first provide lower and upper bounds for $c(z,a)$ and analyze then asymptotic behavior and curvature.

In what follows we assume the agent to be in time t, and we denote future time by T, where $T \geq t$. In order to find bounds for $c(z,a)$, we use that, as the third derivative of utility function (3) is positive, there exists a motive for precautionary saving, see, e.g., Sandmo (1970) and Kimball (1990b). Thus, given current income $z(t) = z$, uncertainty in future labor income $z(T)$ leads for any level of wealth to less consumption expenditure than in a setting in which the agent receives the expected value of $z(T)$, denoted by $\omega^z(t,T)$, rather than the actual, uncertain amount $z(T)$. In Appendix A.1.2 we show that $\omega^z(t,T)$ depends on the current income z and that $\omega^b(t,T) < \omega^w(t,T)$ for all $T \geq t$. The optimal consumption rule in absence of labor income uncertainty is thus different for either state $z \in \{b,w\}$ and reads, see Appendix A.1.3,

$$c_E(b,a) \equiv \frac{\rho - (1-\sigma)r}{\sigma}\left(a + \Omega^b\right) \tag{5}$$

if the agent is currently unemployed (i.e., $z = b$) and

$$c_E(w,a) \equiv \frac{\rho - (1-\sigma)r}{\sigma}\left(a + \Omega^w\right) \tag{6}$$

if he is currently working in job (i.e., $z = w$). Here,

$$\Omega^b \equiv \frac{rb + \lambda_s b + \lambda_m w}{r(r + \lambda_s + \lambda_m)} \tag{7}$$

and

$$\Omega^w \equiv \frac{rw + \lambda_s b + \lambda_m w}{r\left(r + \lambda_s + \lambda_m\right)} \tag{8}$$

stand for the present values of the expected future labor income $\omega^z(t,T)$ conditional on $z(t) = z = b$ and $z(t) = z = w$, respectively. The subscript "E" from c_E shall indicate that expected rather than actual labor income is considered. In order to obtain economically meaningful results, i.e., positive consumption expenditures $c_E(z,a)$, we assume that $\rho - (1-\sigma) r > 0$.

While with (5) and (6) we have obtained upper bounds for $c(b,a)$ and $c(w,a)$, respectively, we get a lower bound by the optimal consumption rule $c_{\min}(a)$ for permanent unemployment. It reads, see also Appendix A.1.3,

$$c_{\min}(a) = \frac{\rho - (1-\sigma) r}{\sigma}\left(a + \frac{b}{r}\right), \tag{9}$$

where b/r stands for the present value of an infinite payment of unemployment benefits. Summarizing, we have found that

$$c_{\min}(a) < c(b,a) < c_E(b,a) \tag{10}$$

and

$$c_{\min}(a) < c(w,a) < c_E(w,a), \tag{11}$$

cf. also Figure 1. Notice that, as $b < w$, we get $b/r < \Omega^b < \Omega^w$, where here the difference between Ω^b and Ω^w is not due to, e.g., more skills or experience the employed agent may have acquired on the job, but rather only due to the underlying search mechanism, cf. Appendix A.1.2.

An immediate conclusion from the fact that the agent if currently working can expect to earn more over his lifetime than if he was currently unemployed is that $c(w,a) > c(b,a)$,[8] and thus

$$\chi(a) \equiv \frac{c(w,a)}{c(b,a)} > 1.$$

[8]Notice that even though expected earnings were equal for each employment status, we would not obtain equality here, since, as also stressed by many authors such as Kimball and Mankiw (1989), Gourinchas and Parker (2002), or Toche (2005), agents cannot fully smooth consumption at points of time at which labor income jumps randomly.

We now turn to the asymptotic behavior and curvature of $c(z,a)$ and $\partial c(z,a)/\partial a$. According to, e.g., Kimball (1990a, 1990b) the precautionary premium[9] is decreasing in the level of consumption c if absolute prudence $-u'''(c)/u''(c)$ is. A simple calculation shows that our utility function (3) exhibits the latter property. Thus, since according to Appendix A.3 the consumption good is normal, the precautionary premium is also decreasing in the level of wealth a. Therefore, as the precautionary premium is the rightward shift of the consumption function $c(z,a)$ in comparison to $c_E(z,a)$ in response to future income risk, we conclude that

$$\lim_{a\to\infty}\left[c_E(z,a)-c(z,a)\right]=0. \tag{12}$$

Graphically, the latter limit means that both $c(z,a)$-curves approach the corresponding $c_E(z,a)$-line as the level of wealth increases, see Figure 1. This result is very intuitive. The agent, when getting wealthier, can use his wealth increasingly as buffer against the uncertainty in labor income, reducing therefore precautionary saving. At high levels of wealth he thus behaves approximately as under certainty.

Strongly related to the latter arguments is a further conclusion drawn from Kimball (1990a, 1990b): The marginal prospensity to consume, $\partial c(z,a)/\partial a$, is decreasing in the level of wealth a. In other words, the optimal consumption rule $c(z,a)$ is concave in a and $\partial^2 c(z,a)/(\partial a)^2<0$, $z\in\{b,w\}$. Additionally, we obtain that the marginal prospensity to consume increases in the level of uncertainty in labor income. Looking at (5) and (6), we thus conclude that

$$\frac{\partial c(z,a)}{\partial a}>\frac{\partial c_E(z,a)}{\partial a}=\frac{\rho-(1-\sigma)r}{\sigma},^{10} \tag{13}$$

and, jointly with the aforementioned monotonicity of $\frac{\partial c(z,a)}{\partial a}$,

$$\frac{\partial c(z,a)}{\partial a}\searrow\frac{\rho-(1-\sigma)r}{\sigma}>0\text{ as }a\to\infty.$$

[9]The precautionary premium is the certain reduction in total wealth that has the same effect on the optimal consumption value as the addition of future labor income uncertainty. It is hence the rightward shift of the consumption function $c(z,a)$ in comparison to $c_E(z,a)$, see, e.g., Kimball (1990a).

[10]Kimball (1990a, 1990b) actually shows that at any given level of consumption c, higher uncertainty raises the marginal prospensity to consume. That is, (13) holds for identical levels of consumption $c=c(z,a_1)=c_E(z,a_2)$ but, presumably, not for the same levels of wealth (i.e., $a_1\neq a_2$). But since the right-hand side of inequality (13) is a constant, the inequality also holds for different levels of consumption and thus for all $a>\underline{a}^b$.

For related setups, though with different modelings of earnings uncertainty, several authors obtain similar results on the optimal consumption rule by either analytical studies (e.g., Carroll and Kimball, 1996, Duffie et al., 1997, and Carroll, 2004) or numerical determinations of the optimal consumption rule (e.g., Zeldes, 1989, Deaton, 1991, and Toche, 2005). Empirical evidence for the concavity of the consumption rule is provided by, e.g., Lusardi (1996) and Souleles (1999).

We now consider the origins of the $c(b,a)$- and $c(w,a)$-curve. Combining (5)-(11), or simply looking at Figure 1, leads to the conclusion that there exist levels of wealth, $\underline{a}^b$ and $\underline{a}^w$ with $-\Omega^w < \underline{a}^w < \underline{a}^b < -\Omega^b$, at which the corresponding consumption expenditures $c\left(b, \underline{a}^b\right)$ and $c(w, \underline{a}^w)$, respectively, are equal to zero. In order to avoid the problem that $c(b,a)$ may not be well defined for $a < \underline{a}^b$ and also since in Section 5 it turns out that, if starting with levels of wealth greater than $\underline{a}^b$, the agent chooses consumption in such a way that wealth always remains above $\underline{a}^b$, we consider throughout the paper only levels of wealth greater than $\underline{a}^b$.

A further conclusion drawn from (5)-(11) is that $\lim_{a\to\infty} c(z,a)/a = [\rho - (1-\sigma)r]/\sigma$ and that therefore,

$$\lim_{a\to\infty} \chi(a) = \lim_{a\to\infty} \frac{c(w,a)/a}{c(b,a)/a} = 1. \tag{14}$$

This result, jointly with $\chi(a) > 1$ for all $a > \underline{a}^b$ and $\lim_{a\to\underline{a}^b} \chi(a) = \infty$, suggests that $\chi'(a) < 0$ for all $a > \underline{a}^b$ which again mirrors the decreasing effect of labor income uncertainty as the level of wealth rises. From $\chi'(a) < 0$ we obtain easily that $\partial c(b,a)/\partial a > \partial c(w,a)/\partial a$, i.e., the marginal prospensity to consume is greater under unemployment than under employment. This again is a very plausible result since an extra Euro loosens the agent's budget constraint more if he is currently unemployed than if he was currently working in a job since, as discussed before, being unemployed means less expected future labor income ($\omega^b(t,T) < \omega^w(t,T)$), cf. also Appendix A.1.2.

4. Derivation of the optimal behavior

4.1. The Keynes-Ramsey rule. Using the Hamilton-Jacobi-Bellman (HJB) equation, we present in the following a stochastic form of the Keynes-Ramsey rule, which tells us how the optimal consumption process changes over time. Supposed that the value function and the optimal consumption rule are sufficiently smooth, the HJB equation reads according to Sennewald (2007)[11]

$$\rho V(z,a) = \max_{c\geq 0} \left\{ \begin{array}{c} u(c) + [ra + z - c] V_a(z,a) \\ +\lambda_s [V(b,a) - V(z,a)] + \lambda_m [V(w,a) - V(z,a)] \end{array} \right\}, \tag{15}$$

where V_a stands for the partial derivative of V with respect to a, and the maximum on the right-hand side is attained by the optimal consumption expenditure $c(z,a)$. The Keynes-Ramsey rule, which is derived in Appendix A.2, reads then

$$\begin{aligned} \frac{dc(z(t),a(t))}{c(z(t_-),a(t))} &= \left\{ \frac{r-\rho}{\sigma} + \lambda_s \frac{\left[\frac{c(z(t),a(t))}{c(b,a(t))}\right]^{\sigma} - 1}{\sigma} + \lambda_m \frac{\left[\frac{c(z(t),a(t))}{c(w,a(t))}\right]^{\sigma} - 1}{\sigma} \right\} dt \\ &\quad + \left[\frac{c(b,a(t))}{c(z(t_-),a(t))} - 1\right] dq_s(t) + \left[\frac{c(w,a(t))}{c(z(t_-),a(t))} - 1\right] dq_m(t). \end{aligned} \tag{16}$$

Thus, the optimal consumption process of the agent when unemployed is described by

$$\frac{dc\left(b,a^b(t)\right)}{c\left(b,a^b(t)\right)} = \left[\frac{r-\rho}{\sigma} - \lambda_m \frac{1-\chi\left(a^b(t)\right)^{-\sigma}}{\sigma}\right] dt + \left[\chi\left(a^b(t)\right) - 1\right] dq_m(t),$$

whereas during employment optimal consumption follows

$$\frac{dc(w,a^w(t))}{c(w,a^w(t))} = \left[\frac{r-\rho}{\sigma} + \lambda_s \frac{\chi(a^w(t))^{\sigma} - 1}{\sigma}\right] dt - \left[1 - \chi(a^w(t))^{-1}\right] dq_s(t).$$

Keynes-Ramsey rule (16) tells us how optimal consumption changes over time. The left-most term on the right-hand side, $(r-\rho)/\sigma$, is the deterministic part of the overall growth rate. It is equal to the growth rate of the consumption process $c_E\left(z, a_E^z(t)\right)$ induced by the optimal consumption rules (5) and (6), respectively, that are obtained for the deterministic setting described in Section 3 and the underlying wealth process $a_E^z(t)$ obeying

$$da_E^z(t) = \left[ra_E^z(t) + r\Omega^z - c_E\left(z, a_E^z(t)\right)\right] dt, \tag{17}$$

cf. Appendix A.1.4. Here, $z = z(t)$ denotes the current employment status. The subscript "E" shall again indicate that we here consider the expected instead of the actual labor income.

The term $(r-\rho)/\sigma$ shows that the usual properties for deterministic setups carry over to the stochastic problem at hand. That is, the higher the interest rate r, or the lower the time preference rate ρ and the risk aversion parameter σ, the more consumption the individual sacrifices today for consumption tomorrow which yields higher consumption

[11] The reader may alternatively resort to Chapter 1 of this thesis, which provides a reproduction of Sennewald (2007).

growth, i.e., $dc(z(t), a(t))/c(z(t_-), a(t))$ goes up. Further economic insights derived from the stochastic parts in Keynes-Ramsey rule (16) are provided later on in Section 5.

4.2. The reduced form. In order to obtain more insights into the optimal behavior, it is useful to distinguish between the two employment states the individual can be in. As long as the employment situation does not change, Keynes-Ramsey rule (16) shows that the optimal consumption path does not jump and that it is differentiable from the right. In case of unemployment and as long as the agent remains unemployed, optimal consumption grows thus with the rate

$$\frac{dc\left(b, a^b(t)\right)}{c\left(b, a^b(t)\right)} = \left[\frac{r-\rho}{\sigma} - \lambda_m \frac{1-\chi\left(a^b(t)\right)^{-\sigma}}{\sigma}\right] dt, \tag{18}$$

where wealth obeys

$$da^b(t) = \left[ra^b(t) + b - c\left(b, a^b(t)\right)\right] dt. \tag{19}$$

Analogously, as long as the agent is working in a job, optimal consumption growth is given by

$$\frac{dc(w, a^w(t))}{c(w, a^w(t))} = \left[\frac{r-\rho}{\sigma} + \lambda_s \frac{\chi(a^w(t))^{\sigma} - 1}{\sigma}\right] dt, \tag{20}$$

and wealth accumulates according to

$$da^w(t) = \left[ra^w(t) + w - c(w, a^w(t))\right] dt. \tag{21}$$

We call (18) and (20) the *reduced forms* of Keynes-Ramsey rule (16). Recalling that $\chi(a) = c(w, a)/c(b, a)$, we see that each pair of corresponding differential equations, (18) together with (19) and (20) joint with (21), represents a under-determined differential system in t with three unknown functions of time, $c\left(b, a^b(t)\right)$, $c\left(w, a^b(t)\right)$, and $a^b(t)$ for (18), (19) and $c(b, a^w(t))$, $c(w, a^w(t))$, and $a^w(t)$ for (20), (21). That means, even if we knew, say, the initial consumption expenditures $c(b, a_0)$ and $c(w, a_0)$, respectively, the systems would not be very helpful in determining the optimal consumption paths or rules. In the following subsection we shall therefore show how on the basis of the differentials (18)-(21) we can derive a differential system with two unknowns and two equations.

Notice that, unfortunately, the systems (18), (19) and (20), (21) cannot be considered simultaneously nor can the reduced forms (18) and (20) be "linked" by, for example, simply equating the ratios $\chi\left(a^b(t)\right)$ and $\chi(a^w(t))$. The reason is that the underlying wealth processes in (18) and (20) follow different laws of motion, namely (19) and (21), respectively. Therefore the consumption process $c\left(w, a^b(t)\right)$ in (18), which is the numerator of $\chi\left(a^b(t)\right)$, does in general not obey (20), as well as $c(b, a^w(t))$ in (20), the denominator of

$\chi(a^w(t))$, is not driven by (18). To be more explicit, consider, for example, $c\left(w, a^b(t)\right)$ in (18). While $c\left(w, a^b(t)\right)$ is the optimal consumption decision under the assumption of employment, wealth $a^b(t)$ changes according to unemployment, hence as in (19). Thus, as the consumption rule $c(w,a)$ is a "fixed" mapping $a \mapsto c(w,a)$, $c\left(w, a^b(t)\right)$ in (18) can, as a function of time, not exhibit the same law of motion as $c(w, a^w(t))$ in (20), where wealth changes as in (21), which finally triggers the aforementioned difficulties.

4.3. Consumption given by a system of deterministic differential equations in a. This subsection provides a method how to condense the under-determined differential systems (18), (19) and (20), (21) in order to obtain a two-dimensional differential system with two unknown functions. More precisely, applying the time-elimination method on (18)-(21), we show that the optimal consumption rules $c(b,a)$ and $c(w,a)$ solve a two-dimensional system of deterministic differential equations in a. Though not being of great use in the following analysis, this result may be the starting point for a numerical approximation to the optimal consumption rules. This step, however, is left for further research, and the reader not interested in this subject may skip the present subsection.

As long as the employment status z does not change and $da^z(t) \neq 0$ (i.e., $c(z, a^z(t)) \neq ra^z(t)+z$), the time-elimination methods yields that the marginal prospensity to consume at time t, $\partial c(z, a^z(t))/\partial a$, is given by the ratio of the time-differentials $dc(z, a^z(t))$ and $da^z(t)$,[12]

$$\frac{\partial c(z, a^z(t))}{\partial a} = \frac{dc(z, a^z(t))}{da^z(t)}. \tag{22}$$

Thus, for the case of unemployment we obtain by inserting the consumption growth rate (18) and the equation of wealth accumulation (19) that for any $a > \underline{a}^b$ with $c(b,a)$ $\neq ra+b$,

$$\frac{\partial c(b,a)}{\partial a} = \frac{\frac{r-\rho}{\sigma} - \lambda_m \frac{1-\chi(a)^{-\sigma}}{\sigma}}{ra+b-c(b,a)} c(b,a). \tag{23}$$

In analogy, now using (20) and (21), the marginal prospensity to consume for the job case reads

$$\frac{\partial c(w,a)}{\partial a} = \frac{\frac{r-\rho}{\sigma} + \lambda_s \frac{\chi(a)^{\sigma}-1}{\sigma}}{ra+w-c(w,a)} c(w,a). \tag{24}$$

[12]The time-elimination method employs the chain rule of differentiation which yields that the time evolution of the optimal consumption process $c(z, a^z(t))$ is given by $dc(z, a^z(t)) = [\partial c(z, a^z(t))/\partial a]\, da^z(t)$. Dividing by $da^z(t) \neq 0$ yields Equation (22).

Hence, recalling that $\chi(a) = c(w,a)/c(b,a)$, the optimal consumption rules $c(b,a)$ and $c(w,a)$ are a solution to the system of deterministic differential equations in a given by (23) and (24). The terminal condition is the convergence property (12).

Note that due to the imposed condition $\dot{a}^z(t) \neq 0$ (or equivalently $c(z,a) \neq ra + z$) the agent must be either saving or dissaving if we wish to apply (22). However, as the consumption good is normal and thus $0 < \partial c(z,a)/\partial a < \infty$ for all $a > \underline{a}^b$, we know that the differential system (23), (24) does not explode at levels a_r^z of wealth at which the optimal consumption spending is equal to total income, i.e., at which $c(z, a_r^z) = ra_r^z + z$.[13] A numerical approach would nevertheless require a very cautious proceeding at these points since the denominators in (23) and (24) tend toward zero as a moves toward a_r^b and a_r^w, respectively, which may lead to numerical distortions.

5. Results I: Saving and dissaving between jumps

Starting from the reduced forms (18) and (20) and the budget constraints (19) and (21) we consider in the present section the saving behavior in each employment status more closely. It turns out that for interest rates less then the time preference rate the agent, while unemployed, always dissaves, whereas, while working in a job, he saves at little wealth but dissaves when wealthy. That means during employment wealth tends toward a certain target level. If the interest rate is equal to the time preference rate, the agent always dissaves while unemployed, but always saves while working in a job. For interest rates above the time preference rate, the agent always saves while working in a job, whereas, while unemployed, he dissaves at low levels of wealth and saves when wealthy. Only for very high interest rates, the agent always saves, for either employment status and any level of wealth.

5.1. First conclusions. A first result is derived directly from the reduced forms (18) and (20). As $\chi(a) > 1$, we deduce that

$$\frac{dc\left(b, a^b(t)\right)}{c\left(b, a^b(t)\right)} < \frac{r-\rho}{\sigma} < \frac{dc\left(w, a^w(t)\right)}{c\left(w, a^w(t)\right)}. \tag{25}$$

Recall from Subsection 4.1 that $(r-\rho)/\sigma$ is the optimal consumption growth rate in absence of labor income uncertainty (i.e., the agent receives the expected rather than the

[13] For the existence of a_r^z, see Proposition 1. There it turns out that both the existence and the level of a_r^z depend on the interest rate r. Therefore, the subscript "r" at a_r^z.

actual, uncertain labor income). With Appendix A.1.4 we can thus conclude that

$$\frac{dc\left(b,a^{b}\left(t\right)\right)}{c\left(b,a^{b}\left(t\right)\right)}<\frac{dc_{E}\left(b,a_{E}^{b}\left(t\right)\right)}{c_{E}\left(b,a_{E}^{b}\left(t\right)\right)}\quad\text{and}\quad\frac{dc\left(w,a^{w}\left(t\right)\right)}{c\left(w,a^{w}\left(t\right)\right)}>\frac{dc_{E}\left(w,a_{E}^{w}\left(t\right)\right)}{c_{E}\left(w,a_{E}^{w}\left(t\right)\right)},\tag{26}$$

where $c_E(z,a)$ is given by (5) and (6), respectively, and $a_E^z(t)$ obeys differential (17). The inequalities in (26) show that in case of unemployment consumption grows more slowly than in the corresponding deterministic setup, while it grows faster when the agent is working in a job. The first result, concerning unemployment, seems to be somehow paradox given the presence of precautionary saving which actually should increase consumption growth due to the following mechanism. As the agent faces uncertainty in labor income, he reduces present consumption (i.e., $c(z,a) < c_E(z,a)$) in order to protect himself against long unemployment spells and short job durations. As a consequence, wealth accumulates faster which yields, jointly with a higher marginal prospensity to consume than in the deterministic setting (see Section 3), higher consumption growth.

But the contribution of precautionary saving is only one part of the story. A second effect stems from the different levels of labor income that contribute to the accumulation of wealth underlying the processes $c(z,a^z(t))$ and $c_E(z,a_E^z(t))$. While $a^z(t)$ accumulates according to (19) (when $z=b$) or (21) (when $z=w$), $a_E^z(t)$ obeys (17). Now, looking at these differentials, we can see that beside the different consumption expenditures also different levels of labor income affect the accumulation of wealth and therefore consumption growth.

To explain things more precisely, consider first the case of unemployment. Here, the agent earns unemployment benefits b. In the corresponding deterministic setup, on the other hand, he would earn the amount $r\Omega^b$, which is, as a simple calculation using (7) shows, greater than b. Thus, looking at the differentials (17) and (19), we can immediately conclude that $a^b(t)$ accumulates ceteris paribus more slowly than $a_E^b(t)$. That in turn leads, again ceteris paribus, to less consumption growth during unemployment compared to the deterministic benchmark case. This effect is so strong that it even outweighs the increase in consumption growth that is due to precautionary saving, which explains the "paradox" result stated in the first inequality of (26).

An analogous story holds for the job case, i.e., the second inequality in (26). But, since here labor income w is greater than average earnings $r\Omega^w$, the precautionary saving effect is even amplified and consumption always grows faster than in the deterministic setting. Later, in Section 6 we shall see that in average risk of unemployment in fact always increases consumption growth.

We now turn back to inequality (25). Since the consumption good is normal and labor income is constant between jumps, consumption growth is positive iff the accumulation of wealth is positive, i.e., iff the agent is saving. Inequality (25) leads therefore directly to the following lemma.

LEMMA 1. *If $r < \rho$, the unemployed agent always dissaves. If $r > \rho$, the agent while working in a job always saves.*

5.2. Deeper results. We now provide a more detailed discussion on the agent's saving behavior. The following proposition first presents more precise analytical results. For the proof we combine the lemmas presented in Appendices A.4.1 and A.4.2.

PROPOSITION 1.

(1) *If $0 < r < \rho$, the agent always dissaves during unemployment. For the job case there exists a target level of wealth, $a_r^w > \underline{a}^b$, which is increasing in the interest rate r and toward which the agent's wealth converges as long as he is working in a job. That means, while working in a job, the agent saves for all $a < a_r^w$, dissaves for all $a > a_r^w$, and spends his total income on consumption at a_r^w, i.e., $c\left(w, a_r^w\right) = ra_r^w + w$. In addition, we find $\lim_{r \searrow 0} a_r^w \geq \underline{a}^b$ and $\lim_{r \nearrow \rho} a_r^w = \infty$.*
(2) *If $r = \rho$, the agent, while unemployed, dissaves, whereas, while working in a job, he saves .*
(3) *If $\rho < r < \rho + \lambda_m$, the agent always saves while working in a job. For the case of unemployment, there exists a level $a_r^b > \underline{a}^b$ of wealth which is decreasing in r and exhibits the following properties. While unemployed, the agent dissaves for all $a < a_r^b$, saves for all $a > a_r^b$, and spends his total income on consumption at a_r^b, i.e., $c\left(b, a_r^b\right) = ra_r^b + b$. In addition, $\lim_{r \searrow \rho} a_r^b = \infty$ and $\lim_{r \nearrow \rho + \lambda_m} a_r^w = \underline{a}^b$.*
(4) *If $r \geq \rho + \lambda_m$, the agent always saves for each employment status and any level of wealth.*

Point 1 from the proposition, where $r < \rho$, is illustrated in Figure 2.[14] Here exists with $\left(a_r^w, c\left(w, a_r^w\right)\right)$ a stable target state for the job case toward which the agent's wealth and consumption converges during employment. The existence of such target levels was also shown by, e.g., Carroll (2001) and Toche (2005), but there, as mentioned before, using models that are not suitable for our purposes.

[14]Using similar arguments as in Remark 3 from Appendix A.4.1 would show that the position of the curves under consideration is indeed as depicted in Figure 2.

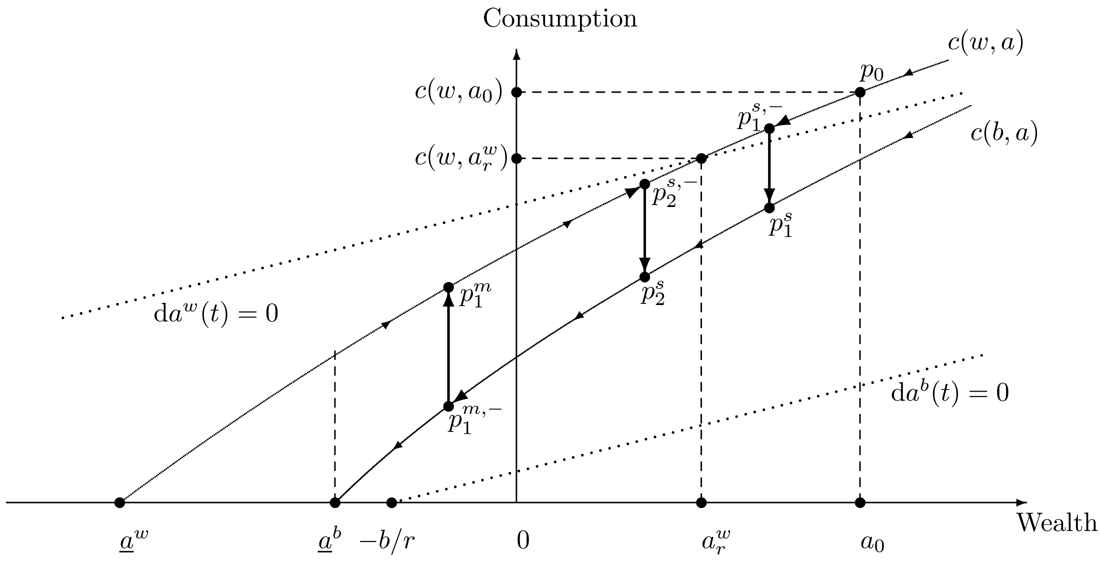

FIGURE 2. Saving behavior if $r < \rho$

Let us consider a typical path for wealth and consumption. Assume the agent initially be endowed with some wealth $a_0 > a_r^w$ and working in a job. Then he chooses consumption $c\,(w, a_0)$ and starts in the a-c space in point $p_0 \equiv (a_0, c\,(w, a_0))$ on the $c\,(w, a)$-curve. As right from the target level a_r^w the zero-motion line for $a^w\,(t)$, depicted by the upper dotted line labeled by $da^w\,(t) = 0$, lies above the $c\,(w, a)$-curve, the agent dissaves at a_0. Thus, wealth and therefore consumption decrease and a and c move left-down on the $c\,(w, a)$-curve until, say, $p_1^{s,-}$ where the agent is separated from his job for the first time. Consumption jumps then downwards and the system jumps to p_1^s on the $c\,(b, a)$-curve. Now, being unemployed, the agent dissaves (the zero-motion line for $a^b\,(t)$, the lower dotted line, always lies below the $c\,(b, a)$-curve). Wealth and consumption therefore decrease further, now along the $c\,(b, a)$-curve, until point $p_1^{m,-}$ where the agent finds a new job. Then consumption jumps upwards and the system jumps from $p_1^{m,-}$ to p_1^m, back on the $c\,(w, a)$-curve. Now, wealth is below the target level a_r^w, and the agent saves so that wealth and consumption move upwards until, say, $p_2^{s,-}$ where he is laid off again. As before, consumption jumps downwards and the system jumps from $p_2^{s,-}$ to p_2^s, and so forth. We see that, even though starting out above the target level a_r^w, the agent finds himself in the space southwest of $(a_r^w, c\,(w, a_r^w))$ after some time. Once arrived there, he always saves while working in a job, moving toward $(a_r^w, c\,(w, a_r^w))$, whereas he always dissaves while unemployed.

We now consider more closely the change in the sign of saving and of consumption growth at the target level a_r^w. An algebraic derivation was provided in Appendix A.4.2. But how can we explain this result economically? For sake of clarity we first focus on saving and turn then to consumption growth. Observe that without labor income uncertainty (i.e., the agent would earn the expected instead of the actual labor income) the agent had little incentive to save since $r < \rho$. For any $a > \underline{a}^b$ and for speculative purposes only he would hence dissave the amount $\|ra + r\Omega^w - c_E(w, a)\|$ and the optimal consumption process $c_E(w, a_E^w(t))$ would decrease with the constant rate $\|r - \rho\| / \sigma$;[15] a target level would not exist. But risk and uncertain spells of unemployment force the agent to precautionary saving. In addition, when employed, his earnings are in each period above the average since $w > r\Omega^w$, which, compared to the deterministic setting, increases saving further by the amount $w - r\Omega^w$.

Now observe that, on the one hand, speculative dissaving due to $r < \rho$ increases as wealth increases. On the other, the wealthier the agent, the lower the amount of precautionary saving, i.e., the less the agent needs to care about the uncertainty in labor income since wealth serves as a buffer against bad income shocks, see Section 3. Summarizing, we thus see that at low levels of wealth the additional saving due to precautionary purposes and excess labor income is large enough in relation to the amount of speculative dissaving such that the agent's total saving becomes positive. However, as wealth increases, the amount of speculative dissaving increases too, while precautionary saving decreases and the impact of the excess labor income $w - r\Omega^w$ diminishes.[16] As a result, there exists a level of wealth, namely a_r^w, at which speculative dissaving is exactly offset by the additional saving due to precautionary purposes and excess labor income. At levels greater than a_r^w speculative dissaving then outweighs the additional saving, and the agent hence dissaves.

A similar story holds for consumption growth $dc(w, a^w(t)) / c(w, a^w(t))$. Here we know from Subsection 5.1 that due to both precautionary saving and excess labor income $w - r\Omega^w$, consumption during employment grows with a rate above the deterministic rate $(r - \rho) / \sigma$. But applying the same arguments as before and considering the positive marginal prospensity to consume, we find that consumption growth is positive at low levels of wealth, where the impact of both precautionary saving and the excess labor income is

[15]See Appendix A.1.4.

[16]Observe here that $w - r\Omega^w$ is a constant with respect to wealth.

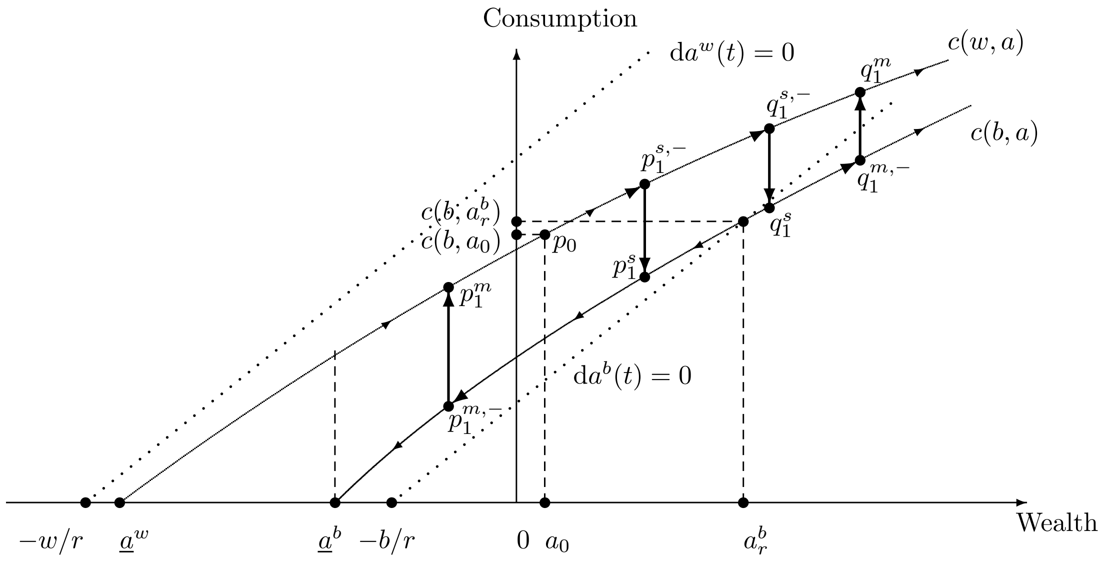

FIGURE 3. Saving behavior if $\rho < r < \rho + \lambda_m$

strong, while consumption growth is negative with a rate close to the deterministic rate $(r - \rho)/\sigma$ if the agent is more wealthy.[17]

Observe that, as the interest rate r approaches the time preference rate ρ, the level a_r^w moves rightward toward ∞. That means that, if r moves toward ρ, which increases the agent's incentive to save, the agent will, unless he is very wealthy, always save while working in a job. This behavior is similar to the one described in point 2 from the proposition, where $r = \rho$. Here the employed agent saves even if he is very wealthy.

We now focus on point 3 where $\rho < r < \rho + \lambda_m$ and which is illustrated in Figure 3.[18] Here we have with a_r^b an unstable steady state for the case of unemployment. Figure 3 shows that the agent can find himself trapped in poverty. Assume he is initially working, but not too wealthy such that $a_0 < a_r^b$. As long as he stays in his job, he saves, starting in $p_0 \equiv (a_0, c(b, a_0))$, and wealth and consumption move up-right on the $c(w, a)$-curve. Assume the agent is laid off before reaching a_r^b, say at $p_1^{s,-}$. Then consumption jumps downwards and the system jumps to p_1^s on the $c(b, a)$-curve. Now, the agent dissaves in order to maintain a certain level of consumption, and wealth as well as consumption decline during the current unemployment spell, until he finds a new job again at, say, $p_1^{m,-}$. Then, consumption jumps upwards and the agent can, starting out of p_1^m, save

[17]Notice that combining (14), (20), and $\chi'(a) < 0$ clearly shows that the consumption growth rate tends from above toward the deterministic rate $(r - \rho)/\sigma$.

[18]Again, using similar arguments as in Remark 3 from Appendix A.4.1 would confirm that the position of the curves under consideration is indeed as drawn in Figure 3.

again. But, as before, he may loose his job before reaching a_r^b. That means, if the agent's employment history turns out to be unfortunate, he does not escape from poverty. On the other hand, looking at another possible sequence, $p_0 \rightarrow q_1^{s,-} \rightarrow q_1^s \rightarrow q_1^{m,-} \rightarrow q_1^m$, we find that once the agent has stayed a sufficiently long time in job, such that he has been able to accumulate wealth beyond a_r^b when being laid off, he will always save, even during unemployment. That means, if the agent's wealth is greater than a_r^b, he still becomes more wealthier, regardless of his job situation. Later on, in the subsequent section we show that in average the agent's consumption and thus wealth always grow if $r \geq \rho$. In the long run, the agent will therefore always escape from poverty.

In analogy to the case of the target level a_r^w from point 1, we here also explain the change in the sign of saving and consumption growth at the unstable steady state a_r^b. As $r > \rho$, the agent's incentive to save is high and he would always save in absence of earnings uncertainty. On the one hand, this speculative saving is amplified by precautionary saving, which is decreasing in the level of wealth. On the other hand, since unemployment benefits b are below the average earnings $r\Omega^b$, saving is reduced by the amount $r\Omega^b - b$. At low levels of wealth, this reduction in saving is large enough to outweigh speculative and precautionary saving, and the agent dissaves. But, as wealth increases the impact of $r\Omega^b - b$ on total saving diminishes. There exists thus a point, namely a_r^b, at which this difference is equal to speculative and precautionary saving, and wealth and consumption remain constant over time. For levels of wealth higher than a_r^b the agent then always saves. In analogy to point 1 a discussion on the change in the sign of consumption growth is now straightforward.

We now turn to point 4 from the proposition. Here, we will hardly observe the required parameter constellation $r \geq \rho + \lambda_m$ in reality. Clark and Summers (1979), for example, suggests that an average spell of unemployment lasts between 3.5 to 4 months, which yields a matching rate of about $\lambda_m = 0.25$ annually. On the other hand, estimated time preference rates typically range between 0.01 (one percent) and 0.05 (five percent) annually, see, e.g., Skinner (1988) or Engen and Gruber (2001). Hence, in order to satisfy $r \geq \rho + \lambda_m$, the real interest rate needs to be greater than about 0.25, i.e., 25 percent. But which safe investment strategy yields such high returns?

Observe that the previous results on the agent's saving behavior differ fundamentally from what we know from deterministic setups. There, the agent always saves (dissaves) if the interest rate r is greater (less) than the time preference rate ρ, while for $r = \rho$

he always spends his total income on consumption, leading to constant levels of both consumption and wealth.

6. Results II: The average consumption growth

In the present section we show that risk of unemployment leads to higher average (or expected) consumption growth than in the deterministic setup. With Keynes-Ramsey rule (16), the expected growth rate of the optimal consumption process at some future time conditional on the initial wealth and employment status reads, see Appendix A.5,

$$E_{t_0}\frac{dc\left(z\left(t\right),a\left(t\right)\right)}{c\left(z\left(t\right),a\left(t\right)\right)}=\left\{\begin{array}{c}\frac{r-\rho}{\sigma}+\lambda_s p_{z_0}^w\left(t_0,t\right)\left\{E_{t_0}\frac{\chi(a(t))^{\sigma}-1}{\sigma}-E_{t_0}\left[1-\chi\left(a\left(t\right)\right)^{-1}\right]\right\}\\+\lambda_m p_{z_0}^b\left(t_0,t\right)\left\{E_{t_0}\left[\chi\left(a\left(t\right)\right)-1\right]-E_{t_0}\frac{1-\chi(a(t))^{-\sigma}}{\sigma}\right\}\end{array}\right\}dt, \tag{27}$$

where $p_{z_0}^z\left(t_0,t\right)$, the probability of being at time t in job status z when being at t_0 in z_0, is given in (28)-(31) in Appendix A.1.1. The terms in braces on the right-hand side that are in addition to the deterministic growth rate $\left(r-\rho\right)/\sigma$ are strictly positive, see Lemma 6 in Appendix A.5. Thus, the average consumption growth under risk of unemployment is greater than for deterministic setups where future labor income is deterministic and given by the expectation of $z\left(t\right)$, $\omega^{z_0}\left(t\right)$.

This finding mirrors the precautionary saving motive. Facing not only the risk but also the uncertain duration of unemployment, a prudent agent sacrifices some consumption today to protect himself against possible future losses in labor income (i.e., when he becomes unemployed or remains a long time in unemployment). This behavior yields in average ever lasting growth of consumption and wealth if the interest rate is equal to (or greater than) the time preference, which in deterministic setups only leads to zero growth. Unfortunately, there is no such an unambiguous result if the interest rate is less than the time preference rate. However, the finding that here the consumption process after some time will only range between zero and the target-level $c\left(w,a_r^w\right)$ (cf. Figure 2 in Subsection 5.2) suggests that there might exists interest rates less than ρ such that average consumption growth becomes zero after some time and that therefore there might exist a steady state distribution for the consumption process.

7. Conclusion

We have studied the optimal saving behavior of an agent who faces not only risk but also uncertain duration of unemployment. We have found that precautionary saving, which is decreasing in the level of wealth, leads to a different saving behavior than in the deterministic setup: (i) If the interest rate is less than the time preference rate, the agent while working in a job saves at little wealth and dissaves when wealthy, toward a target level of wealth, whereas while unemployed he always dissaves. (ii) If the interest rate is equal to the time preference rate, he saves while employed and dissaves while unemployed. (iii) In case of interest rates greater than the time preference rate, the agent while unemployed dissaves at low level of wealth and saves when wealthy, whereas while working in a job, he always saves. Here the agent may be temporarily trapped in poverty.

The average consumption growth turns out to be always greater than in the deterministic setup. That implies that, if the interest rate is equal to (or greater than) the time preference rate — what in deterministic setups leads to zero-growth —, consumption and wealth grow here in average always with a positive rate.

In a next step one could attempt to derive a numerical approximation to the optimal consumption rule, using the differential system presented in Subsection 4.3. Interesting extensions might be to introduce risky assets as investment alternative (that may be correlated with the risk of unemployment and the job matching process) or to endogenize both the agent's labor supply as well as his effort to find a new job while unemployed. These issues, however, are left for further research.

A. Appendix

A.1. The optimal consumption rule if labor income is deterministic . The objective of the present subsection is to find closed-form expressions for the optimal consumption rules $c_E(b,a)$, $c_E(w,a)$, and $c_{\min}(a)$. Here, $c_E(z,a)$ is the optimal consumption expenditure if future labor income is given by the expected value of $z(T)$ conditional on currently being either unemployed $(z=b)$ or working in a job $(z=w)$, while $c_{\min}(a)$ is the optimal consumption rule under the assumption of permanent unemployment. We first determine in Subsection A.1.1 the probabilities of being employed and unemployed in the future, calculate then in Subsection A.1.2 the expected labor income and its present value, and derive finally, in Subsection A.1.3, the closed-form expressions for $c_E(b,a)$, $c_E(w,a)$, and $c_{\min}(a)$. Some important remarks on the deterministic consumption processes induced by $c_E(b,a)$ and $c_E(w,a)$ are added in Subsection A.1.4.

A.1.1. *The Kolmogorov probabilities of* $z(t)$. As mentioned in the main text, $z(t)$ can be considered as a two-state birth-death process. That allows us to apply Kolmogorov's Forward Equation in order to determine the distribution of future earnings. Let $t \geq t_0$

be current and $T \geq t$ future time. Then we denote the probability of receiving in the future (at time T) income $\tilde{z}$ when currently (at t) receiving z by

$$p_z^{\tilde{z}}(t,T) \equiv P(z(T) = \tilde{z}|z(t) = z) \equiv E\left[1_{\{z(T)=\tilde{z}\}}|z(t) = z\right].$$

Following Ross (1983, Ex. 5.4(a)), we find that for the two states of $z(T)$, b and w, these conditional probabilities read

$$p_b^b(t,T) = \frac{\lambda_s}{\lambda_s+\lambda_m} + \frac{\lambda_m}{\lambda_s+\lambda_m} e^{-(\lambda_s+\lambda_m)(T-t)}, \quad (28)$$

$$p_b^w(t,T) = \frac{\lambda_m}{\lambda_s+\lambda_m} - \frac{\lambda_m}{\lambda_s+\lambda_m} e^{-(\lambda_s+\lambda_m)(T-t)} \quad (29)$$

if $z(t) = b$ and

$$p_w^b(t,T) = \frac{\lambda_s}{\lambda_s+\lambda_m} - \frac{\lambda_s}{\lambda_s+\lambda_m} e^{-(\lambda_s+\lambda_m)(T-t)}, \quad (30)$$

$$p_w^w(t,T) = \frac{\lambda_m}{\lambda_s+\lambda_m} + \frac{\lambda_s}{\lambda_s+\lambda_m} e^{-(\lambda_s+\lambda_m)(T-t)} \quad (31)$$

if $z(t) = w$. Notice that, for example, the expression $p_b^w(t,T)$ stands not only for the probability of earning at T wage w when receiving unemployment benefits b at t, but also for the probability of working in a job at T when currently being unemployed. Observe that for any $z \in \{b, w\}$, $p_z^b(t,T) + p_z^w(t,T) = 1$. Furthermore note that, since the future employment status $z(T)$ is independent of the level of wealth at t, $p_z^{\tilde{z}}(t,T) = E_t 1_{\{z(T)=z\}}$. That means it is here the same whether we condition on both $z(t)$ and $a(t)$ or on $z(t)$ only. This result is generalized in the following remark.

Remark 1. *Let $f : z \in \{b, w\} \mapsto f(z) \in \mathbb{R}$ be a measurable function. Then, $E[f(z(T))|z(t)] = E_t[f(z(T))]$.*

A.1.2. *The expected future labor income and its present value.* We now turn to the determination of the expected future labor income and its present value. Let, again, $T \geq t$. Then the expected future labor income (at T) conditional on the current income (at t) is denoted by $\omega^{z(t)}(t,T) \equiv E[z(T)|z(t)] = E_t z(T)$, where for the second equal sign we used Remark 1. As previously shown in Subsection A.1.1, the conditional future labor income is two-state distributed with the conditional probabilities (28)-(31). Thus, the conditional expected labor income reads $\omega^{z(t)}(t,T) = p_{z(t)}^b(t,T)\, b + p_{z(t)}^w(t,T)\, w$ and therefore, if $z(t) = b$,

$$\omega^b(t,T) = \frac{\lambda_s b + \lambda_m w - \lambda_m (w-b)\, e^{-(\lambda_s+\lambda_m)(T-t)}}{\lambda_s+\lambda_m} \quad (32)$$

while, if $z(T) = w$,

$$\omega^{w}(t,T) = \frac{\lambda_s b + \lambda_m w + \lambda_s (w-b)\, e^{-(\lambda_s+\lambda_m)(T-t)}}{\lambda_s + \lambda_m}.^{19} \tag{33}$$

The latter equations show that the agent when currently employed can expect to receive higher labor income in the future than when currently unemployed. Observe that this result is not due to, e.g., more skills or experience he may have achieved on the job, but rather only due to the underlying search mechanism. Consequently, we see that for large time horizons the initial employment status becomes less and less important, and letting T tend toward ∞ even yields

$$\lim_{T\to\infty} \omega^{b}(t,T) = \lim_{T\to\infty} \omega^{w}(t,T) = \frac{\lambda_s b + \lambda_m w}{\lambda_s + \lambda_m}.$$

We now continue with the calculation of the present values at time $T \geq t$, denoted by $\Omega^{b}(t,T)$ and $\Omega^{w}(t,T)$, respectively. The present value of an arbitrary variable, but deterministic flow of labor income $\overline{z}(T)$, amounts to

$$\Omega(T) \equiv \textstyle\int_T^\infty e^{-r(\tau-T)} \overline{z}(\tau)\, d\tau. \tag{34}$$

Inserting (32) and (33) into the latter formula yields (replace in (32) and (33) T with τ)

$$\Omega^{b}(t,T) = \frac{\lambda_s b + \lambda_m w}{r(\lambda_s + \lambda_m)} - \frac{\lambda_m (w-b)}{(\lambda_s+\lambda_m)(r+\lambda_s+\lambda_m)} e^{-(\lambda_s+\lambda_m)(T-t)} \tag{35}$$

and

$$\Omega^{w}(t,T) = \frac{\lambda_s b + \lambda_m w}{r(\lambda_s + \lambda_m)} + \frac{\lambda_s (w-b)}{(\lambda_s+\lambda_m)(r+\lambda_s+\lambda_m)} e^{-(\lambda_s+\lambda_m)(T-t)}. \tag{36}$$

Now observe that the stochastic income process $z(t)$ as defined in (2) is Markovian and that it therefore has no "memory", which, in particular, means that the time elapsed since the agent has become unemployed or employed for the last time is irrelevant for future prospects. Thus, at which time t ever we look at the "system" and observe employment status $z \in \{b, w\}$, the present values of expected labor income are always

$$\Omega^{b} \equiv \Omega^{b}(t,t) = \frac{rb + \lambda_s b + \lambda_m w}{r(r+\lambda_s+\lambda_m)} \tag{37}$$

and

$$\Omega^{w} \equiv \Omega^{w}(t,t) = \frac{rw + \lambda_s b + \lambda_m w}{r(r+\lambda_s+\lambda_m)}, \tag{38}$$

respectively.

---

[19]Interestingly, we arrive at the same results if we form expectation E_t on the stochastic differential (2), apply further the martingale property of the Poisson processes (see Footnote 24 on p. 85), and solve the resulting deterministic linear differential equation for $E_t z(T)$.

We can see that Ω^z does not depend on current time t. In addition, and as already suggested by the expected future labor income, we find that the agent when working in a job can expect to earn more over his lifetime than when currently unemployed. The difference reads $\Omega^w - \Omega^b = \frac{w-b}{r+\lambda_s+\lambda_m}$, and, again, it only stems from the underlying search mechanism.

REMARK 2. *Interestingly, as a simple calculation shows, it is not the same whether the agent who is currently, at* t*, say, working in a job receives either the expected labor income* $\omega^w(t,T)$ *over the life cycle or first wage* w *for the expected duration of a job,* $1/\lambda_s$*, then unemployment benefits* b *for the expected unemployment spell,* $1/\lambda_m$*, afterwards again* w *for a period of length* $1/\lambda_s$*, and so on. We must thus be careful with what we use as the "deterministic world".*

A.1.3. *The optimal consumption rules for the deterministic setting* . Consider the problem

$$\max_{c(T)\geq 0} \int_t^\infty e^{-\rho(T-t)} \frac{c(T)^{1-\sigma}-1}{1-\sigma} dT$$

subject to $da(T) = [ra(T) + \overline{z}(T) - c(T)]\, dT$, where labor income $\overline{z}(T)$ changes deterministically over time. Applying the Hamiltonian approach or the HJB equation yields jointly with the No-Ponzi game condition $\lim_{T\to\infty} e^{-rT} a(T) = 0$ the following closed-form expression for the optimal consumption rule[20]

$$c(T,a) = \frac{\rho - (1-\sigma)r}{\sigma} [a + \Omega(T)], \tag{39}$$

where $\Omega(T)$ stands for the present value of labor income, see (34).

In case of permanent unemployment the agent receives an infinitely lasting payment amounting to b. The present value of this flow reads $\Omega_{\min} \equiv b/r$, which yields upon inserting in (39) consumption rule (9). Analogously, the optimal consumption rule $c_E(t,T,z,a)$ in case that labor income is given by the expected flow $\omega^z(t,T)$ reads then

$$c_E(t,T,z,a) = \frac{\rho-(1-\sigma)r}{\sigma} [a + \Omega^z(t,T)], \tag{40}$$

where $\Omega^z(t,T)$ is given by (35) and (36), respectively. Now, recall that the actual income process $z(t)$ is Markovian. The deterministic consumption rule $c_E(z,a)$ corresponding to the rule under uncertainty, $c(z,a)$ is thus obtained by setting in (40) $T=t$, and it

[20]The No-Ponzi game condition is a sufficient criterion for optimality, see, e.g., Wälde (2006, Sec. 5.4).

reads $c_E(z,a) = \frac{\rho-(1-\sigma)r}{\sigma}[a+\Omega^z]$. Inserting (37) and (38) finally yields the specific rules (5) and (6), respectively.

A.1.4. *The deterministic consumption process.* Using the optimal consumption rule for the deterministic setting, (5) and (6), respectively, we present in the following a budget constraint to which this rule is optimal and which induces the same consumption process as the system for the deterministic setting described in the previous subsections, given by consumption rule (40) and budget constraint

$$d\overline{a}_E^z(T) = \left[r\overline{a}_E^z(T) + \omega^z(t,T) - c_E^z\left(t,T,z,\overline{a}_E^z(T)\right)\right]dT,\ T \geq t. \tag{41}$$

Recall that $z = z(t)$ denotes the employment status at time t. The objective of the introduction of the alternative system is threefold. Observe that, while the actual labor income process is Markovian and constant between jumps, the expectation $\omega^z(t,T)$ depends on the time elapsed since t, which, for example if $z = b$, can be the last time the agent has been laid off. Thus, when we explain the differences between the consumption growth rate obtained in the stochastic setting and the growth rate obtained in the deterministic setting (see, e.g., Subsection 5.1, p. 67), conclusions may be interfered by the following facts: (i) $\omega^z(t,T)$ continuously changes over time T, and it does not "update" information; (ii) the optimal consumption rule (40) depends trough $\omega^z(t,T)$ on the time span $T-t$, which means that in absence of uncertainty and holding all other variables equal, the agent behaves differently at t than at some $T > t$, which is not the case in the stochastic setting; (iii) a positive saving in the deterministic setting (i.e., $d\overline{a}_E^z(T) > 0$) does not necessarily mean positive consumption growth (i.e., $dc_E\left(t,T,z,\overline{a}_E^z(T)\right) > 0$) and vice versa, as we observe in the stochastic setting. In other words, the deterministic system as it stands is not "comparable" with the stochastic system.

The new system solves those problems. It is given by the consumption rules (5) and (6), respectively, and budget constraint (17).[21,22] We use here that under certainty the agent does not consider current labor income but rather its present value when taking a consumption decision. In order to show that the resulting consumption process $c_E\left(z,a_E^z(T)\right)$ is equal to the consumption process $c_E\left(t,T,z,\overline{a}_E^z(T)\right)$ induced by (40) and (41) we notice first that at "initial" time $T = t$ and since by construction $a_E^z(t) = \overline{a}_E^z(t)$,

[21] Proceeding along the lines from Subsection A.1.3, one can show easily that the consumption rules (5) and (6), respectively, are indeed optimal to budget constraint (17).

[22] Since neither the consumption rules (5) and (6) nor budget constraint (17) depend on the time span $T-t$, we do not distinguish anymore between t and T whenever we deal with these formulas, and we denote the time flow in the main text by t.

$c_E(z, a_E^z(t)) = c_E(t, t, z, \overline{a}_E^z(t))$ and, second, that both processes grow with the same rate $(r - \rho)/\sigma$, cf. Barro and Sala-i-Martín (1995, Sec. 2.1). Combining both points yields equality.

A.2. The Derivation of the Keynes-Ramsey rule. Consider HJB equation (15). The first-order condition for the maximum on the right-hand side reads

$$u'(c(z,a)) = V_a(z,a). \tag{42}$$

That is, the optimal consumption for a given level of wealth a is always chosen such that the marginal prospensity to consume is equal to the marginal value of a. Applying the change-of-variables formula from Sennewald (2007, Th. 1), we obtain the time evolution of the marginal value function

$$\begin{aligned} dV_a(z(t), a(t)) &= \{[ra(t) + z(t) - c(t)] V_{aa}(z(t), a(t))\} dt \\ &\quad + [V_a(b, a(t)) - V_a(z(t_-), a(t))] dq_s(t) \\ &\quad + [V_a(w, a(t)) - V_a(z(t_-), a(t))] dq_m(t), \end{aligned} \tag{43}$$

where V_{aa} denotes the second order partial derivative of V with respect to a. Now, we differentiate the maximized HJB equation (15) with respect to a, and we find, using the envelope theorem, that

$$\begin{aligned} \rho V_a(z,a) &= rV_a(z,a) + [ra + z - c(z,a)] V_{aa}(z,a) \\ &\quad + \lambda_s [V_a(b,a) - V_a(z,a)] + \lambda_m [V_a(w,a) - V_a(z,a)], \end{aligned}$$

which gives upon rearranging

$$\begin{aligned} [ra + z - c(z,a)] V_{aa}(z,a) &= [\rho - r] V_a(z,a) - \lambda_s [V_a(b,a) - V_a(z,a)] \\ &\quad - \lambda_m [V_a(w,a) - V_a(z,a)]. \end{aligned}$$

Inserting the latter equation evaluated at $z(t)$ and $a(t)$ into differential (43) yields

$$\begin{aligned} dV_a(z(t), a(t)) &= \left\{ \begin{array}{c} [\rho - r] V_a(z(t), a(t)) - \lambda_s [V_a(b, a(t)) - V_a(z(t), a(t))] \\ -\lambda_m [V_a(w, a(t)) - V_a(z(t), a(t))] \end{array} \right\} dt \\ &\quad + [V_a(b, a(t)) - V_a(z(t_-), a(t))] dq_s(t) \\ &\quad + [V_a(w, a(t)) - V_a(z(t_-), a(t))] dq_m(t). \end{aligned}$$

Now, substituting $V_a(\cdot)$ with $u'(\cdot)$ according to the first-order condition (42), we find that the marginal utility process obeys

$$
\begin{aligned}
-\frac{du'(c(z(t),a(t)))}{u'(c(z(t_-),a(t)))} = & \left\{ \begin{array}{c} r-\rho+\lambda_s\left[\frac{u'(c(b,a(t)))}{u'(c(z(t),a(t)))}-1\right] \\ +\lambda_m\left[\frac{u'(c(w,a(t)))}{u'(c(z(t),a(t)))}-1\right] \end{array} \right\} dt \\
& -\left[\frac{u'(c(b,a(t)))}{u'(c(z(t_-),a(t)))}-1\right] dq_s(t) \\
& -\left[\frac{u'(c(w,a(t)))}{u'(c(z(t_-),a(t)))}-1\right] dq_m(t). \qquad (44)
\end{aligned}
$$

Applying the change-of-variables formula from Sennewald and Wälde (2006, Cor. 3)[23] on the general Keynes-Ramsey rule (44) and the mapping $x \mapsto u'^{-1}(x)$, where u is given by (3), finally yields Keynes-Ramsey (16).

A.3. The consumption good is normal. In the following we show that the consumption good is a normal, i.e., $\partial c(z,a)/\partial a > 0$, $z \in \{b, w\}$. The derivation is in analogy to Chang (2004, Subsec. 4.3.1) who considers a consumption-investment problem with Brownian motion as noise. We exploit that utility function (3) is concave and that budget constraint (1) is linear in a and c.

Let a_1 and a_2 be two different initial levels of wealth, with corresponding optimal wealth and consumption processes $a_i(t)$ and $c_i(t) \equiv c(z(t), a_i(t))$, $i = 1, 2$. Then we define for $\lambda \in [0,1]$, $a_\lambda(t) \equiv \lambda a_1(t) + (1-\lambda) a_2(t)$ and $c_\lambda(t) \equiv \lambda c_1(t) + (1-\lambda) c_2(t)$. As by linearity of budget constraint (1) we obtain $da_\lambda(t) = [ra_\lambda(t) + z(t) - c_\lambda(t)] dt$, we conclude that $a_\lambda(t)$ is the wealth process associated to $c_\lambda(t)$. By definition of the value function V and using the concavity of u we obtain then

$$
\begin{aligned}
V(z, a_\lambda) &\geq E_0 \textstyle\int_{t_0}^{\infty} e^{-\rho(t-t_0)} u(c_\lambda(t)) \, dt \\
&\geq E_0 \textstyle\int_{t_0}^{\infty} e^{-\rho(t-t_0)} [\lambda u(c(z(t), a_1(t))) + (1-\lambda) u(c(z(t), a_2(t)))] \, dt \\
&= \lambda V(z, a_1) + (1-\lambda) V(z, a_2).
\end{aligned}
$$

This chain of inequalities shows that the value function is concave in a, which, under suitable smoothness assumptions, is equivalent to $\partial^2 V(z,a)/(\partial a)^2 < 0$. Now, differentiating the first-order condition (42) for maximizing the HJB equation with respect to a, we find that $u''(c(z,a))\, \partial c(z,a)/\partial a = \partial^2 V(z,a)/(\partial a)^2$. Hence, as both $u''(c(z,a))$ and $\partial^2 V(z,a)/(\partial a)^2$ are negative, we obtain that $\partial c(z,a)/\partial a > 0$.

[23]The reader may also resort to Chapter **??** of the present thesis, which is a slightly modified version of Sennewald and Wälde (2006).

A.4. Saving and Dissaving.

A.4.1. *Saving and dissaving during unemployment.* In this section we consider the saving behavior for either employment status. We start with the case of unemployment. The job case is presented in the following subsection.

LEMMA 2. *While unemployed, the agent*

(1) *dissaves (and* $\frac{dc\left(b,a^b(t)\right)}{c\left(b,a^b(t)\right)} < 0$*) if either*
   (a) $r \leq \rho$ *or*
   (b)
$$\rho < r < \rho + \lambda_m, \tag{45}$$
   and $c(w,a) < \left(1 - \frac{r-\rho}{\lambda_m}\right)^{-1/\sigma} c(b,a)$*;*

(2) *saves (and* $\frac{dc\left(b,a^b(t)\right)}{c\left(b,a^b(t)\right)} > 0$*) if either*
   (a) $r \geq \rho + \lambda_m$ *or*
   (b) *(45) holds and* $c(w,a) > \left(1 - \frac{r-\rho}{\lambda_m}\right)^{-1/\sigma} c(b,a)$*;*

(3) *spends exactly his total income,* $ra+b$*, on consumption if (45) holds and* $c(w,a) = \left(1 - \frac{r-\rho}{\lambda_m}\right)^{-1/\sigma} c(b,a)$*.*

PROOF. First, we recall that, as the consumption good is normal and labor income between jumps constant, $dc\left(b,a^b(t)\right)/c\left(b,a^b(t)\right) > 0$ iff the agent is saving. Using the reduced form (18), we find thus that the agent saves iff

$$r > \rho + \lambda_m \left[1 - \chi(a)^{-\sigma}\right]. \tag{46}$$

Since $\chi(a) > 1$, a sufficient condition for this inequality to hold true is $r \geq \rho + \lambda_m$, whereas a necessary condition is given by $r > \rho$. That is, for $r \geq \rho + \lambda_m$ an unemployed agent always saves, while for $r \leq \rho$ he always dissaves. This proves points 1a) and 2a).

For the remaining constellation (45), the parameters do not provide such a clear distinction between saving and dissaving. However, further rearranging of (46) shows that the agent saves iff $c(w,a) < \left(1 - \frac{r-\rho}{\lambda_m}\right)^{-1/\sigma} c(b,a)$, which yields points 1b), 2b), and 3. Notice that here inequality (45) implies that the term $1 - \frac{r-\rho}{\lambda_m}$ is greater than 0, which in turn ensures that $\left(1 - \frac{r-\rho}{\lambda_m}\right)^{-1/\sigma}$ is well defined. □

The following lemma focuses on the "in-between" cases 1b), 2b) and 3, in which the interest rate satisfies (45) and where we do not know much about whether and when an unemployed agent is saving or dissaving. The results are depicted in Figure 4.

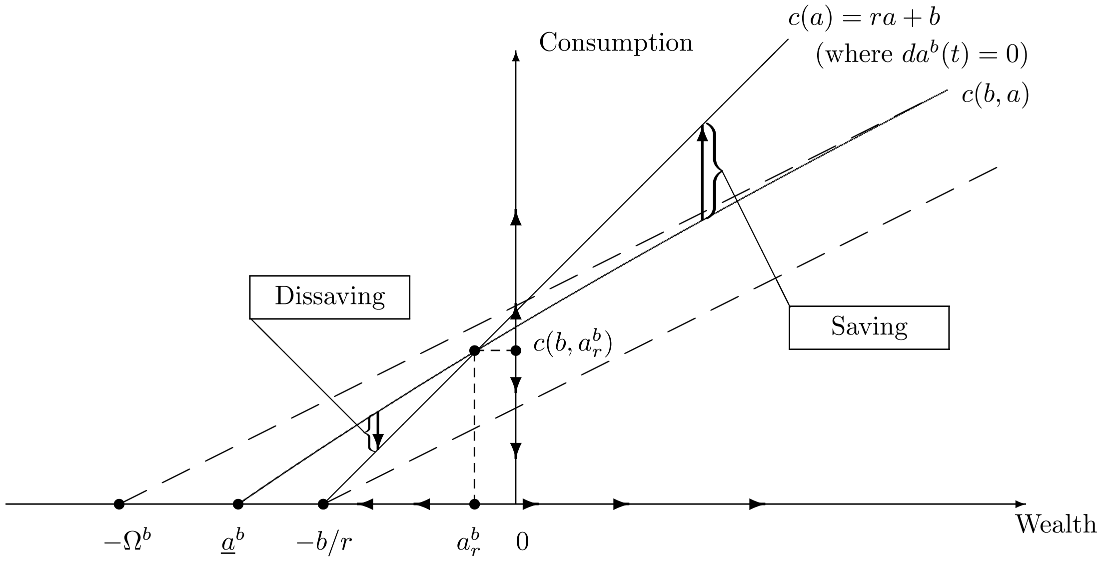

FIGURE 4. Saving behavior during unemployment if $\rho < r < \rho + \lambda_m$.

LEMMA 3. *For any interest rate r satisfying (45) there exists a level of wealth, $a_r^b > \underline{a}^b$, with the following properties. For $a < a_r^b$ an unemployed agent dissaves, for $a > a_r^b$ he saves, while at a_r^b he spends his total income on consumption, i.e., $c\left(b, a_r^b\right) = ra_r^b + b$. Furthermore, the level a_r^b is decreasing in r and $\lim_{r \searrow \rho} a_r^b = \infty$ and $\lim_{r \nearrow \rho+\lambda_m} a_r^b = \underline{a}^b$.*

PROOF. We recall from Lemma 2 that only interest rates satisfying (45) are sensible for the case under consideration. Otherwise the agent would always save or always dissave. We show that (i) for any r satisfying (45) there exists a_r^b, (ii) a_r^b is decreasing in r, and (iii) the limit properties $\lim_{r \searrow \rho} a_r^b = \infty$ and $\lim_{r \nearrow \rho+\lambda_m} a_r^b = \underline{a}^b$ hold.

(i) Define on the interval $[\rho, \rho + \lambda_m] \times (\underline{a}_b, \infty)$ the function $h(r, a) \equiv r - \rho - \lambda_m \left[1 - \chi^r(a)^{-\sigma}\right]$, where the superscript "r" from $\chi^r(a)$ indicates that the optimal consumption rule and thus $\chi(a)$ depend on r. According to (46), the agent saves iff $h(r, a) > 0$. Recalling the properties of $\chi(a)$ stated in Section 3, we find that, for any fixed $r \in (\rho, \rho + \lambda)$, $h(r, a)$ is increasing in a starting from $r - \rho - \lambda_m < 0$ (as $a \searrow \underline{a}_b$) and converging toward $r - \rho > 0$ (as $a \nearrow \infty$). There exists thus a level $a_r^b > \underline{a}_b$ such that $h\left(r, a_r^b\right) = 0$, $h(r, a) < 0$ for $a < a_r^b$, and $h(r, a) > 0$ for $a > a_r^b$.

(ii) Consider two interest rate $r_1 < r_2$ satisfying (45). As shown in (i), there exist a_1^b and a_2^b with $h\left(r_i, a_i^b\right) = 0$, $i = 1, 2$. Then, as $r_2 > r_1$, we conclude that by plausibility — higher interest rates trigger a higher saving rate — for r_2 the agent saves at a_1^b which means that $h\left(r_2, a_1^b\right) > 0$. Hence, again with (i), we now deduce that $a_1^b > a_2^b$, which shows that a_r^b is decreasing in r.

(iii) Consider the limits $A \equiv \lim_{r \searrow \rho} a_r^b$ and $B \equiv \lim_{r \nearrow \rho+\lambda_m} a_r^b$ and assume that $A < \infty$ and $B > \underline{a}^b$. Since r is drawn from a closed interval and since a_r^b is decreasing in r (as shown in (ii)), it follows that $A = \max_{r \in [\rho, \rho+\lambda_m]} a_r^b$ and $h(\rho, A) = 0$ as well as $B = \min_{r \in [\rho, \rho+\lambda_m]} a_r^b$ and $h(\rho + \lambda_m, B) = 0$. Thus, the unemployed agent does not dissave for $r = \rho$ and $a = A$ as well as he does not save for $r = \rho + \lambda_m$ and $a = B$. This behavior is a contradiction to points 1a) and 2a) in Lemma 2. We hence conclude that $A = \infty$ and $B = \underline{a}^b$. □

REMARK 3. *Interestingly, some of the previous results can also be derived graphically. Consider Figure 4 that illustrates the case of unemployment for* $\rho < r < \rho + \lambda_m$*, i.e., points 1b), 2b), and 3 from Lemma 2. We know from Section 3 that the* $c(b,a)$*-curve lies between the* $c_{\min}(a)$*- and the* $c_E(b,a)$*-line (depicted by the lower and upper dashed line, respectively). These lines start in* $-b/r$ *and* $-\Omega^b$*, respectively, and their slope is equal to* $[\rho - (1-\sigma) r]/r$*. The zero-motion line for* $a^b(t)$ *reads* $c = ra + b$*, starts as the* $c_{\min}(a)$*-line in* $-b/r$ *and has slope* r*. Since* $r > \rho$*, the zero-motion line is steeper than both the* $c_{\min}(a)$*- and the* $c_E(b,a)$*-line. Thus, as* $-\Omega^b < -b/r$*, the zero-motion line intersects the* $c_E(b,a)$*-line, while it lies always above that zero-motion line. It therefore intersects the* $c(b,a)$*-curve, which lies between the* $c_{\min}(a)$*- and the* $c_E(b,a)$*-line, at some* $a_r^b > -b/r > \underline{a}^b$*. We can then see that the agent dissaves for* $a < a_r^b$ *(the* $c(b,a)$*-curve lies above the zero-motion line) and saves for* $a > a_r^b$ *(the* $c(b,a)$*-curve lies below the zero-motion line), while at* a_r^b*,* $c\left(b, a_r^b\right) = ra_r^b + b$*.*

A.4.2. *Saving and dissaving while employed in a job.* We now consider the saving behavior for the job case. As in case of unemployment most of the following results could also be derived and illustrated graphically, cf. Remark 3.

LEMMA 4. *While the worker is employed in a job, he*

(1) *dissaves (and* $\frac{dc(w,a^w(t))}{c(w,a^w(t))} < 0$*) if* $r < \rho$ *and* $c(w,a) < \left(1 + \frac{\rho - r}{\lambda_s}\right)^{1/\sigma} c(b,a)$*;*
(2) *saves (and* $\frac{dc(w,a^w(t))}{c(w,a^w(t))} > 0$*) if either*
    (a) $r \geq \rho$ *or*
    (b) $r < \rho$ *and* $c(w,a) > \left(1 + \frac{\rho - r}{\lambda_s}\right)^{1/\sigma} c(b,a)$*;*
(3) *spends exactly his total current income on consumption if* $r < \rho$ *and* $c(w,a) = \left(1 + \frac{\rho - r}{\lambda_s}\right)^{1/\sigma} c(b,a)$*.*

PROOF OF LEMMA 4. According to the reduced form (20) we obtain that the agent saves iff

$$r > \rho - \lambda_s \left[\chi \left(a^w\right)^\sigma - 1\right]. \tag{47}$$

Since $\chi(a) > 1$, a sufficient condition for this inequality to hold true is simply $r \geq \rho$, which yields point 2a). Note that a necessary condition expressed in terms of primitives as in the case of unemployment cannot be given here since the right-hand of (47) tends toward $-\infty$ as $a \searrow \underline{a}_b$ and we assumed that $r > 0$. That means there is no positive lower bound for the right-hand side of (47) such that for interest rates below that bound, (47) does never hold true.

Points 1), 2b), and 3 follow immediately by rearranging (47). □

In the following lemma we present more precise results on the saving behavior if $r < \rho$.

LEMMA 5. *For any $r < \rho$ there exists a level of wealth, $a_r^w > \underline{a}_b$, with the following properties. The worker saves at $a < a_r^w$, dissaves at $a > a_r^w$, and spends his total income on consumption at a_r^w, i.e., $c(w, a_r^w) = ra_r^w + w$. Furthermore, the level a_r^w is increasing in r and* $\lim_{r\searrow 0} a_r^w \geq \underline{a}^b$ *and* $\lim_{r\nearrow\rho} a_r^w = \infty$.

PROOF. The arguments applied here are similar to those in the proof of Lemma 3. First note that, according to Lemma 4, only interest rates $r < \rho$ are sensible for the case under consideration. We show then that (i) for any $r < \rho$ there exists such an a_r^w, (ii) a_r^w is increasing in r, and (iii) the limit properties $\lim_{r\searrow 0} a_r^w \geq \underline{a}^b$ and $\lim_{r\nearrow\rho} a_r^w = \infty$ hold.

(i) In analogy to the case of unemployment, we define on the interval $(0, \rho] \times (\underline{a}_b, \infty)$ a function $h(r, a) \equiv r - \rho + \lambda_s \left[\chi^r \left(a^w\right)^\sigma - 1\right]$. According to (47), the agent saves iff $h(r, a) > 0$. For any fixed r, $h(r, a)$ is decreasing in a and tends toward ∞ as $a \searrow \underline{a}_b$ and toward $r - \rho < 0$ as $a \to \infty$. There exists thus a level a_r^w such that $h(r, a_r^w) = 0$, $h(r, a) > 0$ for $a < a_r^w$, and $h(r, a) < 0$ for $a > a_r^w$.

(ii) Consider two interest rate $r_1 < r_2 < \rho$ and denote for $i = 1, 2$ by a_i^w the level of wealth with $h(r_i, a_i^w) = 0$. Since $r_2 > r_1$, the agent saves for r_2 and at a_1^w, i.e., $h(r_2, a_1^w) > 0$. From (i) we then deduce that $a_1^w < a_2^w$.

(iii) We consider first the limit $A \equiv \lim_{r\nearrow\rho} a_r^w$. Assume that $A < \infty$. As a_r^w is increasing in r, we obtain that $A = \max_{r\in(0,\rho]} a_r^w$ and $h(\rho, A) = 0$. Hence, the agent does not save for $r = \rho$ and at $a = A$, which stands in contradiction to point 2a) in Lemma 4. Thus, $A = \infty$. We now turn to $\lim_{r\searrow 0} a_r^w$. Since according to (i) $a_r^w > \underline{a}^b$ for all $0 < r < \rho$, we conclude that $\lim_{r\searrow 0} a_r^w \geq \underline{a}^b$. Unfortunately, equality cannot be shown since we do not know whether $\lim_{r\searrow 0} h(r,a)$ is negative. (Only negativity would allow to prove equality, similarly to the derivation of $\lim_{r\nearrow\rho+\lambda_m} a_r^b = \underline{a}^b$ in point (iii) of the proof of Lemma 3.) □

A.5. The expected consumption growth. We first present a derivation of the expected consumption growth rate (27) and state then a technical lemma that is useful for the analysis in Section 6.

For sake of notational convenience we set $c(t) \equiv c(z(t), a(t))$. The expected consumption growth rate is given by the limit

$$E_{t_0}\frac{dc(t)}{c(t)} = \lim_{\Delta_t\to 0} E_{t_0}\frac{c(t+\Delta_t)-c(t)}{c(t)} = \lim_{\Delta_t\to 0} E_{t_0}\frac{1}{c(t)}E_t\left[c(t+\Delta_t)-c(t)\right], \tag{48}$$

where for the second step we use the properties of the conditional expectation. We now write Keynes-Ramsey rule (16) in integral notation, which yields

$$\begin{aligned} c(t) = \textstyle\int_{t_0}^t & \left\{\frac{r-\rho}{\sigma} + \lambda_s \frac{\left[\frac{c(\tau)}{c(b,a(\tau))}\right]^\sigma - 1}{\sigma} + \lambda_m \frac{\left[\frac{c(\tau)}{c(w,a(\tau))}\right]^\sigma - 1}{\sigma}\right\} c(\tau)\, d\tau \\ & + \textstyle\int_{t_0}^t \left[c(b,a(\tau)) - c(\tau_-)\right] dq_s(\tau) + \int_{t_0}^t \left[c(w,a(\tau)) - c(\tau_-)\right] dq_m(\tau). \end{aligned}$$

Inserting this expression into (48) and using the martingale property of the compensated Poisson processes gives[24]

$$E_{t_0}\frac{dc(t)}{c(t)} = \lim_{\Delta_t\to 0} E_{t_0}\frac{1}{c(t)}E_t \textstyle\int_t^{t+\Delta t} \left\{\begin{array}{c} \frac{r-\rho}{\sigma} + \lambda_s \frac{\left[\frac{c(\tau)}{c(b,a(\tau))}\right]^\sigma - 1}{\sigma} + \lambda_m \frac{\left[\frac{c(\tau)}{c(w,a(\tau))}\right]^\sigma - 1}{\sigma} \\ \lambda_s \left[\frac{c(b,a(\tau))}{c(\tau)} - 1\right] + \lambda_m \left[\frac{c(w,a(\tau))}{c(\tau)} - 1\right] \end{array}\right\} c(\tau)\, d\tau.$$

By interchanging limit and expectation and applying that $E_t x(t) = x(t)$ for all $(z(t), a(t))$-measurable random variables $x(t)$, we obtain further

$$E_{t_0}\frac{dc(t)}{c(t)} = E_{t_0}\left\{\begin{array}{c} \frac{r-\rho}{\sigma} + \lambda_s\left\{\frac{\left[\frac{c(t)}{c(b,a(t))}\right]^\sigma - 1}{\sigma} + \left[\frac{c(b,a(t))}{c(t)} - 1\right]\right\} \\ +\lambda_m\left\{\frac{\left[\frac{c(t)}{c(w,a(t))}\right]^\sigma - 1}{\sigma} + \left[\frac{c(w,a(t))}{c(t)} - 1\right]\right\} \end{array}\right\} dt.$$

[24]Roughly speaking, the martingale property yields for $t \le T$ that $E_t d\tilde{q}(T) = 0$, where $\tilde{q}(t) = q(t) - \lambda t$ and $q(t)$ is a Poisson process with arrival rate λ. As a consequence, for any integrable stochastic process $x(t)$ one obtains that $E_t \int_t^T x(\tau)\, d\tilde{q}(\tau) = 0$, cf. also García and Griego (1994).

As $z(t)$ in $c(t) = c(z(t), a(t))$ can only take the values b and w, we can write, using the linearity of the expectation operator,

$$E_{t_0}\frac{dc(t)}{c(t)} = \left\{ \begin{array}{c} \frac{r-\rho}{\sigma} + \lambda_s E_{t_0}\left\{1_{\{z(t)=w\}}\left[\frac{\left[\frac{c(w,a(t))}{c(b,a(t))}\right]^{\sigma}-1}{\sigma} + \frac{c(b,a(t))}{c(w,a(t))} - 1\right]\right\} \\ +\lambda_m E_{t_0}\left\{1_{\{z(t)=b\}}\left[\frac{\left[\frac{c(b,a(t))}{c(w,a(t))}\right]^{\sigma}-1}{\sigma} + \frac{c(w,a(t))}{c(b,a(t))} - 1\right]\right\} \end{array} \right\} dt. \tag{49}$$

Since for given a_0 and z_0 the employment status at $t \geq t_0$ is independent of the level of wealth at t, we obtain, for example, further

$$\begin{aligned} E_{t_0}\left\{1_{\{z(t)=w\}}\frac{\left[\frac{c(w,a(t))}{c(b,a(t))}\right]^{\sigma}-1}{\sigma}\right\} &= E_{t_0}\left[1_{\{z(t)=w\}}\right] E_{t_0}\left\{\frac{\left[\frac{c(w,a(t))}{c(b,a(t))}\right]^{\sigma}-1}{\sigma}\right\} \\ &= p^w_{z_0}(t_0,t)\, E_{t_0}\left\{\frac{\left[\frac{c(w,a(t))}{c(b,a(t))}\right]^{\sigma}-1}{\sigma}\right\}, \end{aligned}$$

where for the second equality we refer to Appendix A.1.1. Proceeding analogously with the other conditional expectations in (49), we finally arrive at differential (27) from the text.

The subsequent lemma applies when we show that risk of unemployment yields higher expected consumption growth.

LEMMA 6. *Define for $\sigma, \alpha > 0$ the function $H(x) \equiv \frac{x^{\sigma}-1}{\sigma} - \frac{1-x^{-\alpha}}{\alpha}$, $x > 0$. Then for all $x \neq 1$, $H(x) > 0$.*

PROOF. Obviously, $H(1) = 0$. As the derivative $H'(x) \equiv x^{\sigma-1} - x^{-(\alpha+1)}$ is greater (less) than zero for $x > 1$ ($x < 1$), we find that $H(x)$ is increasing (decreasing) for $x > 1$ ($x < 1$). Jointly with $H(1) = 0$ that means that $H(x)$ is strictly positive for all $x > 0$, $x \neq 1$. $\square$

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