C_0-Semigroup Methods for Delay Equations

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Contents

Introduction

The study of equations describing the time evolution of systems in natural sciences and engineering, in which the history of the system is among the driving forces, has seen a rapid development in the last decades. Typical examples are (possibly controlled) systems with a delayed feedback or with memory effects. In this thesis we will contribute to the theory of linear delay equations. Throughout this work we assume that any operator is linear. Let X be a Banach space, $h \in (0, \infty]$ and $J := [-h, 0]$ if $h < \infty$ and $J := (-\infty, 0]$ otherwise. We are particularly concerned with the well-posedness and asymptotic behaviour of the following linear delay equations

$$\dot{u}(t) = Au(t) + Lu_t, \quad u(0) = x, \quad u_0 = f \quad (t \geq 0), \tag{0.0.1}$$

$$u(t) = Lu_t, \quad u_0 = f \quad (t \geq 0), \tag{0.0.2}$$

$$\dot{u}(t) = Au(t) + \int_0^t d\ell(s)u(t-s), \quad u(0) = x \quad (t \geq 0), \tag{0.0.3}$$

$$\dot{u}(t) = Au(t) + \int_0^t \ell(t-s)\dot{u}(s)\, ds, \quad u(0) = x \quad (t \geq 0), \tag{0.0.4}$$

where $x \in X$ is a given initial state, and $f \colon J \to X$ is the given history of the system. The Banach space X is the space of states of the system. The function $u \colon [-h, \infty) \to X$ (for (0.0.1) and (0.0.2)) and $u \colon [0, \infty) \to X$ (for (0.0.3) and (0.0.4)), respectively, represents the states of the system started at $t = 0$ as a function of time. (For the first two equations we recall that u_t is a notation for the function $s \mapsto u(t+s)$, or shortly $u(t + \cdot)$.) In most cases continuity of $u|_{[0,\infty)}$ is a natural assumption which we will adopt throughout this thesis. The operator A is always assumed to be the generator of a (linear) C_0-semigroup on X. The operator-valued function ℓ maps $[0, \infty)$ into the operators $\mathcal{L}(X)$ or more generally into $\mathcal{L}(Y, Z)$, where Y and Z are Banach spaces related to Sobolev or fractional power spaces associated with A. The delay operator L, occuring in (0.0.1) and (0.0.2), is assumed to act on a function space of X-valued functions with domain J. In applications L can often be written as the Riemann-Stieltjes type integral

$$Lf = \int_{-h}^0 d\eta(s)f(s),$$

where η is an $\mathcal{L}(X)$-valued function (or more generally an $\mathcal{L}(Y, Z)$-valued function with Y and Z as above) of bounded variation and $f \in C_b(J; X)$.

Our main pool of concepts and methods which we are going to use originates in the theory of C_0-semigroups. The structure of C_0-semigroups is the natural mathematical model for an autonomous deterministic system. However, one has to choose a suitable state space before a system can be expressed by an abstract Cauchy problem and thus be tackled by C_0-semigroup methods. In the case of the delay equations listed above one should be aware of the fact that they are not autonomous when considered in the space X, as the history of the system is not included in this state space.

For the equations (0.0.1) and (0.0.3) delay and Volterra semigroups have been utilised. These semigroups act on the product of X and a function space of X-valued functions. In the case of delay semigroups the history of the system is stored in the second component of the product space. For Volterra semigroups in the context of (0.0.3) the second component contains the function $\int_0^t d\ell(s+\cdot)u(t-s)$, so that the Dirac functional δ_0 applied to the second component yields the integral term of (0.0.3).

In both cases the choice of the function space leads to additional regularity conditions on L and ℓ (necessary for the applicability of perturbation theorems) for which the corresponding abstract Cauchy problem becomes well-posed. In this work we use fractional power spaces associated with the (weak) derivative on spaces of continuous and L_p-integrable functions for these semigroups to explore (0.0.1) and (0.0.3). As a preparation we generalise the Miyadera-Voigt and the Desch-Schappacher perturbation theorem by shifting them on the scale of fractional power spaces associated with the generator to be perturbed.

Equation (0.0.4) can be treated by Volterra semigroup methods similar to (0.0.3). In contrast to (0.0.3) we can also write (0.0.4) in the form of (0.0.1) without loosing the differentiability of $u|_{\mathbb{R}_+}$ by choosing the history $u_0 := x \cdot \mathbf{1}_{(-\infty,0]}$. Then the weak derivative $\dot{u}$ exists and becomes zero on the negative time axis. Since we only have the existence as a weak derivative we cannot express it in the framework of continuous functions. Moreover the necessary perturbation arguments only work in the L_p-context.

The equation (0.0.2) is of a different kind. It can be dealt with by left translation semigroups on the space $L_p(-h,0;X)$. These C_0-semigroups are generated by the weak derivative on $L_p(-h,0;X)$ with a boundary condition at 0. They have been studied for operators $L \in \mathcal{L}(L_p(J;X),X)$ and for special cases such as $Lf = \delta_{-h}f$ ($f \in W_p^1(-h,0;X)$) and delay operators associated with flows in networks. We present a general approach unifying these cases by extending the Desch-Schappacher perturbation theorem. The perturbation result for translation semigroups is generalised to the corresponding boundary perturbations of evolution semigroups induced by backward propagators.

As translations are part of delay and Volterra semigroups our investigations fit well into this work. So for example, perturbation arguments for delay and translation semigroups hold for similar delay operators and share common estimates.

For many evolutionary systems the asymptotic behaviour is of great interest. In this thesis we also devote ourselves to topics in the field of the asymptotics of evolution equations. First we will study domination of C_0-semigroups acting on Banach lattices. This notion is of interest for the understanding of the asymptotic behaviour of suitably dominated C_0-semigroups, as the analysis of a C_0-semigroup is often simplified if this semigroup is positive. This can be used to derive asymptotic properties of dominated semigroups from dominating ones. We contribute methods for the determination of smallest dominating C_0-semigroups, so-called modulus semigroups. Besides other examples we apply our results to Volterra semigroups related to integro-differential equations, linking these results with the equations above.

We also derive spectral conditions for the strong stability of solutions of (0.0.4), using the notion of the half line spectrum and other recent methods and results from Laplace transform theory and harmonic analysis. These have been rapidly developed in the last decade and successfully applied to various delay equations. In our studies we will assume that a solution operator family exists for (0.0.4) given by the first part of a delay semigroup.

We could not apply standard C_0-semigroup techniques for these investigations as the involved Volterra and delay semigroups are generally not bounded due to the fact that they contain translations on unbounded intervals. If ℓ or L satisfy additional growth bound conditions, rescaling of the translation parts will be applicable leading to bounded semigroups. In particular the spectral behaviour of solutions of the equations in the neighbourhood of $i\mathbb{R}$ will become visible in the spectrum of the generator of the corresponding delay or Volterra semigroups. However, in the context of aeroelasticity, which was the motivation for our studies, such assumptions do not hold.

Even though the chapters in this thesis are only loosely connected they share a common origin. Our starting point were the two closely related works [48] and [71].

In [48] the authors prove a perturbation theorem for delay equations in the L_p-context which is of interest for equations in aeroelasticity modelling the flutter of aerofoils under aerodynamic load in a subsonic airflow. These equations can be written in the form of (0.0.4). This observation initiated the works on the well-posedness of the equations mentioned above as well as the analysis of the strong stability of solutions of (0.0.4). (Strong stability is the type of stability which engineers in this field of aeroelasticity are striving to understand.) The results are presented in the Chapters 3 and 4.

The paper [71] deals with the problem of determining the modulus semigroup of delay semigroups in the L_p-context and presents a partial answer. In the search for a complete answer questions were raised which are presented and solved in the Chapters 1 and 2.

The thesis is organised as follows.

In Chapter 1 we prove approximation formulas for modulus semigroups and their generators. Our main tool is a sandwiching result for sequences of C_0-semigroups. The second part of this chapter is devoted to various applications.

In Chapter 2 we mainly study translation semigroups on L_p-spaces. We present a unified approach to different boundary perturbations of the weak derivative on $L_p(-h, 0; X)$ with zero boundary condition at 0. As a preparation we generalise the Desch-Schappacher perturbation theorem by closely examining the Volterra operator approach to this perturbation theorem published in [39; Section III.3(a)]. The generalised Desch-Schappacher perturbation theorem is also applied to boundary perturbations of evolution semigroups induced by backward propagators.

We also determine the modulus semigroup of translation semigroups and discuss an application to flows in networks. Last we deal with certain delay semigroups on spaces of continuous functions and their modulus semigroups.

Chapter 3 presents two approaches to (0.0.4) via Volterra and delay semigroups. We investigate the relation to evolutionary integral equations as presented in [58]. Finally we derive conditions for the strong stability of solutions of (0.0.4).

The first part of Chapter 4 is devoted to the development of various perturbation results for operators acting on the scale of fractional power spaces associated with generators of C_0-semigroups. We are particularly concerned with the shifting of the Miyadera-Voigt and the Desch-Schappacher perturbation theorems on these scales. In the second part we apply these perturbation results to Volterra and delay semigroups. This yields numerous well-posedness results for inhomogeneous abstract Cauchy problems and delay equations with fractional regularity conditions on the inhomogeneities and the delay part, respectively.

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Chapter 1

Approximation of Modulus Semigroups and Their Generators

Let T and S be C_0-semigroups on a Banach lattice X. We say that T *is dominated by* S if
$$|T(t)x| \le S(t)|x| \quad (t \ge 0,\ x \in X).$$
If there exists a smallest C_0-semigroup dominating T it is called the *modulus semigroup* and denoted by $T^\sharp$. Similarly, we denote its generator by adding the superscript $\sharp$ to the generator of T. If X has order continuous norm then any dominated C_0-semigroup on X has a modulus semigroup; cf. [11; Theorem 2.1]. If X is a KB-space and T is a C_0-semigroup which is quasi-contractive with respect to the regular norm then T has a modulus semigroup; cf. [71; Proposition A.1].

For a norm continuous semigroup T on an order complete Banach lattice, generated by a regular operator A, it was shown in [11] that
$$A^\sharp = \lim_{t\to 0} \frac{1}{t}(|T(t)| - I), \tag{1.0.1}$$
where the limit exists in operator norm. One of the aims of this chapter is the investigation of the validity of (1.0.1) in a more general context. If T is a C_0-semigroup on a Banach lattice with order continuous norm and T possesses a modulus semigroup then we show that (1.0.1) is valid, where the limit holds in the strong resolvent sense; cf. Corollary 1.3.3. The crucial step for the proof of this fact is the following sandwiching result for sequences of C_0-semigroups. For the notion of convergence for a sequence of C_0-semigroups we refer to Remark B.1(a).

1.0.1 Theorem. *Let X be a Banach lattice with order continuous norm. For $n \in \mathbb{N}$, let T_n, S_n, and U_n be C_0-semigroups on X, T_n consisting of regular operators, and*
$$|T_n(t)| \le U_n(t) \le S_n(t) \quad (t \ge 0).$$
Further we assume that (T_n) converges to a C_0-semigroup T, that T possesses a modulus semigroup, and that (S_n) converges to $T^\sharp$.

Then (U_n) converges to $T^\sharp$.

This result will be proved in Section 1.2.

In the paper [11] mentioned above it was also shown that for norm continuous semigroups the modulus semigroup is given by the Chernoff product formula
$$T^\sharp(t) = \operatorname*{s-lim}_{n\to\infty} |T(t/n)|^n.$$
As a consequence of Theorem 1.0.1 we obtain the Chernoff product formula and similar approximation formulas for the modulus semigroup of (not necessarily norm continuous) C_0-semigroups on a Banach lattice with order continuous norm; cf. Section 1.3.

The chapter is organised as follows.

In Section 1.1 we prove a characterisation of order continuity of the norm in a Banach lattice. This result is used for an improvement of a sandwiching lemma for sequences of operators.

Section 1.2 is devoted to the proof of Theorem 1.0.1, where the characterisation in Proposition 1.1.1 plays a decisive role.

In Section 1.3 we draw conseqences concerning the approximation of the generator of the modulus semigroup and concerning Chernoff product formulas for the modulus semigroup.

In Section 1.4 we present an operator norm version of the Chernoff product formula stated above in the context of norm continuous semigroups.

In Section 1.5 we apply the previously established theory to perturbations of semigroups by bounded operators. The abstract result is then applied to matrix semigroups and multiplication semigroups. At the end of this section we study the modulus of Volterra semigroups.

We also refer to Appendices B and C where we supplement the main body of this chapter. In Appendix B we shortly review the notions of strong resolvent convergence of generators and strong convergence of C_0-semigroups. We further introduce a general type of approximation of the generator of a C_0-semigroup. In Appendix C we review the Chernoff product formula in a general context for which we could not find a reference in the literature. Also, we note an operator norm version of Chernoff's product formula.

1.1 Weakly and Norm Convergent Sequences

We start this section by adding a further property to the numerous properties characterising order continuity of the norm of a Banach lattice; cf. [50; Theorem 2.4.2 and Corollary 2.4.3]. We recall that the norm on a Banach lattice X is called *order continuous* if each downward directed system in X_+ with infimum 0, norm converges to 0.

1.1.1 Proposition. *A Banach lattice X has order continuous norm if and only if any order bounded weak null sequence in X_+ is a norm null sequence.*

Proof. The necessity follows easily from [2; Lemma 4.12.15 and Theorem 4.12.14].

In order to show the sufficiency we use that X has order continuous norm if and only if any order bounded disjoint sequence in X_+ converges to zero (cf. [50; Theorem 2.4.2]). Let $z \in X_+$, and assume that the sequence $(x_n) \subseteq [0, z]$ is disjoint. For $x' \in X'_+$ we conclude

$$0 \le \sum_{n=1}^{k} x'(x_n) = x'\left(\sum_{n=1}^{k} x_n\right) \le x'(z) \quad (k \in \mathbb{N}).$$

Hence $x'(x_n)$ is a null sequence, and thus (x_n) converges weakly to zero. The assumption implies that (x_n) converges to zero in norm. ∎

The property in the following corollary will be needed in the proof of the result given subsequently as well as in the following sections. In fact, this property is equivalent to the order continuity of the norm.

1.1.2 Corollary. *Let X be a Banach lattice with order continuous norm. Let (x_n) and (y_n) be sequences in X, and assume that $0 \leq x_n \leq y_n$ $(n \in \mathbb{N})$. If $x_n \to y \in X$ weakly and $y_n \to y$ in norm as $n \to \infty$ then $x_n \to y$ in norm.*

Proof. Let (y_{n_k}) be a subsequence of (y_n) with $\sum_{k=1}^{\infty} \|y_{n_{k+1}} - y_{n_k}\| < \infty$. Then $z := |y_{n_1}| + \sum_{k=1}^{\infty} |y_{n_{k+1}} - y_{n_k}|$ exists and satisfies $z \geq y_{n_k}$ for all $k \in \mathbb{N}$. For $z_k := y_{n_k} - x_{n_k}$ we have $z_k \to 0$ weakly as $k \to \infty$ and $z_k \in [0, z]$ for all $k \in \mathbb{N}$. Proposition 1.1.1 implies that $z_k \to 0$, and therefore $x_{n_k} \to y$ $(k \to \infty)$ in norm. Since this argument can be applied to any subsequence of (x_n) we see that $x_n \to y$ as $n \to \infty$. ■

From Corollary 1.1.2 we infer the following improvement of the sandwiching result [71; Lemma 3.5] for sequences of operators, giving a positive answer to a question asked in [71; Remark 3.6].

1.1.3 Lemma. *Let X be an Archimedean vector lattice, and let Y be a Banach lattice with order continuous norm. Assume that (A_k), (B_k), (C_k) are sequences of operators in $L^r(X,Y)$, $|A_k| \leq B_k \leq C_k$ $(k \in \mathbb{N})$, $A \in L^r(X,Y)$, and $A_k \to A$, $C_k \to |A|$ $(k \to \infty)$ in the strong operator topology. Then $B_k \to |A|$ $(k \to \infty)$ in the strong operator topology.*

We refer to [61; Chapter IV, §1], [50; Section 1.3] for regular operators in the context of Lemma 1.1.3.

Proof of Lemma 1.1.3. As in [71; Lemma 3.5] we obtain that $B_k \to |A|$ in the weak operator topology. Corollary 1.1.2 then implies $B_k \to |A|$ in the strong operator topology. ■

1.1.4 Remark. (a) If X is a Banach lattice then all the operators occuring in Lemma 1.1.3 are bounded (cf. [61; Theorem II.5.3]).

The reduced assumption on X allows for the application to unbounded operators defined on sublattices of some Banach lattice.

(b) Assume that $X = Y$ in Lemma 1.1.3. The following two examples illustrate that Lemma 1.1.3 cannot be improved in two respects. We cannot omit the order continuity of the norm nor obtain convergence of (B_k) in operator norm by imposing convergence in operator norm on (A_k) and (C_k).

Let $X_n := \mathbb{R}^{2^n}$ be equipped with the Euclidean norm. There exist operators T_n in $\mathcal{L}(X_n)$ with $\|T_n\| = 2^{-n/2}$ and $\|\,|T_n|\,\| = 1$ (cf. [2; Example 5.16.6]). Let

$$X := \{(x_n)_{n\in\mathbb{N}};\, x_n \in X_n, \|(x_n)\|_\infty := \sup_{n\in\mathbb{N}} \|x_n\| < \infty\}$$

and $Y \subseteq X$ the (closed) subspace of all null sequences in X. Then X is an order complete Banach lattice and Y has order continuous norm. For $k \in \mathbb{N}$ we define the operators

$$A := \operatorname{diag}(T_n)_{n\in\mathbb{N}}, \quad A_k := \operatorname{diag}(T_1, T_2, \ldots, T_k, 0 \ldots),$$

on X. We have $|A| = \operatorname{diag}(|T_n|)_{n\in\mathbb{N}}$, $|A_k| = \operatorname{diag}(|T_1|, \ldots, |T_k|, 0, \ldots)$, $A_k \to A$ in operator norm, and $C_k := |A| \geq |A_k|$ $(k \in \mathbb{N})$. However, $(|A_k|)$ does not converge to $|A|$

in the weak operator topology. In order to see this let $x = (x_n) \in X$, $\|x\|_\infty \le 1$, such that $\||T_n|x_n\| = 1$ $(n \in \mathbb{N})$. As Y is a closed subspace of X there exists $x' \in X'$, $x'|_Y = 0$ and $x'(|A|x) = 1$. Since $|A_k|x \in Y$ $(k \in \mathbb{N})$ we have $x'(|A_k|x) = 0$ and thus $(|A_k|x)$ does not converge weakly to $|A|x$.

Now we consider the restrictions $\tilde{A}$, $\tilde{A}_k$, $\tilde{B}_k$ and $\tilde{C}_k$ of A, A_k, B_k and C_k to the subspace Y, respectively. As before $(\tilde{A}_k)$ tends to $\tilde{A}$ in operator norm, $|\tilde{A}_k| \le |\tilde{A}|$. In accordance with Lemma 1.1.3 $(|\tilde{A}_k|)$ converges to $|\tilde{A}|$ in the strong operator topology. However, $(|\tilde{A}_k|)$ does not converge in operator norm.

1.2 Proof of the Sandwiching Result for C_0-Semigroups

The aim of this section is the proof of the sandwiching result Theorem 1.0.1. In order to motivate this result we include the following simple observation.

1.2.1 Remarks. (a) Let X be a Banach lattice. For $n \in \mathbb{N}$, let T_n, U_n, S_n, T, U and S be C_0-semigroups on X,

$$|T_n(t)x| \le U_n(t)|x| \le S_n(t)|x| \quad (x \in X,\ t \ge 0,\ n \in \mathbb{N}),$$

let T, U, and S be C_0-semigroups on X, and assume that (T_n) converges to T, (U_n) to U, and (S_n) to S. Then

$$|T(t)x| \le U(t)|x| \le S(t)|x| \quad (x \in X,\ t \ge 0).$$

In particular, if T possesses a modulus semigroup and (S_n) converges to $T^\sharp$ then (U_n) converges to $T^\sharp$.

(b) Assume additionally that X is order complete. Let T be a C_0-semigroup on X consisting of regular operators, and assume that there exists a sequence $(t_n) \subseteq (0, \infty)$, $t_n \to 0$ $(n \to \infty)$, such that the sequence $(\tilde{A}_n) := \big(\frac{1}{t_n}(|T(t_n)| - I)\big)$ converges to a generator $\tilde{A}$ in the strong resolvent sense, as $n \to \infty$. Then the C_0-semigroup $\tilde{T}$ generated by $\tilde{A}$ is the modulus semigroup of T.

Indeed, let S be a C_0-semigroup dominating T, and define $A_n := \frac{1}{t_n}(T(t_n) - I)$, $B_n := \frac{1}{t_n}(S(t_n) - I)$ $(n \in \mathbb{N})$. Then $|e^{tA_n}| \le e^{t\tilde{A}_n} \le e^{tB_n}$ for all $t \ge 0$, $n \in \mathbb{N}$. Further, the sequences (A_n) and (B_n) converge to A and B in the strong resolvent sense, respectively; cf. Remark B.3(a). Therefore part (a) above implies that $\tilde{T}$ dominates T and is dominated by S, and this implies the assertion.

The important point of Theorem 1.0.1 is that the convergence of the sequence (U_n), which was part of the hypothesis in Remark 1.2.1(a), can in fact be concluded if the Banach lattice has order continuous norm.

For the proof of this sandwiching result we need several preparations. First, for $x, y \in X$, $y \ge 0$ we introduce the *truncation of x by y*, denoted by $\tau(y)x$, defined as the element uniquely determined by the properties

(i) $|\tau(y)x| = |x| \wedge y$,
(ii) $(\operatorname{Re}\gamma\tau(y)x)_+ \le (\operatorname{Re}\gamma x)_+$ for all $\gamma \in \mathbb{K}$, $|\gamma| = 1$
(cf. [71; Section 1]). We recall that, for $x_1, x_2 \in X$, $y_1, y_2 \in X_+$ we have

$$|\tau(y_1)x_1 - \tau(y_2)x_2| \le |x_1 - x_2| + |y_1 - y_2|. \tag{1.2.1}$$

In particular we shall use that if $x \in X$ and $(y_n) \subseteq X_+$ with $y_n \to y \in X$ $(n \to \infty)$ then $\tau(y_n)x \to \tau(y)x$.

A second preparation consists in a formula interchanging the supremum of a set with a positive operator.

1.2.2 Lemma. *Let X be a Banach lattice with order continuous norm. Let $A \in \mathcal{L}(X)_+$, $M \subseteq X_+$ order bounded. Let $\mathcal{P}_{\mathrm{f}}(M) := \{F \subseteq M;\, F \text{ finite}\}$, directed by inclusion. Then*

$$\begin{aligned} A(\sup M) &= A(\sup_{F\in\mathcal{P}_{\mathrm{f}}(M)} (\sup F)) = A(\lim_{F\in\mathcal{P}_{\mathrm{f}}(M)} (\sup F)) \\ &= \lim_{F\in\mathcal{P}_{\mathrm{f}}(M)} A(\sup F) = \sup_{F\in\mathcal{P}_{\mathrm{f}}(M)} A(\sup F). \end{aligned}$$

Proof. The equalities follow from the order continuity of the norm of X, the continuity of A, and the fact that the net $(A(\sup F))_{F\in\mathcal{P}_{\mathrm{f}}(M)}$ is increasing. ■

Finally, we single out the following technical result.

1.2.3 Lemma. *Let X be a Banach lattice with order continuous norm. Let $A_1, \dots, A_m \in \mathcal{L}^r(X)$, for $1 \le j \le m$ let $(A_{jk})_k$ be a sequence in $\mathcal{L}^r(X)$, and $A_{jk} \to A_j$ $(k \to \infty)$ in the strong operator topology. Further, let $x, y, y_k \in X_+$ $(k \in \mathbb{N})$, $y_k \to y$ $(k \to \infty)$ weakly, and*

$$|A_{mk}| \cdots |A_{1k}|x \le y_k$$

for all $k \in \mathbb{N}$. Then

$$|A_m| \cdots |A_1|x \le y. \tag{1.2.2}$$

Proof. For $z \in X_+$, the solid hull of the element z will be denoted by

$$\operatorname{sol}\{z\} := \{x \in X;\, |x| \le z\}.$$

We consider $(m-1)$-tuples $(Z_0, \dots, Z_{m-2})$ of non-empty finite subsets of X with the following property:

$$Z_j \subseteq \operatorname{sol}\{z_j\} \quad \text{for all} \quad j = 0, \dots, m-2, \tag{1.2.3}$$

where

$$\begin{aligned} &z_0 := x, \\ &z_{j+1} := \sup\{|A_{j+1}z|;\, z \in Z_j\} \quad \text{for} \quad j = 0, \dots, m-2. \end{aligned}$$

For an $(m-1)$-tuple satisfying (1.2.3), and for $k \in \mathbb{N}$, $j = 0, \dots, m-2$ we define

$$\begin{aligned} &z_{0k} := z_0 \ (= x), \\ &Z_{jk} := \{\tau(z_{jk})z;\ z \in Z_j\}, \quad z_{j+1,k} := \sup\{|A_{j+1,k}z|;\ z \in Z_{jk}\}. \end{aligned}$$

Using property (i) of the truncation τ we obtain

$$\begin{aligned} Z_{jk} &\subseteq \mathrm{sol}\{z_{jk}\}, \\ z_{j+1,k} &\le \sup\{|A_{j+1,k}|\,|z|;\ z \in Z_{jk}\} \le |A_{j+1,k}|\, z_{jk}, \end{aligned} \tag{1.2.4}$$

for $k \in \mathbb{N}$, $j = 0, \dots, m-2$.

Next we show that

$$z_{jk} \to z_j \quad (k \to \infty) \tag{1.2.5}$$

for all $0 \le j \le m-1$. For $j = 0$ this is trivial. Assume that (1.2.5) is shown for some $0 \le j \le m-2$, and let $z \in Z_j$. Then the properties of τ and the inclusion $Z_j \subseteq \mathrm{sol}\{z_j\}$ imply that $\tau(z_{jk})z \to \tau(z_j)z = z$, $|A_{j+1,k}(\tau(z_{jk})z)| \to |A_{j+1}z|$ $(k \to \infty)$. Hence we obtain (recall that Z_j is finite)

$$\begin{aligned} z_{j+1,k} &= \sup\{|A_{j+1,k}z|;\ z \in Z_{jk}\} = \sup\{|A_{j+1,k}(\tau(z_{jk})z)|;\ z \in Z_j\} \\ &\to \sup\{|A_{j+1}z|;\ z \in Z_j\} = z_{j+1} \quad (k \to \infty). \end{aligned}$$

Now let $z \in \mathrm{sol}\{z_{m-1}\}$. Then, using inequality (1.2.4), we obtain

$$\begin{aligned} |A_{mk}(\tau(z_{m-1,k})z)| &\le |A_{mk}|\, z_{m-1,k} \le |A_{mk}|\,|A_{m-1,k}|\, z_{m-2,k} \\ &\le \cdots \le |A_{mk}|\,|A_{m-1,k}| \cdots |A_{1k}|\, z_{0k} \\ &= |A_{mk}|\,|A_{m-1,k}| \cdots |A_{1k}|\, x \le y_k, \end{aligned} \tag{1.2.6}$$

for all $k \in \mathbb{N}$. Because of (1.2.5), inequality (1.2.6) implies

$$|A_m z| = |A_m(\tau(z_{m-1})z)| \le y.$$

This implies

$$|A_m| z_{m-1} = \sup\{|A_m z|;\ |z| \le z_{m-1}\} \le y.$$

This inequality can also be written as

$$|A_m| \sup\{|A_{m-1}z|;\ z \in Z_{m-2}\} \le y.$$

This holds for arbitrary finite subsets $Z_{m-2} \subseteq \mathrm{sol}\{z_{m-2}\}$. Therefore Lemma 1.2.2 implies

$$\begin{aligned} |A_m|\,|A_{m-1}|\, z_{m-2} &= |A_m| \sup_{Z_{m-2} \in \mathcal{P}_{\mathrm{f}}(\mathrm{sol}\{z_{m-2}\})} \sup\{|A_{m-1}z|;\ z \in Z_{m-2}\} \\ &= \sup_{Z_{m-2} \in \mathcal{P}_{\mathrm{f}}(\mathrm{sol}\{z_{m-2}\})} |A_m| \sup\{|A_{m-1}z|;\ z \in Z_{m-2}\} \le y. \end{aligned}$$

Iterating this argument we finally obtain (1.2.2). ∎

Proof of Theorem 1.0.1. We have to show that for all $t > 0$ the sequence $(U_n(t))_n$ converges to $T^\sharp(t)$, in the strong operator topology. (The fact that the semigroups have a common exponential bound then implies the convergence of (U_n) to $T^\sharp$; cf. [57; Theorem 3.4.2].)

Let $x \in X_+$. The convergence $S_n(t)x \to T^\sharp(t)x$ implies that the solid hull of the set $\{S_n(t)x;\, n \in \mathbb{N}\}$ is relatively weakly compact (cf. [2; Theorem 4.13.8]), and therefore the set $\{U_n(t)x;\, n \in \mathbb{N}\}$ is relatively weakly compact. By the Eberlein-Šmulyan theorem there exists a subsequence $(U_{n_k})_{k\in\mathbb{N}}$ such that $(U_{n_k}(t)x)_{k\in\mathbb{N}}$ is weakly convergent, $y :=$ w-lim $U_{n_k}(t)x \le \lim S_{n_k}(t)x = T^\sharp(t)x$.

Let $m \in \mathbb{N}$, $t_1, \dots, t_m > 0$, $t_1 + t_2 + \cdots + t_m = t$. Then, by hypothesis,

$$|T_{n_k}(t_m)|\,|T_{n_k}(t_{m-1})| \cdots |T_{n_k}(t_1)|x \le y_k := U_{n_k}(t)x,$$

for all $k \in \mathbb{N}$. The application of Lemma 1.2.3 yields

$$|T(t_m)|\,|T(t_{m-1})| \cdots |T(t_1)|x \le y \le T^\sharp(t)x. \tag{1.2.7}$$

The validity of inequality (1.2.7) for arbitrary $m \in \mathbb{N}$, $t_1, \dots, t_m$ as above implies $y = T^\sharp(t)x$ (cf. [11; Theorem 2.1] and the paragraph preceding Corollary 1.3.3). As this argument can be applied to any subsequence of $(U_n(t))$ we conclude that $U_n(t)x \to T^\sharp(t)x$ weakly. Corollary 1.1.2 shows that $U_n(t)x \to T^\sharp(t)x$ in norm. ∎

1.3 Approximation of Modulus Semigroups and their Generators

In this section we assume that X is a Banach lattice with order continuous norm. For a C_0-semigroup T with generator A and growth bound $\omega \in \mathbb{R}$ the bounded operators $n(T(1/n)-I)$ $(n \in \mathbb{N})$ and $n^2R(n,A)-n$ $(n > \omega)$ are generators of norm continuous semigroups which approximate T (cf. Remarks B.3). Moreover, the formation of these operators respect domination in the sense that domination carries over to the generated semigroups. This observation suggests to apply the sandwiching result Theorem 1.0.1 to the sequences of semigroups generated by $(n(|T(1/n)| - I))_{n\in\mathbb{N}}$ and $(n^2|R(n,A)| - n)_{n>\omega}$. This yields approximation formulas for the generator of the modulus semigroup as well as for the modulus semigroup itself.

We shall derive these applications from the more general kind of approximations introduced in Appendix B.

1.3.1 Theorem. *Let T be a C_0-semigroup on X with generator A. Suppose that T possesses a modulus semigroup with exponential estimate $\|T^\sharp(t)\| \le Me^{\omega t}$ $(t \ge 0)$. Let ν be a finite Borel measure on $[0,\infty)$ satisfying*

$$\nu([0,\infty)) = \int_0^\infty \tau \, d\nu(\tau) = 1.$$

If $\omega \le 0$ let $h := \infty$. If $\omega > 0$ we additionally assume that $\int_0^\infty \tau e^{\alpha\tau}\, d\nu(\tau) < \infty$ for some $\alpha > 0$, and we define $h := \alpha/\omega$. We define $W(0) := I$ and

$$W(s) := \Big| \int_0^\infty T(s\tau)\, d\nu(\tau) \Big|, \quad B(s) := \tfrac{1}{s}(W(s) - I) \quad (s \in (0,h)).$$

Then $B(s) \to A^\sharp$ in the strong resolvent sense, as $s \to 0$. Moreover,

$$T^\sharp(t) = \operatorname*{s\text{-}lim}_{n\to\infty} W(\tfrac{t}{n})^n,$$

uniformly for t in compact subsets of $[0,\infty)$.

Proof. In order to derive the first statement from Theorem 1.0.1 we choose a sequence $(s_n)_n \subseteq (0,h)$, $s_n \to 0$ as $n \to \infty$. We define

$$V(s) := \int_0^\infty T(s\tau)\, d\nu(\tau), \quad A(s) := \tfrac{1}{s}(V(s) - I),$$
$$\tilde V(s) := \int_0^\infty T^\sharp(s\tau)\, d\nu(\tau), \quad \tilde A(s) := \tfrac{1}{s}(\tilde V(s) - I) \quad (s \in (0,h)).$$

It is easy to see that $|e^{tA(s_n)}| \le e^{tB(s_n)} \le e^{t\tilde A(s_n)}$ for all $t \ge 0$. In Theorem B.2 it is shown that $(A(s_n))$ and $(\tilde A(s_n))$ converge to A and $A^\sharp$ in the strong resolvent sense, respectively, and thus, by the Trotter-Kato approximation theorem (see Remark B.1), $(e^{tA(s_n)})_{t\ge0}$ and $(e^{t\tilde A(s_n)})_{t\ge0}$ converge to T and $T^\sharp$, respectively. Hence Theorem 1.0.1 implies the convergence of $(e^{tB(s_n)})_{t\ge0}$ to $T^\sharp$. The first assertion of the theorem now follows from the (easy part of the) Trotter-Kato approximation theorem.

The second assertion is a consequence of the Chernoff product formula (Theorem C.1). We note that the boundedness condition required for W follows from $W(s) \le \tilde V(s)$ and the fact that the boundedness condition is satisfied for $\tilde V$, by Theorem B.2. ∎

1.3.2 Remark. In the situation of Theorem 1.3.1 and its proof, the semigroup $(e^{tA(s_n)})$ is dominated by $(e^{tB(s_n)})$ (and, a fortiori, by $(e^{t\tilde A(s_n)})$), for $n \in \mathbb{N}$. Thus, Theorem 1.0.1 implies that the sequence of modulus semigroups $\big((e^{tA(s_n)^\sharp})_{t\ge0}\big)_{n\in\mathbb{N}}$ converges to $T^\sharp$.

As a first application of Theorem 1.3.1 we obtain a formula for the modulus semigroup. In order to put this formula into the proper context we recall that, for a C_0-semigroup T, the modulus semigroup (if it exists) can be obtained by

$$T^\sharp(t) = \sup_{(\gamma_1,\dots,\gamma_n)\in\Gamma} |T(\gamma_1 t)|\cdots|T(\gamma_n t)| = \operatorname*{s\text{-}lim}_{(\gamma_1,\dots,\gamma_n)\in\Gamma} |T(\gamma_1 t)|\cdots|T(\gamma_n t)|, \tag{1.3.1}$$

where $\Gamma = \{\gamma \in (0,1]^n;\ n \in \mathbb{N}, \gamma_1 + \cdots + \gamma_n = 1\}$ (cf. [11; Theorem 2.1] or [46] for a special case).

1.3.3 Corollary. *Let T be a C_0-semigroup with generator A, which possesses a modulus semigroup. Then $\frac{1}{s}(|T(s)| - I) \to A^\sharp$ in the strong resolvent sense as $s \to 0$, and*

$$T^\sharp(t) = \operatorname*{s-lim}_{n\to\infty} |T(t/n)|^n,$$

uniformly for t in compact subsets of $[0,\infty)$.

Proof. With $\nu = \delta_1$ the assertion follows from Theorem 1.3.1; cf. Remark B.3(a). ■

1.3.4 Remark. We could not decide whether, in Corollary 1.3.3, there exists a core D for $A^\sharp$ such that $\frac{1}{s}(|T(s)| - I)x \to A^\sharp x$ $(s \to 0)$ for all $x \in D$.

For a generator A, one of the exponential formulas states that $(n/tR(n/t, A))^n$ tends to the semigroup generated by A in the strong operator topology, uniformly on compact intervals of $[0,\infty)$; cf. Remark B.3(b). As a second consequence of Theorem 1.3.1 we obtain the following approximation for the modulus semigroup.

1.3.5 Corollary. *Let A be the generator of a C_0-semigroup T and suppose that T possesses a modulus semigroup. Then $\mu^2|R(\mu, A)| - \mu \to A^\sharp$ in the strong resolvent sense as $\mu \to \infty$, and*

$$T^\sharp(t) = \operatorname*{s-lim}_{n\to\infty} (n/t|R(n/t, A)|)^n,$$

uniformly for t in compact subsets of $[0,\infty)$.

Proof. With $d\nu(\tau) = e^{-\tau} d\tau$, the assertion follows from Theorem 1.3.1; cf. Remark B.3(b). ■

1.4 The Case of Norm Continuous Semigroups

In this section we assume that X is an order complete Banach lattice. If T is a norm continuous semigroup on X with generator $A \in \mathcal{L}^r(X)$, one obtains stronger results than in the previous section. For $A \in \mathcal{L}^r(X)$ let $A = M + B$ be the unique decomposition of A into $M \in \mathcal{Z}(X)$, the centre of $\mathcal{L}^r(X)$, and $B \in \mathcal{Z}(X)^{\mathrm{d}}$ (cf. [53; C-I, section 9]). Derndinger proved that $A^\sharp = \operatorname{Re} M + |B|$ (cf. [29]). It was shown in [11; Proposition 1.2] that $A^\sharp = \lim_{t\to 0} \frac{1}{t}(|T(t)| - I)$, where the limit is in operator norm. The 'operator norm version' of Chernoff's product formula (cf. Remark C.2) shows that

$$T^\sharp(t) = \lim_{n\to\infty} |T(t/n)|^n$$

in operator norm, uniformly on compact subsets of $[0,\infty)$. The objective of the following theorem is to generalise this result to the kind of approximations of A dealt with in the previous section. We recall that $\|A\|_r = \||A|\|$ denotes the regular norm of $A \in \mathcal{L}^r(X)$.

1.4.1 Theorem. *Let $A \in \mathcal{L}^r(X)$, $A = M + B$, where $M \in \mathcal{Z}(X)$ and $B \in \mathcal{Z}(X)^d$. Let ν be a finite Borel measure on $[0,\infty)$ satisfying*

$$\nu([0,\infty)) = \int_0^\infty \tau \, d\nu(\tau) = 1.$$

We assume that $\int_0^\infty \tau e^{\alpha\tau}\, d\nu(\tau) < \infty$ *for some* $\alpha > 0$. *Let* $h := \alpha/\|A\|_r$, $W(0) := I$ *and*

$$W(s) := \Big|\int\limits_0^\infty e^{s\tau A}\, d\nu(\tau)\Big|, \quad B(s) := \tfrac{1}{s}(W(s) - I) \quad (s \in (0,h)).$$

Then $B(s) \to \operatorname{Re} M + |B|$ *in operator norm, as* $s \to 0$, *and* $W(t/n)^n \to e^{t(\operatorname{Re} M + |B|)}$ *in operator norm, uniformly on compact intervals of* $[0,\infty)$, *as* $n \to \infty$.

Proof. From $e^{tA} = I + tA + \frac{t^2}{2}A^2 + \cdots$ we see that

$$\big\||e^{tA} - (I+tA)|\big\| \le (t\|A\|_r)^2 e^{t\|A\|_r} \quad (t \in [0,\infty)).$$

From $\int_0^\infty \tau^2 e^{c\tau}\, d\nu(\tau) < \infty$ for $c < \alpha$ we obtain

$$\begin{aligned} &\frac{1}{s}\left\| \Big|\int\limits_0^\infty e^{s\tau A}\, d\nu(\tau)\Big| - \Big|\int\limits_0^\infty (I + s\tau A)\, d\nu(\tau)\Big| \right\| \\ &\le \frac{1}{s}\int\limits_0^\infty (s\tau\|A\|_r)^2 e^{s\tau\|A\|_r}\, d\nu(\tau) \to 0 \end{aligned} \tag{1.4.1}$$

as $s \to 0$. Thus $B(s)$ converges if and only if

$$\frac{1}{s}\left(\Big|\int\limits_0^\infty (I + s\tau A)\, d\nu(\tau)\Big| - I\right) = \frac{|I + sA| - I}{s} = \frac{|I + sM| - I}{s} + |B| \tag{1.4.2}$$

converges. From [11; Proof of Proposition 1.2] we know that $\frac{|I+sM|-I}{s} \to \operatorname{Re} M$ $(s \to 0)$ in operator norm. From (1.4.1) and (1.4.2) we conclude $B(s) \to \operatorname{Re} M + |B|$ in operator norm as $s \to 0$. By Remark C.2 this implies that $W(t/n)^n$ converges to $(e^{t(\operatorname{Re} M + |B|)})_{t\ge 0}$ $(n \to \infty)$ in operator norm uniformly on compact subsets of $[0,\infty)$. ∎

1.5 Application to Perturbed Semigroups

1.5.1 Bounded Perturbations

First we will prove an abstract result which deals with the modulus of a semigroup perturbed by a regular operator. As an application we present a new proof concerning the modulus of matrix semigroups.

1.5.1 Proposition. *Let* X *be a Banach lattice with order continuous norm. Let* T_0 *be a* C_0*-semigroup with generator* A_0, *and assume that* T_0 *possesses a modulus semigroup. Let* $B \in \mathcal{L}^r(X)$ *and suppose that one of the following assumptions holds.*

(i) *There exists* $\tau > 0$ *such that*

$$\left| T_0(t) + \int_0^t T_0(t-s)BT_0(s)\,ds \right| = |T_0(t)| + \left| \int_0^t T_0(t-s)BT_0(s)\,ds \right|$$

for all $t \in (0,\tau)$. *(This is satisfied, in particular, if* $\int_0^t T_0(t-s)BT_0(s)\,ds$ *and* $T_0(t)$ *are order disjoint in* $\mathcal{L}^r(X)$ *for all* $t \in (0,\tau)$.*)*

(ii) *There exists* $\tau > 0$ *such that* B *and* $T_0(t)$ *are order disjoint in* $\mathcal{L}^r(X)$ *for all* $t \in (0,\tau)$, *and* $|T_0(s) - I| \to 0$ *in the strong operator topology as* $s \to 0$.

Then the operator $A_0^\sharp + |B|$ *is the generator of the modulus semigroup of the perturbed semigroup* $T := \left(e^{t(A_0+B)}\right)_{t\geq 0}$.

Proof. With $T_1(t) := \int_0^t T_0(t-s)BT_0(s)\,ds$, $R_1(t) := \int_0^t T(t-s)BT_1(s)\,ds$ $(t \geq 0)$ we have the representation

$$T(t) = T_0(t) + T_1(t) + R_1(t) \quad (t \geq 0).$$

It is easy to see that the semigroup $\tilde{T}$ generated by $A_0^\sharp + |B|$ dominates T. For the remainder R_1 we have the estimate

$$|R_1(t)| \leq \int_0^t \tilde{T}(t-s)|B| \int_0^s T_0^\sharp(s-r)|B|T_0^\sharp(r)\,dr\,ds \quad (t \geq 0), \tag{1.5.1}$$

and thus $\big\||R_1(t)|\big\| \leq ct^2$ $(t \in [0,1])$, for some constant $c \geq 0$. This implies that

$$\left| \frac{|T(t)| - I}{t} - \frac{|T_0(t) + T_1(t)| - I}{t} \right| \leq \frac{1}{t}|R_1(t)| \to 0 \quad (t \to 0). \tag{1.5.2}$$

Next, we observe that $\frac{1}{t}T_1(t) \to B$ and

$$\left|\tfrac{1}{t}T_1(t)\right| \leq \frac{1}{t}\int_0^t T_0^\sharp(t-s)|B|\,T_0^\sharp(s)\,ds \to |B|,$$

both in the strong operator topology as $t \to 0$. Thus Lemma 1.1.3 implies that $|\frac{1}{t}T_1(t)| \to |B|$ in the strong operator topology. Let $x \in D(A_0^\sharp)$. By Corollary 1.3.3 and Remark B.1(c) there exists $(x_n) \subseteq X$, $x_n \to x$ such that $n(|T_0(1/n)|x_n - x_n) \to A_0^\sharp x$ as $n \to \infty$, and therefore

$$n(|T_0(1/n)|x_n - x_n) + n|T_1(1/n)|x_n \to (A_0^\sharp + |B|)x \quad (n \to \infty). \tag{1.5.3}$$

From (i), (1.5.2), and (1.5.3) we obtain $n(|T(1/n)|x_n - x_n) \to (A_0^\sharp + |B|)x$. Using Remark B.1(c) we conclude that $(n(|T(1/n)| - I))_{n\in\mathbb{N}}$ converges to $A_0^\sharp + |B|$ in the strong resolvent sense. Remark 1.2.1(b) implies that T possesses a modulus semigroup whose generator is $A_0^\sharp + |B|$.

Now let assumption (ii) be satisfied. Then

$$
\begin{aligned}
\frac{1}{t}\Big||T_0(t)+T_1(t)| - |T_0(t)+tB|\Big| &\le \left|\tfrac{1}{t}T_1(t) - B\right| \\
&\le \frac{1}{t}\int_0^t |T_0(t-s)BT_0(s) - B|\,ds \\
&\le \frac{1}{t}\Big(\int_0^t T_0^\sharp(t-s)\,|B|\,|T_0(s)-I|\,ds + \int_0^t |T_0(t-s)-I|\,|B|\,ds\Big) \\
&\to 0 \quad (t\to 0),
\end{aligned}
\tag{1.5.4}
$$

in the strong operator topology. (We note that, due to the inequality $|T_0(s')-T_0(s)| \le |T_0(s)||T_0(s'-s)-I|$, for $0\le s\le s'$, the integrands in (1.5.4) are strongly continuous.) For $x\in D(A_0^\sharp)$ and (x_n) as above we see that

$$
\begin{aligned}
& n\left(\left|T_0(1/n)+\tfrac{1}{n}B\right|x_n - x_n\right) = n(|T_0(1/n)|x_n - x_n) + |B|x_n \\
& \to (A_0^\sharp + |B|)x \quad (n\to\infty).
\end{aligned}
\tag{1.5.5}
$$

As in the first part of this proof we conclude from (1.5.2), (1.5.4), and (1.5.5) that $A_0^\sharp + |B|$ is the generator of the modulus semigroup of T. ■

1.5.2 Remark. In the proof of Proposition 1.5.1, with condition (iii), the order continuity of the norm was only used for the existence of sequences (x_n) approximating elements of $D(A_0^\sharp)$. If X is order complete, and the semigroup T_0 has the property that $\left(\frac{1}{s}(|T_0(s)| - I)\right)$ converges to $A_0^\sharp$ in the strong resolvent sense then, by the proof given above, $A_0^\sharp+|B|$ is the generator of the modulus semigroup.

1.5.3 Remark. In condition (iii) of Proposition 1.5.1 it is required that $|T(t)-I|\to 0$ in the strong operator topology as $t\to 0$. We could not decide how to characterise semigroups possessing this property. Dominated norm continuous semigroups and multiplication semigroups (see subsection 1.5.2, in particular the proof of Proposition 1.5.5) possess this property. On the other hand, if the semigroup operators $T(t)$ are order disjoint to the centre of $\mathcal{L}^r(X)$ for $t>0$ then $|T(t)-I| = |T(t)|+I$ which does not tend to 0 strongly. An example of such a semigroup is the left translation on $L_p(\mathbb{R})$ $(p\in[1,\infty))$.

1.5.4 Proposition. *(cf. [23], [64]) Let X_1 and X_2 be Banach lattices with order continuous norm, and set $X := X_1\times X_2$. Moreover let $\mathcal{A} := \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{smallmatrix}\right)$, $D(\mathcal{A}) := D(A_{11})\times D(A_{22})$, be an operator matrix on X, where*

(i) *for $j=1,2$, the operator A_{jj} is the generator of a C_0-semigroup T_j on X_j possessing a modulus semigroup,*

(ii) $A_{12} \in \mathcal{L}(X_2, X_1)$ *and* $A_{21} \in \mathcal{L}(X_1, X_2)$ *are regular operators.*

Then the generator of the modulus semigroup of $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ *is* $\mathcal{A}^\sharp = \left(\begin{smallmatrix} A_{11}^\sharp & |A_{12}| \\ |A_{21}| & A_{22}^\sharp \end{smallmatrix}\right)$, $D(\mathcal{A}^\sharp) = D(A_{11}^\sharp) \times D(A_{22}^\sharp)$.

Proof. Let $\mathcal{T}_0(t) := \left(\begin{smallmatrix} T_1(t) & 0 \\ 0 & T_2(t) \end{smallmatrix}\right)$. The assertion follows from Proposition 1.5.1, with condition (ii), and the order disjointness of $\mathcal{T}_0(t)$ and

$$\int_0^t \mathcal{T}_0(t-s) \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} \mathcal{T}_0(s)\, ds = \begin{pmatrix} 0 & B_1(t) \\ B_2(t) & 0 \end{pmatrix} \quad (t > 0),$$

where $B_1(t) := \int_0^t T_1(t-s) A_{12} T_2(s)\, ds$ and $B_2(t) := \int_0^t T_2(t-s) A_{21} T_1(s)\, ds$. ∎

1.5.2 Perturbation of Multiplication Semigroups

A C_0-semigroup T on a (real or complex) order complete Banach lattice X is called a multiplication semigroup if $T(t)$ belongs to the centre $\mathcal{Z}(X)$ for all $t \geq 0$. Multiplication semigroups on real (and to some extent on complex) Banach lattices have been investigated in [53; C-II, Section 5] and [68]. We note that on real Banach lattices all multiplication semigroups are positive; cf. [53; C-II, Corollary 5.14]. Let A be the generator of the multiplication semigroup T. Then $D(A)$ is a dense ideal, A is band preserving, and $A = \operatorname{Re} A + i \operatorname{Im} A$ with both $\operatorname{Re} A$ and $\operatorname{Im} A$ real operators on the domain $D(A)$. Moreover, $\operatorname{Re} A$ is band preserving and *bounded from above* (i.e., there exists $\omega \in \mathbb{R}$ such that $(\operatorname{Re} A)x \leq \omega x$ for all $x \in D(\operatorname{Re} A)_+$). Hence $\operatorname{Re} A$ is closable, and the closure is the generator of a multiplication semigroup ([68; Theorem 1.5]). The modulus semigroup of T is given by $T^\sharp(t) = |T(t)|$ ([53; C-II, Proposition 5.2]). As $T(t) \in \mathcal{Z}(X)$ we have (cf. Section 1.4 for the first equality)

$$\Big(\frac{1}{s}(T(s) - I)\Big)^\sharp x = \frac{1}{s}(\operatorname{Re} T(s) - I)x \to (\operatorname{Re} A)x$$

for $x \in D(\operatorname{Re} A)$ as $s \to 0$. The semigroups generated by $\frac{1}{s}(\operatorname{Re} T(s) - I)$ are dominated by those generated by $\frac{1}{s}(|T(s)| - I) = \frac{1}{s}(T^\sharp(s) - I)$. Using Remark B.3(a) and Remark 1.2.1(b) we obtain $\overline{\operatorname{Re} A} = A^\sharp$. In a similar way, this conclusion can also be inferred from

$$\frac{1}{s}\left(\int_0^\infty T(s\tau)\, d\nu(\tau) - I\right)^\sharp = \frac{1}{s}\left(\int_0^\infty \operatorname{Re} T(s\tau)\, d\nu(\tau) - I\right) \to (\operatorname{Re} A)x \quad (s \to 0),$$

for $x \in D(\operatorname{Re} A)$ and a suitable Borel measure ν.

If we additionally assume order continuity of the norm this result also follows from [53; C-II, Theorem 5.5].

The application of Remark 1.5.2 to multiplication semigroups yields the following generalisation of the result for norm continuous semigroups (cf. Section 1.4).

1.5.5 Proposition. *Let X be an order complete Banach lattice. Let $(e^{tA})_{t\geq 0}$ be a multiplication semigroup, and let $B \in \mathcal{L}^r(X)$ be disjoint to the centre. Then*

$$(A+B)^\sharp = \overline{\operatorname{Re} A} + |B|.$$

Proof. It was noted above that the modulus semigroup of $(e^{tA})_{t\geq 0}$ is generated by $\overline{\operatorname{Re} A}$, and that $(\frac{1}{s}(|e^{sA}| - I))$ converges to $\overline{\operatorname{Re} A}$ in the strong resolvent sense. As $e^{tA} - I$ are disjointness preserving operators for $t \geq 0$ we infer $\||e^{tA} - I|x\| = \|(e^{tA} - I)x\| \to 0$ as $t \to 0$ $(x \in X)$ (cf. [53; C-II, Proposition 5.1]). Since B and e^{tA} are disjoint $(t \geq 0)$, we can apply Remark 1.5.2 to obtain the assertion. ∎

1.5.3 Volterra Semigroups

In our last application we treat Volterra semigroups associated with inhomogeneous abstract Cauchy problems and integro-differential equations.

Let $p \in [1, \infty)$. Let A be the generator of a C_0-semigroup T on a (real or complex) Banach lattice X with order continuous norm possessing a modulus semigroup. The operator $\mathcal{A} := \left(\begin{smallmatrix} A & \delta_0 \\ 0 & \mathcal{D} \end{smallmatrix}\right)$ with domain $D(\mathcal{A}) := D(A) \times W_p^1(\mathbb{R}_+; X)$ on $X \times L_p(\mathbb{R}_+; X)$ is the generator of the C_0-semigroup $\mathcal{T}$ given by $\mathcal{T} := \left(\begin{smallmatrix} T(\cdot) & R(\cdot) \\ 0 & S(\cdot) \end{smallmatrix}\right)$, where S denotes the left translation semigroup on $L_p(\mathbb{R}_+; X)$ and $R(t) \in \mathcal{L}(L_p(\mathbb{R}_+; X), X)$ is defined by $R(t)f := \int_0^t T(t-s)f(s)\,ds$ $(t \in \mathbb{R}_+,\ f \in L_p(\mathbb{R}_+; X))$ (cf. [39; Section VI.7]). The C_0-semigroup $\mathcal{T}$ is related to inhomogeneous abstract Cauchy problems (cf. Section 4.7). We will first determine the modulus semigroup of $\mathcal{T}$.

The main objective of this section is the computation of the modulus of the C_0-semigroup $\mathcal{S}$ generated by $\mathcal{C} := \left(\begin{smallmatrix} A & \delta_0 \\ L & \mathcal{D} \end{smallmatrix}\right)$ with domain $D(\mathcal{C}) = D(\mathcal{A})$ on $X \times L_p(\mathbb{R}_+; X)$, where we assume that $L \in \mathcal{L}_r(X, L_p(\mathbb{R}_+; X))$. We recall that for $L \in \mathcal{L}(X, L_p(\mathbb{R}_+; X))$ the matrix operator $\left(\begin{smallmatrix} 0 & 0 \\ L & 0 \end{smallmatrix}\right) \in \mathcal{L}(X \times L_p(\mathbb{R}_+; X))$ is a bounded perturbation of $\mathcal{A}$ and so $\mathcal{C}$ is indeed a generator. We refer to [39; Section VI.7] for an overview on Volterra semigroups and their relation to integro-differential equations. In Sections 3.1, 4.6 and 4.7 we will also encounter Volterra semigroups in the context of various integro-differential equations.

1.5.6 Proposition. *The modulus semigroup of $\mathcal{T}$ is generated by $\tilde{\mathcal{A}} := \left(\begin{smallmatrix} A^\sharp & \delta_0 \\ 0 & \mathcal{D} \end{smallmatrix}\right)$ with the domain $D(\tilde{\mathcal{A}}) := D(A^\sharp) \times W_p^1(\mathbb{R}_+; X)$.*

Proof. By Corollary 1.3.3 and Remark B.1(c) we know that for $x \in D(A^\sharp)$ there exists $(x_n) \subseteq X$ with $x_n \to x$ such that $n(|T(1/n)|x_n - x_n) \to A^\sharp x$ as $n \to \infty$. For $f \in W_p^1(\mathbb{R}_+; X)$ we therefore obtain

$$n\left(\left|\begin{pmatrix} T(1/n) & 0 \\ 0 & S(1/n) \end{pmatrix}\right| \begin{pmatrix} x_n \\ f \end{pmatrix} - \begin{pmatrix} x_n \\ f \end{pmatrix}\right) = n\left(\begin{pmatrix} |T(1/n)| & 0 \\ 0 & S(1/n) \end{pmatrix} \begin{pmatrix} x_n \\ f \end{pmatrix} - \begin{pmatrix} x_n \\ f \end{pmatrix}\right) \to \begin{pmatrix} A^\sharp x \\ \mathcal{D} f \end{pmatrix}$$

as $n \to \infty$. For $f \geq 0$ we conclude from $\frac{1}{t}R(t)f = \frac{1}{t}\int_0^t T(t-s)f(s)\,ds \to \delta_0 f$ and $\frac{1}{t}|R(t)|f \leq \frac{1}{t}\int_0^t T^\sharp(t-s)f(s)\,ds \to \delta_0 f$ for $t \to 0$ that $\frac{1}{t}|R(t)|f \to \delta_0 f$ $(t \to 0)$. For arbitrary $f \in W_p^1(\mathbb{R}_+;X)$ we see this convergence by writing f as the difference of two positive functions. From

$$\begin{aligned} n\,|\mathcal{T}(1/n)|\left(\begin{smallmatrix} x_n \\ f \end{smallmatrix}\right) - \left(\begin{smallmatrix} x_n \\ f \end{smallmatrix}\right) &= n\left(\left|\begin{pmatrix} T(1/n) & R(1/n) \\ 0 & S(1/n) \end{pmatrix}\right|\left(\begin{smallmatrix} x_n \\ f \end{smallmatrix}\right) - \left(\begin{smallmatrix} x_n \\ f \end{smallmatrix}\right)\right) \\ &= n\left(\begin{pmatrix} |T(1/n)| & |R(1/n)| \\ 0 & S(1/n) \end{pmatrix}\left(\begin{smallmatrix} x_n \\ f \end{smallmatrix}\right) - \left(\begin{smallmatrix} x_n \\ f \end{smallmatrix}\right)\right) \to \begin{pmatrix} A^\sharp x + \delta_0 f \\ \mathcal{D} f \end{pmatrix} \quad (n \to \infty) \end{aligned}$$

we conclude that $\tilde{\mathcal{A}} \subseteq \mathcal{A}^\sharp$. As both operators are generators we see that $\tilde{\mathcal{A}} = \mathcal{A}^\sharp$. ■

1.5.7 Proposition. *Assume that $L \in \mathcal{L}(X, L_p(\mathbb{R}_+;X))$ is a regular operator possessing the modulus $|L| \in \mathcal{L}(X, L_p(\mathbb{R}_+;X))$. Then $\tilde{\mathcal{C}} := \left(\begin{smallmatrix} A^\sharp & \delta_0 \\ |L| & \mathcal{D} \end{smallmatrix}\right)$ with domain $D(A^\sharp) \times W_p^1(\mathbb{R}+;X)$ is the generator of the modulus semigroup of $\mathcal{S}$.*

For the proof we will need the following lemma (cf. [4; Proposition 1.3.4] for the convolution of strongly continuous operator families).

1.5.8 Lemma. *Let Y_1, Y_2 and Y_3 be Banach spaces, $U\colon \mathbb{R}_+ \to \mathcal{L}(Y_2, Y_3)$ and $V\colon \mathbb{R}_+ \to \mathcal{L}(Y_1, Y_2)$. Assume that U and V are strongly continuous. Then $W\colon \mathbb{R}_+ \to \mathcal{L}(Y_1, Y_3)$ defined by $W(t)x := \frac{1}{t}\int_0^t U(t-s)V(s)y\,ds$ ($t \in \mathbb{R}_+$, $y \in Y$) converges to $U(0)V(0)$ in the strong operator topology as $t \to 0$.*

Proof. We first observe that by the principle of uniform boundedness the operators $U(t)$ are uniformly bounded in t in compact intervals of $\mathbb{R}_+$.

For $y \in Y_1$ we have

$$\frac{1}{t}\int_0^t U(t-s)V(0)y\,ds \to U(0)V(0)y \quad (t \to 0) \tag{1.5.6}$$

by the strong continuity of U. Further we obtain

$$\left\|\frac{1}{t}\int_0^t U(t-s)(V(s)y - V(0)y)\,ds\right\| \le \sup_{s\in[0,t]}\|U(s)\| \sup_{s\in[0,t]}\|V(s)y - V(0)y\| \to 0 \quad (t \to 0) \tag{1.5.7}$$

by the uniform boundedness of the operators $U(s)$ for $s \in [0,t]$ and the strong continuity of V. The assertion now follows from (1.5.6) and (1.5.7). ■

Proof of Proposition 1.5.7. By the Dyson-Phillips series for bounded perturbations we have the representation

$$\mathcal{S}(t) = \mathcal{T}(t) + \int_0^t \mathcal{T}(t-s)\left(\begin{smallmatrix} 0 & 0 \\ L & 0 \end{smallmatrix}\right)\mathcal{T}(s)\,ds + \mathcal{R}_1(t) \quad (t \in \mathbb{R}_+),$$

with $\left\| |\mathcal{R}_1(t)| \right\| \le ct^2$ $(t \in [0,1])$ for some constant $c \ge 0$ (for the estimate of $\mathcal{R}_1(t)$ in the regular norm we refer to equation (1.5.1)). In order to deal with the integral

$$\int_0^t \mathcal{T}(t-s) \left(\begin{smallmatrix} 0 & 0 \\ L & 0 \end{smallmatrix}\right) \mathcal{T}(s)\, ds = \int_0^t \begin{pmatrix} R(t-s)LT(s) & R(t-s)LR(s) \\ S(t-s)LT(s) & S(t-s)LR(s) \end{pmatrix} ds$$

let $R^\sharp(t)f := \int_0^t T^\sharp(t-s)f(s)\, ds$ $(t \in \mathbb{R}_+,\ f \in L_p(\mathbb{R}_+; X))$. By [4; Proposition 1.3.4] the operator families R and $R^\sharp$ are strongly continuous. We further have $R(0) = R^\sharp(0) = 0$. From Lemma 1.5.8 we see that

$$\begin{aligned} &\frac{1}{t}\int_0^t R^\sharp(t-s)\, |L|\, T^\sharp(s)\, ds \to 0, \\ &\frac{1}{t}\int_0^t R^\sharp(t-s)\, |L|\, R^\sharp(s)\, ds \to 0, \\ &\frac{1}{t}\int_0^t S(t-s)\, |L|\, R^\sharp(s)\, ds \to 0 \quad (t \to 0) \end{aligned}$$

in the strong operator topology. Since $T^\sharp(t)$ and $R^\sharp(t)$ dominate $T(t)$ and $R(t)$ $(t \in \mathbb{R}_+)$, respectively, we obtain

$$\frac{1}{t}\left| \int_0^t \begin{pmatrix} R(t-s)LT(s) & R(t-s)LR(s) \\ 0 & S(t-s)LR(s) \end{pmatrix} ds \right| \to 0 \quad (t \to 0)$$

in the strong operator topology. This implies that

$$\begin{aligned} &\left| \frac{|\mathcal{S}(t)| - I}{t} - \frac{\left| \mathcal{T}(t) + \begin{pmatrix} 0 & 0 \\ \int_0^t S(t-s)LT(s)\, ds & 0 \end{pmatrix} \right| - I}{t} \right| \\ &\qquad\qquad \le \frac{1}{t}\left| \int_0^t \begin{pmatrix} R(t-s)LT(s) & R(t-s)LR(s) \\ 0 & S(t-s)LR(s) \end{pmatrix} ds \right| + \frac{1}{t}|\mathcal{R}_1(t)| \to 0 \quad (1.5.8) \end{aligned}$$

as $t \to 0$ in the strong operator topology.

From the proof of Proposition 1.5.6 we know that for $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\tilde{\mathcal{C}}) = D(A^\sharp) \times W_p^1(\mathbb{R}_+; X)$ there exists $(x_n) \subseteq X$ with $x_n \to x$ such that

$$n\left(\left|\mathcal{T}(1/n)\right| \left(\begin{smallmatrix} x_n \\ f \end{smallmatrix}\right) - \left(\begin{smallmatrix} x_n \\ f \end{smallmatrix}\right) \right) \to \mathcal{A}^\sharp \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \quad (n \to \infty).$$

From Lemma 1.5.8 we infer that $\frac{1}{t}\int_0^t S(t-s)LT(s)\, ds \to L$ and $\frac{1}{t}\int_0^t S(t-s)\, |L|\, T^\sharp(s)\, ds \to |L|$ both in the strong operator topology as $t \to 0$. As the first term is dominated by the second term we can apply Lemma 1.1.3 to infer that $\frac{1}{t}\left|\int_0^t S(t-s)LT(s)\,ds\right| \to |L|$ in the strong operator topology. Thus we have

$$n\left(\left|\mathcal{T}(1/n) + \begin{pmatrix} 0 & 0 \\ \int_0^{1/n} S(1/n-s)LT(s)\,ds & 0 \end{pmatrix}\right| - I\right)\begin{pmatrix} x_n \\ f \end{pmatrix} \to \tilde{\mathcal{C}}\begin{pmatrix} x \\ f \end{pmatrix} \quad (n \to \infty).$$

Taking into account (1.5.8) we infer that $\tilde{\mathcal{C}} \subseteq \mathcal{C}^\sharp$. As both operators are generators we obtain $\tilde{\mathcal{C}} = \mathcal{C}^\sharp$. ■

Let $\ell\colon \mathbb{R}_+ \to \mathcal{L}(X)$ be an operator-valued function with $\ell(\cdot)x \in L_p(\mathbb{R}_+;X)$ for all $x \in X$. The closed graph theorem implies that $Lx := \ell(\cdot)x$ is a bounded operator from X to $L_p(\mathbb{R}_+;X)$ (cf. Lemma 3.1.1). As this type of operator is particularly interesting in applications to integro-differential equations we provide the following supplement to Proposition 1.5.7.

1.5.9 Proposition. *Let X be a separable (real or complex) Banach lattice with order continuous norm. Let $\ell\colon \mathbb{R}_+ \to \mathcal{L}_r(X)$ with $\ell(\cdot)x \in L_p(\mathbb{R}_+;X)$ $(x \in X)$. Assume that the operator $Lx := \ell(\cdot)x$ $(x \in X)$ is regular. Then L possesses a modulus which is induced by $|\ell(\cdot)|$.*

Proof. Let $x \in X$, $x \geq 0$. By the separability of X there exists $M \subseteq \{y \in X;\, |y| \leq x\}$ which is countable and dense in $\{y \in X;\, |y| \leq x\}$. Let $(y_n)_{n\in\mathbb{N}}$ be an enumeration of $G := \{|Ly|;\, y \in M\} \subseteq L_p(\mathbb{R}_+;X)$. We define the upward directed sequence $z_n := \sup\{y_k;\, 1 \leq k \leq n\}$. Clearly we have

$$|L|x = \sup\{|Ly|;\, |y| \leq x\} = \sup G = \lim_{n\to\infty} z_n,$$
$$|\ell(t)|x = \sup\{|\ell(t)y|;\, |y| \leq x\} = \sup\{g(t);\, g \in G\} = \lim_{n\to\infty} z_n(t) \quad (t \in \mathbb{R}_+),$$

where the last equality in both lines follow from the order continuity of the norm in $L_p(\mathbb{R}_+;X)$ and X, respectively. By choosing a subsequence of (z_n) if necessary we can assume that $z_n \to |L|x$ pointwise almost everywhere. Therefore $|L|x = |\ell(\cdot)|x$ almost everywhere. ■

Chapter 2

A Generalised Desch-Schappacher Perturbation Theorem

In this chapter we are mainly concerned with C_0-semigroups generated by the derivative operator on L_p-spaces with suitable boundary conditions. On $X_p := L_p(-h, 0; X)$, where $h \in (0, \infty]$, $p \in [1, \infty)$ and X is a Banach space, we shall consider the operator

$$A_L f := f', \quad D(A_L) := \{f \in W_p^1(-h, 0; X);\, f(0) = Lf\}. \tag{2.0.1}$$

Here $L\colon W_p^1(-h, 0; X) \to X$ is a suitable linear operator. In the case that A_L is a generator we denote the C_0-semigroup generated by A_L by T_L. Such semigroups will be called translation semigroups (cf. Definition 2.1.1). For $L = 0$ it is well-known that A_0 is the generator of the left translation semigroup T_0 given by $T_0(t)f(s) = f(t + s)$ if $t + s < 0$ and $T_0(t)f(s) = 0$ if $t + s \geq 0$ ($t \geq 0$, $s \in (-h, 0)$).

Translations occur as components in different types of semigroups, as are delay semigroups and semigroups arising from integro-differential equations (see for example Chapter 3 and Sections 4.7 and 4.8).

They are also interesting in their own right, for they are closely related to the equation $u(t) = Lu_t$, $t \geq 0$, with initial value u_0. Further they are suitable for modelling transport processes in networks (cf. Section 2.7). For other applications to population, renewal and Volterra equations we refer to [42]. We also mention that the equation $u(t) = Au(t)+Lu_t$, for a closed operator A with $1 \in \rho(A)$, can be written as $u(t) = R(1, A)Lu_t$ and is therefore not more general (cf. [53; Section C-IV.3.2]).

If L is bounded from X_p to X it has been shown that A_L is the generator of a C_0-semigroup on X_p in [42], see also [53; Corollary C-IV.3.2] and [39; Example III.3.5]. In [42] translation semigroups on the space $L_1(-\infty, 0; X; e^{\eta x}dx)$, with $\eta \in \mathbb{R}$, were considered. (We only mention that our results also hold for such weighted L_p-spaces. This generalisation becomes interesting in the spectral analysis of the operators A_L if $h = \infty$ and η can be chosen to be negative.)

We will mainly investigate operators L for which there is a (finite) Borel measure μ_L on $[-h, 0]$ (respectively $(-\infty, 0]$ for $h = \infty$) and $r \in [1, p]$ such that $\|Lf\| \leq \|f\|_{L_r(\mu;X)}$ ($f \in X_{\mathrm{reg}}$; see Section 2.1 for the space of regulated functions). This class of operators was introduced in [69] as perturbations of the closely related delay semigroups. Most of the interesting operators such as operators given by Riemann-Stieltjes integrals are among this type of operators.

Even though A_L can "almost" be obtained as a Desch-Schappacher perturbation of A_0, these operators are not covered by this kind of perturbation. The boundary perturbation theory developed in [43] and [56] is likewise not applicable to these boundary perturbations. As a preparation we generalise the perturbation theorem of Desch-Schappacher so that A_L can be represented as a generalised Desch-Schappacher perturbation (see [39; Theorem III.3.1] for the general Desch-Schappacher perturbation theorem and Theorem A.2 for a special case). We will also consider the corresponding boundary perturbation of evolution semigroups induced by backward propagators.

A further objective is the determination of the modulus semigroup of translation semigroups on X_p; for the definition of modulus semigroups we refer to the previous chapter. We particularly show that for a bounded operator L in $\mathcal{L}(L_r(\mu_L; X), X)$ possessing a modulus $|L|$, the modulus semigroup $T_L^\sharp$ of T_L is $T_{|L|}$. In [42] this assertion was shown for regular operators $L \in \mathcal{L}_r(X_p, X)$. The technique of associating a delay operator with a dominated translation semigroup as done in [42] however does not work well in this more general setting. We will rather use a sandwiching argument (see Lemma 2.5.3). This idea also works for certain translation semigroups on the space of Banach space valued continuous functions (usually called delay semigroups), which is explored at the end of this chapter.

The chapter is organised as follows.

First we recall regulated functions and define translations on spaces of p-integrable and regulated functions in Section 2.1.

In Section 2.2 we introduce generalised Desch-Schappacher perturbations and prove a generalisation of the Desch-Schappacher perturbation theorem.

In Section 2.3 we introduce the class of delay operators L which we are going to deal with. In order to avoid technical difficulties we only treat the case $h < \infty$.

In Section 2.4 we first show that the operator A_L as defined in (2.0.1) generates a translation semigroup if L is a delay operator and $h < \infty$. We then use approximation techniques to cover the case $h = \infty$.

In Section 2.5 we will determine the modulus semigroup of such translation semigroups.

Section 2.6 is devoted to the perturbation of evolution semigroups arising from backward propagators.

In Section 2.7 we show how the evolution of flows in networks can be described in the framework of translation and evolution semigroups.

Finally we determine the modulus semigroup of delay semigroups on the space of Banach space valued continuous functions for a special case.

2.1 Preliminaries

In this section we recall the definition of regulated functions and introduce the notion of translations.

The space of X-valued regulated functions on an interval $J \subseteq \mathbb{R}$ is defined as the closure of the space of step functions in $\ell_\infty(J; X)$. It is denoted by $\mathrm{Reg}(J; X)$ and abbreviated by $X_{\mathrm{reg}} := \mathrm{Reg}([-h, 0]; X)$ if $h < \infty$. The space $\mathrm{Reg}(J; X)$ contains exactly those functions in $\ell_\infty(J; X)$ which possess right and left hand limits at all points (and which vanish at infinity in the case that J is not a bounded interval). For a regulated function $f\colon J \to X$ we introduce the notation $\overleftarrow{f}\colon J \to X$ for the function $\overleftarrow{f}(t) := \lim_{s \searrow t} f(s)$ $(t \in J)$ (in case J is closed on the right hand side with endpoint a we set $\overleftarrow{f}(a) := f(a)$).

We are interested in translations acting on X_p and/or on X_{reg}. The common notation for a function f translated by $t \in \mathbb{R}$ is f_t. More precisely we define f_t for a function $f\colon J \to X$ on an interval $J \subseteq \mathbb{R}$ and $t \in \mathbb{R}$ as

$$f_t\colon \mathbb{R} \to X, \quad f_t(s) := \begin{cases} f(t+s) & \text{if } t+s \in J, \\ 0 & \text{otherwise.} \end{cases}$$

We can now fix the terminology of a translation semigroup on X_p or X_{reg}.

2.1.1 Definition. Let $\tau > 0$ and $Y \in \{X_p, X_{\mathrm{reg}}\}$ (if $h = \infty$ then $Y = X_p$). A function $F \in \ell_\infty([0,\tau]; \mathcal{L}(Y))$ is called a *translation* if for all $f \in Y$ there is a $g \in L_p(-h,\tau;X)$ (for $Y = X_p$) and $g \in \mathrm{Reg}([-h,\tau];X)$ (for $Y = X_{\mathrm{reg}}$), respectively, such that

$$F(t)f = g_t|_{(-h,0)} \quad (0 \le t \le \tau).$$

A C_0-semigroup T on X_p is called a translation semigroup if $T|_{[0,\tau]}$ is a translation for some (or equivalently all) $\tau > 0$.

An alternative proof of the following fundamental result on translation semigroups can be found in [42; Proposition 1.4].

2.1.2 Proposition. *Let T be a translation semigroup on X_p with generator A. Then $D(A) \subseteq W_p^1(-h,0;X)$ and $Af = f'$.*

Proof. Let $f \in D(A)$ and $\varphi \in C_c^\infty(-h,0)$. For all $\delta > 0$ with $\operatorname{spt}\varphi \subseteq (-h,-\delta)$ we can compute

$$\int_{-h}^{0} \frac{T(\delta)f - f}{\delta}(t)\varphi(t)\,dt = \int_{-h}^{0} f(t)\frac{\varphi(t-\delta) - \varphi(t)}{\delta}\,dt. \tag{2.1.1}$$

The left hand term of (2.1.1) converges to $\int_{-h}^0 Af\varphi$ whereas the right hand term goes to $-\int_{-h}^0 f\varphi'$ as $\delta \to 0$. Hence $f \in W_p^1(-h,0;X)$ and $Af = f'$. ∎

Finally for an operator A whose domain is a function space $F(J;X)$ of functions from an interval $J \subseteq \mathbb{R}$ to X and a function f which is defined on a larger interval than J we write Af as an abbreviation for $A(f|_J)$ provided that $f|_J \in F(J;X)$.

2.2 A generalised Desch-Schappacher Perturbation Theorem

In this section we prove a generalised Desch-Schappacher perturbation theorem. We start with some notations.

For an operator-valued function F and a suitable Banach space valued function g, both defined on some interval of $\mathbb{R}$, the convolution of F and g is (formally) defined by $(F * g)(t) = \int_{\mathbb{R}} F(t-s)g(s)\,ds$ where F and g are taken to be zero outside their domains (cf. [4; Section 1.3] where general conditions on the existence of the integral are given).

By $C(J;\mathcal{L}_s(X))$, where $J \subseteq \mathbb{R}$ is an interval and X a Banach space, we denote the space of strongly continuous operator-valued functions on J.

We also need the Sobolev tower $(X_A^n)_{n\in\mathbb{Z}}$ associated with a generator A on the Banach space $X = X_A^0$ and the induced generators A_n on X_A^n for $n \in \mathbb{Z}$. The definitions and properties can be found in e.g. [39; Section II.5a] or [54] and also in Chapter 4.

Let A be the generator of the C_0-semigroup T acting on the Banach space X. We recall that an operator $B \in \mathcal{L}(X)$ is a multiplicative Desch-Schappacher perturbation of A if there is a $\tau > 0$ such that the Volterra operator V defined by $C([0,\tau];\mathcal{L}_s(X)) \ni F \mapsto T_{-1} * A_{-1}BF$ is a strictly contractive operator in $\mathcal{L}(C([0,\tau];\mathcal{L}_s(X)))$. The Desch-Schappacher perturbation theorem states that if B is a Desch-Schappacher perturbation of A, then $A(I+B)$ is a generator of a C_0-semigroup S satisfying the variation of parameters formula $S = T + VS$. For a proof of this theorem and a discussion of additive versus multiplicative perturbations we refer the reader to [39; Section III.3.a and III.3.d] and Theorem A.2 in the Appendix for a special case of this theorem.

We now introduce our notion of a generalised Desch-Schappacher perturbation. Let Y be a Banach space satisfying $D_A \hookrightarrow Y \hookrightarrow X$ and let $B \in \mathcal{L}(Y)$. Our main observation is that the Volterra operator V might still act on sufficiently nice subspaces of $C([0,\tau];\mathcal{L}_s(X))$.

By $\mathcal{K}$ we denote the space $\mathcal{L}(X) \cap \mathcal{L}(Y)$, i.e. the space of those operators $K \in \mathcal{L}(X)$ for which $K \cap (Y \times Y) = K_{|Y} \in \mathcal{L}(Y)$. We equip $\mathcal{K}$ with the norm $\|K\|_{\mathcal{K}} := \sup\left\{\|K\|_{\mathcal{L}(X)}, \|K_{|Y}\|_{\mathcal{L}(Y)}\right\}$ $(K \in \mathcal{K})$. As there is no danger of confusion we omit the index $|Y$ from now on.

For $\tau > 0$ we introduce the space

$$\mathcal{Z}_0^\tau := \{F \in \ell_\infty([0,\tau];\mathcal{K});\ F \in C([0,\tau];\mathcal{L}_s(X))\}$$

as a subspace of $\ell_\infty([0,\tau];\mathcal{K})$ equipped with the norm $\|F\|_{\infty,\mathcal{K}} := \sup_{s\in[0,\tau]} \|F(s)\|_{\mathcal{K}}$ $(F \in \mathcal{Z}_0^\tau)$. We note that if $(F_n) \subseteq \mathcal{Z}_0^\tau$ converges to F in $\ell_\infty([0,\tau];\mathcal{K})$ then F is already strongly continuous with respect to $\mathcal{L}_s(X)$. Thus $\mathcal{Z}_0^\tau$ is a closed subspace of $\ell_\infty([0,\tau];\mathcal{K})$ and therefore a Banach space.

We say that $B \in \mathcal{L}(Y)$ is a *generalised Desch-Schappacher perturbation* of A if there exist a $\tau > 0$ and a closed subspace $\mathcal{Z} \subseteq \mathcal{Z}_0^\tau$ such that

(i) $T|_{[0,\tau]} \in \mathcal{Z}$;

(ii) for all $F \in \mathcal{Z}$, $y \in Y$ and $t \in [0,\tau]$ we have $BF(\cdot)y \in L_1([0,\tau];X)$, $(T_{-1} * A_{-1}BF(\cdot)y)(t) \in Y$ and $(T_{-1} * A_{-1}BF)(t)$ extends to a bounded operator $U_F(t) \in \mathcal{K}$;

(iii) the Volterra operator V defined by $VF := U_F(\cdot)$ $(F \in \mathcal{Z})$ is a bounded operator in $\mathcal{L}(\mathcal{Z})$, and the Neumann series $\sum_{n=0}^\infty V^n$ converges absolutely in $\mathcal{L}(\mathcal{Z})$;

(iv) $\lambda \in \rho(A(I+B))$ for $\lambda \in \mathbb{R}$ sufficiently large.

2.2.1 Remark. If $B \in \mathcal{L}(X)$ is a (usual) Desch-Schappacher perturbation of A then the norm of $AR(\lambda,A)B$ is smaller than 1 for $\lambda \in \mathbb{R}$ sufficiently large and thus $I - AR(\lambda,A)B$ and $(A-\lambda)(I - AR(\lambda,A)B) = A(I+B) - \lambda$ are bijective mappings on X. For the generalised Desch-Schappacher perturbation the norm of $AR(\lambda,A)B \in \mathcal{L}(Y)$ does not need to get smaller than 1. (For an example we refer to the translation semigroups below.) We were not able to decide whether the bijectivity of $I - AR(\lambda,A)B$ in $\mathcal{L}(Y)$ can be concluded from (i)-(iii). If this is the case then (iv) will become superfluous.

The proof of the following generalisation of the Desch-Schappacher perturbation theorem utilises ideas from [32; Theorem 5] and [39; Theorem III.3.1].

2.2.2 Theorem. *Let A be the generator of a C_0-semigroup T on a Banach space X. Let Y be a Banach space satisfying $D_A \hookrightarrow Y \hookrightarrow X$. If $B \in \mathcal{L}(Y)$ is a generalised Desch-Schappacher perturbation then $A(I+B)$ is the generator of a C_0-semigroup S. The space Y is S-invariant and S satisfies the variation of parameters formula*

$$S(t)y = T(t)y + A\int_0^t T(t-s)BS(s)y\,ds \quad (t \geq 0,\ y \in Y). \tag{2.2.1}$$

Proof. Let $\mathcal{Z}$ and V be as above. By our assumptions the operator $I-V$ is invertible in $\mathcal{L}(\mathcal{Z})$. Let $S := (I-V)^{-1}T$. As in [39; Theorem III.3.1] we verify the formula

$$[V^nT](s+t)y = \sum_{k=0}^{n}[V^{n-k}T](s)\cdot[V^kT](t)y \tag{2.2.2}$$

for all $y \in Y$, $n \geq 0$ and $s,t \in [0,\tau]$ with $s+t \leq \tau$. A denseness argument shows that this formula also holds for all $y \in X$. Considering the absolutely convergent series $S(t) = \sum_0^\infty [V^nT](t)$ for $t \in [0,\tau]$ we verify the semigroup law for S by using the Cauchy product formula and (2.2.2). So we can extend S to a C_0-semigroup on $[0,\infty)$ which we also denote by S.

The S-invariance of Y and the validity of (2.2.1) for $t \in [0,\tau]$ and $y \in Y$ directly follow from the definition of S and the properties of $\mathcal{Z}$. For $t = n\tau + r$ with $n \in \mathbb{N}$, $r \in [0,\tau)$ we see from assumption (ii) and

$$\begin{aligned}\int_0^t T(t-s)BS(s)y\,ds &= \sum_{k=0}^{n-1} T(t-(k+1)\tau)\int_0^\tau T(\tau-s)BS(s)S(k\tau)y\,ds\\ &\quad + \int_0^r T(r-s)BS(s)S(n\tau)y\,ds\end{aligned}$$

that $\int_0^t T(t-s)BS(s)y\,ds \in D(A)$. Similar to [39; Theorem III.3.1] we obtain

$$
\begin{aligned}
A\int_0^t & T(t-s)BS(s)y\,ds \\
&= \sum_{k=0}^{n-1} T(t-(k+1)\tau)A\int_0^\tau T(\tau-s)BS(s)S(k\tau)y\,ds \\
&\quad + A\int_0^r T(r-s)BS(s)S(n\tau)y\,ds \\
&= \sum_{k=0}^{n-1} T(t-(k+1)\tau)(S(\tau)-T(\tau))S(k\tau)y + (S(r)-T(r))S(n\tau)y \\
&= S(t)y - T(t)y.
\end{aligned}
$$

Let C be the generator of S. It remains to show that $C = A(I+B)$ or equivalently $\lambda - C = (\lambda - A)(I - AR(\lambda, A)B)$. To this end let $\lambda \in \mathbb{R}$ be sufficiently large. Taking the Laplace transform of the variation of parameters formula (2.2.1) in the norm of the space X yields

$$
R(\lambda, C)y = R(\lambda, A)y + AR(\lambda, A)BR(\lambda, C)y \quad (y \in Y). \tag{2.2.3}
$$

Let $H_\lambda := (\lambda - A)(I - AR(\lambda, A)B) = \lambda - A(I+B)$. From (2.2.3) we see that $H_\lambda R(\lambda, C)y = y$ $(y \in Y)$. By assumption (iv) the operator H_λ has a bounded inverse in X for $\lambda \in \mathbb{R}$ sufficiently large. Therefore $R(\lambda, C)y = H_\lambda^{-1}y$ $(y \in Y)$ for $\lambda \in \mathbb{R}$ sufficiently large. Since Y is dense in X we infer that $\lambda - C = H_\lambda$. Hence $C = A(I+B)$. ■

2.3 Delay Operators

In order to avoid technical problems we refrain from introducing delay operators for $h = \infty$ and assume $h < \infty$ throughout this section.

Before we can present the definition of a delay operator we need the following lemma. It ensures that the operators $\Lambda(t)$ in (D3) of Definition 2.3.2 below indeed map regulated functions to regulated functions.

2.3.1 Lemma. *Let $h < \infty$, and $L\colon X_{\mathrm{reg}} \to X$ be a bounded linear operator satisfying*

(D1) If $(f_n) \subseteq X_{\mathrm{reg}}$, $f_n \to f \in X_{\mathrm{reg}}$ pointwise as $n \to \infty$ and $\|f_n(\cdot)\| \le g \in L_p(-h, 0)$ almost everywhere then $Lf_n \to Lf$ as $n \to \infty$.

If $f \in \mathrm{Reg}(\mathbb{R}; X)$ then $g(t) := Lf_t$ $(t \in \mathbb{R})$ is a regulated function, $\|g\|_\infty \le \|L\|\|f\|_\infty$ and $\lim_{s \searrow 0} g(s) = L\overleftarrow{f}$.

Proof. We are going to prove that g has left and right hand limits at all points. To this end let $t \in \mathbb{R}$, $\delta > 0$. As $f_{t+\delta} \to \overleftarrow{f_t}$ pointwise as $\delta \searrow 0$ and

$$\sup_{\delta>0} \|f_{t+\delta}\|_{[-h,0],\infty} \le \|f\|_\infty < \infty,$$

(D1) implies that $g(t+\delta) = Lf_{t+\delta} \to L\overleftarrow{f_t}$ as $\delta \searrow 0$. In the same way we see that g possesses left hand limits at all points. Finally as L is a bounded operator and f vanishes at infinity (D1) shows that g vanishes at infinity. Therefore $g \in \mathrm{Reg}(\mathbb{R}; X)$. The norm estimate follows from the definition of g. ■

2.3.2 Definition. Let $h < \infty$ and $c \in (0, 1]$. A bounded linear operator $L\colon X_{\mathrm{reg}} \to X$ is called a *c-delay operator* if it satisfies (D1) above and the following additional properties.

(D2) L has no mass at 0 in the sense that

$$m_L(t) := \sup\{\|L\varphi\|;\, \varphi \in X_{\mathrm{reg}},\, \operatorname{spt}\varphi \subseteq [-t, 0],\, \|\varphi\|_\infty \le 1\}$$

tends to 0 as $t \to 0$.

(D3) There is a $\tau \in (0, h)$ such that the operators

$$\Lambda(t)\colon \mathrm{Reg}([-h,\tau]; X) \to X_{\mathrm{reg}}, \quad \Lambda(t)\varphi(s) := \begin{cases} 0 & \text{for } s \in [-h, -t], \\ L\varphi_{t+s} & \text{for } s \in (-t, 0] \end{cases}$$

are bounded in the norm of $\mathcal{L}(L_p(-h,\tau; X), X_p)$ uniformly in $t \in [0, \tau]$, and if the domain is restricted to functions with support in $[0, \tau]$ then strictly contractive with a common contraction constant less than c.

If $c = 1$ we say that L is a delay operator rather than 1-delay operator. If c can be chosen arbitrarily small then we call L a 0-delay operator.

A large and interesting class of delay operators arises as bounded operators from $L_r(\mu; X)$ to X, where r is in $[1, p]$ and μ is a suitable measure on $[-h, 0]$.

2.3.3 Proposition. *Let $h < \infty$. Let L be a linear operator from X_{reg} to X. Assume there exist $r \in [1, p]$ and a finite Borel measure μ_L on $[-h, 0]$ such that*

$$\|L\varphi\| \le \|\varphi\|_{L_r(\mu_L;X)} \quad (\varphi \in X_{\mathrm{reg}}). \tag{2.3.1}$$

If μ_L has no mass at 0, then L is a 0-delay operator.

Proof. For $\varphi \in X_{\mathrm{reg}}$ the finiteness of the Borel measure implies

$$\|L\varphi\| \le \|\varphi\|_{L_r(\mu_L;X)} \le \mu_L([-h, 0])^{1/r} \|\varphi\|_\infty. \tag{2.3.2}$$

Thus L is a bounded linear operator from X_{reg} to X. By Lebesgue's convergence theorem (D1) holds (observe that $L_p(-h, 0) \subseteq L_r(-h, 0)$). Property (D2) immediately follows from that fact that μ_L is supposed to have no mass at 0. For the verification of (D3) we first estimate the operator norm of $\Lambda(t)$ in $\mathcal{L}(\mathrm{Reg}([-h,\tau];X),X_{\mathrm{reg}})$ and $\mathcal{L}(L_r(-h,\tau;X),X_r)$ for $\tau \in (0,h)$ and $t \in (0,\tau)$. We then use interpolation to show (D3). For $\psi \in L_r(\mathbb{R})$ we compute

$$\begin{aligned}\int_{-t}^{0}\int_{\mathbb{R}} |\psi(\vartheta+s)|^r\,d\mu_L(\vartheta)\,ds &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{[-t,0]}(s)|\psi(\vartheta+s)|^r\,d\mu_L(\vartheta)\,ds\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{[s,s+t]}(\vartheta)|\psi(s)|^r\,d\mu_L(\vartheta)\,ds\\ &\le \|\psi\|_r^r \sup_{s\in\operatorname{spt}\psi} \mu_L([s,s+t]).\end{aligned} \tag{2.3.3}$$

(Here we have used that $\int_{\mathbb{R}}\int_{\mathbb{R}} f(s,\vartheta)\,d\mu_L(\vartheta)\,ds = \int_{\mathbb{R}}\int_{\mathbb{R}} f(s-\vartheta,\vartheta)\,d\mu_L(\vartheta)\,ds$ for an integrable function f, by Fubini's theorem.) Let $\tau \in (0,h)$ and $t \in [0,\tau]$. If $\varphi \in \mathrm{Reg}([-h,\tau];X)$ then (2.3.3) and (2.3.1) yield

$$\begin{aligned}\left(\int_{-t}^{0} \|L\varphi_{t+s}\|^r\,ds\right)^{1/r} &\le \left(\int_{-t}^{0}\int_{\mathbb{R}} \|\varphi_t(\vartheta+s)\|^r\,d\mu_L(\vartheta)\,ds\right)\\ &\le \|\varphi\|_r \sup_{s\in\operatorname{spt}\varphi_t} \mu_L\left([s,s+t]\right)^{1/r}.\end{aligned} \tag{2.3.4}$$

Therefore we have $\|\Lambda(t)\varphi\|_r \le \mu_L([-h,0])^{1/r}\|\varphi\|_r$. Furthermore (2.3.2) provides the estimate $\|\Lambda(t)\varphi\|_\infty \le \mu_L([-h,0])^{1/r}\|\varphi\|_\infty$. The Banach space valued version of the interpolation theorem by Riesz-Thorin implies that

$$\|\Lambda(t)\varphi\|_p \le 2\mu_L([-h,0])^{1/r}\|\varphi\|_p \quad (t \in (0,h)).$$

(In the complex case we can omit the factor 2.) Hence the operators $\Lambda(t)$ extend to a uniformly bounded family of operators in $\mathcal{L}(L_p(-h,\tau;X),X_p)$ for any $\tau \in (0,h)$. From (2.3.2) and (2.3.4) we also see that if we additionally have $\operatorname{spt}\varphi \subseteq [0,\tau]$ then $\|\Lambda(t)\varphi\|_r \le \mu_L([-t,0])^{1/r}\|\varphi\|_r$ and $\|\Lambda(t)\varphi\|_\infty \le \mu_L([-t,0])^{1/r}\|\varphi\|_\infty$. Again by an application of the Riesz-Thorin theorem we infer

$$\|\Lambda(t)\varphi\|_p \le 2\mu_L([-t,0])^{1/r}\|\varphi\|_p \quad (t \in (0,h)). \tag{2.3.5}$$

As μ_L has no mass at 0 we can find $\tau \in (0,h)$ such that the restrictions of $\Lambda(t)$ to $\{f \in \mathrm{Reg}([-h,\tau];X);\ \operatorname{spt} f \subseteq [0,\tau]\}$ become strict contractions in the norm of $\mathcal{L}(L_p(-h,\tau;X),X_p)$ with a common contraction constant less than c for any $c > 0$ and for all $t \in [0,\tau]$. This shows that (D3) holds. ∎

2.3.4 Remarks. There are two important types of delay operators which we like to mention.

(a) Any bounded linear operator $L\colon X_p \to X$ is a 0-delay operator. This is easily seen by setting $r := p$ and $\mu_L := \|L\|\lambda$ where λ denotes the Lebesgue-measure on $[-h,0]$.

(b) The second interesting type of delay operators are operator-valued Riemann-Stieltjes integrals. Let $\eta\colon [-h,0] \to \mathcal{L}(X)$ be of bounded variation without mass at 0. The bounded linear operator $Lf := \int_{-h}^{0} d\eta(s)f(s)$ from X_{reg} to X is a 0-delay operator. We verify this by choosing $r := 1$ and $\mu_L := d|\eta|$. Here $d|\eta|$ denotes the variation of η.

(c) For further reading on operators for delay problems dominated by a measure we refer to [69].

2.3.5 Remark. For our purposes it would suffice to have delay operators defined on the domains $C([-h,0];X)$ or $W_p^1(-h,0;X)$. There are several reasons for considering delay operators defined on spaces of regulated functions. First of all if an operator $L \in \mathcal{L}(C([-h,0];X),X)$ is for example weakly compact (which is for example always true if X is reflexive), if (2.3.1) is satisfied for some $r \in [1,\infty)$ and a finite Borel measure μ, or if L is of finite variation then L always extends to a bounded operator on the regulated functions (cf. [35; Section VI.5], [36; Section III.19] and [10; III.2.a]). Furthermore the definition as well as the perturbation argument carried out in the next section become easier for the larger domain of regulated functions (namely the abstract perturbation argument using Volterra operators would have to be replaced by an explicit fixed point argument). Lastly in the second part of this chapter we deal with translation semigroups on Dedekind-complete Banach lattices and delay operators being regular. Such operators always possess a bounded extension to the space of regular functions. To see this let $L \in \mathcal{L}(C([-h,0];X),X)$ be regular. If L is positive then $\hat{L}f := \sup\{Lf\,;\, g \in C([-h,0];X),\ 0 \le g \le f\}$ $(f \in X_{\mathrm{reg}},\ f \ge 0)$ defines an extension of L in $\mathcal{L}(X_{\mathrm{reg}},X)$. Otherwise there are positive operatos L_+ and L_- with $L = L_+ - L_-$, which extend to bounded operators $\hat{L}_+$ and $\hat{L}_-$ in $\mathcal{L}(X_{\mathrm{reg}},X)$. Now $\hat{L} := \hat{L}_+ - \hat{L}_-$ gives an extension of L to a bounded operator in $\mathcal{L}(X_{\mathrm{reg}},X)$.

2.4 The Generator Property of the Perturbed Weak Derivative

The objective of this section is to show that A_L defined in the introduction generates a C_0-semigroup if L is a delay operator. We first consider the case $h < \infty$. Then we use this result to treat operators $L\colon W_p^1(-\infty,0;X) \to X$ for which there is a suitable Borel measure μ_L on $(-\infty,0]$, such that $\|L\varphi\|_r \le \|\varphi\|_{L_r(\mu;X)}$ for some $r \in [1,p]$ (cf. Corollary 2.4.3).

We now assume that $h < \infty$ and that $L \in \mathcal{L}(X_{\mathrm{reg}},X)$ is a delay operator. The operator $Bf := -Lf \cdot \mathbf{1}_{(-h,0)}$ $(f \in X_{\mathrm{reg}})$ belongs to $\mathcal{L}(X_{\mathrm{reg}})$. We are now going to show that B is a generalised Desch-Schappacher perturbation of A_0. First we observe that by a straightforward computation $A_L = A_0(I+B)$. (We remark that by writing A_L as a multiplicative perturbation of A_0 we avoid extrapolation spaces; cf. [39; Section III.3.d] and [39; Example III.3.5]).

Let $\tau > 0$ such that $m_L(\tau) < 1$ (see (D2)) and such that (D3) holds for this τ (with $c = 1$). Let $\mathcal{K}$ be the space of operators in $\mathcal{L}(X_p) \cap \mathcal{L}(X_{\mathrm{reg}})$ (cf. Section 2.2). By $\mathcal{Z}$ we denote the (closed) subspace of all translations in $\ell_\infty([0,\tau];\mathcal{K})$. Observe that translations in $\ell_\infty([0,\tau];\mathcal{L}(X_p))$ are automatically strongly continuous. Further notice that $T_0|_{[0,\tau]} \in \mathcal{Z}$. For $F \in \mathcal{Z}$ and $f \in X_{\mathrm{reg}}$ we define the function $G(t) := \int_0^t T_0(t-r)BF(r)f\,dr$ $(t \in [0,\tau])$. For $t \in [0,\tau]$ and $s \in [-h,0]$ we obtain

$$G(t)(s) = -\int_0^t LF(r)f \cdot \mathbf{1}_{[-h,r-t]}(s)\,dr = -\int_{\max\{0,t+s\}}^t LF(r)f\,dr.$$

As F is a translation there is a function $g \in \mathrm{Reg}([-h,\tau];X)$ such that $g_r|_{[-h,0]} = F(r)f$. By an application of Lemma 2.3.1 we see that $[0,\tau] \ni r \mapsto LF(r)f = Lg_r$ is again a regulated function. Therefore $G(t)$ is weakly differentiable. The weak derivative of $G(t)$ is given by

$$G(t)' = \left([-h,0] \ni s \mapsto \begin{cases} 0 & \text{if } s \in [-h,-t], \\ LF(s+t)f & \text{if } s \in (-t,0] \end{cases}\right). \tag{2.4.1}$$

As $G(t)' \in X_{\mathrm{reg}} \subseteq X_p$ and $G(t)(0) = 0$ we see that $G(t) \in D(A_0)$. From (2.4.1) we derive $\|A_0 G(\cdot)\|_\infty \leq \|L\|\|F\|\|f\|_\infty$. Thus for $F \in \mathcal{Z}$ and $f \in X_{\mathrm{reg}}$ we can define the Volterra operator $\tilde{V} \in \mathcal{L}(\mathcal{Z}, \ell_\infty([0,\tau];X_{\mathrm{reg}}))$ by

$$(\tilde{V}F)(t)f := A_0 \int_0^t T_0(t-r)BF(r)f\,dr \quad (t \in [0,\tau],\ F \in \mathcal{Z},\ f \in X_{\mathrm{reg}}). \tag{2.4.2}$$

In order to see that $\tilde{V}F$ is a translation for all $F \in \mathcal{Z}$ let $f \in X_{\mathrm{reg}}$, $g(t) := 0$ for $t \in [-h,0]$ and $g(t) := LF(t)f$ for $t \in (0,\tau]$. From (2.4.1) we deduce that $\tilde{V}F(t)f = g_t|_{[-h,0]}$. Thus $\tilde{V}F$ is a translation. From (D3) and (2.4.1) we see that $\tilde{V}$ is continuous in the norm of $\mathcal{L}(\mathcal{Z})$ and so extends to a Volterra operator $V \in \mathcal{L}(\mathcal{Z})$. As $VF(0) = 0$ for all $F \in \mathcal{Z}$ we see that V maps into the closed subspace $\mathcal{Z}_0 := \{F \in \mathcal{Z};\ F(0) = 0\}$. From the assumption $m_L(\tau) < 1$ and the second assumption in (D3) we infer that $V_0 := V|_{\mathcal{Z}_0}$ is strictly contractive in $\mathcal{L}(\mathcal{Z}_0)$.

As $V^n = V_0^{n-1}V$ we infer that the Neumann series $\sum_{n=0}^\infty V^n$ converges absolutely in $\mathcal{L}(\mathcal{Z})$. In order to be able to apply Theorem 2.2.2 it remains to show that $\lambda \in \rho(A_L)$ for $\lambda \in \mathbb{R}$ sufficiently large (which corresponds to assumption (iv)).

2.4.1 Lemma. *(a) For $\lambda \in \mathbb{R}$ we define $L_\lambda x := L(s \mapsto e^{\lambda s}x)$ $(x \in X)$. Then $L_\lambda \in \mathcal{L}(X)$, $L_\lambda \to 0$ as $\lambda \to \infty$ and $1 \in \rho(L_\lambda)$ for λ sufficiently large.*

(b) If $\lambda \in \mathbb{R}$ is sufficiently large, then K_λ, defined by

$$K_\lambda g(s) := e^{\lambda s} R(1, L_\lambda) L R(\lambda, A_0) g \quad (s \in (-h,0),\ g \in X_p),$$

belongs to $\mathcal{L}(X_p)$.

(c) If $\lambda \in \mathbb{R}$ is sufficiently large then $\lambda \in \rho(A_L)$ and

$$R(\lambda, A_L) = R(\lambda, A_0) + K_\lambda.$$

Proof. Assertion (a) follows from (D1) and (D2). Hence $1 - L_\lambda$ is invertible for λ sufficiently large. In order to show the boundedness of K_λ first notice that $R(\lambda, A_0)$ is bounded as a mapping from X_p to $D_{A_0} = (D(A_0), \|\cdot\|_{A_0})$, where $\|\cdot\|_{A_0}$ denotes the graph norm of A_0. As D_{A_0} is continuously embedded into $W^1_p(-h,0;X)$ which again is continuously embedded into $C_b([-h,0];X)$ we see that the operator $LR(\lambda, A_0)$ is bounded from X_p to X. This implies the boundedness of K_λ and shows (b).

In order to prove (c) we first show that $\lambda - A_L$ is surjective for λ sufficiently large (so that $1 \in \rho(L_\lambda)$ by (a)). To this end let $g \in X_p$ and $f := (R(\lambda, A_0) + K_\lambda)g$. Obviously $f \in W^1_p(-h,0;X)$. Differentiation shows that $f' = -g + \lambda f$. As $R(\lambda, A_0)g(0) = 0$ we have $f(0) = (K_\lambda g)(0) = R(1, L_\lambda)LR(\lambda, A_0)g$. Hence

$$L_\lambda(f(0)) = L(s \mapsto e^{\lambda s}R(1, L_\lambda)LR(\lambda, A_0)g) = LK_\lambda g.$$

Moreover we have $(1 - L_\lambda)(f(0)) = LR(\lambda, A_0)g$. From these equations we conclude

$$\begin{aligned} f(0) &= L_\lambda(f(0)) + (I - L_\lambda)(f(0)) = L_\lambda(f(0)) + LR(\lambda, A_0)g \\ &= L(K_\lambda + R(\lambda, A_0))g = Lf. \end{aligned}$$

Thus $f \in D(A_L)$ and $(\lambda - A_L)f = g$. To finish the proof we have to show that $\lambda - A_L$ is injective. To this end we first observe that any solution $f \in D(A_L)$ of $(\lambda - A_L)f = 0$ has the form $f(s) = e^{\lambda s}x$ for some $x \in X$. The boundary condition $f(0) = Lf$ yields $x = L_\lambda x$, which has the unique solution $x = 0$. Thus $\lambda - A_L$ is bijective for λ sufficiently large. This proves assertion (c). ■

We have now shown that the assumptions of Theorem 2.2.2 are met and thus obtain the generator property of A_L.

2.4.2 Corollary. *(a) Let $h < \infty$. If L is a delay operator then A_L is the generator of a C_0-semigroup T_L on X_p.*

(b) T_L maps regulated functions to regulated functions and

$$\lim_{s \searrow -h} T_L(h)f(s) = L\overleftarrow{f} \quad (f \in X_{\mathrm{reg}}). \tag{2.4.3}$$

Proof. It remains to show (b). The X_{reg}-invariance of T_L is stated in Theorem 2.2.2. Let $f \in X_{\mathrm{reg}}$ and define the function $g\colon [-h,\infty) \to X$ by

$$g(t) := \begin{cases} T_L(t)f(0) & \text{if } t > 0, \\ f(t) & \text{if } t \in [-h, 0]. \end{cases} \tag{2.4.4}$$

The variation of parameters formula (2.2.1) and (2.4.1) imply that $g(s) = Lg_s$ for $s \in (0,\tau)$. From Lemma 2.3.1 we infer that

$$\lim_{s \searrow 0} g(s) = \lim_{s \searrow 0} Lg_s = L\overleftarrow{g} = L\overleftarrow{f}.$$ ■

In the second part of this section we deal with the case $h = \infty$. Since $-x \cdot \mathbf{1}_{(-\infty,0)} \notin X_p$ for $x \neq 0$ we cannot obtain A_L as a multiplicative perturbation of A_0 as above. However we have $A_L = (A_0 - \omega)(I + B) + \omega$ for some $\omega > 0$ and $Bf := (t \mapsto -Lf \cdot e^{-\omega t})$. For our purposes it seems easier to use approximation techniques and the translation semigroups already obtained for $h < \infty$.

2.4.3 Corollary. *Let L be a bounded linear operator from $W_p^1(-\infty, 0; X)$ to X. Assume that there exist $r \in [1, p]$ and a Borel measure μ_L on $(-\infty, 0]$ such that $\sup_{t \le 0} \mu_L([t - \delta, t]) < \infty$ for some (or equivalently all) $\delta > 0$ and such that*

$$\|L\varphi\| \le \|\varphi\|_{L_r(\mu_L;X)} \quad (\varphi \in W_p^1(-\infty, 0; X)). \tag{2.4.5}$$

If μ_L has no mass at 0, then A_L is a generator. Moreover the semigroup T_L generated by A_L maps regulated functions with compact support to regulated functions, and (2.4.3) *holds for regulated functions with compact support.*

Proof. Let $\hat{L}$ be the extension of L to a bounded operator in $L_r(\mu_L; X)$. As this space includes regulated functions with compact support we can define $L(n)f := \hat{L}(\mathbf{1}_{[-n,0]}f)$ for $f \in W_p^1(-\infty, 0; X)$ and $n \in \mathbb{N}$ and $\tilde{L}(n)f := \hat{L}\tilde{f}$ for $f \in W_p^1(-n, 0; X)$ and $\tilde{f}(s) := f(s)$ $(s \in (-n, 0))$ and $\tilde{f}(s) = 0$ $(s \in (-\infty, -n))$.

It is easy to see that $\tilde{L}(n)$ are delay operators for $n \in \mathbb{N}$ (cf. Proposition 2.3.3). From Corollary 2.4.2 we see that $A_{\tilde{L}(n)}$ is the generator of a C_0-semigroup $T_{\tilde{L}(n)}$ on $L_p(-n, 0; X)$. We can extend $T_{\tilde{L}(n)}$ to a translation semigroup on $L_p(-\infty, 0; X)$. To this end let $f \in L_p(-\infty, 0; X)$ and $g \in L_p(-\infty, n; X)$ such that $g|_{(-\infty,0)} = f$ and $g|_{(0,n)} = \big(\tilde{T}_{L(n)}(n)f\big)_{-n}$. Then $T_{L(n)}(t)f := g_t|_{(-\infty,0)}$ defines a translation semigroup on $L_p(-\infty, 0; X)$ whose generator is easily identified to be $A_{L(n)}$.

We are now going to show that $\big(A_{L(n)}\big)_{n\in\mathbb{N}}$ approximate A_L in the sense of the Trotter-Kato approximation theorem. As in Lemma 2.4.1 we can show that for $\lambda > 0$ sufficiently large and $n \in \mathbb{N}$ the operators $1 - L_\lambda(n)$ and $1 - L_\lambda$ are invertible, $K_\lambda(n)$ and K_λ (analogously defined as in Lemma 2.4.1 for $L(n)$ and L, respectively) are bounded, $\lambda \in \rho(A_{L(n)})$, $\lambda \in \rho(A_L)$ and $R(\lambda, A_L) = R(\lambda, A_0) + K_\lambda$. From this representation we see that $R(\lambda, A_{L(n)}) \to R(\lambda, A_L)$ in the strong operator topology. In order to obtain a common growth bound for the semigroups $T_{L(n)}$ we first notice that the operators $\tilde{L}(n)$ are delay operators on $\mathrm{Reg}(-n, 0; X)$ satisfying (D3) for a common $\tau \in (0, 1)$. We choose τ such that $2\mu_L([-\tau, 0])^{1/r} < 1$. For this τ let $\tilde{V}(n)$ be the Volterra operator from above corresponding to the operator $\tilde{L}(n)$ on $\mathrm{Reg}(-n, 0; X)$. Using the equalities $T_{\tilde{L}(n)}\Big|_{[0,\tau]} = (I - \tilde{V}(n))^{-1}T_0$ and $(I - \tilde{V}(n))^{-1} = I + \tilde{V}(n)(I - \tilde{V}_0(n))^{-1}$ we obtain the estimate

$$\begin{aligned} \sup_{t\in[0,\tau]} \|T_{L(n)}(t)\| &\le 1 + \sup_{t\in[0,\tau]} \|T_{\tilde{L}(n)}(t)\| \le 2 + \frac{\|V(n)\|}{1 - \|V_0(n)\|} \\ &\le 2 + \frac{2c^{1/r}}{1 - 2\mu_L([-\tau, 0])^{1/r}} \quad (n \in \mathbb{N}), \end{aligned}$$

where $c := \sup_{t \le 0} \mu_L([t - \tau, t]) < \infty$. Thus there exist $M \ge 1$ and $\omega \in \mathbb{R}$ such that $\|T_{L(n)}(t)\| \le Me^{\omega t}$ ($t \ge 0$, $n \in \mathbb{N}$). The second Trotter-Kato Approximation Theorem (cf. [39; Theorem III.4.9]) shows that A_L is the generator of a C_0-semigroup T_L which is the limit of $T_{L(n)}$. Equation (2.4.3) follows from Corollary 2.4.2(b) and the fact that $T_L(t)f = T_{L(n)}(t)f$ for a regulated function f with compact support, $t \ge 0$ and $n \in \mathbb{N}$ sufficiently large. ∎

2.4.4 Remarks. (a) In the proof of Corollary 2.4.2(b) we have derived that $g(s) = Lg_s$ for the function g defined in (2.4.4). In fact g is the unique (locally regulated) solution of the equation $u(s) = Lu_s$, with $s \in \mathbb{R}_+$, $u\colon [-h, \infty) \to X$, and for the initial value $u_0 = f \in X_{\mathrm{reg}}$.

(b) For an investigation of the spectral properties of A_L and the asymptotic behaviour of T_L in the case $L \in \mathcal{L}(X_p, X)$ we refer to [53; Section C-IV.3] and [42]. There the reader can find additional conditions on L such that the assertion $1 \in \rho(L_\lambda)$ if and only if $\lambda \in \rho(A_L)$ holds (cf. Lemma 2.4.1).

2.5 The Modulus Semigroup of Translation Semigroups

In this section we additionally assume that X is a (real or complex) Banach lattice with order continuous norm. Again we distinguish between the cases $h < \infty$ and $h = \infty$.

From [50; Section 1.3] we recall that for Banach lattices Y and Z regular operators in $\mathcal{L}(Y, Z)$ always possess a modulus if Z is Dedekind-complete. Hence if $L_1 \in \mathcal{L}(X_{\mathrm{reg}}, X)$, $L_2 \in \mathcal{L}(L_r(\mu; X), X)$ (for some Borel measure μ on $[-h, 0]$ or $(-\infty, 0]$) are regular operators, then both possess a modulus. Moreover if $h < \infty$ and L_1 is the restriction of L_2 then $|L_1|$ is the restriction of $|L_2|$; see Remark 2.5.5 for details. For the modulus of Riemann-Stieltjes type operators (cf. Remarks 2.3.4) we refer the reader to [71; Section 3].

For the definition and an overview on modulus semigroups we refer to Chapter 1. Here we only recall that on Banach lattices with order continuous norm a dominated C_0-semigroup automatically possesses a modulus semigroup; cf. [11; Theorem 2.1]. We therefore start by showing that translation semigroups induced by a dominated delay operator are dominated and thus possess a modulus semigroup. (We point out that the following lemma does not require order-continuity of the norm of X.)

2.5.1 Lemma. *(a) Assume that $h < \infty$, and let L, $\hat{L}$ be delay operators. If $\hat{L}$ dominates L then $T_{\hat{L}}$ dominates T_L.*

(b) Assume that $h = \infty$. Let $r \in [1, p]$ and let μ be a Borel measure on $(-\infty, 0]$ without mass at 0 and with $\sup_{t \le 0} \mu_L([t - \delta, t]) < \infty$ for some (or equivalently all) $\delta > 0$. Let $L, \hat{L} \in \mathcal{L}(L_r(\mu; X), X)$. If $\hat{L}$ dominates L in $\mathcal{L}(L_r(\mu; X); X)$ for some $r \in [1, p]$ then $T_{\hat{L}}$ dominates T_L.

Proof. (a) By V_L and $V_{\hat{L}}$ we denote the Volterra operators on $\mathcal{Z}$ corresponding to L and $\hat{L}$, respectively (see Section 2.4). Without loss of generality we can assume that V_L and

$V_{\hat{L}}$ are defined for the same $\tau > 0$. As $\hat{L}$ dominates L we see that $V_{\hat{L}}$ dominates V_L. From the representations of T_L and $T_{\hat{L}}$ via the Neumann series of V_L and $V_{\hat{L}}$, respectively, we infer the domination of T_L by $T_{\hat{L}}$.

(b) Let $L(n)$ be defined as in the proof of Corollary 2.4.3, and similarly $\hat{L}(n)$ for the operator $\hat{L}$. From (a) we already know that $T_{L(n)}$ is dominated by $T_{\hat{L}(n)}$. Since $T_{L(n)} \to T_L$ and $T_{\hat{L}(n)} \to T_{\hat{L}}$ in the strong operator topology uniformly on compact intervals the domination property carries over to T_L and $T_{\hat{L}}$. ■

2.5.2 Theorem. *(a) Assume that $h < \infty$, and let L be a delay operator possessing the modulus $|L|$. If $|L|$ is a delay operator then T_L possesses the modulus semigroup $T_{|L|}$.*

(b) Assume that $h = \infty$. Let $r \in [1, p]$ and let μ be a Borel measure on $(-\infty, 0]$ without mass at 0 and with $\sup_{t \leq 0} \mu_L([t-\delta, t]) < \infty$ for some (or equivalently all) $\delta > 0$. If $L \in \mathcal{L}(L_r(\mu; X), X)$ be a regular operator. Then $T_{|L|}$ is the modulus semigroup of T_L.

The proof of this theorem relies on the following two lemmata.

2.5.3 Lemma. *Let $G \subseteq L_p(0, 1; X)_+$ be such that each $g \in G$ has a representative $\tilde{g}$ which is continuous at 0 and assume that $\inf_{g \in G} \tilde{g}(0) = 0$. Then $f := \inf G \in L_p(0, 1; X)$ has a representative $\tilde{f}$ which is continuous at 0 and satisfies $\tilde{f}(0) = 0$.*

Proof. The order continuity of the norm of X implies that each set which possesses a supremum has a countable subset possessing the same supremum (cf. [72; Theorem 8.17.8] or [61; Corollary 1 of Theorem II.5.10]). Therefore we can find a sequence $(g_n)_{n \in \mathbb{N}} \subseteq G$ such that $\inf_{n \in \mathbb{N}} g_n(0) = 0$. The sequence $(h_n)_{n \in \mathbb{N}}$ defined by $h_n := \inf\{g_i;\ 1 \leq i \leq n\}$ is a monotone decreasing sequence of functions which have representatives $\tilde{h}_n$ being continuous at 0. Furthermore $0 \leq f \leq \inf_{n \in \mathbb{N}} h_n$ and $\inf_{n \in \mathbb{N}} \tilde{h}_n(0) = \lim_{n \to \infty} \tilde{h}_n(0) = 0$. Let $\varepsilon > 0$. Then there is an $n \in \mathbb{N}$ such that $\|\tilde{h}_n(0)\| \leq \varepsilon/2$. As h_n is continuous at 0 we can find $\delta > 0$ such that $\|\tilde{h}_n(s) - \tilde{h}_n(0)\| \leq \varepsilon/2$ for all $s \in [0, \delta]$. As $f \leq \tilde{h}_n$ almost everywhere we obtain $\|f(s)\| \leq \|\tilde{h}_n(s)\| \leq \varepsilon$ almost everywhere ($s \in [0, \delta]$). Thus we can find a representative $\tilde{f}$ of f with $\|\tilde{f}(s)\| \leq \varepsilon$ for all $s \in [0, \delta]$. This shows $\tilde{f}(0) = 0$. ■

The statement of the next lemma can be understood as a generalisation of [71; Proposition 9]. In fact we can almost copy the proof of Lemma 8 in [71] as (D1) is the generalisation of the crucial equations (2) and (3) in this paper.

2.5.4 Lemma. *Let $h < \infty$, and let $L \in \mathcal{L}(X_{\mathrm{reg}}, X)$ be a regular operator. If $|L|$ satisfies (D1) in Lemma 2.3.1 then the modulus of $L|_{C([-h,0];X)}$ is the operator $|L|$ restricted to $C([-h, 0]; X)$.*

Proof. We first show that for $\psi \in X_{\mathrm{reg}}$ and $\varepsilon > 0$ there is a $\psi_\varepsilon \in C([-h, 0]; X)$ satisfying

$$\| |L| \, |\psi - \psi_\varepsilon| \| \leq \varepsilon \tag{2.5.1}$$

(this generalises [71; Lemma 8]). It suffices to show the assertion for functions $x \cdot \mathbf{1}_{(a,b)}$ and $x \cdot \mathbf{1}_{\{a\}}$ ($a \in [-h, 0]$, $b \in (a, 0]$, $x \in X$) as step functions on $[-h, 0]$ with values in

X are dense in X_{reg} and as step functions can be represented as linear combinations of such functions.

First let $-h \le a < b \le 0$, $x \in X_+$ and $\varepsilon > 0$. For $\vartheta_0 \in (-h,0]$ and $\vartheta_1 \in [-h,0)$ we conclude from (D1) that

$$\lim_{\vartheta \to \vartheta_0 -} |L|(x \cdot \mathbf{1}_{[\vartheta,\vartheta_0)}) = 0, \quad \lim_{\vartheta \to \vartheta_1 +} |L|(x \cdot \mathbf{1}_{(\vartheta_1,\vartheta]}) = 0 \tag{2.5.2}$$

(these two assertions correspond to equations (2) and (3) in [71]). Therefore there exist $a', b' \in (a,b)$ with $a' \le b'$ such that

$$\big\| |L|(x \cdot \mathbf{1}_{(a,a')}) \big\| \le \frac{\varepsilon}{2}, \quad \big\| |L|(x \cdot \mathbf{1}_{(b',b)}) \big\| \le \frac{\varepsilon}{2}.$$

Thus for $\varphi \in C([-h,0])$ with $\operatorname{spt}\varphi \subseteq [a,b]$, $0 \le \varphi \le 1$ and $\varphi(s) = 1$ for all $s \in [a',b']$ we have

$$\big\| |L|(x \cdot \mathbf{1}_{(a,b)} - x \cdot \varphi) \big\| \le \Big(\big\| |L|(x \cdot \mathbf{1}_{(a,a')}) \big\| + \big\| |L|(x \cdot \mathbf{1}_{(b',b)}) \big\| \Big) \le \varepsilon.$$

Now let $a \in [-h,0]$, $x \in X$ and $\varepsilon > 0$. From (2.5.2) we infer that there is an open interval $J \subseteq [-h,0]$ with $a \in J$ such that $\big\| |L|(x \cdot \mathbf{1}_{\{a\}} - x \cdot \mathbf{1}_J) \big\| \le \varepsilon$. Therefore if $\varphi \in C([-h,0])$ with $\varphi(a) = 1$, $\operatorname{spt}\varphi \subseteq J$ and $0 \le \varphi \le 1$, then $\big\| |L|(x \cdot \mathbf{1}_{\{a\}} - x \cdot \varphi) \big\| \le \varepsilon$. This shows (2.5.1).

In order to show the assertion of this lemma we have to prove that

$$|L|f = \sup\{|Lg|;\, g \in C([-h,0];X), |g| \le f\} \quad (f \in C([-h,0];X)_+). \tag{2.5.3}$$

To this end it suffices to show that the set $\{|Lg|;\, g \in C([-h,0];X), |g| \le f\}$ is dense in $\{|Lg|;\, g \in X_{\mathrm{reg}}, |g| \le f\}$, i.e. for $f \in C([-h,0];X)_+$, $g \in X_{\mathrm{reg}}$ with $|g| \le f$ and $\varepsilon > 0$ we have to find $\psi \in C([-h,0];X)$ satisfying $|\psi| \le f$ and $\big\| L(g - \psi) \big\| \le \varepsilon$.

Let $\varepsilon > 0$ and $g \in X_{\mathrm{reg}}$ with $|g| \le f$. By the first part of the proof there exists $\varphi \in C([-h,0];X)$ so that $\big\| |L|(g - \varphi) \big\| \le \varepsilon$. Let $\psi := \tau(f)\varphi$; cf. Remark 2.5.5 for the definition of the truncation τ. Then we have $|g - \psi| \le |g - \varphi|$ (this follows from property (i) of the truncation; see [48; Section 2]). Therefore

$$\big\| L(g - \psi) \big\| \le \big\| |L||g - \psi| \big\| \le \big\| |L||g - \varphi| \big\| \le \varepsilon. \qquad \blacksquare$$

Proof of Theorem 2.5.2. (a) From [11; Theorem 2.1] and Lemma 2.5.1 we already know that T_L possesses a modulus semigroup $T_L^\sharp$ with generator $A_L^\sharp$ and that $T_{|L|}$ dominates T_L. This implies that the modulus semigroup $T_L^\sharp$ is a translation (cf. [42; proof of Proposition 3.10]). Hence by Proposition 2.1.2 we get $D(A_L^\sharp) \subseteq W_p^1(-h,0;X)$ and $A_L^\sharp f = f'$ for $f \in D(A_L^\sharp)$. Now it suffices to show that $D(A_L^\sharp)_+ \subseteq D(A_{|L|})_+$. As both operators are generators of positive semigroups this implies $D(A_L^\sharp) \subseteq D(A_{|L|})$ and therefore also $A_L^\sharp \subseteq A_{|L|}$. Hence $A_L^\sharp = A_{|L|}$ as both operators are generators.

In order to show the inclusion let $f \in D(A_L^\sharp)_+$. Let $\varphi := T_L^\sharp(h)f \in D(A_L^\sharp)$. From $D(A_L^\sharp) \subseteq W_p^1(-h,0;X)$ we see that φ is a continuous function. The domination of T_L by $T_{|L|}$ implies that

$$\sup\{|T_L(h)g|;\, g \in C([-h,0];X), |g| \le f\} \le \varphi \le T_{|L|}(h)f. \tag{2.5.4}$$

In Corollary 2.4.2(b) it was shown that the right hand limit of $T_L(h)g$ at $-h$ exists for all $g \in C([-h,0];X)$ and $\lim_{s\searrow -h} T_L(h)g(s) = L\overleftarrow{g} = Lg$. Similarly the right hand limit of $T_{|L|}(h)f$ at $-h$ is $|L|f$. From Lemma 2.5.4 we infer that

$$|L|f = \sup\{|Lg|;\, g \in C([-h,0];X)\} = \sup\{|T_L(h)g|(-h);\, g \in C([-h,0];X)\}$$

and therefore

$$T_{|L|}(h)f(-h) - \sup\{|T_L(h)g|(-h);\, g \in C([-h,0];X)\} = 0.$$

Thus from (2.5.4) and Lemma 2.5.3 we obtain $(\varphi - T_{|L|}(h)f)(-h) = 0$. Hence $\varphi(-h) = |L|f = f(0)$ and so $f \in D(A_{|L|})$. This shows that $D(A_L^\sharp)_+ \subseteq D(A_{|L|})_+$.

(b) The case $h = \infty$ is solved in almost the same way as in (a). First we set $\varphi := T_L^\sharp(t)f$ for some arbitrary $t > 0$ (instead of $T_L^\sharp(h)f$ which does not make sense for $h = \infty$). Now the only differences in the proofs of these two cases are that in (2.5.4) we have to replace $g \in C([-h,0];X)$ by $g \in C_c((-\infty,0];X)$, that we need to consider right hand limits at $-t$ instead of $-h$ and that instead of Lemma 2.5.4 we have to invoke [71; Remark 2] (cf. Remark 2.5.5) to infer that $|L|f = \sup\{|Lg|;\, g \in C_c((-\infty,0];X), |g| \le f\}$. ■

2.5.5 Remark. We recall [71; Remark 2]. Let X and Y be (real- or complex) Banach lattices. First, for $x, y \in X$, $y \ge 0$ we need to introduce the *truncation of x by y*, denoted by $\tau(y)x$, defined as the element uniquely determined by the properties

(i) $|\tau(y)x| = |x| \wedge y$,

(ii) $(\operatorname{Re}\gamma\,\tau(y)x)_+ \le (\operatorname{Re}\gamma x)_+$ for all $\gamma \in \mathbb{K}$, $|\gamma| = 1$.

If X is countably order complete and $x \in X$ then the signum operator $\operatorname{sgn} x \in \mathcal{L}(X)$ exists and the truncation can be written as $\tau(y)x = (\operatorname{sgn} x)(|x| \wedge y)$. Also if $X = C(K)$ with K compact, then the formula holds if $\operatorname{sgn} x$ denotes the (possibly discontinuous) pointwise signum of $K \ni t \mapsto x(t)$.

Now let Z be a dense subspace of X enjoying the property that $x, z \in Z$, $x \ge 0$ implies $\tau_x z \in Z$. If $A \in \mathcal{L}(X,Y)$ is a regular operator possessing a modulus satisfying $|A|x = \sup\{|Ay|;\, y \in X,\, |y| \le x\}$ $(x \in X_+)$ then the modulus is already given by

$$|A|x = \sup\{|Az|;\, z \in Z,\, |z| \le x\} \quad (x \in Z_+).$$

2.6 Boundary Perturbations of Evolution Semigroups

Evolution semigroups arising from backward propagators are a natural generalisation of translation semigroups. In this section we consider the corresponding boundary perturbations of evolution semigroups. We refer to [40], [15], [60] and [39; Section VI.9] for propagators, particularly in the context of delay equations.

First we recall the definition of a backward propagator. Let $J \subseteq \mathbb{R}$ be an interval, $J^\Delta := \{(s,t) \in J \times J;\, s \le t\}$. Let X be a Banach space. An operator family $(U(s,t))_{(s,t)\in J^\Delta} \subseteq \mathcal{L}(X)$ is called a *backward propagator* if $U\colon J^\Delta \to \mathcal{L}(X)$ is strongly continuous, $U(s,s) = I$ and $U(r,s)U(s,t) = U(r,t)$ $(r,s,t \in J,\, r \le s \le t)$.

We will restrict ourselves to the case $h < \infty$. Let X be a Banach space and let $(U_0(s,t))_{-h\le s\le t\le 0} \subseteq \mathcal{L}(X)$ be a bounded backward propagator on X. Let $M := \sup_{-h\le\vartheta_1\le\vartheta_2\le 0} \|U_0(\vartheta_1,\vartheta_2)\|$. As in [40] we extend U_0 to a backward propagator on the interval $[-h,\infty)$ by

$$U(s,t) := \begin{cases} U_0(s,t) & \text{if } s \le t \le 0, \\ U_0(s,0) & \text{if } s \le 0 < t, \\ I & \text{if } 0 < s \le t. \end{cases}$$

Let W^+ be the evolution semigroup on $L_p(-h,\infty;X)$ induced by U, i.e.

$$\big(W^+(t)f\big)(\vartheta) := U(\vartheta,\vartheta+t)f(\vartheta+t) \quad (t \in \mathbb{R}_+,\ \vartheta \in (-h,\infty),\ f \in L_p(-h,\infty;X)),$$

and denote by G^+ the generator of W^+. It is well-known that $D(G^+) \subseteq C_0([-h,\infty);X)$ and that G^+ is a local operator. We can therefore define the operator

$$Gf := (G^+f^+)|_{(-h,0)}, \quad D(G) := \{f \in X_p;\ \exists f^+ \in D(G^+) : f^+|_{(-h,0)} = f\}$$

(cf. [40; Definition 2.3]). As $D(G^+) \hookrightarrow C_0([-h,\infty);X)$ we have $D(G) \hookrightarrow C([-h,0];X)$. For $L \in \mathcal{L}(X_{\mathrm{reg}},X)$ we define the restriction

$$G_Lf := Gf, \quad D(G_L) := \{f \in D(G);\ f(0) = Lf\}.$$

The operator G_0 (which is G_L with $L=0$) can be identified as the part of G^+ in $\{f \in L_p(-h,\infty;X);\ f|_{\mathbb{R}_+} = 0\}$, which is a closed and W^+-invariant subspace of $L_p(-h,\infty;X)$. Therefore G_0 is the generator of a C_0-semigroup on X_p, denoted by W_0 and given by

$$(W_0(t)f)(\vartheta) = \begin{cases} U(\vartheta,\vartheta+t)f(\vartheta+t) & \text{if } \vartheta \le -t, \\ 0 & \text{if } -t < \vartheta. \end{cases}$$

As for translation semigroups we write G_L as a perturbation of G_0 and use the generalised Desch-Schappacher perturbation theorem to show that G_L is the generator of a C_0-semigroup on X_p, provided that L is a $\frac{1}{M}$-delay operator.

In order to represent G_L as a multiplicative perturbation of G_0 we introduce the function

$$\psi_\lambda(\vartheta;x) := \begin{cases} e^{-\vartheta}x & \text{if } 0 \le \vartheta, \\ e^{\lambda\vartheta}U(\vartheta,0)x & \text{if } \vartheta < 0, \end{cases}$$

for $\vartheta \in (-h,\infty)$, $x \in X$ and $\lambda \in \mathbb{R}$. Since

$$\big((W^+(t)-I)\psi_\lambda(\,\cdot\,;x)\big)(\vartheta) = \begin{cases} (e^{-\vartheta-t} - e^{-\vartheta})x & \text{if } 0 \le \vartheta, \\ (e^{-\vartheta-t} - e^{\lambda\vartheta})U(\vartheta,0)x & \text{if } -t \le \vartheta < 0, \\ (e^{\lambda(\vartheta+t)} - e^{\lambda\vartheta})U(\vartheta,0)x & \text{if } \vartheta < -t, \end{cases}$$

for $t \in (0,h)$ and $\vartheta \in (-h,\infty)$ we see that

$$\left(\frac{1}{t}(W^+(t)-I)\psi_\lambda(\,\cdot\,;x)\right)(\vartheta) \to \begin{cases} -e^{-\vartheta}x & \text{if } 0 \le \vartheta, \\ \lambda e^{\lambda\vartheta}U(\vartheta,0)x & \text{if } \vartheta < 0, \end{cases} \tag{2.6.1}$$

as $t \to 0$ for $\vartheta \in (-h,\infty)$ pointwise almost everywhere. As this convergence is easily seen to be dominated in $L_p(-h,\infty;X)$ we conclude that $\psi_\lambda(\,\cdot\,;x) \in D(G^+)$, $\psi_\lambda(\,\cdot\,;x)|_{(-h,0)} \in D(G)$ and $G\psi_\lambda(\,\cdot\,;x) = \lambda\psi_\lambda(\,\cdot\,;x)|_{(-h,0)}$.

Let $Bf := \psi_0(\,\cdot\,;-Lf)|_{(-h,0)}$ $(f \in X_{\mathrm{reg}})$. From $G\psi_0(\,\cdot\,;x) = 0$ we derive that $G_L = G_0(I+B)$. (We remark that for evolution semigroups the constant function $\mathbf{1}_{(-h,0)}$, that we used to define the perturbation operator for the translation semigroups, does not yield the proper perturbation generally.)

In order to define the Volterra operator we further need a suitable generalisation of the notion of a translation. Let $\tau > 0$ and $Y \in \{X_p, X_{\mathrm{reg}}\}$. We say that $F \in \ell_\infty([0,\tau];\mathcal{L}(Y))$ is a *U-evolution* if for all $f \in Y$ there exists $g \in L_p(-h,\infty;X)$ (for $Y = X_p$) and $g \in \mathrm{Reg}([-h,\infty);X) \cap L_p(-h,\infty;X)$ (for $Y = X_{\mathrm{reg}}$) such that

$$F(t)f = (W^+(t)g)|_{(-h,0)} \quad (t \in [0,\tau]).$$

From now on we assume that L is a $\frac{1}{M}$-delay operator. Let $\tau > 0$ such that $m_L(\tau) < \frac{1}{M}$ (see (D2)) and such that (D3) holds for this τ. Again let $\mathcal{K}$ be the space of operators in $\mathcal{L}(X_p) \cap \mathcal{L}(X_{\mathrm{reg}})$ (cf. Section 2.2). By $\mathcal{Z}$ we denote the (closed) subspace of all U-evolutions in $\ell_\infty([0,\tau];\mathcal{K})$. Observe that U-evolutions in $\ell_\infty([0,\tau];\mathcal{L}(X_p))$ are automatically strongly continuous as translations and the propagator U are strongly continuous. Further notice that $W_0|_{[0,\tau]} \in \mathcal{Z}$.

For $F \in \mathcal{Z}$ and $f \in X_{\mathrm{reg}}$ we define the function $v(t) := \int_0^t W_0(t-r)BF(r)f\,dr$ $(t \in [0,\tau])$. For $t \in [0,\tau]$ we obtain

$$\begin{aligned} v(t) &= \int_0^t W_0(t-r)\left(\psi(\,\cdot\,;-LF(r)f)|_{(-h,0)}\right)\,dr \\ &= (-h,0) \ni \vartheta \mapsto -\int_{\max\{0,t+\vartheta\}}^{t} U(\vartheta,0)LF(r)f\,dr. \end{aligned}$$

In order to show that $v(t) \in D(G_0)$ we compute for $s \in (0,h)$ and $\vartheta \in (-h,0)$

$$((W_0(s)-I)v(t))(\vartheta) = \begin{cases} 0 & \text{if } \vartheta < -t-s, \\ \int_0^{t+\vartheta+s} U(\vartheta,0)LF(r)f\,dr & \text{if } -t-s \le \vartheta < -t, \\ \int_{t+\vartheta}^{t+\vartheta+s} U(\vartheta,0)LF(r)f\,dr & \text{if } -t \le \vartheta. \end{cases}$$

From the domination of $\frac{1}{s}(W_0(s)-I)v(t)$ by the function $M\|L\|\,\|F\|\,\|f\|_\infty \cdot \mathbf{1}_{(-h,0)}$, from the convergence of

$$\left(\frac{1}{s}(W_0(s)-I)v(t)\right)(\vartheta) \to w(\vartheta) := \begin{cases} 0 & \text{if } \vartheta < -t, \\ U(\vartheta,0)LF(t+\vartheta)f & \text{if } -t \le \vartheta \end{cases} \qquad (s \to 0)$$

for $\vartheta \in (-h,0)$ and from $v(t)(0) = 0$ we infer that $v(t) \in D(G_0)$, $G_0 v(t) = w$ and $\|G_0 v(t)\| \le M\|L\|\,\|F\|\,\|f\|_\infty$.

Let $\mathcal{M}f := ((-h,0) \ni \vartheta \mapsto U(\vartheta,0)f(\vartheta))$ $(f \in X_p)$. The boundedness of U implies that $\mathcal{M} \in \mathcal{K}$. Let $\tilde g\colon [-h,\infty) \to X$ be defined by $\tilde g(t) := 0$ $(t \in [-h,0])$ and $\tilde g(t) := LF(t)f$ $(t \in (0,\infty))$. From Lemma 2.3.1 we see that $[0,\tau] \ni t \mapsto LF(t)f = L\tilde g_t$ is again a regulated function. Since U is strongly continuous and bounded we conclude that $G_0 v(t) = \mathcal{M}\tilde g_t \in X_{\mathrm{reg}}$. Thus for $F \in \mathcal{Z}$ and $f \in X_{\mathrm{reg}}$ we can define the Volterra operator $\tilde V \in \mathcal{L}(\mathcal{Z}, \ell_\infty([0,\tau]; X_{\mathrm{reg}}))$ by

$$(\tilde V F)(t)f := G_0 \int_0^t W_0(t-r)BF(r)f\,dr \quad (t \in [0,\tau],\ F \in \mathcal{Z},\ f \in X_{\mathrm{reg}}). \tag{2.6.2}$$

In order to see that $\tilde V F$ is a U-evolution for all $F \in \mathcal{Z}$ let $f \in X_{\mathrm{reg}}$ and let $\tilde g$ be defined as above. A straightforward computation shows that $\tilde V F(t)f = \mathcal{M}\tilde g_t = (W^+(t)\tilde g)|_{(-h,0)}$. Hence $\tilde V F$ is a U-evolution.

As $\|\mathcal{M}\|_{\mathcal{K}} \le M$ we see from (D3) that $\tilde V F$ is continuous in the norm of $\mathcal{Z}$ and thus has an extension in $\mathcal{Z}$ denoted by VF. The extended Volterra operator V belongs to $\mathcal{L}(\mathcal{Z})$.

As $VF(0) = 0$ for all $F \in \mathcal{Z}$ we further see that V maps into the closed subspace $\mathcal{Z}_0 := \{F \in \mathcal{Z};\ F(0) = 0\}$. From the assumption $m_L(\tau) < \frac{1}{M}$ and the second assumption in (D3) in conjunction with $\|\mathcal{M}\|_{\mathcal{K}} \le M$ we infer that $V_0 := V|_{\mathcal{Z}_0}$ is strictly contractive in $\mathcal{L}(\mathcal{Z}_0)$.

As $V^n = V_0^{n-1}V$ we infer that the Neumann series $\sum_{n=0}^\infty V^n$ converges absolutely in $\mathcal{L}(\mathcal{Z})$. As for translation semigroups it remains to show that $\lambda \in \rho(G_L)$ for $\lambda \in \mathbb{R}$ sufficiently large.

2.6.1 Lemma. *(a) For $\lambda \in \mathbb{R}$ we define $L_\lambda x := L(\psi_\lambda(\,\cdot\,; x))$ $(x \in X)$. Then $L_\lambda \in \mathcal{L}(X)$, $L_\lambda \to 0$ as $\lambda \to \infty$ and $1 \in \rho(L_\lambda)$ for λ sufficiently large.*

(b) If $\lambda \in \mathbb{R}$ is sufficiently large then K_λ, defined by

$$K_\lambda g := \psi_\lambda\,(\,\cdot\,; R(1, L_\lambda)LR(\lambda, A_0)g)\,|_{(-h,0)} \quad (g \in X_p),$$

belongs to $\mathcal{L}(X_p)$.

(c) If $\lambda \in \mathbb{R}$ is sufficiently large then $\lambda \in \rho(A_L)$ and

$$R(\lambda, A_L) = R(\lambda, A_0) + K_\lambda.$$

Proof. Assertion (a) and (b) follow analogously to Lemma 2.4.1(a) and (b).

In order to prove (c) we first show that $\lambda - G_L$ is surjective for λ sufficiently large (so that $1 \in \rho(L_\lambda)$ by (a)). Let $g \in X_p$ and $f := (R(\lambda, G_0) + K_\lambda)g$. As $\psi_\lambda(\,\cdot\,; x) \in D(G^+)$ and therefore $K_\lambda x \in D(G)$ $(x \in X)$ we see that $f \in D(G)$. As G_0 is a restriction of G we can write $GR(\lambda, G_0)g = G_0 R(\lambda, G_0)g = \lambda R(\lambda, G_0)g - g$. From (2.6.1) we obtain $GK_\lambda g = \lambda K_\lambda g$. Therefore we have $Gf = -g + \lambda f$. Since $R(\lambda, G_0)g(0) = 0$ we see that $f(0) =$

$(K_\lambda g)(0) = R(1, L_\lambda)LR(\lambda, G_0)g$. Hence $L_\lambda(f(0)) = L(\psi_\lambda(\,\cdot\,; R(1, L_\lambda)LR(\lambda, G_0)g)) = LK_\lambda g$. Moreover we have $(I - L_\lambda)(f(0)) = LR(\lambda, G_0)g$. Thus

$$\begin{aligned} f(0) &= L_\lambda(f(0)) + (I - L_\lambda)(f(0)) = L_\lambda(f(0)) + LR(\lambda, G_0)g \\ &= L(K_\lambda + R(\lambda, G_0))g = Lf. \end{aligned}$$

We infer that $f \in D(G_L)$ and $(\lambda - G_L)f = g$. To finish the proof we have to show that $\lambda - G_L$ is injective. To this end let $f \in D(G_L)$ be a solution of $(\lambda - G_L)f = 0$. By definition there is a function $f^+ \in D(G^+)$ such that $f^+|_{(-h,0)} = f$. Let $g^+ := (\lambda - G^+)f^+$. The locality of $\lambda - G^+$ implies that $g^+|_{(-h,0)} = (\lambda - G_L)f = 0$. As $\lambda \in \rho(G^+)$ for λ sufficiently large (to be precise for $\lambda > 0$) we can compute

$$\begin{aligned} f(\vartheta) = R(\lambda, G^+)g^+(\vartheta) &= \int_0^\infty e^{-\lambda s}\left(W^+(s)g^+\right)(\vartheta)\,ds \\ &= \int_{-\vartheta}^\infty e^{-\lambda s}U(\vartheta, \vartheta + s)g^+(\vartheta + s)\,ds = e^{\lambda\vartheta}\int_0^\infty e^{-\lambda s}U(\vartheta, s)g^+(s)\,ds \\ &= e^{\lambda\vartheta}\int_0^\infty e^{-\lambda s}U(\vartheta, 0)g^+(s)\,ds = \psi_\lambda(\vartheta; x) \quad (\vartheta \in (-h, 0)), \end{aligned}$$

where $x := \int_0^\infty e^{-\lambda s}g^+(s)\,ds$. The boundary condition $f(0) = Lf$ yields $x = L_\lambda x$. Taking into account that $1 \in \rho(L_\lambda)$ we infer that this equation has the unique solution $x = 0$. Therefore $f = 0$ and the injectivity of $\lambda - G_L$ follows. Thus $\lambda - G_L$ is bijective for λ sufficiently large. This proves assertion (c). ■

By an application of Theorem 2.2.2 we have proved the following corollary.

2.6.2 Corollary. *Let U_0 be a backward propagator on $[-h, 0]$ and assume that $M := \sup_{-h\le s\le t\le 0} \|U(s,t)\| < \infty$. Let L be a $\frac{1}{M}$-delay operator. Then the operator G_L associated with U_0 and L is the generator of a C_0-semigroup on X_p.*

2.7 Flows in Networks

Dynamical networks have attracted the attention of semigroup theorists lately. In [47], [49] and [62] the flow on a network is described by a C_0-semigroup on $L_1(0,1)^n$ (with $n \in \mathbb{N}$ being the number of edges in the network). In this section we extend the results on well-posedness of these C_0-semigroups by allowing that the matter flowing in these networks might take values in an arbitrary Banach space rather than $\mathbb{R}$ or $\mathbb{C}$. By using evolution semigroups we cover networks where evolution of the matter along the edges takes place. We also outline how bounded linear transformations inside vertices are coded into the delay operator of the translation or evolution semigroup used to model the flow.

Let $G = (V, E, \iota, \ell)$ be a directed graph with weighted edges, where V and E are two finite and disjoint sets, $\iota\colon E \to V \times V$ and $\ell\colon E \to \mathbb{R}_+$. The sets V and E are the vertices and the edges in G, respectively. The function ι is the incidence relation giving the start and end vertex of an edge. To be precise $\iota(e) = (v_1, v_2)$ for $e \in E$ and $v_1, v_2 \in V$ means that the edge e starts in v_1 and goes to v_2. The function ℓ gives the weights for each edge. In our case it describes the length of each edge.

We want to model the flow of matter such as fluids or populations, which can be represented as elements of a Banach space X and which "behaves linearly". The matter moves along the edges of G, leaves them at the end points to the corresponding vertices and is redistributed at the vertices to the outgoing edges. We assume that the velocity of the flow on the edges is constant throughout the system, which is not really a restriction as we can adjust the length of each edge separately. (In the mentioned papers the edges all have the same length whereas the velocity might vary.)

To this end we will use translation semigroups or more generally evolution semigroups. Let $p \in [1, \infty)$. The distribution of matter along an edge $e \in E$ is represented by a function $f \in L_p(-\ell(e), 0; X)$, where $f(0)$ is the matter at the starting point of e and $f(-\ell(e))$ is the matter at the endpoint. In order to arrive at our translation semigroups we define $h := \sup_{e\in E} \ell(e)$ and unify the length of the edges so that we can describe our system of edges as an element of the space $L_p(-h, 0; X)^E$. We will use the Dirac functionals $\delta_{-\ell(e)}$ to recover the value of the edge $e \in E$ at the proper endpoint.

Observing that $L_p(-h, 0; X)^E$ can be identified by $L_p(-h, 0; X^E)$ we are now well prepared to introduce different delay operators which will result in a C_0-semigroup modelling the flow in the network G.

First we assume that the matter $x \in X$ leaving the edge $e \in E$ at the vertex $v = P_2\,\iota(e)$ is processed inside the vertex by a transformation $w_{ke} \in \mathcal{L}(X)$ before entering the outgoing edge $k \in \mathrm{Out}(v) := \{k \in E;\ P_1\,\iota(k) = v\}$. So we have a vertex transformation matrix $(w_{ke})_{(k,e)\in E\times E}$ with values in $\mathcal{L}(X)$ and where $w_{ke} = 0$ for all $e \in E$ and $k \notin \mathrm{Out}(P_2\,\iota(e))$.

For example if for each vertex the incoming matter is distributed among the outgoing edges by a fixed ratio so that no matter appears out of nowhere or disappears, we can describe it by a matrix $(w_{ke}) \in [0,1]^{E\times E}$ such that $w_{ke} = 0$ for all $e \in E$ and $k \notin \mathrm{Out}(P_2\,\iota(e))$, and satisfying $\sum_{k\in E} w_{ke} = 1$. Sinks and sources in vertices are realised by weakening the assumption $\sum_{k\in E} w_{ke} = 1$ to $\sum_{k\in E} w_{ke} \le 1$ (for a sink) or $\sum_{k\in E} w_{ke} \ge 1$ (for a source). However only the (dis)appearance of a multiple of the mass at an endpoint of an edge is realised. More sophisticated transformations can be modelled by choosing appropriate operators $w_{jk} \in \mathcal{L}(X)$.

We define the delay operator $L\colon \mathrm{Reg}([-h, 0]; X)^E \to X^E$ as the matrix operator

$$L = \Big(w_{jk}\,\delta_{-\ell(k)}\Big)_{(j,k)\in E\times E}.$$

Since we can write L as $\sum_{j,k\in E} w_{jk} P_k \delta_{-\ell(k)}$ as an operator from $\mathrm{Reg}([-h, 0]; X^E)$ to X^E we immediately see that L satisfies the requirements of Proposition 2.3.3 for any $r \in [1, p]$ and the Borel measure $\mu = \sum_{j,k\in E} \big\|w_{jk}\big\| \delta_{-\ell(k)}$. Thus A_L is the generator of a translation semigroup on $L_p(-h, 0; X^E) = L_p(-h, 0; X)^E$.

Let $g \in D(A_L)$, $t \in \mathbb{R}_+$ and $f := T_L(t)g$. The boundary condition $f(0) = Lf$ in $D(A_L)$ directly leads to the coupling

$$f_j(0) = \sum_{k \in E} w_{jk} f_k(-\ell(k)) \quad (j, k \in E).$$

This is exactly the behaviour which we expect from our network. So T_L indeed describes the desired flow on G.

If (nonautonomous) evolution of the matter takes place during the transportation along the edges we can use the corresponding evolution semigroup generated by G_L (recall that delay operators majorized by a Borel measure as above are 0-delay operators).

We point out that if we need different Banach spaces X_e for each edge e then we can still model the network in our framework by using a translation or evolution semigroup on $L_p\left(-h, 0; \prod_{e \in E} X_e\right)$ and transformations $w_{jk} \in \mathcal{L}(X_k, X_j)$.

2.7.1 Remarks. (a) As values beyond the endpoint $-\ell(e)$ of an edge $e \in E$ do not matter in L and the translation or evolution semigroup T_L we can actually define a C_0-semigroup S on the more natural space $\mathcal{Y} := \prod_{e \in E} L_p(-\ell(e), 0; X)$ by

$$S(t)f := \left((T_L(t)f)_e \Big|_{\left(-\ell(e), 0\right)} \right)_{e \in E} \qquad (f \in \mathcal{Y}),$$

where functions outside their domain are taken to be zero.

2.8 The Modulus of Delay Semigroups in the Space of Continuous Functions

In the last section of this chapter we look at the modulus semigroup of translation semigroups on the space $C([-1, 0]; X)$ (see Chapter 1 for the definition and an overview on modulus semigroups). In the literature these semigroups are called delay semigroup which is the term we will also use. To make it precise we say that $\mathcal{T}$ is a *delay semigroup on* $C([-1, 0]; X)$ if there exists a $\tau > 0$ such that for each $f \in C([-1, 0]; X)$ there is a function $g \in C([-1, \tau]; X)$ so that $\mathcal{T}(t)f = g_t$ (see also Defintition 2.1.1).

We will consider the delay semigroup associated with the delay equation $\dot{u}(t) = Au(t) + Lu_t$ $(t \geq 0)$ on a Dedekind-complete (real or complex) Banach lattice X, where A is the generator of a C_0-semigroup on X and $L \in \mathcal{L}(C([-1, 0]; X), X)$. We assume that the delay operator L has no mass at 0. By this we mean that for each function $f \in C([-1, 0]; X)$ there is a sequence $(\varphi_k) \subseteq C([-1, 0])$ with $0 \leq \varphi_k \leq 1$, $\operatorname{spt} \varphi_k = [-1/k, 0]$ and $\varphi_k(0) = 1$ $(k \in \mathbb{N})$ such that $L(\varphi_k \cdot f) \to 0$ as $k \to \infty$.

The delay equation is solved by the delay semigroup generated by

$$\begin{aligned} \mathcal{B}_{A,L} f &:= f', \\ D(\mathcal{B}_{A,L}) &:= \{ f \in C^1([-1, 0]; X);\ f(0) \in D(A),\ f'(0) = Af(0) + Lf \}, \end{aligned}$$

on $C([-1,0];X)$. In [39; Section VI.6] the generator property of $\mathcal{B}_{A,L}$ is shown. The C_0-semigroup generated by $\mathcal{B}_{A,L}$ is denoted by $\mathcal{T}_{A,L}$. The solution of the delay equation is given by $u(t) = (\mathcal{T}_{A,L}(t)f)(0)$ for the initial value $u_0 = f \in D(\mathcal{B}_{A,L})$.

Let A, $\tilde{A}$ be generators of C_0-semigroup on X, and let L, $\tilde{L} \in \mathcal{L}(C([-1,0];X),X)$. By T and $\tilde{T}$ we denote the C_0-semigroups generated by A and $\tilde{A}$, respectively. In [45] it has been shown that if $\tilde{T}$ dominates T and $\tilde{L}$ dominates L then $\mathcal{T}_{\tilde{A},\tilde{L}}$ dominates $\mathcal{T}_{A,L}$. In general it is open whether $\mathcal{T}_{\tilde{A},\tilde{L}}$ is the modulus semigroup of $\mathcal{T}_{A,L}$, and even whether $\mathcal{T}_{A,L}$ has a modulus semigroup at all. The main obstacle in the search for the modulus semigroup arises from the fact that the Banach lattice $C([-1,0];X)$ does not have order-continuous norm, nor is it Dedekind-complete. In particular we cannot apply [11; Theorem 2.1] to deduce that a modulus semigroup exists.

For $X = \mathbb{R}^n$ it was shown in [11] that the modulus semigroup of $\mathcal{T}_{A,L}$ is given by $\mathcal{T}_{A^\sharp,|L|}$.

In recent years the delay equation has been considered in the L_p-context (cf. [19], [48], [21], [69]). In [14], [71] and [63] the modulus semigroup of a delay semigroup with L_p-history space was determined. In [63] this result was applied to deal with delay semigroups on history spaces of continuous functions and delay operators L given as the Riemann-Stieltjes integral $Lf = \int d\eta\, f$, where $\eta \in BV([-1,0];\mathcal{L}(X))$ is of bounded regular variation and thus L possesses a modulus (cf. [71; Section 3]).

In this section we treat the problem with the additional assumptions $A = 0$ and L has no mass at 0 in the sense that for all $x \in X$ we have

$$\sup\{\|L(x\cdot\varphi)\|;\ \varphi \in C([-1,0]),\ \operatorname{spt}\varphi \subseteq [-t,0]\} \to 0 \quad (t \to 0). \tag{2.8.1}$$

We therefore write $\mathcal{B}_L$ and $\mathcal{T}_L$ instead of $\mathcal{B}_{A,L}$ and $\mathcal{T}_{A,L}$. Our main result is the following theorem.

2.8.1 Theorem. *The C_0-semigroup $\mathcal{T}_L$ possesses the modulus semigroup $\mathcal{T}_L^\sharp = \mathcal{T}_{|L|}$.*

The key observation for the proof of this theorem is that a dominating semigroup $\mathcal{S}$ of $\mathcal{T}_L$ provides a solution $u\colon [-1,\infty) \to X$, continuously differentiable on $[0,\infty)$, of the inequality $\dot{u}(t) \geq |L|u_t$ $(t \geq 0)$ and $u_0 = f$ in the domain of the generator of S. The proof requires some preparation.

2.8.2 Lemma. *Let $g \in C([-1,0];X)$. Then $[0,\infty) \ni t \mapsto \mathcal{T}_L(t)g(0)$ is continuously differentiable with derivative $t \mapsto L\mathcal{T}_L(t)g$.*

Proof. Let $\varphi\colon [0,\infty) \to X$, $\varphi(t) := \mathcal{T}_L(t)g(0)$ $(t \geq 0)$. Further let $(g_n) \subseteq D(\mathcal{B}_L)$, $g_n \to g$ in $C([-1,0];X)$ and $\varphi_n\colon [0,\infty] \to X$, $\varphi_n(t) := \mathcal{T}_L(t)g_n(0)$ $(t \geq 0)$. Then we have $\varphi_n'(t) = L\mathcal{T}_L(t)g_n$ $(t \geq 0)$. Since $\mathcal{T}_L$ is strongly continuous and L is bounded we see that $\varphi_n \to \varphi$ and $\varphi_n' \to L\mathcal{T}_L(\cdot)g$, both uniformly on compact intervals. This shows the differentiability of φ with the continuous derivative $\varphi'(t) = L\mathcal{T}_L(t)g$. ∎

2.8.3 Lemma. *Let $\mathcal{T}$ be a delay semigroup on $C([-1,0];X)$ and $\mathcal{S}$ a positive C_0-semigroup dominating $\mathcal{T}$. Furthermore let $f \in C([-1,0];X)$ and $f \geq 0$. Then the following statements hold.*

(a) $\mathcal{S}(t)f(\tau) \geq \mathcal{S}(t+\tau)f(0)$ *for all* $t \geq 0$ *and* $\tau \in [\max\{-t,-1\},0]$.
(b) Let $g\colon [-1,\infty) \to X$ *be defined by*

$$g(t) := \begin{cases} \mathcal{S}(t)f(0) & \text{for } t > 0, \\ f(t) & \text{for } t \leq 0. \end{cases}$$

Then $\mathcal{S}(t)f \geq g_t$ *for all* $t \geq 0$.

Proof. From $\mathcal{S}(t)f = \mathcal{S}(-\tau)\mathcal{S}(t+\tau)f \geq \mathcal{T}(-\tau)\mathcal{S}(t+\tau)f$ we conclude

$$\mathcal{S}(t)f(\tau) \geq \mathcal{T}(-\tau)(\mathcal{S}(t+\tau)f)(\tau) = \mathcal{S}(t+\tau)f(0),$$

which shows (a). In order to prove (b) let $t \geq 0$. For $\tau \in [-1,-t]$ we have

$$\mathcal{S}(t)f(\tau) \geq \mathcal{T}(t)f(\tau) = f(t+\tau) = g_t(\tau).$$

For $\tau \in [\max\{-t,-1\},0]$ we apply (a) to obtain $\mathcal{S}(t)f(\tau) \geq \mathcal{S}(t+\tau)f(0) = g_t(\tau)$. ∎

2.8.4 Lemma. *Let* $\mathcal{S}$ *be a positive* C_0*-semigroup dominating* $\mathcal{T}_L$*. Let* C *be the generator of* $\mathcal{S}$ *and* $f \in D(C)_+$*. We define* $g\colon [-1,\infty) \to X$,

$$g(t) := \begin{cases} \mathcal{S}(t)f(0) & \text{for } t > 0, \\ f(t) & \text{for } t \leq 0. \end{cases}$$

Then g *is continuously differentiable on* $[0,\infty)$ *and* $g'(t) \geq |L|g_t$.

Proof. The differentiability of g on $[0,\infty)$ and the continuity of g' follow from $f \in D(C)$. Let $t \geq 0$ and $\varphi := \mathcal{S}(t)f$. Since $\mathcal{S}$ is a positive C_0-semigroup we have $\varphi \in D(C)_+$ and thus

$$\frac{g(t+\tau)-g(t)}{\tau} = \frac{\mathcal{S}(\tau)\varphi(0)-\varphi(0)}{\tau} \geq \operatorname{Re}\frac{\mathcal{T}_L(\tau)\psi(0)-\psi(0)}{\tau} \quad (\tau > 0),$$

for all $\psi \in C([-1,0];X)$ with $|\psi| \leq \varphi$ and $\psi(0) = \varphi(0)$. Lemma 2.8.2 shows that the right hand term has the limit $\operatorname{Re} L\psi$, so we see that $g'(t) \geq \operatorname{Re} L\psi$. Taking the supremum on the right hand side we obtain

$$g'(t) \geq \sup\{\operatorname{Re} L\psi;\ \psi \in C([-1,0];X),\ |\psi| \leq \varphi,\ \psi(0) = \varphi(0)\} = |L|\varphi. \tag{2.8.2}$$

(For the equality in (2.8.2) we recall that we suppose that L has no mass at zero; see (2.8.1).) Thus L maps $\{\psi \in C([-1,0];X);\ |\psi| \leq \varphi,\ \psi(0) = \varphi(0)\}$ to a dense subset of $L\big(\{\psi \in C([-1,0];X);\ |\psi| \leq \varphi\}\big)$.) By Lemma 2.8.3(b) we have $\varphi \geq g_t$, and by the positivity of $|L|$ we conclude $g'(t) \geq |L|g_t$. ∎

The inequality obtained in the previous lemma makes it necessary to look at functional-differential inequalities of the form $\dot{u}(t) \geq Lu_t$, with initial value $u_0 \in C([-1,0];X)$. In particular we are interested in the relation between solutions of this inequality and the (unique) solution of the corresponding equality for $L \in \mathcal{L}(C([-1,0];X),X)$ being positive. We say that $u \in C([-1,\infty);X)$ is a *classical solution* of the inequality above if u is continuously differentiable on $[0,\infty)$ and u satisfies the inequality.

2.8.5 Lemma. *Let u be a classical solution of the functional-differential inequality*

$$\dot{u}(t) \geq |L| u_t, \quad u_0 = f \quad (t \geq 0),$$

with initial value $f \in C([-1,0];X)$. Then we have $u(t) \geq \mathcal{T}_{|L|}(t)f(0)$ for $0 \leq t \leq \delta := \max\left\{1, \frac{1}{2}\| \, |L| \, \|^{-1}\right\}$.

Proof. It suffices to consider the case $f = 0$. (Otherwise we can subtract the equation $\frac{d}{dt}\left(\mathcal{T}_{|L|}(t)f(0)\right) = |L|(\mathcal{T}_{|L|}(t)f)$ from the inequality. Notice that by Lemma 2.8.2 the function $t \mapsto \mathcal{T}_{|L|}(t)f(0)$ is differentiable on $[0,\infty)$ with derivative $|L|(\mathcal{T}_{|L|}(t)f)$.)

For the initial value $f = 0$ we simply have to show that any solution of the inequality is positive. To this end we define the operator $\Psi \colon C([0,\delta];X) \to C([0,\delta];X)$ by

$$(\Psi f)(t) := \int_0^t |L|(f_s)\, ds \quad (f \in C([0,\delta];X),\, t \in [0,\delta]).$$

This mapping is strictly contractive because of

$$\|\Psi f(t)\| \leq \int_0^t \| \, |L| f_s\| \, ds \leq \delta \| \, |L| \, \| \|f\|_\infty \leq \frac{1}{2}\|f\|_\infty \quad (t \in [0,\delta]).$$

Let ψ_0 be a solution of the inequality on $[0,\delta]$ and let $\psi_n := \Psi^n(\psi_0)$ $(n \in \mathbb{N})$. From

$$\psi_1(t) = \int_0^t |L|(\psi_0)_s\, ds \leq \int_0^t \psi_0'(s)\, ds = \psi_0(t) \quad (t \in [0,\delta])$$

and the positivity of Ψ we conclude that $\psi_0 \geq \psi_1 \geq \psi_2 \geq \ldots$. As Ψ is strictly contractive we have $\psi_n \to 0$ $(n \to \infty)$ and so we see that $\psi_0 \geq 0$. ■

We are now prepared to prove the main result of this section.

Proof of Theorem 2.8.1. Let $\mathcal{S}$ be a positive C_0-semigroup with generator C, which dominates $\mathcal{T}_L$. Further let δ be as in Lemma 2.8.5 and $t \in [0,\delta]$. For $f \in D(C)_+$ we have $\mathcal{S}(t)f(0) \geq \mathcal{T}_{|L|}(t)f(0)$ (Lemmata 2.8.4 and 2.8.5). Using Lemma 2.8.3 we conclude

$$\mathcal{S}(t)f(\tau) \geq \mathcal{S}(t+\tau)f(0) \geq \mathcal{T}_{|L|}(t+\tau)f(0) = \mathcal{T}_{|L|}(t)f(\tau) \quad (-t \leq \tau \leq 0).$$

Finally for $-1 \leq \tau \leq -t$ we have

$$\mathcal{S}(t)f(\tau) \geq \mathcal{T}_L(t)f(\tau) = f(t+\tau) = \mathcal{T}_{|L|}(t)f(\tau).$$

This proves $\mathcal{S}(t)f \geq \mathcal{T}_{|L|}(t)f$ for all $f \in D(C)_+$ and $0 \leq t \leq \delta$. As $D(C)_+$ is dense in $C([-1,0];X)_+$ we see that $\mathcal{S}$ dominates $\mathcal{T}_{|L|}$. Thus we have shown that any dominating semigroup of $\mathcal{T}_L$ also dominates $\mathcal{T}_{|L|}$. Since $\mathcal{T}_{|L|}$ dominates $\mathcal{T}_L$ we have proven that $\mathcal{T}_L$ possesses a modulus semigroup and $\mathcal{T}_L^\sharp = \mathcal{T}_{|L|}$. ■

Chapter 3

Well-Posedness and Stability for an Integro-Differential Equation with Time Derivative in the Delay Term

In this chapter we treat the integro-differential equation

$$(\mathrm{IDE}^\bullet) \quad \dot u(t) = Au(t) + \int_0^t \ell(t-s)\dot u(s)ds + g(t), \quad u(0) = x \in X \quad (t \in \mathbb{R}_+)$$

on a Banach space X. The operator A is assumed to be the generator of a C_0-semigroup T on X. The function ℓ is a function on $\mathbb{R}_+ = [0, \infty)$ with values in $\mathcal{L}(X)$. We assume that ℓ has the following two properties.

(a) ℓ is strongly Bochner measurable, i.e. $\ell(\cdot)x$ is Bochner measurable for all $x \in X$.

(b) $\|\ell(\cdot)\|_{\mathcal{L}(X)}$ is dominated by a locally integrable function.

These two conditions guarantee that the integral in (IDE$^\bullet$) exists as a Bochner integral if $\dot u$ is a continuous function.

We recall that for a closed operator C the space $D(C)$ equipped with the graph norm coming from C is denoted by D_C. We define classical solutions and well-posedness of (IDE$^\bullet$) as follows.

3.0.6 Definition. (a) A function $u \in C(\mathbb{R}_+; D_A) \cap C^1(\mathbb{R}_+; X)$ is called a classical solution of (IDE$^\bullet$) for the initial value $x \in X$ and inhomogeneity $g \in C(\mathbb{R}_+; X)$, if $u(0) = x$ and (IDE$^\bullet$) holds for all $t \in \mathbb{R}_+$.

(b) A function $u \in C(\mathbb{R}_+; X)$ is called a mild solution of (IDE$^\bullet$) for the initial value $x \in X$ and inhomogeneity $g \in L_{1,loc}(\mathbb{R}_+; X)$ if for all $t \in \mathbb{R}_+$ we have $\int_0^t u(s)\, ds \in D_A$ and

$$u(t) = x + \int_0^t (g(s) - \ell(s)x)\, ds + A\int_0^t u(s)\, ds + \int_0^t \ell(t-s)u(s)\, ds.$$

(c) We say that (IDE$^\bullet$) is well-posed, if for all $x \in D_A$ and $g = 0$ there exists a unique classical solution $u(\cdot\,; x)$ and for any $(x_n)_{n\in\mathbb{N}} \subseteq D_A$, $\lim_{n\to\infty} x_n = 0$ in X we have $\lim_{n\to\infty} u(\cdot\,; x_n) = 0$ uniformly in compact intervals. In this case we say that $\mathcal{S}\colon \mathbb{R}_+ \to \mathcal{L}(X)$ defined as the continuous extension of $\mathcal{S}_0(t)x := u(t; x)$ ($x \in D_A$, $t \in \mathbb{R}_+$) is the *solution operator family* associated with (IDE$^\bullet$).

If ℓ is of bounded variation with respect to $\mathcal{L}(X)$ then integration by parts leads to the inhomogeneous integro-differential equation

$$\dot u(t) = (A + \ell(0))u(t) + \int_0^t d\ell(s)u(t-s)ds + \tilde g(t) - \ell(t)x, \quad u(0) = x \in X \quad (t \in \mathbb{R}_+).$$

This type of integro-differential equations has been dealt with in numerous publications (cf. [58] and the references therein). Our first concern is the presentation of well-posedness conditions for ($\mathrm{IDE}^\bullet$) for which integration by parts is not applicable. Such conditions are obtained by employing the forcing function approach (cf. [58; Section 13.6], [30]) as well as delay semigroups with history function spaces of p-integrable functions (cf. [19], [21], [48], [20], [22]). The author did not succeed in applying the Volterra equation technique developed in [58; Section 0] for evolutionary integral equations. In fact an investigation of the relation of ($\mathrm{IDE}^\bullet$) to such equations (see Section 3.3) reveals that ($\mathrm{IDE}^\bullet$) does not fit well into the notion of a solution operator family for evolutionary intregral equations presented in [58] (see also Remarks 3.4.4(b)).

Further well-posedness results based on Volterra and delay semigroups and involving fractional regularity conditions in time and space are presented in Section 4.7.3 and Corollary 4.8.7.

The investigation of this type of equation was motivated by models describing the phenomenon of flutter of aerofoils under aerodynamic load. We refer the reader to [6], [7], [5], [9], [8] and [37]. Engineers are interested in the characterisation of strong stability of ($\mathrm{IDE}^\bullet$). The second part is devoted to an analysis of this type of stability by means of a spectral analysis via Laplace transform methods (for this concept we refer particularly to the monograph [4] and the references therein, and to [26], [25] for recent developments).

The plan of this chapter is as follows.

Well-posedness conditions for ($\mathrm{IDE}^\bullet$) using Volterra and delay semigroups are presented in Sections 3.1 and 3.2.

Section 3.3 is devoted to the exploration of the relationship of solution operator families of ($\mathrm{IDE}^\bullet$) and resolvents of the corresponding evolutionary integral equation.

Finally in Section 3.4 we present conditions for strong stability of ($\mathrm{IDE}^\bullet$) by means of Laplace transform methods.

3.1 The Forcing Function Approach

In [30] and many other puplications (see [58; Section 13.6] for references) the forcing function approach was used to solve the integro-differential equations without time-derivative of the solution in the integral term. With some modifications this method also works for ($\mathrm{IDE}^\bullet$).

We first derive the Volterra semigroup corresponding to ($\mathrm{IDE}^\bullet$). To this end we start by introducing the spaces $BV_p(\mathbb{R}_+; X)$ (with $p \in [1, \infty)$) of p-integrable X-valued functions of bounded variation equipped with the norm

$$\|f\|_{p,Var} := \|f\|_p + \sup\left\{ \sum_{j=1}^{n} \|\tilde{f}(t_j) - \tilde{f}(t_{j-1})\|;\, n \in \mathbb{N},\, 0 = t_0 < \cdots < t_n \right\}$$

for $f \in BV_p(\mathbb{R}_+; X)$, where $\tilde{f}$ denotes the left continuous representative of f. We also need the following lemma on the boundedness of certain operators which frequently occur in the context of delay equations.

3.1.1 Lemma. *Let X be a Banach space and $p \in [1, \infty)$. Let $k \colon \mathbb{R}_+ \to \mathcal{L}(X)$ and $Kx := k(\cdot)x$ $(x \in X)$.*

(a) If $k(\cdot)x \in BV_p(\mathbb{R}_+; X)$ for all $x \in X$ then $K \in \mathcal{L}(X, BV_p(\mathbb{R}_+; X))$.

(b) If $k(\cdot)x \in L_p(\mathbb{R}_+; X)$ for all $x \in X$ then $K \in \mathcal{L}(X, L_p(\mathbb{R}_+; X))$.

Proof. By the closed graph theorem the proof in both cases is accomplished if we can show that K is a closed operator. To this end let $(x_n) \subseteq X$ be a null sequence such that $Kx_n \to f$ with $f \in BV_p(\mathbb{R}_+; X)$ and $f \in L_p(\mathbb{R}_+; X)$, respectively. In the first case convergence in the variation norm implies pointwise convergence. In the second case we can assume without loss of generality (by choose a subsequence if necessary) that the convergence is pointwise almost everywhere. So in both cases we see that $(Kx_n)(t) = k(t)x_n \to f(t)$ for $t \in \mathbb{R}_+$ (almost everywhere). Since $k(t)$ is a bounded operator we conclude $k(t)x_n \to 0 = f(t)$ (almost everywhere). Hence $f = 0$ (almost everywhere) and so K is closed operator. ∎

From now on we assume that $\ell(\cdot)x \in L_1(\mathbb{R}_+; X)$ $(x \in X)$. Lemma 3.1.1 implies that the operator $Lx := \ell(\cdot)x$ $(x \in X)$ belongs to $\mathcal{L}(X, L_1(\mathbb{R}_+; X))$.

By S we denote the left translation semigroup on $L_1(\mathbb{R}_+; X)$. Its generator, denoted by $\mathcal{D}$, is the weak derivative on $L_1(\mathbb{R}_+; X)$ with maximal domain $W_1^1(\mathbb{R}_+; X)$. Let u be a classical solution of (IDE$^\bullet$) with inhomogeneity $g \in L_1(\mathbb{R}_+; X)$, $F(t) := S(t)g + \int_0^t S(t-s)L\dot{u}(s)\,ds$ and $U(t) := \left(\begin{smallmatrix} u(t) \\ F(t) \end{smallmatrix}\right)$ $(t \in \mathbb{R}_+)$. The function F is called the *forcing function* associated with u. As $L\dot{u}$ is a continuous function on $\mathbb{R}_+$ with values in $L_1(\mathbb{R}_+; X)$ we see that F is the mild solution of the inhomogeneous abstract Cauchy problem associated with $\mathcal{D}$ with initial value $F(0) = g$. Hence F satisfies the equation

$$F(t) = \mathcal{D}\int_0^t F(s)\,ds + \int_0^t L\dot{u}(s)\,ds \quad (t \in \mathbb{R}_+). \tag{3.1.1}$$

As $\delta_0 F(t) = g(t) + \int_0^t \ell(t-s)\dot{u}(s)\,ds$ we further conclude that $\dot{u}(t) = Au(t) + \delta_0 F(t)$. Using this equation in (3.1.1) we obtain $F(t) = LA\int_0^t u(s)\,ds + (\mathcal{D} + L\delta_0)\int_0^t F(s)\,ds$ $(t \in \mathbb{R}_+)$. This shows that $\mathcal{U}$ is a mild solution of the abstract Cauchy problem

$$\text{(FFA)} \quad \begin{cases} \dot{\mathcal{U}}(t) &= \mathcal{A}\mathcal{U}(t), \quad \mathcal{U}(0) = \begin{pmatrix} x \\ g \end{pmatrix} \in X \times L_1(\mathbb{R}_+; X), \\ \mathcal{A} &:= \begin{pmatrix} A & \delta_0 \\ LA & \mathcal{D} + L\delta_0 \end{pmatrix}, \\ D(\mathcal{A}) &:= D_A \times W_1^1(\mathbb{R}_+; X) \end{cases}$$

on $X \times L_1(\mathbb{R}_+; X)$. We have seen that a classical solution of (IDE$^\bullet$) for the initial value x and inhomogeneity g yields a mild solution of (FFA) for the initial value $\left(\begin{smallmatrix} x \\ g \end{smallmatrix}\right)$. We will now show that a classical solution of (FFA) provides a classical solution of (IDE$^\bullet$).

3.1.2 Lemma. *Let $\mathcal{U}(t) = \left(\begin{smallmatrix} u(t) \\ F(t) \end{smallmatrix}\right)$ be a classical solution of (FFA) for the initial value $\mathcal{U}(0) = \left(\begin{smallmatrix} x \\ g \end{smallmatrix}\right) \in D(\mathcal{A})$. Then u is a classical solution of (IDE$^\bullet$).*

Proof. Obviously, u is continuously differentiable and $u(0) = x$. In order to see that u satisfies (IDE$^\bullet$) we infer from the second component of the equation $\dot{\mathcal{U}}(t) = \mathcal{A}\mathcal{U}(t)$ that

$$\dot{F}(t) = LAu(t) + (L\delta_0 + \mathcal{D})\, F(t) = L\dot{u}(t) + \mathcal{D}F(t).$$

Therefore F is a classical solution of the inhomogeneous abstract Cauchy problem associated with the left translation semigroup on $L_1(\mathbb{R}_+; X)$ with initial value g and inhomogeneity $L\dot{u}(\cdot) \in C(\mathbb{R}_+; L_1(\mathbb{R}_+; X))$. Therefore (cf. [39; Section VI.7]) we obtain

$$F(t) = S(t)g + \int_0^t S(t-s)L\dot{u}(s)\, ds \tag{3.1.2}$$

and $\delta_0 F(t) = g(t) + \int_0^t \ell(t-s)\dot{u}(s)\, ds$. Thus the first component of $\dot{U}(t) = \mathcal{A}U(t)$ becomes (IDE$^\bullet$). ■

3.1.3 Lemma. *If (FFA) is well-posed then (IDE$^\bullet$) is well-posed. In this case classical and mild solutions of (IDE$^\bullet$) are given by $t \mapsto P_1 e^{t\mathcal{A}} \binom{x}{g}$ for $\binom{x}{g} \in D(\mathcal{A})$ and $\binom{x}{g} \in X \times L_1(\mathbb{R}_+; X)$, respectively.*

Proof. By Lemma 3.1.2 (IDE$^\bullet$) has a classical solution for all $x \in D_A$. In order to show uniqueness let u be a classical solution of (IDE$^\bullet$) with initial value 0 and F be the forcing function corresponding to u. Then as we have seen above $U(t) := \begin{pmatrix} u(\cdot) \\ F(\cdot) \end{pmatrix}$ is a mild solution of (FFA). The well-posedness of (FFA) implies uniqueness of mild solutions of (FFA) and therefore $U = 0$. This shows $u = 0$.

Finally let $(x_n)_{n\in\mathbb{N}}$ be a sequence in D_A which converges to zero in X. As (FFA) is well-posed $\mathcal{T}(\cdot) \binom{x_n}{0}$ tends to zero uniformly on compact intervals of $\mathbb{R}_+$. Hence solutions of (IDE$^\bullet$) depend continuously on the initial value. This shows that (IDE$^\bullet$) ist well-posed. ■

We can now apply perturbation theory, namely the Desch-Schappacher perturbation theorem, to obtain a well-posedness criterion for (IDE$^\bullet$). To this end we recall that the Favard space $F^1_{\mathcal{D}}$ for the generator $\mathcal{D}$ of the left translation semigroup on $L_1(\mathbb{R}_+; X)$ is the space $BV_1(\mathbb{R}_+; X)$; cf. (A.1) for the general definition of Favard spaces and [55; Proposition 3.6], [16; Proposition A.5] for the Favard space of the generator $\mathcal{D}$.

3.1.4 Theorem. *If $Lx \in BV_1(\mathbb{R}_+; X)$ for all $x \in X$ then (FFA) and hence (IDE$^\bullet$) are well-posed.*

Proof. It is a well-known fact in the theory of Volterra equations (cf. [39; Section VI.7]) that the operator

$$\mathcal{A}_0 = \begin{pmatrix} A & \delta_0 \\ 0 & \mathcal{D} \end{pmatrix}, \quad D(\mathcal{A}_0) = D(A) \times W^1_1(\mathbb{R}_+; X)$$

on $X \times L_1(\mathbb{R}_+; X)$ generates the C_0-semigroup $\mathcal{T}_0$,

$$\mathcal{T}_0(t)\begin{pmatrix} x \\ g \end{pmatrix} := \begin{pmatrix} T(t)x + \int_0^t T(t-s)g(s)\,ds \\ S(t)g \end{pmatrix} \quad (t \in \mathbb{R}_+).$$

This semigroup solves the inhomogeneous Cauchy problem associated with A. The operator $\mathcal{A}$ is the additive perturbation of $\mathcal{A}_0$ with $\mathcal{B} := \left(\begin{smallmatrix} 0 & 0 \\ LA & L\delta_0 \end{smallmatrix}\right)$, $D(\mathcal{B}) := D(\mathcal{A}_0)$. We shall prove that $\mathcal{B}$ is a Desch-Schappacher perturbation of $\mathcal{A}_0$ which shows that $\mathcal{A}$ is a generator.

To this end we first note that by Lemma 3.1.1 the perturbation $\mathcal{B}$ is bounded from $D_{\mathcal{A}}$ to $\{0\} \times F^1_{\mathcal{D}}$. We show that $\{0\} \times F^1_{\mathcal{D}}$ is continuously embedded into $F^1_{\mathcal{A}_0}$. In order to estimate

$$\limsup_{t\to 0} \left\| \frac{1}{t}(\mathcal{T}_0(t) - I)\begin{pmatrix} 0 \\ f \end{pmatrix} \right\| = \limsup_{t\to 0} \left\| \frac{1}{t} \begin{pmatrix} \int_0^t T(t-s)f(s)\,ds \\ (S(t)-I)f \end{pmatrix} \right\| \tag{3.1.3}$$

for $f \in F^1_{\mathcal{D}}$ we first observe that $BV_1(\mathbb{R}_+; X)$ is contractively embedded into $L_\infty(\mathbb{R}_+; X)$. Hence the first component of (3.1.3) is estimated by

$$\limsup_{t\to 0} \left\| \frac{1}{t} \int_0^t T(t-s)f(s)\,ds \right\| \le M\|f\|_{1,Var},$$

where $M := \sup_{0\le t\le 1} \|T(t)\|$. For the second component we have (cf. (A.3))

$$\limsup_{t\to 0} \frac{1}{t}\|(S(t)-I)f\|_1 \le c_1\|f\|_{F^1_{\mathcal{D}}} \le c_2\|f\|_{1,Var}$$

for some $c_1, c_2 \ge 0$. Therefore $\{0\} \times F^1_{\mathcal{D}}$ is continuously embedded into $F^1_{\mathcal{A}_0}$ and thus $\mathcal{B}$ maps $D_{\mathcal{A}_0}$ continuously to $F^1_{\mathcal{A}_0}$. This shows that $\mathcal{A}$ is a generator and (FFA) is well-posed. By Lemma 3.1.3 (IDE$^\bullet$) is well-posed. ∎

3.2 The Delay Semigroup Approach

In this section we shall solve the homogeneous version of (IDE$^\bullet$) using the initial value problem

$$\dot{u}(t) = Au(t) + \int_{-\infty}^{0} \ell(-s)\dot{u}(t+s)\,ds, \quad u(0) = x, u_0 = g \quad (t \ge 0). \tag{3.2.1}$$

If u is a classical solution of (IDE$^\bullet$) then

$$v(t) := \begin{cases} u(t) & \text{if } t \ge 0, \\ u(0) & \text{if } t < 0, \end{cases}$$

is weakly differentiable and

$$\int_{-\infty}^{0} \ell(-s)\dot{v}(t+s)\,ds = \int_{0}^{t} \ell(t-s)\dot{u}(s)\,ds \quad (t \geq 0),$$

where $\dot{v}$ is the weak derivative of v. Thus v satisfies (3.2.1) for all $t \in \mathbb{R}_+$. On the other hand if v solves (3.2.1) (in a suitable sense) for the initial value $u(0) = x$ and $u_0 = x \cdot \mathbf{1}_{(-\infty,0]}$ we can expect that $u|_{\mathbb{R}_+}$ becomes a solution of (IDE$^\bullet$).

There are a number of results for the well-posedness of delay equations (cf. [19], [48], [69], [21], [20], [22] and [39; Section VI.6]). Most of the results are not applicable to (3.2.1) as derivation of u in the delay term is usually not allowed. In [48] well-posedness was deduced for linear delay operators being bounded from $W_p^1(-\infty, 0; X)$ to X. Unfortunately history functions of the form $x \cdot \mathbf{1}_{(-\infty,0)}$ do not belong to $W_p^1(-\infty, 0; X)$ if $x \neq 0$. However solutions of (IDE$^\bullet$) on compact intervals can be obtained (see Theorem 3.2.1 below). The main focus of this section is the construction of a C_0-semigroup similar to [48; Theorem 3.1] solving (IDE$^\bullet$) on $\mathbb{R}_+$. This is achieved by enlarging the space $L_p(-\infty, 0; X)$ using the notion of sum spaces.

3.2.1 Theorem. *Let $p \in [1,\infty)$. Assume that $\ell\colon \mathbb{R}_+ \to \mathcal{L}(X, F_A^1)$ is strongly Bochner measurable with respect to F_A^1 (i.e. $\ell(\cdot)x$ is Bochner measurable with respect to F_A^1 for all $x \in X$) and that $\|\ell(\cdot)\|_{\mathcal{L}(X,F_A^1)}$ is dominated by some $h \in L_{p',loc}(\mathbb{R}_+)$ where p' denotes the conjugate exponent of p. Then (IDE$^\bullet$) is well-posed.*

Proof. We first recall [48; Theorem 3.1] (cf. Theorem 3.2.6 for a similar result and Proposition 4.8.6, where a generalisation of [48; Theorem 3.1] is presented). Let $\tau > 0$. Let $\mathcal{A} := \left(\begin{smallmatrix} A & L \\ 0 & \mathcal{D} \end{smallmatrix}\right)$, $D(\mathcal{A}) := \{(x, f) \in D_A \times W_p^1(-\tau, 0; X);\ f(0) = x\}$, where $\mathcal{D}$ denotes the weak derivative in $L_p(-\tau, 0; X)$ and $L \in \mathcal{L}(W_p^1(-\tau, 0; X), X)$. Theorem 3.1 in [48] states that $\mathcal{A}$ is the generator of a C_0-semigroup on $X \times L_p(-\tau, 0; X)$.

Now let $L\colon W_p^1(-\tau, 0; X) \to X$ be defined by $Lf := \int_{-\tau}^{0} \ell(-\vartheta)\dot{f}(\vartheta)\,d\vartheta$. Then we have $L \in \mathcal{L}(W_p^1(-\tau, 0; X), F_A^1)$ and thus $\mathcal{A}$ becomes a generator. Obviously $[0,\tau] \ni t \mapsto P_1 e^{t\mathcal{A}} \left(\begin{smallmatrix} x \\ x\cdot\mathbf{1}_{(-\tau,0)} \end{smallmatrix}\right)$ is a classical solution of (IDE$^\bullet$) on the interval $[0,\tau]$ for all $x \in D_A$.

On the other hand, if u is a classical solution of (IDE$^\bullet$) with initial value x, then $v(t) := \left(\begin{smallmatrix} u(t) \\ u_t \end{smallmatrix}\right)$ (where we set $u_t(s) := x$ if $t + s < 0$) is a classical solution of the abstract Cauchy problem associated with $\mathcal{A}$ for the initial value $\left(\begin{smallmatrix} x \\ x\cdot\mathbf{1}_{(-\tau,0)} \end{smallmatrix}\right)$. Thus for all $x \in D_A$ there exists a unique classical solution of (IDE$^\bullet$) on $[0,\tau]$.

As τ can be chosen arbitrarily large we obtain a solution of (IDE$^\bullet$) on $\mathbb{R}_+$; the uniqueness property ensures that solutions for different $\tau_1, \tau_2 > 0$ do not differ on $[0,\tau_1] \cap [0,\tau_2]$.

The continuous dependence on the initial values immediately follows from the generator properties of $\mathcal{A}$. This shows the well-posedness of (IDE$^\bullet$). ∎

3.2.1 Sum spaces

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces which are continuously embedded into a Hausdorff topological vector spaces $\mathcal{X}$. Let $p \in [1,\infty)$ and $s\colon X \oplus_p Y \to \mathcal{X}$ be defined as $s(x, y) := x + y$. Then s is a surjective map onto $Z := X + Y$. As s is continuous $\ker s = \{(x, y) \in X \times Y\,;\, x + y = 0\}$ is a closed subspace of $X \oplus_p Y$, and thus the quotient space $\tilde{Z} := (X \oplus_p Y)/_{\ker s}$ equipped with the quotient norm is a Banach space. It is straightforward to check that $\tilde{Z} \ni (x, y) + \ker s \mapsto x + y \in Z$ becomes an isometrical isomorphism if Z is equipped with the norm

$$\|z\| := \inf\{(\|x\|_X^p + \|y\|_Y^p)^{1/p};\, x \in X\,, y \in Y,\, x + y = z\}.$$

We call $Z = X + Y$ the *p-sum space of X and Y*. If X and Y are Hilbert spaces and $p = 2$ then Z is isometrically isomorphic to $(\ker s)^{\perp} \subseteq X \oplus_2 Y$. Therefore Z also becomes a Hilbert space (cf. [28; Theorem I.2.6, Theorem III.4.2]). For sum spaces in interpolation theory we refer to [17; Proposition 2.1.6], [18; Section 3.2], [33; Section 6.1], [12; Section 2.3] and [59; IX.4 Appendix].

3.2.2 The sum space Z_p

Let $X_p := L_p(-\infty, 0; X)$. The space

$$Y_p := \{f \in W^1_{1,loc}(-\infty, 0; X);\, f(0) = 0, f' \in X_p\}$$

becomes a Banach space if equipped with the norm $\|f\|_p' := \|f'\|_p$ $(f \in Y_p)$. The p-sum space $Z_p := X_p + Y_p$ (as a subspace of $L_{1,loc}(-\infty, 0; X)$) endowed with the norm $\|f\|_{+,p} := \inf\{(\|g\|_p^p + \|h'\|_p^p)^{1/p};\, g \in X_p, h \in Y_p, f = g + h\}$ $(f \in Z_p)$ is a Banach space. If $p = 2$ and X is a Hilbert spaces then Z_p becomes a Hilbert space.

Let Z_p^1 be the subspace of all weakly differentiable functions in Z_p whose first derivatives are again in Z_p. The following estimate (3.2.2) applied to the right translation semigroup on Z_p will reveal that a function in Z_p^1 is already p-integrable. This estimate is closely related to the Landau-Kolmogorov inequality (cf. [57; Lemma 1.2.8], [1; Lemma 4.10]).

3.2.2 Proposition. *Let $p \in [1, \infty)$ and let X, Y be Banach spaces as in Section 3.2.1. Let T be a C_0-semigroup on the p-sum space $Z = X + Y$, with generator A. Assume that the following conditions hold.*

(i) T restricted to X is a C_0-semigroup on X.

(ii) $Y \subseteq D(A)$ and $AY \subseteq X$.

Then $A|_Y \in \mathcal{L}(Y, X)$, $\operatorname{rg} A \subseteq X$ and

$$\|Az\|_X \leq c_p \max\{M + 1, M\|A|_Y\|\}\big(\|z\|_Z + \|Az\|_Z\big) \tag{3.2.2}$$

for all $z \in D(A)$, and where $M := \sup_{0\leq s\leq 1} \|T(s)\|_{\mathcal{L}(X)}$ and $c_p := 2^{1-1/p}$.

Proof. First we show that $A|_Y$ is bounded from Y to X. To this end let $(y_n) \subseteq Y$, $y_n \to y \in Y$ and $Ay_n \to z \in X$ $(n \to \infty)$. As X and Y are contractively embedded into

Z we have $y_n \to y$ and $Ay_n \to z$ in Z and the closedness of A implies $(A|_Y)y = Ay = z$. Hence $A|_Y$ is a closed operator. The closed graph theorem implies that $A|_Y$ is bounded.

Let $z \in D(A)$, $z = x_0 + y_0$, $Az = x_1 + y_1$ with $x_0, x_1 \in X$ and $y_0, y_1 \in Y$. Let $t > 0$. Integration by parts yields the formula

$$\int_0^t (t-r)T(r)Ay_1\,dr = -ty_1 + \int_0^t T(r)y_1\,dr. \tag{3.2.3}$$

As $z, y_0 \in D(A)$ we also have $x_0 \in D(A)$ and thus we can write

$$T(t)x_0 - x_0 = \int_0^t T(r)Ax_0\,dr = \int_0^t T(r)(x_1 + y_1 - Ay_0)\,dr. \tag{3.2.4}$$

Using (3.2.3) and (3.2.4) we obtain

$$ty_1 = T(t)x_0 - x_0 - \int_0^t T(r)(x_1 - Ay_0)\,dr - \int_0^t (t-r)T(r)Ay_1\,dr. \tag{3.2.5}$$

As $T(r)$ maps X to X continuously and $x_1 - Ay_0$ and Ay_1 are in X the integrals on the right hand side of (3.2.5) take values in X. Hence $ty_1 \in X$ and

$$\|ty_1\|_X \le (M_t + 1)\|x_0\|_X + tM_t\|x_1 - Ay_0\|_X + \frac{t^2}{2}M_t\|Ay_1\|_X, \tag{3.2.6}$$

where $M_t := \sup_{0\le s\le t}\|T(s)\|_{\mathcal{L}(X)}$. As $Az = x_1 + y_1$ we obtain from (3.2.6) that $Az \in X$ and

$$\|Az\|_X \le \frac{M_t + 1}{t}\|x_0\|_X + M_t\|Ay_0\|_X + (M_t + 1)\|x_1\|_X + \frac{t}{2}M_t\|Ay_1\|_X. \tag{3.2.7}$$

Choosing $t = 1$ in (3.2.7) we obtain

$$\|Az\|_X \le \max\{M + 1, M\|A|_Y\|\}\big(\|x_0\|_X + \|y_0\|_Y + \|x_1\|_X + \|y_1\|_Y\big).$$

As $a + b \le c_p(a^p + b^p)^{1/p}$ $(a, b \in \mathbb{R},\ a, b \ge 0)$ we further obtain

$$\|Az\|_X \le \tilde{M}\big((\|x_0\|_X^p + \|y_0\|_Y^p)^{1/p} + (\|x_1\|_X^p + \|y_1\|_Y^p)^{1/p}\big), \tag{3.2.8}$$

where $\tilde{M} := c_p \max\{M + 1, M\|A|_Y\|\}$. By taking the infimum in (3.2.8) over all decompositions of z and Az we infer (3.2.2). ■

3.2.3 Corollary. *Let $p \in [1, \infty)$. Then the weak derivative on Z_p^1 is a bounded operator in $\mathcal{L}(Z_p^1, X_p)$ and*

$$\|f'\|_p \le 2c_p(\|f\|_{+,p} + \|f'\|_{+,p}) \le 2c_p^2(\|f\|_{+,p}^p + \|f'\|_{+,p}^p)^{1/p} \quad (f \in Z_p^1).$$

Proof. We shall apply Proposition 3.2.2 to the right translation semigroup on Z_p. For $t \geq 0$ and $f \in Z_p$ we define $S(t)f := f(\cdot - t)$. Obviously S satisfies the semigroup law. Let $t \geq 0$. Let $f = g + h$, $g \in X_p$, $h \in Y_p$. We define $\varphi\colon (-\infty, 0) \to \mathbb{R}$ as $\varphi(s) := \max\{0, s+1\}$ $(s < 0)$. Then

$$\begin{aligned} \|S(t)f\|_{+,p} &\leq \|g + h(-t)\varphi\|_p + \|S(t)h - h(-t)\varphi\|_p' \\ &\leq \|g\|_p + \|h'\|_p + \|h(-t)\|(\|\varphi'\|_p + \|\varphi\|_p). \end{aligned}$$

As $\|h(-t)\| \leq \int_{-t}^0 \|h'(s)\|\, ds \leq t^{1-1/p}\|h'\|_p$, $\|\varphi\|_p = (1+p)^{-1/p} \leq 1$ and $\|\varphi'\|_p = 1$ we obtain

$$\|S(t)f\|_{+,p} \leq \|g\|_p + \left(1 + 2t^{1-1/p}\right)\|h\|_p'.$$

Thus $\|S(t)f\|_{+,p} \leq (1 + 2t^{1-1/p})\|f\|_{+,p}$ and so $S(t)$ are bounded operators on Z_p. If we choose h such that $\operatorname{spt} h \subseteq (-\infty, -\delta)$ for some $\delta > 0$ the strong continuity follows from

$$\|S(t)f - f\|_{+,p} \leq \|S(t)g - g\|_p + \|S(t)h' - h'\|_p \to 0 \quad (t \to 0).$$

So S is a C_0-semigroup on Z_p. In order to determine the generator of S, which we denote by $\mathcal{D}$, let $\lambda > 0$ be larger than the growth bound of S. For $f \in Z_p$ we define $F(s) := (R(\lambda, \mathcal{D})f)(s) = \int_0^\infty e^{-\lambda t} f(s-t)\, dt$ $(s \in (-\infty, 0))$. Let $\psi \in C_c^\infty(-\infty, 0)$. Standard computations yield

$$\int \psi'(s)F(s)\, ds = \int \psi(s)(-f(s) + \lambda F(s))\, ds.$$

Thus F is weakly differentiable and $F' = f - \lambda F \in Z_p$. This implies $D(\mathcal{D}) \subseteq Z_p^1$ and $\mathcal{D}f = -f'$ $(f \in D(\mathcal{D}))$. In order to show that $D(\mathcal{D}) = Z_p^1$ we define $\tilde{\mathcal{D}}f := -f'$, $D(\tilde{\mathcal{D}}) := Z_p^1$. As $\lambda - \mathcal{D}$ is bijective and $\lambda - \mathcal{D} \subseteq \lambda - \tilde{\mathcal{D}}$, it suffices to prove that $\lambda - \tilde{\mathcal{D}}\colon Z_p^1 \to Z_p$ is injective. To this end let $f \in Z_p^1$, $\lambda f + f' = 0$. Then there is $x \in X$ such that $f(s) = e^{-\lambda s}x$ $(s \in (-\infty, 0))$. As functions in Z_p cannot grow exponentially (as s goes to $-\infty$) and $\lambda > 0$ we infer $x = 0$ and thus $f = 0$.

We have shown that $D(\mathcal{D}) = Z_p^1$. In particular this implies that $Y_p \subseteq D(\mathcal{D})$ and $\mathcal{D}Y_p \subseteq X_p$. It is well known that S restricted to X_p is a C_0-semigroup. The application of Proposition 3.2.2 yields the assertion. ∎

3.2.4 Remark. The proof of Corollary 3.2.3 shows that Z_p^1 equipped with the norm

$$\|f\|_{+,p,1} := \left(\|f\|_{+,p}^p + \|f'\|_{+,p}^p\right)^{1/p} \quad (f \in Z_p^1)$$

is a Banach space. If $p = 2$ and X a Hilbert space then Z_p^1 is a Hilbert space, too.

3.2.3 Delay Semigroups in the Z_p-Context

Let $p \in [1, \infty)$ and $\mathcal{X}_p := X \times Z_p$. By $\mathcal{D}$ we denote the weak derivative on Z_p^1. In this section we show well-posedness of the abstract Cauchy problem

$$\text{(DE)} \qquad \begin{cases} \mathcal{A} = \begin{pmatrix} A & L \\ 0 & \mathcal{D} \end{pmatrix}, D(\mathcal{A}) = \left\{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix} \right) \in D(A) \times Z_p^1;\ f(0) = x \right\}, \\ \dot{u}(t) = \mathcal{A}u(t), \quad u(0) = \left(\begin{smallmatrix} x \\ f \end{smallmatrix} \right) \in \mathcal{X}_p \quad (t \geq 0) \end{cases}$$

if L satisfies a range condition. This result directly leads to a C_0-semigroup solving (IDE$^\bullet$). In order to derive the generator property of $\mathcal{A}$ we represent this operator as a perturbation of the operator $\mathcal{A}_0 := \left(\begin{smallmatrix} A & 0 \\ 0 & \mathcal{D} \end{smallmatrix} \right)$, $D(\mathcal{A}_0) := D(\mathcal{A})$.

3.2.5 Lemma. *The operator $\mathcal{A}_0$ is the generator of the C_0-semigroup $\mathcal{T}_0$ given by*

$$\mathcal{T}_0(t) \colon \mathcal{X}_p \to \mathcal{X}_p, \quad \mathcal{T}_0(t) := \begin{pmatrix} T(t) & 0 \\ T_t & S(t) \end{pmatrix} \quad (t \geq 0),$$

where $T_t \colon X \to Z_p$ $(t \geq 0)$ is defined as

$$T_t x(s) := \begin{cases} T(t+s)x & \text{if } s \geq -t, \\ 0 & \text{if } s < -t, \end{cases}$$

and $S(t)$ $(t \geq 0)$ denotes the left translation on Z_p (i.e. $S(t)f(s) := f(t+s)$ if $t+s < 0$, otherwise $S(t)f(s) := 0$).

Proof. It is straightforward to see that S and therefore $\mathcal{T}_0$ are C_0-semigroups. Let $\tilde{\mathcal{A}}_0$ be the generator of $\mathcal{T}_0$ and $\lambda > 0$ be greater than the growth bound of $\mathcal{T}_0$. From

$$R(\lambda, \tilde{\mathcal{A}}_0) \begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} R(\lambda, A)x \\ s \mapsto e^{\lambda s}\big(R(\lambda, A)x + \int_s^0 e^{-\lambda r} f(r)\, dr\big) \end{pmatrix} \tag{3.2.9}$$

we infer $R(\lambda, \tilde{\mathcal{A}}_0)(\lambda - \mathcal{A}_0) \left(\begin{smallmatrix} x \\ f \end{smallmatrix} \right) = \left(\begin{smallmatrix} x \\ f \end{smallmatrix} \right)$ for all $\left(\begin{smallmatrix} x \\ f \end{smallmatrix} \right) \in D(\mathcal{A}_0)$. Thus $\mathcal{A}_0 \subseteq \tilde{\mathcal{A}}_0$. From (3.2.9) we also see that the range of $R(\lambda, \tilde{\mathcal{A}}_0)$ is a subset of $D(\mathcal{A}_0)$. This shows $\tilde{\mathcal{A}}_0 = \mathcal{A}_0$. ∎

Presenting $\mathcal{A}$ as the perturbation of $\mathcal{A}_0$ with the operator $\mathcal{B} := \left(\begin{smallmatrix} 0 & L \\ 0 & 0 \end{smallmatrix} \right)$, $D(\mathcal{B}) := D(\mathcal{A}_0)$, we can apply the Desch-Schappacher perturbation theorem.

3.2.6 Theorem. *If $L \colon Z_p^1 \to F_A^1$ is a bounded operator then $\mathcal{A}$ is a generator.*

Proof. We will apply Theorem A.2. Let $Z := F_A^1 \times \{0\}$. Then $\mathcal{B} = \left(\begin{smallmatrix} 0 & L \\ 0 & 0 \end{smallmatrix} \right)$ is a bounded operator from $D_{\mathcal{A}_0}$ to Z. We show that Z satisfies the range condition (RC) of Proposition A.3 with respect to $\mathcal{A}_0$. To this end let $\varphi \colon [0, 1] \to Z$ be continuous, $\varphi(t) = (\varphi_1(t), 0)$ and

$$\psi \colon (-\infty, 0] \to X, \quad \psi(r) := \begin{cases} \int_0^{t+r} T(t + r - s)\varphi_1(s)\, ds & \text{if } r \in [-t, 0], \\ 0 & \text{if } r \in (-\infty, -t). \end{cases}$$

From [48; proof of Theorem 3.1] we adopt the facts that $\psi(0) = \int_0^t T(t-s)\varphi_1(s)\,ds \in D(A)$, $\psi \in W_p^1(-\infty,0;X) \subseteq Z_p^1$, $\int_0^t \mathcal{T}_0(t-s)\varphi(s)\,ds \in D(\mathcal{A}_0)$, and

$$\left\| \mathcal{A}_0 \int_0^t \mathcal{T}_0(t-s)\varphi(s)\,ds \right\|_{X\times X_p} \leq ct^{1/p} \sup_{0\leq s\leq t} \|\varphi_1(s)\|_{F_A^1} \quad (t\in[0,1])$$

for some $c \geq 0$. As X_p is contractively embedded into Z_p we obtain

$$\left\| \mathcal{A}_0 \int_0^t \mathcal{T}_0(t-s)\varphi(s)\,ds \right\|_{X\times Z_p} \leq \left\| \mathcal{A}_0 \int_0^t \mathcal{T}_0(t-s)\varphi(s)\,ds \right\|_{X\times X_p} \qquad (t\in[0,1]).$$

Hence Z fulfils (RC). The application of Theorem A.2 shows that $\mathcal{A}$ is a generator. ■

3.2.7 Corollary. *Assume that $\ell\colon \mathbb{R}_+ \to \mathcal{L}(X, F_A^1)$ is strongly Bochner measurable with respect to F_A^1 (i.e. $\ell(\cdot)x$ is Bochner measurable with respect to F_A^1 for all $x \in X$) and $\|\ell(\cdot)\|_{\mathcal{L}(X,F_A^1)}$ is dominated by some $h \in L_{p'}(\mathbb{R}_+)$, where p' denotes the conjugate exponent of p. Let $Lf := \int_{-\infty}^0 \ell(-\vartheta)\dot f(\vartheta)\,d\vartheta$ $(f \in Z_p^1)$. Then $L \in \mathcal{L}(Z_p^1, F_A^1)$ and $\mathcal{A} = \left(\begin{smallmatrix} A & L \\ 0 & \mathcal{D}\end{smallmatrix}\right)$ is a generator. The classical solution of (IDE$^\bullet$) for the initial value $x \in D_A$ and inhomogeneity $g = 0$ is given by $t \mapsto P_1 e^{t\mathcal{A}} \left(\begin{smallmatrix} x \\ x\cdot\mathbf{1}_{(-\infty,0)}\end{smallmatrix}\right)$.*

Proof. For the proof it suffices to observe that (cf. Corollary 3.2.3)

$$\|Lf\|_{F_A^1} \leq \|h\|_{p'}\|f'\|_p \leq 2c_p^2\|h\|_{p'}\|f\|_{+,p,1}.$$

■

For later use we mention the following delay property of inhomogeneous solutions of delay semigroups.

3.2.8 Proposition. *Assume that $\mathcal{A}$ is a generator. Let $\mathcal{T}$ be the C_0-semigroup generated by $\mathcal{A}$. Let $\left(\begin{smallmatrix} x \\ f\end{smallmatrix}\right) \in X \times Z_p$, $\varphi \in L_{1,loc}(\mathbb{R}_+;X)$, $t \geq 0$. Let*

$$\begin{pmatrix} u(t) \\ F(t) \end{pmatrix} := \mathcal{T}(t)\begin{pmatrix} x \\ f \end{pmatrix} + \int_0^t \mathcal{T}(t-s)\begin{pmatrix} \varphi(s) \\ 0 \end{pmatrix} ds$$

be the (mild) solution of the inhomogeneous Cauchy problem associated with $\mathcal{A}$. Then

$$F(t)(\tau) = \begin{cases} u(t+\tau) & \text{if } t+\tau \geq 0, \\ f(t+\tau) & \text{if } t+\tau < 0, \end{cases} \quad \text{almost everywhere } \tau \in (-\infty,0). \tag{3.2.10}$$

Proof. For $\varphi = 0$ the delay property (3.2.10) is well-known. In order to deal with the general case $\varphi \in L_{1,loc}(\mathbb{R}_+; X)$ we compute (using (3.2.10) with $\varphi = 0$)

$$\begin{aligned} F(t)(\tau) &= P_2 \left(\mathcal{T}(t) \begin{pmatrix} x \\ f \end{pmatrix} + \int_0^t \mathcal{T}(t-s) \begin{pmatrix} \varphi(s) \\ 0 \end{pmatrix} ds \right)(\tau) \\ &= P_1 \mathcal{T}(t+\tau) \begin{pmatrix} x \\ f \end{pmatrix} + \int_0^{t+\tau} P_1 \mathcal{T}(t+\tau-s) \begin{pmatrix} \varphi(s) \\ 0 \end{pmatrix} ds = u(t+\tau) \end{aligned}$$

for $\tau \in (-t, 0)$ almost everywhere. This shows that (3.2.10) holds on the interval $(-t, 0)$. For $\tau \in (-\infty, -t)$ it is straightforward to verify (3.2.10). ■

3.3 Relation to Evolutionary Integral Equations

We shall explore the relation of (IDE$^\bullet$) to the evolutionary integral equation

$$\text{(EIE)} \quad u(t) = f(t) + \int_0^t (A + \ell(t-s))u(s)\, ds \quad (t \geq 0),$$

where $f \in L_{1,loc}(\mathbb{R}_+; X)$. When discussing (EIE) we always assume that ℓ satisfies at least assumptions (a) and (b) in the introduction.

In the theory of evolutionary integral equations one is looking for a resolvent of (EIE), i.e. a strongly continuous solution operator family $\mathcal{R}\colon \mathbb{R}_+ \to \mathcal{L}(X)$ such that for $x \in X$ the function $\mathcal{R}(\cdot)x$ is the unique solution (in a suitable sense) of (EIE) for $f = x \cdot \mathbf{1}_{(-\infty,0)}$. We say that a function $u\colon \mathbb{R}_+ \to X$ is a *mild solution of (EIE) for* $f \in L_{1,loc}(\mathbb{R}_+; X)$ if $\int_0^t u(s)\, ds \in D(A)$ for all $t \geq 0$ and

$$u(t) = f(t) + A \int_0^t u(s)\, ds + \int_0^t \ell(t-s)u(s)\, ds \quad (t \geq 0). \tag{3.3.1}$$

Observe that by definition a mild solution of (IDE$^\bullet$) with inhomogeneity $g \in L_{1,loc}(\mathbb{R}_+; X)$ is a mild solution of (EIE) for $f(t) := x + \int_0^t (g(s) - \ell(s)x)\, ds$ $(t \in \mathbb{R})$. A *resolvent for (EIE)* is a strongly continuous operator family $\mathcal{R}\colon \mathbb{R}_+ \to \mathcal{L}(X)$, so that for all $x \in X$ the function $\mathcal{R}(\cdot)x$ is a mild solution of (EIE) for $f = x \cdot \mathbf{1}_{(-\infty,0)}$.

For various other notions for solutions of evolutionary integral equations, which are not suitable in our situation, we refer to [58; Definition 1.1 and Definition 6.2] and [26; Definition 3.1 and Definition 4.1]

First we show that a resolvent of (EIE) suffices to obtain mild solutions of (EIE) for $f = x + \mathbf{1}_{[0,\infty)} * g$, $g \in L_{1,loc}(\mathbb{R}_+; X)$. A similar result is obtained in [58; Proposition 6.3].

3.3.1 Lemma. *Let $\mathcal{R}$ be a resolvent for (EIE), $g \in L_{1,loc}(\mathbb{R}_+; X)$. Then $(\mathbf{1}_{[0,\infty)} * \mathcal{R} * g)(t) \in D(A)$ for all $t \geq 0$ and*

$$A(\mathbf{1}_{[0,\infty)} * \mathcal{R} * g) = \mathcal{R} * g - \mathbf{1}_{[0,\infty)} * g - \ell * \mathcal{R} * g. \tag{3.3.2}$$

Proof. It suffices to show that (3.3.2) holds for functions $g = x \cdot \mathbf{1}_{[t_1,t_2]}$, $0 \le t_1 < t_2 < \infty$, $x \in X$. By linearity this implies that (3.3.2) holds for step functions. By Young's inequality (cf. [4; Theorem 1.3.5]) and the closedness of A we infer (3.3.2) for all $g \in L_{1,loc}(\mathbb{R}_+; X)$. In order to show (3.3.2) for $g = x \cdot \mathbf{1}_{[t_1,t_2]}$ and $t \ge t_2$ we compute

$$\begin{aligned}\int_0^t \int_0^s \mathcal{R}(s-r)g(r)\,dr\,ds &= \int_0^t \int_r^t \mathcal{R}(s-r)g(r)\,ds\,dr \\ &= \int_0^t \int_0^{t-r} \mathcal{R}(s)g(r)\,ds\,dr = \int_{t_1}^{t_2} \int_0^{t-r} \mathcal{R}(s)x\,ds\,dr.\end{aligned} \tag{3.3.3}$$

As the terms occurring in (3.3.2) do not change their value if g is replaced by $g \cdot \mathbf{1}_{[0,t]} = x \cdot \mathbf{1}_{[t_1,t_2]\cap[0,t]}$ we conclude that (3.3.3) also holds for all $t \in [0, t_2)$. Therefore $\int_0^t \int_0^s \mathcal{R}(s-r)g(r)\,dr\,ds \in D(A)$ $(t \in \mathbb{R}_+)$ and

$$\begin{aligned}A&\int_0^t \int_0^s \mathcal{R}(s-r)g(r)\,dr\,ds \\ &= \int_{t_1}^{t_2} \left(\mathcal{R}(t-r)x - x - \int_0^{t-r} \ell(t-r-s)\mathcal{R}(s)x\,ds \right) dr \\ &= \int_0^t (\mathcal{R}(t-r)g(r) - g(r))\,dr - \int_0^t \ell(t-r) \int_0^r \mathcal{R}(r-s)g(s)\,ds\,dr\end{aligned}$$

establishes (3.3.2). ∎

3.3.2 Proposition. *Let $\mathcal{R}$ be a resolvent for (EIE), $g \in L_{1,loc}(\mathbb{R}_+; X)$. Then $\mathcal{R}(\cdot)x + \mathcal{R}*g$ is a mild solution of (EIE) for $f = x + \mathbf{1}_{[0,\infty)} * g$. In particular $\mathcal{R}(\cdot)x + \mathcal{R}*(g - \ell(\cdot)x)$ is a mild solution of (IDE$^\bullet$) for the initial value $x \in X$ and the inhomogeneity g.*

Proof. Let $t \ge 0$, $x \in X$ and $u := \mathcal{R}(\cdot)x + \mathcal{R}*g$. From our assumptions and Lemma 3.3.1 we infer that $(\mathbf{1}_{[0,\infty)} * u)(t) \in D(A)$ and

$$\begin{aligned}A(\mathbf{1}_{[0,\infty)} * u) &= A(\mathbf{1}_{[0,\infty)} * (\mathcal{R}(\cdot)x + \mathcal{R} * g)) \\ &= (\mathcal{R}(\cdot)x - x - \ell * \mathcal{R}(\cdot)x) + (\mathcal{R} * g - f - \ell * \mathcal{R} * g) \\ &= (\mathcal{R}(\cdot)x + \mathcal{R} * g) - x - \ell * (\mathcal{R}(\cdot)x + \mathcal{R} * g) - f \\ &= u - x - \ell * u - f.\end{aligned}$$

As this is just (3.3.1) we have shown that u is a mild solution of (EIE) for $f = x + \mathbf{1}_{[0,\infty)} * g$. ∎

We now look at the relation of delay semigroups and (EIE). Let $\mathcal{T}$ be the delay semigroup generated by $\mathcal{A}$ from Corollary 3.2.7. Then $\mathcal{S}(\cdot)x := P_1\mathcal{T}(\cdot)\left(\begin{smallmatrix} x \\ x\cdot\mathbf{1}_{(-\infty,0)}\end{smallmatrix}\right)$ is a solution operator family for (IDE$^\bullet$). The delay semigroup also yields a resolvent for (EIE).

3.3.3 Proposition. *Let $\mathcal{T} = \begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{21} & \mathcal{T}_{22} \end{pmatrix}$ be the delay semigroup from Corollary 3.2.7 with generator $\mathcal{A}$. Then $\mathcal{R} := \mathcal{T}_{11}$ is a resolvent for (EIE).*

Proof. Let $x \in X$ and $V : \mathbb{R}_+ \to X$, $V(t) := \mathcal{T}(t) \begin{pmatrix} x \\ 0 \end{pmatrix}$ $(t \in \mathbb{R}_+)$. Then V satisfies

$$V(t) - V(0) = \mathcal{A} \int_0^t V(s)\, ds \qquad (t \in \mathbb{R}_+). \tag{3.3.4}$$

Let $u := \mathcal{R}(\cdot)x = P_1 V(\cdot)$ and $F := P_2 V(\cdot)$. Projection onto the first component in (3.3.4) yields

$$u(t) - x = A \int_0^t u(s)\, ds + L \int_0^t F(s)\, ds, \tag{3.3.5}$$

where $Lf := \int_{-\infty}^0 \ell(-s)\dot{f}(s)\, ds$ $(f \in Z_p^1)$. From Proposition 3.2.8 we know that

$$F(s)(\tau) = \begin{cases} u(s+\tau) & \text{if } s+\tau \geq 0, \\ 0 & \text{otherwise,} \end{cases} \qquad (\tau \in (-\infty, 0) \text{ almost everywhere}).$$

Therefore $L \int_0^t F(s)\, ds = \int_0^t \ell(t-s)u(s)\, ds$. From (3.3.5) we see that $u = \mathcal{R}(\cdot)x$ is a mild solution of (EIE) for $f = x \cdot \mathbf{1}_{[0,\infty)}$. Hence $\mathcal{R}$ is a resolvent for (EIE). ■

If well-posedness of (IDE$^\bullet$) is obtained by the forcing function approach we have the following result.

3.3.4 Proposition. *Assume that $\mathcal{A}$ from (FFA) in Section 3.1 is a generator. Let $\mathcal{T}$ be the C_0-semigroup generated by $\mathcal{A}$. Let $\begin{pmatrix} x \\ g \end{pmatrix} \in X \times L_1(\mathbb{R}_+; X)$ and $u(t) := P_1 \mathcal{T}(t) \begin{pmatrix} x \\ g \end{pmatrix}$. Then*

$$u(t) = x + \int_0^t (g(s) - \ell(s)x)\, ds + A \int_0^t u(s)\, ds + \int_0^t \ell(t-s)u(s)\, ds \quad (t \geq 0).$$

In particular $\mathcal{R} : \mathbb{R}_+ \to \mathcal{L}(X)$, $\mathcal{R}(\cdot)x := P_1 \mathcal{T}(\cdot) \begin{pmatrix} x \\ \ell(\cdot)x \end{pmatrix}$ $(x \in X)$ is a resolvent for (EIE).

Proof. If $\begin{pmatrix} x \\ g \end{pmatrix} \in D(\mathcal{A})$ the equation is easily verified. Otherwise an approximation by a sequence of initial values in $D(\mathcal{A})$ and the closedness of A prove the assertion. ■

3.4 Strong Stability of (IDE$^\bullet$)

Let $\mathcal{S} : \mathbb{R}_+ \to \mathcal{L}(X)$ be a strongly continuous family of operators. We say that $\mathcal{S}$ is strongly stable if $\mathcal{S}(t) \to 0$ in the strong operator topology as $t \to \infty$. Further $\mathcal{S}$ is said to be strongly integrable if $\mathcal{S}(\cdot)x$ is integrable for all $x \in X$. We call (IDE$^\bullet$) strongly stable (strongly integrable) if a solution operator family exists, and if this family is strongly stable (strongly integrable). For models describing the phenomenon of flutter strong stability is the type of stability which engineers are interested in.

We start with a remark concerning integrability of a resolvent for (EIE) and the relation to strong stability of (IDE$^\bullet$). Assume that $\mathcal{R}$ is a strongly integrable resolvent for (EIE), that $\mathcal{S}$ is a solution operator family for (IDE$^\bullet$) and that $\ell(\cdot)x \in L_1(\mathbb{R}_+; X)$ $(x \in X)$. Then for $x \in X$ the function $\mathcal{S}(\cdot)x$ is the sum of the integrable functions $\mathcal{R}(\cdot)x$ and $-\mathcal{R} * \ell(\cdot)x$ (cf. [58; Section 10.1]). Therefore if we additionally assume that $\mathcal{S}$ is uniformly continuous then $\mathcal{S}(t)x \to 0$ as $t \to \infty$. Hence (IDE$^\bullet$) is strongly stable.

We point out that general conditions for strong integrability of resolvents of non-scalar evolutionary integral equations seem to be available only for special cases (see [58; Section 10]). One reason is the fact that the conditions on Laplace transforms ensuring integrability of the transformed functions are generally difficult to check (cf. [58; Theorem 0.3]).

As a motivation for the main result of this section Theorem 3.4.3 we recall a well-known condition for strong stability of C_0-semigroups: A bounded C_0-semigroup is strongly stable if the spectrum of its generator on $i\mathbb{R}$ is countable and the residual spectrum of its generator on $i\mathbb{R}$ is empty. Two different proofs can be found in [4; Theorem 5.5.5] and [39; Theorem V.2.21]. In both cases the algebraic structure of C_0-semigroup plays a crucial role.

For Volterra equations similiar results have been derived using Laplace transform methods. However, for such equations the uniform continuity and the uniform ergodicity of individual solutions can not generally be derived from the boundedness of resolvents as in the case of C_0-semigroups (cf. [26]).

In the case that (IDE$^\bullet$) is solved by the delay semigroup approach from Section 3.2 more can be said. From now on we suppose that the assumptions of Corollary 3.2.7 hold. Let $p \in (1, \infty)$ and $\mathcal{T} = \left(\begin{smallmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{21} & \mathcal{T}_{22} \end{smallmatrix}\right)$ be the delay semigroup on $X \times Z_p$ obtained in this corollary (our method does not work for $p = 1$, see Remarks 3.4.4). We assume that $P_1\mathcal{T}$ is a bounded operator family (or equivalently $\mathcal{T}_{11}$ and $\mathcal{T}_{12}$ are bounded operator families). We recall that solutions of (IDE$^\bullet$) for the initial value $x \in X$ are given by $\mathbb{R}_+ \ni t \mapsto P_1\mathcal{T}(t) \left(\begin{smallmatrix} x \\ x \cdot \mathbf{1}_{(-\infty,0)} \end{smallmatrix}\right)$. Let $\mathcal{S} \colon \mathbb{R}_+ \to \mathcal{L}(X)$ be defined as the solution operator family $\mathcal{S}(\cdot)x := P_1\mathcal{T}(\cdot) \left(\begin{smallmatrix} x \\ x \cdot \mathbf{1}_{(-\infty,0)} \end{smallmatrix}\right)$ $(x \in X)$. Taking the Laplace transform of (IDE$^\bullet$) (strongly as a Bochner integral and with $g = 0$) we obtain

$$(\lambda - A - \lambda\hat{\ell}(\lambda))\hat{\mathcal{S}}(\lambda) = I - \hat{\ell}(\lambda) \tag{3.4.1}$$

for $\lambda \in \mathbb{C}_+$ (as we assume that $\|\ell(\cdot)\|_{\mathcal{L}(X,F_A^1)}$ is bounded by a function in $L_{p'}(\mathbb{R}_+)$, the Laplace transform $\hat{\ell}$ exists on $\mathbb{C}_+$ strongly as a Bochner integral in X).

We make the following assumptions additional to the boundedness of $P_1\mathcal{T}$:

(a) $\ell(\cdot)x$, $(s \mapsto s\ell(s)x) \in L_1(\mathbb{R}_+; X)$ for all $x \in X$;

(b) $U \colon \overline{\mathbb{C}_+} \to \mathcal{L}(D_A, X)$, defined by $U(\lambda) := \lambda - A - \lambda\hat{\ell}(\lambda)$, is invertible in $\mathcal{L}(X)$ for $\lambda \in \mathbb{C}_+$.

We point out that by (a) the Laplace transform of $\ell(\cdot)x$ exists as a Bochner integral on $\overline{\mathbb{C}_+}$ and thus the definition of U in (b) makes sense.

For the following lemma in the context of C_0-semigroups we refer to [4; Proposition 4.3.1].

3.4.1 Lemma. *Let $\eta \in \mathbb{R}$ and $x \in X$. If $(I - \hat{\ell}(i\eta))x \in \overline{\operatorname{rg} U(i\eta)}$ then*

$$\alpha\hat{\mathcal{S}}(\alpha + i\eta)x \to 0 \qquad (\alpha \to 0+).$$

Proof. As $\mathcal{T}_{11}$ is a bounded resolvent for (EIE) (cf. Proposition 3.3.3) its Laplace transform exists on $\mathbb{C}_+$ (strongly as a Bochner integral) and is given by $\widehat{\mathcal{T}_{11}}(\lambda) = U(\lambda)^{-1}$ for $\lambda \in \mathbb{C}_+$. Moreover $\alpha U(\alpha + i\eta)^{-1}$ is uniformly bounded in $\alpha \in (0, \infty)$ as $e^{i\eta}\mathcal{T}_{11}$ is bounded.

First let $y = U(i\eta)z$ for some $z \in D(A)$. Assumption (a) implies that $\alpha\hat{\ell}(\alpha + i\eta) \to 0$ strongly as $\alpha \to 0+$. From this convergence, the boundedness of $\alpha U(\alpha+i\eta)^{-1}$ uniformly in $\alpha \in (0, \infty)$ and

$$\begin{aligned} U(\alpha + i\eta)^{-1}U(i\eta) &= U(\alpha + i\eta)^{-1}(U(i\eta) - U(\alpha + i\eta)) + I \\ &= \alpha U(\alpha + i\eta)^{-1}(\hat{\ell}(\alpha + i\eta) - I) \\ &\quad + i\eta U(\alpha + i\eta)^{-1}(\hat{\ell}(\alpha + i\eta) - \hat{\ell}(i\eta)) + I \end{aligned} \tag{3.4.2}$$

we conclude that $\alpha U(\alpha + i\eta)^{-1}y = \alpha U(\alpha + i\eta)^{-1}U(i\eta)z \to 0$ as $\alpha \to 0+$. (We remark that (3.4.2) is a variant of the well-known relation $R(\lambda, A)A = \lambda R(\lambda, A) - I$.)

For $y \in \overline{\operatorname{rg} U(i\eta)}$ we choose a sequence $(y_n) \subseteq \operatorname{rg} U(i\eta)$ converging to y as $n \to \infty$. The boundedness of $\alpha U(\alpha + i\eta)^{-1}$ uniformly in $\alpha \in (0, \infty)$ implies that for any $\varepsilon > 0$ we can find $n \in \mathbb{N}$ and $\alpha_0 > 0$ such that $\|\alpha U(\alpha + i\eta)^{-1}(y_n - y)\| \le \varepsilon/2$ $(\alpha \in (0, \alpha_0))$ and $\|\alpha U(\alpha + i\eta)^{-1}y_n\| \le \varepsilon/2$ $(\alpha \in (0, \alpha_0))$. Hence $\alpha U(\alpha + i\eta)^{-1}y \to 0$ as $\alpha \to 0+$ for all $y \in \overline{\operatorname{rg} U(i\eta)}$.

From the continuity of $\alpha \mapsto \hat{\ell}(\alpha + i\eta)x$ at $\alpha = 0$ (which follows from (a)) and the uniform boundedness of $\alpha U(\alpha + i\eta)^{-1}$ we conclude that

$$\begin{aligned} \alpha\hat{\mathcal{S}}(\alpha + i\eta)x &= \alpha U(\alpha + i\eta)^{-1}(\hat{\ell}(i\eta) - \hat{\ell}(\alpha + i\eta))x \\ &\quad + \alpha U(\alpha + i\eta)^{-1}(I - \hat{\ell}(i\eta))x \to 0 \qquad (\alpha \to 0+). \end{aligned}$$ ■

3.4.2 Lemma. *Let $f \in Y_p$ and $g(t) := \int_0^\infty \ell(t+s)\dot{f}(-s)\,ds$ $(t \ge 0)$. Then $P_1\mathcal{T}(\cdot)\binom{0}{f} = P_1(\mathcal{T} * \binom{g(\cdot)}{0})$.*

Proof. Let

$$u(s) := \begin{cases} P_1\mathcal{T}(s)\binom{0}{f} & \text{if } s \ge 0, \\ f(s) & \text{if } s < 0, \end{cases} \qquad v(s) := \begin{cases} P_1\mathcal{T}(s)\binom{0}{f} & \text{if } s \ge 0, \\ 0 & \text{if } s < 0. \end{cases}$$

As $\binom{0}{f} \in D(\mathcal{A})$ and $u(0) = v(0) = 0$ both functions are weakly differentiable on $\mathbb{R}$ and

$$\begin{aligned}
\dot{u}(t) &= Au(t) + \int_{-\infty}^{t} \ell(t-s)\dot{u}(s)\,ds \\
&= Au(t) + \int_{0}^{t} \ell(t-s)\dot{u}(s)\,ds + \int_{0}^{\infty} \ell(t+s)\dot{f}(-s)\,ds \\
&= Au(t) + \int_{-\infty}^{t} \ell(t-s)\dot{v}(s)\,ds + \int_{0}^{\infty} \ell(t+s)\dot{f}(-s)\,ds \quad (t \geq 0).
\end{aligned}$$

Therefore v is the solution to the inhomogeneous abstract Cauchy problem associated with $\mathcal{A}$ for the initial value 0 and the inhomogeneity g. Hence $u|_{\mathbb{R}_+} = v|_{\mathbb{R}_+} = P_1(\mathcal{T} * \binom{g}{0})$. ■

We are now prepared to prove a stability results for solutions of (IDE$^\bullet$) in the case that we have a delay semigroup solving (IDE$^\bullet$).

3.4.3 Theorem. *Assume that*

$$\Sigma := \mathbb{R} \setminus \{\eta \in \mathbb{R};\ \hat{\mathcal{S}} \textit{ has a holomorphic extension to a neighbourhood of } i\eta\}$$

is countable and that the range of $U(i\eta)$ is dense in X for all $\eta \in \mathbb{R}$. Then $\mathcal{S}$ is strongly stable.

Proof. We refer the reader to [4; Chapter 4] for the notions of ergodicity and frequency we are now going to employ.

Let $x \in X$. We first show that $u := \mathcal{S}(\cdot)x \in C_{ub}(\mathbb{R}_+; X)$. Let $t, h \in \mathbb{R}_+$. From

$$\mathcal{S}(t+h)x - \mathcal{S}(t)x = P_1\mathcal{T}(t)\big(\mathcal{T}(h) - I\big)\begin{pmatrix} x \\ x \cdot \mathbf{1}_{(-\infty,0)} \end{pmatrix},$$

the boundedness of the operator family $P_1\mathcal{T}$ and the convergence $\mathcal{T}(h) - I \to 0$ strongly as $h \to 0$ we infer the uniform continuity of $\mathcal{S}(\cdot)x$. Hence $\mathcal{S}(\cdot)x \in C_{ub}(\mathbb{R}_+; X)$.

Next we show that u is totally ergodic with respect to the left translation semigroup (which we denote by S) on $C_{ub}(\mathbb{R}_+; X)$ and the set of frequencies is empty, i.e. $\frac{1}{\tau}\int_0^\tau e^{-i\eta s}S(s)u\,ds \to 0$ $(\tau \to \infty)$ in $C_{ub}(\mathbb{R}_+; X)$ for all $\eta \in \mathbb{R}$. To this end we extend u and set $u(\vartheta) := x$ for $\vartheta \in (-\infty, 0)$. Further let $F(s) := u_s \in Z_p$ $(s \in \mathbb{R}_+)$. Since we can write

$$\begin{aligned}
\frac{1}{\tau}\int_0^\tau e^{-i\eta s}S(s)u\,ds &= \frac{1}{\tau}\int_0^\tau e^{-i\eta s}u(s+\cdot)\,ds \\
&= \mathbb{R}_+ \ni t \mapsto \frac{1}{\tau}\int_0^\tau e^{-i\eta s}P_1\mathcal{T}(t)\begin{pmatrix} u(s) \\ u_s \end{pmatrix} ds
\end{aligned}$$

we will first show that

$$\alpha \left(P_1 \mathcal{T}(t) \begin{pmatrix} u(\cdot) \\ F(\cdot) \end{pmatrix} \right)^{\wedge} (\alpha + i\eta) = \alpha P_1 \mathcal{T}(t) \begin{pmatrix} \hat{u}(\alpha + i\eta) \\ \hat{F}(\alpha + i\eta) \end{pmatrix} \to 0 \quad (\alpha \to 0+)$$

uniformly in $t \in \mathbb{R}_+$; then the total ergodicity of u and the emptiness of the set of frequencies follow from [4; Theorem 4.2.7]), applied to the function

$$(\mathbb{R}_+ \ni s \mapsto e^{-i\eta s} S(s)u) \in C_b(\mathbb{R}_+; C_{bu}(\mathbb{R}_+; X)).$$

For $\lambda \in \mathbb{C}$ let $\varepsilon_\lambda(\vartheta) := e^{\lambda\vartheta}$ $(\vartheta \in (-\infty, 0))$. As $\operatorname{rg} U(i\eta)$ is dense in X for all $\eta \in \mathbb{R}$ the application of Lemma 3.4.1 yields that $\alpha\hat{u}(\alpha + i\eta) \to 0$ $(\alpha \to 0+)$. The boundedness of $P_1\mathcal{T}$ implies that $\alpha P_1 \mathcal{T}(t) \begin{pmatrix} \hat{u}(\alpha + i\eta) \\ 0 \end{pmatrix} \to 0$ $(\alpha \to 0+)$ uniformly in $t \in \mathbb{R}_+$. For the Laplace transform of F we compute

$$\begin{aligned} \hat{F}(\alpha + i\eta) &= \left((-\infty, 0) \ni \vartheta \mapsto \int_0^{-\vartheta} e^{-(\alpha+i\eta)t} x \, dt + \int_{-\vartheta}^{\infty} e^{-(\alpha+i\eta)t} u(t+\vartheta) \, dt \right) \\ &= \frac{x}{\alpha + i\eta} \left(\mathbf{1}_{(-\infty,0)} - \varepsilon_{\alpha+i\eta} \right) + \hat{u}(\alpha + i\eta)\varepsilon_{\alpha+i\eta} \\ &= \hat{u}(\alpha + i\eta) \cdot \mathbf{1}_{(-\infty,0)} + \left(\frac{x}{\alpha + i\eta} - \hat{u}(\alpha + i\eta) \right) \left(\mathbf{1}_{(-\infty,0)} - \varepsilon_{\alpha+i\eta} \right). \end{aligned} \tag{3.4.3}$$

For the first summand in (3.4.3) we observe that $\alpha\hat{u}(\alpha + i\eta) \cdot \mathbf{1}_{(-\infty,0)} \to 0$ in Z_p as $\alpha \to 0+$. Hence $\alpha P_1 \mathcal{T}(t) \begin{pmatrix} 0 \\ \hat{u}(\alpha+i\eta)\cdot\mathbf{1}_{(-\infty,0)} \end{pmatrix} \to 0$ uniformly in $t \in \mathbb{R}_+$.

Let $x_{\alpha,\eta} := \frac{x}{\alpha+i\eta} - \hat{u}(\alpha + i\eta)$. In order to deal with the second summand we first assume that $\eta = 0$. As $\alpha x_{\alpha,0} = x + \alpha\hat{u}(\alpha + i\eta) \to x$ $(\alpha \to 0+)$ the elements $\alpha x_{\alpha,0}$ are uniformly bounded for $\alpha \in (0, 1]$. Since

$$\left\| y(\mathbf{1}_{(-\infty,0)} - \varepsilon_\alpha) \right\|_{Z_p} \le \left\| y(\mathbf{1}_{(-\infty,0)} - \varepsilon_\alpha) \right\|_{Y_p} \le \alpha \|y\| \, \|\varepsilon_\alpha\|_p = \alpha^{1-1/p} p^{-1/p} \|y\| \tag{3.4.4}$$

for $y \in X$ we see that

$$\alpha x_{\alpha,0} \left(\mathbf{1}_{(-\infty,0)} - \varepsilon_\alpha \right) \to 0 \quad (\alpha \to 0+)$$

in Z_p. Hence by the boundedness of $P_1\mathcal{T}$ we conclude that $\alpha P_1 \mathcal{T}(t) \begin{pmatrix} \hat{u}(\alpha) \\ \hat{F}(\alpha) \end{pmatrix} \to 0$ $(\alpha \to 0+)$ uniformly in $t \in \mathbb{R}_+$.

Now we assume that $\eta \neq 0$. By using Lemma 3.4.2 we can write the second summand in (3.4.3) as

$$\begin{aligned}
&\left\| \alpha P_1 \mathcal{T}(t) \begin{pmatrix} 0 \\ x_{\alpha,\eta}\left(\mathbf{1}_{(-\infty,0)} - \varepsilon_{\alpha+i\eta}\right) \end{pmatrix} \right\| \\
&\qquad = \left\| \alpha \int_0^t \int_0^\infty \mathcal{T}_{11}(t-s)\ell(s+r)x_{\alpha,\eta}(\alpha+i\eta)e^{(\alpha+i\eta)r}\,dr\,ds \right\| \\
&\qquad \le M\alpha|\alpha+i\eta| \int_0^\infty \int_0^\infty \|\ell(s+r)x_{\alpha,\eta}\|\,dr\,ds \\
&\qquad = M\alpha|\alpha+i\eta| \int_0^\infty \int_0^\rho \|\ell(\tau+(\rho-\tau))x_{\alpha,\eta}\|\,d\tau\,d\rho \\
&\qquad = M\alpha|\alpha+i\eta| \int_0^\infty \rho\|\ell(\rho)x_{\alpha,\eta}\|\,d\rho,
\end{aligned}$$

where $M := \sup_{t\in\mathbb{R}_+} \|\mathcal{T}_{11}(t)\| < \infty$. Let $k(\rho) := \rho\ell(\rho)$ $(\rho \in \mathbb{R}_+)$. By assumption (a) and the closed graph theorem the operator $(y \mapsto k(\cdot)y)$ belongs to $\mathcal{L}(X, L_1(\mathbb{R}_+; X))$. Therefore

$$\left\| \alpha P_1 \mathcal{T}(t) \begin{pmatrix} 0 \\ x_{\alpha,\eta}\left(\mathbf{1}_{(-\infty,0)} - \varepsilon_{\alpha+i\eta}\right) \end{pmatrix} \right\| \le M|\alpha+i\eta|\,\|k\|_{\mathcal{L}(X,L_1(\mathbb{R}_+;X))}\,\alpha\|x_{\alpha,\eta}\|.$$

Since $\eta \neq 0$ we have $\alpha x_{\alpha,\eta} \to 0$ as $\alpha \to 0+$. Thus $\alpha P_1 \mathcal{T}(t) \left(\begin{smallmatrix} 0 \\ x_{\alpha,\eta}(\mathbf{1}_{(-\infty,0)} - \varepsilon_{\alpha+i\eta}) \end{smallmatrix} \right)$ converges to 0 as $\alpha \to 0+$ uniformly in $t \in \mathbb{R}_+$. Hence we see that $\alpha P_1 \mathcal{T}(t) \left(\begin{smallmatrix} \hat{u}(\alpha+i\eta) \\ \hat{F}(\alpha+i\eta) \end{smallmatrix} \right) \to 0$ $(\alpha \to 0+)$ uniformly in $t \in \mathbb{R}_+$.

So we have shown that u is totally ergodic with respect to the left translation semigroup on $C_{ub}(\mathbb{R}_+; X)$ with an empty set of frequencies. Now [4; Corollary 4.7.8] (using the countability of Σ) implies that $\mathcal{S}(\cdot)x \in C_0(\mathbb{R}_+; X)$. ■

We conclude with some remarks.

3.4.4 Remarks. (a) If T is a C_0-semigroup then the function $T(\cdot)x$ is automatically uniformly continuous for all $x \in X$ provided that T is bounded. This easily follows from the semigroup law. For solution operator families of integro-differential equations boundedness generally does not imply uniform continuity.

(b) Assume that $\mathcal{R}$ is a resolvent for (EIE) with growth bound $\omega \in \mathbb{R}$. Then $\hat{\mathcal{R}}(\lambda)$ exists for $\operatorname{Re}\lambda > \omega$ and $U(\lambda)\hat{\mathcal{R}}(\lambda) = I$. Thus if $U(\lambda)$ is invertible then $U(\lambda)^{-1} = \hat{\mathcal{R}}(\lambda)$. We note that, unlike to the evolutionary integral equations dealt with in [58], we cannot deduce the equation $\hat{\mathcal{R}}(\lambda)U(\lambda) = I$ to obtain invertibility of $U(\lambda)$ for $\operatorname{Re}\lambda > \omega$ due to missing commutativity; see in particular equations (6.2) and (6.3) or (6.5) and (6.6) in [58] for the relevant commutativity type properties in the context of evolutionary integral equations. If we additionally assume that $\mathcal{R} * (A \cdot \mathbf{1}_{[0,\infty)} + \ell) = (A \cdot \mathbf{1}_{[0,\infty)} + \ell) * \mathcal{R}$ on D_A, then it is easy to show that $\hat{\mathcal{R}}(\lambda)U(\lambda) = I$ and invertibility of $U(\lambda)$ follows.

(c) Unlike to the spaces Z_p for $p \in (1, \infty)$ we do not have the convergence of ε_α to $\varepsilon_0 = \mathbf{1}_{(-\infty,0)}$ in Z_1, cf. the estimate (3.4.4). (In fact, one can show that $\varepsilon_0 \notin \overline{X_1}^{Z_1}$.) For this reason we cannot prove Theorem 3.4.3 for $p = 1$.

(d) In the proof of Theorem 3.4.3 we have to distinguish the cases $\eta = 0$ and $\eta \neq 0$ partly due to the fact that $x \cdot \varepsilon_{i\eta}$ belongs to Z_p if and only if $x = 0$ or $\eta = 0$. Assume that $x \neq 0$ and $f := x \cdot \varepsilon_{i\eta} \in Z_p$. Then $f' = i\eta f \in Z_p$ and therefore $f \in Z_p^1$. By Corollary 3.2.3 we see that $f' \in L_p$. From this we immediately conclude that $\eta = 0$.

Chapter 4

The Fractional Power Tower in Perturbation Theory of C_0-semigroups

The Sobolev Tower $(X_A^n)_{n\in\mathbb{Z}}$ is a family of spaces associated with a generator A of a C_0-semigroup on a Banach space $X = X_A^0$. For $n > 0$ the space X_A^n is the domain of A^n equipped with the graph norm coming from A^n. For $n < 0$ the space X_A^n is the completion of X equipped with the norm $(A-\omega)^n$ for some $\omega \in \mathbb{R}$ larger than the growth bound of the C_0-semigroup generated by A. In the perturbation theory of C_0-semigroups the "floor" $(X_A^n)_{n\in\{-1,0,1\}}$ is of particular interest. For example these three spaces remain invariant (up to an equivalent norm) under a bounded perturbation of A. This stability property allows the transfer of the well-known bounded perturbation theorem to the spaces X_A^1 and X_A^{-1}: If $B_1 \in \mathcal{L}(X_A^1)$ and $B_2 \in \mathcal{L}(X_A^{-1})$ then $A+B_1$ and $(A_{-1}+B_2)_{|X}$ are generators of C_0-semigroups on X; cf. [39; Corollary III.1.5]. (For the definition of A_α for $\alpha \in \mathbb{R}$ see Proposition 4.1.2 and [39; Definition II.5.4]. The part of an operator C in X is the operator $C_{|X}$ defined by $C_{|X}\, x := Cx$ for $x \in D(C_{|X}) := \{x \in D(C) \cap X\,;\, Cx \in X\}$. Equivalently we can define $C_{|X} := C \cap (X \times X)$.)

The idea of shifting perturbation theorems on the Sobolev tower also occurs in [39; Corollary III.3.22] and [32; Theorem 1].

In [39; Exercise VI.7.10(3)] this method was applied to inhomogeneous abstract Cauchy problems using abstract Hölder spaces, which extend the Sobolev Tower to a continuous scale of interpolation and extrapolation spaces.

The objective of this chapter is the application of the concept to the scale $(X_A^\gamma)_{\gamma\in\mathbb{R}}$ of fractional power spaces related to A; see Section 4.1 for their definition. As we will see this scale is more suitable for it has a better iteration property (cf. Theorem 4.1.4 and [39; Proposition II.5.35]).

For a general introduction to Banach scales and in particular fractional power scales we refer to [3; Chapter V]. There one can also find a more complicated proof of the iteration property in a more general context (cf. [3; Theorem V.1.5.4]).

Besides the iteration property our main abstract tool is a stability property for certain fractional power spaces X_A^γ under perturbations of A. Namely if A and C are generators of C_0-semigroups and $C = (A_{-1} + B)_{|X}$ for some $B \in \mathcal{L}(X_A^{\gamma_1}, X_A^{\gamma_2})$ with $\gamma_1, \gamma_2 \in (-1, 1)$ and $\gamma_1 - \gamma_2 < 1$ then we will see that for $\alpha \in (\gamma_1 - 1, \gamma_2 + 1]$ the spaces X_A^α and X_C^α are equal with equivalent norms.

This stability property together with the iteration property allows us to shift perturbation theorems (mainly the Desch-Schappacher and the Miyadera-Voigt perturbation theorem) on the continuous scale of fractional power spaces. In applications to inhomogeneous abstract Cauchy problems, various integro-differential equations as well as delay equations in the L_p-context, this notion yields well-posedness conditions with mixed fractional time and space regularity conditions on the inhomogeneities and delay terms, respectively, in these equations.

The chapter is organised as follows. In Section 4.1 we introduce fractional powers for generators of C_0-semigroups and for slightly more general operators. We also show the iteration property, determine Favard spaces associated to the scale of generators induced by the power spaces and provide a preparative estimate for certain compositions of fractional powers.

Sections 4.2 and 4.3 are devoted to the proof of our main tool on the stability of fractional power spaces under perturbation of the underlying generator. In Section 4.3

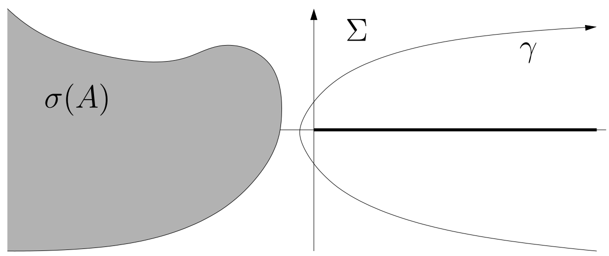

Figure 1

we also show stability of certain Favard spaces, which we will use in our applications.

In Section 4.4 we use the previously developed tools to shift perturbation theorems on the scale of fractional power spaces.

In Section 4.5 we provide two conditions on perturbing operators related to Desch-Schappacher and Miyadera-Voigt perturbations which ensure that the perturbing operator belongs to the class of operators covered in the previous sections.

Applications are presented in Sections 4.6, 4.7 and 4.8. Our main results for inhomogeneous abstract Cauchy problems are Propositions 4.6.1 and 4.6.3; see also Remarks 4.6.2(c). For integro-differential equations we mention Propositions 4.7.2, 4.7.3, 4.7.5, 4.7.7, 4.7.9 and 4.7.10 and Corollary 4.8.7; see also Remark 4.7.4. The perturbation results for delay semigroups are stated in Propositions 4.8.5 and 4.8.6.

4.1 Fractional Power Spaces

We recall the definition and elementary properties of fractional powers of generators of C_0-semigroups and similar operators. We also prove some elementary properties of the induced fractional power spaces. In particular we show that fractional power spaces of generators of C_0-semigroups possess the iteration property.

Let $\mathcal{K}(X)$ be the set of closed and densely defined operators A on the Banach space X whose resolvent set $\rho(A)$ contains an open sector Σ such that $\mathbb{R}_+ := [0,\infty) \subseteq \Sigma \subseteq \mathbb{C}$ and $\|R(\lambda, A)\| \leq \frac{M}{1+|\lambda|}$ for all $\lambda \in \Sigma$ and some $M \geq 0$. We mention that generators of C_0-semigroups with negative growth bound belong to $\mathcal{K}(X)$.

Let $A \in \mathcal{K}(X)$. The fractional power A^α for $\alpha < 0$ is defined by

$$A^\alpha := \frac{1}{2\pi i} \int_\gamma \lambda^\alpha R(\lambda, A)\, d\lambda$$

for a suitable path $\gamma \in \Sigma \setminus \mathbb{R}_+$ (cf. Figure 1). and with $\lambda \mapsto \lambda^\alpha = e^{\alpha \ln \lambda}$ defined on $\mathbb{C}\setminus\mathbb{R}_+$ (here ln denotes a branch of the logarithm on $\mathbb{C}\setminus\mathbb{R}_+$) (cf. [27; Definition III.2.18]). The required estimate for the resolvent ensures that the integral exists. By Cauchy's integral theorem the integral is independent of the path γ. For details on this approach to fractional powers we refer to [39; Section II.5].

For $\alpha \in (0,1)$ we also mention the formulas

$$A^{-\alpha} = c_\alpha \int_0^\infty s^{-\alpha} R(s,A)\,ds, \quad c_\alpha := \frac{1}{2\pi i}(1 - e^{-2\pi i\alpha}), \tag{4.1.1}$$

$$A^{-\alpha} = \tilde{c}_\alpha \int_0^\infty s^{\alpha-1} T(s)\,ds, \quad \tilde{c}_\alpha := \frac{(-1)^{-\alpha}}{\Gamma(\alpha)}, \tag{4.1.2}$$

where in the second formula we assume that A is the generator of a C_0-semigroup T (cf. [39; Corollary II.5.28, Exercise II.5.36(2)]).

The operators A^α are injective for $\alpha < 0$. For $\alpha > 0$ we define A^α on X as the inverse of $A^{-\alpha}$ with domain $D(A^\alpha) := \operatorname{rg} A^{-\alpha}$. We set $A^0 := I$. To justify the terminology we note that for $\alpha, \beta \in \mathbb{R}$ the operators $A^\alpha A^\beta$ and $A^{\alpha+\beta}$ agree on $D(A^\gamma)$ with $\gamma := \max\{\alpha, \beta, \alpha+\beta\}$ (for the proof of these properties we refer to [39; Proposition II.5.30, Theorem II.5.32]).

For $\alpha \geq 0$ the norm $\|x\|_\alpha := \|A^\alpha x\|$ $(x \in D(A^\alpha))$ makes $X^\alpha := (D(A^\alpha), \|\cdot\|_\alpha)$ a Banach space. For $\alpha < 0$ the space X equipped with the norm $\|x\|_\alpha := \|A^\alpha x\|$ $(x \in X)$ is not complete in general. By X^α we denote the completion of X with respect to $\|\cdot\|_\alpha$. This scale of Banach spaces includes the Sobolev tower $(X_n)_{n\in\mathbb{Z}}$ (where $X_n = X^n$). We again refer to [39; Proposition II.5.33] for details on this scale and on the (close) relation to the abstract Hölder spaces. We call $(X^\alpha)_{\alpha\in\mathbb{R}}$ the fractional power tower. (We often write X_A^α instead of X^α to highlight the associated operator. When considering iterated fractional power spaces we sometimes need to write $\|\cdot\|_{X_A^\alpha}$ or $\|\cdot\|_{X^\alpha}$ for the norm on X^α.)

4.1.1 Lemma. *Let X be a Banach space and $A \in \mathcal{K}(X)$.*

(a) For $\alpha, \beta \geq 0$ we have $A^{-\beta}X^\alpha = X^{\alpha+\beta}$.

(b) Let $\alpha, \beta \in \mathbb{R}$, $\alpha \geq \beta$. Then X^α is densely embedded in X^β.

Proof. In order to proof (a) we observe that $x \in X^{\alpha+\beta}$ if and only if $x = A^{-(\alpha+\beta)}y = A^{-\beta}(A^{-\alpha}y)$ for some $y \in X$ if and only if $x = A^{-\beta}z$ for some $z \in X^\alpha$.

We prove (b) for $\beta \geq 0$ first. To this end we choose $n \in \mathbb{N}$ such that $\beta + n \geq \alpha$. Let $x \in X^\beta$ and $y := A^\beta x \in X$. As X^n is densely embedded in X there exists a sequence $(y_m) \subseteq X^n$ which converges to y as $m \to \infty$. As $A^{-\beta}$ is a bounded operator we conclude that $(A^{-\beta}y_m) \subseteq X^{\beta+n} \subseteq X^\alpha$ tends to $A^{-\beta}y = x$.

If $\beta \leq 0$ and $\alpha \leq 0$ then the assertion follows from $X \subseteq X^\alpha \subseteq X^\beta$ and the fact that X is densely embedded in X^β.

Last let $\beta \leq 0$ and $\alpha > 0$ and choose $n \in \mathbb{N}$, $n \geq \alpha$. For $x \in X$ there exists a sequence $(x_m) \subseteq X^n$, converging to x in X. The boundedness of A^β in X implies that (x_m) converges to x in X^β. As X is densely embedded in X^β this shows assertion (b). ■

4.1.2 Proposition. *Let A be the generator of a C_0-semigroup T on a Banach space X with negative growth bound.*

(a) The operators $T(t)$ $(t \in \mathbb{R}_+)$, $R(\lambda, A)$ $(\lambda \geq 0)$ and the fractional powers A^α $(\alpha \in \mathbb{R})$ commute with each other.

(b) For $\alpha \geq 0$ the operators $T(t)$ leave the spaces X^α invariant ($t \in \mathbb{R}_+$). Moreover they are continuous with respect to the norm $\|\cdot\|_\alpha$. The restrictions of $T(t)$ to X^α are denoted by $T_\alpha(t)$.

(c) For $\alpha < 0$ the operators $T(t)$ are continuous with respect to the norm $\|\cdot\|_\alpha$ and therefore extend to operators $T_\alpha(t)$ on X^α ($t \in \mathbb{R}_+$).

(d) T_α is a C_0-semigroup on X^α for all $\alpha \in \mathbb{R}$.

(e) For $\alpha \geq 0$ the generator A_α of T_α is the part of A in X^α. Its domain is $X^{\alpha+1}$.

(f) For $\alpha < 0$ the generator A_α of T_α is the closure of A in X^α. Its domain is $X^{\alpha+1}$.

(g) For any $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$ the operator A_β is the part of A_α in X^β and A_α is the continuous extension of A_β to an isomorphism of $X^{\alpha+1}$ to X^α.

Proof. For $\alpha < 0$, assertion (a) follows from the definition of fractional powers. In particular this implies that $T(t)X^{-\alpha} \subseteq X^{-\alpha}$. Therefore we obtain for $\alpha > 0$ from $T(t)A^{-\alpha} = A^{-\alpha}T(t)$ and the definition of A^α as the inverse of $A^{-\alpha}$ that $T(t)$ and A^α commute. The commutativity of the fractional powers and the resolvents of A follows from the integral representation for the resolvents of A.

We conclude the continuity of $T(t)$ with respect to $\|\cdot\|_\alpha$ for $\alpha \in \mathbb{R}$ from

$$\|T(t)x\|_\alpha = \|A^\alpha T(t)x\| = \|T(t)A^\alpha x\| \leq \|T(t)\| \, \|A^\alpha x\| = \|T(t)\| \, \|x\|_\alpha \tag{4.1.3}$$

($x \in X^{\max\{0,\alpha\}}$). This shows (b) and (c). In order to proof (d) we need to show that $T_\alpha(t)$ is strongly continuous. For $x \in X^{\max\{0,\alpha\}}$ we conclude the strong continuity from

$$\|T(t)x - x\|_\alpha = \|A^\alpha(T(t)x - x)\| = \|(T(t) - I)(A^\alpha x)\| \to 0 \quad (t \to 0).$$

For $\alpha \geq 0$ this shows assertion (d). For $\alpha < 0$ the strong continuity of T_α follows from the denseness of X in X^α and the uniform boundedness of $T_\alpha(t)$ for $t \in \mathbb{R}_+$ (note that (4.1.3) implies $\|T(t)\|_{\mathcal{L}(X^\alpha)} \leq \|T(t)\|_{\mathcal{L}(X)}$).

Let $\alpha \in \mathbb{R}$ and $x, y \in X^{\max\{0,\alpha\}}$. In order to determine the generator of $T_\alpha(t)$ we compute

$$\left\| \frac{T_\alpha(t)x - x}{t} - y \right\|_\alpha = \left\| \frac{T(t)(A^\alpha x) - (A^\alpha x)}{t} - A^\alpha y \right\|. \tag{4.1.4}$$

First assume that $\alpha \geq 0$. Then (4.1.4) converges to 0 if and only if $A^\alpha x \in X^1$ (i.e. $x \in X^{\alpha+1}$; cf. Lemma 4.1.1(a)) and $A^\alpha y = A(A^\alpha x) = A^{\alpha+1}x$, thus $y = Ax$. This shows (e). If $\alpha < 0$ then (4.1.4) converges to 0 if and only if $A^{\alpha+1}x = A^\alpha y$, thus $x = A^{-1}y$. Therefore A is the part of A_α in X. We already have shown that X^1 is dense in X^α and invariant under T_α. By Nelson's Lemma X^1 is a core for A_α. Hence A_α is the closure of A in X^α. As the graph norm of A_α on X^1 is equivalent to $\|\cdot\|_{\alpha+1}$ and X^1 is dense in $X^{\alpha+1}$ we see that $D(A_\alpha) = X^{\alpha+1}$. This shows (f).

Let $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$. From (d) and (e) it is clear that A_β is the part of A_α in X^β and A_α is the unique continuous extension of A_β to a bounded operator from $X^{\alpha+1}$ to X^α. It remains to show that A_α is an isomorphism. To this end we first observe that by (4.1.3) the C_0-semigroups T_α have negative growth bound for all $\alpha \in \mathbb{R}$. Therefore A_α is a bijective mapping from $X^{\alpha+1}$ to X^α. As $\|A_\alpha x\|_\alpha = \|A^{\alpha+1}x\| = \|x\|_{\alpha+1}$ ($x \in X^{\alpha+1}$) we see that A_α is isometric. ■

In applications we need iterated fractional power spaces. For the corresponding result on iterated abstract Hölder spaces we refer to [39; Proposition II.5.35]. We note that in contrast to abstract Hölder spaces the iteration of fractional power spaces works for all orders (cf. Theorem 4.1.4). This is in fact a crucial advantage which will allow us to derive new perturbation theorems from known ones. First we provide an embedding lemma for iterated fractional power spaces. We write A_α^β as an abbreviation for $(A_\alpha)^\beta$. We also omit the generator in the index of fractional power spaces whenever we think that the corresponding generator is clear from the context.

4.1.3 Lemma. *Let A be the generator of a C_0-semigroup on a Banach space X with negative growth bound. Further let $\alpha, \beta, \gamma \in \mathbb{R}$. If one of the following two conditions is met, then X_A^γ is dense in $(X_A^\alpha)_{A_\alpha}^\beta$.*
(a) $\beta \leq 0$ and $\gamma \geq \alpha$.
(b) $\beta \geq 0$ and $\gamma \geq \max\{\beta, \alpha+\beta\}$.

Proof. If assumption (a) holds, then $(X^\alpha)^\beta$ is the completion of X^α with respect to the norm $\|\cdot\|_{(X^\alpha)^\beta}$. Therefore it suffices to show that X^γ is dense in X^α with respect to $\|\cdot\|_{(X^\alpha)^\beta}$. To this end let $x \in X^\alpha$. As X^γ is dense in X^α there exists a sequence $(x_n) \subseteq X^\gamma$ such that $x_n \to x$ in X^α. The continuity of A_α^β implies that $A_\alpha^\beta x_n \to A_\alpha^\beta x$ in X^α. This is equivalent to $x_n \to x$ with respect to $\|\cdot\|_{(X^\alpha)^\beta}$ and shows the assertion for assumption (a).

Now suppose that (b) holds. First we show that $X^\gamma \subseteq (X^\alpha)^\beta$. To this end let $x \in X^\gamma$ and $y \in X$ such that $x = A^{-\gamma}y$. From $x = A^{-\beta}(A^{\beta-\gamma}y)$ and $\beta - \gamma \leq \min\{0, -\alpha\}$ we conclude that $A^{\beta-\gamma}y \in X^\alpha$ and thus $x \in (X^\alpha)^\beta$. Next let $x \in (X^\alpha)^\beta$ and $y \in X^\alpha$ such that $x = A_\alpha^{-\beta}y$. The denseness of $X^{\gamma-\beta}$ in X^α allows us to choose a sequence $(y_n) \subseteq X^{\gamma-\beta}$ such that $y_n \to y$ in X^α. As $A_\alpha^{-\beta}$ is continuous on X^α we infer that $X^\gamma \ni A_\alpha^{-\beta}y_n \to A_\alpha^{-\beta}y = x$. Thus X^γ is densely embedded in $(X^\alpha)^\beta$. ∎

For the following theorem we also refer to [3; Theorem V.1.5.4].

4.1.4 Theorem. *Let A be the generator of a C_0-semigroup on a Banach space X with negative growth bound and $\alpha, \beta \in \mathbb{R}$. Then*

$$(X_A^\alpha)_{A_\alpha}^\beta = X_A^{\alpha+\beta} \quad \text{and} \quad (A_\alpha)_\beta = A_{\alpha+\beta}.$$

Proof. Let $\gamma := \max\{0, \alpha, \beta, \alpha+\beta\}$. By Lemma 4.1.3 we know that X^γ is dense in $(X^\alpha)^\beta$. So it suffices to show that the norms $\|\cdot\|_{X^{\alpha+\beta}}$ and $\|\cdot\|_{(X^\alpha)^\beta}$ agree on X^γ. This is done by computing

$$\|x\|_{(X^\alpha)^\beta} = \|A_\alpha^\beta x\|_\alpha = \|A^\beta x\|_\alpha = \|A^{\alpha+\beta}x\| = \|x\|_{\alpha+\beta} \quad (x \in X^\gamma).$$

In order to prove the equality of the two generators we first observe that A_α and A coincide on the domain $X^1 \cap X^{\alpha+1}$. Therefore $(A_\alpha)_\beta$ and A agree on $X^1 \cap (X^\alpha)^{\beta+1}$. This implies that $(A_\alpha)_\beta$ and $A_{\alpha+\beta}$ agree on $X^1 \cap X^{\alpha+\beta+1}$. As this is a core for both operators we obtain the equality. ∎

We mention that if the norm on X is replaced by an equivalent norm then the fractional power spaces only change by equivalent norms. We also point out that $X^\alpha_{A-\lambda} = X^\alpha_A$ with equivalent norms for all $\lambda \geq 0$. We therefore always write X^α_A, even if the semigroup generated by A does not have negative growth bound. The particular choice of λ will not be relevant in the situations we shall encounter except for a few estimates where we will mention the λ being used.

For the applications we have in mind we need the extrapolated Favard space $F^0_{A_\alpha}$ of A_α, which we will denote by F^α_A (see (A.1) in the appendix for details). (The space F^α_A is not to be confused with the Favard space of fractional order α.)

4.1.5 Proposition. *Let A be the generator of a C_0-semigroup T on X with growth bound less than $\omega \in \mathbb{R}$. Let $\alpha \in \mathbb{R}$. For the extrapolated Favard space F^α_A of A_α the equality $F^\alpha_A = (A_{\min\{-1,\alpha-1\}} - \omega)^{-\alpha} F^0_A$ holds.*

Proof. Without loss of generality we assume that T has negative growth bound and $\omega = 0$. Let $x \in X^{\alpha-1}$ and $y := A^\alpha_{\min\{-1,\alpha-1\}} x \in X^{-1}$. The assertion now follows from the bijectivity of $A^\alpha_{\min\{-1,\alpha-1\}}$ from $X^{\alpha-1}$ to X^{-1}, the definition of the Favard space (A.1) and

$$\begin{aligned} \|\lambda R(\lambda, A_{\alpha-1})x\|_\alpha &= \|\lambda A_{\min\{0,\alpha\}} R(\lambda, A_{\alpha-1})x\| \\ &= \|\lambda R(\lambda, A_{-1}) A^\alpha_{\min\{-1,\alpha-1\}} x\| = \|\lambda R(\lambda, A_{-1}) y\|. \end{aligned}$$ ■

Last we show that for an operator $A \in \mathcal{K}(X)$ and $0 \leq \alpha < \beta$, the operators $(A-r)^\alpha (A-s)^{-\beta}$ are bounded, uniformly for $0 \leq r \leq s$. To this end let Σ be a suitable open sector and $M \geq 0$ such that $\|R(\lambda, A)\| \leq \frac{M}{1+|\lambda|}$ $(\lambda \in \Sigma)$. For $0 \leq \alpha < \beta$ and $0 \leq r \leq s$ we define

$$A(\alpha, \beta, r, s) := \frac{1}{2\pi i} \int_\gamma \frac{\lambda^\alpha}{(\lambda - (s-r))^\beta} R(\lambda + r, A)\, d\lambda, \tag{4.1.5}$$

where γ is a path as in Figure 1. As $\alpha < \beta$ and $\|R(\lambda, A)\| \leq \frac{M}{1+|\lambda|}$ for some $M \geq 0$ and $\lambda \in \Sigma$ the integral exists. Moreover, by Cauchy's integral theorem the expression is independent of the particular choice of γ.

4.1.6 Lemma. *Let $A \in \mathcal{K}(X)$, $0 \leq \alpha < \beta$ and $0 \leq r \leq s$. Then*

$$A(\alpha, \beta, r, s) = (A-r)^\alpha (A-s)^{-\beta}. \tag{4.1.6}$$

Proof. First we choose paths γ_1 and γ_2 in $\Sigma \setminus \mathbb{R}_+$, such that γ_1 lies to the right of γ_2. Replacing A with $A - r$ we can assume that $r = 0$ without loss of generality. We start by computing

$$\begin{aligned}
A^{-\alpha}A(\alpha,\beta,0,s) &= \frac{1}{(2\pi i)^2}\int_{\gamma_1}\int_{\gamma_2}\mu^{-\alpha}\frac{\lambda^\alpha}{(\lambda-s)^\beta}R(\mu,A)R(\lambda,A)\,d\lambda\,d\mu \\
&= \frac{1}{(2\pi i)^2}\int_{\gamma_1}\int_{\gamma_2}\mu^{-\alpha}\frac{\lambda^\alpha}{(\lambda-s)^\beta}\left[\frac{R(\mu,A)}{\lambda-\mu}+\frac{R(\lambda,A)}{\mu-\lambda}\right]d\lambda\,d\mu \\
&= \frac{1}{2\pi i}\int_{\gamma_1}\mu^{-\alpha}\left[\frac{1}{2\pi i}\int_{\gamma_2}\frac{\lambda^\alpha}{(\lambda-s)^\beta}\frac{1}{\lambda-\mu}\,d\lambda\right]R(\mu,A)\,d\mu \\
&\quad+\frac{1}{2\pi i}\int_{\gamma_2}\frac{\lambda^\alpha}{(\lambda-s)^\beta}\left[\frac{1}{2\pi i}\int_{\gamma_1}\frac{\mu^{-\alpha}}{\mu-\lambda}\,d\mu\right]R(\lambda,A)\,d\lambda \\
&= \frac{1}{2\pi i}\int_{\gamma_2}\frac{\lambda^\alpha}{(\lambda-s)^\beta}\lambda^{-\alpha}R(\lambda,A)\,d\lambda \\
&= \frac{1}{2\pi i}\int_{\gamma_2-s}\lambda^{-\beta}R(\lambda,A-s)\,d\lambda = (A-s)^{-\beta}.
\end{aligned}$$

Here we have used that by Cauchy's integral theorem we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\gamma_2}\frac{\lambda^\alpha}{(\lambda-s)^\beta}\frac{1}{\lambda-\mu}\,d\lambda &= 0 \quad (\mu\in\gamma_1), \\
\frac{1}{2\pi i}\int_{\gamma_1}\frac{\mu^{-\alpha}}{\mu-\lambda}\,d\mu &= \lambda^{-\alpha} \quad (\lambda\in\gamma_2).
\end{aligned}$$

From $A^{-\alpha}A(\alpha,\beta,0,s)=(A-s)^{-\beta}$ and the definition of A^α as the inverse of $A^{-\alpha}$ we obtain the assertion. ■

4.1.7 Proposition. *Let $A\in\mathcal{K}(X)$ and $0\le\alpha<\beta$. Then there exists $K\ge 0$, depending only on α and β, such that*

$$\|(A-r)^\alpha(A-s)^{-\beta}\|\le K \quad (0\le r\le s).$$

Proof. We use (4.1.6) and the integral definition (4.1.5) of $A(\alpha,\beta,r,s)$ to show the assertion. To this end we choose $\delta>0$ and $m>0$, both sufficiently small, such that $\gamma_{1/2}(x):=x\pm im(x+\delta)$ ($x\in[-\delta,\infty)$) are in Σ. Then for $\lambda\in\gamma_{1/2}(\mathbb{R}_+)$ we obtain (with $s':=s-r$)

$$\left|\frac{\lambda^\alpha}{(\lambda-s')^\beta}\right| = |\lambda|^{\alpha-\beta}\left|\frac{\lambda}{\lambda-s'}\right|^\beta \le |\lambda|^{\alpha-\beta}\left(\frac{|\lambda|}{|\operatorname{Im}\lambda|}\right)^\beta \le |\lambda|^{\alpha-\beta}\left(\frac{m^2+1}{m^2}\right)^{\beta/2}$$

and $\|R(\lambda + r, A)\| \le \frac{M}{1+|\lambda+r|} \le \frac{M}{1+|\lambda|}$, whereas for $\lambda \in \gamma_{1/2}([-\delta, 0))$ we have

$$\left|\frac{\lambda^\alpha}{(\lambda - s)^\beta}\right| \le |\lambda|^{\alpha-\beta} \left(\frac{|\lambda|}{|\lambda|}\right)^\beta = |\lambda|^{\alpha-\beta}$$

and $\|R(\lambda + r, A)\| \le M'$ for some $M' \ge 0$ independent of r. Therefore the norm of the integrand in (4.1.5) is bounded uniformly in s and r by an integrable function for the path consisting of $\gamma_1([-\delta, \infty))$ and $\gamma_2([-\delta, \infty))$. Thus the operators $A(\alpha, \beta, r, s)$ are bounded uniformly in s and r. ■

4.1.8 Remark. We were not able to show the assertion in Proposition 4.1.7 for $\alpha = \beta$. This improvement would at least simplify the proof of Lemma 4.3.1.

4.2 Preliminary Estimates

Throughout this section we assume that A and C are operators in $\mathcal{K}(X)$ for some Banach space X. Let $M \ge 0$ such that $\|R(s, A)\|, \|R(s, C)\| \le M(1+s)^{-1}$ for all $s \ge 0$. Further we assume that there are $\gamma_1 \in [0, 1)$ and $\gamma_2 \in (-1, 0]$ with $\gamma_1 - \gamma_2 < 1$, and a bounded operator $B\colon X_A^{\gamma_1} \to X_A^{\gamma_2}$ such that $C = (A_{\gamma_2} + B)_{|X}$ (here the index "$|X$" denotes the part of $A_{\gamma_2} + B$ in X). We also define the important quantity $\Gamma := 1 - \gamma_1 + \gamma_2$, which is by assumption strictly positive.

4.2.1 Lemma. *Let Y be a Banach space. Let $E \in \mathcal{K}(Y)$, $\alpha, \beta \ge 0$ and $\varepsilon > 0$ such that $\alpha + \varepsilon < \beta$. There exists $K \ge 0$, only depending on α, β and ε, such that the following assertions hold.*

(a) $\|(E - s)^{-\alpha}\| \le K(1 + s)^{-\alpha} \quad (0 \le s)$.

(b) $\|(E - r)^\alpha (E - s)^{-\beta}\| \le K(1 + s - r)^{\alpha+\varepsilon-\beta} \quad (0 \le r \le s)$.

Proof. Let $M' \ge 0$ be such that $\|R(\lambda, E)\| \le \frac{M'}{1+\lambda}$ for all $\lambda \ge 0$. For $\alpha \in \mathbb{N} \cup \{0\}$ the first statement is a well-known fact. For $\alpha \in (0, 1)$ we verify it by computing (using (4.1.1))

$$\begin{aligned}\|(E - s)^{-\alpha}\| &\le |c_\alpha| \int_0^\infty r^{-\alpha} \|R(r + s, E)\|\, dr \\ &\le |c_\alpha| M' \int_0^\infty r^{-\alpha}(1 + r + s)^{-1}\, dr = |c_\alpha| M'(1 + s)^{-\alpha} \int_0^\infty t^{-\alpha}(1 + t)^{-1}\, dt,\end{aligned}$$

where in the last step we have used the substitution $t = r/(1 + s)$. If $\alpha = k + \alpha_0$ with $k \in \mathbb{N}$ and $\alpha_0 \in (0, 1)$ assertion (a) follows from $(E - s)^{-\alpha} = (E - s)^{-k}(E - s)^{-\alpha_0}$ and the estimates above.

The second assertion follows from the first one and Proposition 4.1.7 by writing

$$(E - r)^\alpha (E - s)^{-\beta} = (E - r)^\alpha (E - s)^{-\alpha-\varepsilon}(E - s)^{\alpha+\varepsilon-\beta}. \quad ■$$

For later use we single out the following consequence of Lemma 4.2.1 (it also follows from [39; Proposition II.5.33 and Lemma III.2.13]).

4.2.2 Lemma. *Let Y be a Banach space and $E \in \mathcal{K}(Y)$. Let $\gamma \in [0,1)$ and $B\colon Y_E^\gamma \to Y$ be a bounded operator. Then B has E-bound 0.*

Proof. Let $\beta \in (\gamma, 1)$ arbitrary. The assertion follows from Lemma 4.2.1 and

$$\|Bx\| \le \|BE^{-\gamma}\|_{\mathcal{L}(Y)} \|E^\gamma (E-\lambda)^{-\beta}\|_{\mathcal{L}(Y)} \|(E-\lambda)^{\beta-1}\|_{\mathcal{L}(Y)} \|(E-\lambda)x\|$$

which holds for all $x \in D(E)$ and $\lambda \ge 0$. ■

Our next aim is a formula for the resolvents of C in terms of A and B. For $\lambda \ge 0$, $0 \le \varepsilon < \Gamma/2$ and $\delta \in [-\gamma_2, 1-\gamma_1]$ we define the operators

$$\begin{aligned} G_{\lambda,\varepsilon} &:= (A_{\gamma_2} - \lambda)^{\gamma_2 - \varepsilon}\, B\, (A-\lambda)^{-\gamma_1-\varepsilon}, \\ H_{\lambda,\delta} &:= -(A_{-\delta} - \lambda)^{-\delta}\, B\, (A-\lambda)^{\delta-1} \end{aligned}$$

on X. These operators are bounded by the assumptions on B. From Lemma 4.2.1(b) and

$$\begin{aligned} G_{\lambda,\varepsilon} &= \left[(A_{\gamma_2}-\lambda)^{\gamma_2-\varepsilon} A_{\gamma_2}^{-\gamma_2}\right] G_{0,0} \left[A^{\gamma_1}(A-\lambda)^{-\gamma_1-\varepsilon}\right] \\ &= \left[A^{-\gamma_2}(A-\lambda)^{\gamma_2-\varepsilon}\right] G_{0,0} \left[A^{\gamma_1}(A-\lambda)^{-\gamma_1-\varepsilon}\right] \end{aligned}$$

we see that $G_{\lambda,\varepsilon}$ are uniformly bounded in $\lambda \ge 0$, provided $\varepsilon > 0$. Therefore if $\delta \in (-\gamma_2+\varepsilon, 1-\gamma_1-\varepsilon)$ there exists $K \ge 0$, depending only on ε such that

$$\|H_{\lambda,\delta}\| = \|(A-\lambda)^{-\delta-\gamma_2+\varepsilon}\, G_{\lambda,\varepsilon}\, (A-\lambda)^{\delta-1+\gamma_1+\varepsilon}\| \le K(1+\lambda)^{2\varepsilon-\Gamma}. \tag{4.2.1}$$

As $2\varepsilon - \Gamma < 0$ we conclude $H_{\lambda,\delta} \to 0$ as $\lambda \to \infty$. Hence $I - H_{\lambda,\delta}$ becomes invertible for λ sufficiently large.

For $\delta \in [-\gamma_2, 1-\gamma_1]$ and $\lambda \ge 0$ the operators $\tilde{C}_{-\delta} := A_{-\delta} + B$ on $X_A^{-\delta}$ with domain $X_A^{1-\delta}$ satisfy

$$\begin{aligned} \tilde{C}_{-\delta} &= A_{-\delta}^{\delta}(I - H_{0,\delta})A^{1-\delta}, \\ C &= A^{\delta}(I - H_{0,\delta})A^{1-\delta}, \\ \lambda - \tilde{C}_{-\delta} &= -(A_{-\delta}-\lambda)^{\delta}\,(I - H_{\lambda,\delta})\,(A-\lambda)^{1-\delta}. \end{aligned} \tag{4.2.2}$$

We already have seen that $I - H_{\lambda,\delta}$ is a bijective operator on X for λ sufficiently large and $\delta \in (-\gamma_2, 1-\gamma_1)$. Moreover $(A_{-\delta}-\lambda)^{\delta}$ and $(A-\lambda)^{1-\delta}$ are bijective mappings from X to $X_A^{-\delta}$ and from $X_A^{1-\delta}$ to X, respectively. Hence $\lambda - \tilde{C}_{-\delta}$ is a bijective mapping from $X_A^{1-\delta}$ to $X_A^{-\delta}$ and the resolvent is given by

$$R(\lambda, \tilde{C}_{-\delta}) = -(A-\lambda)^{\delta-1}(I - H_{\lambda,\delta})^{-1}(A_{-\delta}-\lambda)^{-\delta}. \tag{4.2.3}$$

This almost proves the following representation formula for the resolvent of C.

4.2.3 Proposition. *Let $\delta \in (-\gamma_2, 1-\gamma_1)$. There exists $h \geq 0$ (depending on δ) such that for all $\lambda \geq h$ the resolvent $R(\lambda, C)$ is given by*

$$R(\lambda, C) = -(A-\lambda)^{\delta-1}(I - H_{\lambda,\delta})^{-1}(A-\lambda)^{-\delta}. \tag{4.2.4}$$

Proof. We know that $\lambda - C$ is invertible for all $\lambda \geq 0$. As $\lambda - C$ is the part of $\lambda - \tilde{C}_{-\delta}$ in X, which means that the graph of $\lambda - C$ is the graph of $\lambda - \tilde{C}_{-\delta}$ restricted to the space $X \times X$, we see that $R(\lambda, C)$ is the part of $R(\lambda, \tilde{C}_{-\delta})$ in X. Let $h \geq 0$ such that $H_{\lambda,\delta}$ is invertible for all $\lambda \geq h$. The proof is done by observing that for $\lambda \geq h$ the part of $R(\lambda, \tilde{C}_{-\delta})$ in X is obviously given by (4.2.4). ■

Formula (4.2.3) for the resolvent of $\tilde{C}_{-\delta}$ implies that there is a $K \geq 0$ such that for all $\lambda \geq h$ we have $\|R(\lambda, \tilde{C}_{-\delta})\| \leq K(1+\lambda)^{-1}$. This means that $\tilde{C}_{-\delta} - h$ belongs to $\mathcal{K}(X_A^{-\delta})$. We will use fractional powers of this operator in the next section.

We now turn our attention to the difference of the resolvents $R(\lambda, C)$ and $R(\lambda, A)$. We recall that for a bounded perturbation B the norm of this difference can be estimated by $K(1+\lambda)^{-2}$ for some $K \geq 0$ and all $\lambda \geq 0$ (cf. Remarks 4.5.2(a)).

4.2.4 Lemma. *Let $\delta \in (-\gamma_2, 1-\gamma_1)$. Let $h \geq 0$ be as in Proposition 4.2.3. For $\lambda \geq h$ the following assertions hold.*

(a) $R(\lambda, C) - R(\lambda, A)$ maps X into X^{γ_2+1} and

$$R(\lambda, C) - R(\lambda, A) = -(A-\lambda)^{\delta-1}H_{\lambda,\delta}(1 - H_{\lambda,\delta})^{-1}(A-\lambda)^{-\delta}. \tag{4.2.5}$$

(b) Let $\beta_1 \in [0, \delta]$, $\beta_2 \in [0, \gamma_2+1)$. For $\varepsilon > 0$ there exist $K, L \geq 0$ such that

$$\|(R(\lambda, \tilde{C}_{-\delta}) - R(\lambda, A_{-\delta}))(A_{-\delta} - \lambda)^{\beta_1}\| \leq K(1+\lambda)^{\beta_1 - 1 - \Gamma + \varepsilon}, \tag{4.2.6}$$

$$\|(A-\lambda)^{\beta_2}(R(\lambda, C) - R(\lambda, A))\| \leq L(1+\lambda)^{\beta_2 - 1 - \Gamma + \varepsilon}. \tag{4.2.7}$$

Proof. Equation (4.2.5) is obtained by computing

$$\begin{aligned} R(\lambda, C) - R(\lambda, A) &= -(A-\lambda)^{\delta-1}(I - H_{\lambda,\delta})^{-1}(A-\lambda)^{-\delta} + (A-\lambda)^{(\delta-1)-\delta} \\ &= -(A-\lambda)^{\delta-1}H_{\lambda,\delta}(1 - H_{\lambda,\delta})^{-1}(A-\lambda)^{-\delta}. \end{aligned}$$

Using

$$(A-\lambda)^{\delta-1}H_{\lambda,\delta} = -(A-\lambda)^{-1-\gamma_2}G_{\lambda,0}(A-\lambda)^{\delta-1+\gamma_1}$$

we infer that the range of $R(\lambda, C) - R(\lambda, A)$ is contained in X^{γ_2+1}. In order to show (4.2.6) we choose $\varepsilon \in (0, (\delta+\gamma_2)/2)$ and compute

$$\begin{aligned} &(R(\lambda, \tilde{C}_{-\delta}) - R(\lambda, A_{-\delta}))(A_{-\delta} - \lambda)^{\beta_1} \\ &\qquad = -(A-\lambda)^{\delta-1}(1 - H_{\lambda,\delta})^{-1}H_{\lambda,\delta}(A_{-\delta} - \lambda)^{-\delta}(A_{-\delta} - \lambda)^{\beta_1} \\ &\qquad = (A-\lambda)^{\delta-1}\,(1 - H_{\lambda,\delta})^{-1}\,(A-\lambda)^{-\delta-\gamma_2+\varepsilon}\,G_{\lambda,\varepsilon}\,(A-\lambda)^{\beta_1+\gamma_1-1+\varepsilon} \end{aligned}$$

Using Lemma 4.2.1(a) we obtain (4.2.6). The estimate (4.2.7) is derived similarly. ■

4.3 Perturbation of the Fractional Power Tower

In this section we investigate the stability of the fractional power spaces while perturbing the underlying semigroup generator. Besides the iteration property the stability properties will be our main tool in the remaining sections.

4.3.1 Lemma. *Let A and C be generators of C_0-semigroups on a Banach space X with negative growth bound. Assume that there exists $\gamma_1 \in [0,1)$, $\gamma_2 \in (-1,0]$ with $\Gamma := 1-\gamma_1+\gamma_2 > 0$, and an operator $B\colon X_A^{\gamma_1} \to X_A^{\gamma_2}$ such that $C = (A_{\gamma_2}+B)_{|X}$. Then the followings assertions hold.*

(a) Let $h \geq 0$ such that (4.2.7) holds for all $\lambda \geq h$. If $\alpha \in [0, \gamma_2+1)$ then $X_C^\alpha \subseteq X_A^\alpha$ and $(A-r)^\alpha(C-r)^{-\alpha}$ are bounded uniformly in $r \geq h$.

(b) If $\alpha \in [0, 1-\gamma_1)$, $\delta \in (-\gamma_2, 1-\gamma_1) \cap [\alpha, 1-\gamma_1)$ and $h \geq 0$ such that (4.2.6) holds for all $\lambda \geq h$, then $(\tilde{C}_{-\delta}-r)^{-\alpha}(A_{-\delta}-r)^\alpha$ is bounded on X uniformly in $r \geq h$.

(c) If $\alpha \in [0, \gamma_2+1)$ then $X_A^\alpha \subseteq X_C^\alpha$ and $(C-h)^\alpha(A-h)^{-\alpha}$ is bounded for $h \geq 0$ sufficiently large.

(d) If $\alpha \in [0, 1-\gamma_1)$, $\delta \in (-\gamma_2, 1-\gamma_1) \cap [\alpha, 1-\gamma_1)$ and $h \geq 0$ such that (4.2.6) holds for all $\lambda \geq h$, then $(A_{-\delta}-h)^{-\alpha}(\tilde{C}_{-\delta}-h)^\alpha$ is bounded on X.

Proof. In order to show (a) we first write

$$\begin{aligned}(C-r)^{-\alpha} &= (A-r)^{-\alpha} + ((C-r)^{-\alpha} - (A-r)^{-\alpha}) \\ &= (A-r)^{-\alpha} + c_\alpha \int_0^\infty s^{-\alpha}(R(s+r,C) - R(s+r,A))\,ds.\end{aligned} \tag{4.3.1}$$

From (4.2.7) and Lemma 4.2.1(b) we infer that for $\varepsilon > 0$ sufficiently small

$$\begin{aligned}I + (A-r)^\alpha\, c_\alpha \int_0^t s^{-\alpha}(R(s+r,C) - R(s+r,A))\,ds \\ = I + c_\alpha \int_0^t s^{-\alpha}(A-r)^\alpha(A-s-r)^{-\alpha-\varepsilon}(A-s-r)^{\alpha+\varepsilon} \\ (R(s+r,C) - R(s+r,A))\,ds\end{aligned}$$

is a bounded operator on X and converges in operator norm as $t \to \infty$. The closedness of $(A-r)^\alpha$ implies that for $x \in X$ we have $(C-r)^{-\alpha}x \in X_A^\alpha$ and thus $X_C^\alpha \subseteq X_A^\alpha$. From

$$(A-r)^\alpha(C-r)^{-\alpha} = I + c_\alpha \int_0^\infty s^{-\alpha}(A-r)^\alpha(R(s+r,C) - R(s+r,A))\,ds,$$

where the integral exists in operator norm, we see that $(A-r)^\alpha(C-r)^{-\alpha}$ are bounded uniformly in $r \geq h$.

Assertion (b) immediately follows from (4.2.6) and Lemma 4.2.1(b) if we write

$$\begin{aligned}(\tilde{C}_{-\delta} - r)^{-\alpha}(A_{-\delta} - r)^{\alpha} \\ &= I + c_\alpha \int_0^\infty s^{-\alpha}(R(s+r, \tilde{C}_{-\delta}) - R(s+r, A_{-\delta})) \\ &\qquad (A_{-\delta} - s - r)^{\alpha+\varepsilon}(A_{-\delta} - s - r)^{-\alpha-\varepsilon}(A_{-\delta} - r)^{\alpha}\, ds,\end{aligned}$$

for some $\varepsilon > 0$ sufficiently small.

The proof of (c) requires more labour. First we choose an arbitrary $\delta \in (-\gamma_2, 1-\gamma_1)$ and $h \geq 0$ such that (4.2.6) and (4.2.7) hold. As in (4.3.1) we start by writing

$$(A - h)^{-\alpha} = (C - h)^{-\alpha} + c_\alpha \int_0^\infty s^{-\alpha}(R(s+h, A) - R(s+h, C))\, ds.$$

We need to reason that $(C-h)^{\alpha}(R(s+h, A) - R(s+h, C))$ is a bounded operator on X and that

$$I + c_\alpha \int_0^t s^{-\alpha}(C-h)^{\alpha}(R(s+h, A) - R(s+h, C))\, ds \tag{4.3.2}$$

converges in operator norm as $t \to \infty$. To this end we expand

$$\begin{aligned}R(s+h, A) - R(s+h, C) \\ &= (\tilde{C}_{-\delta} - s - h)^{\delta-1}\left[(\tilde{C}_{-\delta} - s - h)^{-\delta}(A_{-\delta} - s - h)^{\delta}\right] \\ &\qquad (I - H_{s+h,\delta})\left[(A - s - h)^{1-\delta}(R(s+h, A) - R(s+h, C))\right],\end{aligned} \tag{4.3.3}$$

where we have used

$$\tilde{C}_{-\delta} - s - h = (A_{-\delta} - s - h)^{\delta}(I - H_{s+h,\delta})(A - s - h)^{1-\delta}.$$

By (b) and (4.2.7) the second, third and fourth bracketed expression in (4.3.3) are bounded operators on X. Thus we can restrict the operator $(\tilde{C}_{-\delta} - s - h)^{\delta-1}$ in this composition to the space X. As $R(\lambda, C)$ is the part of $R(\lambda, \tilde{C}_{-\delta})$ in X we see that the restriction of $(\tilde{C}_{-\delta} - s - h)^{\delta-1}$ is $(C - s - h)^{\delta-1}$. As $(C - s - h)^{\delta-1}$ maps X into $X_C^{1-\delta}$ and $1 - \delta > \alpha$ we conclude that $R(s+h, A) - R(s+h, C)$ maps X into X_C^{α}. We infer that $(C-h)^{\alpha}(R(s+h, A) - R(s+h, C))$ is a bounded operator.

In order to derive assertion (c) with a closedness argument as in (a) we now need to show that the integral (4.3.2) exists in operator norm. To this end we use (4.3.3) to rewrite the integrand of (4.3.2) as

$$\begin{aligned}s^{-\alpha}(C-h)^{\alpha}(R(s+h, A) - R(s+h, C)) \\ &= s^{-\alpha}\left[(C-h)^{\alpha}(C - s - h)^{\delta-1}\right]\left[(\tilde{C}_{-\delta} - s - h)^{-\delta}(A_{-\delta} - s - h)^{\delta}\right] \\ &\qquad (I - H_{s+h,\delta})\left[(A - s - h)^{1-\delta}(R(s+h, A) - R(s+h, C))\right].\end{aligned}$$

We already have shown that $(\tilde{C}_{-\delta} - s - h)^{-\delta}(A_{-\delta} - s - h)^{\delta}$ are uniformly bounded in $s \geq 0$. Applying the usual suspects Lemma 4.2.1(b) and (4.2.7) we obtain the integrable bound $Ks^{-\alpha}(1+s)^{\alpha-1-\Gamma+\varepsilon}$ for some $\varepsilon \in (0, \Gamma)$ and $K \geq 0$. We now infer (c) as in (a) by a closedness argument.

The proof of (d) is done very similarly to (c), so we only sketch it. We start with

$$(A_{-\delta} - h)^{-\alpha}(\tilde{C}_{-\delta} - h)^{\alpha} = I + c_\alpha \int_0^\infty s^{-\alpha}(R(s+h, A_{-\delta}) - R(s+h, \tilde{C}_{-\delta}))(\tilde{C}_{-\delta} - h)^{\alpha}\, ds,$$

and then rewrite the integrand as

$$\begin{aligned} &s^{-\alpha}(R(s+h, A_{-\delta}) - R(s+h, \tilde{C}_{-\delta}))(\tilde{C}_{-\delta} - h)^{\alpha} \\ &\quad = s^{-\alpha}\left[(R(s+h, A_{-\delta}) - R(s+h, \tilde{C}_{-\delta}))(A_{-\delta} - s - h)^{\delta}\right](I - H_{s+h,\delta}) \\ &\qquad \left[(A - s - h)^{1-\delta}(C - s - h)^{\delta-1}\right]\left[(\tilde{C}_{-\delta} - s - h)^{-\delta}(\tilde{C}_{-\delta} - h)^{\alpha}\right]. \end{aligned}$$

First observe that by assumption $1 - \delta \in (\gamma_1, \min\{\gamma_2 + 1, 1 - \alpha\}) \subseteq [0, \gamma_2 + 1)$. Thus we can apply (a) to see that $(A - s - h)^{1-\delta}(C - s - h)^{\delta-1}$ is bounded uniformly in $s \geq 0$. The remaining steps of the proof of (d) are now done as for (c), except for the fact that (4.2.6) has to be invoked instead of (4.2.7). This yields the different restriction on α compared to (c). ■

After this trudge through tedious estimates we are ready to proof our main tool.

4.3.2 Theorem. *Let A and C be generators of C_0-semigroups on X with negative growth bound. Assume that there exists $B\colon X_A^{\gamma_1} \to X_A^{\gamma_2}$, with $-1 < \gamma_2 \leq \gamma_1 < 1$ and $\gamma_1 - \gamma_2 < 1$, such that*

$$C = (A_{-\delta} + B)_{|X} = A^{\delta}(I + A_{-\delta}^{-\delta} B A^{\delta-1})A^{1-\delta}$$

where $\delta := -\min\{0, \gamma_2\}$. Then $X_A^\alpha = X_C^\alpha$ with equivalent norms for all $\alpha \in (\gamma_1 - 1, \gamma_2 + 1)$.

Proof. As $X_A^\alpha = X_{A-h}^\alpha$ with equivalent norms for any $h \geq 0$ (and similarly $X_C^\alpha = X_{C-h}^\alpha$) it suffices to show $X_{A-h}^\alpha = X_{C-h}^\alpha$ for some $h \geq 0$ sufficiently large.

First assume that $\alpha \in (-1, 1)$. Then without loss of generality we can assume that $\gamma_1 \geq 0$ and $\gamma_2 \leq 0$. If $\alpha \geq 0$ the assertion follows immediately from (a) and (c) of Lemma 4.3.1, whereas for $\alpha < 0$ we infer the assertion from (b) and (d).

Next we suppose that $\gamma_2 > 0$ and $\alpha \in [1, \gamma_2 + 1)$. We do the proof in two steps using the iteration property of fractional power spaces (cf. Theorem 4.1.4). To this end we set $\alpha' := \gamma_2$, $\alpha'' := \alpha - \alpha'$, $\gamma_1' = \gamma_1$, $\gamma_1'' := \gamma_1 - \alpha'$, $\gamma_2' := 0$ and $\gamma_2'' := \gamma_2 - \alpha' = 0$. As B is a bounded operator from $X_A^{\gamma_1'}$ to $X_A^{\gamma_2'}$ we first observe that $Y := X_A^{\alpha'} = X_C^{\alpha'}$ with equivalent norms by the first part of this proof. Now B becomes a bounded operator from $Y^{\gamma_1''}_{A_{\alpha'}} \to Y^{\gamma_2''}_{A_{\alpha'}}$. As $\alpha'' \in [0, \gamma_2'' + 1)$ we see that

$$X^\alpha_A = (X^{\alpha'}_A)^{\alpha''}_{A_{\alpha'}} = Y^{\alpha''}_{A_{\alpha'}} = Y^{\alpha''}_{C_{\alpha'}} = (X^{\alpha'}_C)^{\alpha''}_{C_{\alpha'}} = X^\alpha_C.$$

The case $\gamma_1 < 0$ and $\alpha \in (\gamma_1 - 1, -1]$ is done similarly. ∎

4.3.3 Remark. The assumption $\Gamma > 0$ (i.e. $\gamma_1 - \gamma_2 < 1$) in Theorem 4.3.2 cannot be dropped. For example let A be the generator of a C_0-semigroup with negative growth bound and assume that A is unbounded. Let $B := -A_\gamma$ with $\gamma \in [-1, 0]$. Then $B \in \mathcal{L}(X^{\gamma+1}, X^\gamma)$, and therefore $\Gamma = 0$. As $C := (A_* + B)_{|X} = 0$ we see that the spaces X^α_A and X^α_C coincide only for $\alpha = 0$.

4.3.4 Corollary. *Let A, B and C be as in Theorem 4.3.2. If $\alpha \in (\gamma_1 - 1, \gamma_2 + 1)$ then $C_\alpha = (A_{-\delta} + B)_{|X^\alpha_A}$ is a generator on X^α_A, where $\delta := -\min\{\alpha, \gamma_2\}$.*

Proof. By definition C_α is a generator on $X^\alpha_C = X^\alpha_A$. It remains to show that $C_\alpha = (A_{-\delta} + B)_{|X^\alpha_A}$. To this end we first assume that $\alpha \geq 0$. Then

$$C_\alpha = C_{|X^\alpha_C} = ((A_{-\delta} + B)_{|X})_{|X^\alpha_A} = (A_{-\delta} + B)_{|X^\alpha_A}.$$

Now let $\alpha \in (\gamma_1 - 1, \gamma_2)$. Then $D(C_\alpha) = X^{\alpha+1}_C = X^{\alpha+1}_A$. The operator C_α is the closure of $\{(x, Cx);\, x \in X^1_C\}$ in $X^\alpha_A \times X^\alpha_A$. Let $x \in X^{\alpha+1}_A$. There exists $(x_n) \subseteq X^1_C$ such that $x_n \to x$ and $Cx_n \to C_\alpha x$ both in X^α_A. As C and $A_\alpha + B$ agree on X^1_C we see from the closedness of $A_\alpha + B$ (cf. Lemma 4.2.2 and [44; Theorem IV.1.1]) that $(A_\alpha + B)x = C_\alpha x$. In order to show the assertion for $\alpha \in [\gamma_2, 0)$ we observe that $C_\alpha = (C_\beta)_{\alpha-\beta}$ for some arbitrary $\beta \in (\gamma_1 - 1, \gamma_2)$. Hence this case follows from the first two. ∎

The fact that the A_{γ_2}-bound of the considered type of perturbation is 0 allows the following extension (cf. Lemma 4.2.2).

4.3.5 Corollary. *The assertions of Theorem 4.3.2 and Corollary 4.3.4 also hold for $\alpha = \gamma_2 + 1$.*

Proof. In Corollary 4.3.4 we have seen that $C_{\gamma_2} = A_{\gamma_2} + B$ on $X^{\gamma_2}_C = X^{\gamma_2}_A$. As B has A_{γ_2}-bound 0 we infer that $D(A_{\gamma_2}) = X^{\gamma_2+1}_A$ and $D(C_{\gamma_2}) = X^{\gamma_2+1}_C$ are equal with equivalent (graph) norms. The equality of C_{γ_2+1} and $(A + B)_{|X^{\gamma_2+1}_A}$ and the generator property of C_{γ_2+1} follow as in the proof of Corollary 4.3.4. ∎

For later use we show that certain extrapolated Favard spaces also remain preserved.

4.3.6 Corollary. *Let A, B and C be as in Theorem 4.3.2. If $\alpha \in (\gamma_1 - 1, \gamma_2 + 1]$ then $F^\alpha_A = F^\alpha_C$.*

Proof. By Theorem 4.3.2 and Corollary 4.3.5 it suffices to show that $F_A^0 \subseteq F_C^0$ (note that $A = C - B$ and $-B$ is bounded from $X_C^{\gamma_1}$ to $X_C^{\gamma_2}$). Hence we also can assume that $\gamma_1 \geq 0$ and $\gamma_2 \leq 0$. Let $x \in F_A^0$, then $x \in X^\alpha$ for any $\alpha < 0$ (cf. [39; Proposition II.5.33]). For $\delta \in (-\gamma_2, 1 - \gamma_1)$ and λ sufficiently large we have (cf. Lemma 4.2.4)

$$\begin{aligned}\|\lambda R(\lambda, C_{-\delta})x\| &\leq \|\lambda R(\lambda, A_{-\delta})x\| \\ &\quad + \|\lambda^{\delta-1}(A-\lambda)^{\delta-1}(I - H_{\lambda,\delta})^{-1} H_{\lambda,\delta} \lambda^{-\delta}(A_{-\delta} - \lambda)^{-\delta} x\|.\end{aligned}$$

The first expression on the right hand side is bounded by assumption. The second expression becomes bounded if we rewrite

$$(A_{-\delta} - \lambda)^{-\delta} x = A^{2\varepsilon}(A-\lambda)^{-\delta} A^{\varepsilon}(A-\lambda)^{-2\varepsilon} A_{-\delta}^{-\varepsilon} x$$

for $\varepsilon > 0$ sufficiently small such that $H_{\lambda,\delta} A^\varepsilon$ remain uniformly bounded operators in λ. This shows the inclusion $F_A^0 \subseteq F_C^0$. ■

4.4 Perturbation Theorems

In this section we use the results on iterated and perturbed fractional power spaces to derive new perturbation theorems from known ones. In order to slightly curb the ongoing blizzard of indices we introduce (for the remaining part of this section!) the notation A_* as an abbreviation of A_β for some $\beta \in \mathbb{R}$ sufficiently small (in most situations $\beta = -2$ or $\beta = -1$ is suitable).

4.4.1 Theorem. *Let A be a generator of a C_0-semigroup on a Banach space X. Further let $-1 \leq \gamma_2 \leq \gamma_1 \leq 1$, $\gamma_1 - \gamma_2 < 1$ and $B\colon X_A^{\gamma_1} \to X_A^{\gamma_2}$ be a bounded operator. If $(A_* + (A_* - \omega)^\alpha B (A_* - \omega)^{-\alpha})_{|X}$ is a generator of a C_0-semigroup on X for some $\alpha \in [\gamma_1 - 1, \gamma_2 + 1]$ and $\omega \in \mathbb{R}$ sufficiently large then $(A_* + B)_{|X}$ is a generator.*

Proof. Let $\tilde{B} := (A_* - \omega)^\alpha B (A_* - \omega)^{-\alpha} \in \mathcal{L}(X^{\gamma_1 - \alpha}, X^{\gamma_2 - \alpha})$. Without loss of generality we can assume that the C_0-semigroups generated by A and $C := (A_* + \tilde{B})_{|X}$ both have negative growth bound. (Otherwise we choose $\omega_1 \geq 0$ sufficiently large and consider $A - \omega_1$ instead of A.)

First we assume that $-1 < \gamma_2 \leq \gamma_1 < 1$ and $\alpha \in (\gamma_1 - 1, \gamma_2 + 1)$. From Corollary 4.3.4 we know that $C_{\gamma_2 - \alpha} = (A_* + \tilde{B})_{|X^{\gamma_2 - \alpha}}$ is a generator of a C_0-semigroup on $X^{\gamma_2 - \alpha}$. Let T denote the C_0-semigroup generated by $C_{\gamma_2 - \alpha}$. Observe that $V := (A_{\min\{\gamma_2, \gamma_2 - \alpha\}} - \omega)^\alpha$ is an isomorphism from X^{γ_2} to $X^{\gamma_2 - \alpha}$. Thus $V^{-1} T(\cdot) V$ generates a C_0-semigroup on X^{γ_2} similar to T (for the notion of similarity of C_0-semigroups we refer to [39; Section II.2.1]). Its generator is given by

$$V^{-1} C_{\gamma_2 - \alpha} V = V^{-1}\left(A_{\gamma_2 - \alpha} + V B V^{-1}\right) V = A_{\gamma_2} + B$$

(with domain $X^{\gamma_2 + 1}$). Applying Corollary 4.3.4 once more we see that $(A_{\gamma_2} + B)_{-\gamma_2} = (A_* + B)_{|X}$ is a generator.

We next consider the case $-1 < \gamma_2 \le \gamma_1 < 1$ and $\alpha \in \{\gamma_1 - 1, \gamma_2 + 1\}$. For $\alpha = \gamma_2 + 1$ let $Q := I + (A_{-1} - \omega)^{-1}\tilde{B} \in \mathcal{L}(X)$. As $(A - \omega)Q + \omega = (A_{-1} + \tilde{B})_{|X}$ is a generator by our assumptions we see from [39; Theorem 3.20(ii)] that

$$Q(A - \omega) + \omega = A + (A_* - \omega)^{\gamma_2} B (A_* - \omega)^{-\gamma_2}$$

is also a generator (observe that $\rho(Q(A-\omega)) \neq \emptyset$ as $(A_* - \omega)^{\gamma_2} B (A_* - \omega)^{-\gamma_2}$ has A-bound 0; cf. [39; Lemma III.2.6]). As $\gamma_2 \in (\gamma_1 - 1, \gamma_2 + 1)$ we obtain the generator property of $(A_* + B)_{|X}$ from the first part of this proof.

Similarly if $\alpha = \gamma_1 - 1$ then [39; Theorem 3.20(i)] implies that $\big(A_{-1} + (A_* - \omega)^{\gamma_1} B (A_* - \omega)^{-\gamma_1}\big)_{|X}$ is a generator. As $\gamma_1 \in (\gamma_1 - 1, \gamma_2 + 1)$ we obtain the generator property of $(A_* + B)_{|X}$ again from the first part of this proof.

Now assume that $\gamma_1 = 1$ and $\alpha \in [\gamma_1 - 1, \gamma_2 + 1]$. Let $Q := I + B(A - \omega)^{-1} \in \mathcal{L}(X)$ and $B' := (A_* - \omega)B(A - \omega)^{-1} \in \mathcal{L}(X, X^{\gamma_2 - 1})$. As we have $(A_* - \omega)^{\alpha} B (A_* - \omega)^{-\alpha} = (A_* - \omega)^{\alpha - 1} B' (A_* - \omega)^{1-\alpha}$ we infer from the parts of the assertion already proved that $(A_* + B')_{|X} = (A - \omega)Q + \omega$ is a generator. From [39; Theorem III.3.20(ii)] we see that $Q(A - \omega) + \omega = A + B$ is a generator (observe that $\rho(A + B) \neq \emptyset$ as B has A-bound 0).

Last we assume that $\gamma_2 = -1$ and $\alpha \in [\gamma_1 - 1, \gamma_2 + 1]$. Let $Q := I + (A_{-1} - \omega)^{-1} B \in \mathcal{L}(X)$. Similarly as above we conclude that $A + (A_{-1} - \omega)^{-1} B (A - \omega) = Q(A - \omega) + \omega$ is a generator. From [39; Theorem III.3.20(i)] we infer that $(A - \omega)Q + \omega = (A_{-1} + B)_{|X}$ is a generator. ∎

We now apply our technique to some of the more prominent perturbation theorems. They can all be found in [39; Chapter III]. In the appendix we recall the variants of the Miyadera-Voigt and the Desch-Schappacher perturbation theorem which we use here.

4.4.2 Corollary. *Let A be the generator of a C_0-semigroup T on X, $-1 \le \gamma_2 \le \gamma_1 \le 1$, $\gamma_1 - \gamma_2 < 1$ and $B\colon X^{\gamma_1} \to X^{\gamma_2}$ a bounded operator. Let $\alpha \in [\gamma_1 - 1, \gamma_2 + 1]$, $\omega \ge 0$ sufficiently large and $\tilde{B} := (A_* - \omega)^{\alpha} B (A_* - \omega)^{-\alpha}$. If one of the following additional assumptions hold, then $(A_* + B)_{|X}$ is a generator.*

(a) $\alpha \le \gamma_2$ and $\tilde{B}$ is a Miyadera-Voigt perturbation of A.
(b) $\alpha \ge \gamma_1$ and $\tilde{B}$ is a Desch-Schappacher perturbation of A.
(c) T is an analytic semigroup.
(d) T is a contraction semigroup, $\alpha \le \gamma_2$ and $\tilde{B}$ is dissipative in X^{α}.

Proof. Assertions (a) and (b) are easily deduced from Theorem 4.4.1. In order to obtain (c) we choose $\alpha := \gamma_2$ and observe that $\tilde{B}$ has A-bound zero (cf. Lemma 4.2.2). For (d) we note that the dissipativity of B in X^{α} implies the dissipativity of $\tilde{B}$ in X. ∎

As a special case of Corollary 4.4.2(b) we state a perturbation theorem for perturbations satisfying a range condition (cf. Proposition A.3).

4.4.3 Corollary. *Let A be the generator of a C_0-semigroup. Let $\alpha \in [-1, 1]$ and Y a Banach space satisfying (RC) in Proposition A.3 with respect to A_{α}. Further assume that there exists $\gamma \in [-1, 1] \cap (\alpha - 1, \alpha]$ such that $Y \hookrightarrow X^{\gamma}$. If $B \in \mathcal{L}(X^{\alpha}, Y)$ then $(A_* + B)_{|X}$ is a generator.*

Proof. We observe that $\tilde{B} := (A_\beta - \omega)^\alpha B(A_\beta - \omega)^{-\alpha}$, where $\beta := \min\{0, \alpha\}$ and $\omega \geq 0$ sufficiently large, is a bounded operator from X to $Z := (A_\beta - \omega)^\alpha Y$. From Proposition A.4 and Proposition A.3 we see that $\tilde{B}$ is a Desch-Schappacher perturbation of A. The assertion now follows from Corollary 4.4.2(b). ∎

4.4.4 Remarks. (a) Arbitrary Miyadera-Voigt and Desch-Schappacher perturbations are not covered by Corollary 4.4.2. In [39; Corollary III.3.22] these perturbations are treated for $\alpha = -1$ and $\alpha = 1$, respectively.

(b) The statement of Corollary 4.4.3 for $\alpha = 1$ was obtained in [30]; also cf. [48; Theorem A.1].

(c) We note that most of the common regularity properties of a C_0-semigroup T such as analyticity and (immediate, eventual) differentiability hold for the whole scale of C_0-semigroups $(T_\alpha)_{\alpha\in\mathbb{R}}$ and remain preserved under similarity constructions. Preservation of such regularity properties in Theorem 4.4.1, Corollary 4.4.3 and Corollary 4.4.2 thus depends on the perturbation theorem invoked.

(d) Theorem 4.3.2 can be slightly improved if $B \in \mathcal{L}(X, X^{-1})$ is a Desch-Schappacher perturbation of A. In (4.2.2) we have derived the representation $\lambda - (A_{-1} + B) = (\lambda - A_{-1})(I - R(\lambda, A_{-1})B)$ for $\lambda \in \mathbb{R}$ sufficiently large. Further in [39; Equation (III.3.6)] it was shown that the norm of $(I - R(\lambda, A_{-1})B)$ becomes smaller than 1 for λ sufficiently large. Hence $(I - R(\lambda, A_{-1})B)$ has a bounded inverse and so the norms $\|R(\lambda - (A_{-1} + B)) \cdot \|$ and $\|R(\lambda, A_{-1}) \cdot \|$ on X are equivalent. It shows that $X_A^{-1} = X_{A_{-1}+B} = X^{-1}_{(A_{-1}+B)_{|X}}$. This implies (together with Corollary 4.3.5) that if $B \in \mathcal{L}(X^{\gamma_1}, X^{\gamma_2})$ is a Desch-Schappacher perturbation of A_{γ_1} then the assertion of Theorem 4.3.2 holds for $\alpha \in [\gamma_1 - 1, \gamma_2 + 1]$.

4.5 Perturbations with a Growth Condition

We investigate Miyadera-Voigt and Desch-Schappacher type perturbations with an additional growth condition. We will see that such perturbations are among the type of perturbation we have considered in the previous sections. This will greatly help to apply the perturbation theorems presented in the last section. For examples of such perturbations we refer to the Sections 4.6, 4.7 and 4.8 and to [39; Corollary III.3.4, Example III.3.5, Exercise III.3.8(5)(iv)].

4.5.1 Proposition. *Let A be the generator of a C_0-semigroup T on X.*

(a) (Miyadera-Voigt type perturbation) Assume that $B_1 \in \mathcal{L}(X^1, X)$ satisfies

$$\left\| \int_0^t B_1 T(r)x \, dr \right\| \leq K t^{1-\beta} \|x\| \quad (t \in [0, t_0],\ x \in X^1) \tag{4.5.1}$$

for some $K \geq 0$, $t_0 \geq 0$ and $\beta \in (0, 1)$. Then B_1 extends to a bounded operator in $\mathcal{L}(X^\alpha, X)$ for all $\alpha > \beta$.

(b) (Desch-Schappacher type perturbation) Let $B_2 \in \mathcal{L}(X, X^{-1})$. Assume that there is a $t_0 \geq 0$ so that $\int_0^t T_{-1}(r)B_2 x\, dr \in X$ for all $t \in [0, t_0]$ and $x \in X$. If

$$\left\| \int_0^t T_{-1}(r)B_2 x\, dr \right\| \leq K t^{1-\beta} \|x\| \quad (t \in [0, t_0],\ x \in X) \tag{4.5.2}$$

holds for some $K \geq 0$ and $\beta \in (0,1)$, then $B_2 \in \mathcal{L}(X, X^{\alpha})$ for all $\alpha < \beta - 1$.

Proof. Without loss of generality we may assume that T has negative growth bound. Let $\alpha \in (\beta, 1)$, $x \in X^1$ and $t > 0$. First we observe that by the semigroup law the operator $\int_0^t B_1 T(s)\, ds \in \mathcal{L}(X^1, X)$ extends to a bounded operator on X for all $t \in \mathbb{R}_+$. Let $t \geq t_0$ and choose $n \in \mathbb{N}$ such that $\tau := t/n \in [\frac{1}{2}\min\{1, t_0\}, \min\{1, t_0\}]$. Further let $M \geq 1$ and $\omega < 0$ such that $\|T(t)\| \leq M e^{\omega t}$ $(t \in \mathbb{R}_+)$. Using

$$\int_0^t B_1 T(r)\, dr = \sum_{k=0}^{n-1} \int_0^{\tau} B_1 T(r) T(k\tau)\, dr$$

we infer that for $x \in X^1$ we have

$$\begin{aligned}\left\| \int_0^t B_1 T(r) x\, dr \right\| &\leq \sum_{k=0}^{n-1} M \tau^{1-\beta} e^{\omega k \tau} \|x\| \leq M \int_0^n e^{\omega \tau \vartheta}\, d\vartheta\, \|x\| \\ &\leq -\frac{L}{\omega \tau} \|x\| \leq -\frac{2L}{\omega \min\{1, t_0\}} \|x\|.\end{aligned}$$

Thus $\sup_{t \in \mathbb{R}_+} \| \int_0^t B_1 T(r) x\, dr \| \leq K \|x\|$ $(x \in X^1)$ for some $K \geq 0$. In order to reason that $B_1 A^{-\alpha}$ extends to a bounded operator on X we approximate $B_1 A^{-\alpha} x$ for $x \in X^1$ by

$$\begin{aligned}&\tilde{c}_{\alpha} \int_t^{\infty} s^{\alpha-1} B_1 T(s) x\, ds \\ &\qquad = \tilde{c}_{\alpha} \left(s^{\alpha-1} \int_0^s B_1 T(r) x\, dr \Bigg|_t^{\infty} + (1-\alpha) \int_t^{\infty} s^{\alpha-2} \int_0^s B_1 T(r) x\, dr\, ds \right),\end{aligned}$$

where $t > 0$ (cf. (4.1.2)). First note that the integrals exist as $\int_0^s B_1 T(r) x\, dr$ is uniformly bounded in $s \in \mathbb{R}_+$ for $x \in X^1$. The first expression in the parenthesis converges to 0 as $t \to 0$ for we have $\|t^{\alpha-1} \int_0^t B_1 T(r) x\, dr\| \leq K t^{\alpha-\beta} \|x\|$ and $\alpha > \beta$. The second expression converges as $t^{\alpha-\beta-1}$ is integrable on $[0, t_0]$. Hence there exists $K \geq 0$ such that $\|B_1 A^{-\alpha} x\| \leq K \|x\|$ for all $x \in X^1$. This is equivalent to $\|B_1 y\| \leq K \|A^{\alpha} y\| = K \|y\|_{\alpha}$ for all $y \in A^{-\alpha} X^1 = X^{\alpha+1}$. Since $X^{\alpha+1}$ is dense in X^{α} we see that B_1 extends to a bounded operator in $\mathcal{L}(X^{\alpha}, X)$.

If the assumptions in (b) hold we infer from $A_{-1}\int_0^t T_{-1}(r)B_2x\,dr = (T_{-1}-I)B_2x$ and (4.5.2) that B_2 maps continuously into the Favard space of fractional order $\beta-1$ with respect to A (cf. [39; Definition II.5.10] or [18; Proposition 3.1.3] for Favard spaces of fractional order). Assertion (b) now follows from the embedding properties of Favard spaces, Hölder spaces and fractional power spaces; cf. [39; Proposition II.5.33]. ■

4.5.2 Remarks. (a) The assumptions (a) and (b) in Proposition 4.5.1 in the stronger sense of Propositions A.1 and A.2 imply that $C_1 := A + B_1$ and $C_2 := (A_{-1}+B_2)_{|X}$ are generators of C_0-semigroups, denoted by S_1 and S_2, respectively. Moreover $\|S_i(t) - T(t)\| \le Lt^{1-\beta}$ and $\|R(\lambda, C_i) - R(\lambda, A)\| \le L(1+\lambda)^{\beta-2}$ ($i \in \{1,2\}$, $t \in [0, t_0]$, $\lambda \ge \omega$) for some $L \ge 0$ and $\omega \ge 0$ sufficiently large.

In [31] and [34] (see also [39; Theorem III.3.9]) it was shown that if a densely defined operator C with $[\omega,\infty) \subseteq \rho(A)\cap\rho(C)$ for some $\omega \ge 0$ satisfies $\|R(\lambda, C) - R(\lambda, A)\| \le L(1+\lambda)^{-2}$ ($\lambda \ge \omega$) for some $L \ge 0$, then there is a bounded operator $B\colon X \to F_A^0$ such that $C = (A_{-1}+B)_{|X}$. (Here F_A^0 denotes the extrapolated Favard space of A; cf. (A.1).)

(b) Perturbations satisfying Condition (4.5.1) with $\beta = 1$ have recently been explored in [65].

4.6 Inhomogeneous Abstract Cauchy Problems

Our first application are inhomogeneous abstract Cauchy problems. The function spaces for the inhomogeneities will be fractional power spaces associated with the left translation semigroup on spaces of continuous and p-integrable functions.

This section also serves as a preparation for Section 4.7 where we extensively use the Volterra semigroups constructed in this section in the treatment of various integro-differential equations.

Let A be the generator of a C_0-semigroup T on a Banach space X. In order to solve the inhomogeneous abstract Cauchy problem

$$\text{(iACP)}\quad \dot{u}(t) = Au(t) + f(t), \quad u(0) = x \in X, \quad f \in L_{1,loc}(\mathbb{R}_+; X^{-1}),$$

we use the Volterra semigroup approach (cf. [39; Section VI.7.a]) with different function spaces. The inhomogeneities will either take values in X^α or the extrapolated Favard space F_A^α for some suitable α; cf. Proposition 4.1.5. We recall that if X is reflexive then $F_A^\alpha = X^\alpha$. A further relaxation on the range of the inhomogeneities is presented in Remarks 4.6.2(c).

Let Y be a Banach space with $X \hookrightarrow Y \hookrightarrow X^{-1}$. In the following we say that $u \in C^1(\mathbb{R}_+; X)$ is a classical solution of (iACP) in Y if $A_*u(t) \in Y$ for all $t \in \mathbb{R}_+$ and $\dot{u}(t) = A_*u(t) + f(t)$. For $Y = X$ this corresponds to the usual definition of a classical solution of (iACP) (cf. [39; Definition VI.7.2]).

4.6.1 Inhomogeneities in Spaces of Continuous Functions

Let $\alpha \in (-1, 1]$ and $\mathcal{Z} := X \times C_{bu}^{-\alpha}(\mathbb{R}_+; F_A^\alpha)$, where $C_{bu}^{-\alpha}(\mathbb{R}_+; F_A^\alpha)$ denotes the fractional power space of order $-\alpha$ with respect to the left translation semigroup on $C_{bu}(\mathbb{R}_+; F_A^\alpha)$, which we denote by S and its generator by $\mathcal{D}$; cf. Remarks 4.6.2(d) for embeddings of the fractional power spaces associated with $\mathcal{D}$. On $\mathcal{Z}$ we consider the operator

$$D(\mathcal{A}) := X^1 \times C^{1-\alpha}_{bu}(\mathbb{R}_+; F^\alpha_A), \quad \mathcal{A} := \begin{pmatrix} A & 0 \\ 0 & \mathcal{D}_{-\alpha} \end{pmatrix},$$

which is obviously a generator. The operator $\mathcal{B} := \left(\begin{smallmatrix} 0 & \delta_0 \\ 0 & 0 \end{smallmatrix}\right)$ on $\mathcal{Z}^\alpha = X^\alpha \times C_{bu}(\mathbb{R}_+; F^\alpha_A)$ is a bounded operator from $\mathcal{Z}^\alpha$ into the Favard space F^α_A. Therefore $\mathcal{B}$ satisfies the assumptions of Corollary 4.4.3 and we infer the generator property of $\mathcal{C} := (\mathcal{A}_* + \mathcal{B})_{|\mathcal{Z}}$.

4.6.1 Proposition. *(a) For $\alpha \in (0,1]$ we obtain classical solutions of (iACP) whenever $x \in X^1$ and $f \in C^{1-\alpha}_{bu}(\mathbb{R}_+; F^\alpha_A)$.*

(b) For $\alpha \in (-1,0]$ we obtain classical solutions of (iACP) in F^α_A, whenever $x \in F^{\alpha+1}_A$, $f \in C^{1-\alpha}_{bu}(\mathbb{R}_+; F^\alpha_A)$ and $A_* x + f(0) \in X$.

Proof. We first note that for $f \in C^{1-\alpha}_{bu}(\mathbb{R}_+; F^\alpha_A)$ we have $\delta_0 S_{-\alpha}(t) f = f(t)$ $(t \in \mathbb{R}_+)$. Thus the first component of $t \mapsto e^{t\mathcal{C}} \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)$ is differentiable in X for $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{C})$ and solves (iACP) in F^α_A. If $\alpha > 0$ this means that we obtain classical solutions whenever $x \in X^1$ and $f \in C^{1-\alpha}_{bu}(\mathbb{R}_+; F^\alpha_A)$. If $\alpha \leq 0$ the domain of $D(\mathcal{C})$, for which we obtain solutions differentiable in X, is given by the part of $\mathcal{A}_* + \mathcal{B}$ in $\mathcal{Z}$. This yields

$$D(\mathcal{C}) = \{\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in F^{\alpha+1}_A \times C^{1-\alpha}_{bu}(\mathbb{R}_+; F^\alpha_A);\ A_* x + f(0) \in X\}.$$

Hence for $x \in F^{\alpha+1}_A$, $f \in C^{1-\alpha}_{bu}$ with $A_* x + f(0) \in X$ we obtain classical solutions in F^α_A. ∎

4.6.2 Remarks. (a) A similar result can be obtained by using abstract Hölder spaces; see [39; Exercise VI.7.10(3)] for details. For the case $\alpha = 1$ we refer to [39; Corollary VI.7.8].

(b) If we assume that the inhomogeneities only take values in X^α instead of F^α_A, then the perturbation $\mathcal{B}$ becomes a bounded operator in $\mathcal{Z}^\alpha_C$ for all $\alpha \in [-1,1]$. For $\alpha \in [0,1]$ we obtain classical solutions.

(c) Let Y be a Banach space satisfying (RC) in Proposition A.3 with respect to A_α; cf. Proposition A.4. Assume that there exists $\gamma > \max\{-1, \alpha - 1\}$ such that $Y \hookrightarrow X^\gamma$. Using $\mathcal{Z} = X \times C^{-\alpha}_{bu}(\mathbb{R}_+; Y)$ instead of $X \times C^{-\alpha}_{bu}(\mathbb{R}_+; F^\alpha_A)$ in the computations above we still obtain the generator property of $\mathcal{C}$ by Corollary 4.4.3.

For $x \in X^\alpha$, $f \in C^{1-\alpha}_{bu}(\mathbb{R}_+; Y)$ with $A_* x \in Y$ and $A_* x + f(0) \in X$ we get a classical solution of (iACP) in Y. If additionally $Y \subseteq X$ we obtain classical solutions of (iACP) for all $x \in X^1$ and $f \in C^{1-\alpha}_{bu}(\mathbb{R}_+; Y)$.

If A generates an analytic semigroup this generalisation becomes particularly interesting as any $Y = X^\gamma$ with $\gamma > \max\{-1, \alpha - 1\}$ fulfils condition (RC).

(d) Let Y be a Banach space. For $\beta, \gamma \in (0,1)$, $\beta > \gamma$ we have the embeddings

$$h^\beta(\mathbb{R}_+; Y) \hookrightarrow C^\gamma_{bu}(\mathbb{R}_+; Y) \hookrightarrow h^\gamma(\mathbb{R}_+; Y),$$

where $h^\beta(\mathbb{R}_+; Y)$ denotes the space of Y-valued little Hölder functions of order β. (cf. [39; Proposition II.5.33 and Exercise II.5.23(5)]).

4.6.2 Inhomogeneities in Spaces of p-integrable Functions

We now solve the inhomogeneous abstract Cauchy problem in the space $\mathcal{Z} := X \times W_p^{-\alpha}(\mathbb{R}_+; X^\alpha)$ where $p \in (1, \infty)$ and $\alpha \in [-1, 1 - 1/p)$. Here $W_p^{-\alpha}(\mathbb{R}_+; X^\alpha)$ denotes the fractional power space of order $-\alpha$ associated with the left translation on $L_p(\mathbb{R}_+; X^\alpha)$. The generator of the left translation is again denoted by $\mathcal{D}$. On $\mathcal{Z}$ we consider the generator

$$D(\mathcal{A}) := X^1 \times W_p^{1-\alpha}(\mathbb{R}_+; X^\alpha), \quad \mathcal{A} := \begin{pmatrix} A & 0 \\ 0 & \mathcal{D}_{-\alpha} \end{pmatrix}.$$

Let $\omega > 0$ be larger than the growth bound of T. (The growth bound of the translation semigroup is 0 and thus in any case smaller than ω.) In the following estimates we assume that X^γ and $W_p^\gamma(\mathbb{R}_+; X^\alpha)$ ($\gamma \in \mathbb{R}$) are equipped with the norms $x \mapsto \|(A_* - \omega)^\gamma x\|$ and $f \mapsto \|(\mathcal{D}_* - \omega)^\gamma f\|_p$, respectively. (The index p refers to the usual p-norm of Banach space valued p-integrable functions, so there will be no danger of confusing it with the norm of fractional power spaces.) Let $\mathcal{B} := \left(\begin{smallmatrix} 0 & \delta_0 \\ 0 & 0 \end{smallmatrix}\right)$ with domain $\mathcal{Z}^{\alpha+1} = X^{\alpha+1} \times W_p^1(\mathbb{R}_+; X^\alpha)$. It is not difficult to see that the operator

$$\tilde{\mathcal{B}} := (\mathcal{A}_* - \omega)^\alpha \mathcal{B} (\mathcal{A}_* - \omega)^{-\alpha} = \begin{pmatrix} 0 & (A_* - \omega)^\alpha \delta_0 (\mathcal{D}_* - \omega)^{-\alpha} \\ 0 & 0 \end{pmatrix}$$

with domain $\mathcal{Z}^1$ is a Miyadera-Voigt perturbation of $\mathcal{A}$. In fact for $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in \mathcal{Z}^1$ and $g := (\mathcal{D}_* - \omega)^{-\alpha} f \in W_p^1(\mathbb{R}_+; X^\alpha)$ we have

$$\begin{aligned} \int_0^t \left\| \tilde{\mathcal{B}} T_\alpha(r) \begin{pmatrix} x \\ f \end{pmatrix} \right\|_{\mathcal{Z}} dr &= \int_0^t \left\| \begin{pmatrix} (A_* - \omega)^\alpha \delta_0 (\mathcal{D}_* - \omega)^{-\alpha} f(r + \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{Z}} dr \\ &= \int_0^t \|g(r)\|_{X^\alpha} \, dr \le t^{1-1/p} \|g\|_p \le t^{1-1/p} \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|_{\mathcal{Z}}. \end{aligned}$$

Since $\alpha + 1/p < 1$ the generator property for $\mathcal{C} := (\mathcal{A}_* + \mathcal{B})_{|X}$ follows from Proposition 4.5.1(a) and Corollary 4.4.2(a). (We also infer that $\mathcal{B}$ extends to a bounded operator from $\mathcal{Z}^{\alpha+1/p+\varepsilon}$ to $\mathcal{Z}^\alpha$ for any $\varepsilon > 0$.) After this preparation the proof of the next result can be followed through as in Proposition 4.6.1.

4.6.3 Proposition. *(a) For $\alpha \in [0, 1 - 1/p)$ we obtain classical solutions of (iACP) whenever $x \in X^1$ and $f \in W_p^{1-\alpha}(\mathbb{R}_+; X^\alpha)$.*

(b) For $\alpha \in [-1, 0)$ we obtain classical solutions of (iACP) in X^α, whenever $x \in X^{\alpha+1}$, $f \in W_p^{1-\alpha}(\mathbb{R}_+; X^\alpha)$ and $A_* x + f(0) \in X$.

4.6.4 Remarks. (a) For the case $p = 1$ and inhomogeneities with values in F_A^0 we refer to [39; Proposition VI.7.12]. We cannot expect that our method works for $p = 1$.

(b) There are a number of fractional order Sobolev spaces, and numerous embedding theorems concerning these spaces; cf. e.g. [1; Chapter VII] or [66] for real- and complex-valued Sobolev spaces. In our applications we need the embedding

$$W_p^\alpha(\mathbb{R}_+; Y) \hookrightarrow C_0(\mathbb{R}_+; Y) \quad (\alpha > 1/p),$$

where Y is some Banach space (for the real- and complex-valued case see [1; Theorem 7.57], [66; Theorem 1.15.2(d) and Section 2.8]).

4.7 Integro-Differential Equations

Let A be the generator of a C_0-semigroup T on the Banach space X. The forcing function method utilises Volterra semigroups to solve the equation

$$\text{(IDE)} \quad \dot{u}(t) = A_* u(t) + \int_0^t \ell(t-s)u(s)\,ds + f(t), \;\; u(0) = x \in X \quad (t \in \mathbb{R}_+),$$

where f is a locally integrable X^α- or F_A^α-valued function and ℓ is a function defined on $\mathbb{R}_+$ with values in $\mathcal{L}(Y, Z)$, where Y and Z are Banach spaces with $Y \hookrightarrow X$ and $Z \hookrightarrow X^{-1}$. We additionally assume that $\ell(\cdot)x$ is a locally integrable function for all $x \in Y$. This assumptions guarantees that the integral in (IDE) exists for all $t \in \mathbb{R}_+$ whenever $u \in C(\mathbb{R}_+; Y)$.

A function u is called a classical solution of (IDE) if $u \in C(\mathbb{R}_+; Y) \cap C^1(\mathbb{R}_+; X)$ and u satisfies (IDE) in X^{-1}. Further a function $u \in C(\mathbb{R}_+; X)$ is a mild solution of (IDE) if $\int_0^t u(s)\,ds \in Y$ for all $t \in \mathbb{R}_+$, $\int_0^{\cdot} u(s)\,ds$ is in $L_{1,loc}(\mathbb{R}_+; Y)$ and the integrated equation of (IDE)

$$u(t) = u(0) + A_* \int_0^t u(s)\,ds + \int_0^t \ell(t-r) \int_0^r u(s)\,ds\,dr + \int_0^t f(s)\,ds$$

holds for all $t \in \mathbb{R}_+$.

We say that (IDE) is well-posed if for all $x \in X^1$ a unique classical solution of (IDE) with $f = 0$ exists, continuously depending (in the norm of X) on the initial value uniformly in compact intervals.

We mention that the approach via Volterra semigroups also yields classical or mild solutions of (IDE) for certain inhomogeneities, depending on the forcing-function space chosen for the Volterra semigroup.

In Section 4.7.3 we look at a variant of (IDE) where instead of u the derivative $\dot{u}$ occurs in the integral.

For the forcing-function approach we refer the reader to [30] and [58; Section 13.6].

4.7.1 (IDE) in the Context of Continuous Functions

We use the generators $\mathcal{A}$ and $\mathcal{C}$ on $\mathcal{Z} = X \times C_{bu}^{-\alpha}(\mathbb{R}_+; F_A^\alpha)$ from Section 4.6.1, with $\alpha \in (-1, 1]$. We are going to perturb $\mathcal{C}$ in such a way that the first component of the obtained C_0-semigroup solves (IDE).

Let $\beta \in (\alpha - 1, \alpha + 1) \cap (-1, 1]$. Then $\mathcal{Z}_{\mathcal{C}}^\beta = \mathcal{Z}_{\mathcal{A}}^\beta$ and $F_{\mathcal{C}}^\beta = F_{\mathcal{A}}^\beta$ (cf. Theorem 4.3.2 and Corollary 4.3.6). As $\mathcal{A}$ is a diagonal matrix with no coupling in the domain we have $\mathcal{Z}_{\mathcal{A}}^{\beta} = X^{\beta} \times C_{bu}^{\beta-\alpha}(\mathbb{R}_+; F_A^{\alpha})$ and $F_{\mathcal{A}}^{\beta} = F_A^{\beta} \times F_{\mathcal{D}}^{\beta-\alpha}$. Thus if $L \in \mathcal{L}(X^{\beta}, F_{\mathcal{D}}^{\beta-\alpha})$ then $\mathcal{Q} := \left(\begin{smallmatrix} 0 & 0 \\ L & 0 \end{smallmatrix}\right)$ is a bounded operator from $\mathcal{Z}_{\mathcal{C}}^{\beta}$ to $F_{\mathcal{C}}^{\beta}$ and $(\mathcal{C}_* + \mathcal{Q})_{|\mathcal{Z}}$ becomes a generator by Corollary 4.4.3. For $\beta > 0$ the domain of $(\mathcal{C}_* + \mathcal{Q})_{|\mathcal{Z}} = \mathcal{C} + \mathcal{Q}$ is $D(\mathcal{C})$. If $\beta \leq 0$ then

$$D\big((\mathcal{C}_* + \mathcal{Q})_{|\mathcal{Z}}\big) = \big\{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in F_A^{\alpha+1} \times F_{\mathcal{D}}^{\beta-\alpha+1};\ A_* x + f(0) \in X, \\ Lx + \mathcal{D}_* f \in C_{bu}^{-\alpha}(\mathbb{R}_+; F_A^{\alpha}) \big\}.$$

So in this case we can expect only to obtain mild solutions of (IDE).

For $\alpha \in (-1, 0]$ and $\beta \in [\alpha+1, 1]$ we first observe that

$$\mathcal{Z}_{\mathcal{C}}^{\beta} = D(\mathcal{C}_{\beta-1}) = \big\{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in F_A^{\alpha+1} \times C_{bu}^{\beta-\alpha}(\mathbb{R}_+; F_A^{\alpha});\ A_* x + f(0) \in X^{\beta-1} \big\}.$$

From this we see that for $\beta > \alpha + 1$ the operator L would still have to continuously map $F_A^{\alpha+1}$ to some function space related to $C_{bu}(\mathbb{R}_+; F_A^{\alpha})$. So it seems that only the case $\beta = \alpha + 1$ is worth to be considered. Next we show that $\{0\} \times F_{\mathcal{D}}^1$ is continuously embedded into the Favard space $F_{\mathcal{C}}^{\alpha+1}$.

4.7.1 Lemma. *The space $\{0\} \times F_{\mathcal{D}}^1$ is continuously embedded into $F_{\mathcal{C}}^{\alpha+1}$.*

Proof. Let $f \in F_{\mathcal{D}}^1$. It is easy to see that $F_{\mathcal{D}}^1 = Lip(\mathbb{R}_+; F_A^{\alpha})$, by which we denote the Banach space of uniformly Lipschitz-continuous functions on $\mathbb{R}_+$ with values in F_A^{α}. In order to show the assertion we use the alternative definition of the Favard space given in (A.2). First we note that $e^{t\mathcal{C}} \left(\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right) = \left(\begin{smallmatrix} \int_0^t T_\alpha(t-s) f(s)\, ds \\ S(t) f \end{smallmatrix}\right)$ $(t \in \mathbb{R}_+)$. As the norm of $\mathcal{Z}_{\mathcal{C}}^{\alpha}$ is equivalent to $\mathcal{Z}_{\mathcal{A}}^{\alpha} = X^{\alpha} \times C_{bu}(\mathbb{R}_+; F_A^{\alpha})$ we obtain for $\omega \in \mathbb{R}$ sufficiently large

$$\begin{aligned}
\left|\!\left|\!\left| \left(\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right) \right|\!\right|\!\right|_{F_{\mathcal{C}}^{\alpha+1}} &= \sup_{t>0} \frac{1}{t} \left\| e^{t(\mathcal{C}-\omega)} \left(\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right) \right\|_{\mathcal{Z}_{\mathcal{C}}^{\alpha}} \\
&\leq c \sup_{t>0} \left(\left\| \frac{1}{t} e^{-\omega t} \int_0^t T_\alpha(t-s) f(s)\, ds \right\|_{X^{\alpha}} + \left\| \frac{1}{t} (e^{-\omega t} S(t) f - f) \right\|_{\infty} \right) \\
&\leq c \sup_{t>0} \left\| \frac{1}{t} e^{-\omega t} \int_0^t T(t-s) A^{\alpha} f(s)\, ds \right\|_X + c |\!|\!| f |\!|\!|_{F_{\mathcal{D}}^1} \\
&\leq c \Big(\sup_{t \in \mathbb{R}_+} \| e^{-\omega t} T(t) \| \, \| f \|_{\infty} + |\!|\!| f |\!|\!|_{F_{\mathcal{D}}^1} \Big) \\
&\leq c' |\!|\!| f |\!|\!|_{F_{\mathcal{D}}^1}
\end{aligned}$$

for some $c, c' \geq 0$. ■

Now if $L \in \mathcal{L}(F_A^{\alpha+1}, F_{\mathcal{D}}^1)$ then $\mathcal{Q} := \left(\begin{smallmatrix} 0 & 0 \\ L & 0 \end{smallmatrix}\right)$ is a bounded operator from $\mathcal{Z}_{\mathcal{C}}^{\beta}$ to $F_{\mathcal{C}}^{\alpha+1}$ by Lemma 4.7.1. Again from Corollary 4.4.2(b) we infer the generator property of $\mathcal{C} + \mathcal{Q}$.

As a last preperation we identify the Favard space $F_{\mathcal{D}}^0$. If F_A^{α} has the Radon-Nikodym property (cf. [4; Section 1.2], [35; Section VII.6]), then it was shown in [55; Proposition 3.4] that $F_{\mathcal{D}}^0$ can be identified with $L_\infty(\mathbb{R}_+; F_A^{\alpha})$. However in general we cannot expect that F_A^α possesses the Radon-Nikodym property. At least we can show that $L_\infty(\mathbb{R}_+;F_A^\alpha)$ is a subspace of $F_\mathcal{D}^0$. To this end let $\omega>0$. From $(\mathcal{D}_{-1}-\omega)^{-1}L_\infty(\mathbb{R}_+;F_A^\alpha)\hookrightarrow Lip(\mathbb{R}_+;F_A^\alpha)=F_\mathcal{D}^1$ and Proposition 4.1.5 we conclude that for the general case we have $L_\infty(\mathbb{R}_+;F_A^\alpha)\hookrightarrow F_\mathcal{D}^0$.

Now we can give conditions on ℓ so that (IDE) becomes well-posed.

4.7.2 Proposition. *(a) Assume that $\alpha\in(-1,1]$ and $\beta\in(0,1]\cap[\alpha,\alpha+1)$. If $\ell(t)\in\mathcal{L}(X^\beta,F_A^\alpha)$ $(t\in\mathbb{R}_+)$, and $\ell(\cdot)x\in F_\mathcal{D}^{\beta-\alpha}$ for all $x\in X^\beta$, then (IDE) becomes well-posed. In particular for $\beta=\alpha$ this holds if $\ell(\cdot)x\in L_\infty(\mathbb{R}_+;F_A^\alpha)$.*

(b) Assume that $\alpha\in(-1,0]$ (and $\beta=\alpha+1$). If $\ell(t)\in\mathcal{L}(F_A^{\alpha+1},F_A^\alpha)$ $(t\in\mathbb{R}_+)$, and $\ell(\cdot)x\in Lip(\mathbb{R}_+;F_A^\alpha)$ for all $x\in X^{\alpha+1}$, then (IDE) becomes well-posed.

Proof. We only show (b), the proof of (a) is done similarly.

If the assumptions in (b) hold then we observe that L, defined by $Lx:=\ell(\cdot)x$ $(x\in F_A^{\alpha+1})$, is a bounded operator from $F_A^{\alpha+1}$ to $F_\mathcal{D}^1=Lip(\mathbb{R}_+;F_A^\alpha)$ by the closed graph theorem (see Lemma 3.1.1 for similar cases). Hence $\mathcal{C}+\mathcal{Q}$ becomes a generator. Let $\binom{x}{f}\in D(\mathcal{C}+\mathcal{Q})=D(\mathcal{C})$ and $\binom{u(t)}{F(t)}:=\exp(t(\mathcal{C}+\mathcal{Q}))\binom{x}{f}$ $(t\in\mathbb{R}_+)$ be the classical solution of the abstract Cauchy problem associated with $\mathcal{C}+\mathcal{Q}$. Thus we have

$$\dot u(t)=A_*u(t)+\delta_0F(t), \tag{4.7.1}$$

$$\dot F(t)=Lu(t)+\mathcal{D}_{-\alpha}F(t)\quad(t\in\mathbb{R}_+). \tag{4.7.2}$$

From (4.7.2) we infer that F is the classical solution of the inhomogeneous abstract Cauchy problem associated with the left translation semigroup on $C_{bu}(\mathbb{R}_+;F_A^\alpha)$ with inhomogeneity $Lu(\cdot)$. This gives us

$$F(t)=S(t)f+\int_0^t S(t-s)Lu(s)\,ds=S(t)f+\int_0^t \ell(t-s+\cdot)u(s)\,ds\quad(t\in\mathbb{R}_+).$$

Therefore we obtain $\delta_0F(t)=f(t)+\int_0^t\ell(t-s)u(s)\,ds$ $(t\in\mathbb{R}_+)$. This shows that u indeed solves (IDE). In order to show that solutions of (IDE) are unique assume that u is a solution for the initial value $u(0)=0$. Let $F(t):=\int_0^tS(t-s)Lu(s)\,ds$ $(t\in\mathbb{R}_+)$. Then F is a mild solution of the inhomogeneous abstract Cauchy problem associated with S and with the continuous inhomogeneity $Lu(\cdot)\in C(\mathbb{R}_+;Lip(\mathbb{R}_+;F_A^\alpha))$. Therefore F satisfies the integrated version of (4.7.2). Since $\delta_0F(t)=\int_0^t\ell(t-s)u(s)\,ds$ equation (4.7.1) is also met and thus $\binom{u(\cdot)}{F(\cdot)}$ is a mild solution of the abstract Cauchy problem associated with $\mathcal{C}+\mathcal{Q}$. As mild solutions are unique we conclude $u=0$. The continuous dependence on the initial value follows from the uniform boundedness of the operators of the semigroup generated by $\mathcal{C}+\mathcal{Q}$ in compact intervals. This shows the well-posedness of (IDE). ∎

For analytic semigroups T we derive the following result, this time starting with the Volterra semigroup outlined in Remarks 4.6.2(c).

4.7.3 Proposition. *Assume that T generates an analytic semigroup. Let $\alpha \in (-1,1]$, $\gamma \in [-1,1] \cap (\alpha-1,\alpha]$ and $\beta \in [\alpha, \max\{1,\gamma+1\}]$. If $\ell(t) \in \mathcal{L}(X^\beta, X^\gamma)$ $(t \in \mathbb{R}_+)$ and $\ell(\cdot)x \in F_{\mathcal{D}}^{\beta-\alpha}$ $(x \in X^\beta)$ then (IDE) is well-posed. For $\beta = \alpha$ this particularly holds if $\ell(\cdot)x \in L_\infty(\mathbb{R}_+; X^\gamma)$ $(x \in X^\beta)$.*

Proof. From Remarks 4.6.2(c) we take the space $\mathcal{Z} = X \times C_{bu}(\mathbb{R}_+; X^\gamma)$ and the corresponding generators $\mathcal{A} = \left(\begin{smallmatrix} A & 0 \\ 0 & \mathcal{D}_{-\alpha}\end{smallmatrix}\right)$ and $\mathcal{C} = \left(\begin{smallmatrix} A & \delta_0 \\ 0 & \mathcal{D}_{-\alpha}\end{smallmatrix}\right)$. The generator $\mathcal{C}$ is obtained by perturbing $\mathcal{A}$ with $\left(\begin{smallmatrix} 0 & \delta_0 \\ 0 & 0\end{smallmatrix}\right) \in \mathcal{L}(\mathcal{Z}_{\mathcal{A}}^\alpha, \mathcal{Z}_{\mathcal{A}}^\gamma)$. Therefore $\mathcal{Z}_{\mathcal{A}}^\beta = \mathcal{Z}_{\mathcal{C}}^\beta$ and $F_{\mathcal{A}}^\beta = F_{\mathcal{C}}^\beta$ for all $\beta \in (\alpha-1,\gamma+1]$; cf. Theorem 4.3.2, Corollary 4.3.5 and Corollary 4.3.6. Hence $\left(\begin{smallmatrix} 0 & 0 \\ L & 0\end{smallmatrix}\right)$ with $Lx := \ell(\cdot)x$ $(x \in X^\beta)$ becomes a bounded operator from $\mathcal{Z}_{\mathcal{C}}^\beta$ to $F_{\mathcal{C}}^\beta$. The proof is now accomplished as in Proposition 4.7.2. ∎

4.7.4 Remarks. (a) The case $\beta = 0$ in Proposition 4.7.2(a) can be included by demanding the stronger condition $\ell(\cdot)x \in C_{bu}^{-\alpha}(\mathbb{R}_+; F_A^\alpha)$.

(b) If we replace the space $C_{bu}(\mathbb{R}_+; F_A^\alpha)$ by $C_{bu}(\mathbb{R}_+; X^\alpha)$ then Proposition 4.7.2(b) holds with the conditions $\ell(t) \in \mathcal{L}(X^{\alpha+1}, X^\alpha)$ $(t \in \mathbb{R}_+)$ and $\ell(\cdot)x \in Lip(\mathbb{R}_+; X^\alpha)$ $(x \in X^{\alpha+1})$.

(c) Proposition 4.7.2 can be generalised using the idea being at the bottom of Proposition 4.7.3. Let T be an arbitrary semigroup, $\alpha \in (-1,1]$, $\gamma \in (\max\{-1,\alpha-1\},\alpha]$, and let $Y \hookrightarrow X^\gamma$ be a Banach space satisfying (RC) with respect to A_α; cf. Proposition A.4. Let $\beta \in [\alpha,\gamma+1] \cap (0,1]$. Then (IDE) becomes well-posed if $\ell(t) \in \mathcal{L}(X^\beta, Y)$ $(t \in \mathbb{R}_+)$ and $\ell(\cdot)x \in F_{\mathcal{D}}^{\beta-\alpha}$ $(x \in X^\beta)$. This particularly holds if $\ell(t) \in \mathcal{L}(X^\alpha, Y)$ $(t \in \mathbb{R}_+)$ and $\ell(\cdot)x \in L_\infty(\mathbb{R}_+; Y)$ $(x \in X^\alpha)$.

4.7.2 (IDE) in the Context of p-integrable Functions

If we use the Volterra semigroup from Section 4.6.2 with inhomogeneities in Sobolev spaces of fractional order, a similar result to Proposition 4.7.2 can be deduced (however the generalisation in Proposition 4.7.3 and Remarks 4.7.4 do not apply). We only sketch the proof of this analogous result in the first part of this section.

In the second part we deal with assumptions involving that ℓ is p-integrable and of bounded variation with respect to either $\mathcal{L}(\mathcal{Z}^\alpha)$ or $\mathcal{L}(\mathcal{Z}^{\alpha+1}, \mathcal{Z}^\alpha)$. This will extend known results for integro-differential equations obtained by using the forcing function approach on $X \times L_1(\mathbb{R}_+; X)$; cf. [30], see also [58; Theorem II.6.1 and Corollary II.6.1] for a different approach.

Let $p \in (1,\infty)$ and $\alpha \in [-1, 1-1/p)$. We start with $\mathcal{Z} := X \times W_p^{-\alpha}(\mathbb{R}_+; X^\alpha)$ and the generator $\mathcal{C}$ from Section 4.6.2. Again we perturb $\mathcal{C}$ in such a way that the first component of the obtained C_0-semigroup solves (IDE).

Let $\beta \in (\alpha+1/p-1, \alpha+1]$. Then $\mathcal{Z}_{\mathcal{C}}^\beta = X^\beta \times W_p^{\beta-\alpha}(\mathbb{R}_+; X^\alpha)$. Hence if $L \in \mathcal{L}(X^\beta, W_p^{\beta-\alpha}(\mathbb{R}_+; X^\alpha))$ then $\mathcal{Q} := \left(\begin{smallmatrix} 0 & 0 \\ L & 0\end{smallmatrix}\right) \in \mathcal{L}(\mathcal{Z}_{\mathcal{C}}^\beta)$. If additionally $\beta \in [-1,1]$ then $(\mathcal{C}_* + \mathcal{Q})_{|\mathcal{Z}}$ becomes a generator. This yields the following well-posedness criteria for (IDE), where the proof of the well-posedness of (IDE) is done as in the proof of Proposition 4.7.2.

4.7.5 Proposition. *Let $p \in (1,\infty)$. Assume that $\alpha \in [-1,1-1/p)$, $\beta \in [0,1] \cap [\alpha,\alpha+1]$. If $\ell(t) \in \mathcal{L}(X^\beta, X^\alpha)$ $(t \in \mathbb{R}_+)$, and $\ell(\cdot)x \in W_p^{\beta-\alpha}(\mathbb{R}_+; X^\alpha)$ for all $x \in X^\beta$, then (IDE) is well-posed.*

We are now going to improve Proposition 4.7.5 considerably. To this end we assume that $\alpha \in [-1, 0]$, $\mathcal{Z}$ and $\mathcal{C}$ are as above and $\ell \in BV_p(\mathbb{R}_+; \mathcal{L}(X^{\alpha+1}, X^\alpha))$. (For a Banach space Y we denote the space of p-integrable Y-valued functions being of bounded variation by $BV_p(\mathbb{R}_+; Y)$. The norm is the sum of the p-norm and the variation norm of the function and will be denoted by $\|\cdot\|_{p,Var}$; also see the paragraph before Lemma 3.1.1.) We also note that in contrast to Proposition 4.7.5 it will not be sufficient to require that $\ell(\cdot)x \in BV_p(\mathbb{R}_+; X^\alpha)$ for all $x \in X^{\alpha+1}$.

Let $Lx := \ell(\cdot)x$, $(x \in X^{\alpha+1})$, then $\mathcal{Q} := \left(\begin{smallmatrix} 0 & 0 \\ L & 0 \end{smallmatrix}\right) \in \mathcal{L}(\mathcal{Z}_{\mathcal{C}}^{\alpha+1}, \mathcal{Z}_{\mathcal{C}}^\alpha)$, since we have $\mathcal{Z}_{\mathcal{C}}^\alpha = X^\alpha \times L_p(\mathbb{R}_+; X^\alpha)$ and $\mathcal{Z}_{\mathcal{C}}^{\alpha+1} = X^{\alpha+1} \times W_p^1(\mathbb{R}_+; X^\alpha)$. We first show that $\mathcal{Q}$ is a Desch-Schappacher perturbation of $\mathcal{C}_{\alpha+1}$. Let $\mathcal{T}_\alpha$ denote the semigroup generated by $\mathcal{C}_\alpha$ on $\mathcal{Z}_{\mathcal{C}}^\alpha = X^\alpha \times L_p(\mathbb{R}_+; X^\alpha)$. We recall that $\mathcal{T}_\alpha = \left(\begin{smallmatrix} T_\alpha(\cdot) & R_\alpha(\cdot) \\ 0 & S(\cdot) \end{smallmatrix}\right)$, where T_α and S are the semigroups generated by A_α and $\mathcal{D}$, respectively, and $R_\alpha(t)f = \int_0^t T_\alpha(t-s)f(s)\,ds$ ($f \in L_p(\mathbb{R}_+; X^\alpha)$, $t \in \mathbb{R}_+$). In the following computations we need the operator-valued Riemann-Stieltjes measure $d\ell$ and its variation $d|\ell|$; we refer the reader to [36; Section III.17.2], a treatment of the vector-valued Riemann-Stieltjes integral, which can be extended without much change to the operator-valued case, can also be found in [4; Section 1.9].

4.7.6 Lemma. *Let Y be a Banach space and $\eta \in BV_p(\mathbb{R}_+; \mathcal{L}(Y, X^\alpha))$. For $t \in \mathbb{R}_+$ and $u \in C_c(\mathbb{R}_+; Y)$ the following assertions hold.*

(a) $\int_0^t R_\alpha(t-s)(\eta(\cdot)u(s))\,ds \in X^{\alpha+1}$ and $\|\int_0^t R_\alpha(t-s)(\eta(\cdot)u(s))\,ds\|_{\alpha+1} \leq ct\|u\|_\infty$ for some $c \geq 0$.

(b) $\int_0^t S(t-s)(\eta(\cdot)u(s))\,ds \in W_p^1(\mathbb{R}_+; X^\alpha)$ and $\|\int_0^t S(t-s)(\eta(\cdot)u(s))\,ds\|_{p,1} \leq ct^{1/p}\|u\|_\infty$ for some $c \geq 0$.

Proof. Let $Ey := \eta(\cdot)y$ $(y \in Y)$. Let $\varphi(t) := \int_0^t \eta(s)u(t-s)\,ds = (\eta * u)(t)$ $(t \in \mathbb{R}_+)$ (as usual $*$ denotes the convolution where the convoluted functions are taken to be zero outside their domains). A straightforward computation yields

$$\int_0^{\cdot} R_\alpha(t-s)Eu(s)\,ds = R_\alpha * Eu(\cdot) = (T_\alpha * \eta) * u = T_\alpha * \varphi = R_\alpha(\cdot)\varphi.$$

In order to estimate the norm of $R_\alpha(t)\varphi$ we first observe that by the bounded variation of η we have $\varphi \in C^1(\mathbb{R}_+; X^\alpha)$, where the derivative of φ is given by $\varphi'(t) = \eta(0)u(t) + \int_0^t d\eta(s)u(t-s)$ $(t \in \mathbb{R}_+)$. Since $\eta \in BV_p(\mathbb{R}_+; \mathcal{L}(Y, X^\alpha))$ Young's inequality implies that $\varphi \in W_p^1(\mathbb{R}_+; X^\alpha)$. Therefore

$$\begin{pmatrix} R_\alpha(t)\varphi \\ S(t)\varphi \end{pmatrix} = \mathcal{T}_\alpha(t)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in D(\mathcal{C}_\alpha) = X^{\alpha+1} \times W_p^1(\mathbb{R}_+; X^\alpha) \quad (t \in \mathbb{R}_+).$$

In particular $R_\alpha(t)\varphi \in X^{\alpha+1}$ $(t \in \mathbb{R}_+)$. In order to obtain an estimate for the norm we infer from

$$\begin{aligned} \begin{pmatrix} A_\alpha R_\alpha(t)\varphi + \varphi(t) \\ S(t)\varphi' \end{pmatrix} &= \mathcal{C}_\alpha \mathcal{T}_\alpha(t)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \mathcal{T}_\alpha(t)\mathcal{C}_\alpha\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \\ &= \mathcal{T}_\alpha(t)\begin{pmatrix} \varphi(0) \\ \varphi' \end{pmatrix} = \begin{pmatrix} T_\alpha(t)\varphi(0) + R_\alpha(t)\varphi' \\ S(t)\varphi' \end{pmatrix}. \end{aligned}$$

and $\varphi(0) = 0$ that $A_\alpha R_\alpha(t)\varphi = R_\alpha(t)\varphi' - \varphi(t)$. Hence we obtain

$$\|R_\alpha(t)\varphi\|_{\alpha+1} = \|A_\alpha R_\alpha(t)\varphi\|_\alpha \leq t \sup_{0\leq s\leq t} \|T_\alpha(s)\|\|\varphi'\|_\infty + \|\varphi(t)\|.$$

Assertion (a) follows from $\|\varphi\|_\infty \leq t\|\eta\|_\infty\|u\|_\infty$ and $\|\varphi'\|_\infty \leq \big(d|\eta|(\mathbb{R}_+) + \|\eta(0)\|\big)\|u\|_\infty$.

In order to show (b) we observe that the derivative of $\int_0^t S(t-s)Eu(s)\,ds = \big(\mathbb{R}_+ \ni \vartheta \mapsto \int_0^t \eta(t-s+\vartheta)u(s)\,ds\big)$ is the continuous function

$$\left(\vartheta \mapsto \int_{s=0}^{t} d\eta(t-s+\vartheta)u(s)\right) = \left(\vartheta \mapsto \int_{\vartheta}^{t+\vartheta} d\eta(s)u(t-s+\vartheta)\right).$$

Thus $\int_0^t S(t-s)Eu(s)\,ds \in C^1(\mathbb{R}_+; X^\alpha)$. The p-norm of the derivative can be estimated by

$$\begin{aligned}
\left\|\mathcal{D}\int_0^t S(t-s)Eu(s)\,ds\right\|_p^p &= \int_0^\infty \left\|\int_\vartheta^{t+\vartheta} d\eta(s)u(t-s+\vartheta)\right\|^p d\vartheta \\
&\leq \|u\|_\infty^p \int_0^\infty \left(\int_\vartheta^{t+\vartheta} d|\eta|(s)\right)^p d\vartheta \\
&\leq \|u\|_\infty^p (d|\eta|(\mathbb{R}_+))^{p-1} \int_0^\infty \int_\vartheta^{t+\vartheta} d|\eta|(s)\,d\vartheta \\
&\leq \|u\|_\infty^p (d|\eta|(\mathbb{R}_+))^{p-1} \int_0^\infty \int_{s-t}^{t} d\vartheta\,d|\eta|(s) \\
&= \left(t^{1/p} d|\eta|(\mathbb{R}_+)\|u\|\right)^p.
\end{aligned}$$

For the p-norm of the function itself we have

$$\begin{aligned}
\left\|\int_0^t S(t-s)Eu(s)\,ds\right\|_p^p &= \int_0^\infty \left\|\int_0^t \eta(t-s+\vartheta)u(s)\,ds\right\|^p d\vartheta \\
&\leq t^{p-1}\|u\|_\infty^p \int_0^\infty \int_0^t \|\eta(t-s+\vartheta)\|^p\,ds\,d\vartheta \\
&\leq (t\|u\|_\infty\|\eta\|_p)^p.
\end{aligned}$$

This shows assertion (b). ∎

We can now reason that $\mathcal{Q}$ is a Desch-Schappacher perturbation of $\mathcal{C}_{\alpha+1}$. By Theorem A.2 it suffices to show that for t_0 sufficiently small and for all $U \in C([0,t_0]; \mathcal{Z}_{\mathcal{C}}^{\alpha+1})$

we have

$$\int_0^t \mathcal{T}_\alpha(t-s)\mathcal{Q}U(s)\,ds = \int_0^t \begin{pmatrix} R_\alpha(t-s)Lu(s) \\ S(t-s)Lu(s) \end{pmatrix} ds \in \mathcal{Z}_{\mathcal{C}}^{\alpha+1},$$

(by u we denote the first component of U) and the norm of the integral is bounded by $q\|U\|_\infty$ for some $q \in [0,1)$. By the fact that $\mathcal{Z}_{\mathcal{A}}^{\alpha+1} = \mathcal{Z}_{\mathcal{C}}^{\alpha+1}$ with equivalent norms this immediately follows from Lemma 4.7.6 showing that $(\mathcal{C}_\alpha + \mathcal{Q})_{|\mathcal{Z}_{\mathcal{C}}^{\alpha+1}}$ is a generator of a C_0-semigroup on $\mathcal{Z}_{\mathcal{C}}^{\alpha+1}$. Even more can be deduced from Lemma 4.7.6. The estimates show that the assumptions of Proposition 4.5.1(b) are satisfied for $\beta = 1/p$. Hence $\mathcal{Q}$ maps $\mathcal{Z}_{\mathcal{C}}^{\alpha+1}$ continuously to $\mathcal{Z}_{\mathcal{C}}^{\alpha+1/p-\varepsilon}$ for all $\varepsilon > 0$. From Corollary 4.3.4 we infer that $(\mathcal{C}_\alpha + \mathcal{Q})_{|\mathcal{Z}}$ is a generator. This yields the following well-posedness criteria for (IDE).

4.7.7 Proposition. *Let $p \in (1,\infty)$, $\alpha \in [-1,0]$ and $\ell \in BV_p(\mathbb{R}_+; \mathcal{L}(X^{\alpha+1}, X^\alpha))$. Then $(\mathcal{C}_* + \mathcal{Q})_{|\mathcal{Z}}$ is the generator of a C_0-semigroup on $\mathcal{Z} = X \times W_p^{-\alpha}(\mathbb{R}_+; X^\alpha)$. We further have:*

(a) If $-1/p < \alpha \le 0$ then $\mathcal{Q} \in \mathcal{L}(\mathcal{Z}_{\mathcal{C}}^{\alpha+1}, \mathcal{Z})$ and

$$D(\mathcal{C} + \mathcal{Q}) = \{\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in X^{\alpha+1} \times W_p^1(\mathbb{R}_+; X^\alpha);\ A_*x + f(0) \in X\}.$$

In particular $X^1 \times \{0\} \subseteq D(\mathcal{C} + \mathcal{Q})$ and thus (IDE) is well-posed.

(b) If $-1 \le \alpha \le -1/p$ then

$$\begin{aligned} D\left((\mathcal{C}_\alpha + \mathcal{Q})_{|\mathcal{Z}}\right) = \{&\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in X^{\alpha+1} \times W_p^1(\mathbb{R}_+; X^\alpha); \\ &A_*x + f(0) \in X,\ \ell(\cdot)x + \mathcal{D}f \in W_p^{-\alpha}(\mathbb{R}_+; X^\alpha)\}. \end{aligned}$$

Unique classical solutions of (IDE) exist for all $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D\left((\mathcal{C}_\alpha + \mathcal{Q})_{|\mathcal{Z}}\right)$, continuously depending (in the norm of X) on the initial value. Mild solutions of (IDE) exist for all $x \in X$ and $f \in W_p^{-\alpha}(\mathbb{R}_+; X^\alpha)$.

Proof. In both cases we already have seen that $\mathcal{E} := (\mathcal{C}_* + \mathcal{Q})_{|\mathcal{Z}}$ generates a C_0-semigroup. So it remains to show that the first component of the generated C_0-semigroup indeed solves (IDE). The well-posedness assertion in (a) and the statement about unique classical solutions in (b) are shown as in the proof of Proposition 4.7.2 and therefore omitted.

In order to treat the assertion on mild solutions in (b) let $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in \mathcal{Z}$. Let $\left(\begin{smallmatrix} x_n \\ f_n \end{smallmatrix}\right) \in D(\mathcal{E})$ ($n \in \mathbb{N}$) be a sequence converging to $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)$ in $\mathcal{Z}$ as $n \to \infty$. By $\left(\begin{smallmatrix} u(\cdot) \\ F(\cdot) \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} u_n(\cdot) \\ F_n(\cdot) \end{smallmatrix}\right)$ we denote the solutions of the abstract Cauchy problem associated with $\mathcal{E}$ for the initial values $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} x_n \\ f_n \end{smallmatrix}\right)$ ($n \in \mathbb{N}$), respectively. For $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$ we have

$$\dot{u}_n(t) = Au_n(t) + \delta_0 F_n(s), \tag{4.7.3}$$

$$\dot{F}_n(t) = Lu_n(t) + \mathcal{D}_{-\alpha}F_n(t). \tag{4.7.4}$$

As in the proof of Proposition 4.7.2 we see from (4.7.4) that $\delta_0 F_n(t) = \int_0^t \ell(t-s)u_n(s)\,ds + f_n(t)$ ($t \in \mathbb{R}_+$). Therefore we obtain

$$\delta_0 \int_0^t F_n(s)\,ds = \int_0^t \ell(t-r)\int_0^r u_n(s)\,ds\,dr + \int_0^t f_n(s)\,ds \quad (t \in \mathbb{R}_+). \tag{4.7.5}$$

In order to show that (4.7.5) also holds for u and F we first observe that $\mathcal{Z}_{\mathcal{E}}^{\alpha+1} = \mathcal{Z}_{\mathcal{C}}^{\alpha+1} = \mathcal{Z}_{\mathcal{A}}^{\alpha+1}$ with equivalent norms, where the first equation follows from Theorem 4.3.2 using $\mathcal{Q} \in \mathcal{L}(\mathcal{Z}_{\mathcal{C}}^{\alpha+1}, \mathcal{Z}_{\mathcal{C}}^{\alpha+1/p-\varepsilon})$ $(\varepsilon > 0)$. We also have $\mathcal{Z}_{\mathcal{E}}^{\alpha} = \mathcal{Z}_{\mathcal{C}}^{\alpha} = \mathcal{Z}_{\mathcal{A}}^{\alpha}$ with equivalent norms, the first equality being a consequence of $R(\lambda, \mathcal{E}) = (I - R(\lambda, \mathcal{C}_\alpha)\mathcal{Q})^{-1} R(\lambda, \mathcal{C}_\alpha)$ and the boundedness of the operators $(I - R(\lambda, \mathcal{C}_\alpha)\mathcal{Q})^{-1}$ and $I - R(\lambda, \mathcal{C}_\alpha)\mathcal{Q}$ in $\mathcal{Z}_{\mathcal{C}}^{\alpha}$ for $\lambda \in \mathbb{R}$ sufficiently large; cf. the proof of the Desch-Schappacher perturbation theorem [39; Equation (III.3.9)]. Let $\omega \geq 0$ be sufficiently large and assume that $\mathcal{Z}_{\mathcal{E}}^{\gamma}$ is equipped with the norm $\|(\mathcal{E} - \omega)^\gamma \cdot\|_{\mathcal{Z}}$ for $\gamma \in \mathbb{R}$. As $\begin{pmatrix} u(\cdot) \\ F(\cdot) \end{pmatrix}$ is a mild solution of the Cauchy problem associated to $\mathcal{E}$ (and therefore also to $\mathcal{E}_\alpha$) we obtain for $t \in \mathbb{R}_+$ the estimate

$$
\begin{aligned}
&\left\| \int_0^t u(s)\,ds \right\|_{X_A^{\alpha+1}} + \left\| \int_0^t F(s)\,ds \right\|_{W_p^1(\mathbb{R}_+;X^\alpha)} = \left\| \int_0^t \begin{pmatrix} u(s) \\ F(s) \end{pmatrix} ds \right\|_{\mathcal{Z}_{\mathcal{A}}^{\alpha+1}} \\
&\quad \leq c_1 \left\| \int_0^t \begin{pmatrix} u(s) \\ F(s) \end{pmatrix} ds \right\|_{\mathcal{Z}_{\mathcal{E}}^{\alpha+1}} = c_1 \left\| (\mathcal{E}_\alpha - \omega) \int_0^t \begin{pmatrix} u(s) \\ F(s) \end{pmatrix} ds \right\|_{\mathcal{Z}_{\mathcal{E}}^{\alpha}} \\
&\quad = c_1 \left\| \begin{pmatrix} u(t) \\ F(t) \end{pmatrix} - \begin{pmatrix} u(0) \\ F(0) \end{pmatrix} - \omega \int_0^t \begin{pmatrix} u(s) \\ F(s) \end{pmatrix} ds \right\|_{\mathcal{Z}_{\mathcal{E}}^{\alpha}} \\
&\quad \leq c_2 \left(t \sup_{s \in [0,t]} \left\| \begin{pmatrix} u(s) \\ F(s) \end{pmatrix} \right\|_{\mathcal{Z}_{\mathcal{E}}^{\alpha}} + \left\| \begin{pmatrix} u(t) - u(0) \\ F(t) - F(0) \end{pmatrix} \right\|_{\mathcal{Z}_{\mathcal{E}}^{\alpha}} \right) \leq c_3 \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|_{\mathcal{Z}_{\mathcal{A}}^{\alpha}} \leq c_4 \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|_{\mathcal{Z}}
\end{aligned}
$$

for constants $c_1, c_2, c_3, c_4 \geq 0$ independent of t in compact intervals in $\mathbb{R}_+$. This estimate implies that $\int_0^t u_n(s)\,ds \to \int_0^t u(s)\,ds$ $(n \to \infty)$ in $X_A^{\alpha+1}$ uniformly for t in compact intervals in $\mathbb{R}_+$. Hence for $t \in \mathbb{R}_+$ we have

$$
\int_0^t \ell(t-r) \int_0^r u_n(s)\,ds\,dr \to \int_0^t \ell(t-r) \int_0^r u(s)\,ds\,dr \quad (n \to \infty),
$$

where the convergence is in X_A^α. As $W_p^1(\mathbb{R}_+; X^\alpha) \hookrightarrow C_0(\mathbb{R}_+; X^\alpha)$ we also infer that $\delta_0 \int_0^t F_n(s) \to \delta_0 \int_0^t F(s)\,ds$ $(n \to \infty)$. This shows that

$$
\delta_0 \int_0^t F(s)\,ds = \int_0^t \ell(t-r) \int_0^r u(s)\,ds\,dr + \int_0^t f(s)\,ds \quad (t \in \mathbb{R}_+). \tag{4.7.6}
$$

Now using (4.7.6) and the closedness of A we infer from the integrated equation of (4.7.3) that u is indeed a mild solution of (IDE). ■

For our last result in this section we first note that by the same estimate obtained in the proof of Lemma 4.7.6(b) we have the following embedding.

4.7.8 Lemma. *Let $p \in [1, \infty)$. Let Y be a Banach space. For $\beta < 1/p$ we have $BV_p(\mathbb{R}_+; Y) \hookrightarrow W_p^\beta(\mathbb{R}_+; Y)$.*

Proof. By S we denote the left translation semigroup on $L_p(\mathbb{R}_+; Y)$. As in the proof of Lemma 4.7.6(b) we obtain $\| \int_0^t S(s)f\, ds\|_{p,1} \le ct^{1/p}\|f\|_{p,Var}$ $(f \in BV_p(\mathbb{R}_+; Y))$ for some $c \ge 0$. This shows that $BV_p(\mathbb{R}_+; Y)$ is continuously embedded into the Favard space of fractional order $1/p$ corresponding to S, which is further continuously embedded into $W_p^\beta(\mathbb{R}_+; Y)$ for any $\beta < 1/p$ by [39; Proposition II.5.33]. ■

We are now going to deal with

$$\text{(IDE')} \quad \dot u(t) = A_* u(t) + \int_0^t d\ell(s)u(t-s) + f(t), \quad u(0) = x \in X \quad (t \in \mathbb{R}_+),$$

where we assume that $\alpha \in (-1/p, 1-1/p)$ and ℓ is a p-integrable $\mathcal{L}(X^\alpha)$-valued function of bounded variation, which is left-continuous on $\mathbb{R}_+$ and additionally satisfies $\ell(0) = 0$. We will see that classical solutions of (IDE') only exists for $x \in X^1$ and $f \in W_p^{1-\alpha}(\mathbb{R}_+; X^\alpha)$ satisfying the coupling condition $f + \ell(\cdot)x \in W_p^{1-\alpha}(\mathbb{R}_+; X^\alpha)$ (if $\alpha < 0$ even a second coupling occurs). We therefore consider the integrated version of (IDE')

$$\text{(IE)} \quad u(t) = x + \int_0^t f(s)\, ds + \int_0^t (A_* + \ell(t-s))u(s)\, ds \quad (t \in \mathbb{R}_+).$$

A function u is called a solution of (IE) if $u \in C(\mathbb{R}_+; X^{\max\{0,\alpha\}})$ and (IE) holds for all $t \in \mathbb{R}_+$. We say that (IDE') is well-posed if a unique solution $u \in C(\mathbb{R}_+; X^{\max\{0,\alpha\}})$ of (IE) exists for all $x \in X^{\max\{0,\alpha\}}$ and $f = 0$, depending continuously (in the norm of X) on the initial value.

In Section 4.7.3 we will look at the equation (IDE$^\bullet$) obtained from (IDE') by applying integration by parts; see Remarks 4.7.11 for a comparison of the results for the two equations.

Let $\alpha \in (-1/p, 1-1/p)$. Let $\mathcal{Z}$, $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{T}$ be as above. We define $Lx := \mathcal{D}_*\ell(\cdot)x$ and $\tilde L x := \ell(\cdot)x$ $(x \in X^\alpha)$. Further let $\tilde{\mathcal{Q}} := \left(\begin{smallmatrix} 0 & 0 \\ \tilde L & 0 \end{smallmatrix}\right)$ with domain $\mathcal{Z}_\mathcal{C}^\alpha$. Since $\tilde L \in \mathcal{L}(X^\alpha, L_p(\mathbb{R}_+; X^\alpha))$ we have $\mathcal{Q} := \mathcal{C}_{\alpha-1}\tilde{\mathcal{Q}} \in \mathcal{L}(\mathcal{Z}_\mathcal{C}^\alpha, \mathcal{Z}_\mathcal{C}^{\alpha-1})$. As $\ell(0) = 0$ we obtain

$$\mathcal{Q} = \begin{pmatrix} \delta_0 \tilde L & 0 \\ L & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix}.$$

We show that $\mathcal{Q}$ is a Desch-Schappacher perturbation of $\mathcal{C}_\alpha$. To this end we have to reason that there is a $t_0 > 0$ such that for all $U \in C([0, t_0]; \mathcal{Z}_\mathcal{C}^\alpha)$ we have

$$\int_0^t \mathcal{T}_{\alpha-1}(t-s)\mathcal{Q}U(s)\, ds \in \mathcal{Z}_\mathcal{C}^\alpha \quad (t \in [0, t_0])$$

and the norm of the integral is bounded by $q\|U\|_\infty$ for some $q \in [0,1)$. For $\lambda > 0$ sufficiently large we have

$$R(\lambda, \mathcal{C}_{\alpha-1}) = \begin{pmatrix} R(\lambda, A_{\alpha-1}) & -R(\lambda, A_{\alpha-1})\delta_0 R(\lambda, \mathcal{D}_{-1}) \\ 0 & R(\lambda, \mathcal{D}_{-1}) \end{pmatrix}.$$

Let u be the first component of U. With $L_\lambda := R(\lambda, \mathcal{D}_{-1})L$ we can write

$$\begin{aligned}
&\int_0^t \mathcal{T}_{\alpha-1}(t-s)\mathcal{Q}U(s)\,ds \\
&\qquad = (\lambda - \mathcal{C}_{\alpha-1})\int_0^t \mathcal{T}_\alpha(t-s)R(\lambda, \mathcal{C}_{\alpha-1})\begin{pmatrix} 0 \\ Lu(s)\end{pmatrix} ds \\
&\qquad = (\lambda - \mathcal{C}_{\alpha-1})\int_0^t \mathcal{T}_\alpha(t-s)\begin{pmatrix} -R(\lambda, A_{\alpha-1})\delta_0 L_\lambda u(s) \\ L_\lambda u(s)\end{pmatrix} ds \\
&\qquad = (\lambda - \mathcal{C}_{\alpha-1})\int_0^t \begin{pmatrix} -R(\lambda, A_\alpha)T_\alpha(t-s)\delta_0 L_\lambda u(s) + R_\alpha(t-s)L_\lambda u(s) \\ S(t-s)L_\lambda u(s)\end{pmatrix} ds.
\end{aligned}$$

We first observe that $\lambda - C_{\alpha-1}$ is a bounded operator from $\mathcal{Z}_{\mathcal{A}}^{\alpha+1}$ to $\mathcal{Z}_{\mathcal{C}}^{\alpha}$. Hence it suffices to show that the integral term belongs to $\mathcal{Z}_{\mathcal{A}}^{\alpha+1}$ and that there is a $c \geq 0$ so that

$$\left\| \int_0^t \left(\begin{pmatrix} -R(\lambda, A_\alpha)T_\alpha(t-s)\delta_0 L_\lambda u(s) \\ 0 \end{pmatrix} + \begin{pmatrix} R_\alpha(t-s)L_\lambda u(s) \\ S(t-s)L_\lambda u(s)\end{pmatrix}\right) ds \right\|_{\mathcal{Z}_{\mathcal{A}}^{\alpha+1}}$$

is bounded by $ct^{1/p}\|u\|_\infty$ for all $t \in [0, t_0]$ and $u \in C([0, t_0]; X^\alpha)$. The norm of the integral of the first summand is easily seen to be bounded by $c_1 t$ for some $c_1 \geq 0$. As $R(\lambda, \mathcal{D}_*)\ell$ and therefore also $R(\lambda, \mathcal{D}_*)\mathcal{D}_*\ell = \lambda R(\lambda, \mathcal{D}_*)\ell - \ell$ are in $BV_p(\mathbb{R}_+; \mathcal{L}(X^\alpha))$ the necessary estimate for the second summand follows from Lemma 4.7.6. So we infer that $(\mathcal{C}_{\alpha-1} + \mathcal{Q})_{|\mathcal{Z}_{\mathcal{C}}^{\alpha}}$ is a generator.

From the estimates we also infer that $\mathcal{Q}$ is a bounded operator from $\mathcal{Z}_{\mathcal{C}}^{\alpha}$ to $\mathcal{Z}_{\mathcal{C}}^{\beta}$ for any $\beta < \alpha + 1/p - 1$. As β can be chosen to be $\alpha - 1$ we conclude by Corollary 4.3.4 the generator property of

$$\mathcal{E} := (\mathcal{C}_{\alpha-1} + \mathcal{Q})_{|\mathcal{Z}} = \big(\mathcal{C}_{\alpha-1}(I + \tilde{\mathcal{Q}})\big)_{|\mathcal{Z}}. \tag{4.7.7}$$

Taking into account that $I + \tilde{\mathcal{Q}} \in \mathcal{L}(\mathcal{Z}_{\mathcal{C}}^{\alpha})$ the domain of $\mathcal{E}$ is given by

$$D(\mathcal{E}) = \{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in \mathcal{Z}_{\mathcal{C}}^{\alpha};\, (I + \tilde{\mathcal{Q}})\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in \mathcal{Z}_{\mathcal{C}}^{1}\}.$$

If $\alpha \geq 0$ we have $\mathcal{Z}_{\mathcal{C}}^{1} = X^1 \times W_p^{1-\alpha}(\mathbb{R}_+; X^\alpha)$ and therefore

$$D(\mathcal{E}) = \{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in X^1 \times L_p(\mathbb{R}_+; X^\alpha);\, f + \tilde{L}x \in W_p^{1-\alpha}(\mathbb{R}_+; X^\alpha)\}.$$

(Note that $1-\alpha > 1/p$ and $\tilde{L}x \notin W_p^{1-\alpha}(\mathbb{R}_+; X^\alpha)$ generally.) If $\alpha < 0$ then $\mathcal{Z}_{\mathcal{C}}^1 = \{\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in X^{\alpha+1} \times W_p^{1-\alpha}(\mathbb{R}_+; X^\alpha);\ A_* x + f(0) \in X\}$. Together with $\ell(0) = 0$ this leads to

$$D(\mathcal{E}) = \{\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in X^{\alpha+1} \times L_p(\mathbb{R}_+; X^\alpha);\ f + \tilde{L}x \in W_p^{1-\alpha}(\mathbb{R}_+; X^\alpha),$$
$$A_* x + f(0) \in X\}.$$

In general $X^1 \times \{0\}$ does not belong to the domain of $\mathcal{E}$.

4.7.9 Proposition. *Let $p \in (1,\infty)$ and $\alpha \in (-1/p, 1-1/p)$. Assume that $\ell \in BV_p(\mathbb{R}_+; \mathcal{L}(X^\alpha))$. Then (IDE') is well-posed.*

Proof. Let $x \in X^{\max\{0,\alpha\}}$. We have to to show that the function u defined by $\begin{pmatrix} u(t) \\ F(t) \end{pmatrix} := e^{t\mathcal{E}} \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$ $(t \in \mathbb{R}_+)$ indeed solves (IE). We first observe that $u \in C(\mathbb{R}_+; X^\alpha)$ as $\left(\begin{smallmatrix} \in \\ \mathcal{Z} \end{smallmatrix}\right)_{\mathcal{E}}^{\alpha} = \mathcal{Z}_{\mathcal{A}}^{\alpha}$. From

$$\begin{pmatrix} u(t) - u(0) \\ F(t) - F(0) \end{pmatrix} = \begin{pmatrix} A_* & \delta_0 \\ L & \mathcal{D}_{-1} \end{pmatrix} \int_0^t \begin{pmatrix} u(s) \\ F(s) \end{pmatrix} ds \quad (t \in \mathbb{R}_+) \tag{4.7.8}$$

we see that $F(t) = \mathcal{D}_{-1} \int_0^t F(s)\, ds + \int_0^t \mathcal{D}_{-1} \tilde{L} u(s)\, ds$. Considering F as a mild solution of the inhomogeneous abstract Cauchy problem associated with $\mathcal{D}_{-1}$ on $W_p^{-1}(\mathbb{R}_+; X^\alpha)$ with inhomogeneity $Lu(\cdot)$ (where we have $\tilde{L}u(\cdot) \in C(\mathbb{R}_+; BV_p(\mathbb{R}_+; X^\alpha))$) we infer that

$$\begin{aligned} F(t) &= \int_0^t S_{-1}(s)(Lu(t-s))\, ds = \mathcal{D}_{-1} \int_0^t \ell(s+\cdot) u(t-s)\, ds \\ &= \int_0^t d\ell(s+\cdot) u(t-s). \end{aligned}$$

In particular $\delta_0 F(t) = \int_0^t d\ell(s) u(t-s)\, ds$. Now the first line of (4.7.8) reads as

$$u(t) = x + \int_0^t A_* u(s)\, ds + \int_0^t \int_0^s d\ell(r) u(s-r)\, dr\, ds \quad (t \in \mathbb{R}_+).$$

Using $\ell(0) = 0$ standard computations show that u is a solution of (IE).

In order to show that solutions are unique assume that $u \in C(\mathbb{R}_+; X^{\max\{0,\alpha\}})$ solves (IE) for the initial value $u(0) = 0$ and $f = 0$. For $t \in \mathbb{R}_+$ we define the function

$$F(t) := \int_0^t d\ell(s+\cdot) u(t-s)\, ds = \int_0^t S_{-1}(s) \mathcal{D}_{-1} \tilde{L} u(t-s)\, ds.$$

As $\tilde{L}u(\cdot) \in C(\mathbb{R}_+; BV_p(\mathbb{R}_+; X^\alpha))$ we see that F is unique mild solution of the inhomogeneous abstract Cauchy problem associated with $\mathcal{D}_{-1}$ for the initial value 0 and inhomogeneity $Lu(\cdot) \in C(\mathbb{R}_+; W_p^{-1}(\mathbb{R}_+; X^\alpha))$. Taking into account that $\delta_0 F(t) = \int_0^t d\ell(s)u(t-s)$ $(t \in \mathbb{R}_+)$ by definition we conculude that $\left(\begin{smallmatrix} u(\cdot) \\ F(\cdot) \end{smallmatrix}\right)$ solves the abstract Cauchy problem associated with $\mathcal{E}$ for the initial value $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. Since $\mathcal{E}$ is a generator we have $u = 0$.

The continuous dependency of the solution on the initial value directly follows from the semigroup properties. ■

4.7.3 An Integro-Differential Equation with Time-Derivative in the Delay Term

We now come back to the equation

$$\text{(IDE}^\bullet\text{)} \quad \dot{u}(t) = A_* u(t) + \int_0^t \ell(t-s)\dot{u}(s)ds + f(t), \quad u(0) = x \in X, \quad (t \in \mathbb{R}_+),$$

which we already have dealt with in Chapter 3. Now we assume that X is a Banach space, A is the generator of a C_0-semigroup on X and ℓ is an operator-valued function on $\mathbb{R}_+$ with values in $\mathcal{L}(Y, X^{-1})$, where Y is a Banach space satisfying $X \hookrightarrow Y \hookrightarrow X^{-1}$. The inhomogeneity f is supposed to belong to $L_{1,loc}(\mathbb{R}_+; X^{-1})$.

In contrast to the previous chapter we now treat (IDE$^\bullet$) in the larger space X^{-1}. This requires the weakening of the notions introduced in Definition 3.0.6. Now we call a function u a classical solution of (IDE$^\bullet$) if $u \in C^1(\mathbb{R}_+; X)$ and u satisfies (IDE$^\bullet$) in the space X^{-1}. Further a function $u \in C(\mathbb{R}_+; X)$ is a mild solution of (IDE$^\bullet$) if

$$\text{(IE')} \quad u(t) = x + \int_0^t \big(f(s) - \ell(s)x\big)\, ds + \int_0^t (A_* + \ell(t-s))u(s)\, ds \quad (t \in \mathbb{R}_+)$$

holds for all $t \in \mathbb{R}_+$. We say that (IDE$^\bullet$) is well-posed if for all $x \in X^1$ a unique classical solution $u \in C^1(\mathbb{R}_+; X)$ exists, which depends continuously (in the norm of X) on the initial value uniformly in compact intervals.

In Chapter 3 well-posedness of (IDE$^\bullet$) has been shown under conditions on ℓ which do not allow integration by parts. We supplement these results with further well-posedness conditions involving mixed regularity conditions on ℓ. In Corollary 4.8.7 we obtain similar conditions by employing delay semigroups.

In order to obtain solutions of (IDE$^\bullet$) we again start with the generator $\mathcal{C} = \left(\begin{smallmatrix} A & \delta_0 \\ 0 & \mathcal{D}_{-\alpha} \end{smallmatrix}\right)$ on either $\mathcal{Z} = X \times C_{bu}^{-\alpha}(\mathbb{R}_+; F_A^\alpha)$ or $\mathcal{Z} = X \times W_p^{-\alpha}(\mathbb{R}_+; X^\alpha)$ from Sections 4.6.1 and 4.6.2. We shall perturb $\mathcal{C}$ with the operator $\mathcal{Q} := \left(\begin{smallmatrix} 0 & 0 \\ \ell(\cdot)A_* & \ell(\cdot)\delta_0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ \ell(\cdot) & 0 \end{smallmatrix}\right)\mathcal{C}_*$ with a domain belonging to the scale $\big(\mathcal{Z}_\mathcal{C}^\gamma\big)_{\gamma\in\mathbb{R}}$. (We hope that the reader does not get confused by the fact that we use $\mathcal{Z}$, $\mathcal{C}$ and $\mathcal{Q}$ in two different contexts.) As for the other integro-differential equations the solutions of (IDE$^\bullet$) are given by the first component of the obtained Volterra semigroup.

4.7.10 Proposition. *(a) Assume that* $\alpha \in (-1,1]$, $\beta \in [\alpha,\alpha+1] \cap (0,1]$, $\ell(t) \in \mathcal{L}(X^{\beta-1}, F_A^\alpha)$ $(t \in \mathbb{R}_+)$ *and* $\ell(\cdot)x \in F_{\mathcal{D}}^{\beta-\alpha}$ $(x \in X^{\beta-1})$*. Then (IDE$^\bullet$) is well-posed. In particular this holds if* $\alpha \in (-1,0]$ *(and* $\beta = \alpha+1$*),* $\ell(t) \in \mathcal{L}(X^\alpha, F_A^\alpha)$ *and* $\ell(\cdot)x \in Lip(\mathbb{R}_+; F_A^\alpha)$ $(x \in X^\alpha)$*, or if* $\alpha \in (0,1]$ *(and* $\beta = \alpha$*),* $\ell(t) \in \mathcal{L}(X^{\alpha-1}, F_A^\alpha)$ *and* $\ell(\cdot)x \in L_\infty(\mathbb{R}_+; F_A^\alpha)$ $(x \in X^{\alpha-1})$*.*

(b) Assume that $p \in (1,\infty)$, $\alpha \in [-1, 1-1/p)$, $\beta \in (\alpha+1/p, \alpha+1] \cap [0,1]$, $\ell(t) \in \mathcal{L}(X^{\beta-1}, X^\alpha)$ $(t \in \mathbb{R}_+)$ *and* $\ell(\cdot)x \in W_p^{\beta-\alpha}(\mathbb{R}_+; X^\alpha)$ $(x \in X^{\beta-1})$*. Then (IDE$^\bullet$) is well-posed.*

(c) Assume that $p \in (1,\infty)$, $\alpha \in [-1,0]$ *and* $\ell \in BV_p(\mathbb{R}_+; \mathcal{L}(X^\alpha))$ *being left-continuous and satisfying* $\ell(0) = 0$*. If* $\alpha \in (-1/p, 0]$ *then (IDE$^\bullet$) is well-posed. If* $\alpha \in [-1,-1/p]$ *then a mild solution of (IDE$^\bullet$) exists for all* $x \in X$ *and* $g \in W_p^{-\alpha}(\mathbb{R}_+; X^\alpha)$*.*

Proof. For the cases (a) and (b) let $Lx := \ell(\cdot)x$ $(x \in X^{\beta-1})$.

(a) Let $\mathcal{Z} := X \times C_{bu}^{-\alpha}(\mathbb{R}_+; F_A^\alpha)$. The closed graph theorem implies that L belongs to $\mathcal{L}(X^{\beta-1}, F_{\mathcal{D}}^{\beta-\alpha})$ (see Lemma 3.1.1 for similar cases). From Section 4.6.1 we use that $\mathcal{Z}_{\mathcal{C}}^\gamma = X^\gamma \times C_{bu}^{\gamma-\alpha}(\mathbb{R}_+; F_A^\alpha)$ and $F_{\mathcal{C}}^\gamma = F_A^\gamma \times F_{\mathcal{D}}^{\gamma-\alpha}$ $(\gamma \in (\alpha-1, \alpha+1))$.

For $\beta \in [\alpha, \alpha+1)$ we see that $\mathcal{Q} \in \mathcal{L}(\mathcal{Z}_{\mathcal{C}}^\beta, F_{\mathcal{C}}^\beta)$. If $\beta = \alpha+1$ then we infer from $\left(\begin{smallmatrix} 0 & 0 \\ L & 0 \end{smallmatrix}\right) \in \mathcal{L}(\mathcal{Z}_{\mathcal{C}}^\alpha, \mathcal{F})$, where $\mathcal{F} := \{0\} \times F_{\mathcal{D}}^1$, and the embedding $\mathcal{F} \hookrightarrow F_{\mathcal{C}}^{\alpha+1}$ shown in Lemma 4.7.1, that $\mathcal{Q} = \left(\begin{smallmatrix} 0 & 0 \\ L & 0 \end{smallmatrix}\right)\mathcal{C}_\alpha \in \mathcal{L}(\mathcal{Z}_{\mathcal{C}}^{\alpha+1}, F_{\mathcal{C}}^{\alpha+1})$. By Corollary 4.4.3 we infer that $\mathcal{C} + \mathcal{Q}$ is a generator.

In order to show the well-posedness of (IDE$^\bullet$) we first show existence of solutions. Let $x \in X^1$. As $\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) \in D(\mathcal{C}+\mathcal{Q}) = D(\mathcal{C})$ we see that $\left(\begin{smallmatrix} u(t) \\ F(t) \end{smallmatrix}\right) := \exp(t(\mathcal{C}+\mathcal{Q}))\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)$ $(t \in \mathbb{R}_+)$ is the classical solution of the abstract Cauchy problem associated with $\mathcal{C}+\mathcal{Q}$. Thus we have

$$\dot{u}(t) = A_* u(t) + \delta_0 F(t), \tag{4.7.9}$$

$$\dot{F}(t) = L(A_* u(t) + \delta_0 F(t)) + \mathcal{D}_{-\alpha} F(t) \quad (t \in \mathbb{R}_+). \tag{4.7.10}$$

Using (4.7.9) we can write (4.7.10) as

$$\dot{F}(t) = L\dot{u}(t) + \mathcal{D}_{-\alpha} F(t) \quad (t \in \mathbb{R}_+). \tag{4.7.11}$$

From (4.7.11) we infer that F is the classical solution of the inhomogeneous abstract Cauchy problem associated with the left translation semigroup on $C_{bu}^{-\alpha}(\mathbb{R}_+; F_A^\alpha)$ with inhomogeneity $L\dot{u}(\cdot)$. This gives us

$$F(t) = \int_0^t S_{-1}(t-s) L\dot{u}(s)\, ds = \int_0^t \ell(t-s+\cdot)\dot{u}(s)\, ds \quad (t \in \mathbb{R}_+).$$

Therefore we obtain $\delta_0 F(t) = \int_0^t \ell(t-s)\dot{u}(s)\, ds$. This shows that u solves (IDE$^\bullet$).

In order to show that solutions of (IDE$^\bullet$) are unique assume that u is a solution of (IDE$^\bullet$) for the initial value $u(0) = 0$. Let $F(t) := \int_0^t S_{-1}(t-s) L\dot{u}(s)\, ds$ $(t \in \mathbb{R}_+)$. As $\delta_0 F(t) = \int_0^t \ell(t-s)\dot{u}(s)\, ds$ we see that equation (4.7.9) holds. This implies that F is a mild solution of the inhomogeneous abstract Cauchy problem associated with S_{-1} and with the continuous inhomogeneity $L\dot{u}(\cdot) = L(A_* u(\cdot) + \delta_0 F(\cdot)) \in C(\mathbb{R}_+; F_d^{\beta-\alpha})$. Hence F satisfies the integrated version of (4.7.10). Therefore $\left(\begin{smallmatrix} u(\cdot) \\ F(\cdot) \end{smallmatrix}\right)$ is a mild solution of the abstract Cauchy problem associated with $\mathcal{C} + \mathcal{Q}$. As mild solutions are unique we conclude $u = 0$.

The continuous dependence on the initial value follows from the uniform boundedness of the operators of the semigroup generated by $\mathcal{C} + \mathcal{Q}$ in compact intervals. This shows the well-posedness of (IDE$^\bullet$).

(b) Let $\mathcal{Z} := X \times W_p^{-\alpha}(\mathbb{R}_+; X^\alpha)$. The assumptions on ℓ imply that L belongs to $\mathcal{L}(X^{\beta-1}, W_p^{\beta-\alpha}(\mathbb{R}_+; X^\alpha))$ by the closed graph theorem (similarly as in Lemma 3.1.1). From Section 4.6.2 we use that $\mathcal{Z}_{\mathcal{C}}^\gamma = X^\gamma \times W_p^{\gamma-\alpha}(\mathbb{R}_+; X^\alpha)$ for all $\gamma \in (\alpha + 1/p - 1, \alpha + 1]$. Hence we have $\mathcal{Q} \in \mathcal{L}(\mathcal{Z}_{\mathcal{C}}^\beta)$. Again Corollary 4.4.3 shows that $\mathcal{C} + \mathcal{Q}$ is a generator. The well-posedness of (IDE$^\bullet$) is shown similarly as in (a).

(c) Once more we start with the generator $\mathcal{C}$ on $\mathcal{Z} := X \times W_p^{-\alpha}(\mathbb{R}_+; X^\alpha)$ from Section 4.6.2. In (4.7.7) we have concluded that $\mathcal{C}_{\alpha-1}(I + \tilde{\mathcal{Q}})$ is a generator on $\mathcal{Z}_{\mathcal{C}}^{\alpha-1}$, where $\tilde{\mathcal{Q}} = \left(\begin{smallmatrix} 0 & 0 \\ L & 0 \end{smallmatrix}\right)$ with $Lx := \ell(\cdot)x$ $(x \in X^\alpha)$. From [39; Theorem III.3.20(ii)] we infer that (and by using $\tilde{\mathcal{Q}} \in \mathcal{L}(\mathcal{Z}_{\mathcal{C}}^\alpha)$)

$$\Big((I + \tilde{\mathcal{Q}})\mathcal{C}_{\alpha-1}\Big)_{\big|(\mathcal{Z}_{\mathcal{C}}^{\alpha-1})^1_{\mathcal{C}_{\alpha-1}}} = \Big((I + \tilde{\mathcal{Q}})\mathcal{C}_{\alpha-1}\Big)_{|\mathcal{Z}_{\mathcal{C}}^\alpha} = (I + \tilde{\mathcal{Q}})\mathcal{C}_\alpha = \mathcal{C}_\alpha + \tilde{\mathcal{Q}}\mathcal{C}_\alpha =: \mathcal{E}$$

is a generator on $\left(\mathcal{Z}_{\mathcal{C}}^{\alpha-1}\right)^1_{\mathcal{C}_{\alpha-1}} = \mathcal{Z}_{\mathcal{C}}^\alpha$. If $\alpha = 0$ we see that $\mathcal{E} = \mathcal{C} + \mathcal{Q}$ is a generator on $\mathcal{Z}$ (observe that $\mathcal{Q} = \tilde{\mathcal{Q}}\mathcal{C}_\alpha$). For $\alpha \in [-1, 0)$ let $\lambda > 0$ be larger than the growth bound of C_α as a generator on $\mathcal{Z}_{\mathcal{C}}^\alpha$. The operator $\tilde{\mathcal{E}} := (I + \tilde{Q})(C_\alpha - \lambda) = \mathcal{E} - \lambda(I + \tilde{\mathcal{Q}})$ (with domain $\mathcal{Z}_{\mathcal{C}}^{\alpha+1}$) is a generator on $\mathcal{Z}_{\mathcal{C}}^\alpha$ since $\mathcal{E}$ is a generator on this space and $I + \tilde{\mathcal{Q}}$ is a bounded operator. From the boundedness of $I + \tilde{\mathcal{Q}}$ we also conclude that the norm $\|\cdot\|_{\mathcal{Z}_{\mathcal{C}}^{\alpha+1}}$ is finer than the graph norm $\|\cdot\|_{\tilde{\mathcal{E}}}$ on $\mathcal{Z}_{\mathcal{C}}^{\alpha+1}$. Hence by the open mapping theorem the two norms are equivalent (see the proof of [39; Theorem III.3.20(i)]). Therefore $\mathcal{E}_1 = \mathcal{E}_{|\mathcal{Z}_{\mathcal{C}}^{\alpha+1}}$ is a generator on $\mathcal{Z}_{\mathcal{C}}^{\alpha+1}$. Now we can infer from Corollary 4.3.4, taking into account that $\tilde{\mathcal{Q}}\mathcal{C}_\alpha \in \mathcal{L}(\mathcal{Z}_{\mathcal{C}}^{\alpha+1}, \mathcal{Z}_{\mathcal{C}}^{\alpha+1/p-\varepsilon})$ for any $\varepsilon > 0$ (see above the equation (4.7.7)), that $\mathcal{E}_{-\alpha} = \mathcal{E}_{|\mathcal{Z}}$ is a generator of a C_0-semigroup on $\mathcal{Z}$.

If $\alpha > -1/p$ then $\mathcal{E}_{|\mathcal{Z}} = \mathcal{C} + \tilde{\mathcal{Q}}\mathcal{C}$ and the domain of $\mathcal{E}$ is

$$D(\mathcal{E}) = \left\{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in X^{\alpha+1} \times W_p^{1-\alpha}(\mathbb{R}_+; X^\alpha);\ A_\alpha x + f(0) \in X \right\}.$$

Again the well-posedness of (IDE$^\bullet$) is shown similarly as in (a).

If $\alpha \in [-1, -1/p]$ we have

$$\begin{aligned} \mathcal{E}_{-\alpha} &= \mathcal{E}_{|\mathcal{Z}} \\ &= \left\{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in \mathcal{Z}_{\mathcal{C}}^{\alpha+1};\ (I + \tilde{\mathcal{Q}})\mathcal{C}_\alpha \in \mathcal{Z} \right\} \\ &= \Big\{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in X^{\alpha+1} \times W_p^1(\mathbb{R}_+; X^\alpha);\ A_\alpha x + f(0) \in X, \\ &\qquad\qquad \mathcal{D}f + L(A_* x + f(0)) \in W_p^{-\alpha}(\mathbb{R}_+; X^\alpha) \Big\}. \end{aligned}$$

Let $\binom{x}{f} \in \mathcal{Z}$ and $\binom{u(t)}{F(t)} := e^{t\mathcal{E}} \binom{x}{f}$ $(t \in \mathbb{R}_+)$. If $\binom{x}{f} \in D(\mathcal{E})$ then we see similar as in (a) that $u \in C(\mathbb{R}_+; X^{\alpha+1}) \cap C^1(\mathbb{R}_+; X)$ satisfies (IDE$^\bullet$). Integrating (IDE$^\bullet$) we obtain

$$u(t) = x + \int_0^t \big(f(s) - \ell(s)x\big)\, ds + \int_0^t (A_\alpha + \ell(t-s))u(s)\, ds \quad (t \in \mathbb{R}_+).$$

As $D(\mathcal{E})$ is dense in $\mathcal{Z}$ and as u depends continuously on the initial value in the norm of X (and therefore also in the norm of X^α) in compact intervals in $\mathbb{R}_+$ we see by an approximation argument that u is a mild solution of (IDE$^\bullet$) for all initial values $\binom{x}{f} \in \mathcal{Z}$. This shows assertion (c) for $\alpha \in [-1, -1/p]$. ■

4.7.11 Remarks. (a) As in Remarks 4.7.4(c) the assertion of Proposition 4.7.10(a) can be generalised. Let $\alpha \in (-1, 1]$, $\gamma \in (\max\{-1, \alpha-1\}, \alpha]$ and let $Y \hookrightarrow X^\gamma$ a Banach space satisfying (RC) with respect to A_α; cf. Proposition A.4. Let $\beta \in [\alpha, \gamma+1] \cap (0, 1]$. Then (IDE$^\bullet$) becomes well-posed if $\ell(t) \in \mathcal{L}(X^{\beta-1}, Y)$ $(t \in \mathbb{R}_+)$ and $\ell(\cdot)x \in F_{\mathcal{D}}^{\beta-\alpha}$ $(x \in X^{\beta-1})$.

(b) If we assume that $\ell \in BV_p(\mathbb{R}_+; \mathcal{L}(X^\alpha))$ as in Proposition 4.7.10(c) then integration by parts is applicable to (IDE$^\bullet$) leading to the equation (IDE') with inhomogeneity $f - \ell(\cdot)x$. Surprisingly the conditions under which (IDE') with inhomogeneity $f - \ell(\cdot)x$ possesses classical or mild solutions do not cover Proposition 4.7.10(c). So even for conditions for which integration by parts is applicable it is worse investigating (IDE$^\bullet$) rather than the corresponding inhomogeneous version of (IDE').

4.8 Delay Semigroups in the L_p-Context

Let $h \in (0, \infty]$ and $J := (-h, 0)$. In this section we are going to treat the equation

$$\text{(DE)} \quad \dot{u}(t) = Au(t) + Lu_t, \quad u(0) = x \in X, \quad u_0 = f \in L_p(J; X^\alpha)$$

where $p \in [1, \infty)$, $\alpha \in (-1/p, 1]$, X is a Banach space, A is the generator of a C_0-semigroup T and L is a delay operator on a function space related to $L_p(J; X^\alpha)$.

For delay semigroups in the L_p-context we refer to [19], [48], [69], [21], [20] and [22], see also Section 3.2.

First we introduce delay semigroups. In the second part we will perturb these C_0-semigroups to solve (DE).

4.8.1 Delay Semigroups

We start by introducing fractional order Sobolev spaces for the interval J, where (in contrast to the previous sections) we now need to take care of the zero boundary condition at 0 of the left translation semigroup on this interval. Let $p \in [1, \infty)$ and Y be a Banach space. We define $W_p^\gamma(-h, \infty; Y)$ and $V_p^\gamma(J; Y)$ as the fractional power spaces of order $\gamma \in \mathbb{R}$ with respect to the left translation semigroup $\check{S}$ on $L_p(-h, \infty; Y)$ and the left translation semigroup S on $L_p(J; Y)$ with zero boundary condition at

0, respectively. As $L_p(J;Y)$ can be identified with the $\check{S}$-invariant closed subspace $\{f \in L_p(-h,\infty;Y);\ \operatorname{spt} f \subseteq (-h,0]\}$, we see that S is the restriction of $\check{S}$ to $L_p(J;Y)$. This immediately shows that

$$V_p^\gamma(J;Y) = \{f \in W_p^\gamma(-h,\infty;Y);\ \operatorname{spt} f \subseteq (-h,0]\}.$$

We denote the generator of the left translation semigroup S on $L_p(J;Y)$ by $\mathcal{D}$. We point out that for $h=\infty$ the translation semigroup S has growth bound 0. So the fractional derivatives $(\mathcal{D}-\omega)^\alpha$ can only be evaluated for $\omega > 0$. For $h \in (0,\infty)$ the semigroup S is nilpotent and so has growth bound $-\infty$. Fractional derivatives exist for all $\omega \in \mathbb{R}$. Let $\omega_h := 0$ if $h < \infty$ and $\omega_h > 0$ if $h = \infty$. On $V_p^\gamma(J;X)$ ($\gamma \in \mathbb{R}$) we use the norm $\|\cdot\|_{p,\gamma} := \|(\mathcal{D}-\omega_h)^\gamma \cdot\|_p$.

For $\lambda \in \mathbb{C}$ we denote by ε_λ the function $(-\infty,0] \ni \vartheta \mapsto e^{\lambda\vartheta}$. For the definition of the fractional order Sobolev space $W_p^\gamma(J;Y)$ without a boundary condition at 0 we need the fractional derivative of $x \cdot \varepsilon_\lambda$ ($x \in X$, $\lambda > 0$) with respect to S.

4.8.1 Lemma. *Let $p \in [1,\infty)$. Let Y be a Banach space, $x \in Y$, $\alpha \in (0,1)$ and $\lambda,\omega \in \mathbb{R}$. If $h=\infty$ we require that $\lambda > \omega > 0$. Then $x \cdot \varepsilon_\lambda \in V_p^\alpha(J;Y)$ if and only if $\alpha < 1/p$ or $x = 0$ and in this case we have*

$$\begin{aligned}(\mathcal{D}-\omega)^\alpha&(x\cdot\varepsilon_\lambda)\\ &= J \ni \vartheta \mapsto \tilde{c}_{1-\alpha}\left((\lambda-\omega)e^{\lambda\vartheta}\int_0^{-\vartheta} e^{(\lambda-\omega)s}s^{-\alpha}\,ds - e^{\omega\vartheta}(-\vartheta)^{-\alpha}\right)x \end{aligned}\qquad (4.8.1)$$

(cf. (4.1.2) for the constant $\tilde{c}_{1-\alpha}$).

Proof. Using (4.1.2) we compute

$$(\mathcal{D}-\omega)^{\alpha-1}(x\cdot\varepsilon_\lambda) = \left(J \ni \vartheta \mapsto \tilde{c}_{1-\alpha}e^{\lambda\vartheta}\int_0^{-\vartheta} e^{(\lambda-\omega)s}s^{-\alpha}\,ds \cdot x\right).$$

Taking the derivative in $L_{1,loc}(J;X)$ we see that (4.8.1) holds in $L_{1,loc}(J;X)$. For the first term in (4.8.1) we easily obtain the estimate

$$|\tilde{c}_{1-\alpha}|(\lambda-\omega)e^{\lambda\vartheta}\int_0^{-\vartheta} e^{(\lambda-\omega)s}s^{-\alpha}\,ds \le |\tilde{c}_{1-\alpha}|\frac{\lambda-\omega}{1-\alpha}(-\vartheta)^{1-\alpha}e^{\lambda\vartheta} \quad (\vartheta \in J),$$

which belongs to $L_p(J)$ as a function in ϑ. The second part of (4.8.1), which is the function $\vartheta \mapsto -\tilde{c}_{1-\alpha}e^{\omega\vartheta}(-\vartheta)^{-\alpha}\cdot x$, is in $L_p(J;X)$ if and only if $\alpha < 1/p$ or $x=0$. This shows the assertion. ■

Let $\lambda_h := 0$ if $h \in (0,\infty)$ and $\lambda_h > \omega_h$ if $h=\infty$. For $\gamma \ge 1/p$ we define the Banach space

$$W_p^\gamma(J;Y) := V_p^\gamma(J;Y) \oplus \{x \cdot \varepsilon_{\lambda_h};\ x \in X\}.$$

By Lemma 4.8.1 we know that $W_p^\gamma(J;Y) \ni (f, x \cdot \varepsilon_{\lambda_h}) \mapsto f + x \cdot \varepsilon_{\lambda_h} \in L_p(J;Y)$ is injective. We therefore identify $W_p^\gamma(J;Y)$ with the space of all functions $f + x \cdot \varepsilon_{\lambda_h}$ with $f \in V_p^\gamma(J;X)$ and $x \in X$. We further remark that for $g \in W_p^\gamma(J;Y)$ the decomposition $g = f + x \cdot \varepsilon_{\lambda_h}$ is unique. We write $g(0) := x$. On $W_p^\gamma(J;X)$ we will use norm

$$\|g\|_{p,\gamma} := \|g(0)\| + \|g - g(0) \cdot \varepsilon_{\lambda_h}\|_{V_p^\gamma(J;Y)}.$$

We point out that for $\gamma > 1/p$ the mapping $W_p^\gamma(J;X) \ni g \mapsto g(0) \in X$ is bounded as $W_p^\gamma(-h,\infty;Y)$ is continuously embedded into $C_0([-h,\infty);Y)$, cf. Remarks 4.6.4(b). (We have again denoted the norm on $W_p^\gamma(J;Y)$ by $\|\cdot\|_{p,\gamma}$ as $V_p^\gamma(J;Y) \subseteq W_p^\gamma(J;Y)$ and both norms agree on $V_p^\gamma(J;Y)$.)

We now generalise delay semigroups introduced in [19]. Let $\alpha \in (-1/p, 1]$ and $\mathcal{Z} := X \times V_p^{-\alpha}(J;X^\alpha)$. On $\mathcal{Z}$ we consider the generator $\tilde{\mathcal{A}} := \left(\begin{smallmatrix} A & 0 \\ 0 & \mathcal{D}_{-\alpha} \end{smallmatrix}\right)$, $D(\tilde{\mathcal{A}}) := D(A) \times D(\mathcal{D}_{-\alpha})$. Let $\mathcal{B}\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) := \left(\begin{smallmatrix} 0 \\ -(\mathcal{D}_{-1}-\omega_h)(x \cdot \varepsilon_{\lambda_h}) \end{smallmatrix}\right)$ with domain $D(\mathcal{B}) := D(\tilde{\mathcal{A}}_{\alpha-1}) = \mathcal{Z}_{\tilde{\mathcal{A}}}^\alpha$. From Lemma 4.8.1 we conclude that $\mathcal{B} \in \mathcal{L}\big(\mathcal{Z}_{\tilde{\mathcal{A}}}^\alpha, \mathcal{Z}_{\tilde{\mathcal{A}}}^{\alpha+1/p-1-\varepsilon}\big)$ for any $\varepsilon > 0$. (We also infer that this assertion is not true for $\varepsilon = 0$.) Moreover $\mathcal{B}$ is a Desch-Schappacher perturbation of $\tilde{\mathcal{A}}_\alpha$ as we will show next (also cf. [39; Exercise III.3.8(5)(iv)]).

4.8.2 Lemma. *The operator $\mathcal{B}$ is a Desch-Schappacher perturbation of $\tilde{\mathcal{A}}_\alpha - \omega_h$.*

Proof. We will invoke [39; Corollary III.3.4] in order to prove the assertion. We have to show that

$$\int_0^t e^{r(\tilde{\mathcal{A}}_{\alpha-1}-\omega_h)} \mathcal{B} U(t-r)\, dr \in \mathcal{Z}_{\tilde{\mathcal{A}}}^\alpha = X^\alpha \times L_p(J;X) \tag{4.8.2}$$

for a fixed $t > 0$ and all $U \in L_p\big(0,t;\mathcal{Z}_{\tilde{\mathcal{A}}}^\alpha\big)$. Let $f := P_1 U(\cdot) \in L_p(0,t;X^\alpha)$. The integral in (4.8.2) can be written as

$$\begin{aligned}
\int_0^t e^{r(\tilde{\mathcal{A}}_{\alpha-1}-\omega_h)} \mathcal{B} U(t-r)\, dr &= (\tilde{\mathcal{A}}_{\alpha-1} - \omega_h) \int_0^t e^{r(\tilde{\mathcal{A}}_\alpha - \omega_h)} \begin{pmatrix} 0 \\ f(t-r) \cdot \varepsilon_{\lambda_h} \end{pmatrix} dr \\
&= \begin{pmatrix} A_{\alpha-1} - \omega_h & 0 \\ 0 & \mathcal{D}_{-1} - \omega_h \end{pmatrix} \begin{pmatrix} 0 \\ \int_0^t e^{-\omega_h r} S(r)(f(t-r) \cdot \varepsilon_{\lambda_h})\, dr \end{pmatrix} \\
&= \begin{pmatrix} A_{\alpha-1} - \omega_h & 0 \\ 0 & \mathcal{D}_{-1} - \omega_h \end{pmatrix} \begin{pmatrix} 0 \\ J \ni \vartheta \mapsto \int_0^{\min\{-\vartheta,t\}} e^{-\omega_h r} e^{\lambda_h(r+\vartheta)} f(t-r)\, dr \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ (\mathcal{D}_{-1} - \omega_h) \Big(J \ni \vartheta \mapsto \int_0^{\min\{-\vartheta,t\}} e^{-\omega_h r} e^{\lambda_h(r+\vartheta)} f(t-r)\, dr \Big) \end{pmatrix}.
\end{aligned}$$

For the second component a straightforward computation yields

$$\begin{aligned}(\mathcal{D}_{-1}-\omega_h)&\left(J\ni\vartheta\mapsto\int_0^{\min\{-\vartheta,t\}} e^{-\omega_h r}e^{\lambda_h(r+\vartheta)}f(t-r)\,dr\right)\\&=J\ni\vartheta\mapsto\lambda e^{\omega\vartheta}\int_0^{\min\{-\vartheta,t\}} e^{(\lambda_h-\omega_h)(r+\vartheta)}f(t-r)\,dr-e^{\omega\vartheta}f(t+\vartheta)\end{aligned}\tag{4.8.3}$$

(where we set $f(r):=0$ for $r<0$). For $h=\infty$ we infer that (using the Hölder-inequality)

$$\left\|\int_0^{\min\{-\vartheta,t\}} e^{(\lambda_h-\omega_h)(r+\vartheta)}f(t-r)\,dr\right\|\le\frac{\|f\|_p}{\big(p'(\lambda_h-\omega_h)\big)^{1/p'}}\quad(\vartheta\in J),$$

where p' denotes the conjugate exponent of p. For $h\in(0,\infty)$ the norm of this integral is bounded by $t^{1/p'}\|f\|_p$. Thus we see that the function in (4.8.3) is in $L_p(J;X)$. This shows that (4.8.2) holds and so the assumptions of [39; Corollary III.3.4] are met. ∎

Invoking Corollary 4.4.2(b) and Remarks 4.4.4(a) we now see that $\mathcal{A}:=(\tilde{\mathcal{A}}_{\alpha-1}+\mathcal{B})_{|\mathcal{Z}}$ is a generator of a C_0-semigroup on $\mathcal{Z}$. By $\mathcal{T}$ we denote the C_0-semigroup generated by $\mathcal{A}$. We call this C_0-semigroup the *delay semigroup*. We first give a description of the fractional power spaces associated with $\mathcal{A}$.

4.8.3 Lemma. *Let $p\in[1,\infty)$ and $\alpha\in(-1/p,1]$.*
(a) If $\beta\in(\alpha-1,\alpha+1/p)$ then $\mathcal{Z}^\beta_{\mathcal{A}}=X^\beta\times V_p^{\beta-\alpha}(J;X^\alpha)$.
(b) If $\beta\in[\alpha+1/p,\alpha+1/p+1)$ then

$$\mathcal{Z}^\beta_{\mathcal{A}}=\{\left(\begin{smallmatrix}x\\f\end{smallmatrix}\right)\in X^\beta\times W_p^{\beta-\alpha}(J;X^\alpha);\ f(0)=x\}$$

and the norm of $\mathcal{Z}^\beta_{\mathcal{A}}$ is equivalent to the norm $\mathcal{Z}^\beta_{\mathcal{A}}\ni\left(\begin{smallmatrix}x\\f\end{smallmatrix}\right)\mapsto\|x\|_\beta+\|f\|_{W_p^{\beta-\alpha}(J;X^\alpha)}$.
(c) For the domain of $D(\mathcal{A})$ we have

$$D(\mathcal{A})=\begin{cases}X^1\times V_p^{1-\alpha}(J;X^\alpha) & \text{if } \alpha\in(1-1/p,1],\\ \big\{\left(\begin{smallmatrix}x\\f\end{smallmatrix}\right)\in X^1\times W_p^{1-\alpha}(J;X^\alpha);\ f(0)=x\big\} & \text{if } \alpha\in(-1/p,1-1/p].\end{cases}$$

Proof. Assertion (a) follows from Theorem 4.3.2 and $\mathcal{Z}^\beta_{\tilde{\mathcal{A}}}=X^\beta\times V_p^{\beta-\alpha}(J;X^\alpha)$.

In order to show assertion (b) we first observe that $\mathcal{Z}^\beta_{\mathcal{A}}=D(\mathcal{A}_{\beta-1})$. From Corollary 4.3.4 we conclude that $\mathcal{A}_{\beta-1}=(\tilde{\mathcal{A}}_{\alpha-1}+\mathcal{B})_{|\mathcal{Z}^{\beta-1}_{\tilde{\mathcal{A}}}}$. Using (a) this yields

$$\begin{aligned}D(\mathcal{A}_{\beta-1})&=\big\{\left(\begin{smallmatrix}x\\f\end{smallmatrix}\right)\in\mathcal{Z}^\alpha_{\tilde{\mathcal{A}}};\ (\tilde{\mathcal{A}}_{\alpha-1}+\mathcal{B})\left(\begin{smallmatrix}x\\f\end{smallmatrix}\right)\in\mathcal{Z}^{\beta-1}_{\tilde{\mathcal{A}}}\big\}\\&=\big\{\left(\begin{smallmatrix}x\\f\end{smallmatrix}\right)\in X^\alpha\times L_p(J;X^\alpha);\ A_*x\in X^{\beta-1},\\&\qquad\qquad(\mathcal{D}_*-\omega_h)(f-x\cdot\varepsilon_{\lambda_h})\in V_p^{\beta-1-\alpha}(J;X^\alpha)\big\}\\&=\big\{\left(\begin{smallmatrix}x\\f\end{smallmatrix}\right)\in X^\beta\times L_p(J;X^\alpha);\ f-x\cdot\varepsilon_{\lambda_h}\in V_p^{\beta-\alpha}(J;X^\alpha)\big\}\\&=\big\{\left(\begin{smallmatrix}x\\f\end{smallmatrix}\right)\in X^\beta\times W_p^{\beta-\alpha}(J;X^\alpha);\ f(0)=x\big\}.\end{aligned}$$

Let $\omega > 0$ be larger than the growth bound of T and $\mathcal{T}$. In order to show the equivalence of the norms we assume that $\mathcal{Z}^{\beta-1}_{\tilde{\mathcal{A}}}$ is equipped with the norm $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \mapsto \|(A_* - \omega)^\beta\| + \|(\mathcal{D}_* - \omega)^{\beta-1-\alpha} f\|_p$. From (a) we know that $\mathcal{Z}^{\beta-1}_{\tilde{\mathcal{A}}} = \mathcal{Z}^{\beta-1}_{\mathcal{A}}$ with equivalent norms. Hence the norm of $\mathcal{Z}^{\beta}_{\mathcal{A}}$ is equivalent to the graph norm

$$\mathcal{Z}^{\beta}_{\mathcal{A}} \ni \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \mapsto \|(\mathcal{A}_{\beta-1} - \omega)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\|_{\mathcal{Z}^{\beta-1}_{\tilde{\mathcal{A}}}}. \tag{4.8.4}$$

From

$$\|(\mathcal{A}_{\beta-1} - \omega)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\|_{\mathcal{Z}^{\beta-1}_{\tilde{\mathcal{A}}}} = \|(A-\omega)^\beta x\| + \|(\mathcal{D} - \omega)^{\beta-\alpha}(f - x \cdot \varepsilon_{\lambda_h})\|_p$$

for $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in \mathcal{Z}^{\beta}_{\mathcal{A}}$, the boundedness of $(\mathcal{D}-\omega)^{\beta-\alpha}(\mathcal{D}-\omega_h)^{\alpha-\beta}$ and $(\mathcal{D}-\omega_h)^{\beta-\alpha}(\mathcal{D}-\omega)^{\alpha-\beta}$ as operators on $L_p(J; X^\alpha)$ and the inequality $\|x\|_\alpha \le \|(A-\omega)^{\alpha-\beta}\|_{\mathcal{L}(X^\alpha)}\|x\|_\beta$ we see that

$$\begin{aligned} \|(A-\omega)^\beta x\| + \|(\mathcal{D} - \omega)^{\beta-\alpha}(f - x \cdot \varepsilon_{\lambda_h})\|_p &\le c_1 \left(\|x\|_\beta + \|f\|_{W_p^{\beta-\alpha}(J;X^\alpha)}\right) \\ &\le c_2 \left(\|(A-\omega)^\beta x\| + \|(\mathcal{D} - \omega)^{\beta-\alpha}(f - x \cdot \varepsilon_{\lambda_h})\|_p\right) \end{aligned}$$

for some $c_1, c_2 \ge 0$. Therefore the norm in (4.8.4) is further equivalent to the norm $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \mapsto \|x\|_\beta + \|f\|_{W_p^{\beta-\alpha}(J;X^\alpha)}$. This shows assertion (b).

Assertion (c) follows from (a) and (b) by observing that $D(\mathcal{A}) = \mathcal{Z}^1_{\mathcal{A}}$. ∎

For $\beta \in [\alpha, \alpha + 1/p + 1)$ the semigroup operators are given by $\mathcal{T}_\beta(t) = \left(\begin{smallmatrix} T_\beta(t) & 0 \\ V_\beta(t) & S_{\beta-\alpha}(t) \end{smallmatrix}\right)$, where

$$(V_\beta(t)x)(\vartheta) := \begin{cases} 0 & \text{if } t + \vartheta < 0, \\ T_\beta(t+\vartheta)x & \text{if } t + \vartheta \ge 0; \end{cases}$$

cf. [19]. (Recall that for $\beta \ge \alpha + 1/p$ the space $\mathcal{Z}^\beta_{\mathcal{A}}$ contains a coupling.)

Before we perturb the delay semigroup we present a result concerning time and space regularity properties of semigroup trajectories. The delay semigroup $\mathcal{T}$ has the interesting feature that the second component (to be precise the operator family V_β) "records" the function $T(\cdot)x$ in a fractional order Sobolev space. The result is a corollary of Lemma 4.8.3.

4.8.4 Corollary. *Let $h < \infty$, $p \in [1, \infty)$, $\alpha \in (-1/p, 1]$ and $\beta \in [\alpha, \alpha + 1/p + 1)$. If $x \in X^\beta$ then*

$$\big(J \ni \tau \mapsto T(\tau + h)x\big) \in \begin{cases} W_p^{\beta-\alpha}(J; X^\alpha) & \text{if } \beta - \alpha \ge 1/p, \\ V_p^{\beta-\alpha}(J; X^\alpha) & \text{if } \beta - \alpha < 1/p, \end{cases}$$

and $\|T(\cdot + h)x\|_{W_p^{\beta-\alpha}(J,X^\alpha)} \le c\|x\|_\beta$, where $c \ge 0$ depends on h.

Proof. First assume that $\beta \in [\alpha, \alpha + 1/p)$. Then the assertion follows by observing that $V_\beta(h)x$ is the desired function and $V_\beta(h)$ is a bounded operator from X^β to $V_p^{\beta-\alpha}(J; X^\alpha)$.

If $\beta \in [\alpha + 1/p, \alpha + 1/p + 1)$ then we have $\left(\begin{smallmatrix} x \\ x\cdot \mathbf{1}_J \end{smallmatrix}\right) \in \mathcal{Z}_\mathcal{A}^\beta$. Therefore

$$\begin{pmatrix} T_\beta(h)x \\ V_\beta(h)x \end{pmatrix} = \mathcal{T}_\beta(h) \begin{pmatrix} x \\ x \cdot \mathbf{1}_J \end{pmatrix} \in \mathcal{Z}_\mathcal{A}^\beta.$$

Using Lemma 4.8.3(b) we infer

$$\|V_\beta(h)x\|_{W_p^{\beta-\alpha}(J,X^\alpha)} \leq c_1 \|\mathcal{T}_\beta(h)\| \left\| \begin{pmatrix} x \\ x \cdot \mathbf{1}_J \end{pmatrix} \right\|_{\mathcal{Z}_\mathcal{A}^\beta} \leq c_2 \|\mathcal{T}_\beta(h)\| \, \|x\|_\beta$$

for some constants $c_1, c_2 \geq 0$. ∎

4.8.2 Perturbation of Delay Semigroups

We are now going to perturb $\mathcal{A}$ with $\left(\begin{smallmatrix} 0 & L \\ 0 & 0 \end{smallmatrix}\right)$ defined on a suitable domain. We only give conditions on L for which this perturbation yields a C_0-semigroup. For the arguments showing that the first component of this semigroup indeed solves (DE) we refer to [19; Proposition 2.3].

We begin with a result obtained by applying the Miyadera-Voigt type perturbation theorem. To this end let $\bar{J}$ be the closure of J in $\mathbb{R}$. For a Borel measure μ on $\bar{J}$ and $t \geq 0$ we define the function

$$\nu_{\mu,t} : \mathbb{R} \to \mathbb{R}_+, \quad \nu_{\mu,t}(\vartheta) := \mu\big((\vartheta - t, \vartheta] \cap \bar{J}\big) \quad (\vartheta \in \mathbb{R})$$

(cf .[69; Sections 3 and 4] for details).

4.8.5 Proposition. *Let $p \in (1, \infty)$, $\alpha \in (-1/p, 1 - 1/p)$ and $L \in \mathcal{L}(W_p^1(J; X^\alpha), X^\alpha)$. Assume that there is a Borel measure μ on $\bar{J}$ and $r \in [1, p]$ such that $\nu_{\mu,1} \in L_{\frac{p}{p-r}}(-h, 1)$ and*

$$\|Lf\| \leq \|f\|_{L_r(\mu;X^\alpha)} \quad (f \in W_p^1(J; X^\alpha)).$$

Let $\mathcal{Q} := \left(\begin{smallmatrix} 0 & L \\ 0 & 0 \end{smallmatrix}\right)$, $D(\mathcal{Q}) := D(\mathcal{A}_\alpha)$. Then $\mathcal{C} := (\mathcal{A}_* + \tilde{\mathcal{Q}})_{|\mathcal{Z}}$ is the generator of a C_0-semigroup on $\mathcal{Z}$. The domain of $\mathcal{C}$ is given by

$$D(\mathcal{C}) = \begin{cases} \big\{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in X^1 \times W_p^{1-\alpha}(J; X^\alpha);\ f(0) = x \big\} & \text{if } \alpha \geq 0, \\ \big\{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in X^{\alpha+1} \times W_p^{1-\alpha}(J; X^\alpha);\ A_*x + Lf \in X,\ f(0) = x \big\} & \text{if } \alpha < 0. \end{cases}$$

Proof. In [69; Theorem 3.1] it was shown that $\mathcal{Q}$ is a Miyadera-Voigt perturbation of $\mathcal{A}_\alpha$ and

$$\int_0^t \|\mathcal{Q}\mathcal{T}_\alpha(s)U\| \, ds \leq ct^{1-1/p} \|U\|_{\mathcal{Z}_\mathcal{A}^\alpha} \quad (U \in \mathcal{Z}_\mathcal{A}^{\alpha+1},\ t \in [0, 1]),$$

for some $c \geq 0$. From Proposition 4.5.1(a) we see that $\mathcal{Q}$ extends to a bounded operator $\tilde{\mathcal{Q}}$ in $\mathcal{L}(\mathcal{Z}_{\mathcal{A}}^{\alpha+1/p+\varepsilon}, \mathcal{Z}_{\mathcal{A}}^{\alpha})$ for any $\varepsilon > 0$. From Corollary 4.3.4 we conclude the generator property of $\mathcal{C}$. If $\alpha \geq 0$ then the domain of $D(\mathcal{C})$ and $D(\mathcal{A})$ coincide and so the assertion on the domain of $D(\mathcal{C})$ follows from Lemma 4.8.3. If $\alpha < 0$ then the domain of $\mathcal{C}$ is the set of all $\binom{x}{f} \in \mathcal{Z}_{\mathcal{A}}^{\alpha+1}$ for which $\begin{pmatrix} A_\alpha x + Lf \\ \mathcal{D}(f - f(0)\cdot\varepsilon_{\lambda_h}) \end{pmatrix} \in \mathcal{Z}$. Hence $x \in X^{\alpha+1}$, $f \in W_p^{1-\alpha}(J; X^\alpha)$, $f(0) = x$ and $A_* x + Lf \in X$. ■

Our next aim is the proof of our second perturbation result which is an application of the Desch-Schappacher perturbation theorem. We point out that the result particularly holds for $Y = X^\beta$ (and with $\gamma = \beta$, that is the simplest case), for $Y = F_A^\beta$ (and with $\gamma < \beta$) and for $Y = X^\gamma$ (with $\gamma > \beta - 1$) provided that T is an analytic semigroup.

4.8.6 Proposition. *Let $p \in [1, \infty)$, $\alpha \in (-1/p, 1]$, $\beta \in [-1, 1] \cap (\alpha - 1, \alpha + 1]$ and $\gamma \in [-1, 1] \cap (\beta - 1, 1]$. Let $Y \hookrightarrow X^\gamma$ be a Banach space satisfying (RC) with respect to A_β. Assume that*

$$L \in \begin{cases} \mathcal{L}(V_p^{\beta-\alpha}(J; X^\alpha), Y) & \text{if } \beta - \alpha < 1/p, \\ \mathcal{L}(W_p^{\beta-\alpha}(J; X^\alpha), Y) & \text{if } \beta - \alpha \geq 1/p. \end{cases}$$

Let $\mathcal{Q} := \left(\begin{smallmatrix} 0 & L \\ 0 & 0 \end{smallmatrix}\right)$ with domain $\mathcal{Z}_{\mathcal{A}}^\beta$. Then $\mathcal{C} := (\mathcal{A}_* + \mathcal{Q})_{|\mathcal{Z}}$ is a generator on $\mathcal{Z} = X \times V_p^{-\alpha}(\mathbb{R}_+; X^\alpha)$. The domain of $\mathcal{C}$ is given by

$$\begin{array}{ll} \left\{ \binom{x}{f} \in X^{\gamma+1} \times W_p^{1-\alpha}(J; X^\alpha);\ A_* x + Lf \in X,\ f(0) = x \right\} & \text{if } \gamma < 0,\ \alpha \leq 1 - 1/p, \\ \left\{ \binom{x}{f} \in X^{\gamma+1} \times V_p^{1-\alpha}(J; X^\alpha);\ A_* x + Lf \in X \right\} & \text{if } \gamma < 0,\ \alpha > 1 - 1/p, \\ \left\{ \binom{x}{f} \in X^{1} \times W_p^{1-\alpha}(J; X^\alpha);\ f(0) = x \right\} & \text{if } \gamma \geq 0,\ \alpha \leq 1 - 1/p, \\ X^1 \times V_p^{1-\alpha}(J; X^\alpha) & \text{if } \gamma \geq 0,\ \alpha > 1 - 1/p. \end{array}$$

Before we present the lengthy proof we give a corollary, which was the motivation for this proposition. In the corollary we obtain a well-posedness condition for (IDE$^\bullet$) generalising Theorem 3.2.1; cf. Chapter 3 and Section 4.7.3.

4.8.7 Corollary. *Let $p \in [1, \infty)$ and $\alpha \in (-1/p, 0]$. Let $Y \hookrightarrow X^\alpha$ be a Banach space satisfying (RC) with respect to $A_{\alpha+1}$. Assume that $\ell \colon \mathbb{R}_+ \to \mathcal{L}(X^\alpha, Y)$ is strongly Bochner measurable with respect to Y (i.e. $\ell(\cdot)x$ is Bochner measurable with respect to Y for all $x \in X$) and $\|\ell(\cdot)\|_{\mathcal{L}(X,Y)}$ is dominated by some $h \in L_{p',loc}(\mathbb{R}_+)$ where p' denotes the conjugate exponent of p. Then (IDE$^\bullet$) is well-posed (in the sense of Section 4.7.3).*

Proof. Let $h > 0$ and $J := (-h, 0)$. By our assumptions $Lf := \int_{-h}^0 \ell(-s)\dot{f}(s)\,ds$ ($f \in W_p^1(J; X^\alpha)$) is an operator in $\mathcal{L}(W_p^1(J; X^\alpha), Y)$. Thus we can apply Proposition 4.8.6 with $\beta = \alpha + 1$. Observing that $x \cdot \mathbf{1}_J \in W_p^{1-\alpha}(J; X^\alpha)$ for all $x \in X^\alpha$ we can literally copy the proof of Theorem 3.2.1 in order to obtain the well-posedness of (IDE$^\bullet$). ■

In order to be able to apply the Desch-Schappacher perturbation theorem to prove Proposition 4.8.6 we need some preparation. Namely we will show in Corollary 4.8.12 that if a Banach space Y satisfies (RC) in Proposition A.3 with respect to A_β for a

$\beta \in [\alpha, \alpha+1]$ then $Y \times \{0\}$ satisfies (RC) with respect to $\mathcal{A}_\beta$. After Corollary 4.8.12 has been shown we are prepared for the proof of Proposition 4.8.6.

In some of the following lemmata we assume that $h < \infty$ and that T has negative growth bound. As the translation semigroup S is nilpotent with these additional assumptions this implies that $\mathcal{T}$ and $\tilde{\mathcal{T}}$ both have negative growth bound. (In fact if $\mathcal{Z}$ is equipped with e.g. the sum norm then both have the same growth bound as T.) These two assumptions simplify the computations considerably and does not restrict the applicability of Corollary 4.8.12. (For C_0-semigroups T not having a negative growth bound we will later use a rescaling argument. The case $h = \infty$ will be dealt with by using estimates obtained in the case $h < \infty$.)

4.8.8 Lemma. *Let $p \in [1, \infty)$ and $\alpha \in (-1/p, 1]$. Let $Y_1 \hookrightarrow X^{\alpha-1}$ and $Y_2 \hookrightarrow X^\alpha$ be Banach spaces satisfying (RC) for A_α and $A_{\alpha+1}$, respectively. Then $Y_1 \times \{0\}$ and $Y_2 \times \{0\}$ satisfy (RC) for $\mathcal{A}_\alpha$ and $\mathcal{A}_{\alpha+1}$, respectively.*

Proof. Let (RC) for Y_1 with respect to A_α be satisfied for $t_0 \in (0, h]$ and $a \in C([0, t_0])$. Let $\varphi \in C([0, t_0]; Y_1)$ and assume that $\operatorname{rg}\varphi \subseteq X^\alpha$. For $t \in [0, t_0]$ we can write

$$\int_0^t \mathcal{T}_{\alpha-1}(t-s) \begin{pmatrix} \varphi(s) \\ 0 \end{pmatrix} ds = \begin{pmatrix} \int_0^t T_{\alpha-1}(t-s)\varphi(s)\, ds \\ \vartheta \mapsto \int_0^t T_{\alpha-1}(t-s+\vartheta)\varphi(s)\, ds \end{pmatrix},$$

where we set $T_{\alpha-1}(s) := 0$ for $s < 0$. By assumption the first component belongs to X^α and its norm is bounded by $a(t)\|\varphi\|_{C([0,t_0];Y_1)}$. We also see that the second component is a function with values in X^α. Its norm in $L_p(J; X^\alpha)$ can be estimated by

$$\begin{aligned} \Big\| \int_0^t T_{\alpha-1}(s+\cdot)\varphi(t-s)\, ds \Big\|_p^p &= \int_{-h}^0 \Big\| \int_0^{\max\{t+\vartheta,0\}} T_{\alpha-1}(s)\varphi(t-s+\vartheta)\, ds \Big\|_\alpha^p d\vartheta \\ &\le \int_{-h}^0 a(\max\{t+\vartheta, 0\})^p\, d\vartheta\, \|\varphi\|^p_{C([0,t_0];Y_1)} \\ &\le \Big(t \sup_{s\in[0,t]} a(s) \|\varphi\|_{C([0,t_0];Y_1)} \Big)^p. \end{aligned}$$

Since $t \sup_{s\in[0,t]} a(s)$ goes to zero as $t \to 0$ condition (RC) is satisfied for all $\varphi \in C([0, t_0]; Y_1)$ with values in X^α. A continuity argument shows the assertion for all $\varphi \in C([0, t_0]; Y_1)$.

For the proof of the case of the space Y_2 we refer to [48; Theorem 3.1]. The computations done there carry over to this slightly more general case without changes. ■

For the following computations we recall the definition of the incomplete Beta-function

$$B(t, \gamma_1, \gamma_2) := \int_0^t \tau^{\gamma_1-1}(1-\tau)^{\gamma_2-1}\, d\tau \quad (t \in [0,1], \gamma_1, \gamma_2 > 0).$$

We also recall that $B(1, 1-\gamma, \gamma) = \Gamma(1-\gamma)\Gamma(\gamma)$ for $\gamma \in (0,1)$. Finally we define the function

$$B^\gamma(s,\vartheta) := \begin{cases} B\big(-\frac{\vartheta}{s-\vartheta}, 1-\gamma, \gamma\big) & \text{if } \vartheta < 0, \\ 0 & \text{if } \vartheta \geq 0, \end{cases}$$

for $(s,\vartheta) \in \mathbb{R}_+ \times (-h, \infty)$ and $\gamma \in (0,1)$. Properties of this function relevant for our purposes are explored in Lemma 4.8.10.

4.8.9 Lemma. *Let $p \in [1,\infty)$ and $\alpha \in (-1/p, 1]$. Assume that $h \in (0,\infty)$ and that T has negative growth bound. Let $Y \hookrightarrow X^{\alpha-1}$ be a Banach space satisfying (RC) with respect to A_α. Let $\gamma \in (0,1)$. For $x \in Y$ we define the operator*

$$Kx := P_2\big(\mathcal{A}_{\alpha-1}^{-\gamma} - \tilde{\mathcal{A}}_{\alpha-1}^{-\gamma}\big)\begin{pmatrix} x \\ 0 \end{pmatrix}.$$

Then $\mathcal{D}^{\gamma-1}K \in \mathcal{L}(Y, C_b(J; X^\alpha))$ and

$$\mathcal{D}^{\gamma-1}Kx = \tilde{c}_{1-\gamma}\tilde{c}_\gamma \int_0^\infty B^\gamma(s,\cdot)T_{\alpha-1}(s)x\,ds \quad (x \in Y). \tag{4.8.5}$$

Proof. Let $t_0 \in (0,h]$ and $a \in C([0,t_0])$ with $a(0) = 0$ such that Y satisfies (RC) with respect to A_α with t_0 and a. Let $\omega > 0$ and $M \geq 1$ such that $\|T_{\alpha-1}(t)\| \leq Me^{-\omega t}$ $(t \in \mathbb{R}_+)$. Using (4.1.2) we infer for $x \in X^\alpha$ and $\vartheta \in J$ the formula

$$(Kx)(\vartheta) = \tilde{c}_\gamma \int_{-\vartheta}^\infty s^{\gamma-1}T_{\alpha-1}(s+\vartheta)x\,ds = \tilde{c}_\gamma \int_0^\infty (s-\vartheta)^{\gamma-1}T_{\alpha-1}(s)x\,ds.$$

Again from (4.1.2) we obtain for $f \in L_p(J; X^\alpha)$ the formula

$$\mathcal{D}^{\gamma-1}f = \left(J \ni \vartheta \mapsto \tilde{c}_{1-\gamma}\int_0^{-\vartheta} r^{-\gamma}f(r+\vartheta)\,dr\right).$$

As $\mathcal{A}_\alpha^{-\gamma}$ and $\tilde{\mathcal{A}}_\alpha^{-\gamma}$ are bounded operators on $X^\alpha \times L_p(J; X^\alpha)$ we see that $Kx \in L_p(J; X^\alpha)$ and thus

$$\begin{aligned}\mathcal{D}^{\gamma-1}Kx &= \left(J \ni \vartheta \mapsto \tilde{c}_{1-\gamma}\tilde{c}_\gamma \int_0^\infty \left(\int_0^{-\vartheta} r^{-\gamma}(s-r-\vartheta)^{\gamma-1}\,dr\right) T_{\alpha-1}(s)x\,ds\right) \\ &= \tilde{c}_{1-\gamma}\tilde{c}_\gamma \int_0^\infty B^\gamma(s,\cdot)T_{\alpha-1}(s)x\,ds.\end{aligned}$$

For $u > 0$ small enough and $\vartheta \in J$ we obtain

$$\big(\mathcal{D}^{\gamma-1}Kx\big)(\vartheta) = \tilde{c}_{1-\gamma}\tilde{c}_\gamma \sum_{k=0}^{\infty} \int_0^u T_{\alpha-1}(s)\big(B^\gamma(ku+s,\vartheta)T_{\alpha-1}(ku)x\big)\,ds.$$

As $B^\gamma(\cdot,\cdot)$ is bounded by $\Gamma(1-\gamma)\Gamma(\gamma)$ a straightforward estimate yields

$$\big\|\big(\mathcal{D}^{\gamma-1}Kx\big)(\vartheta)\big\|_\alpha \leq c\|x\|_Y, \quad c := M\,|\tilde{c}_{1-\gamma}\tilde{c}_\gamma|\,\Gamma(1-\gamma)\Gamma(\gamma)a(u)\sum_{k=0}^{\infty} e^{-ku\omega}$$

(here we have used condition (RC) with the function $s \mapsto B^\gamma(ku+(u-s))T_{\alpha-1}(ku)x$). For $x \in X^{\alpha+1}$ we know that $\mathcal{D}^{\gamma-1}Kx \in W_p^1(J;X^\alpha) \hookrightarrow C_b(J;X^\alpha)$. A continuity argument shows that (4.8.5) holds for all $x \in Y$, that $\mathcal{D}^{\gamma-1}Kx \in C_b(J;X^\alpha)$, and that $\|\mathcal{D}^{\gamma-1}Kx\|_{C_b(J;X^\alpha)} \leq c\|x\|_Y$. ∎

4.8.10 Lemma. *The function B^γ has the following properties.*

(a) Let $s > 0$. Then $B^\gamma(s,\cdot)$ is weakly differentiable and

$$\|\partial_2 B^\gamma(s,\cdot)\|_{L_1(\vartheta_1,\vartheta_2)} = B^\gamma(s,\vartheta_1) - B^\gamma(s,\vartheta_2) \quad (-h < \vartheta_1 < \vartheta_2).$$

(b) Let $t \in (0,h)$ and $s_0 \in \mathbb{R}_+$. Then

$$\sup_{s\in[s_0,\infty),\vartheta\in J} \big(B^\gamma(s,\vartheta) - B^\gamma(s,\vartheta+t)\big) = B^\gamma(s_0,-t).$$

(c) Let $\vartheta_1,\vartheta_2 \in J$ and $s \in \mathbb{R}_+$. Then

$$\|\partial_2 B^\gamma(s,\vartheta_1+\cdot) - \partial_2 B^\gamma(s,\vartheta_2+\cdot)\|_{L_1(0,t)} \leq B^\gamma(s,-|\vartheta_1-\vartheta_2|).$$

(d) For $0 < \delta < \frac{1-\gamma}{2-\gamma}$ we have $\frac{B^\gamma(t^\delta,-t)}{t^\delta} \to 0$ as $t \to 0$.

Proof. For $(s,\vartheta) \in \mathbb{R}_+ \times (-h,\infty)$ we define the function

$$f(s,\vartheta) := \begin{cases} -s^\gamma(-\vartheta)^{-\gamma}(s-\vartheta)^{-1} & \text{if } \vartheta < 0 \text{ and } s > 0,\\ 0 & \text{if } \vartheta \geq 0. \end{cases}$$

As $B^\gamma(s,\vartheta) = \int_0^\vartheta f(s,\tau)\,d\tau$ ($s \in (0,\infty)$, $\vartheta \in (-h,\infty)$) we see that $B^\gamma(s,\cdot)$ is weakly differentiable and its derivative is $f(s,\cdot)$ for all $s > 0$. Moreover as $f(s,\vartheta) \leq 0$ we have

$$\|\partial_2 B^\gamma(s,\cdot)\|_{L_1(\vartheta_1,\vartheta_2)} = -\int_{\vartheta_1}^{\vartheta_2} \partial_2 B^\gamma(s,\vartheta)\,d\vartheta = B^\gamma(s,\vartheta_1) - B^\gamma(s,\vartheta_2).$$

In order to show (b) we observe that $B^\gamma(s,\vartheta)$ is monoton decreasing in ϑ, therefore it suffices to consider the case $\vartheta \in (-h,-t]$. From

$$\frac{d}{d\vartheta}(B^\gamma(s,\vartheta) - B^\gamma(s,\vartheta+t)) = s^\gamma((-\vartheta-t)^{-\gamma}(s-\vartheta-t)^{-1} - (-\vartheta)^{-\gamma}(s-\vartheta)^{-1})$$

for $\vartheta \in (-h,-t)$ and $s \in [s_0,\infty)$ we learn that $B^\gamma(s,\vartheta) - B^\gamma(s,\vartheta+t)$ is monoton increasing in ϑ. Since $B^\gamma(s,0) = 0$ we conclude

$$\sup_{\vartheta\in J} \big(B^\gamma(s,\vartheta) - B^\gamma(s,\vartheta+t)\big) = B^\gamma(s,-t).$$

Furthermore as $B^\gamma(\cdot,-t)$ is decreasing we see that the function attains its maximum at $(s_0,-t)$, which shows (b).

In order to prove (c) we assume that $\vartheta_1 < \vartheta_2$. As

$$\partial_2 B^\gamma(s,\vartheta_1+\tau) - \partial_2 B^\gamma(s,\vartheta_2+\tau) = f(s,\vartheta_1+\tau) - f(s,\vartheta_2+\tau) \geq 0$$

for all $\tau \in (0,t)$ we obtain from (b)

$$\begin{aligned}
&\|f(s,\vartheta_1+\cdot) - f(s,\vartheta_2+\cdot)\|_{L_1(0,t)} \\
&\qquad\qquad = (B^\gamma(s,\vartheta_1+t) - B^\gamma(s,\vartheta_2+t)) - (B^\gamma(s,\vartheta_1) - B^\gamma(s,\vartheta_2)) \\
&\qquad\qquad \leq B^\gamma(s,\vartheta_1-\vartheta_2).
\end{aligned}$$

We show (d) by invoking l'Hôpital's rule. To this end we first compute

$$\begin{aligned}
\frac{d}{dt}B^\gamma(t^\delta,-t) &= \frac{d}{dt}\int_0^{\frac{1}{t^{\delta-1}+1}} \tau^{-\gamma}(1-\tau)^{\gamma-1}\,d\tau \\
&= (t^{\delta-1}+1)^\gamma \left(\frac{t^{\delta-1}+1}{t^{\delta-1}}\right)^{1-\gamma} \frac{(1-\delta)t^{2-\delta}}{(t^{\delta-1}+1)^2} \\
&= (1-\delta)(t^{\delta-1}+1)^{-1} t^{(1-\delta)(1-\gamma)+\delta-2}.
\end{aligned}$$

Now l'Hôpital's rule yields

$$\begin{aligned}
\lim_{t\to 0}\frac{B^\gamma(t^\delta,-t)}{t^\delta} &= \lim_{t\to 0}\frac{1-\delta}{\delta}(t^{\delta-1}+1)^{-1}\,t^{(1-\delta)(1-\gamma)-1} \\
&= \frac{1-\delta}{\delta}\lim_{t\to 0}\frac{1}{t^{1-\delta}+1}\,t^{(1-\delta)(2-\gamma)-1} = 0
\end{aligned}$$

whenever $(1-\delta)(2-\gamma)-1 > 0$. This holds if $0 < \delta < \frac{1-\gamma}{2-\gamma}$. ∎

4.8.11 Lemma. *Let $p \in [1,\infty)$ and $\alpha \in (-1/p,1]$. Assume that $h \in (0,\infty)$ and that T has negative growth bound. Let Y, γ and K be as in Lemma 4.8.9. There is a $t_0 > 0$ and a positive function $b \in C([0,t_0])$ with $b(0) = 0$, such that for all $\varphi \in C([0,t_0];Y)$ we have $\int_0^t S(\tau)(\mathcal{D}^{\gamma-1}K\varphi(t-\tau))\,d\tau \in V_p^1(J;X^\alpha)$ and*

$$\left\| \int_0^t S(\tau)(\mathcal{D}^{\gamma-1}K\varphi(t-\tau))\,d\tau \right\|_{V_p^1(J;X^\alpha)} \leq b(t)\|\varphi\|_{C([0,t_0];Y)} \quad (t \in [0,t_0]).$$

Proof. Let $t_0 \in (0, h]$ and $a \in C([0, t_0])$ with $a(0) = 0$ such that Y satisfies (RC) with respect to A_α with t_0 and a. Let $\varphi \in C([0, t_0]; Y)$ and $u, t \in (0, t_0]$. From (4.8.5) we obtain

$$
\begin{aligned}
f &:= \int_0^t S(\tau)(\mathcal{D}^{\gamma-1} K\varphi(t-\tau))\, d\tau \\
&= \tilde{c}_{1-\gamma}\tilde{c}_\gamma \int_0^t \int_0^\infty T_{\alpha-1}(s)\left(B^\gamma(s, \cdot + \tau)\varphi(t-\tau)\right)\, ds\, d\tau \qquad (4.8.6)\\
&= \tilde{c}_{1-\gamma}\tilde{c}_\gamma \sum_{k=0}^\infty \int_0^u T_{\alpha-1}(ku+s) \int_0^t B^\gamma(ku+s, \cdot + \tau)\varphi(t-\tau)\, d\tau\, ds.
\end{aligned}
$$

Using the fact that $B^\gamma(s, \cdot) \in W_1^1(-h, \infty)$ for all s with norm uniformly bounded in s (cf. Lemma 4.8.10(a)) we see from (4.8.6) that

$$
\begin{aligned}
&\int_{-h}^0 \int_0^t \int_0^\infty T_{\alpha-1}(s)\left(B^\gamma(s, \vartheta + \tau)\varphi(t-\tau)\right)\, ds\, d\tau\, \psi'(\vartheta)\, d\vartheta \\
&\qquad = \int_0^t \int_0^\infty T_{\alpha-1}(s)\left(\varphi(t-\tau)\int_{-h}^0 B^\gamma(s, \vartheta+\tau)\psi'(\vartheta)\, d\vartheta\right)\, ds\, d\tau \\
&\qquad = -\int_0^t \int_0^\infty T_{\alpha-1}(s)\left(\varphi(t-\tau)\int_{-h}^0 \partial_2 B^\gamma(s, \vartheta+\tau)\psi(\vartheta)\, d\vartheta\right)\, ds\, d\tau \\
&\qquad = -\int_{-h}^0 \int_0^t \int_0^\infty T_{\alpha-1}(s)\left(\partial_2 B^\gamma(s, \vartheta+\tau)\varphi(t-\tau)\right)\, ds\, d\tau\, \psi(\vartheta)\, d\vartheta
\end{aligned}
$$

for all $\psi \in C_c^\infty(-h, 0)$. Therefore f is weakly differentiable in $L_{1,loc}(J; Y)$ and

$$
f' = \tilde{c}_{1-\gamma}\tilde{c}_\gamma \sum_{k=0}^\infty \int_0^u T_{\alpha-1}(ku+s) \int_0^t \partial_2 B^\gamma(ku+s, \cdot+\tau)\varphi(t-\tau)\, d\tau\, ds.
$$

Using (RC) and Lemma 4.8.10(a) and (b) we infer that

$$
\|f'(\vartheta)\|_\alpha \le |\tilde{c}_{1-\gamma}\tilde{c}_\gamma| \sum_{k=0}^\infty M e^{-ku\omega} a(u) B^\gamma(ku, -t)\|\varphi\|_{C([0,t_0];Y)} \quad (\vartheta \in J).
$$

For $k = 0$ the summand is bounded by $M\,|\tilde{c}_{1-\gamma}\tilde{c}_\gamma|\,\Gamma(1-\gamma)\Gamma(\gamma)a(u)\|\varphi\|_{C([0,t_0];Y)}$. For $k \ge 1$ we have the bound $M\,|\tilde{c}_{1-\gamma}\tilde{c}_\gamma|\, a(u)B^\gamma(u, -t)e^{-ku\omega}\|\varphi\|_{C([0,t_0];Y)}$ (note that $B^\gamma(ku, -t) \le$

$B^\gamma(u,-t))$. As $\sum_{k=1}^\infty e^{-ku\omega} \le \frac{1}{u\omega}$ we get

$$\|f'(\vartheta)\|_\alpha \le c'' a(u) \left(1 + \frac{B^\gamma(u,-t)}{u}\right) \|\varphi\|_{C([0,t_0];Y)} \quad (\vartheta \in J) \tag{4.8.7}$$

for some $c'' \ge 0$. By the same arguments we see from Lemma 4.8.10(c) that

$$\|f'(\vartheta_1) - f'(\vartheta_2)\|_\alpha \le c'' a(u) \left(1 + \frac{B^\gamma(u, -|\vartheta_1 - \vartheta_2|)}{u}\right) \|\varphi\|_{C([0,t_0];Y)} \tag{4.8.8}$$

for $\vartheta_1, \vartheta_2 \in J$. Let $\delta \in (0, \frac{1-\gamma}{2-\gamma})$ and $u(t) := t^\delta$ ($t \in [0, \min\{t_0, t_0^{1/\delta}\}]$). Then $u(t) \to 0$ and therefore $a(u(t)) \to 0$ as $t \to 0$. Moreover $\frac{B^\gamma(u(t),-t)}{u(t)} \to 0$ as $t \to 0$ by Lemma 4.8.10(d). Thus if we put $u := u(|\vartheta_1 - \vartheta_2|)$ in (4.8.8) we see that $\|f'(\vartheta_1) - f'(\vartheta_2)\|_\alpha \to 0$ as $|\vartheta_1 - \vartheta_2| \to 0$. Hence $f' \in C_b(J; X^\alpha)$, in particular it belongs to $L_p(J; X^\alpha)$. Using $u(t)$ in (4.8.7) we obtain $\|f'\|_{L_p(J;X^\alpha)} \le b(t)\|\varphi\|_{C([0,t_0];Y)}$ with

$$b(t) := c'' h^{1/p} a(u(t)) \left(1 + \frac{B^\gamma(u(t),-t)}{u(t)}\right) \to 0 \quad (t \to 0).$$

Since we also have $f(0) = 0$ we see that $f \in V_p^1(J; X^\alpha)$ and that $f' = \mathcal{D}f$. This proves the assertion. ■

We are now well prepared to prove our main tool for applying the Desch-Schappacher perturbation theorem.

4.8.12 Corollary. *Let $p \in [1,\infty)$, $\alpha \in (-1/p, 1]$ and $\beta \in (\alpha - 1, \alpha + 1]$. Assume that T has negative growth bound. Let Y be a Banach space satisfying (RC) in Proposition A.3 with respect to A_β. Then $Y \times \{0\}$ satisfies (RC) with respect to $\mathcal{A}_\beta$. This assertion particularly holds for $Y = F_A^\beta$.*

Proof. The cases $\beta \in \{\alpha, \alpha+1\}$ were dealt with in Lemma 4.8.8. Let $\gamma := \beta - \alpha$.

First we assume that $h \in (0,\infty)$ and $\beta \in (\alpha, \alpha+1)$. Let $\tilde{Y} := A_{\alpha-1}^\gamma Y$ be equipped with the norm $\|x\|_{\tilde{Y}} := \|A_{\alpha-1}^{-\gamma} x\|_Y$ ($x \in \tilde{Y}$). Then $\tilde{Y}$ satisfies (RC) with respect to A_α (cf. Proposition A.4). Let $t_0 > 0$ be sufficiently small. Let $\varphi \in C([0,t_0];Y)$ and $\tilde{\varphi} := A_{\alpha-1}^\gamma \varphi(\cdot)$. Then we have $\|\varphi\|_{C([0,t_0];Y)} = \|\tilde{\varphi}\|_{C([0,t_0];\tilde{Y})}$ and $\tilde{\mathcal{A}}_{\alpha-1}^{\gamma} \binom{\varphi(\cdot)}{0} = \binom{\tilde{\varphi}(\cdot)}{0} \in C([0,t_0]; \tilde{Y} \times \{0\})$. For $t \in [0,t_0]$ we write

$$\begin{aligned}\int_0^t \mathcal{T}_{\beta-1}(t-s) \begin{pmatrix}\varphi(s)\\0\end{pmatrix} ds &= \mathcal{A}_{\alpha-1}^{-\gamma} \int_0^t \mathcal{T}_{\alpha-1}(t-s) \begin{pmatrix}\tilde{\varphi}(s)\\0\end{pmatrix} ds \\ &\quad + \int_0^t \mathcal{T}_{\beta-1}(t-s)\big(\tilde{\mathcal{A}}_{\alpha-1}^{-\gamma} - \mathcal{A}_{\alpha-1}^{-\gamma}\big) \begin{pmatrix}\tilde{\varphi}(s)\\0\end{pmatrix} ds.\end{aligned} \tag{4.8.9}$$

For the norm of the first expression on the right hand side in (4.8.9) we obtain from Lemma 4.8.8 the estimate

$$\begin{aligned}\left\| \mathcal{A}_{\alpha-1}^{-\gamma} \int_0^t \mathcal{T}_{\alpha-1}(t-s)\begin{pmatrix}\tilde\varphi(s)\\ 0\end{pmatrix} ds\right\|_{\mathcal{Z}_{\mathcal{A}}^{\beta}} &= \left\| \int_0^t \mathcal{T}_{\alpha-1}(t-s)\begin{pmatrix}\tilde\varphi(s)\\ 0\end{pmatrix} ds\right\|_{\mathcal{Z}_{\mathcal{A}}^{\alpha}} \\ &\le a(t)\|\tilde\varphi\|_{C([0,t_0];\tilde Y)} = a(t)\|\varphi\|_{C([0,t_0];Y)},\end{aligned}$$

for a positive function $a \in C([0,t_0])$, $a(0)=0$. The second expression on the right hand side of (4.8.9) can be written as (using the operator K introduced in Lemma 4.8.9)

$$\int_0^t \mathcal{T}_{\beta-1}(t-s)\begin{pmatrix}0\\ K\tilde\varphi(s)\end{pmatrix} ds = \begin{pmatrix}0\\ \mathcal{D}^{1-\gamma}\int_0^t S(t-s)\big(\mathcal{D}^{\gamma-1}K\tilde\varphi\big)(s)\,ds\end{pmatrix}.$$

Let $f := \int_0^t S(t-s)\big(\mathcal{D}^{\gamma-1}K\tilde\varphi\big)(s)\,ds$. From Lemma 4.8.11 we know that $f \in V_p^1(J;X^\alpha)$ and

$$\|f\|_{V_p^1(J;X^\alpha)} \le b(t)\|\tilde\varphi\|_{C([0,t_0];\tilde Y)} = b(t)\|\varphi\|_{C([0,t_0];Y)}$$

for some function $b \in C([0,t_0])$ with $b(0)=0$. As we have $\{0\}\times V_p^\gamma(J;X^\alpha) \hookrightarrow \mathcal{Z}_{\mathcal{A}}^{\beta}$ by Lemma 4.8.3(a) and (b) we infer $\begin{pmatrix}0\\ \mathcal{D}^{1-\gamma}f\end{pmatrix} \in \mathcal{Z}_{\mathcal{A}}^{\beta}$. Moreover Lemma 4.8.3(a) and (b) imply that

$$\left\|\begin{pmatrix}0\\ \mathcal{D}^{1-\gamma}f\end{pmatrix}\right\|_{\mathcal{Z}_{\mathcal{A}}^{\beta}} \le c\|\mathcal{D}^{1-\gamma}f\|_{V_p^{\beta-\alpha}(J;X^\alpha)} = c\|\mathcal{D}f\|_{L_p(J;X^\alpha)} \le c\,b(t)\|\tilde\varphi\|_{C([0,t_0];\tilde Y)}$$

for some constant $c \ge 0$. Therefore the second expression on the right hand side of (4.8.9) belongs to $\mathcal{Z}_{\mathcal{A}}^{\beta}$ and its norm is bounded by $c\,b(t)\|\varphi\|_{C([0,t_0];Y)}$. Hence $Y\times\{0\}$ fulfils condition (RC) with respect to $\mathcal{A}_\beta$.

We now treat the case $h=\infty$ and $\beta \in (\alpha,\alpha+1)$. Let $\hat h < \infty$ arbitrary. By $\hat{\mathcal{T}}$ and $\hat{\mathcal{A}}$ we denote the delay semigroup and its generator on $X \times W_p^\alpha(-\hat h,0;X^{-\alpha})$. By the previous case there is a $t_0 \in (0,\hat h/2)$ and a function $a \in C([0,t_0])$ with $a(0)=0$ such that $Y\times\{0\}$ satisfies (RC) with respect to $\hat{\mathcal{A}}_\beta$ with t_0 and a.

In order to show (RC) for $\mathcal{A}_\beta$ let $\varphi \in C([0,t_0];Y)$, $\begin{pmatrix}u(t)\\ F(t)\end{pmatrix} := \int_0^t \mathcal{T}_{\beta-1}(t-s)\begin{pmatrix}\varphi(s)\\ 0\end{pmatrix} ds$ and $\begin{pmatrix}v(t)\\ G(t)\end{pmatrix} := \int_0^t \hat{\mathcal{T}}_{\beta-1}(t-s)\begin{pmatrix}\varphi(s)\\ 0\end{pmatrix} ds$ ($t\in[0,t_0]$). For $t \in [0,t_0]$ we have $u(t)=v(t)$ and $F(t)|_{(-\hat h,0)} = G(t)$. Hence

$$\begin{aligned}\|u(t)\|_{X^\beta} = \|v(t)\|_{X^\beta} &\le a(t)\|\varphi\|_{C([0,t_0];Y)},\\ \|G(t)\|_{W_p^{\beta-\alpha}(-\hat h,0;X^\alpha)} &\le a(t)\|\varphi\|_{C([0,t_0];Y)}.\end{aligned}$$

So we already have a suitable estimate for $u(t)$. In order to obtain such an estimate for $f := F(t)$ we define $\psi\colon(-\infty,0)\to\mathbb{R}$ by $\psi(\vartheta) := \max\{0,\vartheta/t_0+1\}$ ($\vartheta\in(-\infty,0)$).

Taking into account that $f(0)\cdot(\varepsilon_{\lambda_\infty}-\psi)\in W_p^1(-\infty,0;X^\alpha)$ we obtain the estimate

$$\begin{aligned}\|f\|_{W_p^{\beta-\alpha}(-\infty,0;X^\alpha)} &= \|f(0)\|_{X^\alpha} + \|(\mathcal{D}-\omega_\infty)^\gamma\left(f-f(0)\cdot\varepsilon_{\lambda_\infty}\right)\|_{L_p(-\infty,0;X^\alpha)}\\ &\le c_1\left(\|f(0)\|_{X^\beta} + \|(\mathcal{D}-\omega_\infty)^{\gamma-1}\left(f-f(0)\cdot\varepsilon_{\lambda_\infty}\right)\|_{W_p^1(-\infty,0;X^\alpha)}\right)\\ &\le c_2\left(\|f(0)\|_{X^\beta} + \|(\mathcal{D}-\omega_\infty)^{\gamma-1}\left(f-f(0)\cdot\psi\right)\|_{W_p^1(-\infty,0;X^\alpha)}\right)\end{aligned}$$

for some $c_1,c_2\ge 0$. Since $u(t)=F(t)(0)=f(0)$ we already know that $\|f(0)\|_{X^\beta}\le a(t)\|\varphi\|_{C([0,t_0];Y)}$. Let $\tilde{f}:=f-f(0)\cdot\psi$. As $\operatorname{spt} f\subseteq[-t,0)$ and $\operatorname{spt}\psi=[-t_0,0)$ we have $\operatorname{spt}\tilde{f}\subseteq[-t_0,0)$. Therefore

$$\begin{aligned}(\mathcal{D}-\omega_\infty)^{\gamma-1}\tilde{f} &= \left((-\infty,0)\ni\vartheta\mapsto\tilde{c}_{1-\gamma}\int_0^{-\vartheta} r^{-\gamma}e^{-\omega_\infty r}S(r)\tilde{f}(\vartheta)\,dr\right)\\ &= (-\infty,0)\ni\vartheta\mapsto\tilde{c}_{1-\gamma}\int_{\max\{0,-\vartheta-t_0\}}^{-\vartheta} r^{-\gamma}e^{-\omega_\infty r}\tilde{f}(r+\vartheta)\,dr.\end{aligned}\tag{4.8.10}$$

If $\vartheta\in(-\hat{h},0)$ we see that

$$\left((\mathcal{D}-\omega_\infty)^{\gamma-1}\tilde{f}\right)(\vartheta) = \left((\hat{\mathcal{D}}-\omega_\infty)^{\gamma-1}\left(G(t)-G(t)(0)\cdot\psi\right)\right)(\vartheta)$$

(where $\hat{\mathcal{D}}$ denotes the generator of the left translation semigroup on $L_p(-\hat{h},0;X^\alpha)$ with zero boundary condition at 0). This gives the estimate

$$\left\|\left((\mathcal{D}-\omega_\infty)^{\gamma-1}\tilde{f}\right)\Big|_{(-\hat{h},0)}\right\|_{W_p^1(-\hat{h},0;X^\alpha)} \le c a(t)\|\varphi\|_{C([0,t_0];Y)}\tag{4.8.11}$$

for some $c\ge 0$. The proof is finished if we can show that $\left((\mathcal{D}-\omega_\infty)^{\gamma-1}\tilde{f}\right)\Big|_{(-\infty,-\hat{h}/2)}$ is in $W_p^1(-\infty,-\hat{h}/2;X^\alpha)$ with a similar estimate as in (4.8.11). To this end we compute (using (4.8.10))

$$\begin{aligned}&\left((\mathcal{D}-\omega_\infty)^{\gamma}\tilde{f}\right)\Big|_{(-\infty,-\hat{h}/2)}\\ &\quad= \left(\frac{d}{d\vartheta}-\omega_\infty\right)\left((-\infty,-\hat{h}/2)\ni\vartheta\mapsto\tilde{c}_{1-\gamma}\int_{-\vartheta-t_0}^{-\vartheta} r^{-\gamma}e^{-\omega_\infty r}\tilde{f}(r+\vartheta)\,dr\right)\\ &\quad= \left(\frac{d}{d\vartheta}-\omega_\infty\right)\left((-\infty,-\hat{h}/2)\ni\vartheta\mapsto\tilde{c}_{1-\gamma}\,e^{\omega_\infty\vartheta}\int_{-t_0}^{0}(r-\vartheta)^{-\gamma}e^{-\omega_\infty r}\tilde{f}(r)\,dr\right)\\ &\quad= (-\infty,-\hat{h}/2)\ni\vartheta\mapsto -\tilde{c}_{1-\gamma}\gamma\,e^{\omega_\infty\vartheta}\int_{-t_0}^{0}(r-\vartheta)^{-\gamma-1}e^{-\omega_\infty r}\tilde{f}(r)\,dr.\end{aligned}$$

Since $r-\vartheta > \hat{h}/2-t_0 > 0$ for $r \in (-t_0, 0)$ and $\vartheta \in (-\infty, -\hat{h}/2)$ we obtain $(r-\vartheta)^{-\gamma-1} < (\hat{h}/2-t_0)^{-\gamma-1}$. Further as

$$\|f(r)\|_{X^\alpha} = \|u(t+r)\|_{X^\alpha} \le c\|u(t+r)\|_{X^\beta} \le c\, a(t+r)\|\varphi\|_{C([0,t_0];Y)} \quad (r \in (-t, 0))$$

for some $c \ge 0$ we obtain

$$\sup_{r\in(-t_0,0)} \|\tilde{f}(r)\|_{X^\alpha} \le \sup_{r\in[-t,0]} \|f(r)\|_{X^\alpha} + \|f(0)\|_{X^\alpha} \le b(t)\|\varphi\|_{C([0,t_0];Y)}$$

with $b(t) := a(t) + \sup_{r\in[0,t]} a(r)$. These considerations yield the estimate

$$\begin{aligned}
&\left\| \left((\mathcal{D}-\omega_\infty)^\gamma \tilde{f}\right)\Big|_{(-\infty,-\hat{h}/2)} \right\|_{L_p(-\infty,-\hat{h}/2;X^\alpha)} \\
&\qquad \le |\tilde{c}_{1-\gamma}|\,\gamma\, \|\varepsilon_{\omega_\infty}|_{(-\infty,-\hat{h}/2)}\|_p\, t_0 e^{\omega_\infty t_0}(\hat{h}/2-t_0)^{-\gamma-1}\, b(t)\|\varphi\|_{C([0,t_0];Y)} \\
&\qquad \le c\, b(t)\|\varphi\|_{C([0,t_0];Y)}
\end{aligned}$$

for some $c \ge 0$. Thus $\left((\mathcal{D}-\omega_\infty)^{\gamma-1}\tilde{f}\right)\Big|_{(-\infty,-\hat{h}/2)}$ is in $W_p^1(-\infty,-\hat{h}/2;X^\alpha)$. Moreover as $b(t) \to 0$ $(t \to 0)$ we finally see that

$$\|f\|_{W_p^{\beta-\alpha}(-\infty,0;X^\alpha)} \le \tilde{a}(t)\|\varphi\|_{C([0,t_0];Y)}$$

for a function $\tilde{a} \in C([0,t_0])$ with $\tilde{a}(0) = 0$. This shows that $Y \times \{0\}$ fulfils condition (RC) with respect to $\mathcal{A}_\beta$ also in the case $h = \infty$.

Last we deal with the case $\beta \in (\alpha-1, \alpha)$ (and $h \in (0,\infty]$). Let $Y_1 := A_\beta^{-1}Y$ with the corresponding norm. Then Y_1 satisfies (RC) with respect to $A_{\beta+1}$. Hence by the previous cases $Y_1 \times \{0\}$ satisfies (RC) with respect to $\mathcal{A}_{\beta+1}$. Now the assertion follows from Proposition A.4 and Lemma 4.8.3(a) by observing that $Y\times\{0\} = \mathcal{A}_\beta(Y_1\times\{0\})$. ■

We can now prove Proposition 4.8.6.

Proof of Proposition 4.8.6. Without loss of generality we can also assume that T has negative growth bound, otherwise we consider the delay semigroup with $A-\omega$ instead of A for ω sufficiently large, and at the very end we perturb the obtained generator with the bounded operator $\left(\begin{smallmatrix}\omega & 0\\ 0 & 0\end{smallmatrix}\right)$.

Corollary 4.8.12 and Proposition A.3 imply that $\mathcal{Q} \in \mathcal{L}(\mathcal{Z}_\mathcal{A}^\beta, Y \times \{0\})$ is a Desch-Schappacher perturbation of $\mathcal{A}_\beta$. As $\mathcal{Z}_\mathcal{A}^{\beta-1+\varepsilon} = X^{\beta-1+\varepsilon} \times V_p^{\beta-\alpha-1+\varepsilon}(J;X^\alpha)$ for $\varepsilon \in (0,1/p)$ (cf. Lemma 4.8.3(a)) we conclude that the space $Y \times \{0\}$ is continuously embedded in $\mathcal{Z}_\mathcal{A}^{\beta-1+\varepsilon}$ if $\varepsilon \le \gamma-(\beta-1)$. Thus the generator property of $\mathcal{C}$ follows from Corollary 4.3.4.

In order to determine the domain of $D(\mathcal{C})$ we observe that for $\gamma \ge 0$ the perturbation $\mathcal{Q}$ maps into $\mathcal{Z}$ and thus $D(\mathcal{C}) = D(\mathcal{A})$. Now the assertion follows from Lemma 4.8.3(c). For the case $\gamma < 0$ we first observe that if $1-\alpha < 1/p$ then $f - x\varepsilon_{\lambda_h} \in V_p^{1-\alpha}(J;X^\alpha)$ if and only if $f \in V_p^{1-\alpha}(J;X^\alpha)$ $(x \in X^{\gamma+1},\ f \in L_p(J;X^\alpha))$ by Lemma 4.8.1. Now we can proceed as in the proof of Lemma 4.8.3(b). ■

We conclude this section with some remarks.

4.8.13 Remarks. (a) The case $\alpha = 0$, $\beta = 1$ and $Y = F_A^1$ in Proposition 4.8.6 was shown in [48; Theorem 3.1].

(b) The case $\alpha = 0$ in Proposition 4.8.5 was treated in [69; Theorem 3.1].

(c) Let $p \in (1, \infty)$, $\alpha \in (-1/p, 1 - 1/p)$ and $Lf := \int_{-h}^0 d\eta(s) f(s)$ $(f \in C([-h, 0]; X^\alpha))$ for some $\eta \in BV(J; \mathcal{L}(X^\alpha))$. Then L satisfies the assumptions of Proposition 4.8.5; cf. [19] for the case $\alpha = 0$. Except for the different type of delay this result compares to Proposition 4.7.9.

(d) It can be expected that a result analogous to Corollary 4.8.4 holds in spaces of continuous functions. However delay semigroups on $C([-h, 0]; X)$ as presented in [39; Section VI.7] are not obtained by a suitable Desch-Schappacher perturbation of a translation semigroup on $C([-h, 0]; X)$. This makes it difficult to apply the technique used in the proof of Corollary 4.8.4.

Appendix

A Appendix. Desch-Schappacher and Miyadera-Voigt Perturbation Theorem

In this appendix we recall (variants of) the Desch-Schappacher and the Miyadera-Voigt perturbation theorems; cf. [39; Section III.3(a) and (c)], [51], [52], [70], [32] and Section 2.2 where we prove a generalisation of the Desch-Schappacher perturbation theorem. We also recall the definition of Favard spaces.

A.1 Theorem. *(Miyadera-Voigt perturbation) Let X be a Banach space, T the C_0-semigroup generated by A. Let $B \in \mathcal{L}(X^1, X)$. Assume there exist $t > 0$ and $q \in [0,1)$ such that*

$$\int_0^t \|BT(s)x\|_X \, ds \le q\|x\|_X \quad (x \in X^1).$$

Then $A + B$ is a generator.

A.2 Theorem. *(Desch-Schappacher perturbation) Let X be a Banach space, and let T the C_0-semigroup generated by A. Let $B \in \mathcal{L}(X, X^{-1})$. Assume there exist $t > 0$ and $q \in [0,1)$ such that $\int_0^t T_{-1}(t-s)Bu(s)\,ds \in X$ and*

$$\left\| \int_0^t T_{-1}(t-s)Bu(s)\,ds \right\| \le q\|u\|_\infty \quad (u \in C([0,t];X)).$$

Then $(A_{-1} + B)_{|X}$ is a generator.

The assumptions of the Desch-Schappacher perturbation theorem are met if the perturbing operator satisfies a range condition; cf. [30], [32; Definition 4], [48; Theorem A.1] and [39; Corollary III.3.6].

A.3 Proposition. *Let A be the generator of the C_0-semigroup T on a Banach space X, and let $Y \hookrightarrow X_A^{-1}$ be a Banach space. The operator B is a Desch-Schappacher perturbation of A (i.e. satisfies the assumptions of Theorem A.2) if $B \in \mathcal{L}(X, Y)$ and Y satisfies the following range condition.*

(RC) *There is a $t_0 > 0$ and a positive function $a \in C([0,t_0])$, $a(0) = 0$ such that for any $\varphi \in C([0,t_0];Y)$ we have $\int_0^t T_{-1}(t-s)\varphi(s)\,ds \in X$ and*

$$\left\| \int_0^t T_{-1}(t-s)\varphi(s)\,ds \right\| \le a(t)\|\varphi\|_\infty \quad (t \in [0,t_0]).$$

The most prominent extrapolation space satisfying the range condition (RC) is the Favard space F_A^0 associated with a generator A on a Banach space X. Let $\omega \in \mathbb{R}$ be the growth bound of the C_0-semigroup generated by A. The Favard space is defined by

$$F_A^0 := \{x \in X^{-1};\ \|x\|_{F_A^0} := \sup_{\lambda > \omega} \|\lambda R(\lambda, A_{-1})x\| < \infty\}. \tag{A.1}$$

The space does not depend on the particular choice of ω. We also mention that

$$F_A^0 = \Big\{x \in X^{-1};\ |||x|||_{F_A^0} := \sup_{t>0} \frac{1}{t}\|e^{-\omega t}T(t)x - x\|_{X^{-1}} < \infty\Big\} \tag{A.2}$$

$$= \Big\{x \in X^{-1};\ |||x|||'_{F_A^0} := \|x\|_{X^{-1}} + \limsup_{t\to 0} \frac{1}{t}\|T(t)x - x\|_{X^{-1}} < \infty\Big\} \tag{A.3}$$

and $\|\cdot\|_{F_A^0}$, $|||\cdot|||_{F_A^0}$ and $|||\cdot|||'_{F_A^0}$ are equivalent norms (cf. [39; Definition II.5.10, Proposition II.5.12]).

For the applications we have in mind we need the extrapolated Favard space $F^0_{A_\alpha}$ of the generator A_α, where $\alpha \in \mathbb{R}$ (cf. Proposition 4.1.2 for the definition of A_α). This Favard space is denote by F_A^α. The space F_A^α is not to be confused with the Favard space of fractional order α; cf. [39; Definition II.5.10] or [18; Proposition 3.1.3] for the definition and embedding properties (we will encounter fractional order Favard spaces only in the proof of Proposition 4.5.1). We recall that if X is reflexive then X^α, F_A^α and the Favard space of fractional order α coincide ([67; Corollary 3.2.4], [39; Corollary II.5.21]).

The condition (RC) fits well into the notion of the fractional power tower, which is explored in the next proposition.

A.4 Proposition. *Let A be the generator of the C_0-semigroup T on X with growth bound less than $\omega \in \mathbb{R}$. Let $\alpha, \beta \in \mathbb{R}$ and $\delta := \min\{\alpha, \beta\} - 1$. If the Banach space Y satisfies (RC) with respect to A_α then the Banach space $Z := (A_\delta - \omega)^{\alpha-\beta}Y$, equipped with the norm $Z \ni z \mapsto \|(A_\delta - \omega)^{\beta-\alpha}z\|_Y$, satisfies (RC) with respect to A_β.*

Proof. By Theorem 4.1.4 it suffices to consider the case $\beta = 0$. By V we denote the isomorphism $(A_\delta - \omega)^\alpha$ from X^α to X. We assume that X^α is equipped with the norm $x \mapsto \|Vx\|$. Let t_0 and a be as in (RC) for Y and A_α. Let $\psi \in C([0, t_0]; Z)$ and $\varphi := V^{-1} \circ \psi \in C([0, t_0]; Y)$. Then

$$\int_0^t T_{-1}(t-s)\psi(s)\,ds = V \int_0^t T_{\alpha-1}(t-s)\varphi(s)\,ds \in VX^\alpha = X$$

and the norm of the integral is bounded by $a(t)\|\varphi\|_\infty = a(t)\|\psi\|_\infty$. Thus Z fulfils (RC) with respect to A. ■

B Appendix. Convergence of C_0-semigroups and exponential formulas

Here we recall the usual notion of convergence for C_0-semigroups. We also introduce a general type of approximation for C_0-semigroups and their generators.

B.1 Remarks. Let T_n ($n \in \mathbb{N}$), T be C_0-semigroups on a Banach space X. Let A_n be the generator of T_n ($n \in \mathbb{N}$), and let A be the generator of T.

(a) We shortly recall the notion of convergence of a sequence of C_0-semigroups. We say that (T_n) *converges* to T if $T(t) = \text{s-lim}_{n\to\infty} T_n(t)$ uniformly for t in compact subsets of $[0,\infty)$. It is equivalent to assume that there exist $M \geq 0$, $\omega \in \mathbb{R}$ such that $\|T_n(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, $n \in \mathbb{N}$, and $T(t) = \text{s-lim}_{n\to\infty} T_n(t)$ for all $t \geq 0$; cf. [57; Theorem 3.4.2].

(b) We say that the sequence (A_n) *converges to A in the strong resolvent sense* if there exist $M \geq 1$, $\omega \in \mathbb{R}$ such that $(\omega,\infty) \subseteq \rho(A_n)$ $(n \in \mathbb{N})$, $(\omega,\infty) \subseteq \rho(A)$,

$$\|R(\lambda, A_n)^k\| \leq M(\lambda - \omega)^{-k} \quad \text{for all} \quad \lambda > \omega,\ k, n \in \mathbb{N}, \tag{B.1}$$

and $R(\lambda, A_n) \to R(\lambda, A)$ $(n \to \infty)$ in the strong operator topology, for some (or equivalently all) $\lambda > \omega$.

(c) We note that, under the boundedness assumption (B.1), the strong resolvent convergence of (A_n) to A is equivalent to the *graph convergence* (cf. [41]), i.e.,

$$\mathrm{gr}(A) = \{(x,y);\ \text{there exist } (x_n, y_n) \in \mathrm{gr}(A_n)\ (n \in \mathbb{N}),\ x_n \to x,\ y_n \to y\}.$$

In fact, it is furthermore equivalent to this statement that for all x in a core of A there exist $x_n \in D(A_n)$ $(n \in \mathbb{N})$, $x_n \to x$ such that $A_n x_n \to Ax$ $(n \to \infty)$.

(d) It is the content of the *first Trotter-Kato approximation theorem* that (A_n) converges to A in the strong resolvent sense if and only if $T = \text{s-lim}_{n\to\infty} T_n$; cf. [39; Theorem III.4.8], [57; Theorem 3.4.2].

For the motivation of the following theorem we refer to the subsequent Remarks B.3.

B.2 Theorem. *Let T be a C_0-semigroup on the Banach space X. Let A be the generator of T, and let $M \geq 1$, $\omega \in \mathbb{R}$ be such that $\|T(t)\| \leq Me^{\omega t}$ $(t \geq 0)$. Let ν be a finite Borel measure on $[0,\infty)$ satisfying*

$$\nu([0,\infty)) = \int_0^\infty \tau \, d\nu(\tau) = 1.$$

If $\omega \leq 0$ let $h := \infty$. If $\omega > 0$ we additionally assume that $\int_0^\infty \tau e^{\alpha\tau} \, d\nu(\tau) < \infty$ for some $\alpha > 0$, and we define $h := \alpha/\omega$.

We define $V(0) := I$ and

$$V(s) := \int_0^\infty T(s\tau)\, d\nu(\tau), \quad A(s) := \tfrac{1}{s}(V(s) - I) \quad (s \in (0,h)).$$

Then $A(s)x \to Ax$ for all $x \in D(A)$, and $A(s) \to A$ in the strong resolvent sense, as $s \to 0$. There exists $\omega' \geq 0$ such that $\|V(\frac{t}{n})^n\| \leq Me^{\omega' t}$, for all $t \geq 0$, $n \in \mathbb{N}$ such that $t/n < h$. Moreover, $T(t) = \text{s-lim}_{n\to\infty} V(\frac{t}{n})^n$, uniformly for t in compact subsets of $[0,\infty)$.

B.3 Remarks. The motivation for Theorem B.2 was to formulate a result covering the following two cases.

(a) Setting $\nu := \delta_1$ (unit mass at the point 1) we obtain $V(s) = T(s)$. Then Theorem B.2 yields the known strong resolvent convergence of $\frac{1}{s}(T(s) - I)$ to A (cf. [38; Theorem VIII.1.10], [39; subsection III.4.12]) and the trivial formula $T(t) = \text{s-lim}_{n\to\infty} T(\frac{t}{n})^n$. In Section 1.3 it is shown that this ν yields a formula for the generator of the modulus semigroup.

(b) Setting $d\nu(\tau) := e^{-\tau} d\tau$ we obtain $V(s) = \frac{1}{s} R(\frac{1}{s}, A)$. The operators $A(s) = \frac{1}{s^2} R(\frac{1}{s}, A) - \frac{1}{s} I$ are the Yosida approximants of A, and the last formula of Theorem B.2 is the known exponential formula $T(t) = \text{s-lim}_{n\to\infty} \left(\frac{n}{t} R(\frac{n}{t}, A)\right)^n = \text{s-lim}_{n\to\infty} \left(I - \frac{t}{n} A\right)^{-n}$ (cf. [38; proof of Theorem VIII.1.13] for the strong resolvent convergence of $(A(s))$ to A, and [39; Corollary III.5.5] for the exponential formula).

Proof of Theorem B.2. If $x \in D(A)$ then

$$A(s)x = \int_0^\infty \tfrac{1}{s}\big(T(s\tau)x - x\big)\, d\nu(\tau) \to \int_0^\infty \tau Ax \, d\nu(\tau) = Ax \quad (s \to 0),$$

by dominated convergence. (Using $T(t)x - x = \int_0^t T(r)Ax\, dr$ one obtains the ν-integrable bound $\tau M\|Ax\|$ if $\omega \le 0$, and $\tau e^{\alpha\tau} M \|Ax\|$ if $\omega > 0$.)

Thus it remains to obtain uniform bounds for the semigroups $(e^{tA(s)})_{t\ge 0}$. Let $c_s := \int_0^\infty e^{\omega s \tau}\, d\nu(\tau)$ $(s \in [0, h))$. Using

$$e^{tA(s)} = e^{-\frac{t}{s}} \sum_{n=0}^\infty \frac{1}{n!} \Big(\frac{t}{s} V(s)\Big)^n,$$

$$\begin{aligned} \|V(s)^n\| &= \Big\|\Big(\int_0^\infty T(s\tau)\, d\nu(\tau)\Big)^n\Big\| \\ &= \Big\| \int_{[0,\infty)^n} T(s(\tau_1 + \cdots + \tau_n))\, d\nu(\tau_1) \cdots d\nu(\tau_n)\Big\| \\ &\le M \int_{[0,\infty)^n} e^{\omega s(\tau_1 + \cdots + \tau_n)}\, d\nu(\tau_1) \cdots d\nu(\tau_n) \\ &= M\Big(\int_0^\infty e^{\omega s \tau}\, d\nu(\tau)\Big)^n = M c_s^n \end{aligned} \tag{B.2}$$

we obtain

$$\big\|e^{tA(s)}\big\| \le M \exp\Big(t\frac{c_s - 1}{s}\Big).$$

If $\omega \le 0$ one obtains the uniform bound M. If $\omega > 0$ the estimate

$$\frac{1}{s}\big(e^{\omega s\tau} - 1\big) \le \omega\tau e^{\omega s \tau} \le \omega \tau e^{\alpha\tau}$$

shows $\frac{1}{s}(c_s - 1) \le \omega \int_0^\infty \tau e^{\alpha\tau}\, d\nu(\tau)$. (And in both cases we have $\frac{1}{s}(c_s - 1) \to \omega$ as $s \to 0$).

From the previous considerations we conclude that there exists $\omega' \ge 0$ such that $c_s \le 1 + \omega' s$ for all $s \in [0, h)$. Therefore (B.2) implies $\|V(\frac{t}{n})^n\| \le M c_{t/n}^n \le M\big(1 + \frac{\omega' t}{n}\big)^n \le M e^{\omega' t}$, for all $t \ge 0$, $n \in \mathbb{N}$ with $t/n < h$.

Now the last assertion follows from Theorem C.1. ∎

C Appendix. A generalised Chernoff product formula

In this appendix we assume that X is a Banach space. The following generalised version of the Chernoff product formula was shown in [24; Theorem 1.1] for the case of contractions, i.e., for $M = 1$, $\omega = 0$.

C.1 Theorem. *Let $M \ge 0$, $\omega \in \mathbb{R}$, $h \in (0, \infty]$ and assume that the function $V\colon [0, h) \to \mathcal{L}(X)$ satisfies $V(0) = I$, $\|V(t/n)^n\| \le M e^{\omega t}$ for all $t \ge 0$, $n \in \mathbb{N}$ with $t/n < h$. Let A be the generator of a C_0-semigroup T satisfying $\|T(t)\| \le M e^{\omega t}$ $(t \ge 0)$.*

For $s \in (0, h)$ we define

$$A(s) := \frac{1}{s}(V(s) - I),$$

and we assume that $A(s)$ converges to A in the strong resolvent sense as $s \to 0$.

Then

$$T(t) = \operatorname*{s\text{-}lim}_{n\to\infty} V(t/n)^n,$$

uniformly for t in compact subsets of $[0, \infty)$.

Proof. We note that a straightforward computation shows that for each $\omega' > \omega$ there exist $M' \ge 0$, $\delta \in (0, h)$ such that

$$\|e^{tA(s)}\| \le M' e^{\omega' t} \quad (t \ge 0,\ 0 < s \le \delta).$$

Thus, for each $\lambda > \omega$, one has $\lambda \in \rho(A(s))$ for small s, and the hypothesis means that $R(\lambda, A(s)) \to R(\lambda, A)$ strongly, as $s \to 0$.

First we show that, without loss of generality, we may assume $\omega = 0$. Rescaling

$$\tilde{V}(s) := e^{-\omega s} V(s), \quad \tilde{T}(s) := e^{-\omega s} T(s) \quad (s \in (0, h))$$

we obtain

$$(\tilde{V}(t/n))^n = e^{-\omega t}(V(t/n))^n \quad (t \ge 0, n \in \mathbb{N}, t/n \in (0, h)).$$

Thus, with $\tilde{A}(s) := \frac{1}{s}(\tilde{V}(s) - I)$ $(s \in (0, h))$, we have to show that, for all $\tilde{\lambda} > 0$, one has $(\tilde{\lambda} - \tilde{A}(s))^{-1} \to (\tilde{\lambda} - \tilde{A})^{-1}$ strongly, as $s \to 0$, where $\tilde{A} := A - \omega$ is the generator of $\tilde{T}$. Let $\tilde{\lambda} > 0$. Noting $\tilde{A}(s) = e^{-\omega s}(A(s) - \frac{1}{s}(e^{\omega s} - 1))$ we obtain

$$\tilde{\lambda} - \tilde{A}(s) = e^{-\omega s}(e^{\omega s}\tilde{\lambda} + \frac{1}{s}(e^{\omega s} - 1) - A(s)).$$

Using standard arguments one concludes $(\tilde{\lambda} - \tilde{A}(s))^{-1} \to (\tilde{\lambda} + \omega - A)^{-1} = (\tilde{\lambda} - \tilde{A})^{-1}$ strongly, as $s \to 0$.

We now assume $\omega = 0$. As a consequence of the Trotter-Kato approximation theorem, the convergence

$$\operatorname*{s\text{-}lim}_{n\to\infty}(T(t) - e^{tA(t/n)}) = 0,$$

uniformly for t in compact subsets of $[0, \infty)$, is obtained as in [39; proof of Theorem III.5.2].

Let $\lambda > 0$. Then $(\lambda - A(s))^{-1} \to (\lambda - A)^{-1}$ strongly $(s \to 0)$, by hypothesis. Let $x \in D(A)$. For $s \in (0, h)$ we define

$$x(s) := (\lambda - A(s))^{-1}(\lambda - A)x.$$

Then $x(s) \to x$, $A(s)x(s) \to Ax$ $(s \to 0)$. Recall that, for $S \in \mathcal{L}(X)$ satisfying $\|S^m\| \leq M$ for all $m \in \mathbb{N}$, the estimate

$$\left\|e^{n(S-I)}x - S^n x\right\| \leq \sqrt{n}M\|Sx - x\| \tag{C.1}$$

holds for every $n \in \mathbb{N}$ and $x \in X$ (cf. [39; Lemma III.5.1]). Applying this estimate with $S := V(t/n)$ we obtain

$$\begin{aligned}\|e^{tA(t/n)}x(t/n) - V(t/n)^n x(t/n)\| &= \|e^{n(V(t/n)-I)}x(t/n) - V(t/n)^n x(t/n)\| \\ &\leq \sqrt{n}M\,\|V(t/n)x(t/n) - x(t/n)\| = \frac{tM}{\sqrt{n}}\,\|A(t/n)x(t/n)\| \to 0\end{aligned}$$

as $n \to \infty$, uniformly for t in compact subsets of $[0, \infty)$. Observing $\|e^{tA(t/n)} - V(t/n)^n\| \leq 2M$ $(t \geq 0, n \in \mathbb{N})$ we conclude

$$\begin{aligned}\|T(t)x - V(t/n)^n x\| &\leq \|(T(t) - e^{tA(t/n)})x\| + \|(e^{tA(t/n)} - V(t/n)^n)x\| \\ &\leq \|(T(t) - e^{tA(t/n)})x\| + 2M\|x - x(t/n)\| + \|(e^{tA(t/n)} - V(t/n)^n)x(t/n)\| \to 0,\end{aligned}$$

as $n \to \infty$, uniformly for t in compact subsets of $[0, \infty)$. Now the fact that $D(A)$ is dense in X implies the assertion. ■

C.2 Remark. If in Theorem C.1 one assumes $A \in \mathcal{L}(X)$ and $A(s) \to A$ in operator norm then the conclusion is that

$$T(t) = \lim_{n\to\infty} V(t/n)^n$$

(operator norm limit!), uniformly for t in compact subsets of $[0, \infty)$.

This fact, except for the uniformity of the convergence, can be obtained as a consequence of [13; Theorem 1.1]. We include a simple proof for our special case. We note that, for small s, the logarithm of $V(s)$ exists and is given by

$$\ln V(s) = \ln(I + sA(s)) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{s^n}{n}A(s)^n.$$

Obviously $\frac{1}{s}\ln V(s) \to A$ as $s \to 0$. Therefore

$$V(t/n)^n = e^{n\ln V(t/n)} = e^{t(n/t)\ln V(t/n)} \to e^{tA}$$

as $n \to \infty$, uniformly for t in compact subsets of $[0, \infty)$. This shows the assertion.

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