Multivariate Mixed Poisson Processes

DISSERTATION

zur Erlangung des akademischen Grades

Doctor rerum naturalium
(Dr. rer. nat.)

vorgelegt

der Fakultät Mathematik und Naturwissenschaften
der Technischen Universität Dresden

von

Dipl.-Math. Mathias Zocher
geboren am 01. August 1977 in Dresden

Gutachter:

Prof. Dr. K.D. Schmidt, Technische Universität Dresden
Prof. Dr. R. Kühne, Technische Universität Dresden
Prof. Dr. G. Alsmeyer, Universität Münster

Eingereicht am: 20.06.2005

Tag der Verteidigung: 02.12.2005

Contents

Introduction

Univariate mixed Poisson distributions and univariate mixed Poisson processes are widely used for modelling the occurrence of rare events. This dates back to the twenties of the last century and an enormous amount of work in various scientific areas has been published since then and is based on a solid fundament of theoretical results.

The tradition of using multivariate mixed Poisson distributions and multivariate mixed Poisson processes is almost as long. Bates and Neyman [1952], Consael [1952], and Hofmann [1955] have to be mentioned in this context. But in contrast to the univariate case the number of publications is relatively small. Nevertheless, different areas are covered by the work published so far, as aerial accidents (Bates and Neyman [1952]), working and non–working accidents (Hofmann [1955]), motor car insurance (Picard [1976], Partrat [1994], Lemaire [1995], Walhin and Paris [2001], Zocher [2005]), victimizations (Nelson [1984]), hurricanes (Partrat [1994]), image detecting in astro physics (Ferrari et al. [2004]), and loss reserving (Schmidt and Zocher [2005]).

Since the theoretical background has not yet been developed to the same extent as in the univariate case, there exists a gap between desired practical applications and available theoretical results. The aim of the present work is to close this gap a little bit. The basis of this study is the multivariate counting process, which is related to the birth process. The model of multivariate counting processes will be specified by different assumptions leading to different models of multivariate mixed Poisson processes, which are, however, connected with each other. Starting with the most general model and specifying it step by step, this work is organized as follows:

Chapter 1 provides some definitions and propositions of auxiliary character on multivariate counting distributions which will be needed in the subsequent chapters and for which no citable reference was found. This involves multivariate versions of the probability generating function (Section 1.1), the moment generating function (Section 1.2), and the Bernstein–Widder theorem (Section 1.3). The reader who is primarily interested in the results for multivariate mixed Poisson processes may skip this chapter at the first reading.

Multivariate counting processes are the subject of Chapter 2. First, these processes are introduced (Section 2.1) and then some properties, which counting processes may have and which are related to mixed Poisson processes, are presented (Section 2.2). The relations between such properties, like for example stationary increments, the multinomial property, and the Markov property, are also studied in detail. Furthermore, the concept of regularity, which is closely connected with transition intensities, is introduced (Section 2.3). This section also contains a characterization of regularity in terms of the system of Kolmogorov forward differential equations and in terms of the system of Kolmogorov backward differential equations.

Chapter 3 is devoted to multivariate mixed Poisson processes with an arbitrary mixing distribution. Again some properties of these processes are derived and it is shown that the one–dimensional distributions are sufficient to determine the distribution of a multivariate mixed Poisson process (Section 3.1). The use of the multivariate setting is justified in this section by Theorem 3.1.4 which asserts that the coordinates of a multivariate mixed Poisson process are independent if, and only if, the mixing distribution has a representation as a product measure. Moreover, multivariate mixed Poisson processes are characterized as multivariate counting processes having the multinomial property (Section 3.2). Upon this result it is shown that a multivariate mixed Poisson process with independent increments even is a multivariate Poisson process in the sense that the coordinates are independent and each coordinate is a univariate Poisson process. Properties of the moment structure of multivariate mixed Poisson processes are given as well (Section 3.3). If such a process has a finite moment of first order then, and only then, it is a regular process. This result and some properties of transition probabilities and transition intensities of multivariate mixed Poisson processes (Section 3.4) conclude the study of multivariate mixed Poisson processes with an arbitrary mixing distribution.

An alternative way to model multivariate mixed Poisson processes within the class of multivariate counting processes is to assume the existence of a random vector on the same probability space and to consider conditional probabilities of the process with respect to this random vector, such that the process is still a multivariate mixed Poisson process. This yields multivariate mixed Poisson processes with a random parameter, which are discussed in Chapter 4, and their mixing distribution is then given by the distribution of the random vector. With the mixing distribution originating from a random vector, simpler representations of some results are obtained whereas the use of conditional probabilities offers new questions. Similar to the previous chapter some basic properties (Section 4.1) and the moment structure (Section 4.2) of multivariate mixed Poisson processes are studied. Because of the additional assumption a characterization analogous to that of Section 3.2 is not possible. Since the model considered in this chapter requires the existence of a random parameter, posterior distributions of the parameter with respect to the process can be considered (Section 4.3). Again, the chapter is concluded by a look at properties of transition probabilities and transition intensities (Section 4.4).

The model in Chapter 5 originates from the multivariate mixed Poisson process with random parameter by adding an additional assumption on the parameter. This assumption, which is implicitly made in most of the models discussed in literature, leads to the multivariate mixed Poisson process with special parameter and offers at the same time new results and simpler representations of distributions (Section 5.1), moments (Section 5.2), posterior distributions (Section 5.3), and transition intensities (Section 5.4). Although the specification made in this chapter reduces the complexity of the multivariate modelling, it still allows for a large variety of correlation structures between the coordinates.

Throughout this work it is studied for each property if this property is transferred from the original multivariate process to processes obtained by certain linear transformations. For example, coordinates and the sum of all coordinates of a multivariate mixed Poisson process are again a mixed Poisson process. Moreover, it is shown with the help of the incremental process that all models under consideration are in a certain sense stable over time. Thus, in order to be able to accept one of these models it is not crucial to know when the process started, which has a substantial impact on possible applications.

It is possible to initially skip the sections on regularity and read these sections consecutively since they do not influence the other sections. Further it has to be mentioned that, of course, the publications concerning the univariate setting, like Schmidt [1996] and Grandell [1997] to name just two of them, also offer relations between properties, questions to ask, and ideas for some proofs in the multivariate case. This influence will not be pronounced at every possible occasion, but whenever the ideas of univariate setting are also essential in the multivariate setting the references will be given.

Throughout this work $(\Omega, \mathcal{F}, \mathrm{P})$ is the underlying probability space. Every bold letter represents a vector or a random vector. Special notation for vectors can be found in the list of symbols. It should be pointed out that every sum in which the summation is restricted to a multivariate interval is understood to be also restricted to $\mathbb{N}_0^k$.

Acknowledgement

I would like to thank all members of the Institut für Mathematische Stochastik for their manifold support; in particular Dr. Lothar Partzsch and Dr. Klaus-Thomas Hess for fruitful conversations about this work. Especially, I would like to express my gratitude to my supervisor Prof. Dr. Klaus D. Schmidt for the valuable comments and the inspiring discussions.

Chapter 1

Multivariate Counting Distributions

1.1 Probability Generating Function

In this section we will state some properties of the probability generating function, which we need analyzing multivariate mixed Poisson processes. The probability generating function belongs to a distribution. Since we only need this function in connection with moments of random vectors, we keep the notation simple by defining the probability generating function for random vectors.

But before we define the probability generating function we introduce a notation concerning derivatives of functions with a multivariate argument.

$$D^{\mathbf{n}} f(\mathbf{t}) \quad := \quad \frac{\partial^{\mathbf{1}'\mathbf{n}} f}{\partial {t_1}^{n^{(1)}} \cdots \partial {t_k}^{n^{(k)}}}(\mathbf{t})$$

Of course this notation will only be used when the partial derivatives are continuous and so the order of execution of the derivatives is negligible. Due to linear transformations, which will occur, we will use another notation for derivatives. For example consider $g : \mathbb{R}^k \to \mathbb{R}^k$ with $g(\mathbf{t}) = r\,(\mathbf{t} - \mathbf{1})$ for $r \in \mathbb{R}_+$. Then, for the sake of clearness, we will use

$$\left.\frac{\partial^{\mathbf{1}'\mathbf{n}} f\,(r\,(\mathbf{t} - \mathbf{1}))}{\partial^{\mathbf{n}} \mathbf{t}}\right|_{\mathbf{t}=\mathbf{0}}$$

instead of $D^{\mathbf{n}}\,(f \circ g)\,(\mathbf{0})$. Other notations concerning vectors, which will always be written in bold letters, can be found in the list of symbols.

Let $\mathbf{X} : \Omega \to \mathbb{N}_0^k$ be a random vector. The function $g_{\mathbf{X}} : [\mathbf{0}, \mathbf{1}] \to \mathbb{R}$ with

$$g_{\mathbf{X}}(\mathbf{r}) \quad := \quad \mathrm{E}\left[\mathbf{r}^{\mathbf{X}}\right] \quad = \sum_{\mathbf{n}\,\in\,\mathbb{N}_0^k} \mathbf{r}^{\mathbf{n}}\,\mathrm{P}\left[\{\mathbf{X} = \mathbf{n}\}\right]$$

is called the ***probability generating function*** of $\mathbf{X}$.

The probability generating function is therefore a power series in k coordinates. The theorems we need to prove the propositions in this section are taken from Dieudonné [1971] Chapter 9 (power series in k coordinates) and Heuser [2003a] Chapter 103 (series of functions). The treatment of the probability generating function of a random vectors utilizes ideas used for the treatment of probability generating functions of random variables as given in Schmidt [2002].

1.1.1 Lemma. *The probability generating function $g_{\mathbf{X}}$ of a random vector $\mathbf{X} : \Omega \to \mathbb{N}_0^k$ possesses the following properties.*

(1) *$g_{\mathbf{X}}$ is increasing and*

$$0 \;\leq\; g_{\mathbf{X}}(\mathbf{r}) \;\leq\; g_{\mathbf{X}}(\mathbf{1}) \;=\; 1$$

holds for all $\mathbf{r} \in [\mathbf{0}, \mathbf{1}]$.

(2) *$g_{\mathbf{X}}$ is continuous on $[\mathbf{0}, \mathbf{1}]$.*

(3) *$g_{\mathbf{X}}$ is infinitely often differentiable on $[\mathbf{0}, \mathbf{1})$.*

(4) *For all $\mathbf{l} \in \mathbb{N}_0^k$ and $\mathbf{r} \in [\mathbf{0}, \mathbf{1})$ the probability generating function fulfils*

$$D^{\mathbf{l}} g_{\mathbf{X}}(\mathbf{r}) \;=\; \sum_{\mathbf{n} \in [\mathbf{l}, \boldsymbol{\infty})} \mathbf{r}^{\mathbf{n}-\mathbf{l}} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{l})!} \, \mathrm{P}[\{\mathbf{X} = \mathbf{n}\}]$$

(5) *For all $\mathbf{l} \in \mathbb{N}_0^k$ the derivative $D^{\mathbf{l}} g_{\mathbf{X}}$ is increasing on $[\mathbf{0}, \mathbf{1})$ and*

$$\sup_{\mathbf{r} \in [\mathbf{0}, \mathbf{1})} D^{\mathbf{l}} g_{\mathbf{X}}(\mathbf{r}) \;=\; \sum_{\mathbf{n} \in [\mathbf{l}, \boldsymbol{\infty})} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{l})!} \, \mathrm{P}[\{\mathbf{X} = \mathbf{n}\}]$$

is valid.

Proof:
(1): obvious

(2): For all $\mathbf{r} \in [\mathbf{0}, \mathbf{1}]$ and $\mathbf{m} \in \mathbb{N}_0^k$ we get

$$\begin{aligned}
\left| \sum_{\mathbf{n} \in \mathbb{N}_0^k} \mathbf{r}^{\mathbf{n}} \, \mathrm{P}[\{\mathbf{X} = \mathbf{n}\}] - \sum_{\mathbf{n} \in [\mathbf{0}, \mathbf{m}]} \mathbf{r}^{\mathbf{n}} \, \mathrm{P}[\{\mathbf{X} = \mathbf{n}\}] \right| &= \sum_{\mathbf{n} \in \mathbb{N}_0^k \setminus [\mathbf{0}, \mathbf{m}]} \mathbf{r}^{\mathbf{n}} \, \mathrm{P}[\{\mathbf{X} = \mathbf{n}\}] \\
&\leq \sum_{\mathbf{n} \in \mathbb{N}_0^k \setminus [\mathbf{0}, \mathbf{m}]} \mathrm{P}[\{\mathbf{X} = \mathbf{n}\}]
\end{aligned}$$

Since this last series is a tail of a convergent series, we get for every $\varepsilon > 0$ the existence of some $\mathbf{m} \in \mathbb{N}_0^k$ such that

$$\left| \sum_{\mathbf{n} \in \mathbb{N}_0^k} \mathbf{r}^{\mathbf{n}} \, \mathrm{P}[\{\mathbf{X} = \mathbf{n}\}] - \sum_{\mathbf{n} \in [\mathbf{0}, \mathbf{m}]} \mathbf{r}^{\mathbf{n}} \, \mathrm{P}[\{\mathbf{X} = \mathbf{n}\}] \right| \;<\; \varepsilon$$

holds for all $\mathbf{r} \in [\mathbf{0}, \mathbf{1}]$. Hence, the power series converges uniformly on $[\mathbf{0}, \mathbf{1}]$. Since every partial sum of the power series is a polynomial and therefore continuous the theory of series of functions yields, the continuity of the power series and thus of $g_{\mathbf{X}}$.

(3): Since the power series

$$\sum_{\mathbf{n} \in \mathbb{N}_0^k} \mathbf{r}^{\mathbf{n}} \, \mathrm{P}[\{\mathbf{X} = \mathbf{n}\}]$$

is absolutely convergent on $[-\mathbf{1}, \mathbf{1}]$, the theory of power series yields that the power series is infinitely often differentiable on $(-\mathbf{1}, \mathbf{1})$ and thus the probability generating function $g_{\mathbf{X}}$ is infinitely often differentiable on $[\mathbf{0}, \mathbf{1})$.

(4): From Dieudonné [1971] we have

$$D^{\mathbf{e}_j} g_{\mathbf{X}}(\mathbf{r}) \;=\; \sum_{\mathbf{n} \in [\mathbf{e}_j, \boldsymbol{\infty})} n^{(j)} \, \mathbf{r}^{\mathbf{n}-\mathbf{e}_j} \, \mathrm{P}[\{\mathbf{X} = \mathbf{n}\}]$$

and induction yields the assertion.

(5): Let $\mathbf{l} \in \mathbb{N}_0^k$. It is obvious that $D^{\mathbf{l}} g_{\mathbf{X}}$ is increasing on $[\mathbf{0}, \mathbf{1})$. Furthermore, we set

$$c_{\mathbf{l}} \;:=\; \sup_{\mathbf{r} \in [\mathbf{0}, \mathbf{1})} D^{\mathbf{l}} g_{\mathbf{X}}(\mathbf{r})$$

For all $\mathbf{r} \in [\mathbf{0}, \mathbf{1})$ we obtain

$$\begin{aligned} D^{\mathbf{l}} g_{\mathbf{X}}(\mathbf{r}) &= \sum_{\mathbf{n} \in [\mathbf{l}, \boldsymbol{\infty})} \mathbf{r}^{\mathbf{n}-\mathbf{l}} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{l})!} \mathrm{P}[\{\mathbf{X} = \mathbf{n}\}] \\ &\leq \sum_{\mathbf{n} \in [\mathbf{l}, \boldsymbol{\infty})} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{l})!} \mathrm{P}[\{\mathbf{X} = \mathbf{n}\}] \end{aligned}$$

and therefore

$$c_{\mathbf{l}} \;\leq\; \sum_{\mathbf{n} \in [\mathbf{l}, \boldsymbol{\infty})} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{l})!} \mathrm{P}[\{\mathbf{X} = \mathbf{n}\}]$$

On the other hand we get for all $\mathbf{m} \in \mathbb{N}_0^k$ and all $\mathbf{r} \in [\mathbf{0}, \mathbf{1})$

$$\begin{aligned} \sum_{\mathbf{n} \in [\mathbf{l}, \mathbf{m}]} \mathbf{r}^{\mathbf{n}-\mathbf{l}} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{l})!} \mathrm{P}[\{\mathbf{X} = \mathbf{n}\}] &\leq \sum_{\mathbf{n} \in [\mathbf{l}, \boldsymbol{\infty})} \mathbf{r}^{\mathbf{n}-\mathbf{l}} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{l})!} \mathrm{P}[\{\mathbf{X} = \mathbf{n}\}] \\ &= D^{\mathbf{l}} g_{\mathbf{X}}(\mathbf{r}) \\ &\leq c_{\mathbf{l}} \end{aligned}$$

As a consequence of the continuity of polynomials in k coordinates the inequality

$$\sum_{\mathbf{n} \in [\mathbf{l}, \mathbf{m}]} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{l})!} \mathrm{P}[\{\mathbf{X} = \mathbf{n}\}] \;\leq\; c_{\mathbf{l}}$$

holds for all $\mathbf{m} \in \mathbb{N}_0^k$. This yields

$$\sum_{\mathbf{n} \in [\mathbf{l},\boldsymbol{\infty})} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{l})!} \, \mathrm{P}[\{\mathbf{X}=\mathbf{n}\}] \;\; \leq \;\; c_{\mathbf{l}}$$

and so together with the inequality shown before we get

$$\sum_{\mathbf{n} \in [\mathbf{l},\boldsymbol{\infty})} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{l})!} \, \mathrm{P}[\{\mathbf{X}=\mathbf{n}\}] \;\; = \;\; c_{\mathbf{l}}$$

which completes the proof. ■

1.1.2 Corollary. *Let $\mathbf{X} : \Omega \to \mathbb{N}_0^k$ be a random vector. Then*

$$\mathrm{P}[\{\mathbf{X}=\mathbf{l}\}] \;\; = \;\; \frac{1}{\mathbf{l}!} \, D^{\mathbf{l}} g_{\mathbf{X}}(\mathbf{0})$$

holds for all $\mathbf{l} \in \mathbb{N}_0^k$.

The name probability generating function is therewith justified. We also see that the distribution of the random vector $\mathbf{X}$ is uniquely determined by its probability generating function.

Let $\mathbf{X} : \Omega \to \mathbb{N}_0^k$ be a random vector and $\mathbf{l} \in \mathbb{N}_0^k$. Then

$$\mathrm{E}\left[\binom{\mathbf{X}}{\mathbf{l}}\right] \;\; = \;\; \sum_{\mathbf{n} \in [\mathbf{l},\boldsymbol{\infty})} \binom{\mathbf{n}}{\mathbf{l}} \, \mathrm{P}[\{\mathbf{X}=\mathbf{n}\}]$$

is called ***binomial moment*** of order $\mathbf{l}$ of $\mathbf{X}$. From Lemma 1.1.1 we have

$$\mathrm{E}\left[\binom{\mathbf{X}}{\mathbf{l}}\right] \;\; = \;\; \sup_{\mathbf{r} \in [\mathbf{0},\mathbf{1})} \frac{1}{\mathbf{l}!} \, D^{\mathbf{l}} g_{\mathbf{X}}(\mathbf{r})$$

The binomial moment of order $\mathbf{l}$ of $\mathbf{X}$ exists as an expectation of a positive random vector but need not to be finite.

1.1.3 Lemma. *Let $\mathbf{X} : \Omega \to \mathbb{N}_0^k$ be a random vector and $\mathbf{l} \in \mathbb{N}_0^k$. Then the following are equivalent.*

(a) *The binomial moment of order $\mathbf{l}$ fulfils*

$$\mathrm{E}\left[\binom{\mathbf{X}}{\mathbf{l}}\right] \;\; < \;\; \infty$$

(b) *The inequality*

$$\lim_{\mathbf{r} \to \mathbf{s}} D^{\mathbf{l}} g_{\mathbf{X}}|_{[\mathbf{0},\mathbf{1})}(\mathbf{r}) \;\; < \;\; \infty$$

holds for all $\mathbf{s} \in [\mathbf{0},\mathbf{1}]$.

If $\mathbf{X}$ *satisfies one and hence all preceding items, then the binomial moment can be expressed as*

$$\mathrm{E}\left[\binom{\mathbf{X}}{\mathbf{l}}\right] \;=\; \frac{1}{\mathbf{l}!}\lim_{\mathbf{r}\uparrow\mathbf{1}} D^{\mathbf{l}} g_{\mathbf{X}}|_{[\mathbf{0},\mathbf{1})}(\mathbf{r})$$

Proof:
(a) $\Rightarrow$ (b): Since (a) holds we have

$$\sum_{\mathbf{n}\in[\mathbf{l},\infty)} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{l})!}\,\mathrm{P}[\{\mathbf{X}=\mathbf{n}\}] \;=\; \mathbf{l}!\,\mathrm{E}\left[\binom{\mathbf{X}}{\mathbf{l}}\right] \;<\; \infty$$

With the same argumentation as in the proof of 1.1.1 (2) we have for all $\varepsilon > 0$ the existence of some $\mathbf{q} \geq \mathbf{l}$ such that

$$\begin{aligned}
&\left|\sum_{\mathbf{n}\in[\mathbf{l},\infty)} \mathbf{r}^{\mathbf{n}-\mathbf{l}}\frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{l})!}\,\mathrm{P}[\{\mathbf{X}=\mathbf{n}\}] - \sum_{\mathbf{n}\in[\mathbf{l},\mathbf{q}]} \mathbf{r}^{\mathbf{n}-\mathbf{l}}\frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{l})!}\,\mathrm{P}[\{\mathbf{X}=\mathbf{n}\}]\right| \\
&\qquad\leq\; \sum_{\mathbf{n}\in[\mathbf{l},\infty)\setminus[\mathbf{l},\mathbf{q}]} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{l})!}\,\mathrm{P}[\{\mathbf{X}=\mathbf{n}\}] \\
&\qquad<\; \varepsilon
\end{aligned}$$

holds for all $\mathbf{r} \in [\mathbf{0},\mathbf{1}]$. So the power series

$$\sum_{\mathbf{n}\in[\mathbf{l},\infty)} \mathbf{r}^{\mathbf{n}-\mathbf{l}}\frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{l})!}\,\mathrm{P}[\{\mathbf{X}=\mathbf{n}\}]$$

converges uniformly on $[\mathbf{0},\mathbf{1}]$ to a continuous function. As a consequence of Lemma 1.1.1 (4)

$$D^{\mathbf{l}} g_{\mathbf{X}}(\mathbf{r}) \;=\; \sum_{\mathbf{n}\in[\mathbf{l},\infty)} \mathbf{r}^{\mathbf{n}-\mathbf{l}}\frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{l})!}\,\mathrm{P}[\{\mathbf{X}=\mathbf{n}\}]$$

holds for $\mathbf{r} \in [\mathbf{0},\mathbf{1})$. Thus (b) follows.
In particular, we have

$$\mathrm{E}\left[\binom{\mathbf{X}}{\mathbf{l}}\right] \;=\; \frac{1}{\mathbf{l}!}\lim_{\mathbf{r}\uparrow\mathbf{1}} D^{\mathbf{l}} g_{\mathbf{X}}|_{[\mathbf{0},\mathbf{1})}(\mathbf{r})$$

(b) $\Rightarrow$ (a): The assumption yields the finiteness of $\sup_{\mathbf{r}\in[\mathbf{0},\mathbf{1})} D^{\mathbf{l}} g_{\mathbf{X}}(\mathbf{r})$. Thus

$$\begin{aligned}
\mathrm{E}\left[\binom{\mathbf{X}}{\mathbf{l}}\right] \;&=\; \sup_{\mathbf{r}\in[\mathbf{0},\mathbf{1})} \frac{1}{\mathbf{l}!} D^{\mathbf{l}} g_{\mathbf{X}}(\mathbf{r}) \\
&<\; \infty
\end{aligned}$$

and therefore (a) is valid. ■

Contrary to the one–dimensional case of positive discrete random variables (compare Schmidt [2002]) the finiteness of the binomial moment of order $\mathbf{l}$ is not equivalent to the finiteness of the moment of order $\mathbf{l}$. Furthermore, we can not conclude if Lemma 1.1.3 (b) holds for $\mathbf{l} \in \mathbb{N}_0^k$, that there exists some $\mathbf{m} \in [\mathbf{0}, \mathbf{l}]$ with $\mathbf{m} \neq \mathbf{l}$ such that (b) holds for $\mathbf{m}$. Therefore, we are not able to use the term $D^{\mathbf{l}} g_{\mathbf{X}}(\mathbf{1})$. The succeeding example will illustrate the issue.

Example: We consider a bivariate random vector $\mathbf{X}$ satisfying

$$\mathrm{P}\left[\{\mathbf{X} = \mathbf{n}\}\right] \;=\; \begin{cases} (2\,c\,n^2)^{-1} & \text{if } \mathbf{n} = (n, 0)' \text{ or } \mathbf{n} = (1, n)' \text{ with } n \in \mathbb{N} \\ 0 & \text{else} \end{cases}$$

with $c = \sum_{n=1}^{\infty} 1/n^2$. Then we have

$$\mathrm{E}\left[X^{(1)}\left(X^{(1)} - 1\right) X^{(2)}\right] \;=\; 0$$

and on the other hand

$$\begin{aligned} \mathrm{E}\left[X^{(1)}\right] &\geq \sum_{n=2}^{\infty} \frac{1}{2\,c} \frac{1}{n} \\ \mathrm{E}\left[X^{(2)}\right] &= \sum_{n=1}^{\infty} \frac{1}{2\,c} \frac{1}{n} \\ \mathrm{E}\left[X^{(1)} X^{(2)}\right] &= \sum_{n=1}^{\infty} \frac{1}{2\,c} \frac{1}{n} \end{aligned}$$

where the sum $\sum_{n=2}^{\infty}(2\,c\,n)^{-1}$ is infinite. Thus, the binomial moment of order $(2,1)'$ is finite, but no other binomial moment of order $\mathbf{l}$ with $\mathbf{l} \leq (2,1)'$ is finite. Furthermore, we have

$$\mathrm{E}\left[\left(X^{(1)}\right)^2 X^{(2)}\right] \;\geq\; \mathrm{E}\left[X^{(1)} X^{(2)}\right]$$

and therefore the moment of order $(2,1)'$ is also not finite. In terms of the probability generating function it looks like

$$g_{\mathbf{X}}(\mathbf{r}) \;=\; \sum_{n=1}^{\infty} \frac{1}{2\,c} \frac{(r_1)^n}{n^2} + \sum_{n=1}^{\infty} \frac{1}{2\,c} \frac{r_1\,(r_2)^n}{n^2}$$

for $\mathbf{r} \in [\mathbf{0}, \mathbf{1}]$ and

$$\begin{aligned} D^{\mathbf{e}_1} g_{\mathbf{X}}(\mathbf{r}) &= \sum_{n=1}^{\infty} \frac{1}{2\,c} \frac{(r_1)^{n-1}}{n} + \sum_{n=1}^{\infty} \frac{1}{2\,c} \frac{(r_2)^n}{n^2} \\ D^{\mathbf{e}_2} g_{\mathbf{X}}(\mathbf{r}) &= \sum_{n=1}^{\infty} \frac{1}{2\,c} \frac{r_1\,(r_2)^{n-1}}{n} \\ D^{\mathbf{e}_1} D^{\mathbf{e}_2} g_{\mathbf{X}}(\mathbf{r}) &= \sum_{n=1}^{\infty} \frac{1}{2\,c} \frac{(r_2)^{n-1}}{n} \\ D^{\mathbf{e}_1} D^{\mathbf{e}_1} D^{\mathbf{e}_2} g_{\mathbf{X}}(\mathbf{r}) &= 0 \end{aligned}$$

for $\mathbf{r} \in [\mathbf{0}, \mathbf{1})$ where we can see that

$$\lim_{\mathbf{r}\uparrow\mathbf{1}} D^{\mathbf{e}_1} g_{\mathbf{X}}|_{[\mathbf{0},\mathbf{1})}(\mathbf{r}) \;=\; \lim_{\mathbf{r}\uparrow\mathbf{1}} D^{\mathbf{e}_2} g_{\mathbf{X}}|_{[\mathbf{0},\mathbf{1})}(\mathbf{r}) \;=\; \lim_{\mathbf{r}\uparrow\mathbf{1}} D^{\mathbf{e}_1} D^{\mathbf{e}_2} g_{\mathbf{X}}|_{[\mathbf{0},\mathbf{1})}(\mathbf{r}) \;=\; \infty$$

Therefore, we are not able to use the term $D^{\mathbf{l}} g_{\mathbf{X}}(\mathbf{1})$. This shows that the theory of probability generating functions for one–dimensional random variables cannot be carried over to the multivariate case unmodified. To get the desired equivalences we have to strengthen the requirements. □

1.1.4 Lemma. *Let $\mathbf{X} : \Omega \to \mathbb{N}_0^k$ be a random vector and $\mathbf{l} \in \mathbb{N}_0^k$. Then the following are equivalent.*

(a) *For all $\mathbf{m} \leq \mathbf{l}$ the binomial moment of order $\mathbf{m}$ fulfils*

$$\mathrm{E}\left[\binom{\mathbf{X}}{\mathbf{m}}\right] \;<\; \infty$$

(b) *For all $\mathbf{m} \leq \mathbf{l}$ the moment of order $\mathbf{m}$ fulfils*

$$\mathrm{E}\left[\mathbf{X}^{\mathbf{m}}\right] \;<\; \infty$$

(c) *For all $\mathbf{m} \leq \mathbf{l}$ the inequality*

$$\lim_{\mathbf{r}\to\mathbf{s}} D^{\mathbf{m}} g_{\mathbf{X}}|_{[\mathbf{0},\mathbf{1})}(\mathbf{r}) \;<\; \infty$$

holds for all $\mathbf{s} \in [\mathbf{0}, \mathbf{1}]$.

(d) *For all $\mathbf{m} \leq \mathbf{l}$ the $\mathbf{m}$–th derivative of $g_{\mathbf{X}}$ is continuous on $[\mathbf{0}, \mathbf{1}]$.*

If $\mathbf{X}$ satisfies one and hence all preceding items, then

$$\mathrm{E}\left[\binom{\mathbf{X}}{\mathbf{l}}\right] \;=\; \frac{1}{\mathbf{l}!} D^{\mathbf{l}} g_{\mathbf{X}}(\mathbf{1})$$

holds.

Proof: The equivalence of (a) and (c) holds due to Lemma 1.1.3. The remaining assertions are proven according to the following scheme: (a) ⇔ (b) and (a) ⇔ (d).

(a) ⇒ (b): By induction we are able to show that for all $\mathbf{m} \in \mathbb{N}_0^k$

$$\mathbf{X}^{\mathbf{m}} \in \operatorname{span}\left\{\binom{\mathbf{X}}{\mathbf{j}} \;:\; \mathbf{j} \in \mathbb{N}_0^k, \mathbf{j} \leq \mathbf{m}\right\}$$

and so all moments of order $\mathbf{m}$ with $\mathbf{m} \leq \mathbf{l}$ have a representation of the form

$$\mathrm{E}\left[\mathbf{X}^{\mathbf{m}}\right] \;=\; \sum_{\mathbf{j}\in[\mathbf{0},\mathbf{m}]} a_{\mathbf{mj}} \,\mathrm{E}\left[\binom{\mathbf{X}}{\mathbf{j}}\right]$$

with $a_{\mathbf{mj}} \in \mathbb{R}$ and thus

$$\mathrm{E}[\mathbf{X}^{\mathbf{m}}] \;<\; \infty$$

holds for all $\mathbf{m} \le \mathbf{l}$.

(b) $\Rightarrow$ (a): Since

$$\mathrm{E}\left[\binom{\mathbf{X}}{\mathbf{m}}\right] \;\le\; \mathrm{E}[\mathbf{X}^{\mathbf{m}}]$$

holds for all $\mathbf{m} \in \mathbb{N}_0^k$, the assertion follows.

(a) $\Rightarrow$ (d): We consider $\mathbf{m} \in \mathbb{N}_0^k$ with $\mathbf{m} \le \mathbf{l}$. Due to the assumption, we can show with the same argumentation as in the proof of 1.1.1 (2) for all $\varepsilon > 0$ the existence of some $\mathbf{q} \ge \mathbf{l}$ such that

$$\begin{aligned} &\left| \sum_{\mathbf{n} \in [\mathbf{m},\boldsymbol{\infty})} \mathbf{r}^{\mathbf{n}-\mathbf{m}} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{m})!} \mathrm{P}[\{\mathbf{X}=\mathbf{n}\}] - \sum_{\mathbf{n} \in [\mathbf{m},\mathbf{q}]} \mathbf{r}^{\mathbf{n}-\mathbf{m}} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{m})!} \mathrm{P}[\{\mathbf{X}=\mathbf{n}\}] \right| \\ &\qquad \le \sum_{\mathbf{n} \in [\mathbf{m},\boldsymbol{\infty}) \setminus [\mathbf{m},\mathbf{q}]} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{m})!} \mathrm{P}[\{\mathbf{X}=\mathbf{n}\}] \\ &\qquad < \varepsilon \end{aligned}$$

holds for all $\mathbf{r} \in [\mathbf{0},\mathbf{1}]$. So the power series

$$\sum_{\mathbf{n} \in [\mathbf{m},\boldsymbol{\infty})} \mathbf{r}^{\mathbf{n}-\mathbf{m}} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{m})!} \mathrm{P}[\{\mathbf{X}=\mathbf{n}\}]$$

converges uniformly on $[\mathbf{0},\mathbf{1}]$ to a continuous function we will give the name $f_{\mathbf{m}}$. Let $h \in \{1,\ldots,k\}$ with $\mathbf{m} - \mathbf{e}_h \ge \mathbf{0}$. Inserting $\mathbf{r} = (r - s^{(h)})\mathbf{e}_h + \mathbf{s}$ for arbitrary $\mathbf{s} \in [\mathbf{0},\mathbf{1}]$ in $f_{\mathbf{m}}$ and $f_{\mathbf{m}-\mathbf{e}_h}$ we obtain two power series $f_{\mathbf{m},h}$ and $f_{\mathbf{m}-\mathbf{e}_h,h}$ in r which are uniformly convergent on $[0,1]$. The theory of one–dimensional power series yields

$$f_{\mathbf{m},h}(r) \;=\; f'_{\mathbf{m}-\mathbf{e}_h,h}(r)$$

for all $r \in [0,1]$. As $\mathbf{s}$ was arbitrary we get

$$f_{\mathbf{m}}(\mathbf{r}) \;=\; D^{\mathbf{e}_h} f_{\mathbf{m}-\mathbf{e}_h}(\mathbf{r})$$

for all $\mathbf{r} \in [\mathbf{0},\mathbf{1}]$. By induction, $f_{\mathbf{m}} = D^{\mathbf{m}} g_{\mathbf{X}}$ and the assertion follows.

(d) $\Rightarrow$ (a): The continuity of the derivative and Lemma 1.1.1 (5) yield for all $\mathbf{m} \le \mathbf{l}$

$$\mathrm{E}\left[\binom{\mathbf{X}}{\mathbf{m}}\right] \;=\; \sup_{\mathbf{r} \in [\mathbf{0},\mathbf{1})} \frac{1}{\mathbf{m}!} D^{\mathbf{m}} g_{\mathbf{X}}(\mathbf{r}) \;=\; \frac{1}{\mathbf{m}!} D^{\mathbf{m}} g_{\mathbf{X}}(\mathbf{1})$$

and therefore (a) holds. ∎

Now we are able to give a compact form of the first two central moments of a random vector.

1.1.5 Corollary. *Let* $\mathbf{X} : \Omega \to \mathbb{N}_0^k$ *be a random vector.*

(1) *If* $X^{(i)} \in \mathcal{L}^1(\mathbb{N}_0)$ *for all* $i \in \{1, \ldots, k\}$, *then*

$$\mathrm{E}[\mathbf{X}] \;=\; \mathrm{grad} g_{\mathbf{X}}(\mathbf{1})$$

(2) *If* $X^{(i)} \in \mathcal{L}^2(\mathbb{N}_0)$ *for all* $i \in \{1, \ldots, k\}$, *then*

$$\mathrm{Var}[\mathbf{X}] \;=\; \mathrm{Hess} g_{\mathbf{X}}(\mathbf{1}) - \mathrm{grad} g_{\mathbf{X}}(\mathbf{1})\, \mathrm{grad} g_{\mathbf{X}}(\mathbf{1})' + \mathrm{Diag}\left(\mathrm{grad} g_{\mathbf{X}}(\mathbf{1})\right)$$

Proof:
(1): The assertion immediately follows from Lemma 1.1.4.

(2): Let $i \in \{1, \ldots, k\}$. As a consequence of the assumption $\mathrm{E}[(X^{(i)})^2]$ and therewith $\mathrm{E}[X^{(i)}]$ and $\mathrm{E}[X^{(i)}\,(X^{(i)} - 1)]$ are finite. From the Cauchy–Schwarz inequality we also get the finiteness of $\mathrm{E}[X^{(i)}\,X^{(j)}]$ for $j \in \{1, \ldots, k\}$, $j \neq i$. Now, using Lemma 1.1.4 we obtain

$$\begin{aligned}
\mathrm{Var}\left[X^{(i)}\right] &= \mathrm{E}\left[\left(X^{(i)}\right)^2\right] - \left(\mathrm{E}\left[X^{(i)}\right]\right)^2 \\
&= \mathrm{E}\left[X^{(i)}\left(X^{(i)} - 1\right)\right] + \mathrm{E}\left[X^{(i)}\right] - \left(\mathrm{E}\left[X^{(i)}\right]\right)^2 \\
&= D^{2\mathbf{e}_i} g_{\mathbf{X}}(\mathbf{1}) + D^{\mathbf{e}_i} g_{\mathbf{X}}(\mathbf{1}) - \left(D^{\mathbf{e}_i} g_{\mathbf{X}}(\mathbf{1})\right)^2
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Cov}\left[X^{(i)}, X^{(j)}\right] &= \mathrm{E}\left[X^{(i)}\,X^{(j)}\right] - \mathrm{E}\left[X^{(i)}\right]\,\mathrm{E}\left[X^{(j)}\right] \\
&= D^{\mathbf{e}_i + \mathbf{e}_j} g_{\mathbf{X}}(\mathbf{1}) - D^{\mathbf{e}_i} g_{\mathbf{X}}(\mathbf{1})\, D^{\mathbf{e}_j} g_{\mathbf{X}}(\mathbf{1})
\end{aligned}$$

Combining these two identities yields the assertion. ■

1.2 Moment Generating Function

In this section we introduce another auxiliary tool which can be applied to arbitrary distributions on $\mathcal{B}(\mathbb{R}^k)$.

The ***moment generating function*** $M_U : \mathbb{R}^k \to [0, \infty]$ of a distribution $U : \mathcal{B}(\mathbb{R}^k) \to [0, 1]$ is defined as

$$M_U(\mathbf{s}) \;:=\; \int_{\mathbb{R}^k} e^{\,\mathbf{s}'\mathbf{x}}\, dU(\mathbf{x})$$

The subsequent lemma and its proof is derived from the univariate setting as carried out in Billingsley [1995].

1.2.1 Lemma. *Consider a distribution* $U : \mathcal{B}(\mathbb{R}^k) \to [0,1]$ *and assume that* $M_U(\mathbf{s})$ *is finite in a neighbourhood* B *of* $\mathbf{s} \in \mathbb{R}^k$. *Then*

$$D^{\mathbf{n}} M_U(\mathbf{s}) \;=\; \int_{\mathbb{R}^k} \mathbf{x}^{\mathbf{n}}\, e^{\,\mathbf{s}'\mathbf{x}}\, dU(\mathbf{x})$$

holds for all $\mathbf{n} \in \mathbb{N}_0^k$. *Furthermore, the moment generating function has a Taylor expansion around* $\mathbf{s}$ *of the kind*

$$M_U(\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{N}_0^k} \frac{(\mathbf{t}-\mathbf{s})^{\mathbf{n}}}{\mathbf{n}!} \int_{\mathbb{R}^k} \mathbf{x}^{\mathbf{n}}\, e^{\,\mathbf{s}'\mathbf{x}}\, dU(\mathbf{x})$$

for all $\mathbf{t} \in B$.

Proof: First, we assume that the moment generating function M_U is finite in a neighbourhood $B := (-\mathbf{s}_0, \mathbf{s}_0)$ of $\mathbf{0}$ with $\mathbf{s}_0 > \mathbf{0}$. Since the inequality $e^{\,|\mathbf{t}'\mathbf{x}|} \le e^{\,\mathbf{t}'\mathbf{x}} + e^{\,-\mathbf{t}'\mathbf{x}}$ holds and the right hand side has a finite integral with respect to U so has $\sum_{n=0}^{\infty} |\mathbf{t}'\mathbf{x}|^{\,n}/\, n! \;=\; e^{\,|\mathbf{t}'\mathbf{x}|}$. Thus, we can apply dominated convergence (see also Billingsley [1995] Theorem 16.7) and obtain

$$\begin{aligned} M_U(\mathbf{t}) &= \int_{\mathbb{R}^k} e^{\,\mathbf{t}'\mathbf{x}}\, dU(\mathbf{x}) \\ &= \int_{\mathbb{R}^k} \sum_{n=0}^{\infty} \frac{(\mathbf{t}'\mathbf{x})^{\,n}}{n!}\, dU(\mathbf{x}) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{R}^k} \frac{(\mathbf{t}'\mathbf{x})^{n}}{n!}\, dU(\mathbf{x})\,. \end{aligned}$$

for all $\mathbf{t} \in B$.

Similar to the binomial theorem we rewrite $(\mathbf{t}'\mathbf{x})^n = (\sum_{i=1}^{k} t_i\, x_i)^n$ and get a power series representation with k coordinates for the moment generating function.

$$\begin{aligned} M_U(\mathbf{t}) &= \sum_{n=0}^{\infty} \int_{\mathbb{R}^k} \sum_{\substack{\mathbf{n}\in\mathbb{N}_0^k \\ \mathbf{1}'\mathbf{n}=n}} \prod_{i=1}^{k} \frac{(t_i x_i)^{n^{(i)}}}{n^{(i)}!}\, dU(\mathbf{x}) \\ &= \sum_{n=0}^{\infty} \sum_{\substack{\mathbf{n}\in\mathbb{N}_0^k \\ \mathbf{1}'\mathbf{n}=n}} \frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} \int_{\mathbb{R}^k} \mathbf{x}^{\mathbf{n}}\, dU(\mathbf{x}) \\ &= \sum_{\mathbf{n}\in\mathbb{N}_0^k} \frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} \int_{\mathbb{R}^k} \mathbf{x}^{\mathbf{n}}\, dU(\mathbf{x}) \end{aligned}$$

The Taylor expansion around $\mathbf{0}$ also yields a power series representation

$$M_U(\mathbf{t}) \;=\; \sum_{\mathbf{n}\in\mathbb{N}_0^k} \frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}\, D^{\mathbf{n}} M_U(\mathbf{0}) \qquad (+)$$

By the uniqueness of powers series representation (see Dieudonné [1971] 9.1.6.) we have for all $\mathbf{n} \in \mathbb{N}_0^k$

$$D^{\mathbf{n}} M_U(\mathbf{0}) = \int_{\mathbb{R}^k} \mathbf{x}^{\mathbf{n}} \, dU(\mathbf{x}) \qquad (*)$$

Now let the moment generating function be finite in a neighbourhood B of $\mathbf{s} \in \mathbb{R}^k$. Consider the distribution $V : \mathcal{B}(\mathbb{R}^k) \to [0,1]$ such that

$$V(A) := \int_A \frac{e^{\,\mathbf{s}'\mathbf{x}}}{M_U(\mathbf{s})} \, dU(\mathbf{x})$$

holds for all $A \in \mathcal{B}(\mathbb{R}^k)$. Then V has a finite moment generating function

$$\begin{aligned} M_V(\mathbf{v}) &= \int_{\mathbb{R}^k} e^{\,\mathbf{v}'\mathbf{x}} \, dV(\mathbf{x}) \\ &= \int_{\mathbb{R}^k} \frac{e^{\,(\mathbf{v}+\mathbf{s})'\mathbf{x}}}{M_U(\mathbf{s})} \, dU(\mathbf{x}) \\ &= \frac{M_U(\mathbf{v}+\mathbf{s})}{M_U(\mathbf{s})} \end{aligned}$$

for $\mathbf{v}$ in a neighbourhood of $\mathbf{0}$. Let $\mathbf{n} \in \mathbb{N}_0^k$. From $(*)$ we get

$$\begin{aligned} D^{\mathbf{n}} M_V(\mathbf{0}) &= \int_{\mathbb{R}^k} \mathbf{x}^{\mathbf{n}} \, dV(\mathbf{x}) \\ &= \int_{\mathbb{R}^k} \mathbf{x}^{\mathbf{n}} \frac{e^{\,\mathbf{s}'\mathbf{x}}}{M_U(\mathbf{s})} \, dU(\mathbf{x}) \end{aligned}$$

On the other hand $D^{\mathbf{n}} M_V(\mathbf{0}) = D^{\mathbf{n}} M_U(\mathbf{s}) \,/\, M_U(\mathbf{s})$ and therefore

$$D^{\mathbf{n}} M_U(\mathbf{s}) = \int_{\mathbb{R}^k} \mathbf{x}^{\mathbf{n}} \, e^{\,\mathbf{s}'\mathbf{x}} \, dU(\mathbf{x})$$

The function M_V is finite in a neighbourhood of $\mathbf{0}$ and has a Taylor expansion of form $(+)$. So we get for $\mathbf{v}$ in that neighbourhood

$$\begin{aligned} \frac{M_U(\mathbf{v}+\mathbf{s})}{M_U(\mathbf{s})} &= M_V(\mathbf{v}) \\ &= \sum_{\mathbf{n} \in \mathbb{N}_0^k} \frac{\mathbf{v}^{\mathbf{n}}}{\mathbf{n}!} D^{\mathbf{n}} M_V(\mathbf{0}) \\ &= \sum_{\mathbf{n} \in \mathbb{N}_0^k} \frac{\mathbf{v}^{\mathbf{n}}}{\mathbf{n}!} \int_{\mathbb{R}^k} \mathbf{x}^{\mathbf{n}} \frac{e^{\,\mathbf{s}'\mathbf{x}}}{M_U(\mathbf{s})} \, dU(\mathbf{x}) \end{aligned}$$

With $\mathbf{t} = \mathbf{v} + \mathbf{s}$ the last formula leads to the assertion. ■

1.2.2 Lemma. *Let $U : \mathcal{B}(\mathbb{R}^k) \to [0,1]$ be a distribution and $A : \mathbb{R}^k \to \mathbb{R}^d$ be a matrix. Then*

$$M_{U_A}(\mathbf{t}) \;=\; M_U(A'\mathbf{t})$$

holds for all $\mathbf{t} \in \mathbb{R}^d$.

Proof: As A is a measurable function and the exponential function is positive, integration theory yields

$$\begin{aligned} M_{U_A}(\mathbf{t}) &= \int_{\mathbb{R}^d} e^{\,\mathbf{t}'\mathbf{x}}\, dU_A(\mathbf{x}) \\ &= \int_{\mathbb{R}^k} e^{\,\mathbf{t}'A\mathbf{x}}\, dU(\mathbf{x}) \\ &= \int_{\mathbb{R}^k} e^{\,(A'\mathbf{t})'\mathbf{x}}\, dU(\mathbf{x}) \\ &= M_U(A'\mathbf{t}) \end{aligned}$$

for all $\mathbf{t} \in \mathbb{R}^d$. ■

After introducing the moment generating function we can state a characterization of independence for some special positive random variables in terms of this function and in terms of some moments. Furthermore, we also will state a corresponding result concerning conditional independence. Both results will be used in Section 5.3. In order to keep the notation simple, we use the symbol M_X instead of M_{P_X} for the moment generating function (of the distribution) of an arbitrary random variable X.

Theorem 1.2.3 *Let $X : \Omega \to \mathbb{R}_+$ be a random variable and let $\mathbf{Y} : \Omega \to \mathbb{R}_+^k$ be a bounded random vector. Then the following are equivalent.*

(a) *X and $\mathbf{Y}$ are independent.*

(b) *The moment generating functions satisfy*

$$M_{(X,\mathbf{Y})}(t,\mathbf{s}) \;=\; M_X(t)\, M_{\mathbf{Y}}(\mathbf{s})$$

for all $t < 0$ and all $\mathbf{s} \in \mathbb{R}^k$.

(c) *There exists some $t > 0$ such that*

$$\mathrm{E}\left[e^{-X\,t}\, X^n\, \mathbf{Y}^{\mathbf{l}}\right] \;=\; \mathrm{E}\left[e^{-X\,t}\, X^n\right]\, \mathrm{E}\left[\mathbf{Y}^{\mathbf{l}}\right]$$

holds for all $n \in \mathbb{N}_0$ and all $\mathbf{l} \in \mathbb{N}_0^k$.

(d) *The identity*

$$\mathrm{E}\left[e^{-X\,t}\, X^n\, e^{\mathbf{s}'\mathbf{Y}}\, \mathbf{Y}^{\mathbf{l}}\right] \;=\; \mathrm{E}\left[e^{-X\,t}\, X^n\right]\, \mathrm{E}\left[e^{\mathbf{s}'\mathbf{Y}}\, \mathbf{Y}^{\mathbf{l}}\right]$$

holds for all $t \in \mathbb{R}_+$ and all $\mathbf{s} \in \mathbb{R}^k$ as well as for all $n \in \mathbb{N}_0$ and all $\mathbf{l} \in \mathbb{N}_0^k$.

Proof: We prove the assertion according to the following scheme: (a) ⇒ (d) ⇒ (c) ⇒ (b) ⇒ (a).

(a) ⇒ (d): obvious

(d) ⇒ (c): obvious

(c) ⇒ (b): Since X is positive and $\mathbf{Y}$ is bounded there exists an open set $B = B_t \times B_{\mathbf{s}}$ such that $-t \in B_t \subseteq (-\infty, 0)$ and $\mathbf{0} \in B_{\mathbf{s}} \subseteq \mathbb{R}^k$ and that the moment generating functions M_X, $M_{\mathbf{Y}}$, and $M_{(X,\mathbf{Y})}$ are finite on B_t, $B_{\mathbf{s}}$, and B, respectively. Now, consider arbitrary $\hat{t} \in B_t$ and $\hat{\mathbf{s}} \in B_{\mathbf{s}}$. Then from Lemma 1.2.1 we obtain

$$\begin{aligned}
M_{(X,\mathbf{Y})}(\hat{t}, \hat{\mathbf{s}}) &= \sum_{n \in \mathbb{N}_0} \sum_{\mathbf{l} \in \mathbb{N}_0^k} \frac{(\hat{t}+t)^n}{n!} \frac{\hat{\mathbf{s}}^{\mathbf{l}}}{\mathbf{l}!} \,\mathrm{E}\left[e^{-X\,t} X^n\, \mathbf{Y}^{\mathbf{l}}\right] \\
&= \sum_{n \in \mathbb{N}_0} \sum_{\mathbf{l} \in \mathbb{N}_0^k} \frac{(\hat{t}+t)^n}{n!} \frac{\hat{\mathbf{s}}^{\mathbf{l}}}{\mathbf{l}!} \,\mathrm{E}\left[e^{-X\,t} X^n\right] \,\mathrm{E}\left[\mathbf{Y}^{\mathbf{l}}\right] \\
&= \left(\sum_{n \in \mathbb{N}_0} \frac{(\hat{t}+t)^n}{n!} \,\mathrm{E}\left[e^{-X\,t} X^n\right] \right) \left(\sum_{\mathbf{l} \in \mathbb{N}_0^k} \frac{\hat{\mathbf{s}}^{\mathbf{l}}}{\mathbf{l}!} \,\mathrm{E}\left[\mathbf{Y}^{\mathbf{l}}\right] \right) \\
&= M_X(\hat{t})\, M_{\mathbf{Y}}(\hat{\mathbf{s}})
\end{aligned}$$

So the desired identity is valid on B. The analyticity of the moment generating functions M_X, $M_{\mathbf{Y}}$, and $M_{(X,\mathbf{Y})}$ on $(-\infty, 0)$, $\mathbb{R}^k$, and $(-\infty, 0) \times \mathbb{R}^k$, respectively, and the principle of analytic continuation (see Dieudonné [1971] 9.4.2) yield the assertion.

(b) ⇒ (a): For all $t < 0$ and all $\mathbf{s} \in \mathbb{R}^k$ we obtain with the help of Tonelli's theorem

$$\begin{aligned}
\int_{\mathbb{R}^{1+k}} e^{\,t\,x+\mathbf{s}'\mathbf{y}}\, d\mathrm{P}_{(X,\mathbf{Y})}(x, \mathbf{y}) &= M_{(X,\mathbf{Y})}(t, \mathbf{s}) \\
&= M_X(t)\, M_{\mathbf{Y}}(\mathbf{s}) \\
&= \int_{\mathbb{R}} e^{\,t\,x}\, d\mathrm{P}_X(x) \int_{\mathbb{R}^k} e^{\,\mathbf{s}'\mathbf{y}}\, d\mathrm{P}_{\mathbf{Y}}(\mathbf{y}) \\
&= \int_{\mathbb{R}^{1+k}} e^{\,t\,x+\mathbf{s}'\mathbf{y}}\, dP_X \otimes P_{\mathbf{Y}}(x, \mathbf{y})
\end{aligned}$$

and thus

$$\int_{\mathbb{R}^{1+k}} e^{\,t\,x+\mathbf{s}'\mathbf{y}}\, d\mathrm{P}_{(X,\mathbf{Y})}(x, \mathbf{y}) = \int_{\mathbb{R}^{1+k}} e^{\,t\,x+\mathbf{s}'\mathbf{y}}\, dP_X \otimes P_{\mathbf{Y}}(x, \mathbf{y})$$

The last equation also is true if we choose $t = 0$. Then the uniqueness of the Laplace transform of measures concentrated on $\mathbb{R}^d_+$ (see Kallenberg [2002] Theorem 5.3) yields $P_{(X,\mathbf{Y})} = P_X \otimes P_{\mathbf{Y}}$ and therefore the independence of X and $\mathbf{Y}$. ■

Now, we turn to the conditional independence. To prove a corresponding result to the previous theorem we first state two preliminary results.

1.2.4 Lemma. *Let* $X : \Omega \to \mathbb{R}_+$ *and* $Y : \Omega \to \mathbb{R}_+$ *be two random variables. Additionally, let* $\mathbf{Z} : \Omega \to \mathbb{N}_0^d$ *be a random vector such that* $\mathrm{P}\left[\{\mathbf{Z} = \mathbf{n}\}\right] > 0$ *holds for all* $\mathbf{n} \in \mathbb{N}_0^d$. *Then the following are equivalent.*

(a) *The identity*

$$\mathrm{E}\big(X\,Y\big|\mathbf{Z}\big) \;=\; \mathrm{E}\big(X\big|\mathbf{Z}\big)\;\mathrm{E}\big(Y\big|\mathbf{Z}\big)$$

is valid.

(b) *The identity*

$$\mathrm{E}\big[X\,Y\big|\,\{\mathbf{Z}=\mathbf{n}\}\big] \;=\; \mathrm{E}\big[X\big|\,\{\mathbf{Z}=\mathbf{n}\}\big]\;\mathrm{E}\big[Y\big|\,\{\mathbf{Z}=\mathbf{n}\}\big]$$

holds for all $\mathbf{n} \in \mathbb{N}_0^d$.

Proof: By the Fourier expansion for conditional expectation we have

$$\mathrm{E}\big(X\,Y\big|\mathbf{Z}\big) \;=\; \sum_{\mathbf{n}\,\in\,\mathbb{N}_0^k} \mathrm{E}\big[X\,Y\big|\,\{\mathbf{Z}=\mathbf{n}\}\big]\;\chi_{\{\mathbf{Z}=\mathbf{n}\}}$$

as well as

$$\begin{aligned}
&\mathrm{E}\big(X\big|\mathbf{Z}\big)\;\mathrm{E}\big(Y\big|\mathbf{Z}\big)\\
&\qquad= \left(\sum_{\mathbf{n}\,\in\,\mathbb{N}_0^k} \mathrm{E}\big[X\big|\,\{\mathbf{Z}=\mathbf{n}\}\big]\;\chi_{\{\mathbf{Z}=\mathbf{n}\}}\right)\left(\sum_{\mathbf{n}\,\in\,\mathbb{N}_0^k} \mathrm{E}\big[Y\big|\,\{\mathbf{Z}=\mathbf{n}\}\big]\;\chi_{\{\mathbf{Z}=\mathbf{n}\}}\right)\\
&\qquad= \sum_{\mathbf{n}\,\in\,\mathbb{N}_0^k} \mathrm{E}\big[X\big|\,\{\mathbf{Z}=\mathbf{n}\}\big]\;\mathrm{E}\big[Y\big|\,\{\mathbf{Z}=\mathbf{n}\}\big]\;\chi_{\{\mathbf{Z}=\mathbf{n}\}}
\end{aligned}$$

which yields the assertion. ■

1.2.5 Corollary. *Let* $X : \Omega \to \mathbb{R}_+$ *a random variable and let* $\mathbf{Y} : \Omega \to \mathbb{R}_+^k$ *be a random vector. Additionally, let* $\mathbf{Z} : \Omega \to \mathbb{N}_0^d$ *be a random vector such that* $\mathrm{P}\left[\{\mathbf{Z} = \mathbf{n}\}\right] > 0$ *holds for all* $\mathbf{n} \in \mathbb{N}_0^d$. *Then the following are equivalent.*

(a) X *and* $\mathbf{Y}$ *are conditionally independent with respect to* $\mathbf{Z}$.

(b) *The identity*

$$\mathrm{P}\big(\{X \in B\} \cap \{\mathbf{Y} \in C\}\,\big|\mathbf{Z}\big) \;=\; \mathrm{P}\big(\{X \in B\}\,\big|\mathbf{Z}\big)\;\mathrm{P}\big(\{\mathbf{Y} \in C\}\,\big|\mathbf{Z}\big)$$

holds for all $B \in \mathcal{B}(\mathbb{R})$ *and* $C \in \mathcal{B}(\mathbb{R}^k)$.

(c) *For all* $\mathbf{n} \in \mathbb{N}_0^d$ *the random variable* X *and the random vector* $\mathbf{Y}$ *are independent with respect to the measure* $P\left[\,\cdot\,|\{\mathbf{Z}=\mathbf{n}\}\right]$.

Proof:
(a) $\Leftrightarrow$ (b): obvious

(b) $\Leftrightarrow$ (c): Condition (c) is valid if, and only if, for all $\mathbf{n} \in \mathbb{N}_0^d$

$$\begin{aligned} &\mathrm{P}\left[\{X \in B\} \cap \{\mathbf{Y} \in C\} \,\middle|\, \{\mathbf{Z} = \mathbf{n}\}\right] \\ &\qquad = \mathrm{P}\left[\{X \in B\} \,\middle|\, \{\mathbf{Z} = \mathbf{n}\}\right] \, \mathrm{P}\left[\{\mathbf{Y} \in C\} \,\middle|\, \{\mathbf{Z} = \mathbf{n}\}\right] \end{aligned}$$

holds for all $B \in \mathcal{B}(\mathbb{R})$ and $C \in \mathcal{B}(\mathbb{R}^k)$. Considering the random variables $\chi_B \circ X$ and $\chi_C \circ \mathbf{Y}$ for arbitrary $B \in \mathcal{B}(\mathbb{R})$ and $C \in \mathcal{B}(\mathbb{R}^k)$, Lemma 1.2.4 yields now the assertion. ∎

1.2.6 Corollary. *Let $X : \Omega \to \mathbb{R}_+$ a random variable and let $\mathbf{Y} : \Omega \to \mathbb{R}_+^k$ be a bounded random vector. Additionally, let $\mathbf{Z} : \Omega \to \mathbb{N}_0^d$ be a random vector such that $\mathrm{P}\left[\{\mathbf{Z} = \mathbf{n}\}\right] > 0$ holds for all $\mathbf{n} \in \mathbb{N}_0^d$. Then the following are equivalent.*

(a) *X and $\mathbf{Y}$ are conditionally independent with respect to $\mathbf{Z}$.*

(b) *For all $\mathbf{n} \in \mathbb{N}_0^d$*

$$M_{P_{(X,\mathbf{Y})|\{\mathbf{Z}=\mathbf{n}\}}}(t, \mathbf{s}) \;=\; M_{P_{X|\{\mathbf{Z}=\mathbf{n}\}}}(t) \, M_{P_{\mathbf{Y}|\{\mathbf{Z}=\mathbf{n}\}}}(\mathbf{s})$$

holds for all $t < 0$ and $\mathbf{s} \in \mathbb{R}^k$.

(c) *There exists some $t > 0$ such that*

$$\mathrm{E}\left(e^{-X\,t} X^n \, \mathbf{Y}^{\mathbf{l}} \middle| \mathbf{Z}\right) \;=\; \mathrm{E}\left(e^{-X\,t} X^n \middle| \mathbf{Z}\right) \, \mathrm{E}\left(\mathbf{Y}^{\mathbf{l}} \middle| \mathbf{Z}\right)$$

holds for all $n \in \mathbb{N}_0$ and all $\mathbf{l} \in \mathbb{N}_0^k$.

(d) *For all $t \in \mathbb{R}_+$ and all $\mathbf{s} \in \mathbb{R}^k$*

$$\mathrm{E}\left(e^{-X\,t} X^n \, e^{\mathbf{s}'\mathbf{Y}} \, \mathbf{Y}^{\mathbf{l}} \middle| \mathbf{Z}\right) \;=\; \mathrm{E}\left(e^{-X\,t} X^n \middle| \mathbf{Z}\right) \, \mathrm{E}\left(e^{\mathbf{s}'\mathbf{Y}} \, \mathbf{Y}^{\mathbf{l}} \middle| \mathbf{Z}\right)$$

holds for all $n \in \mathbb{N}_0$ and all $\mathbf{l} \in \mathbb{N}_0^k$.

Proof: Aggregating for all $\mathbf{n} \in \mathbb{N}_0^d$ condition (b) of Theorem 1.2.3 under the measure $P\left[\,\cdot\,|\{\mathbf{Z} = \mathbf{n}\}\right]$ gives condition (b) of this theorem. Therefore, using additionally on the one hand Corollary 1.2.5 gives the equivalence of (b) and (a) and on the other hand Lemma 1.2.4 gives the equivalence of (b) and (c) and of (b) and (d). ∎

1.3 Bernstein–Widder Theorem

For the main result in Section 3.2 we need a multivariate extension of the famous Bernstein–Widder theorem, which states that a completely monotone function has a representation as Laplace–transform of a distribution. The Bernstein–Widder theorem possesses a lot of different proofs from various fields of mathematics. However, the proof of the multivariate extension is often taken for granted and therefore not carried out (compare Bochner [1955] Theorem 4.2.1 and Berg et al. [1984] Exercise 4.6.27). So in this section we state the multivariate Bernstein–Widder theorem in a fashion fitting our purpose and give a proof, which is based on Berg et al. [1984].

1.3.1 Theorem (Multivariate Bernstein–Widder). *Let $f : \mathbb{R}_+^k \to \mathbb{R}$ be a continuous function with $f(\mathbf{0}) = 1$ and*

$$(-1)^{\mathbf{1}'\mathbf{n}} D^{\mathbf{n}} f(\mathbf{t}) \;\; \geq \;\; 0$$

for all $\mathbf{n} \in \mathbb{N}_0^k$. Then there exists a distribution U on $\mathcal{B}(\mathbb{R}^k)$ with $U\left[\mathbb{R}_+^k\right] = 1$ such that

$$f(\mathbf{t}) \;\; = \;\; \int_{\mathbb{R}^k} e^{-\mathbf{t}'\mathbf{x}} \, dU(\mathbf{x})$$

holds for all $\mathbf{t} \in \mathbb{R}_+^k$.

Proof: Every numeration used in this proof refers to Berg et al. [1984].
First, we show that f is completely monotone in the sense of Definition 4.6.1, which states that a function has to be nonnegative and fulfils for all finite sets $\{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subseteq \mathbb{R}_+^k$ and all $\mathbf{s} \in \mathbb{R}_+^k$ the inequality $\nabla_{\mathbf{a}_1} \cdots \nabla_{\mathbf{a}_n} f(\mathbf{s}) \geq 0$ in order to be completely monotone, where $\nabla_{\mathbf{a}}$ is defined by $\nabla_{\mathbf{a}} f(\mathbf{s}) := f(\mathbf{s}) - f(\mathbf{s}+\mathbf{a})$. Thus, we generalize a part of the proof of Theorem 4.6.13. Consider $\mathbf{a} \in \mathbb{R}_+^k$, then the function $\nabla_{\mathbf{a}} f$ is continuous on $\mathbb{R}_+^k$. Furthermore, we have for all $\mathbf{n} \in \mathbb{N}_0^k$ and $\mathbf{t} > \mathbf{0}$ with the mean value theorem (see Heuser [2003b] Section 167)

$$\begin{aligned}
(-1)^{\mathbf{1}'\mathbf{n}} D^{\mathbf{n}} (\nabla_{\mathbf{a}} f)(\mathbf{t}) &= (-1)^{\mathbf{1}'\mathbf{n}} \nabla_{\mathbf{a}} D^{\mathbf{n}} f(\mathbf{t}) \\
&= (-1)^{\mathbf{1}'\mathbf{n}} \big(D^{\mathbf{n}} f(\mathbf{t}) - D^{\mathbf{n}} f(\mathbf{t}+\mathbf{a}) \big) \\
&= (-1)^{\mathbf{1}'\mathbf{n}+1} \sum_{i=1}^{k} a_i \, D^{\mathbf{n}+\mathbf{e}_i} f(\boldsymbol{\xi})
\end{aligned}$$

with $\boldsymbol{\xi} \in [\mathbf{t}, \mathbf{t}+\mathbf{a}]$. And so we have $(-1)^{\mathbf{1}'\mathbf{n}} D^{\mathbf{n}} (\nabla_{\mathbf{a}} f)(\mathbf{t}) \geq 0$. By iteration we get for all $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}_+^k$, $n \in \mathbb{N}$ that the function $\nabla_{\mathbf{a}_1} \dots \nabla_{\mathbf{a}_n} f$ is continuous on $\mathbb{R}_+^k$ and fulfils $(-1)^{\mathbf{1}'\mathbf{n}} D^{\mathbf{n}} (\nabla_{\mathbf{a}_1} \dots \nabla_{\mathbf{a}_n} f)(\mathbf{t}) \geq 0$ for all $\mathbf{n} \in \mathbb{N}_0^k$ and $\mathbf{t} > \mathbf{0}$. In particular, $\nabla_{\mathbf{a}_1} \dots \nabla_{\mathbf{a}_n} f(\mathbf{t}) \geq 0$ for all $\mathbf{t} > \mathbf{0}$ and by continuity $\nabla_{\mathbf{a}_1} \dots \nabla_{\mathbf{a}_n} f(\mathbf{t}) \geq 0$ for all $\mathbf{t} \geq \mathbf{0}$. As f is by assumption nonnegative it is completely monotone.
It follows from 4.6.5 that f is positive definite and bounded (in notation of Berg et al. [1984] $f \in \mathscr{P}^b(\mathbb{R}_+^k)$). Thus, the continuity of f in connection with Proposition 4.4.7. yields the existence of a finite, nonnegative measure U on $\mathcal{B}(\mathbb{R}_+^k)$ with

$$f(\mathbf{t}) \;\; = \;\; \int_{\mathbb{R}_+^k} e^{-\mathbf{t}'\mathbf{x}} \, dU(\mathbf{x})$$

for all $\mathbf{t} \in \mathbb{R}_+^k$. Finally

$$\begin{aligned} U\left[\mathbb{R}_+^k\right] &= \int_{\mathbb{R}_+^k} dU(\mathbf{x}) \\ &= f(\mathbf{0}) \\ &= 1 \end{aligned}$$

and the assertion is shown. ■

1.3.2 Corollary. *Let $f : \mathbb{R}_+^k \to \mathbb{R}$ be a continuous function with $f(\mathbf{0}) = 1$ and $(-1)^{\mathbf{1}'\mathbf{n}} D^{\mathbf{n}} f(\mathbf{t}) \geq 0$ for all $\mathbf{n} \in \mathbb{N}_0^k$. Then there exists a distribution U on $\mathcal{B}(\mathbb{R}^k)$ with $U\left[\mathbb{R}_+^k\right] = 1$ such that $f(\mathbf{t}) = M_U(-\mathbf{t})$.*

Chapter 2

Multivariate Counting Processes

2.1 The Model

A stochastic process $\{N_t\}_{t \in \mathbb{R}_+}$ is said to be a ***counting process (without explosion)*** if there exists a null set $N \in \mathcal{F}$ (called the exceptional null set) such that the following properties are satisfied for every $\omega \in \Omega \backslash N$:

(i) $N_0(\omega) = 0$,

(ii) $N_t(\omega) \in \mathbb{N}_0$ for all $t > 0$,

(iii) $N_t(\omega) = \inf_{s \in (t,\infty)} N_s(\omega)$ for all $t \in \mathbb{R}_+$,

(iv) $\sup_{s \in [0,t)} N_s(\omega) \leq N_t(\omega) \leq \sup_{s \in [0,t)} N_s(\omega) + 1$ for all $t \in \mathbb{R}_+$, and

(v) $\sup_{t \in \mathbb{R}_+} N_t(\omega) = \infty$.

N_t can be interpreted as the number of events occurring in the interval $(0, t]$. The above definition excludes the positive probability of infinitely many events occurring in a finite time interval as well as the possibility of a finite number of events occurring in an infinite time interval. Some results in this work are related to Schmidt [1996]. There, a counting process is allowed to explode, but here, talking about a counting process, we always refer to a counting process without explosion.

A multivariate stochastic process $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ in k dimensions is said to be a ***multivariate counting process*** if every coordinate $\{N_t^{(i)}\}_{t \in \mathbb{R}_+}$, $i \in \{1, \ldots, k\}$, and the sum $\{N_t\}_{t \in \mathbb{R}_+} := \{\mathbf{1}'\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ of all coordinates is a counting process. Thus, there exists a null set $M \in \mathcal{F}$ (called the exceptional null set of the multivariate counting process) such that for all $\omega \in \Omega \backslash M$ properties (i)–(v) are fulfilled by all coordinates $\{N_t^{(i)}(\omega)\}_{t \in \mathbb{R}_+}$, $i \in \{1, \ldots, k\}$, and the sum $\{N_t(\omega)\}_{t \in \mathbb{R}_+}$ of all coordinates. As a consequence, simultaneous jumps of different coordinates are almost surely excluded. From now on k will always be the dimension of the multivariate counting process we are working with.

To see how multivariate counting processes can be transformed, we define different sets of matrices. Firstly, let us consider permutation matrices, matrices which select coordinates, and matrices which cumulate coordinates according to some rules.

– Let $\mathcal{A}_P$ be the set consisting of all $A \in \{0,1\}^{k\times k}$ with $k \in \mathbb{N}$ such that the identities $\mathbf{1}'A\mathbf{e}_j = 1 = \mathbf{e}_i{}'A\mathbf{1}$ hold for all $i,j \in \{1,\dots,k\}$.
– Let $\mathcal{A}_S$ be the set consisting of all $A = (I_d, 0) \in \{0,1\}^{d\times k}$ with $d,k \in \mathbb{N}$ such that $d < k$ and I_d is the identity matrix of dimension d.
– Let $\mathcal{A}_C$ be the set consisting of all $A \in \{0,1\}^{d\times k}$ with $d,k \in \mathbb{N}$ and $d \le k$ such that there exist $k_i \in \mathbb{N}$ for $i \in \{1,\dots,d\}$ with $\sum_{i=1}^{d} k_i = k$ and $A = (A_1,\dots,A_d)$ where $A_i := (e_i,\dots,e_i) \in \mathbb{R}^{d\times k_i}$ for $i \in \{1,\dots,d\}$.

Now the set of possible transformation matrices can be defined as the set $\mathcal{A}$ consisting of all $A \in \{0,1\}^{d\times k}$ with $d,k \in \mathbb{N}$ and $d \le k$ such that there exists some $m \in \mathbb{N}$ and $A_i \in \mathcal{A}_P \cup \mathcal{A}_S \cup \mathcal{A}_C$, $i \in \{1,\dots,m\}$, with $A = A_m A_{m-1} \cdots A_1$. Thus, $\mathcal{A}$ consists of matrices which have entries of 0 or 1, at least one 1 per line, and at most one 1 per column. That $\mathcal{A}$ includes all such matrices is shown within the proof of the following lemma.

2.1.1 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process and $A \in \mathbb{R}^{d\times k}$. Then $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ is a multivariate counting process if, and only if, $A \in \mathcal{A}$.*

Proof:
Assume $A \in \mathcal{A}$. To show that $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ is a multivariate counting process we have to prove that a multivariate counting process is stable under every transformation from each of the three sets $\mathcal{A}_P$, $\mathcal{A}_S$, and $\mathcal{A}_C$. It is obvious that $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ is a counting process for $A \in \mathcal{A}_P \cup \mathcal{A}_S$. Now, let $A \in \mathcal{A}_{\mathcal{C}}$ and consider $\omega \in \Omega\backslash M$. Due to the assumption the coordinates of $\{\mathbf{N}_t(\omega)\}_{t\in\mathbb{R}_+}$ cannot jump simultaneously. Every coordinate of the transformed process $\{A\mathbf{N}_t(\omega)\}_{t\in\mathbb{R}_+}$ is a sum of coordinates of the original process $\{\mathbf{N}_t(\omega)\}_{t\in\mathbb{R}_+}$ and therefore fulfils properties (i)–(v) of a counting process. Since $\{\mathbf{1}'A\mathbf{N}_t(\omega)\}_{t\in\mathbb{R}_+} = \{\mathbf{1}'\mathbf{N}_t(\omega)\}_{t\in\mathbb{R}_+}$, the sum of all coordinates of $\{A\mathbf{N}_t(\omega)\}_{t\in\mathbb{R}_+}$ fulfills properties (i)–(v) of a counting process, too. Thus, M serves as an exceptional null set for the transformed process $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ which is therefore a multivariate counting process.
Now consider $A \in \mathbb{R}^{d\times k}$ and that $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ is a multivariate counting process and let M_A be the exceptional null set of $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+}$. Consider $\omega \in \Omega\backslash(M \cup M_A)$. Since every coordinate of $\{A\mathbf{N}_t(\omega)\}_{t\in\mathbb{R}_+}$ has jumps of height one and no simultaneous jumps of coordinates of $\{\mathbf{N}_t(\omega)\}_{t\in\mathbb{R}_+}$ are allowed, all entries of A are either 0 or 1. Furthermore, every coordinate of $\{A\mathbf{N}_t(\omega)\}_{t\in\mathbb{R}_+}$ has paths which increase to infinity and we get $A\mathbf{1} \ge \mathbf{1}$. Additionally, there exist no simultaneous jumps of coordinates of $\{A\mathbf{N}_t(\omega)\}_{t\in\mathbb{R}_+}$ and thus $\mathbf{1}'A \le \mathbf{1}'$. These three arguments yield the existence of a matrix $A_P \in \mathcal{A}_P$ such that

$$A\,A_P \;=\; \begin{pmatrix} 1\dots1 & 0 & & & \\ & & 1\dots1 & 0 & \\ 0 & & \ddots & & 0\dots0 \\ & 0 & & 1\dots1 & \end{pmatrix}$$

where the last part (all zeros) may or may not be existent. In the first case there exists $A_C \in \mathbb{R}^{(d+1)\times k}$ and $A_S \in \mathbb{R}^{d\times(d+1)}$ with $A_C \in \mathcal{A}_C$ and $A_S \in \mathcal{A}_S$ such that $A\,A_P = A_S\,A_C$. In the second case we already have $A\,A_P \in \mathcal{A}_C$. Since $(A_P)^{-1} \in \mathcal{A}_P$ we obtain $A \in \mathcal{A}$. ■

Examples of useful transformations are

- $A = \mathbf{1}'$, in which case $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+} = \{N_t\}_{t\in\mathbb{R}_+}$ is the sum of all coordinates,
- $A = \mathbf{e}_i{}'$, in which case $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+} = \left\{N_t^{(i)}\right\}_{t\in\mathbb{R}_+}$ is the i–th coordinate,
- $A \in \mathcal{A}_S$, in which case $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ consists of the first d coordinates of the original process, and
- $A \in \mathcal{A}_P$, in which case $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ permutes the coordinates of the original process.

For a practical use of transformation we introduce the following notation. A property (P) of counting processes is said to be ***$\mathcal{A}$–stable*** if, for each $A \in \mathcal{A}$, the counting process $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ has property (P) whenever $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ has property (P).

The next lemma states some properties of the one–dimensional probabilities of multivariate counting processes we need later.

2.1.2 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process. Then*

(1) *The identity*

$$\lim_{\mathbf{t}\downarrow\mathbf{s}} \mathrm{P}\left[\bigcap_{i=1}^{k}\left\{N_{t_i}^{(i)} = n^{(i)}\right\}\right] \;=\; \mathrm{P}\left[\bigcap_{i=1}^{k}\left\{N_{s_i}^{(i)} = n^{(i)}\right\}\right]$$

holds for all $\mathbf{s} \in \mathbb{R}_+^k$ and all $\mathbf{n} \in \mathbb{N}_0^k$.

(2) *The identity*

$$\lim_{t\downarrow s} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \;=\; \mathrm{P}\left[\{\mathbf{N}_s = \mathbf{n}\}\right]$$

holds for all $s \in \mathbb{R}_+$ and all $\mathbf{n} \in \mathbb{N}_0^k$.

(3) *The identity*

$$\lim_{t\downarrow 0} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \;=\; \begin{cases} 1 & \text{if } \mathbf{n} = \mathbf{0} \\ 0 & \text{else} \end{cases}$$

holds for all $\mathbf{n} \in \mathbb{N}_0^k$.

(4) *The identity* $\lim_{t\uparrow\infty} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] = 0$ *holds for all $\mathbf{n} \in \mathbb{N}_0^k$.*

(5) *The identity* $\lim_{t\uparrow\infty} \mathrm{P}\left[\{\mathbf{N}_t \geq \mathbf{n}\}\right] = 1$ *holds for all $\mathbf{n} \in \mathbb{N}_0^k$.*

Proof:

(1): By definition, every coordinate of $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ is a counting process and has therefore almost surely right continuous and increasing paths. So we get for arbitrary $m \in \mathbb{N}$ and $\mathbf{n}_j \in \mathbb{Z}^k$, $j \in \{1, \ldots, m\}$,

$$
\begin{aligned}
\lim_{\mathbf{t} \downarrow \mathbf{s}} \mathrm{P}\left[\bigcup_{j=1}^{m} \bigcap_{i=1}^{k} \left\{N_{t_i}^{(i)} \le n_j^{(i)}\right\}\right] &= \sup_{\mathbf{t} \in (\mathbf{s}, \boldsymbol{\infty})} \mathrm{P}\left[\bigcup_{j=1}^{m} \bigcap_{i=1}^{k} \left\{N_{t_i}^{(i)} \le n_j^{(i)}\right\}\right] \\
&= \mathrm{P}\left[\bigcup_{\mathbf{t} \in (\mathbf{s}, \boldsymbol{\infty})} \bigcup_{j=1}^{m} \bigcap_{i=1}^{k} \left\{N_{t_i}^{(i)} \le n_j^{(i)}\right\}\right] \\
&= \mathrm{P}\left[\bigcup_{j=1}^{m} \bigcup_{\mathbf{t} \in (\mathbf{s}, \boldsymbol{\infty})} \bigcap_{i=1}^{k} \left\{N_{t_i}^{(i)} \le n_j^{(i)}\right\}\right] \\
&= \mathrm{P}\left[\bigcup_{j=1}^{m} \bigcap_{i=1}^{k} \left\{\inf_{t_i \in (s_i, \infty)} N_{t_i}^{(i)} \le n_j^{(i)}\right\}\right] \\
&= \mathrm{P}\left[\bigcup_{j=1}^{m} \bigcap_{i=1}^{k} \left\{N_{s_i}^{(i)} \le n_j^{(i)}\right\}\right]
\end{aligned}
$$

Now, consider $\mathbf{n} \in \mathbb{N}_0^k$. Then we obtain with the previous identity (considering the case $m = 1$ as well as $m = k$)

$$
\begin{aligned}
&\lim_{\mathbf{t} \downarrow \mathbf{s}} \mathrm{P}\left[\bigcap_{i=1}^{k} \left\{N_{t_i}^{(i)} = n^{(i)}\right\}\right] \\
&\quad = \lim_{\mathbf{t} \downarrow \mathbf{s}} \mathrm{P}\left[\bigcap_{i=1}^{k} \left\{N_{t_i}^{(i)} \le n^{(i)}\right\} \Big\backslash \bigcup_{j=1}^{k} \bigcap_{i=1}^{k} \left\{N_{t_i}^{(i)} \le n^{(i)} - \delta_{ij}\right\}\right] \\
&\quad = \lim_{\mathbf{t} \downarrow \mathbf{s}} \left(\mathrm{P}\left[\bigcap_{i=1}^{k} \left\{N_{t_i}^{(i)} \le n^{(i)}\right\}\right] - \mathrm{P}\left[\bigcup_{j=1}^{k} \bigcap_{i=1}^{k} \left\{N_{t_i}^{(i)} \le n^{(i)} - \delta_{ij}\right\}\right]\right) \\
&\quad = \mathrm{P}\left[\bigcap_{i=1}^{k} \{N_{s_i}^{(i)} \le n^{(i)}\}\right] - \mathrm{P}\left[\bigcup_{j=1}^{k} \bigcap_{i=1}^{k} \{N_{s_i}^{(i)} \le n^{(i)} - \delta_{ij}\}\right] \\
&\quad = \mathrm{P}\left[\bigcap_{i=1}^{k} \{N_{s_i}^{(i)} = n^{(i)}\}\right]
\end{aligned}
$$

(2): Considering only vectors $\mathbf{s}$ with equal coordinates, the assertion immediately follows from (1).

(3): Since all coordinates have paths which almost surely start at zero, setting $s = 0$ in (2) gives the assertion.

(4): By definition, the sum $\{N_t\}_{t \in \mathbb{R}_+}$ of all coordinates is a counting process and has therefore paths which have no upper limit. This yields

$$\begin{aligned}
\lim_{t \uparrow \infty} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] &\le \lim_{t \uparrow \infty} \mathrm{P}\left[\{N_t \le \mathbf{1}'\mathbf{n}\}\right] \\
&= \inf_{t \in (0,\infty)} \mathrm{P}\left[\{N_t \le \mathbf{1}'\mathbf{n}\}\right] \\
&= \mathrm{P}\left[\bigcap_{t \in (0,\infty)} \{N_t \le \mathbf{1}'\mathbf{n}\}\right] \\
&= \mathrm{P}\left[\left\{\sup_{t \in (0,\infty)} N_t \le \mathbf{1}'\mathbf{n}\right\}\right] \\
&= 0
\end{aligned}$$

(5): By definition, all coordinates are counting processes and have therefore paths which increase and have no upper limit. Thus

$$\begin{aligned}
\lim_{t \uparrow \infty} \mathrm{P}\left[\{\mathbf{N}_t \ge \mathbf{n}\}\right] &\ge \lim_{t \uparrow \infty} \mathrm{P}\left[\left\{N_t^{(1)} \ge n^{(1)}\right\}\right] \\
&= \sup_{t \in (0,\infty)} \mathrm{P}\left[\left\{N_t^{(1)} \ge n^{(1)}\right\}\right] \\
&\ge \mathrm{P}\left[\bigcup_{t \in (0,\infty)} \left\{N_t^{(1)} \ge n^{(1)}\right\}\right] \\
&= \mathrm{P}\left[\left\{\sup_{t \in (0,\infty)} N_t^{(1)} \ge n^{(1)}\right\}\right] \\
&= 1
\end{aligned}$$

and the assertion follows. ■

The second item seems to be the natural version regarding continuity of the probability as a function of time. But for the characterization of multivariate mixed Poisson processes we need as many different time variables as the process has coordinates. Thus, Lemma 2.1.2 (1) is also necessary and will be used in the proof of Lemma 3.2.1.

We will also study so called posterior distributions and processes. To this end we introduce for $t \in \mathbb{R}_+$ the ***incremental process*** $\{\mathbf{K}_{t,h}\}_{h \in \mathbb{R}_+}$ with

$$\mathbf{K}_{t,h} := \mathbf{N}_{t+h} - \mathbf{N}_t$$

for all $h \in \mathbb{R}_+$. Since all trajectory properties carry over from $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ to $\{\mathbf{K}_{t,h}\}_{h \in \mathbb{R}_+}$, the next lemma is obvious.

2.1.3 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process. Then the process $\{\mathbf{K}_{t,h}\}_{h\in\mathbb{R}_+}$ is a multivariate counting process for all $t\in\mathbb{R}_+$.*

Latter, the treatment of the incremental process will require the restriction of the probability measure. Hence, we also define for $t\in\mathbb{R}_+$ and $\mathbf{n}\in\mathbb{N}_0^k$ with $\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}\}\right]>0$ a new probability measure with

$$\mathrm{P}_{t,\mathbf{n}}[B] := \mathrm{P}\left[B \,\middle|\, \{\mathbf{N}_t=\mathbf{n}\}\right]$$

for $B\in\mathcal{F}$ and modify the previous lemma, such that it can be directly used in the subsequent chapters.

2.1.4 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process. Then for all $t\in\mathbb{R}_+$ and all $\mathbf{n}\in\mathbb{N}_0^k$ with $\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}\}\right]>0$ the process $\{\mathbf{K}_{t,h}\}_{h\in\mathbb{R}_+}$ is a multivariate counting process on the probability space $(\Omega,\mathcal{F},\mathrm{P}_{t,\mathbf{n}})$.*

2.2 The Multinomial Property

In the present section we introduce several properties, which a multivariate counting process may possess. All of them are related to the multinomial property. We start with two properties which are just concerned with the increments.

A multivariate counting process $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ has ***independent increments*** if

$$\mathrm{P}\left[\bigcap_{j=1}^m \left\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\right\}\right] \;=\; \prod_{j=1}^m \mathrm{P}\left[\left\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\right\}\right]$$

holds for all $m\in\mathbb{N}$ and $t_0,t_1,\ldots,t_m\in\mathbb{R}_+$ with $0=t_0<t_1<\ldots<t_m$ and for all $\mathbf{n}_j\in\mathbb{N}_0^k$, $j\in\{1,\ldots,m\}$.
A multivariate counting process $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ has ***stationary increments*** if

$$\mathrm{P}\left[\bigcap_{j=1}^m \left\{\mathbf{N}_{t_j+h}-\mathbf{N}_{t_{j-1}+h}=\mathbf{n}_j\right\}\right] \;=\; \mathrm{P}\left[\bigcap_{j=1}^m \left\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\right\}\right]$$

holds for all $m\in\mathbb{N}$ and $t_0,t_1,\ldots,t_m,h\in\mathbb{R}_+$ with $0=t_0<t_1<\ldots<t_m$ and for all $\mathbf{n}_j\in\mathbb{N}_0^k$, $j\in\{1,\ldots,m\}$.

As can be seen from the next lemma, both the property of independent increments and the property of stationary increments are stable under certain transformations.

2.2.1 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process. Then*

(1) *The property of having independent increments is $\mathcal{A}$–stable.*
(2) *The property of having stationary increments is $\mathcal{A}$–stable.*

Proof:
(1): Consider $m \in \mathbb{N}$ and $t_0, t_1, \ldots, t_m \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_m$ and $\mathbf{l}_j \in \mathbb{N}_0^d$, $j \in \{1, \ldots, m\}$. Then we obtain

$$\begin{aligned}
\mathrm{P}\left[\bigcap_{j=1}^{m}\left\{A\mathbf{N}_{t_j} - A\mathbf{N}_{t_{j-1}} = \mathbf{l}_j\right\}\right] &= \mathrm{P}\left[\bigcap_{j=1}^{m}\left\{A\left(\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}}\right) = \mathbf{l}_j\right\}\right] \\
&= \sum_{\mathbf{n}_1 \in A^{-1}(\{\mathbf{l}_1\})} \cdots \sum_{\mathbf{n}_m \in A^{-1}(\{\mathbf{l}_m\})} \mathrm{P}\left[\bigcap_{j=1}^{m}\left\{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\right\}\right] \\
&= \sum_{\mathbf{n}_1 \in A^{-1}(\{\mathbf{l}_1\})} \cdots \sum_{\mathbf{n}_m \in A^{-1}(\{\mathbf{l}_m\})} \prod_{j=1}^{m} \mathrm{P}\left[\left\{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\right\}\right] \\
&= \prod_{j=1}^{m} \sum_{\mathbf{n}_j \in A^{-1}(\{\mathbf{l}_j\})} \mathrm{P}\left[\left\{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\right\}\right] \\
&= \prod_{j=1}^{m} \mathrm{P}\left[\left\{A\mathbf{N}_{t_j} - A\mathbf{N}_{t_{j-1}} = \mathbf{l}_j\right\}\right]
\end{aligned}$$

which proves the assertion.

(2): Consider $m \in \mathbb{N}$ and $t_0, t_1, \ldots, t_m, h \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_m$ and $\mathbf{l}_j \in \mathbb{N}_0^d$, $j \in \{1, \ldots, m\}$. Then we get

$$\begin{aligned}
\mathrm{P}\left[\bigcap_{j=1}^{m}\left\{A\mathbf{N}_{t_j+h} - A\mathbf{N}_{t_{j-1}+h} = \mathbf{l}_j\right\}\right] &= \mathrm{P}\left[\bigcap_{j=1}^{m}\left\{A\left(\mathbf{N}_{t_j+h} - \mathbf{N}_{t_{j-1}+h}\right) = \mathbf{l}_j\right\}\right] \\
&= \sum_{\mathbf{n}_1 \in A^{-1}(\{\mathbf{l}_1\})} \cdots \sum_{\mathbf{n}_m \in A^{-1}(\{\mathbf{l}_m\})} \mathrm{P}\left[\bigcap_{j=1}^{m}\left\{\mathbf{N}_{t_j+h} - \mathbf{N}_{t_{j-1}+h} = \mathbf{n}_j\right\}\right] \\
&= \sum_{\mathbf{n}_1 \in A^{-1}(\{\mathbf{l}_1\})} \cdots \sum_{\mathbf{n}_m \in A^{-1}(\{\mathbf{l}_m\})} \mathrm{P}\left[\bigcap_{j=1}^{m}\left\{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\right\}\right] \\
&= \mathrm{P}\left[\bigcap_{j=1}^{m}\left\{A\mathbf{N}_{t_j} - A\mathbf{N}_{t_{j-1}} = \mathbf{l}_j\right\}\right]
\end{aligned}$$

and the prove is completed. ■

The next properties we introduce are dealing with inverse transition probabilities. This means, probabilities of increments which occurred before a certain state of the process.

A multivariate counting process $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ has

– the ***multinomial property*** if the identity

$$
\begin{aligned}
&\mathrm{P}\left[\bigcap_{j=1}^{m}\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\}\right] \\
&\quad = \left(\prod_{i=1}^{k}\frac{(\sum_{j=1}^{m} n_j^{(i)})!}{\prod_{j=1}^{m} n_j^{(i)}!}\prod_{j=1}^{m}\left(\frac{t_j-t_{j-1}}{t_m}\right)^{n_j^{(i)}}\right)\mathrm{P}\left[\left\{\mathbf{N}_{t_m}=\sum_{j=1}^{m}\mathbf{n}_j\right\}\right]
\end{aligned}
$$

holds for all $m\in\mathbb{N}$ and $t_0,t_1,\ldots,t_m\in\mathbb{R}_+$ with $0=t_0<t_1<\ldots<t_m$ and for all $\mathbf{n}_j\in\mathbb{N}_0^k$, $j\in\{1,\ldots,m\}$.

– the ***extended binomial property*** if the identity

$$
\begin{aligned}
&\mathrm{P}\left[\bigcap_{i=1}^{k}\left\{N_{t_i}^{(i)}=l^{(i)}\right\}\cap\left\{N_t^{(i)}-N_{t_i}^{(i)}=n^{(i)}\right\}\right] \\
&\quad = \left(\prod_{i=1}^{k}\binom{n^{(i)}+l^{(i)}}{l^{(i)}}\left(\frac{t_i}{t}\right)^{l^{(i)}}\left(1-\frac{t_i}{t}\right)^{n^{(i)}}\right)\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}+\mathbf{l}\}\right]
\end{aligned}
$$

holds for all $\mathbf{t}\in\mathbb{R}_+^k$, $t\in\mathbb{R}_+$ with $\mathbf{t}\in(\mathbf{0},t\mathbf{1})$ and for all $\mathbf{l}$, $\mathbf{n}\in\mathbb{N}_0^k$.

– the ***binomial property*** if the identity

$$
\begin{aligned}
&\mathrm{P}\left[\{\mathbf{N}_s=\mathbf{l}\}\cap\{\mathbf{N}_t-\mathbf{N}_s=\mathbf{n}\}\right] \\
&\quad = \left(\prod_{i=1}^{k}\binom{n^{(i)}+l^{(i)}}{l^{(i)}}\left(\frac{s}{t}\right)^{l^{(i)}}\left(1-\frac{s}{t}\right)^{n^{(i)}}\right)\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}+\mathbf{l}\}\right]
\end{aligned}
$$

holds for all $s,t\in\mathbb{R}_+$ with $0<s<t$ and all $\mathbf{l}$, $\mathbf{n}\in\mathbb{N}_0^k$.

For a multivariate counting process having the multinomial property, the finite–dimensional distributions are completely determined by the one–dimensional distributions. Furthermore, given the number of events at some time t_m the partitioning of the events into disjoint time intervals in the past is due to sampling with replacement. As this sampling is independent for the coordinates of the process, every coordinate could be sampled separately. The meaning of definition of the multinomial property would stay unchanged, if we allow equal times (i.e. $0=t_0\le t_1\le\ldots\le t_m$). Without loss of generality consider $t_m=t_{m-1}$. If $\mathbf{n}_m=\mathbf{0}$ we can ignore t_m and consider $m-1$ intervals. If $\mathbf{n}_m\neq\mathbf{0}$ both sides of the definition equal zero and the identity holds as well.

The binomial property is the natural extension of its one–dimensional counterpart in relation to the multinomial property, that means only two different times are considered. The extended binomial property considers a different time in the past for every coordinate. It can be seen from Lemma 2.2.7 that given the Markov property the binomial property is equal to the extended binomial property.

However, first we have a look at how our transformation works according to the above properties.

2.2.2 Lemma. *Let* $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *be a multivariate counting process. Then*
(1) *The multinomial property is* $\mathcal{A}$*–stable.*
(2) *The extended binomial property is* $\mathcal{A}$*–stable.*
(3) *The binomial property is* $\mathcal{A}$*–stable.*

Proof:
(1) and (3): Consider $m \in \mathbb{N}$ and $t_0, t_1, \dots, t_m \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \dots < t_m$ and $\mathbf{l}_j \in \mathbb{N}_0^d$, $j \in \{1, \dots, m\}$.
Assume that

$$\mathrm{P}\left[\bigcap_{j=1}^{m} \{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right] = \left(\prod_{i=1}^{k} \frac{n^{(i)}!}{\prod_{j=1}^{m} n_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m}\right)^{n_j^{(i)}}\right) \mathrm{P}\left[\left\{\mathbf{N}_{t_m} = \sum_{j=1}^{m} \mathbf{n}_j\right\}\right]$$

holds for all $\mathbf{n}_j \in \mathbb{N}_0^k$, $j \in \{1, \dots, m\}$. We want to show that

$$\mathrm{P}\left[\bigcap_{j=1}^{m} \{A\mathbf{N}_{t_j} - A\mathbf{N}_{t_{j-1}} = \mathbf{l}_j\}\right] = \left(\prod_{i=1}^{d} \frac{(\sum_{j=1}^{m} l_j^{(i)})!}{\prod_{j=1}^{m} l_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m}\right)^{l_j^{(i)}}\right) \mathrm{P}\left[\{A\mathbf{N}_{t_m} = \mathbf{l}\}\right] \qquad (+)$$

holds for all $A \in \mathcal{A}$ with $A \in \mathbb{R}^{d\times k}$.
- Let $A \in \mathcal{A}_P$. Then (+) holds obviously.
- Let $A \in \mathcal{A}_S$. We obtain

$$\begin{aligned}
\mathrm{P}\left[\bigcap_{j=1}^{m} \{A\mathbf{N}_{t_j} - A\mathbf{N}_{t_{j-1}} = \mathbf{l}_j\}\right] &= \mathrm{P}\left[\bigcap_{j=1}^{m} \{A\left(\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}}\right) = \mathbf{l}_j\}\right] \\
&= \sum_{\mathbf{n}_1 \in A^{-1}(\{\mathbf{l}_1\})} \cdots \sum_{\mathbf{n}_m \in A^{-1}(\{\mathbf{l}_m\})} \mathrm{P}\left[\bigcap_{j=1}^{m} \{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right] \\
&= \sum_{\mathbf{n}_1 \in A^{-1}(\{\mathbf{l}_1\})} \cdots \sum_{\mathbf{n}_m \in A^{-1}(\{\mathbf{l}_m\})} \left(\prod_{i=1}^{k} \frac{(\sum_{j=1}^{m} n_j^{(i)})!}{\prod_{j=1}^{m} n_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m}\right)^{n_j^{(i)}}\right) \\
&\qquad \cdot \mathrm{P}\left[\left\{\mathbf{N}_{t_m} = \sum_{j=1}^{m} \mathbf{n}_j\right\}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \left(\prod_{i=1}^{d} \frac{(\sum_{j=1}^{m} l_j^{(i)})!}{\prod_{j=1}^{m} l_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m} \right)^{l_j^{(i)}} \right) \\
&\quad \cdot \sum_{\mathbf{n}_1 \in A^{-1}(\{\mathbf{l}_1\})} \cdots \sum_{\mathbf{n}_m \in A^{-1}(\{\mathbf{l}_m\})} \left(\prod_{i=d+1}^{k} \frac{(\sum_{j=1}^{m} n_j^{(i)})!}{\prod_{j=1}^{m} n_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m} \right)^{n_j^{(i)}} \right) \\
&\qquad \cdot \mathrm{P}\left[\left\{ \mathbf{N}_{t_m} = \sum_{j=1}^{m} \mathbf{n}_j \right\} \right] \\
&= \left(\prod_{i=1}^{d} \frac{(\sum_{j=1}^{m} l_j^{(i)})!}{\prod_{j=1}^{m} l_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m} \right)^{l_j^{(i)}} \right) \sum_{\mathbf{n} \in A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{ \mathbf{N}_{t_m} = \mathbf{n} \} \right] \\
&\quad \cdot \sum_{\mathbf{n}_1 \in A^{-1}(\{\mathbf{l}_1\})} \cdots \sum_{\substack{\mathbf{n}_m \in A^{-1}(\{\mathbf{l}_m\}) \\ \sum_{j=1}^{m} \mathbf{n}_j = \mathbf{n}}} \left(\prod_{i=d+1}^{k} \frac{(\sum_{j=1}^{m} n_j^{(i)})!}{\prod_{j=1}^{m} n_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m} \right)^{n_j^{(i)}} \right) \\
&= \left(\prod_{i=1}^{d} \frac{(\sum_{j=1}^{m} l_j^{(i)})!}{\prod_{j=1}^{m} l_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m} \right)^{l_j^{(i)}} \right) \sum_{\mathbf{n} \in A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{ \mathbf{N}_{t_m} = \mathbf{n} \} \right] \\
&= \left(\prod_{i=1}^{d} \frac{(\sum_{j=1}^{m} l_j^{(i)})!}{\prod_{j=1}^{m} l_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m} \right)^{l_j^{(i)}} \right) \mathrm{P}\left[\{ A\mathbf{N}_{t_m} = \mathbf{l} \} \right]
\end{aligned}
$$

where the equality preceding the last one is simply the use of the multinomial distribution. So (+) holds for $A \in \mathcal{A}_S$.
- Let $A \in \mathcal{A}_C$. Setting $I(i) := \{h \in \{1, \dots, k\} \, : \, \mathbf{e}_i{}' A \mathbf{e}_h = 1\}$ (the set of coordinates cumulated in the i–th coordinate of the transformed process) we get

$$
\begin{aligned}
\mathrm{P}\left[\bigcap_{j=1}^{m} \{ A\mathbf{N}_{t_j} - A\mathbf{N}_{t_{j-1}} = \mathbf{l}_j \} \right] &= \mathrm{P}\left[\bigcap_{j=1}^{m} \{ A\left(\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} \right) = \mathbf{l}_j \} \right] \\
&= \sum_{\mathbf{n}_1 \in A^{-1}(\{\mathbf{l}_1\})} \cdots \sum_{\mathbf{n}_m \in A^{-1}(\{\mathbf{l}_m\})} \mathrm{P}\left[\bigcap_{j=1}^{m} \{ \mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j \} \right] \\
&= \sum_{\mathbf{n}_1 \in A^{-1}(\{\mathbf{l}_1\})} \cdots \sum_{\mathbf{n}_m \in A^{-1}(\{\mathbf{l}_m\})} \left(\prod_{i=1}^{k} \frac{(\sum_{j=1}^{m} n_j^{(i)})!}{\prod_{j=1}^{m} n_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m} \right)^{n_j^{(i)}} \right) \\
&\qquad \cdot \mathrm{P}\left[\left\{ \mathbf{N}_{t_m} = \sum_{j=1}^{m} \mathbf{n}_j \right\} \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\prod_{i=1}^{d} \frac{(\sum_{j=1}^{m} l_j^{(i)})!}{\prod_{j=1}^{m} l_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m}\right)^{l_j^{(i)}}\right) \\
&\quad \cdot \sum_{\mathbf{n}_1 \in A^{-1}(\{\mathbf{l}_1\})} \cdots \sum_{\mathbf{n}_m \in A^{-1}(\{\mathbf{l}_m\})} \left(\prod_{i=1}^{d} \frac{\prod_{j=1}^{m} l_j^{(i)}!}{(\sum_{j=1}^{m} l_j^{(i)})!}\right) \left(\prod_{i=1}^{k} \frac{(\sum_{j=1}^{m} n_j^{(i)})!}{\prod_{j=1}^{m} n_j^{(i)}!}\right) \\
&\qquad \cdot \mathrm{P}\left[\left\{\mathbf{N}_{t_m} = \sum_{j=1}^{m} \mathbf{n}_j\right\}\right] \\
&= \left(\prod_{i=1}^{d} \frac{(\sum_{j=1}^{m} l_j^{(i)})!}{\prod_{j=1}^{m} l_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m}\right)^{l_j^{(i)}}\right) \sum_{\mathbf{n} \in A^{-1}(\{\mathbf{l}\})} \mathrm{P}[\{\mathbf{N}_{t_m} = \mathbf{n}\}] \\
&\quad \cdot \sum_{\mathbf{n}_1 \in A^{-1}(\{\mathbf{l}_1\})} \cdots \sum_{\substack{\mathbf{n}_m \in A^{-1}(\{\mathbf{l}_m\}) \\ \sum_{j=1}^{m} \mathbf{n}_j = \mathbf{n}}} \left(\prod_{i=1}^{d} \frac{\prod_{j=1}^{m} l_j^{(i)}!}{(\sum_{j=1}^{m} l_j^{(i)})!}\right) \left(\prod_{i=1}^{k} \frac{n^{(i)}!}{\prod_{j=1}^{m} n_j^{(i)}!}\right) \\
&= \left(\prod_{i=1}^{d} \frac{(\sum_{j=1}^{m} l_j^{(i)})!}{\prod_{j=1}^{m} l_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m}\right)^{l_j^{(i)}}\right) \sum_{\mathbf{n} \in A^{-1}(\{\mathbf{l}\})} \mathrm{P}[\{\mathbf{N}_{t_m} = \mathbf{n}\}] \\
&\quad \cdot \left(\prod_{i=1}^{d} \frac{\prod_{h \in I(i)} n^{(h)}!}{(\sum_{j=1}^{m} l_j^{(i)})!}\right) \sum_{\mathbf{n}_1 \in A^{-1}(\{\mathbf{l}_1\})} \cdots \sum_{\substack{\mathbf{n}_m \in A^{-1}(\{\mathbf{l}_m\}) \\ \sum_{j=1}^{m} \mathbf{n}_j = \mathbf{n}}} \prod_{i=1}^{d} \prod_{j=1}^{m} \frac{l_j^{(i)}!}{\prod_{h \in I(i)} n_j^{(h)}!} \\
&= \left(\prod_{i=1}^{d} \frac{(\sum_{j=1}^{m} l_j^{(i)})!}{\prod_{j=1}^{m} l_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m}\right)^{l_j^{(i)}}\right) \\
&\quad \cdot \sum_{\mathbf{n} \in A^{-1}(\{\mathbf{l}\})} \mathrm{P}[\{\mathbf{N}_{t_m} = \mathbf{n}\}] \left(\prod_{i=1}^{d} \frac{\prod_{h \in I(i)} n^{(h)}!}{(\sum_{j=1}^{m} l_j^{(i)})!}\right) \left(\prod_{i=1}^{d} \frac{(\sum_{j=1}^{m} l_j^{(i)})!}{\prod_{h \in I(i)} n^{(h)}!}\right) \\
&= \left(\prod_{i=1}^{d} \frac{(\sum_{j=1}^{m} l_j^{(i)})!}{\prod_{j=1}^{m} l_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m}\right)^{l_j^{(i)}}\right) \mathrm{P}[\{A\mathbf{N}_{t_m} = \mathbf{l}\}]
\end{aligned}
$$

where the identity previous to the last one is due to the d–fold use of the multinomial coefficient formula, which states that for arbitrary $u, v \in \mathbb{N}$, $z_j \in \mathbb{N}_0$, $j \in \{1, \ldots, u\}$, with $z := \sum_{j=1}^{u} z_j$ and arbitrary $\mathbf{x} \in \mathbb{N}_0^v$ with $\mathbf{1}'\mathbf{x} = z$ the identity

$$
\binom{z}{\mathbf{x}} = \sum_{\substack{\mathbf{x}_1 \in \mathbb{N}_0^v \\ \mathbf{1}'\mathbf{x}_1 = z_1}} \cdots \sum_{\substack{\mathbf{x}_u \in \mathbb{N}_0^v \\ \mathbf{1}'\mathbf{x}_u = z_u \\ \sum_{j=1}^{u} \mathbf{x}_j = \mathbf{x}}} \prod_{j=1}^{u} \binom{z_j}{\mathbf{x}_j}
$$

is valid. So (+) holds for $A \in \mathcal{A}_C$, too.
As m and $t_0, t_1, \ldots, t_m$ with $0 = t_0 < t_1 < \ldots < t_m$ and $\mathbf{l}_j$, $j \in \{1, \ldots, m\}$, have

been arbitrary, we have shown that the multinomial as well as the binomial property is $\mathcal{A}$–stable.
(2): Now, we assume that $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ has the extended binomial property. Consider $\mathbf{t}\in\mathbb{R}_+^d$, $t\in\mathbb{R}_+$ with $\mathbf{t}\in(\mathbf{0},t\mathbf{1})$ and $\mathbf{l}_1$, $\mathbf{l}_2\in\mathbb{N}_0^d$. We want to show that

$$\mathrm{P}\left[\bigcap_{i=1}^{d}\left\{\mathbf{e}_i{}'A\mathbf{N}_{t_i}=l_1^{(i)}\right\}\cap\left\{\mathbf{e}_i{}'A\mathbf{N}_t-\mathbf{e}_i{}'A\mathbf{N}_{t_i}=l_2^{(i)}\right\}\right]$$
$$=\left(\prod_{i=1}^{d}\binom{l_1^{(i)}+l_2^{(i)}}{l_1^{(i)}}\left(\frac{t_i}{t}\right)^{l_1^{(i)}}\left(1-\frac{t_i}{t}\right)^{l_2^{(i)}}\right)\mathrm{P}\left[\{A\mathbf{N}_t=\mathbf{l}_1+\mathbf{l}_2\}\right]\qquad(*)$$

holds for all $A\in\mathcal{A}$ with $A\in\mathbb{R}^{d\times k}$.
- Let $A\in\mathcal{A}_P$. Then $(*)$ holds obviously.
- Let $A\in\mathcal{A}_S$. We consider $\mathbf{s}\in\mathbb{R}_+^k$ with $\mathbf{s}\in(\mathbf{0},t\mathbf{1})$ and $A\mathbf{s}=\mathbf{t}$. Then, using the same argumentation as in part (1), we get

$$\mathrm{P}\left[\bigcap_{i=1}^{d}\left\{\mathbf{e}_i{}'A\mathbf{N}_{t_i}=l_1^{(i)}\right\}\cap\left\{\mathbf{e}_i'A\mathbf{N}_t-\mathbf{e}_i{}'A\mathbf{N}_{t_i}=l_2^{(i)}\right\}\right]$$
$$=\sum_{\mathbf{n}_1\in A^{-1}(\{\mathbf{l}_1\})}\ \sum_{\mathbf{n}_2\in A^{-1}(\{\mathbf{l}_2\})}\mathrm{P}\left[\bigcap_{i=1}^{k}\left\{N_{s_i}^{(i)}=n_1^{(i)}\right\}\cap\left\{N_t^{(i)}-N_{s_i}^{(i)}=n_2^{(i)}\right\}\right]$$
$$=\sum_{\mathbf{n}_1\in A^{-1}(\{\mathbf{l}_1\})}\ \sum_{\mathbf{n}_2\in A^{-1}(\{\mathbf{l}_2\})}\left(\prod_{i=1}^{k}\binom{n_1^{(i)}+n_2^{(i)}}{n_1^{(i)}}\left(\frac{s_i}{t}\right)^{n_1^{(i)}}\left(1-\frac{s_i}{t}\right)^{n_2^{(i)}}\right)\cdot\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}_1+\mathbf{n_2}\}\right]$$
$$=\left(\prod_{i=1}^{d}\binom{l_1^{(i)}+l_2^{(i)}}{l_1^{(i)}}\left(\frac{t_i}{t}\right)^{l_1^{(i)}}\left(1-\frac{t_i}{t}\right)^{l_2^{(i)}}\right)$$
$$\cdot\sum_{\mathbf{n}_1\in A^{-1}(\{\mathbf{l}_1\})}\ \sum_{\mathbf{n}_2\in A^{-1}(\{\mathbf{l}_2\})}\left(\prod_{i=d+1}^{k}\binom{n_1^{(i)}+n_2^{(i)}}{n_1^{(i)}}\left(\frac{s_i}{t}\right)^{n_1^{(i)}}\left(1-\frac{s_i}{t}\right)^{n_2^{(i)}}\right)\cdot\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}_1+\mathbf{n_2}\}\right]$$
$$=\left(\prod_{i=1}^{d}\binom{l_1^{(i)}+l_2^{(i)}}{l_1^{(i)}}\left(\frac{t_i}{t}\right)^{l_1^{(i)}}\left(1-\frac{t_i}{t}\right)^{l_2^{(i)}}\right)\sum_{\mathbf{n}\in A^{-1}(\{\mathbf{l}_1+\mathbf{l}_2\})}\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}\}\right]$$
$$\cdot\sum_{\substack{\mathbf{n}_1\in A^{-1}(\{\mathbf{l}_1\})\\ \mathbf{n}_1\in[\mathbf{0},\mathbf{n}]}}\left(\prod_{i=d+1}^{k}\binom{n^{(i)}}{n_1^{(i)}}\left(\frac{s_i}{t}\right)^{n_1^{(i)}}\left(1-\frac{s_i}{t}\right)^{n^{(i)}-n_1^{(i)}}\right)$$

$$
\begin{aligned}
&= \left(\prod_{i=1}^{d} \binom{l_1^{(i)} + l_2^{(i)}}{l_1^{(i)}} \left(\frac{t_i}{t}\right)^{l_1^{(i)}} \left(1 - \frac{t_i}{t}\right)^{l_2^{(i)}} \right) \sum_{\mathbf{n} \in A^{-1}(\{\mathbf{l}_1 + \mathbf{l}_2\})} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \\
&= \left(\prod_{i=1}^{d} \binom{l_1^{(i)} + l_2^{(i)}}{l_1^{(i)}} \left(\frac{t_i}{t}\right)^{l_1^{(i)}} \left(1 - \frac{t_i}{t}\right)^{l_2^{(i)}} \right) \mathrm{P}\left[\{A\mathbf{N}_t = \mathbf{l}_1 + \mathbf{l}_2\}\right]
\end{aligned}
$$

where the equality preceding the last one is simply the use of binomial distribution. So $(*)$ holds for $A \in \mathcal{A}_S$.
- Let $A \in \mathcal{A}_C$. Setting $I(i) := \{h \in \{1, \ldots, k\} \,:\, \mathbf{e}_i{}' A \mathbf{e}_h = 1\}$ and $\mathbf{s} \in \mathbb{R}_+^k$ such that the identity $s_h = t_i$ holds for all $h \in I(i)$ and $i \in \{1, \ldots, d\}$. Then, by using the same argumentation as in part (1), we obtain

$$
\begin{aligned}
&\mathrm{P}\left[\bigcap_{i=1}^{d} \left\{ \mathbf{e}_i{}' A \mathbf{N}_{t_i} = l_1^{(i)} \right\} \cap \left\{ \mathbf{e}_i{}' A \mathbf{N}_t - \mathbf{e}_i{}' A \mathbf{N}_{t_i} = l_2^{(i)} \right\} \right] \\
&= \sum_{\mathbf{n}_1 \in A^{-1}(\{\mathbf{l}_1\})} \sum_{\mathbf{n}_2 \in A^{-1}(\{\mathbf{l}_2\})} \mathrm{P}\left[\bigcap_{i=1}^{k} \left\{ N_{s_i}^{(i)} = n_1^{(i)} \right\} \cap \left\{ N_t^{(i)} - N_{s_i}^{(i)} = n_2^{(i)} \right\} \right] \\
&= \sum_{\mathbf{n}_1 \in A^{-1}(\{\mathbf{l}_1\})} \sum_{\mathbf{n}_2 \in A^{-1}(\{\mathbf{l}_2\})} \left(\prod_{i=1}^{k} \binom{n_1^{(i)} + n_2^{(i)}}{n_1^{(i)}} \left(\frac{s_i}{t}\right)^{n_1^{(i)}} \left(1 - \frac{s_i}{t}\right)^{n_2^{(i)}} \right) \\
&\qquad \cdot \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}_1 + \mathbf{n_2}\}\right] \\
&= \left(\prod_{i=1}^{d} \binom{l_1^{(i)} + l_2^{(i)}}{l_1^{(i)}} \left(\frac{t_i}{t}\right)^{l_1^{(i)}} \left(1 - \frac{t_i}{t}\right)^{l_2^{(i)}} \right) \\
&\quad \cdot \sum_{\mathbf{n}_1 \in A^{-1}(\{\mathbf{l}_1\})} \sum_{\mathbf{n}_2 \in A^{-1}(\{\mathbf{l}_2\})} \left(\prod_{i=1}^{d} \binom{l_1^{(i)} + l_2^{(i)}}{l_1^{(i)}}^{-1} \right) \left(\prod_{i=1}^{k} \binom{n_1^{(i)} + n_2^{(i)}}{n_1^{(i)}} \right) \\
&\qquad \cdot \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}_1 + \mathbf{n_2}\}\right] \\
&= \left(\prod_{i=1}^{d} \binom{l_1^{(i)} + l_2^{(i)}}{l_1^{(i)}} \left(\frac{t_i}{t}\right)^{l_1^{(i)}} \left(1 - \frac{t_i}{t}\right)^{l_2^{(i)}} \right) \sum_{\mathbf{n} \in A^{-1}(\{\mathbf{l}_1 + \mathbf{l}_2\})} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \\
&\quad \cdot \sum_{\substack{\mathbf{n}_1 \in A^{-1}(\{\mathbf{l}_1\}) \\ \mathbf{n}_1 \in [\mathbf{0}, \mathbf{n}]}} \left(\prod_{i=1}^{d} \binom{l_1^{(i)} + l_2^{(i)}}{l_1^{(i)}}^{-1} \right) \left(\prod_{i=1}^{k} \binom{n^{(i)}}{n_1^{(i)}} \right) \\
&= \left(\prod_{i=1}^{d} \binom{l_1^{(i)} + l_2^{(i)}}{l_1^{(i)}} \left(\frac{t_i}{t}\right)^{l_1^{(i)}} \left(1 - \frac{t_i}{t}\right)^{l_2^{(i)}} \right) \sum_{\mathbf{n} \in A^{-1}(\{\mathbf{l}_1 + \mathbf{l}_2\})} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]
\end{aligned}
$$

$$\cdot \left(\prod_{i=1}^{d} \frac{\prod_{h \in I(i)} n^{(h)}!}{(l_1^{(i)} + l_2^{(i)})!} \right) \sum_{\substack{\mathbf{n}_1 \in A^{-1}(\{\mathbf{l}_1\}) \\ \mathbf{n}_1 \in [\mathbf{0},\mathbf{n}]}} \prod_{i=1}^{d} \frac{l_1^{(i)}!}{\prod_{h \in I(i)} n_1^{(h)}!} \, \frac{l_2^{(i)}!}{\prod_{h \in I(i)} (n^{(i)} - n_1^{(h)})!}$$

$$= \left(\prod_{i=1}^{d} \binom{l_1^{(i)} + l_2^{(i)}}{l_1^{(i)}} \left(\frac{t_i}{t}\right)^{l_1^{(i)}} \left(1 - \frac{t_i}{t}\right)^{l_2^{(i)}} \right) \sum_{\mathbf{n} \in A^{-1}(\{\mathbf{l}_1 + \mathbf{l}_2\})} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]$$

$$\cdot \left(\prod_{i=1}^{d} \frac{\prod_{h \in I(i)} n^{(h)}!}{(l_1^{(i)} + l_2^{(i)})!} \right) \left(\prod_{i=1}^{d} \frac{(l_1^{(i)} + l_2^{(i)})!}{\prod_{h \in I(i)} n^{(h)}!} \right)$$

$$= \left(\prod_{i=1}^{d} \binom{l_1^{(i)} + l_2^{(i)}}{l_1^{(i)}} \left(\frac{t_i}{t}\right)^{l_1^{(i)}} \left(1 - \frac{t_i}{t}\right)^{l_2^{(i)}} \right) \mathrm{P}\left[\{A\mathbf{N}_t = \mathbf{l}_1 + \mathbf{l}_2\}\right]$$

Hence, equation $(*)$ holds for $A \in \mathcal{A}_\mathcal{S}$, too.
So the extended binomial property is also $\mathcal{A}$–stable. ■

The next lemma states an implication of the binomial property which is derived in interaction with properties of the paths of multivariate counting processes.

2.2.3 Lemma. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process. If $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ has the binomial property, then*

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] > 0$$

holds for all $t > 0$ and all $\mathbf{n} \in \mathbb{N}_0^k$.

Proof: First, we assume there exists some $\mathbf{m} \in \mathbb{N}_0^k$ such that

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] = 0$$

holds for all $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$ with $\mathbf{n} \geq \mathbf{m}$. Then we have $\mathrm{P}\left[\{\mathbf{N}_t \geq \mathbf{m}\}\right] = 0$, which is a contradiction to $\lim_{t \uparrow \infty} \mathrm{P}\left[\{\mathbf{N}_t \geq \mathbf{n}\}\right] = 1$ for all $\mathbf{n} \in \mathbb{N}_0^k$ (Lemma 2.1.2 (5)).
Now, consider $\mathbf{m} \in \mathbb{N}_0^k$. By the first part of the proof there exists some $t > 0$ and some $\mathbf{n} \in \mathbb{N}_0^k$ with $\mathbf{n} \geq \mathbf{m}$ such that

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] > 0$$

The binomial property leads to

$$\begin{aligned} \mathrm{P}\left[\{\mathbf{N}_s = \mathbf{l}\}\right] &\geq \mathrm{P}\left[\{\mathbf{N}_s = \mathbf{l}\} \cap \{\mathbf{N}_t - \mathbf{N}_s = \mathbf{n} - \mathbf{l}\}\right] \\ &= \left(\prod_{i=1}^{k} \binom{n^{(i)}}{l^{(i)}} \left(\frac{s}{t}\right)^{l^{(i)}} \left(1 - \frac{s}{t}\right)^{n^{(i)}} \right) \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \end{aligned}$$

and hence

$$\mathrm{P}\left[\{\mathbf{N}_s = \mathbf{l}\}\right] > 0$$

for all $s \in (0, t)$ and all $\mathbf{l} \in \mathbb{N}_0^k$ with $\mathbf{l} \leq \mathbf{n}$. Moreover, for all $u \in (t, \infty)$ the identity $\sum_{\mathbf{p} \geq \mathbf{n}} \mathrm{P}[\{\mathbf{N}_u = \mathbf{p}\} \,|\, \{\mathbf{N}_t = \mathbf{n}\}] = 1$ yields the existence of some $\mathbf{p} \in \mathbb{N}_0^k$ with $\mathbf{n} \leq \mathbf{p}$ such that

$$\begin{aligned} \mathrm{P}[\{\mathbf{N}_u = \mathbf{p}\}] &\geq \mathrm{P}[\{\mathbf{N}_u = \mathbf{p}\} \cap \{\mathbf{N}_t = \mathbf{n}\}] \\ &= \mathrm{P}[\{\mathbf{N}_u = \mathbf{p}\} \,|\, \{\mathbf{N}_t = \mathbf{n}\}]\ \mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}] \\ &> 0 \end{aligned}$$

Replacing t and $\mathbf{n}$ by u and $\mathbf{p}$ in the preceding argument, we get

$$\mathrm{P}[\{\mathbf{N}_s = \mathbf{l}\}] > 0$$

for all $s > 0$ and all $\mathbf{l} \in \mathbb{N}_0^k$ with $\mathbf{l} \leq \mathbf{n}$.
Since $\mathbf{m} \in \mathbb{N}_0^k$ was arbitrary the assertion is shown. ■

2.2.4 Corollary. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process. If $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ has the multinomial property, then* $\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}] > 0$ *holds for all* $t > 0$ *and* $\mathbf{n} \in \mathbb{N}_0^k$.

The property of counting processes having the binomial property that all states possess strictly positive probability will be subsequently used quite often and is of special interest in Chapter 2.3. After studying each property alone, we state a first relation between the properties introduced so far.

2.2.5 Lemma. *If a multivariate counting process has the multinomial property, then it has stationary increments.*

Proof: Consider $m \in \mathbb{N}$ and $t_0, t_1, \ldots, t_m, h \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_m$ and $\mathbf{n}_j \in \mathbb{N}_0^k$, $j \in \{1, \ldots, m\}$. Setting $t_{-1} := -h$ and $\mathbf{l}_m := \sum_{j=1}^m \mathbf{n}_j$ we get

$$\begin{aligned} \mathrm{P}\left[\bigcap_{j=1}^m \{\mathbf{N}_{t_j+h} - \mathbf{N}_{t_{j-1}+h} = \mathbf{n}_j\}\right] &= \sum_{\mathbf{n}_0 \in \mathbb{N}_0^k} \mathrm{P}\left[\bigcap_{j=0}^m \{\mathbf{N}_{t_j+h} - \mathbf{N}_{t_{j-1}+h} = \mathbf{n}_j\}\right] \\ &= \sum_{\mathbf{n}_0 \in \mathbb{N}_0^k} \left(\prod_{i=1}^k \frac{(l_m^{(i)} + n_0^{(i)})!}{\prod_{j=0}^m n_j^{(i)}!} \prod_{j=0}^m \left(\frac{t_j - t_{j-1}}{t_m + h}\right)^{n_j^{(i)}}\right) \mathrm{P}[\{\mathbf{N}_{t_m+h} = \mathbf{l}_m + \mathbf{n}_0\}] \\ &= \left(\prod_{i=1}^k \frac{l_m^{(i)}!}{\prod_{j=1}^m n_j^{(i)}!} \prod_{j=1}^m \left(\frac{t_j - t_{j-1}}{t_m}\right)^{n_j^{(i)}}\right) \sum_{\mathbf{n}_0 \in \mathbb{N}_0^k} \mathrm{P}[\{\mathbf{N}_{t_m+h} = \mathbf{l}_m + \mathbf{n}_0\}] \\ &\qquad \cdot \left(\prod_{i=1}^k \binom{l_m^{(i)} + n_0^{(i)}}{l_m^{(i)}} \left(\frac{t_m}{t_m + h}\right)^{l_m^{(i)}} \left(\frac{h}{t_m + h}\right)^{n_0^{(i)}}\right) \\ &= \left(\prod_{i=1}^k \frac{l_m^{(i)}!}{\prod_{j=1}^m n_j^{(i)}!} \prod_{j=1}^m \left(\frac{t_j - t_{j-1}}{t_m}\right)^{n_j^{(i)}}\right) \end{aligned}$$

$$
\begin{aligned}
&\cdot \sum_{\mathbf{n}_0 \in \mathbb{N}_0^k} \mathrm{P}\left[\{\mathbf{N}_{t_m} = \mathbf{l}_m\} \cap \{\mathbf{N}_{t_m+h} - \mathbf{N}_{t_m} = \mathbf{n}_0\}\right] \\
= & \left(\prod_{i=1}^k \frac{l_m^{(i)}!}{\prod_{j=1}^m n_j^{(i)}!} \prod_{j=1}^m \left(\frac{t_j - t_{j-1}}{t_m} \right)^{n_j^{(i)}} \right) \mathrm{P}\left[\{\mathbf{N}_{t_m} = \mathbf{l}_m\}\right] \\
= & \ \mathrm{P}\left[\bigcap_{j=1}^m \left\{ \mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j \right\} \right]
\end{aligned}
$$

and the assertion is shown. ■

The third and last set of properties we introduce in this section is related to transition probabilities. That means, they are dealing with probabilities of increments which occur after certain events of the process.

A multivariate counting process $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ has the ***Markov property*** (is a ***Markov process***) if the identity

$$
\begin{aligned}
&\mathrm{P}\left[\bigcap_{j=1}^{m+1} \left\{ \mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j \right\} \right] \mathrm{P}\left[\{\mathbf{N}_{t_m} = \mathbf{l}_m\}\right] \\
&\qquad = \mathrm{P}\left[\bigcap_{j=1}^{m} \left\{ \mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j \right\} \right] \mathrm{P}\left[\{\mathbf{N}_{t_m} = \mathbf{l}_m\} \cap \{\mathbf{N}_{t_{m+1}} - \mathbf{N}_{t_m} = \mathbf{n}_{m+1}\} \right]
\end{aligned}
$$

holds for all $m \in \mathbb{N}$ and $t_0, t_1, \ldots, t_{m+1} \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_{m+1}$ and for all $\mathbf{n}_1, \ldots, \mathbf{n}_{m+1} \in \mathbb{N}_0^k$ with $\mathbf{l}_m := \sum_{j=1}^m \mathbf{n}_j$.

If $\mathrm{P}\left[\cap_{j=1}^m \left\{ \mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j \right\} \right] > 0$ the previous identities are equivalent to

$$
\begin{aligned}
&\mathrm{P}\left[\left\{ \mathbf{N}_{t_{m+1}} - \mathbf{N}_{t_m} = \mathbf{n}_{m+1} \right\} \,\middle|\, \bigcap_{j=1}^{m} \left\{ \mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j \right\} \right] \\
&\qquad = \mathrm{P}\left[\left\{ \mathbf{N}_{t_{m+1}} - \mathbf{N}_{t_m} = \mathbf{n}_{m+1} \right\} \,\middle|\, \{\mathbf{N}_{t_m} = \mathbf{l}_m\} \right]
\end{aligned}
$$

The first identities are more useful for technical reasons whereas the second ones offer an interpretation of the Markov property. Roughly speaking, the future increment of a Markov process only depends on the total increment up to the present and not on the partitioning of the increment in the past.

A multivariate counting process $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ has the ***Chapman–Kolmogorov property*** if the identity

$$
\begin{aligned}
&\mathrm{P}\left[\{\mathbf{N}_t - \mathbf{N}_r = \mathbf{m}\} \,|\, \{\mathbf{N}_r = \mathbf{n}\}\right] \\
&\quad = \sum_{\substack{\mathbf{l} \in [\mathbf{0}, \mathbf{m}] \\ \mathrm{P}[\{\mathbf{N}_s = \mathbf{n}+\mathbf{l}\}] > 0}} \mathrm{P}\left[\{\mathbf{N}_s - \mathbf{N}_r = \mathbf{l}\} \,|\, \{\mathbf{N}_r = \mathbf{n}\}\right] \mathrm{P}\left[\{\mathbf{N}_t - \mathbf{N}_s = \mathbf{m} - \mathbf{l}\} \,|\, \{\mathbf{N}_s = \mathbf{n} + \mathbf{l}\}\right]
\end{aligned}
$$

holds for all $r, t \in \mathbb{R}_+$, $\mathbf{n}, \mathbf{m} \in \mathbb{N}_0^k$ with $r \le t$ and $\mathrm{P}[\mathbf{N}_r = \mathbf{n}] > 0$ as well as for all $s \in [r, t]$.

Since the multivariate counting process has increasing paths, strictly negative increments have probability zero. Therefore, the above identities are equivalent to

$$\begin{aligned}
&\mathrm{P}[\{\mathbf{N}_t = \mathbf{m}\} \,|\, \{\mathbf{N}_r = \mathbf{n}\}] \\
&\qquad = \sum_{\substack{\mathbf{l} \in \mathbb{N}_0^k \\ \mathrm{P}[\{\mathbf{N}_s = \mathbf{l}\}] > 0}} \mathrm{P}[\{\mathbf{N}_s = \mathbf{l}\} \,|\, \{\mathbf{N}_r = \mathbf{n}\}] \; \mathrm{P}[\{\mathbf{N}_t = \mathbf{m}\} \,|\, \{\mathbf{N}_s = \mathbf{l}\}]
\end{aligned}$$

These are the general Chapman–Kolmogorov equations often found in literature. The advantage of the use of the first identities is a finite sum and the use of increments, which fits right into the definitions of the other properties.

Since in general settings of stochastic processes there exist examples of multivariate Markov processes with coordinates having not the Markov property it seems likely that the Markov property is not $\mathcal{A}$–stable in the setting of counting processes.

2.2.6 Lemma. *If a multivariate counting process is a Markov process, then it has the Chapman–Kolmogorov property.*

Proof: Consider $r, t \in \mathbb{R}_+$, $\mathbf{n}, \mathbf{m} \in \mathbb{N}_0^k$ with $r \le t$ and $\mathrm{P}[\mathbf{N}_r = \mathbf{n}] > 0$ as well as an arbitrary $s \in [r, t]$. Setting $B := \{\mathbf{N}_s - \mathbf{N}_r = \mathbf{l}\} \cap \{\mathbf{N}_r = \mathbf{n}\}$ we obtain

$$\begin{aligned}
&\mathrm{P}[\{\mathbf{N}_t - \mathbf{N}_r = \mathbf{m}\} \,|\, \{\mathbf{N}_r = \mathbf{n}\}] \\
&= \sum_{\mathbf{l} \in [\mathbf{0}, \mathbf{m}]} \mathrm{P}[\{\mathbf{N}_t - \mathbf{N}_s = \mathbf{m} - \mathbf{l}\} \cap \{\mathbf{N}_s - \mathbf{N}_r = \mathbf{l}\} \,|\, \{\mathbf{N}_r = \mathbf{n}\}] \\
&= \sum_{\substack{\mathbf{l} \in [\mathbf{0}, \mathbf{m}] \\ \mathrm{P}[B] > 0}} \mathrm{P}[\{\mathbf{N}_t - \mathbf{N}_s = \mathbf{m} - \mathbf{l}\} \,|\, \{\mathbf{N}_s - \mathbf{N}_r = \mathbf{l}\} \cap \{\mathbf{N}_r = \mathbf{n}\}] \\
&\qquad\qquad \cdot \mathrm{P}[\{\mathbf{N}_s - \mathbf{N}_r = \mathbf{l}\} \,|\, \{\mathbf{N}_r = \mathbf{n}\}] \\
&= \sum_{\substack{\mathbf{l} \in [\mathbf{0}, \mathbf{m}] \\ \mathrm{P}[B] > 0}} \mathrm{P}[\{\mathbf{N}_t - \mathbf{N}_s = \mathbf{m} - \mathbf{l}\} \,|\, \{\mathbf{N}_s = \mathbf{n} + \mathbf{l}\}] \; \mathrm{P}[\{\mathbf{N}_s - \mathbf{N}_r = \mathbf{l}\} \,|\, \{\mathbf{N}_r = \mathbf{n}\}] \\
&= \sum_{\substack{\mathbf{l} \in [\mathbf{0}, \mathbf{m}] \\ \mathrm{P}[\{\mathbf{N}_s = \mathbf{n} + \mathbf{l}\}] > 0}} \mathrm{P}[\{\mathbf{N}_t - \mathbf{N}_s = \mathbf{m} - \mathbf{l}\} \,|\, \{\mathbf{N}_s = \mathbf{n} + \mathbf{l}\}] \; \mathrm{P}[\{\mathbf{N}_s - \mathbf{N}_r = \mathbf{l}\} \,|\, \{\mathbf{N}_r = \mathbf{n}\}]
\end{aligned}$$

and thus the assertion. ∎

Our next aim is to show relations between properties concerning inverse transition probabilities and properties concerning transition probabilities.

2.2.7 Lemma. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process. Then the following are equivalent:*

(a) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *has the multinomial property.*
(b) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *has the extended binomial property and the Markov property.*
(c) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *has the binomial property and the Markov property.*

Proof: We prove the assertion according to the following scheme: (b) $\Rightarrow$ (c) $\Rightarrow$ (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (c): obvious

(c) $\Rightarrow$ (a): We use the induction method for the number m of time periods in the equation

$$\begin{aligned} &\mathrm{P}\left[\bigcap_{j=1}^{m}\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\}\right] \\ &\qquad = \left(\prod_{i=1}^{k}\frac{(\sum_{j=1}^{m}n_j^{(i)})!}{\prod_{j=1}^{m}n_j^{(i)}!}\prod_{j=1}^{m}\left(\frac{t_j-t_{j-1}}{t_m}\right)^{n_j^{(i)}}\right)\mathrm{P}\left[\left\{\mathbf{N}_{t_m}=\sum_{j=1}^{m}\mathbf{n}_j\right\}\right] \qquad (*) \end{aligned}$$

for all $t_0, t_1, \ldots, t_m \in \mathbb{R}_+$, $0 = t_0 < t_1 < \ldots < t_m$ and all $\mathbf{n}_j \in \mathbb{N}_0^k$, $j \in \{1, \ldots, m\}$.
For $m = 1$ $(*)$ is evidently satisfied.
Now, assume that $(*)$ holds for $m \in \mathbb{N}$. Consider $t_0, t_1, \ldots, t_m, t_{m+1} \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_m < t_{m+1}$ and $\mathbf{n}_j \in \mathbb{N}_0^k$, $j \in \{1, \ldots, m+1\}$. Setting $\mathbf{l}_j := \sum_{h=1}^{j}\mathbf{n}_h$ for $j \in \{1, \ldots, m+1\}$ we get

$$\begin{aligned} &\mathrm{P}\left[\bigcap_{j=1}^{m+1}\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\}\right]\mathrm{P}[\{\mathbf{N}_{t_m}=\mathbf{l}_m\}] \\ &= \mathrm{P}\left[\bigcap_{j=1}^{m}\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\}\right]\mathrm{P}\left[\{\mathbf{N}_{t_m}=\mathbf{l}_m\}\cap\{\mathbf{N}_{t_{m+1}}-\mathbf{N}_{t_m}=\mathbf{n}_{m+1}\}\right] \\ &= \left(\prod_{i=1}^{k}\frac{l_m^{(i)}!}{\prod_{j=1}^{m}n_j^{(i)}!}\prod_{j=1}^{m}\left(\frac{t_j-t_{j-1}}{t_m}\right)^{n_j^{(i)}}\right)\mathrm{P}[\{\mathbf{N}_{t_m}=\mathbf{l}_m\}] \\ &\qquad \cdot\left(\prod_{i=1}^{k}\binom{l_{m+1}^{(i)}}{l_m^{(i)}}\left(\frac{t_m}{t_{m+1}}\right)^{l_m^{(i)}}\left(\frac{t_{m+1}-t_m}{t_{m+1}}\right)^{n_{m+1}^{(i)}}\right)\mathrm{P}\left[\{\mathbf{N}_{t_{m+1}}=\mathbf{l}_{m+1}\}\right] \\ &= \left(\prod_{i=1}^{k}\frac{l_{m+1}^{(i)}!}{\prod_{j=1}^{m+1}n_j^{(i)}!}\prod_{j=1}^{m+1}\left(\frac{t_j-t_{j-1}}{t_{m+1}}\right)^{n_j^{(i)}}\right)\mathrm{P}\left[\{\mathbf{N}_{t_{m+1}}=\mathbf{l}_{m+1}\}\right]\mathrm{P}[\{\mathbf{N}_{t_m}=\mathbf{l}_m\}] \end{aligned}$$

Since we obtain from the binomial property $\mathrm{P}[\{\mathbf{N}_{t_m}=\mathbf{l}_m\}] > 0$ (see Lemma 2.2.3), the above identity yields that $(*)$ is valid for $m+1$ time periods. Hence, the binomial property and the Markov property imply the multinomial property.

(a) $\Rightarrow$ (b): Consider $m \in \mathbb{N}$ and $t_0, t_1, \ldots, t_{m+1} \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_{m+1}$ and $\mathbf{n}_0, \mathbf{n}_1, \ldots, \mathbf{n}_{m+1} \in \mathbb{N}_0^k$. Setting $\mathbf{l}_j := \sum_{h=1}^{j} \mathbf{n}_h$ for $j \in \{1, \ldots, m+1\}$ we obtain

$$
\begin{aligned}
& \mathrm{P}\left[\bigcap_{j=1}^{m+1} \{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right] \mathrm{P}[\{\mathbf{N}_{t_m} = \mathbf{l}_m\}] \\
&= \left(\prod_{i=1}^{k} \frac{l_{m+1}^{(i)}!}{\prod_{j=1}^{m+1} n_j^{(i)}!} \prod_{j=1}^{m+1} \left(\frac{t_j - t_{j-1}}{t_{m+1}}\right)^{n_j^{(i)}}\right) \mathrm{P}\left[\left\{\mathbf{N}_{t_{m+1}} = \mathbf{l}_{m+1}\right\}\right] \mathrm{P}[\{\mathbf{N}_{t_m} = \mathbf{l}_m\}] \\
&= \left(\prod_{i=1}^{k} \frac{l_m^{(i)}!}{\prod_{j=1}^{m} n_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m}\right)^{n_j^{(i)}}\right) \mathrm{P}[\{\mathbf{N}_{t_m} = \mathbf{l}_m\}] \\
&\qquad \cdot \left(\prod_{i=1}^{k} \binom{l_{m+1}^{(i)}}{l_m^{(i)}} \left(\frac{t_m}{t_{m+1}}\right)^{l_m^{(i)}} \left(1 - \frac{t_m}{t_{m+1}}\right)^{n_{m+1}^{(i)}}\right) \mathrm{P}\left[\left\{\mathbf{N}_{t_{m+1}} = \mathbf{l}_{m+1}\right\}\right] \\
&= \mathrm{P}\left[\bigcap_{j=1}^{m} \{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right] \mathrm{P}\left[\{\mathbf{N}_{t_m} = \mathbf{l}_m\} \cap \{\mathbf{N}_{t_{m+1}} - \mathbf{N}_{t_m} = \mathbf{n}_{m+1}\}\right]
\end{aligned}
$$

Thus, $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ has the Markov property.

Now, we turn to the extended binomial property. Consider $\mathbf{t} \in \mathbb{R}_+^k$, $t \in \mathbb{R}_+$ with $\mathbf{t} \in (\mathbf{0}, t\mathbf{1})$ and $\mathbf{l}, \mathbf{n} \in \mathbb{N}_0^k$. Since the extended binomial property is $\mathcal{A}$–stable, thus especially stable under permutation ($A \in \mathcal{A}_{\mathcal{P}}$), we can without loss of generality assume $t_1 \leq t_2 \leq \ldots \leq t_k < t$. Furthermore, we use the multinomial property, as mentioned before, in a way that equal times are allowed. Finally, putting $M(i) := \left\{n_j^{(i)} \,:\, n_j^{(i)} \in \mathbb{N}_0\,,\, j \in \{1, \ldots, k+1\}, \sum_{j=1}^{i} n_j^{(i)} = l^{(i)}, \sum_{j=i+1}^{k+1} n_j^{(i)} = n^{(i)}\right\}$ and $t_{k+1} := t$ we get

$$
\begin{aligned}
& \mathrm{P}\left[\bigcap_{i=1}^{k} \left\{N_{t_i}^{(i)} = l^{(i)}\right\} \cap \left\{N_t^{(i)} - N_{t_i}^{(i)} = n^{(i)}\right\}\right] \\
&= \sum_{M(1)} \cdots \sum_{M(k)} \mathrm{P}\left[\bigcap_{j=1}^{k+1} \{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right] \\
&= \sum_{M(1)} \cdots \sum_{M(k)} \left(\prod_{i=1}^{k} \frac{(n^{(i)} + l^{(i)})!}{\prod_{j=1}^{k+1} n_j^{(i)}!} \prod_{j=1}^{k+1} \left(\frac{t_j - t_{j-1}}{t_{k+1}}\right)^{n_j^{(i)}}\right) \mathrm{P}\left[\left\{\mathbf{N}_{t_{k+1}} = \mathbf{n} + \mathbf{l}\right\}\right] \\
&= \mathrm{P}[\{\mathbf{N}_t = \mathbf{n} + \mathbf{l}\}] \prod_{i=1}^{k} \sum_{M(i)} \frac{(n^{(i)} + l^{(i)})!}{\prod_{j=1}^{k+1} n_j^{(i)}!} \prod_{j=1}^{k+1} \left(\frac{t_j - t_{j-1}}{t}\right)^{n_j^{(i)}}
\end{aligned}
$$

$$
\begin{aligned}
&= \ \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}+\mathbf{l}\}\right] \prod_{i=1}^{k} \frac{(n^{(i)}+l^{(i)})!}{l^{(i)}!\, n^{(i)}!} \left(\frac{1}{t}\right)^{l^{(i)}+n^{(i)}} t_i^{\,l^{(i)}} \, (t-t_i)^{n^{(i)}} \\
&\quad \cdot \sum_{M(i)} \left[\frac{l^{(i)}!}{\prod_{j=1}^{i} n_j^{(i)}!} \prod_{j=1}^{i} \left(\frac{t_j - t_{j-1}}{t_i}\right)^{n_j^{(i)}} \right] \left[\frac{n^{(i)}!}{\prod_{j=i+1}^{k+1} n_j^{(i)}!} \prod_{j=i+1}^{k+1} \left(\frac{t_j - t_{j-1}}{t - t_i}\right)^{n_j^{(i)}} \right] \\
&= \ \left(\prod_{i=1}^{k} \binom{n^{(i)}+l^{(i)}}{l^{(i)}} \left(\frac{t_i}{t}\right)^{l^{(i)}} \left(1-\frac{t_i}{t}\right)^{n^{(i)}} \right) \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}+\mathbf{l}\}\right]
\end{aligned}
$$

So the multinomial property implies the extended binomial property. ■

Having proven the equivalence, we know that the multinomial property implies the Chapman–Kolmogorov property. Our next aim is to show that just having the binomial property is a sufficient condition for a counting process to possess the Chapman–Kolmogorov property.

2.2.8 Lemma. *If a multivariate counting process $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ has the binomial property, then it possesses the Chapman–Kolmogorov property.*

Proof: Due to Lemma 2.2.3 we have $\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}] > 0$ for all $t > 0$ and all $\mathbf{n} \in \mathbb{N}_0^k$.
Consider $r, t \in \mathbb{R}_+$, $r \le t$, and $s \in [r,t]$ as well as $\mathbf{n}, \mathbf{m} \in \mathbb{N}_0^k$ with $\mathrm{P}[\{\mathbf{N}_r = \mathbf{n}\}] > 0$. For $s > 0$ we get

$$
\begin{aligned}
&\sum_{\substack{\mathbf{l}\in[\mathbf{0},\mathbf{m}] \\ \mathrm{P}[\{\mathbf{N}_s = \mathbf{n}+\mathbf{l}\}]>0}} \mathrm{P}\left[\{\mathbf{N}_s - \mathbf{N}_r = \mathbf{l}\} \,|\, \{\mathbf{N}_r = \mathbf{n}\}\right] \mathrm{P}\left[\{\mathbf{N}_t - \mathbf{N}_s = \mathbf{m}\} \,|\, \{\mathbf{N}_s = \mathbf{n}+\mathbf{l}\}\right] \\
&= \sum_{\mathbf{l}\in[\mathbf{0},\mathbf{m}]} \frac{\mathrm{P}\left[\{\mathbf{N}_s - \mathbf{N}_r = \mathbf{l}\} \cap \{\mathbf{N}_r = \mathbf{n}\}\right]}{\mathrm{P}\left[\{\mathbf{N}_r = \mathbf{n}\}\right]} \frac{\mathrm{P}\left[\{\mathbf{N}_t - \mathbf{N}_s = \mathbf{m}\} \cap \{\mathbf{N}_s = \mathbf{n}+\mathbf{l}\}\right]}{\mathrm{P}\left[\{\mathbf{N}_s = \mathbf{n}+\mathbf{l}\}\right]} \\
&= \sum_{\mathbf{l}\in[\mathbf{0},\mathbf{m}]} \left(\prod_{i=1}^{k} \binom{n^{(i)}+l^{(i)}}{n^{(i)}} \left(\frac{r}{s}\right)^{n^{(i)}} \left(1-\frac{r}{s}\right)^{l^{(i)}} \right) \frac{\mathrm{P}\left[\{\mathbf{N}_s = \mathbf{n}+\mathbf{l}\}\right]}{\mathrm{P}\left[\{\mathbf{N}_r = \mathbf{n}\}\right]} \\
&\qquad \cdot \left(\prod_{i=1}^{k} \binom{n^{(i)}+m^{(i)}}{n^{(i)}+l^{(i)}} \left(\frac{s}{t}\right)^{n^{(i)}+l^{(i)}} \left(1-\frac{s}{t}\right)^{m^{(i)}-l^{(i)}} \right) \frac{\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}+\mathbf{m}\}\right]}{\mathrm{P}\left[\{\mathbf{N}_s = \mathbf{n}+\mathbf{l}\}\right]} \\
&= \frac{\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}+\mathbf{m}\}\right]}{\mathrm{P}\left[\{\mathbf{N}_r = \mathbf{n}\}\right]} \left(\prod_{i=1}^{k} \binom{n^{(i)}+m^{(i)}}{n^{(i)}} r^{n^{(i)}}\, t^{-(n^{(i)}+m^{(i)})} \right) \\
&\qquad \cdot \sum_{\mathbf{l}\in[\mathbf{0},\mathbf{m}]} \prod_{i=1}^{k} \binom{m^{(i)}}{l^{(i)}} (s-r)^{l^{(i)}} \, (t-s)^{m^{(i)}-l^{(i)}} \\
&= \frac{\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}+\mathbf{m}\}\right]}{\mathrm{P}\left[\{\mathbf{N}_r = \mathbf{n}\}\right]} \left(\prod_{i=1}^{k} \binom{n^{(i)}+m^{(i)}}{n^{(i)}} \left(\frac{r}{t}\right)^{n^{(i)}} \left(1-\frac{r}{t}\right)^{m^{(i)}} \right)
\end{aligned}
$$

$$
\cdot \sum_{l=0}^{\mathbf{1}'\mathbf{m}} \left(\frac{s-r}{t-r}\right)^{l} \left(\frac{t-s}{t-r}\right)^{\mathbf{1}'\mathbf{m}-l} \sum_{\substack{\mathbf{l}\in[\mathbf{0},\mathbf{m}] \\ \mathbf{1}'\mathbf{l}=l}} \prod_{i=1}^{k} \binom{m^{(i)}}{l^{(i)}}
$$

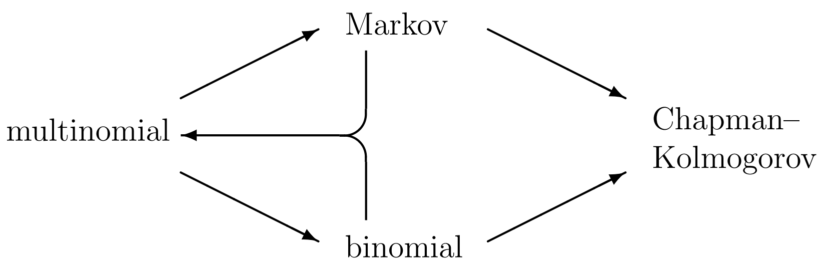
$$
\begin{aligned}
&= \frac{\mathrm{P}\left[\{\mathbf{N}_t - \mathbf{N}_r = \mathbf{m}\} \cap \{\mathbf{N}_r = \mathbf{n}\}\right]}{\mathrm{P}\left[\{\mathbf{N}_r = \mathbf{n}\}\right]} \sum_{l=0}^{\mathbf{1}'\mathbf{m}} \left(\frac{s-r}{t-r}\right)^{l} \left(\frac{t-s}{t-r}\right)^{\mathbf{1}'\mathbf{m}-l} \binom{\mathbf{1}'\mathbf{m}}{l} \\
&= \mathrm{P}\left[\{\mathbf{N}_t - \mathbf{N}_r = \mathbf{m}\} \,|\, \{\mathbf{N}_r = \mathbf{n}\}\right]
\end{aligned}
$$

The case $s = 0$ can only occur if $r = 0$ and $\mathbf{n} = \mathbf{0}$. So then

$$
\begin{aligned}
&\sum_{\substack{\mathbf{l}\in[\mathbf{0},\mathbf{m}] \\ \mathrm{P}[\{\mathbf{N}_0=\mathbf{n}+\mathbf{l}\}]>0}} \mathrm{P}\left[\{\mathbf{N}_0 - \mathbf{N}_r = \mathbf{l}\} \,|\, \{\mathbf{N}_r = \mathbf{n}\}\right] \mathrm{P}\left[\{\mathbf{N}_t - \mathbf{N}_0 = \mathbf{m}-\mathbf{l}\} \,|\, \{\mathbf{N}_0 = \mathbf{n}+\mathbf{l}\}\right] \\
&\qquad = \mathrm{P}\left[\{\mathbf{N}_0 - \mathbf{N}_0 = \mathbf{0}\} \,|\, \{\mathbf{N}_0 = \mathbf{0}\}\right] \mathrm{P}\left[\{\mathbf{N}_t - \mathbf{N}_0 = \mathbf{m}\} \,|\, \{\mathbf{N}_0 = \mathbf{0}\}\right] \\
&\qquad = \mathrm{P}\left[\{\mathbf{N}_t - \mathbf{N}_r = \mathbf{m}\} \,|\, \{\mathbf{N}_r = \mathbf{n}\}\right]
\end{aligned}
$$

holds, where the right hand side, and hence the left hand side, is in fact nothing else than $\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{m}\}\right]$. Thus, the proof is completed. ■

The following pictures recapitulates the relations between the properties of multivariate counting processes stated in the last lemmas.



On the basis of this picture we immediately see that the Chapman–Kolmogorov property cannot imply the binomial property. If the Chapman–Kolmogorov property implies the binomial property, then the Markov property would be identical with the multinomial property. Since there exist Markov processes which do not have the multinomial property, e.g. the univariate inhomogeneous Poisson process (see Schmidt [1996]) the assumption cannot be fulfilled.

In contrast to the multinomial property the Markov property, as stated after the definition, does not seem to be $\mathcal{A}$–stable. However, with Lemma 2.2.7 we can easily derive a corollary from Lemma 2.2.2 which provides a sufficient condition for the stability of the Markov property.

2.2.9 Corollary. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process having the Markov property. If $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ has the binomial property, then $\{A\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ is a Markov process for all $A \in \mathcal{A}$.*

We have shown so far that all considered properties, except the independent increments, are linked in a more or less direct way to the multinomial property. The remaining gap will be filled right now.

2.2.10 Lemma. *If a multivariate counting process has independent increments, then it has the Markov property.*

Proof: Consider $m \in \mathbb{N}$ and $t_0, t_1, \ldots, t_{m+1} \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_{m+1}$ and $\mathbf{n}_0, \mathbf{n}_1, \ldots, \mathbf{n}_{m+1} \in \mathbb{N}_0^k$ with $\mathbf{l}_m := \sum_{j=1}^m \mathbf{n}_j$. Then we have

$$
\begin{aligned}
&P\left[\bigcap_{j=1}^{m+1} \{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right] P[\{\mathbf{N}_{t_m} = \mathbf{l}_m\}] \\
&\quad = \left(\prod_{j=1}^{m+1} P\left[\{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right]\right) P[\{\mathbf{N}_{t_m} = \mathbf{l}_m\}] \\
&\quad = P\left[\bigcap_{j=1}^{m} \{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right] P\left[\{\mathbf{N}_{t_m} = \mathbf{l}_m\} \cap \{\mathbf{N}_{t_{m+1}} - \mathbf{N}_{t_m} = \mathbf{n}_{m+1}\}\right]
\end{aligned}
$$

Therefore, $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ is a Markov process. ■

So with a process with independent increments we are in a familiar setting. The next corollary is obvious.

2.2.11 Corollary. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process with independent increments and the binomial property. Then $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ has the multinomial property.*

In Theorem 3.2.6 we will see that a process with independent increments and the binomial property even has independent coordinates.

2.3 Regularity

This section is devoted to transition probabilities and transition intensities of multivariate counting processes. To proceed we need a few new notations.

Let $Z := \{(\mathbf{0}, 0)\} \cup \big(\mathbb{N}_0^k \times (0, \infty)\big)$. Then each pair $(\mathbf{n}, r) \in Z$ is called admissible. For $r, t \in \mathbb{R}_+$, $\mathbf{n}, \mathbf{m} \in \mathbb{N}_0^k$ with $r \le t$ and $(\mathbf{n}, r) \in Z$

$$
p_{\mathbf{n},\mathbf{m}}(r,t) := \begin{cases} P[\{\mathbf{N}_t = \mathbf{m}\} \mid \{\mathbf{N}_r = \mathbf{n}\}] & if \;\; P[\{\mathbf{N}_r = \mathbf{n}\}] > 0 \\ f_{\mathbf{n},\mathbf{m}}(r,t) & if \;\; P[\{\mathbf{N}_r = \mathbf{n}\}] = 0 \end{cases}
$$

are called ***transition probabilities*** of the multivariate counting process $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ where the functions $f_{\mathbf{n},\mathbf{m}}$ fulfil $f_{\mathbf{n},\mathbf{m}}(r,t) \geq 0$ and $\sum_{\mathbf{l} \in \mathbb{N}_0^k} f_{\mathbf{n},\mathbf{l}}(r,t) \leq 1$ for all $r,t \in \mathbb{R}_+$, $\mathbf{n},\mathbf{m} \in \mathbb{N}_0^k$ with $r \leq t$ and $(\mathbf{n},r) \in Z$.

The set of transition probabilities of a multivariate counting process need not to be unique. However, all subsequent results are independent of the choice of the functions $f_{\mathbf{n},\mathbf{m}}$.

Before we go deeper into the matter we want to provide the link between transition probabilities and the incremental process. It is easy to see that for $r,t \in \mathbb{R}_+$ and $\mathbf{n}$, $\mathbf{m} \in \mathbb{N}_0^k$ with $r \leq t$, $\mathbf{n} \leq \mathbf{m}$, and $\mathrm{P}\,[\{\mathbf{N}_r = \mathbf{n}\}] > 0$ the identity

$$p_{\mathbf{n},\mathbf{m}}(r,t) \quad = \quad \mathrm{P}_{r,\mathbf{n}}\,[\{\mathbf{K}_{r,t-r} = \mathbf{m} - \mathbf{n}\}]$$

is valid.

In order to formulate some results in a more uncomplicated style we extend the Chapman–Kolmogorov property to the set of transition probabilities. A set of transitions probabilities has the ***Chapman–Kolmogorov property*** if

$$p_{\mathbf{n},\mathbf{m}}(r,t) \quad = \quad \sum_{\substack{\mathbf{l} \in [\mathbf{n},\mathbf{m}] \\ (\mathbf{l},s) \in Z}} p_{\mathbf{n},\mathbf{l}}(r,s)\, p_{\mathbf{l},\mathbf{m}}(s,t)$$

holds for all $r,t \in \mathbb{R}_+$, $\mathbf{n},\mathbf{m} \in \mathbb{N}_0^k$ with $r \leq t$, $\mathbf{n} \leq \mathbf{m}$, and $(\mathbf{n},r) \in Z$ as well as for all $s \in [r,t]$.

It is obvious that if a set of transition probabilities has the Chapman–Kolmogorov property, then so has the underlying process. On the other hand, if the process has the Chapman–Kolmogorov property, then we obtain by setting all occurring $f_{\mathbf{n},\mathbf{m}} \equiv 0$ a set of transition probabilities which has the Chapman–Kolmogorov property. If $\mathrm{P}\,[\{\mathbf{N}_r = \mathbf{n}\}] > 0$ holds for all $(\mathbf{n},r) \in Z$, then both definitions are equivalent.

Before we turn to the property called regularity we state a few lemmas concerning transition probabilities.

2.3.1 Lemma. *Let $p_{\mathbf{n},\mathbf{m}}(r,t)$ be a set of transition probabilities of a multivariate counting process $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$. If $p_{\mathbf{n},\mathbf{m}}(r,t)$ has the Chapman–Kolmogorov property, then*

(1) *The inequality*

$$\big|p_{\mathbf{n},\mathbf{m}}(r,t) - p_{\mathbf{n},\mathbf{m}}(s,t)\big| \quad \leq \quad 1 - p_{\mathbf{n},\mathbf{n}}(r,s)$$

holds for all $r,s,t \in \mathbb{R}_+$, $\mathbf{n},\mathbf{m} \in \mathbb{N}_0^k$ with $r \leq s \leq t$, $\mathbf{n} \leq \mathbf{m}$, and $(\mathbf{n},r) \in Z$.

(2) *The inequality*

$$\left|p_{\mathbf{n},\mathbf{m}}(r,t) - p_{\mathbf{n},\mathbf{m}}(r,s)\right| \;\leq\; \sum_{\substack{\mathbf{l}\,\in\,[\mathbf{n},\mathbf{m}]\\ (\mathbf{l},s)\,\in\, Z}} (1 - p_{\mathbf{l},\mathbf{l}}(s,t))$$

holds for all $r,s,t \in \mathbb{R}_+$, $\mathbf{n},\mathbf{m} \in \mathbb{N}_0^k$ *with* $r \leq s \leq t$, $\mathbf{n} \leq \mathbf{m}$ *as well as* $(\mathbf{n},r) \in Z$ *and* $(\mathbf{m},s) \in Z$.

Proof:
(1): Consider $r,s,t \in \mathbb{R}_+$, $\mathbf{n},\mathbf{m} \in \mathbb{N}_0^k$ with $r \leq s \leq t$, $\mathbf{n} \leq \mathbf{m}$, and $(\mathbf{n},r) \in Z$. With $(\mathbf{n},r) \in Z$ and $r \leq s$ the relation $(\mathbf{n},s) \in Z$ follows. Thus, all occurring transition probabilities are well defined and we have

$$\begin{aligned}
p_{\mathbf{n},\mathbf{m}}(r,t) - p_{\mathbf{n},\mathbf{m}}(s,t) &\leq \sum_{\substack{\mathbf{l}\,\in\,[\mathbf{n},\mathbf{m}]\\ (\mathbf{l},s)\,\in\, Z}} p_{\mathbf{n},\mathbf{l}}(r,s)\, p_{\mathbf{l},\mathbf{m}}(s,t) - p_{\mathbf{n},\mathbf{n}}(r,s)\, p_{\mathbf{n},\mathbf{m}}(s,t) \\
&= \sum_{\substack{\mathbf{l}\,\in\,[\mathbf{n},\mathbf{m}]\setminus\{\mathbf{n}\}\\ (\mathbf{l},s)\,\in\, Z}} p_{\mathbf{n},\mathbf{l}}(r,s)\, p_{\mathbf{l},\mathbf{m}}(s,t) \\
&\leq \sum_{\substack{\mathbf{l}\,\in\,[\mathbf{n},\mathbf{m}]\setminus\{\mathbf{n}\}\\ (\mathbf{l},s)\,\in\, Z}} p_{\mathbf{n},\mathbf{l}}(r,s) \\
&\leq 1 - p_{\mathbf{n},\mathbf{n}}(r,s)
\end{aligned}$$

as well as

$$\begin{aligned}
p_{\mathbf{n},\mathbf{m}}(r,t) - p_{\mathbf{n},\mathbf{m}}(s,t) &= \sum_{\substack{\mathbf{l}\,\in\,[\mathbf{n},\mathbf{m}]\\ (\mathbf{l},s)\,\in\, Z}} p_{\mathbf{n},\mathbf{l}}(r,s)\, p_{\mathbf{l},\mathbf{m}}(s,t) - p_{\mathbf{n},\mathbf{m}}(s,t) \\
&= \sum_{\substack{\mathbf{l}\,\in\,[\mathbf{n},\mathbf{m}]\setminus\{\mathbf{m}\}\\ (\mathbf{l},s)\,\in\, Z}} p_{\mathbf{n},\mathbf{l}}(r,s)\, p_{\mathbf{l},\mathbf{m}}(s,t) + (p_{\mathbf{n},\mathbf{n}}(r,s) - 1)\, p_{\mathbf{n},\mathbf{m}}(s,t) \\
&\geq (p_{\mathbf{n},\mathbf{n}}(r,s) - 1)\, p_{\mathbf{n},\mathbf{m}}(s,t) \\
&\geq -(1 - p_{\mathbf{n},\mathbf{n}}(r,s))
\end{aligned}$$

(2): Consider $r,s,t \in \mathbb{R}_+$, $\mathbf{n},\mathbf{m} \in \mathbb{N}_0^k$ with $r \leq s \leq t$, $\mathbf{n} \leq \mathbf{m}$ as well as $(\mathbf{n},r) \in Z$ and $(\mathbf{m},s) \in Z$. Then we obtain

$$\begin{aligned}
p_{\mathbf{n},\mathbf{m}}(r,t) - p_{\mathbf{n},\mathbf{m}}(r,s) &\leq \sum_{\substack{\mathbf{l}\,\in\,[\mathbf{n},\mathbf{m}]\\ (\mathbf{l},s)\,\in\, Z}} p_{\mathbf{n},\mathbf{l}}(r,s)\, p_{\mathbf{l},\mathbf{m}}(s,t) \\
&\leq \sum_{\substack{\mathbf{l}\,\in\,[\mathbf{n},\mathbf{m}]\\ (\mathbf{l},s)\,\in\, Z}} p_{\mathbf{l},\mathbf{m}}(s,t) \\
&\leq \sum_{\substack{\mathbf{l}\,\in\,[\mathbf{n},\mathbf{m}]\\ (\mathbf{l},s)\,\in\, Z}} (1 - p_{\mathbf{l},\mathbf{l}}(s,t))
\end{aligned}$$

and

$$\begin{aligned}
p_{\mathbf{n},\mathbf{m}}(r,t) - p_{\mathbf{n},\mathbf{m}}(r,s) &= \sum_{\substack{\mathbf{l}\in[\mathbf{n},\mathbf{m}]\\(\mathbf{l},s)\in Z}} p_{\mathbf{n},\mathbf{l}}(r,s)\,p_{\mathbf{l},\mathbf{m}}(s,t) - p_{\mathbf{n},\mathbf{m}}(r,s)\\
&= \sum_{\substack{\mathbf{l}\in[\mathbf{n},\mathbf{m}]\setminus\{\mathbf{m}\}\\(\mathbf{l},s)\in Z}} p_{\mathbf{n},\mathbf{l}}(r,s)\,p_{\mathbf{l},\mathbf{m}}(s,t) + p_{\mathbf{n},\mathbf{m}}(r,s)\,(p_{\mathbf{m},\mathbf{m}}(s,t)-1)\\
&\geq -p_{\mathbf{n},\mathbf{m}}(r,s)\,(1-p_{\mathbf{m},\mathbf{m}}(s,t))\\
&\geq -(1-p_{\mathbf{m},\mathbf{m}}(s,t))\\
&\geq -\sum_{\substack{\mathbf{l}\in[\mathbf{n},\mathbf{m}]\\(\mathbf{l},s)\in Z}} (1-p_{\mathbf{l},\mathbf{l}}(s,t))
\end{aligned}$$

Hence, the proof is completed. ■

2.3.2 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process and let $(\mathbf{n},r)\in Z$ and $\mathbf{m}\in\mathbb{N}_0^k$. If $\mathrm{P}[\{\mathbf{N}_r=\mathbf{n}\}]>0$, then the function $p_{\mathbf{n},\mathbf{m}}(r,\cdot)\ :\ [r,\infty)\to\mathbb{R}_+$ is right continuous.*

Proof: We divide the proof into two parts. First, assume $\mathbf{n}\not\leq\mathbf{m}$. In this case we get

$$\begin{aligned}
p_{\mathbf{n},\mathbf{m}}(r,t) &= \mathrm{P}[\{\mathbf{N}_t=\mathbf{m}\}\,|\,\{\mathbf{N}_r=\mathbf{n}\}]\\
&= 0
\end{aligned}$$

for all $t\in[r,\infty)$ and $p_{\mathbf{n},\mathbf{m}}(r,\cdot)$ even is continuous.
Now, assume $\mathbf{n}\leq\mathbf{m}$. As a consequence of

$$p_{\mathbf{n},\mathbf{m}}(r,t) = \mathrm{P}_{r,\mathbf{n}}[\{\mathbf{K}_{r,t-r}=\mathbf{m}-\mathbf{n}\}]$$

Lemma 2.1.4 and Lemma 2.1.2 (2) yield the right continuity of $p_{\mathbf{n},\mathbf{m}}(r,t)$ in t on $[r,\infty)$. ■

2.3.3 Corollary. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process having the Chapman–Kolmogorov property and let $\mathrm{P}[\{\mathbf{N}_r=\mathbf{n}\}]>0$ for all $(\mathbf{n},r)\in Z$. Then the following are equivalent.*

(a) *The functions*
- $p_{\mathbf{0},\mathbf{m}}(\cdot,t)\ :\ [0,t]\to\mathbb{R}_+$ *with* $\mathbf{m}\in\mathbb{N}_0^k$, $t>0$,
- $p_{\mathbf{n},\mathbf{m}}(\cdot,t)\ :\ (0,t]\to\mathbb{R}_+$ *with* $\mathbf{n}\in\mathbb{N}_0^k\setminus\{\mathbf{0}\}$, $\mathbf{m}\in\mathbb{N}_0^k$, $t>0$, *and*
- $p_{\mathbf{n},\mathbf{m}}(r,\cdot)\ :\ [r,\infty)\to\mathbb{R}_+$ *with* $(\mathbf{n},r)\in Z$, $\mathbf{m}\in\mathbb{N}_0^k$

are continuous.

(b) *The identity*

$$\lim_{h\downarrow 0} p_{\mathbf{n},\mathbf{n}}(t-h,t) = 1$$

holds for all $t>0$ and $\mathbf{n}\in\mathbb{N}_0^k$.

Proof:
(a) $\Rightarrow$ (b): obvious

(b) $\Rightarrow$ (a): We obtain from the assumption

$$\lim_{h\downarrow 0} p_{\mathbf{n},\mathbf{n}}(t-h,t) \;=\; 1 \;=\; p_{\mathbf{n},\mathbf{n}}(t,t)$$

for $\mathbf{n} \in \mathbb{N}_0^k$ and $t > 0$ as well as from Lemma 2.3.2

$$\lim_{h\downarrow 0} p_{\mathbf{n},\mathbf{n}}(t,t+h) \;=\; p_{\mathbf{n},\mathbf{n}}(t,t) \;=\; 1$$

for all $(\mathbf{n},t) \in Z$.
Now, Lemma 2.3.1 yields the asserted continuity of the functions $p_{\mathbf{n},\mathbf{m}}(\cdot,t)$ and $p_{\mathbf{n},\mathbf{m}}(r,\cdot)$ for $\mathbf{n} \le \mathbf{m}$. Since for $r \le t$ the transition probabilities satisfy $p_{\mathbf{n},\mathbf{m}}(r,t) = 0$ whenever $\mathbf{n} \not\le \mathbf{m}$, the assertion follows. ∎

Now, we introduce the concept of regularity, which differs a little bit from the concept used in Schmidt [1996], but still permits a characterization in terms of the systems of Kolmogorov differential equations.

A counting process $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ is called ***regular*** if there exists a family $\{\boldsymbol{\kappa}_\mathbf{n}\}_{\mathbf{n}\in\mathbb{N}_0^k}$ of continuous functions with $\boldsymbol{\kappa}_\mathbf{0} : \mathbb{R}_+ \to (\mathbf{0},\boldsymbol{\infty})$ and $\boldsymbol{\kappa}_\mathbf{n} : (0,\infty) \to (\mathbf{0},\boldsymbol{\infty})$ for $\mathbf{n} \neq \mathbf{0}$ such that

(i) for all $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$ the inequality

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] > 0$$

is valid,

(ii) the functions
- $p_{\mathbf{0},\mathbf{m}}(\cdot,t) : [0,t] \to \mathbb{R}_+$ with $\mathbf{m} \in \mathbb{N}_0^k$, $t > 0$,
- $p_{\mathbf{n},\mathbf{m}}(\cdot,t) : (0,t] \to \mathbb{R}_+$ with $\mathbf{n} \in \mathbb{N}_0^k \backslash \{\mathbf{0}\}$, $\mathbf{m} \in \mathbb{N}_0^k$, $t > 0$, and
- $p_{\mathbf{n},\mathbf{m}}(r,\cdot) : [r,\infty) \to \mathbb{R}_+$ with $(\mathbf{n},r) \in Z$, $\mathbf{m} \in \mathbb{N}_0^k$

are continuous, and

(iii) for all $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$ all limits used below exist and the identities

$$\lim_{h\downarrow 0} \frac{1}{h}\Big(1 - p_{\mathbf{n},\mathbf{n}}(t-h,t)\Big) \;=\; \lim_{h\downarrow 0} \frac{1}{h}\Big(1 - p_{\mathbf{n},\mathbf{n}}(t,t+h)\Big) \;=\; \sum_{j=1}^{k} \kappa_\mathbf{n}^{(j)}(t)$$

$$\lim_{h\downarrow 0} \frac{1}{h}\, p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(t-h,t) \;=\; \lim_{h\downarrow 0} \frac{1}{h}\, p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(t,t+h) \;=\; \kappa_\mathbf{n}^{(i)}(t)$$

are fulfilled for all $i \in \{1,\ldots,k\}$, whereas $\kappa_\mathbf{n}^{(i)}$ is the i–th coordinate of $\boldsymbol{\kappa}_\mathbf{n}$.

As a consequence of (i) the transition probabilities of a regular process are unique. The functions $\boldsymbol{\kappa}_\mathbf{n}$ are called ***(transition) intensities*** of the counting process.

They show the tendency of the counting process, being in state $\mathbf{n}$, to have had a jump of height one at the corresponding coordinate in an infinitesimal time interval previous to time t as well as the tendency to have a jump of height one at the corresponding coordinate in an infinitesimal time interval after time t. Condition (iii) further means, that in any state the tendency of a jump of height one at an arbitrary coordinate is equal to the tendency of any jump.

Characterization of Regularity

The definition of regularity handles both relevant times occurring in the transition probabilities equally. Under the Chapman–Kolmogorov property it is sufficient to concentrate on one side.

2.3.4 Theorem (Characterization of regularity I). *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process having the Chapman–Kolmogorov property and let $\{\boldsymbol{\kappa}_\mathbf{n}\}_{\mathbf{n}\in\mathbb{N}_0^k}$ be a family of continuous functions with $\boldsymbol{\kappa}_\mathbf{0} : \mathbb{R}_+ \to (\mathbf{0},\boldsymbol{\infty})$ and $\boldsymbol{\kappa}_\mathbf{n} : (0,\infty) \to (\mathbf{0},\boldsymbol{\infty})$ for $\mathbf{n} \neq \mathbf{0}$. Then the following are equivalent.*

(a) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *is regular with intensities* $\{\boldsymbol{\kappa}_\mathbf{n}\}_{\mathbf{n}\in\mathbb{N}_0^k}$.

(b) *For all $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$*
   - (i) *the inequality* $\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] > 0$ *is valid,*
   - (ii) $\lim_{h\downarrow 0} p_{\mathbf{n},\mathbf{n}}(t-h,t) = 1$ *holds, and*
   - (iii) *all limits used below exist and the identities*

$$\begin{aligned}
\lim_{h\downarrow 0} \frac{1}{h}\Big(1 - p_{\mathbf{n},\mathbf{n}}(t-h,t)\Big) &= \sum_{j=1}^{k} \kappa_\mathbf{n}^{(j)}(t) \\
\lim_{h\downarrow 0} \frac{1}{h}\, p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(t-h,t) &= \kappa_\mathbf{n}^{(i)}(t)
\end{aligned}$$

   are fulfilled for all $i \in \{1,\ldots,k\}$.

(c) *For all $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$*
   - (i) *the inequality* $\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] > 0$ *is valid,*
   - (ii) *the function $p_{\mathbf{n},\mathbf{n}}(r,\cdot) : [r,\infty) \to \mathbb{R}_+$ with $(\mathbf{n},r) \in Z$ is left continuous, and*
   - (iii) *all limits used below exist and the identities*

$$\begin{aligned}
\lim_{h\downarrow 0} \frac{1}{h}\Big(1 - p_{\mathbf{n},\mathbf{n}}(t,t+h)\Big) &= \sum_{j=1}^{k} \kappa_\mathbf{n}^{(j)}(t) \\
\lim_{h\downarrow 0} \frac{1}{h}\, p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(t,t+h) &= \kappa_\mathbf{n}^{(i)}(t)
\end{aligned}$$

   are fulfilled for all $i \in \{1,\ldots,k\}$.

The conditions (bii) and (cii) are not quite symmetric. The aim is to get the continuity of the transition probabilities $p_{\mathbf{n},\mathbf{m}}(r,t)$ in r and in t, respectively. Due to the right continuity of the paths of a counting process and the upper bounds in Lemma 2.3.1, there are different assumptions which lead to the desired continuity in r and in t. The precise argumentation can be seen in the proof, which will be carried out after the two other characterizations of regularity.

The concept of regularity is closely linked with the Kolmogorov systems of differential equations. There are two systems, the backward and the forward system. As we will see, the existence of each of the systems is adjoint to the existence of a system of integral equations and is also necessary and sufficient for regularity if the process has the Chapman–Kolmogorov property. For a compact notation we define the index set $I(\mathbf{n},\mathbf{m}) := \big\{i \in \{1,\ldots,k\} \,:\, n^{(i)} < m^{(i)}\big\}$.

2.3.5 Theorem (Characterization of regularity II). *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process having the Chapman–Kolmogorov property and let $\{\boldsymbol{\kappa}_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}_0^k}$ be a family of continuous functions with $\boldsymbol{\kappa}_{\mathbf{0}} : \mathbb{R}_+ \to (\mathbf{0},\boldsymbol{\infty})$ and $\boldsymbol{\kappa}_{\mathbf{n}} : (0,\infty) \to (\mathbf{0},\boldsymbol{\infty})$ for $\mathbf{n} \neq \mathbf{0}$. Then the following are equivalent.*

(a) *$\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ is regular with intensities $\{\boldsymbol{\kappa}_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}_0^k}$.*

(b) **(backward differential equations)** *There exists a set of transition probabilities of the process $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ such that for all $t > 0$, $\mathbf{n},\mathbf{m} \in \mathbb{N}_0^k$ with $\mathbf{n} \le \mathbf{m}$ the differential equation*

$$\frac{d}{dr} p_{\mathbf{n},\mathbf{m}}(r,t) \;=\; p_{\mathbf{n},\mathbf{m}}(r,t) \sum_{i=1}^{k} \kappa_{\mathbf{n}}^{(i)}(r) - \sum_{i\in I(\mathbf{n},\mathbf{m})} \kappa_{\mathbf{n}}^{(i)}(r)\, p_{\mathbf{n}+\mathbf{e}_i,\mathbf{m}}(r,t)$$

with the final conditions

$$p_{\mathbf{n},\mathbf{m}}(t,t) \;=\; \begin{cases} 1 & \text{if } \mathbf{n}=\mathbf{m} \\ 0 & \text{if } \mathbf{n}\neq\mathbf{m} \end{cases}$$

holds for $r \in [0,t]$ if $\mathbf{n}=\mathbf{m}=\mathbf{0}$ and $r \in (0,t]$ otherwise.

(c) **(backward integral equations)** *There exists a set of transition probabilities of the process $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ such that for all $t > 0$, $\mathbf{n},\mathbf{m} \in \mathbb{N}_0^k$ with $\mathbf{n} \le \mathbf{m}$ the integral equation*

$$p_{\mathbf{n},\mathbf{m}}(r,t) \;=\; \begin{cases} e^{-\int_r^t \sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(s)\,ds} & \text{if } \mathbf{n}=\mathbf{m} \\[2ex] \displaystyle\int_r^t p_{\mathbf{n},\mathbf{n}}(r,s)\Bigg(\sum_{i\in I(\mathbf{n},\mathbf{m})} \kappa_{\mathbf{n}}^{(i)}(s)\, p_{\mathbf{n}+\mathbf{e}_i,\mathbf{m}}(s,t)\Bigg)ds & \text{if } \mathbf{n}\neq\mathbf{m} \end{cases}$$

holds for $r \in [0,t]$ if $\mathbf{n}=\mathbf{m}=\mathbf{0}$ and $r \in (0,t]$ otherwise.

2.3.6 Theorem (Characterization of regularity III). *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process having the Chapman–Kolmogorov property and let $\{\boldsymbol{\kappa}_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}_0^k}$ be a family of continuous functions with $\boldsymbol{\kappa}_{\mathbf{0}} : \mathbb{R}_+ \to (\mathbf{0},\boldsymbol{\infty})$ and $\boldsymbol{\kappa}_{\mathbf{n}} : (0,\infty) \to (\mathbf{0},\boldsymbol{\infty})$ for $\mathbf{n} \neq \mathbf{0}$. Then the following are equivalent.*

(a) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *is regular with intensities* $\{\boldsymbol{\kappa}_\mathbf{n}\}_{\mathbf{n}\in\mathbb{N}_0^k}$.

(b) **(forward differential equations)** *There exists a set of transition probabilities of the process* $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *such that for all* $r\in\mathbb{R}_+$, $\mathbf{n},\mathbf{m}\in\mathbb{N}_0^k$ *with* $\mathbf{n}\le\mathbf{m}$ *if* $r>0$ *and* $\mathbf{n}=\mathbf{m}=\mathbf{0}$ *if* $r=0$ *the differential equation*

$$\frac{d}{dt}p_{\mathbf{n},\mathbf{m}}(r,t) = \sum_{i\in I(\mathbf{n},\mathbf{m})} p_{\mathbf{n},\mathbf{m}-\mathbf{e}_i}(r,t)\,\kappa^{(i)}_{\mathbf{m}-\mathbf{e}_i}(t) - p_{\mathbf{n},\mathbf{m}}(r,t)\sum_{i=1}^{k}\kappa^{(i)}_{\mathbf{m}}(t)$$

with the initial conditions

$$p_{\mathbf{n},\mathbf{m}}(r,r) = \begin{cases} 1 & \textit{if} \quad \mathbf{n}=\mathbf{m} \\ 0 & \textit{if} \quad \mathbf{n}\neq\mathbf{m}\end{cases}$$

holds for $t\in[r,\infty)$.

(c) **(forward integral equations)** *There exists a set of transition probabilities of the process* $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *such that for all* $r\in\mathbb{R}_+$, $\mathbf{n},\mathbf{m}\in\mathbb{N}_0^k$ *with* $\mathbf{n}\le\mathbf{m}$ *if* $r>0$ *and* $\mathbf{n}=\mathbf{m}=\mathbf{0}$ *if* $r=0$ *the integral equation*

$$p_{\mathbf{n},\mathbf{m}}(r,t) = \begin{cases} e^{-\int_r^t \sum_{i=1}^k \kappa^{(i)}_{\mathbf{n}}(s)\,ds} & \textit{if} \quad \mathbf{n}=\mathbf{m} \\ \displaystyle\int_r^t \Bigg(\sum_{i\in I(\mathbf{n},\mathbf{m})} p_{\mathbf{n},\mathbf{m}-\mathbf{e}_i}(r,s)\kappa^{(i)}_{\mathbf{m}-\mathbf{e}_i}(s)\Bigg)p_{\mathbf{m},\mathbf{m}}(s,t)ds & \textit{if} \quad \mathbf{n}\neq\mathbf{m}\end{cases}$$

holds for $t\in[r,\infty)$.

The backward and the forward integral equations show that, for a regular process which has the Chapman–Kolmogorov property, the intensities do uniquely define the transition probabilities and therefore the one–dimensional distributions of the process. We will go a little bit deeper into this point after the proof. Before the proof we state a lemma we need therein.

2.3.7 Lemma. *Let* $a,b\in\mathbb{R}$ *and let* $f:[a,b]\to\mathbb{R}$ *be continuous.*

(1) *If the right derivative of* f *is continuous at* $x\in(a,b)$ *then the derivative of* f *at* x *exists and is continuous as well.*

(2) *Let the right derivative of* f *exist for all* $x\in(a,b)$. *Furthermore, let the limit of the right derivative of* f *for* $x\uparrow b$ *exist and let its value be* c. *Then the left derivative of* f *exists at* $x=b$ *and its value is* c.

Proof:

(1): see Kannan and Krueger [1996] Theorem 3.4.6

(2): see Bourbaki [2003] p.18 Proposition 6 ■

Proof: (Theorem 2.3.4, 2.3.5, and 2.3.6)

The assertions 2.3.4 (a), 2.3.5 (a), and 2.3.6 (a) are identical. So we can prove the theorems simultaneously according to the following scheme: 2.3.4 (a) $\Rightarrow$ 2.3.4 (c) $\Rightarrow$ 2.3.6 (b) $\Rightarrow$ 2.3.6 (c) $\Rightarrow$ 2.3.4 (b) $\Rightarrow$ 2.3.5 (b) $\Rightarrow$ 2.3.5 (c) $\Rightarrow$ 2.3.4 (a).

2.3.4 (a) $\Rightarrow$ 2.3.4 (c): obvious

2.3.4 (c) $\Rightarrow$ 2.3.6 (b): Let $\mathbf{n} \in \mathbb{N}_0^k$ and $r \in \mathbb{R}_+$ if $\mathbf{n} = \mathbf{0}$ and $r > 0$ otherwise as well as $t > 0$ with $t \geq r$.
By the Chapman–Kolmogorov property we get with $h > 0$

$$\begin{aligned} p_{\mathbf{n},\mathbf{n}}(r,t+h) - p_{\mathbf{n},\mathbf{n}}(r,t) &= p_{\mathbf{n},\mathbf{n}}(r,t)p_{\mathbf{n},\mathbf{n}}(t,t+h) - p_{\mathbf{n},\mathbf{n}}(r,t) \\ &= -\,p_{\mathbf{n},\mathbf{n}}(r,t)\,(1 - p_{\mathbf{n},\mathbf{n}}(t,t+h)) \end{aligned}$$

and therefore

$$\begin{aligned} \lim_{h\downarrow 0} \frac{1}{h}\Big(p_{\mathbf{n},\mathbf{n}}(r,t+h) - p_{\mathbf{n},\mathbf{n}}(r,t)\Big) &= -\,p_{\mathbf{n},\mathbf{n}}(r,t)\, \lim_{h\downarrow 0} \frac{1}{h}\Big(1 - p_{\mathbf{n},\mathbf{n}}(t,t+h)\Big) \\ &= -\,p_{\mathbf{n},\mathbf{n}}(r,t) \sum_{i=1}^{k} \kappa_{\mathbf{n}}^{(i)}(t) \end{aligned}$$

It follows from (ci) and Lemma 2.3.2 that $p_{\mathbf{n},\mathbf{n}}(r,\cdot)$ is right continuous and as a consequence of (cii) it is left continuous, too. Hence, $p_{\mathbf{n},\mathbf{n}}(r,\cdot)$ is a continuous function on $[r,\infty) \cap (0,\infty)$ with a continuous right derivative thereon. Thus, the derivative of $p_{\mathbf{n},\mathbf{n}}(r,\cdot)$ exists (see Lemma 2.3.7) and satisfies the differential equation

$$\frac{d}{dt} p_{\mathbf{n},\mathbf{n}}(r,t) \;=\; -\,p_{\mathbf{n},\mathbf{n}}(r,t) \sum_{i=1}^{k} \kappa_{\mathbf{n}}^{(i)}(t)$$

with initial condition $p_{\mathbf{n},\mathbf{n}}(r,r) = 1$.
Since the functions $p_{\mathbf{0},\mathbf{0}}(0,\cdot)$ and $\boldsymbol{\kappa}_{\mathbf{0}}$ are continuous on $[0,\infty)$ the limit for $t \downarrow 0$ of $\frac{d}{dt} p_{\mathbf{0},\mathbf{0}}(0,t)$ exists. With Lemma 2.3.7 we obtain

$$\begin{aligned} \frac{d}{dt} p_{\mathbf{0},\mathbf{0}}(0,t)\Big|_{t=0} &= \lim_{t\downarrow 0} \frac{d}{dt} p_{\mathbf{0},\mathbf{0}}(0,t) \\ &= \lim_{t\downarrow 0} \left(-\,p_{\mathbf{0},\mathbf{0}}(0,t) \sum_{i=1}^{k} \kappa_{\mathbf{0}}^{(i)}(t) \right) \\ &= -\,p_{\mathbf{0},\mathbf{0}}(0,0) \sum_{i=1}^{k} \kappa_{\mathbf{n}}^{(i)}(0) \\ &= -\sum_{i=1}^{k} \kappa_{\mathbf{n}}^{(i)}(0) \end{aligned}$$

This proves the assertion for $\mathbf{n} = \mathbf{0}$. In particular, we have

$$\begin{aligned} p_{\mathbf{n},\mathbf{n}}(r,t) &= e^{-\int_r^t \sum_{i=1}^{k} \kappa_{\mathbf{n}}^{(i)}(s)\,ds} \\ &> 0 \end{aligned}$$

Now, consider $r > 0$ and $\mathbf{n}, \mathbf{m} \in \mathbb{N}_0^k$ with $\mathbf{n} \leq \mathbf{m}$ as well as $t \in [r, \infty)$. Assume $h > 0$. For all $\mathbf{l} \leq \mathbf{m}$ with $\mathbf{1}'\mathbf{m} - \mathbf{1}'\mathbf{l} \geq 2$ we have

$$p_{\mathbf{l},\mathbf{m}}(t, t+h) \;\leq\; 1 - p_{\mathbf{l},\mathbf{l}}(t, t+h) - \sum_{i=1}^{k} p_{\mathbf{l},\mathbf{l}+\mathbf{e}_i}(t, t+h)$$

which leads together with the assumption to

$$\lim_{h \downarrow 0} \frac{1}{h}\, p_{\mathbf{l},\mathbf{m}}(t, t+h) \;=\; 0$$

Using the Chapman–Kolmogorov property we get

$$\begin{aligned}
p_{\mathbf{n},\mathbf{m}}(r, t+h) - p_{\mathbf{n},\mathbf{m}}(r, t) \;&=\; \sum_{\mathbf{l} \in [\mathbf{n},\mathbf{m}]} p_{\mathbf{n},\mathbf{l}}(r, t)\, p_{\mathbf{l},\mathbf{m}}(t, t+h) - p_{\mathbf{n},\mathbf{m}}(r, t) \\
&=\; -\, p_{\mathbf{n},\mathbf{m}}(r, t)\, (1 - p_{\mathbf{m},\mathbf{m}}(t, t+h)) + \sum_{i \in I(\mathbf{n},\mathbf{m})} p_{\mathbf{n},\mathbf{m}-\mathbf{e}_i}(r, t)\, p_{\mathbf{m}-\mathbf{e}_i,\mathbf{m}}(t, t+h) \\
&\quad + \sum_{\substack{\mathbf{l} \in [\mathbf{n},\mathbf{m}] \\ \mathbf{1}'\mathbf{m} - \mathbf{1}'\mathbf{l} \geq 2}} p_{\mathbf{n},\mathbf{l}}(r, t)\, p_{\mathbf{l},\mathbf{m}}(t, t+h)
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{h \downarrow 0} \frac{1}{h} \left(p_{\mathbf{n},\mathbf{m}}(r, t+h) - p_{\mathbf{n},\mathbf{m}}(r, t)\right) \;&=\; -\, p_{\mathbf{n},\mathbf{m}}(r, t) \lim_{h \downarrow 0} \frac{1}{h} \left(1 - p_{\mathbf{m},\mathbf{m}}(t, t+h)\right) \\
&\quad + \sum_{i \in I(\mathbf{n},\mathbf{m})} p_{\mathbf{n},\mathbf{m}-\mathbf{e}_i}(r, t) \lim_{h \downarrow 0} \frac{1}{h}\, p_{\mathbf{m}-\mathbf{e}_i,\mathbf{m}}(t, t+h) \\
&\quad + \sum_{\substack{\mathbf{l} \in [\mathbf{n},\mathbf{m}] \\ \mathbf{1}'\mathbf{m} - \mathbf{1}'\mathbf{l} \geq 2}} p_{\mathbf{n},\mathbf{l}}(r, t) \lim_{h \downarrow 0} \frac{1}{h}\, p_{\mathbf{l},\mathbf{m}}(t, t+h) \\
&=\; -\, p_{\mathbf{n},\mathbf{m}}(r, t) \sum_{i=1}^{k} \kappa_{\mathbf{m}}^{(i)}(t) + \sum_{i \in I(\mathbf{n},\mathbf{m})} p_{\mathbf{n},\mathbf{m}-\mathbf{e}_i}(r, t) \kappa_{\mathbf{m}-\mathbf{e}_i}^{(i)}(t)
\end{aligned}$$

Hence, the right derivative of $p_{\mathbf{n},\mathbf{m}}(r, \cdot)$ exists on $[r, \infty)$ and it follows that the function $p_{\mathbf{n},\mathbf{m}}(r, \cdot)$ is right continuous thereon. Furthermore, for $s \in [r, t]$ Lemma 2.3.1 and the strict positivity of $p_{\mathbf{n},\mathbf{n}}(r, t)$ for all $\mathbf{n} \in \mathbb{N}_0^k$ and $r, t > 0$ with $r \leq t$ together with the Chapman–Kolmogorov property yield

$$\begin{aligned}
\left| p_{\mathbf{n},\mathbf{m}}(r, t) - p_{\mathbf{n},\mathbf{m}}(r, s) \right| \;&\leq\; \sum_{\mathbf{l} \in [\mathbf{n},\mathbf{m}]} \left(1 - p_{\mathbf{l},\mathbf{l}}(s, t)\right) \\
&=\; \sum_{\mathbf{l} \in [\mathbf{n},\mathbf{m}]} \left(1 - \frac{p_{\mathbf{l},\mathbf{l}}(r, t)}{p_{\mathbf{l},\mathbf{l}}(r, s)}\right)
\end{aligned}$$

and thus

$$\lim_{s\uparrow t}\left|p_{\mathbf{n},\mathbf{m}}(r,t)-p_{\mathbf{n},\mathbf{m}}(r,s)\right| \;=\; 0$$

So the function $p_{\mathbf{n},\mathbf{m}}(r,\cdot)$ is also left continuous on (r,∞) and therefore continuous on $[r,\infty)$. Since we can assume that the functions $p_{\mathbf{n},\mathbf{m}-\mathbf{e}_i}(r,\cdot)$ are continuous for $i\in I(\mathbf{n},\mathbf{m})$ (which is possible since $p_{\mathbf{n},\mathbf{n}}(r,\cdot)$ is continuous and we can go up inductively by looking first at $\mathbf{m}\in\mathbb{N}_0^k$ with $\mathbf{1}'\mathbf{m}-\mathbf{1}'\mathbf{n}=1$, then at $\mathbf{m}\in\mathbb{N}_0^k$ with $\mathbf{1}'\mathbf{m}-\mathbf{1}'\mathbf{n}=2$ etc.), the representation of the right derivative of $p_{\mathbf{n},\mathbf{m}}(r,\cdot)$ yields its continuity and as consequence of Lemma 2.3.7 the derivative exist and satisfies the differential equation

$$\frac{d}{dt}p_{\mathbf{n},\mathbf{m}}(r,t) \;=\; \sum_{i\in I(\mathbf{n},\mathbf{m})} p_{\mathbf{n},\mathbf{m}-\mathbf{e}_i}(r,t)\kappa^{(i)}_{\mathbf{m}-\mathbf{e}_i}(t)-p_{\mathbf{n},\mathbf{m}}(r,t)\sum_{i=1}^{k}\kappa^{(i)}_{\mathbf{m}}(t)$$

with initial condition $p_{\mathbf{n},\mathbf{m}}(r,r)=0$.

2.3.6 (b) $\Rightarrow$ 2.3.6 (c): Let $r\in\mathbb{R}_+$ and $\mathbf{n},\mathbf{m}\in\mathbb{N}_0^k$ with $\mathbf{n}\le\mathbf{m}$.
Firstly, we notice that $p_{\mathbf{n},\mathbf{n}}(r,\cdot)$ with $\mathbf{n}=0$ if $r=0$ satisfies the differential equation

$$\frac{d}{dt}p_{\mathbf{n},\mathbf{n}}(r,t) \;=\; -p_{\mathbf{n},\mathbf{n}}(r,t)\sum_{i=1}^{k}\kappa^{(i)}_{\mathbf{n}}(t)$$

with initial condition $p_{\mathbf{n},\mathbf{n}}(r,r)=1$ for $t\in[r,\infty)$. This differential equation has the unique solution

$$p_{\mathbf{n},\mathbf{n}}(r,t) \;=\; e^{-\int_r^t \sum_{i=1}^{k}\kappa^{(i)}_{\mathbf{n}}(s)\,ds}$$

for $t\in[r,\infty)$.
If $\mathbf{n}\neq\mathbf{m}$ and $r>0$, then the function $p_{\mathbf{n},\mathbf{m}}(r,\cdot)$ with $p_{\mathbf{n},\mathbf{m}}(r,t)=0$ for all $t\in[r,\infty)$ is the unique solution of the homogeneous differential equation

$$\frac{d}{dt}p_{\mathbf{n},\mathbf{m}}(r,t) \;=\; -p_{\mathbf{n},\mathbf{m}}(r,t)\sum_{i=1}^{k}\kappa^{(i)}_{\mathbf{m}}(t)$$

for $t\in[r,\infty)$ with initial condition $p_{\mathbf{n},\mathbf{m}}(r,r)=0$. This implies that the inhomogeneous differential equation

$$\frac{d}{dt}p_{\mathbf{n},\mathbf{m}}(r,t) \;=\; \sum_{i\in I(\mathbf{n},\mathbf{m})} p_{\mathbf{n},\mathbf{m}-\mathbf{e}_i}(r,t)\kappa^{(i)}_{\mathbf{m}-\mathbf{e}_i}(t)-p_{\mathbf{n},\mathbf{m}}(r,t)\sum_{i=1}^{k}\kappa^{(i)}_{\mathbf{m}}(t)$$

for $t\in[r,\infty)$ with initial condition $p_{\mathbf{n},\mathbf{m}}(r,r)=0$ has at most one solution.
We assume that the functions $p_{\mathbf{n},\mathbf{m}-\mathbf{e}_i}(r,\cdot)$, fulfilling the inhomogeneous differential

equations, are already known for $i \in I(\mathbf{n},\mathbf{m})$ (which is possible since $p_{\mathbf{n},\mathbf{n}}(r,\cdot)$ is known and we can go up inductively with the succeeding argument by looking first at $\mathbf{m} \in \mathbb{N}_0^k$ with $\mathbf{1}'\mathbf{m} - \mathbf{1}'\mathbf{n} = 1$, then at $\mathbf{m} \in \mathbb{N}_0^k$ with $\mathbf{1}'\mathbf{m} - \mathbf{1}'\mathbf{n} = 2$ etc.). We define for $t \in [r,\infty)$

$$\hat{p}_{\mathbf{n},\mathbf{m}}(r,t) \;\; := \;\; \int_r^t \Bigg(\sum_{i\,\in\,I(\mathbf{n},\mathbf{m})} p_{\mathbf{n},\mathbf{m}-\mathbf{e}_i}(r,s)\, \kappa^{(i)}_{\mathbf{m}-\mathbf{e}_i}(s) \Bigg) p_{\mathbf{m},\mathbf{m}}(s,t)ds$$

and obtain

$$\begin{aligned}
\frac{d}{dt}\hat{p}_{\mathbf{n},\mathbf{m}}(r,t) \;\; &= \;\; \int_r^t \Bigg(\sum_{i\,\in\,I(\mathbf{n},\mathbf{m})} p_{\mathbf{n},\mathbf{m}-\mathbf{e}_i}(r,s)\, \kappa^{(i)}_{\mathbf{m}-\mathbf{e}_i}(s) \Bigg) \frac{d}{dt} p_{\mathbf{m},\mathbf{m}}(s,t)ds \\
&\quad\;\; + \sum_{i\,\in\,I(\mathbf{n},\mathbf{m})} p_{\mathbf{n},\mathbf{m}-\mathbf{e}_i}(r,t)\, \kappa^{(i)}_{\mathbf{m}-\mathbf{e}_i}(t) \\
&= \;\; \int_r^t \Bigg(\sum_{i\,\in\,I(\mathbf{n},\mathbf{m})} p_{\mathbf{n},\mathbf{m}-\mathbf{e}_i}(r,s)\, \kappa^{(i)}_{\mathbf{m}-\mathbf{e}_i}(s) \Bigg) \Bigg(- p_{\mathbf{n},\mathbf{m}}(s,t) \sum_{i=1}^k \kappa^{(i)}_{\mathbf{m}}(t) \Bigg) ds \\
&\quad\;\; + \sum_{i\,\in\,I(\mathbf{n},\mathbf{m})} p_{\mathbf{n},\mathbf{m}-\mathbf{e}_i}(r,t)\, \kappa^{(i)}_{\mathbf{m}-\mathbf{e}_i}(t) \\
&= \;\; -\,\hat{p}_{\mathbf{n},\mathbf{m}}(r,t) \sum_{i=1}^k \kappa^{(i)}_{\mathbf{m}}(t) + \sum_{i\,\in\,I(\mathbf{n},\mathbf{m})} p_{\mathbf{n},\mathbf{m}-\mathbf{e}_i}(r,t) \kappa^{(i)}_{\mathbf{m}-\mathbf{e}_i}(t)
\end{aligned}$$

as well as $\hat{p}_{\mathbf{n},\mathbf{m}}(r,r) = 0$. Thus, the function $\hat{p}_{\mathbf{n},\mathbf{m}}(r,\cdot)$ is the unique solution of the preceding inhomogeneous differential equation and so

$$p_{\mathbf{n},\mathbf{m}}(r,t) \;\; = \;\; \int_r^t \Bigg(\sum_{i\,\in\,I(\mathbf{n},\mathbf{m})} p_{\mathbf{n},\mathbf{m}-\mathbf{e}_i}(r,s)\, \kappa^{(i)}_{\mathbf{m}-\mathbf{e}_i}(s) \Bigg) p_{\mathbf{m},\mathbf{m}}(s,t)ds$$

holds for all $t \in [r,\infty)$.

2.3.6 (c) $\Rightarrow$ 2.3.4 (b): For $r,t \in \mathbb{R}_+$ with $r \le t$ we have

$$p_{\mathbf{0},\mathbf{0}}(r,t) \;\; = \;\; e^{-\int_r^t \sum_{i=1}^k \kappa^{(i)}_{\mathbf{0}}(s)\,ds}$$

Thus, the function $p_{\mathbf{0},\mathbf{0}}(r,\cdot) \,:\, [r,\infty) \to [0,1]$ is continuous. Additionally, we obtain

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{0}\}\right] \;=\; p_{\mathbf{0},\mathbf{0}}(0,t) \;>\; 0$$

for $t \in \mathbb{R}_+$. So $\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{0}\}\right] > 0$ holds for all $t \in \mathbb{R}_+$.
Now, let $r,s,t \in \mathbb{R}_+$ with $r \le s \le t$ and $\mathbf{m} \in \mathbb{N}_0^k$. From Lemma 2.3.1 (1) we get

$$\big| p_{\mathbf{0},\mathbf{m}}(r,t) - p_{\mathbf{0},\mathbf{m}}(s,t) \big| \;\; \le \;\; 1 - p_{\mathbf{0},\mathbf{0}}(r,s)$$

and thus the right continuity of the function $p_{\mathbf{0},\mathbf{m}}(\cdot,t) \,:\, [0,t) \to [0,1]$.
The next item is to show that $p_{\mathbf{n},\mathbf{m}}(r,t) > 0$ holds for all $r,t > 0$, $\mathbf{n},\mathbf{m} \in \mathbb{N}_0^k$ with $r < t$ and $\mathbf{n} \le \mathbf{m}$. Firstly, we consider $t > 0$. Then

$$p_{\mathbf{n},\mathbf{n}}(r,t) \;=\; e^{-\int_r^t \sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(s)\,ds} \;>\; 0$$

holds for all $r \in (0,t)$ and $\mathbf{n} \in \mathbb{N}_0^k$. Now, additionally choose $\mathbf{n} \in \mathbb{N}_0^k$ and assume that $p_{\mathbf{n},\mathbf{m}-\mathbf{e}_i}(r,t) > 0$ already holds for $i \in I(\mathbf{n},\mathbf{m})$ and $r \in (0,t)$ (which is possible since $p_{\mathbf{n},\mathbf{n}}(r,t) > 0$ holds for all $r \in (0,t)$ and we can go up inductively with the succeeding argument by looking first at $\mathbf{m} \in \mathbb{N}_0^k$ with $\mathbf{1}'\mathbf{m} - \mathbf{1}'\mathbf{n} = 1$, then at $\mathbf{m} \in \mathbb{N}_0^k$ with $\mathbf{1}'\mathbf{m} - \mathbf{1}'\mathbf{n} = 2$ etc.). This yields

$$p_{\mathbf{n},\mathbf{m}}(r,t) \;=\; \int_r^t \Bigg(\sum_{i \in I(\mathbf{n},\mathbf{m})} p_{\mathbf{n},\mathbf{m}-\mathbf{e}_i}(r,s)\,\kappa_{\mathbf{m}-\mathbf{e}_i}^{(i)}(s) \Bigg) p_{\mathbf{m},\mathbf{m}}(s,t)ds \;>\; 0$$

for all $r \in (0,t)$. As t and $\mathbf{n}$ have been arbitrary, $p_{\mathbf{n},\mathbf{m}}(r,t) > 0$ holds for all $r,t > 0$, $\mathbf{n},\mathbf{m} \in \mathbb{N}_0^k$ with $r < t$ and $\mathbf{n} \le \mathbf{m}$.
Now, consider $r,t > 0$, $\mathbf{m} \in \mathbb{N}_0^k$ with $r < t$ and $\mathbf{m} \ne \mathbf{0}$. Then

$$\Bigg(\sum_{i \in I(\mathbf{0},\mathbf{m})} p_{\mathbf{0},\mathbf{m}-\mathbf{e}_i}(r,s)\,\kappa_{\mathbf{m}-\mathbf{e}_i}^{(i)}(s) \Bigg) p_{\mathbf{m},\mathbf{m}}(s,t) \;>\; 0$$

holds for all $s \in (r,t)$. So the function $p_{\mathbf{0},\mathbf{m}}(\cdot,t) \,:\, [0,t) \to [0,1]$ with

$$p_{\mathbf{0},\mathbf{m}}(r,t) \;=\; \int_r^t \Bigg(\sum_{i \in I(\mathbf{0},\mathbf{m})} p_{\mathbf{0},\mathbf{m}-\mathbf{e}_i}(r,s)\,\kappa_{\mathbf{m}-\mathbf{e}_i}^{(i)}(s) \Bigg) p_{\mathbf{m},\mathbf{m}}(s,t)ds$$

for $r > 0$ is strictly positive and decreasing on the interval $(0,t)$. Thus, we get as a result of the shown right continuity of this function

$$\begin{aligned}
\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{m}\}\right] &= p_{\mathbf{0},\mathbf{m}}(0,t) \\
&= \lim_{r \downarrow 0} p_{\mathbf{0},\mathbf{m}}(r,t) \\
&= \lim_{r \downarrow 0} \int_r^t \Bigg(\sum_{i \in I(\mathbf{0},\mathbf{m})} p_{\mathbf{0},\mathbf{m}-\mathbf{e}_i}(r,s)\,\kappa_{\mathbf{m}-\mathbf{e}_i}^{(i)}(s) \Bigg) p_{\mathbf{m},\mathbf{m}}(s,t)ds \\
&> 0
\end{aligned}$$

Hence, the process fulfils (bi).
Consider $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$. Then the integral representation yields

$$p_{\mathbf{n},\mathbf{n}}(r,t) \;=\; e^{-\int_r^t \sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(s)\,ds}$$

for $r \in (0, t]$. Therefore, the function is continuous in r and (bii) is fulfilled. Now, consider $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$. Then we have

$$\lim_{h \downarrow 0} \frac{1}{h} \Big(1 - p_{\mathbf{n},\mathbf{n}}(t-h,t) \Big) = \lim_{h \downarrow 0} \frac{1}{h} \Big(1 - e^{-\int_{t-h}^{t} \sum_{i=1}^{k} \kappa_{\mathbf{n}}^{(i)}(s)\,ds} \Big) = \sum_{i=1}^{k} \kappa_{\mathbf{n}}^{(i)}(t)$$

and as a consequence of

$$\begin{aligned} p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(t-h,t) &= \int_{t-h}^{t} p_{\mathbf{n},\mathbf{n}}(t-h,s)\,\kappa_{\mathbf{n}}^{(i)}(s)\,p_{\mathbf{n}+\mathbf{e}_i,\mathbf{n}+\mathbf{e}_i}(s,t)\,ds \\ &= \int_{t-h}^{t} e^{-\int_{t-h}^{s} \sum_{i=1}^{k} \kappa_{\mathbf{n}}^{(i)}(u)\,du}\,\kappa_{\mathbf{n}}^{(i)}(s)\,e^{-\int_{s}^{t} \sum_{i=1}^{k} \kappa_{\mathbf{n}+\mathbf{e}_i}^{(i)}(u)\,du}\,ds \\ &= e^{-\int_{t-h}^{t} \sum_{i=1}^{k} \kappa_{\mathbf{n}}^{(i)}\,du} \int_{t-h}^{t} e^{\int_{s}^{t} \sum_{i=1}^{k} \left(\kappa_{\mathbf{n}}^{(i)}(u) - \kappa_{\mathbf{n}+\mathbf{e}_i}^{(i)}(u)\right) du}\,\kappa_{\mathbf{n}}^{(i)}(s)\,ds \end{aligned}$$

for $h > 0$ with $t - h > 0$ we additionally obtain

$$\lim_{h \downarrow 0} \frac{1}{h}\, p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(t-h,t) = e^0\, e^0\, \kappa_{\mathbf{n}}^{(i)}(t) = \kappa_{\mathbf{n}}^{(i)}(t)$$

Thus, the process fulfils (biii).

2.3.4 (b) $\Rightarrow$ 2.3.5 (b): Let $t > 0$, $\mathbf{n}, \mathbf{m} \in \mathbb{N}_0^k$ with $\mathbf{n} \le \mathbf{m}$ and $r \in (0, t]$. Now consider $h > 0$. For all $\mathbf{l} \ge \mathbf{n}$ with $\mathbf{1}'\mathbf{l} - \mathbf{1}'\mathbf{n} \ge 2$ we have

$$p_{\mathbf{n},\mathbf{l}}(r-h,r) \le 1 - p_{\mathbf{n},\mathbf{n}}(r-h,r) - \sum_{i=1}^{k} p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(r-h,r)$$

which leads together with the assumption to

$$\lim_{h \downarrow 0} \frac{1}{h}\, p_{\mathbf{n},\mathbf{l}}(r-h,r) = 0$$

Using the Chapman–Kolmogorov property we get

$$\begin{aligned} p_{\mathbf{n},\mathbf{m}}(r,t) - p_{\mathbf{n},\mathbf{m}}(r-h,t) &= p_{\mathbf{n},\mathbf{m}}(r,t) - \sum_{\mathbf{l} \in [\mathbf{n},\mathbf{m}]} p_{\mathbf{n},\mathbf{l}}(r-h,r)\,p_{\mathbf{l},\mathbf{m}}(r,t) \\ &= p_{\mathbf{n},\mathbf{m}}(r,t)\,(1 - p_{\mathbf{n},\mathbf{n}}(r-h,r)) - \sum_{i \in I(\mathbf{n},\mathbf{m})} p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(r-h,r)\,p_{\mathbf{n}+\mathbf{e}_i,\mathbf{m}}(r,t) \\ &\quad - \sum_{\substack{\mathbf{l} \in [\mathbf{n},\mathbf{m}] \\ \mathbf{1}'\mathbf{l} - \mathbf{1}'\mathbf{n} \ge 2}} p_{\mathbf{n},\mathbf{l}}(r-h,r)\,p_{\mathbf{l},\mathbf{m}}(r,t) \end{aligned}$$

Thus

$$
\begin{aligned}
\lim_{h\downarrow 0}\frac{1}{h}\left(p_{\mathbf{n},\mathbf{m}}(r,t)-p_{\mathbf{n},\mathbf{m}}(r-h,t)\right) &= p_{\mathbf{n},\mathbf{m}}(r,t)\lim_{h\downarrow 0}\frac{1}{h}\left(1-p_{\mathbf{n},\mathbf{n}}(r-h,r)\right)\\
&\quad -\sum_{i\in I(\mathbf{n},\mathbf{m})}\lim_{h\downarrow 0}\frac{1}{h}\,p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(r-h,r)\,p_{\mathbf{n}+\mathbf{e}_i,\mathbf{m}}(r,t)\\
&\quad -\sum_{\substack{\mathbf{l}\in[\mathbf{n},\mathbf{m}]\\ \mathbf{1}'\mathbf{l}-\mathbf{1}'\mathbf{n}\geq 2}}\lim_{h\downarrow 0}\frac{1}{h}\,p_{\mathbf{n},\mathbf{l}}(r-h,r)\,p_{\mathbf{l},\mathbf{m}}(r,t)\\
&= p_{\mathbf{n},\mathbf{m}}(r,t)\sum_{i=1}^{k}\kappa_{\mathbf{n}}^{(i)}(r)-\sum_{i\in I(\mathbf{n},\mathbf{m})}\kappa_{\mathbf{n}}^{(i)}(r)p_{\mathbf{n}+\mathbf{e}_i,\mathbf{m}}(r,t)
\end{aligned}
$$

The assumption also yields the continuity of $p_{\mathbf{u},\mathbf{v}}(\cdot,t)$ (see Corollary 2.3.3) for any $\mathbf{u},\mathbf{v}\in\mathbb{N}_0^k$. So $p_{\mathbf{n},\mathbf{m}}(\cdot,t)$ is a continuous function on $(0,t]$ with a continuous left derivative thereon. Thus, the derivative of $p_{\mathbf{n},\mathbf{m}}(\cdot,t)$ exists (see Lemma 2.3.7) and satisfies the differential equation

$$
\frac{d}{dr}p_{\mathbf{n},\mathbf{m}}(r,t) = p_{\mathbf{n},\mathbf{m}}(r,t)\sum_{i=1}^{k}\kappa_{\mathbf{n}}^{(i)}(r)-\sum_{i\in I(\mathbf{n},\mathbf{m})}\kappa_{\mathbf{n}}^{(i)}(r)p_{\mathbf{n}+\mathbf{e}_i,\mathbf{m}}(r,t)
$$

with the final condition

$$
p_{\mathbf{n},\mathbf{m}}(t,t) = \begin{cases}1 & \text{if } \mathbf{n}=\mathbf{m}\\ 0 & \text{if } \mathbf{n}\neq\mathbf{m}\end{cases}
$$

As the functions $p_{\mathbf{0},\mathbf{0}}(\cdot,t)$ and $\boldsymbol{\kappa}_{\mathbf{0}}$ are continuous on $[0,t]$, the limit of $\frac{d}{dr}p_{\mathbf{0},\mathbf{0}}(r,t)$ for $r\downarrow 0$ exists. With Lemma 2.3.7 we obtain

$$
\begin{aligned}
\frac{d}{dr}p_{\mathbf{0},\mathbf{0}}(r,t)\Big|_{r=0} &= \lim_{r\downarrow 0}\frac{d}{dr}p_{\mathbf{0},\mathbf{0}}(r,t)\\
&= \lim_{r\downarrow 0}p_{\mathbf{0},\mathbf{0}}(r,t)\sum_{i=1}^{k}\kappa_{\mathbf{0}}^{(i)}(r)\\
&= p_{\mathbf{0},\mathbf{0}}(0,t)\sum_{i=1}^{k}\kappa_{\mathbf{0}}^{(i)}(0)
\end{aligned}
$$

This proves the assertion for $\mathbf{n}=\mathbf{m}=\mathbf{0}$.

2.3.5 (b) $\Rightarrow$ 2.3.5 (c): Let $t>0$ and $\mathbf{n},\mathbf{m}\in\mathbb{N}_0^k$ with $\mathbf{n}\leq\mathbf{m}$.
Firstly, we notice that $p_{\mathbf{m},\mathbf{m}}(\cdot,t)$ satisfies the differential equation

$$
\frac{d}{dr}p_{\mathbf{m},\mathbf{m}}(r,t) = p_{\mathbf{m},\mathbf{m}}(r,t)\sum_{i=1}^{k}\kappa_{\mathbf{m}}^{(i)}(r)
$$

for $r \in [0,t]$ if $\mathbf{m} = \mathbf{0}$ and $r \in (0,t]$ otherwise with final condition $p_{\mathbf{m},\mathbf{m}}(t,t) = 1$. This differential equation has a unique solution and we get

$$p_{\mathbf{m},\mathbf{m}}(r,t) \quad = \quad e^{-\int_r^t \sum_{i=1}^k \kappa_{\mathbf{m}}^{(i)}(s)\,ds}$$

for $r \in [0,t]$ if $\mathbf{m} = \mathbf{0}$ and $r \in (0,t]$ otherwise.
If $\mathbf{n} \neq \mathbf{m}$, then the function $p_{\mathbf{n},\mathbf{m}}(\cdot,t)$ with $p_{\mathbf{n},\mathbf{m}}(r,t) = 0$ for all $r \in (0,t]$ is the unique solution of the homogeneous differential equation

$$\frac{d}{dr} p_{\mathbf{n},\mathbf{m}}(r,t) \quad = \quad p_{\mathbf{n},\mathbf{m}}(r,t) \sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(r)$$

for $r \in (0,t]$ with final condition $p_{\mathbf{n},\mathbf{m}}(t,t) = 0$. This implies that the inhomogeneous differential equation

$$\frac{d}{dr} p_{\mathbf{n},\mathbf{m}}(r,t) \quad = \quad p_{\mathbf{n},\mathbf{m}}(r,t) \sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(r) - \sum_{i \in I(\mathbf{n},\mathbf{m})} \kappa_{\mathbf{n}}^{(i)}(r) p_{\mathbf{n}+\mathbf{e}_i,\mathbf{m}}(r,t)$$

for $r \in (0,t]$ with final condition $p_{\mathbf{n},\mathbf{m}}(t,t) = 0$ has at most one solution.
We assume that the functions $p_{\mathbf{n}+\mathbf{e}_i,\mathbf{m}}(\cdot,t)$, fulfilling the inhomogeneous differential equations, are already known for $i \in I(\mathbf{n},\mathbf{m})$ (which is possible since $p_{\mathbf{m},\mathbf{m}}(\cdot,t)$ is known and we can go down inductively with the succeeding argument by looking first at $\mathbf{n} \in \mathbb{N}_0^k$ with $\mathbf{1}'\mathbf{m} - \mathbf{1}'\mathbf{n} = 1$, then at $\mathbf{n} \in \mathbb{N}_0^k$ with $\mathbf{1}'\mathbf{m} - \mathbf{1}'\mathbf{n} = 2$ etc.). We define for $r \in (0,t]$

$$\hat{p}_{\mathbf{n},\mathbf{m}}(r,t) \quad := \quad \int_r^t p_{\mathbf{n},\mathbf{n}}(r,s) \Bigg(\sum_{i \in I(\mathbf{n},\mathbf{m})} \kappa_{\mathbf{n}}^{(i)}(s)\, p_{\mathbf{n}+\mathbf{e}_i,\mathbf{m}}(s,t) \Bigg) ds$$

and get

$$\begin{aligned}
\frac{d}{dr} \hat{p}_{\mathbf{n},\mathbf{m}}(r,t) \quad &= \quad \int_r^t \frac{d}{dr} p_{\mathbf{n},\mathbf{n}}(r,s) \Bigg(\sum_{i \in I(\mathbf{n},\mathbf{m})} \kappa_{\mathbf{n}}^{(i)}(s)\, p_{\mathbf{n}+\mathbf{e}_i,\mathbf{m}}(s,t) \Bigg) ds \\
&\quad - \sum_{i \in I(\mathbf{n},\mathbf{m})} \kappa_{\mathbf{n}}^{(i)}(r)\, p_{\mathbf{n}+\mathbf{e}_i,\mathbf{m}}(r,t) \\
&= \quad \int_r^t p_{\mathbf{n},\mathbf{n}}(r,s) \sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(r) \Bigg(\sum_{i \in I(\mathbf{n},\mathbf{m})} \kappa_{\mathbf{n}}^{(i)}(s)\, p_{\mathbf{n}+\mathbf{e}_i,\mathbf{m}}(s,t) \Bigg) ds \\
&\quad - \sum_{i \in I(\mathbf{n},\mathbf{m})} \kappa_{\mathbf{n}}^{(i)}(r)\, p_{\mathbf{n}+\mathbf{e}_i,\mathbf{m}}(r,t) \\
&= \quad \hat{p}_{\mathbf{n},\mathbf{m}}(r,t) \sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(r) - \sum_{i \in I(\mathbf{n},\mathbf{m})} \kappa_{\mathbf{n}}^{(i)}(r) p_{\mathbf{n}+\mathbf{e}_i,\mathbf{m}}(r,t)
\end{aligned}$$

as well as $\hat{p}_{\mathbf{n},\mathbf{m}}(t,t) = 0$. Thus, the function $\hat{p}_{\mathbf{n},\mathbf{m}}(\cdot,t)$ is the unique solution of the preceding inhomogeneous differential equation and so

$$p_{\mathbf{n},\mathbf{m}}(r,t) \;=\; \int_r^t p_{\mathbf{n},\mathbf{n}}(r,s)\Biggl(\sum_{i\,\in\, I(\mathbf{n},\mathbf{m})} \kappa_{\mathbf{n}}^{(i)}(s)\, p_{\mathbf{n}+\mathbf{e}_i,\mathbf{m}}(s,t)\Biggr)ds$$

holds for $r \in (0,t]$.

2.3.5 (c) $\Rightarrow$ 2.3.4 (a): Consider $r,t \in \mathbb{R}_+$ with $r \le t$. We have $p_{\mathbf{0},\mathbf{0}}(r,r) = 1$ and with the integral equation

$$p_{\mathbf{0},\mathbf{0}}(r,t) \;=\; e^{-\int_r^t \sum_{i=1}^k \kappa_{\mathbf{0}}^{(i)}(s)\,ds}$$

holds for $t > r$. Thus, the function $p_{\mathbf{0},\mathbf{0}}(r,\cdot) \;:\; [r,\infty) \to [0,1]$ is continuous. Additionally, we obtain

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{0}\}\right] \;=\; p_{\mathbf{0},\mathbf{0}}(0,t) \;>\; 0$$

for $t > 0$. So $\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{0}\}\right] > 0$ holds for all $t \in \mathbb{R}_+$.
Now, let $r,s,t \in \mathbb{R}_+$ with $r \le s \le t$ and $\mathbf{m} \in \mathbb{N}_0^k$. From Lemma 2.3.1 (1) we get

$$\Bigl|p_{\mathbf{0},\mathbf{m}}(r,t) - p_{\mathbf{0},\mathbf{m}}(s,t)\Bigr| \;\le\; 1 - p_{\mathbf{0},\mathbf{0}}(r,s)$$

and thus the right continuity of the function $p_{\mathbf{0},\mathbf{m}}(\cdot,t) \;:\; [0,t) \to [0,1]$.
The next item is to show that $p_{\mathbf{n},\mathbf{m}}(r,t) > 0$ holds for all $r,t > 0$, $\mathbf{n},\mathbf{m} \in \mathbb{N}_0^k$ with $r < t$ and $\mathbf{n} \le \mathbf{m}$. Firstly, we consider $t > 0$. Then

$$p_{\mathbf{m},\mathbf{m}}(r,t) \;=\; e^{-\int_r^t \sum_{i=1}^k \kappa_{\mathbf{m}}^{(i)}(s)\,ds} \;>\; 0$$

holds for all $r \in (0,t)$ and $\mathbf{m} \in \mathbb{N}_0^k$. Now, additionally choose $\mathbf{m} \in \mathbb{N}_0^k$ and assume that $p_{\mathbf{n}+\mathbf{e}_i,\mathbf{m}}(r,t) > 0$ already holds for $i \in I(\mathbf{n},\mathbf{m})$ and $r \in (0,t)$ (which is possible since $p_{\mathbf{m},\mathbf{m}}(r,t) > 0$ holds for all $r \in (0,t)$ and we can go down inductively with the succeeding argument by looking first at $\mathbf{n} \in \mathbb{N}_0^k$ with $\mathbf{1}'\mathbf{m} - \mathbf{1}'\mathbf{n} = 1$, then at $\mathbf{n} \in \mathbb{N}_0^k$ with $\mathbf{1}'\mathbf{m} - \mathbf{1}'\mathbf{n} = 2$ etc.). This yields

$$p_{\mathbf{n},\mathbf{m}}(r,t) \;=\; \int_r^t p_{\mathbf{n},\mathbf{n}}(r,s)\Biggl(\sum_{i\,\in\, I(\mathbf{n},\mathbf{m})} \kappa_{\mathbf{n}}^{(i)}(s)\, p_{\mathbf{n}+\mathbf{e}_i,\mathbf{m}}(s,t)\Biggr)ds \;>\; 0$$

for all $r \in (0,t)$. As t and $\mathbf{m}$ have been arbitrary, $p_{\mathbf{n},\mathbf{m}}(r,t) > 0$ holds for all $r,t > 0$, $\mathbf{n},\mathbf{m} \in \mathbb{N}_0^k$ with $r < t$ and $\mathbf{n} \le \mathbf{m}$.
Now, consider $r,t > 0$, $\mathbf{m} \in \mathbb{N}_0^k$ with $r < t$ and $\mathbf{m} \ne \mathbf{0}$. Then

$$p_{\mathbf{0},\mathbf{0}}(r,s)\Biggl(\sum_{i\,\in\, I(\mathbf{0},\mathbf{m})} \kappa_{\mathbf{0}}^{(i)}(s)\, p_{\mathbf{e}_i,\mathbf{m}}(s,t)\Biggr) \;>\; 0$$

holds for all $s \in [r, t)$. So the function $p_{\mathbf{0},\mathbf{m}}(\cdot, t) \,:\, [0, t) \to [0, 1]$ with

$$p_{\mathbf{0},\mathbf{m}}(r, t) \;=\; \int_r^t p_{\mathbf{0},\mathbf{0}}(r, s)\Bigg(\sum_{i\,\in\, I(\mathbf{0},\mathbf{m})} \kappa_{\mathbf{0}}^{(i)}(s)\, p_{\mathbf{e}_i,\mathbf{m}}(s, t)\Bigg) ds$$

for $r > 0$ is strictly positive and decreasing on the interval $(0, t)$. Thus, we get as a result of the shown right continuity of this function

$$\begin{aligned}
\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{m}\}\right] &= p_{\mathbf{0},\mathbf{m}}(0, t)\\
&= \lim_{r\downarrow 0} p_{\mathbf{0},\mathbf{m}}(r, t)\\
&= \lim_{r\downarrow 0} \int_r^t p_{\mathbf{0},\mathbf{0}}(r, s)\Bigg(\sum_{i\,\in\, I(\mathbf{0},\mathbf{m})} \kappa_{\mathbf{0}}^{(i)}(s)\, p_{\mathbf{e}_i,\mathbf{m}}(s, t)\Bigg) ds\\
&> 0
\end{aligned}$$

Therefore, the process fulfils condition (i) of regularity.
Consider $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$. Then by the integral representation

$$p_{\mathbf{n},\mathbf{n}}(r, t) \;=\; e^{-\int_r^t \sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(s)\, ds}$$

holds for $r \in [0, t]$ if $\mathbf{n} = \mathbf{0}$ and $r \in (0, t]$ otherwise. This function is continuous in r and Corollary 2.3.3 yields condition (ii) of regularity.
Now, consider $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$. Then we have

$$\begin{aligned}
\lim_{h\downarrow 0} \frac{1}{h}\Big(1 - p_{\mathbf{n},\mathbf{n}}(t - h, t)\Big) &= \lim_{h\downarrow 0} \frac{1}{h}\Big(1 - e^{-\int_{t-h}^t \sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(s)\, ds}\Big)\\
&= \sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(t)
\end{aligned}$$

as well as

$$\begin{aligned}
\lim_{h\downarrow 0} \frac{1}{h}\Big(1 - p_{\mathbf{n},\mathbf{n}}(t, t + h)\Big) &= \lim_{h\downarrow 0} \frac{1}{h}\Big(1 - e^{-\int_t^{t+h} \sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(s)\, ds}\Big)\\
&= \sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(t)
\end{aligned}$$

Due to

$$\begin{aligned}
p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(t - h, t) &= \int\limits_{t-h}^{t} p_{\mathbf{n},\mathbf{n}}(t - h, s)\, \kappa_{\mathbf{n}}^{(i)}(s)\, p_{\mathbf{n}+\mathbf{e}_i,\mathbf{n}+\mathbf{e}_i}(s, t)\, ds\\
&= \int\limits_{t-h}^{t} e^{-\int_{t-h}^s \sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(u)\, du}\, \kappa_{\mathbf{n}}^{(i)}(s)\, e^{-\int_s^t \sum_{i=1}^k \kappa_{\mathbf{n}+\mathbf{e}_i}^{(i)}(u)\, du}\, ds\\
&= e^{-\int_{t-h}^t \sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}\, du} \int\limits_{t-h}^{t} e^{\int_s^t \sum_{i=1}^k \left(\kappa_{\mathbf{n}}^{(i)}(u) - \kappa_{\mathbf{n}+\mathbf{e}_i}^{(i)}(u)\right) du}\, \kappa_{\mathbf{n}}^{(i)}(s)\, ds
\end{aligned}$$

for $h > 0$ we additionally obtain

$$\begin{aligned}\lim_{h\downarrow 0}\frac{1}{h}\,p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(t-h,t) &= e^0\,e^0\,\kappa_{\mathbf{n}}^{(i)}(t)\\ &= \kappa_{\mathbf{n}}^{(i)}(t)\end{aligned}$$

On the other hand

$$\begin{aligned}p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(t,t+h) &= \int_t^{t+h} p_{\mathbf{n},\mathbf{n}}(t,s)\,\kappa_{\mathbf{n}}^{(i)}(s)\,p_{\mathbf{n}+\mathbf{e}_i,\mathbf{n}+\mathbf{e}_i}(s,t+h)\,ds\\ &= \int_t^{t+h} e^{-\int_t^s \sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(u)\,du}\,\kappa_{\mathbf{n}}^{(i)}(s)\,e^{-\int_s^{t+h}\sum_{i=1}^k \kappa_{\mathbf{n}+\mathbf{e}_i}^{(i)}(u)\,du}\,ds\\ &= e^{-\int_t^{t+h}\sum_{i=1}^k \kappa_{\mathbf{n}+\mathbf{e}_i}^{(i)}\,du}\int_t^{t+h} e^{-\int_t^s \sum_{i=1}^k \left(\kappa_{\mathbf{n}}^{(i)}(u)-\kappa_{\mathbf{n}+\mathbf{e}_i}^{(i)}(u)\right)du}\,\kappa_{\mathbf{n}}^{(i)}(s)\,ds\end{aligned}$$

for $h > 0$ gives

$$\begin{aligned}\lim_{h\downarrow 0}\frac{1}{h}\,p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(t,t+h) &= e^0\,e^0\,\kappa_{\mathbf{n}}^{(i)}(t)\\ &= \kappa_{\mathbf{n}}^{(i)}(t)\end{aligned}$$

This proves condition (iii) of regularity. ■

The one–dimensional probabilities $\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}] = p_{\mathbf{0},\mathbf{n}}(0,t)$ of a regular process which has the Chapman–Kolmogorov property do also satisfy (forward) differential equations. However, they are not necessary in the characterization of regularity and also have not for sure a counterpart considering proper integral equations. Therefore, they are stated in a corollary after the characterization.

2.3.8 Corollary. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process having the Chapman–Kolmogorov property. If the process is regular with intensities $\{\boldsymbol{\kappa}_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}_0^k}$, then the differential equation*

$$\frac{d}{dt}\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}] = \sum_{i=1}^k \mathrm{P}[\{\mathbf{N}_t = \mathbf{n}-\mathbf{e}_i\}]\,\kappa_{\mathbf{n}-\mathbf{e}_i}^{(i)}(t) - \mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}]\sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(t)$$

holds for all $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$.

Proof: Let $\mathbf{n} \in \mathbb{N}_0^k$ and $t > 0$.
Assume $h > 0$ such that $t - h > 0$. For all $\mathbf{l}$ with $\mathbf{1}'\mathbf{n} - \mathbf{1}'\mathbf{l} \geq 2$ we have

$$p_{\mathbf{l},\mathbf{n}}(t-h,t) \leq 1 - p_{\mathbf{l},\mathbf{l}}(t-h,t) - \sum_{i=1}^k p_{\mathbf{l},\mathbf{l}+\mathbf{e}_i}(t-h,t)$$

which leads together with the regularity to

$$\lim_{h\downarrow 0} \frac{1}{h}\, p_{\mathbf{l},\mathbf{n}}(t-h,t) \;=\; 0$$

Using the Chapman–Kolmogorov property we get

$$\begin{aligned}
p_{\mathbf{0},\mathbf{n}}(0,t) - p_{\mathbf{0},\mathbf{n}}(0,t-h) \;&=\; \sum_{\mathbf{l}\in[\mathbf{0},\mathbf{n}]} p_{\mathbf{0},\mathbf{l}}(0,t-h)\, p_{\mathbf{l},\mathbf{n}}(t-h,t) - p_{\mathbf{0},\mathbf{n}}(0,t-h) \\
&= \; -\, p_{\mathbf{0},\mathbf{n}}(0,t-h)\,(1 - p_{\mathbf{n},\mathbf{n}}(t-h,t)) \\
&\quad + \sum_{i\in I(\mathbf{0},\mathbf{n})} p_{\mathbf{0},\mathbf{n}-\mathbf{e}_i}(0,t-h)\, p_{\mathbf{n}-\mathbf{e}_i,\mathbf{n}}(t-h,t) \\
&\quad + \sum_{\substack{\mathbf{l}\in[\mathbf{0},\mathbf{n}] \\ \mathbf{1}'\mathbf{n}-\mathbf{1}'\mathbf{l}\geq 2}} p_{\mathbf{0},\mathbf{l}}(0,t-h)\, p_{\mathbf{l},\mathbf{n}}(t-h,t)
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{h\downarrow 0}\frac{1}{h}\,(p_{\mathbf{0},\mathbf{n}}(0,t) - p_{\mathbf{0},\mathbf{n}}(0,t-h)) \;&=\; -\lim_{h\downarrow 0} p_{\mathbf{0},\mathbf{n}}(0,t-h)\,\frac{1}{h}\,(1 - p_{\mathbf{n},\mathbf{n}}(t-h,t)) \\
&\quad + \sum_{i\in I(\mathbf{0},\mathbf{n})} \lim_{h\downarrow 0} p_{\mathbf{0},\mathbf{n}-\mathbf{e}_i}(0,t-h)\;\frac{1}{h}\, p_{\mathbf{n}-\mathbf{e}_i,\mathbf{n}}(t-h,t) \\
&\quad + \sum_{\substack{\mathbf{l}\in[\mathbf{0},\mathbf{n}] \\ \mathbf{1}'\mathbf{n}-\mathbf{1}'\mathbf{l}\geq 2}} \lim_{h\downarrow 0} p_{\mathbf{0},\mathbf{l}}(0,t-h)\,\frac{1}{h}\, p_{\mathbf{l},\mathbf{n}}(t-h,t) \\
&= \; -\, p_{\mathbf{0},\mathbf{n}}(0,t)\sum_{i=1}^{k}\kappa_{\mathbf{n}}^{(i)}(t) + \sum_{i\in I(\mathbf{0},\mathbf{n})} p_{\mathbf{0},\mathbf{n}-\mathbf{e}_i}(0,t)\kappa_{\mathbf{n}-\mathbf{e}_i}^{(i)}(t)
\end{aligned}$$

Hence, the left derivative of $p_{\mathbf{0},\mathbf{n}}(0,\cdot)$ exists on $(0,\infty)$. Under the assumption of the corollary the function $p_{\mathbf{0},\mathbf{m}}(0,\cdot)$, $\mathbf{m}\in\mathbb{N}_0^k$, is continuous on $(0,\infty)$, hence the left derivative of $p_{\mathbf{0},\mathbf{n}}(0,\cdot)$ is continuous thereon, and therefore (see Lemma 2.3.7) the derivative exist and satisfies the differential equation

$$\begin{aligned}
\frac{d}{dt}\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \;&=\; \frac{d}{dt}p_{\mathbf{0},\mathbf{n}}(0,t) \\
&= \sum_{i\in I(\mathbf{0},\mathbf{n})} p_{\mathbf{0},\mathbf{n}-\mathbf{e}_i}(0,t)\kappa_{\mathbf{n}-\mathbf{e}_i}^{(i)}(t) - p_{\mathbf{0},\mathbf{n}}(0,t)\sum_{i=1}^{k}\kappa_{\mathbf{n}}^{(i)}(t) \\
&= \sum_{i=1}^{k}\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}-\mathbf{e}_i\}\right]\,\kappa_{\mathbf{n}-\mathbf{e}_i}^{(i)}(t) - \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]\sum_{i=1}^{k}\kappa_{\mathbf{n}}^{(i)}(t)
\end{aligned}$$

where the first sum can be extended since $\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] = 0$ whenever $\mathbf{n}\not\geq\mathbf{0}$. ■

The integral equations are of some interest as they allow the inductive determination of the transition probabilities from the intensities. The direction of the recursion illustrates why the systems are called backward and forward. In both systems the transition probabilities $p_{\mathbf{n},\mathbf{n}}(r,t)$ are solely given by the intensities. Using the backward system we choose $\mathbf{m} \in \mathbb{N}_0^k$ and $t > 0$ and evaluate $p_{\mathbf{m},\mathbf{m}}(r,t)$ for all $r \in (0,t]$. Then it is possible to evaluate the function $p_{\mathbf{m}-\mathbf{e}_i,\mathbf{m}}(\cdot,t)$ for $i \in \{1,\dots,k\}$ just in terms of the intensities and $p_{\mathbf{m},\mathbf{m}}$. Thus, by reducing the first number of events in one coordinate we can inductively obtain the transition probabilities $p_{\mathbf{n},\mathbf{m}}(r,t)$ for all $\mathbf{n} \leq \mathbf{m}$. This means, going backward from the chosen number of events.

Naturally, the forward system offers another way of obtaining the transition probabilities from the intensities. There we choose $\mathbf{n} \in \mathbb{N}_0^k$ and $r > 0$ and first evaluate $p_{\mathbf{n},\mathbf{n}}(r,t)$ for $t \in [r,\infty)$. The next step is to compute the functions $p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(r,\cdot)$ for $i \in \{1,\dots,k\}$. Following this way we can inductively evaluate the transition probabilities $p_{\mathbf{n},\mathbf{m}}(r,t)$ just with the help of the intensities for all $\mathbf{m} \geq \mathbf{n}$ by going forward from the chosen number of events through increasing the second number of events in one coordinate.

Since the transition probabilities of a regular process are continuous, we also can obtain with the help of $\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}] = p_{\mathbf{0},\mathbf{n}}(0,t)$ the one–dimensional distributions of the process just in terms of the intensities. However, the iteration necessary for that purpose does not provide a short explicit representation of these probabilities.

Using the same ideas for a regular Markov process yields the following result.

2.3.9 Corollary. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process which is a regular Markov process. Then the intensities $\{\boldsymbol{\kappa}_\mathbf{n}\}_{\mathbf{n}\in\mathbb{N}_0^k}$ do uniquely determine the finite–dimensional probabilities of the process.*

Proof: Every regular Markov process has the Chapman–Kolmogorov property and thus $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ fulfils the Kolmogorov systems of differential equations. Therefore, the intensities do uniquely determine the transition probabilities, in particular the one–dimensional distributions of the process. With the help of the Markov property we have for all $m \in \mathbb{N}$ and $t_0, t_1, \dots, t_m \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \dots < t_m$ and all $\mathbf{n}_0, \mathbf{n}_1, \dots, \mathbf{n}_m \in \mathbb{N}_0^k$ with $\mathbf{l}_j := \sum_{h=1}^j \mathbf{n}_h$

$$\begin{aligned}
\mathrm{P}\left[\bigcap_{j=1}^m \{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right] &= \prod_{j=1}^m \mathrm{P}\left[\{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\} \,\middle|\, \{\mathbf{N}_{t_{j-1}} = \mathbf{l}_{j-1}\}\right] \\
&= \mathrm{P}[\{\mathbf{N}_{t_1} = \mathbf{n}_1\}] \prod_{j=2}^m p_{\mathbf{l}_{j-1},\mathbf{l}_j}(t_{j-1}, t_j)
\end{aligned}$$

and hence the intensities do uniquely determine the finite–dimensional distributions of the process. ■

The specified iteration for the transition probabilities also allows to check that every transition probability of a regular process which has the Chapman–Kolmogorov property is strictly positive.

2.3.10 Corollary. *Let* $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *be a multivariate counting process which has the Chapman–Kolmogorov property and which is regular with intensities* $\{\boldsymbol{\kappa}_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}_0^k}$. *Then*

$$p_{\mathbf{n},\mathbf{m}}(r,t) \;>\; 0$$

holds for all $r,t\in\mathbb{R}_+$, $\mathbf{n},\mathbf{m}\in\mathbb{N}_0^k$ *with* $r<t$, $\mathbf{n}\le\mathbf{m}$ *and* $(\mathbf{n},r)\in Z$.

2.3.11 Corollary. *Let* $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *be a multivariate counting process which has the Chapman–Kolmogorov property and which is regular with intensities* $\{\boldsymbol{\kappa}_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}_0^k}$. *Then*

$$\lim_{t\uparrow\infty}\int_r^t \sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(s)\,ds \;=\; \infty$$

holds for all $(\mathbf{n},r)\in Z$.

Proof: Let $(\mathbf{n},r)\in Z$ and $t\in(r,\infty)$. Due to $p_{\mathbf{n},\mathbf{n}}(r,t) = \mathrm{P}_{r,\mathbf{n}}\left[\{\mathbf{K}_{r,t-r}=\mathbf{0}\}\right]$ as well as Lemma 2.1.4 (the incremental process is a multivariate counting process, too) and Lemma 2.1.2 (4) we obtain

$$\lim_{t\uparrow\infty} p_{\mathbf{n},\mathbf{n}}(r,t) \;=\; 0$$

Since the transition probabilities fulfil under the assumption of the corollary the integral equations linked to the Kolmogorov systems of differential equations, we get

$$0 \;=\; \lim_{t\uparrow\infty} p_{\mathbf{n},\mathbf{n}}(r,t) \;=\; \lim_{t\uparrow\infty} e^{-\int_r^t \sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(s)\,ds}$$

which leads to the assertion. ■

The definition of regularity contains relations between the intensities and the transition probabilities. For $t>0$ every intensity $\kappa_{\mathbf{n}}^{(i)}(t)$ is defined as derivative of the transition probability $p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(r,t)$. But no relation to the transition probabilities is specified for $\boldsymbol{\kappa}_{\mathbf{0}}(0)$. From the integral equation we get

$$\sum_{i=1}^k \kappa_{\mathbf{0}}^{(i)}(0) \;=\; \lim_{h\downarrow 0}\frac{1}{h}\Big(1-p_{\mathbf{0},\mathbf{0}}(0,h)\Big)$$

However, to specify a relation for $\kappa_{\mathbf{0}}^{(i)}(0)$ an additional assumption is necessary.

2.3.12 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process which has the Chapman–Kolmogorov property and which is regular with intensities $\{\boldsymbol{\kappa}_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}_0^k}$. Let $i\in\{1,\dots,k\}$. If the limit $\lim_{t\downarrow 0}\boldsymbol{\kappa}_{\mathbf{e}_i}(t)$ is finite, then*

$$\kappa_{\mathbf{0}}^{(i)}(0) \;=\; \lim_{h\downarrow 0}\frac{1}{h}p_{\mathbf{0},\mathbf{e}_i}(0,h)$$

is valid.

Proof: Let $i\in\{1,\dots,k\}$. Since $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ is a counting process, the identity $p_{\mathbf{0},\mathbf{e}_i}(0,0)=0$ is valid. Furthermore, the assumptions assure that all limits used below exist and are finite and so the forward differential equations in connection with Lemma 2.3.7 yield

$$\begin{aligned}
\kappa_{\mathbf{0}}^{(i)}(0) &= \lim_{t\downarrow 0}\left(p_{\mathbf{0},\mathbf{0}}(0,t)\,\kappa_{\mathbf{0}}^{(i)}(t) - p_{\mathbf{0},\mathbf{e}_i}(0,t)\sum_{i=1}^{k}\kappa_{\mathbf{e}_i}^{(i)}(t)\right)\\
&= \lim_{t\downarrow 0}\frac{d}{dt}p_{\mathbf{0},\mathbf{e}_i}(0,t)\\
&= \lim_{h\downarrow 0}\frac{1}{h}\Big(p_{\mathbf{0},\mathbf{e}_i}(0,h) - p_{\mathbf{0},\mathbf{e}_i}(0,0)\Big)\\
&= \lim_{h\downarrow 0}\frac{1}{h}p_{\mathbf{0},\mathbf{e}_i}(0,h)
\end{aligned}$$

which proves the assertion. ■

Some aspects of regularity

Our next aim is to look whether the property of being a regular process is $\mathcal{A}$–stable. Therefore, we introduce some additional notation. The transition probabilities of the transformed process will be denoted by ${}_Ap_{\mathbf{l},\mathbf{u}}(r,t)$ whereas the intensities of the transformed process (in the case of their existence) will be denoted by $\{{}_A\boldsymbol{\kappa}_{\mathbf{l}}\}_{\mathbf{l}\in\mathbb{N}_0^d}$. Before we state the appropriate theorem just a necessary lemma.

2.3.13 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process which fulfils condition (i) and (ii) of regularity and let $A\in\mathcal{A}$. Then the inequality*

$$\mathrm{P}\left[\{A\mathbf{N}_t=\mathbf{l}\}\right] > 0$$

holds for all $t>0$ and $\mathbf{l}\in\mathbb{N}_0^d$ and the functions
– ${}_Ap_{\mathbf{0},\mathbf{u}}(\cdot,t)\;:\;[0,t]\to\mathbb{R}_+$ *with* $\mathbf{u}\in\mathbb{N}_0^d$, $t>0$,
– ${}_Ap_{\mathbf{l},\mathbf{u}}(\cdot,t)\;:\;(0,t]\to\mathbb{R}_+$ *with* $\mathbf{l}\in\mathbb{N}_0^d\backslash\{\mathbf{0}\}$, $\mathbf{u}\in\mathbb{N}_0^d$, $t>0$, *and*
– ${}_Ap_{\mathbf{l},\mathbf{u}}(r,\cdot)\;:\;[r,\infty)\to\mathbb{R}_+$ *with* $(\mathbf{l},r)\in Z$, $\mathbf{u}\in\mathbb{N}_0^d$
are continuous. This means, that the property of fulfilling condition (i) and (ii) of regularity is $\mathcal{A}$–stable.

Furthermore, the transition probabilities of the transformed process can be expressed as convex combinations of sums of the original transition probabilities such that

$$_A p_{\mathbf{l},\mathbf{u}}(r,t) \;=\; \sum_{\mathbf{n}\,\in\, A^{-1}(\{\mathbf{l}\})} \frac{\mathrm{P}\left[\{\mathbf{N}_r = \mathbf{n}\}\right]}{\sum\limits_{\mathbf{v}\,\in\, A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_r = \mathbf{v}\}\right]} \sum_{\mathbf{m}\,\in\, A^{-1}(\{\mathbf{u}\})} p_{\mathbf{n},\mathbf{m}}(r,t)$$

holds for all $r,t \in \mathbb{R}_+$, $\mathbf{l},\mathbf{u} \in \mathbb{N}_0^d$ *with* $(\mathbf{l},r) \in Z$ *and* $r \le t$.

Proof: Firstly, we notice that by condition (i) of regularity the inequality

$$\mathrm{P}\left[\{A\mathbf{N}_t = \mathbf{l}\}\right] \;=\; \sum_{\mathbf{n}\,\in\, A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \;>\; 0$$

holds for all $t > 0$ and $\mathbf{l} \in \mathbb{N}_0^d$. Thus, $\{A\mathbf{N}_t\}_{t\,\in\,\mathbb{R}_+}$ fulfils condition (i) of regularity. Now let $(\mathbf{l},r) \in Z$, $\mathbf{u} \in \mathbb{N}_0^d$, and $t \ge r$. Then

$$\begin{aligned}
_A p_{\mathbf{l},\mathbf{u}}(r,t) \;&=\; \frac{\mathrm{P}\left[\{A\mathbf{N}_r = \mathbf{l}\} \cap \{A\mathbf{N}_t - A\mathbf{N}_r = \mathbf{u} - \mathbf{l}\}\right]}{\mathrm{P}\left[\{A\mathbf{N}_r = \mathbf{l}\}\right]} \\
&=\; \sum_{\mathbf{n}\,\in\, A^{-1}(\{\mathbf{l}\})} \sum_{\mathbf{m}\,\in\, A^{-1}(\{\mathbf{u}\})} \mathrm{P}\left[\{\mathbf{N}_r = \mathbf{n}\} \cap \{\mathbf{N}_t - \mathbf{N}_r = \mathbf{m} - \mathbf{n}\}\right] \frac{1}{\mathrm{P}\left[\{A\mathbf{N}_r = \mathbf{l}\}\right]} \\
&=\; \sum_{\mathbf{n}\,\in\, A^{-1}(\{\mathbf{l}\})} \sum_{\mathbf{m}\,\in\, A^{-1}(\{\mathbf{u}\})} \mathrm{P}\left[\{\mathbf{N}_r = \mathbf{n}\}\right] p_{\mathbf{n},\mathbf{m}}(r,t) \frac{1}{\mathrm{P}\left[\{A\mathbf{N}_r = \mathbf{l}\}\right]} \\
&=\; \sum_{\mathbf{n}\,\in\, A^{-1}(\{\mathbf{l}\})} \frac{\mathrm{P}\left[\{\mathbf{N}_r = \mathbf{n}\}\right]}{\sum\limits_{\mathbf{v}\,\in\, A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_r = \mathbf{v}\}\right]} \sum_{\mathbf{m}\,\in\, A^{-1}(\{\mathbf{u}\})} p_{\mathbf{n},\mathbf{m}}(r,t)
\end{aligned}$$

holds.

If $A \in \mathcal{A}_P \cup \mathcal{A}_C$, then the transition probabilities $_A p_{\mathbf{l},\mathbf{u}}(r,t)$ are finite compositions of continuous functions regarding r as well as t and therefore $\{A\mathbf{N}_t\}_{t\,\in\,\mathbb{R}_+}$ fulfils condition (ii) of regularity.

If $A \in \mathcal{A}_S$, then we have to sum infinite many functions. For each $t \in \mathbb{R}_+$ and each $\varepsilon > 0$ there exists $\mathbf{n}^* \in \mathbb{N}_0^k$ such that $\sum_{\mathbf{n}\,\in\,[\mathbf{n}^*,\boldsymbol{\infty})} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] < \varepsilon$. Thus, with the almost surely increasing paths of a counting process we obtain for all $s \in [0,t]$

$$\begin{aligned}
\sum_{\mathbf{n}\,\in\,[\mathbf{n}^*,\boldsymbol{\infty})} \mathrm{P}\left[\{\mathbf{N}_s = \mathbf{n}\}\right] \;&=\; \mathrm{P}\left[\{\mathbf{N}_s \ge \mathbf{n}^*\}\right] \\
&\le\; \mathrm{P}\left[\{\mathbf{N}_t \ge \mathbf{n}^*\}\right] \\
&<\; \varepsilon
\end{aligned}$$

Hence $\sum_{\mathbf{n}\,\in\,\mathbb{N}_0^k} \mathrm{P}\left[\{\mathbf{N}_s = \mathbf{n}\}\right]$ and therewith $\sum_{\mathbf{n}\,\in\,A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_s = \mathbf{n}\}\right]$ for all $\mathbf{l} \in \mathbb{N}_0^d$ converge uniformly on $[0,t]$. Using condition (ii) of regularity for $\{\mathbf{N}_t\}_{t\,\in\,\mathbb{R}_+}$ we get the continuity of

$$P\left[\{A\mathbf{N}_t = \mathbf{l}\}\right] \;=\; \sum_{\mathbf{n}\,\in\,A^{-1}(\{\mathbf{l}\})} P\left[\{\mathbf{N}_t = \mathbf{n}\}\right]$$

for all $t \in \mathbb{R}_+$.
Now consider $(\mathbf{l},\mathbf{r}) \in Z$ and $\mathbf{u} \in \mathbb{N}_0^d$. For each $r \in \mathbb{R}_+$ and each $\varepsilon > 0$ there exists $\mathbf{n}^* \in \mathbb{N}_0^k$ such that $\sum_{\mathbf{n}\,\in\,[\mathbf{n}^*,\boldsymbol{\infty})} P\left[\{\mathbf{N}_r = \mathbf{n}\}\right] < \varepsilon$. Hence

$$\begin{aligned}
&\sum_{\substack{\mathbf{n}\,\in\,A^{-1}(\{\mathbf{l}\})\\ \mathbf{n}\,\in\,[\mathbf{n}^*,\boldsymbol{\infty})}} \sum_{\mathbf{m}\,\in\,A^{-1}(\{\mathbf{u}\})} P\left[\{\mathbf{N}_r = \mathbf{n}\} \cap \{\mathbf{N}_t = \mathbf{m}\}\right] \\
&\qquad = \sum_{\substack{\mathbf{n}\,\in\,A^{-1}(\{\mathbf{l}\})\\ \mathbf{n}\,\in\,[\mathbf{n}^*,\boldsymbol{\infty})}} P\left[\{\mathbf{N}_r = \mathbf{n}\} \cap \{A\mathbf{N}_t = \mathbf{u}\}\right] \\
&\qquad \le \sum_{\substack{\mathbf{n}\,\in\,A^{-1}(\{\mathbf{l}\})\\ \mathbf{n}\,\in\,[\mathbf{n}^*,\boldsymbol{\infty})}} P\left[\{\mathbf{N}_r = \mathbf{n}\}\right] \\
&\qquad \le \sum_{\mathbf{n}\,\in\,[\mathbf{n}^*,\boldsymbol{\infty})} P\left[\{\mathbf{N}_r = \mathbf{n}\}\right] \\
&\qquad < \varepsilon
\end{aligned}$$

holds for all $t \in [r,\infty)$. Thus $\sum_{\mathbf{n}\,\in\,A^{-1}(\{\mathbf{l}\})} \sum_{\mathbf{m}\,\in\,A^{-1}(\{\mathbf{u}\})} P\left[\{\mathbf{N}_r = \mathbf{n}\} \cap \{\mathbf{N}_t = \mathbf{m}\}\right]$ converges uniformly for $t \in [r,\infty)$. A similar argument yields the uniform convergence for $r \in [0,t]$. Using condition (ii) again for $\{\mathbf{N}_t\}_{t\,\in\,\mathbb{R}_+}$ gives the continuity of

$$\begin{aligned}
P\left[\{A\mathbf{N}_r = \mathbf{l}\} \cap \{A\mathbf{N}_t = \mathbf{u}\}\right] &= \sum_{\mathbf{n}\,\in\,A^{-1}(\{\mathbf{l}\})} \sum_{\mathbf{m}\,\in\,A^{-1}(\{\mathbf{u}\})} P\left[\{\mathbf{N}_r = \mathbf{n}\} \cap \{\mathbf{N}_t = \mathbf{m}\}\right] \\
&= \sum_{\mathbf{n}\,\in\,A^{-1}(\{\mathbf{l}\})} \sum_{\mathbf{m}\,\in\,A^{-1}(\{\mathbf{u}\})} P\left[\{\mathbf{N}_r = \mathbf{n}\}\right] p_{\mathbf{n},\mathbf{m}}(r,t)
\end{aligned}$$

regarding r and t. Altogether

$$_{A}p_{\mathbf{l},\mathbf{u}}(r,t) \;=\; \frac{P\left[\{A\mathbf{N}_r = \mathbf{l}\} \cap \{A\mathbf{N}_t = \mathbf{u}\}\right]}{P\left[\{A\mathbf{N}_t = \mathbf{l}\}\right]}$$

has the desired continuity and $\{A\mathbf{N}_t\}_{t\,\in\,\mathbb{R}_+}$ fulfils condition (ii). ∎

2.3.14 Theorem. *Let $\{\mathbf{N}_t\}_{t\,\in\,\mathbb{R}_+}$ be a multivariate counting process which is regular with intensities $\{\boldsymbol{\kappa}_\mathbf{n}\}_{\mathbf{n}\,\in\,\mathbb{N}_0^k}$ and let $A \in \mathcal{A}$.*

(1) *If $A \in \mathcal{A}_P \cup \mathcal{A}_C$, then the process $\{A\mathbf{N}_t\}_{t\,\in\,\mathbb{R}_+}$ is regular.*

(2) *If $A \in \mathcal{A}_S$ and for all $\mathbf{l} \in \mathbb{N}_0^d$*

$$\sum_{\mathbf{n}\,\in\,A^{-1}(\{\mathbf{l}\})} P\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \sum_{j=1}^{d} \kappa_\mathbf{n}^{(j)}(t)$$

converges uniformly for $t \in \mathbb{R}_+$, then the process $\{A\mathbf{N}_t\}_{t\,\in\,\mathbb{R}_+}$ is regular.

In the case of their existence the intensities of the transformed process also have a representation as convex combination of sums of the original intensities such that

$$ {}_A\kappa_{\mathbf{l}}^{(i)}(t) \;=\; \sum_{\mathbf{n}\,\in\,A^{-1}(\{\mathbf{l}\})} \frac{\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}\}\right]}{\sum\limits_{\mathbf{v}\,\in\,A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_t=\mathbf{v}\}\right]} \sum_{\substack{j\,\in\,\{1,\ldots,k\}\\ \mathbf{e}_i{}'A\mathbf{e}_j\,=\,1}} \kappa_{\mathbf{n}}^{(j)}(t) $$

holds for all $(\mathbf{l},t)\in Z$ *and* $i\in\{1,\ldots,d\}$*.*

Proof: Due to Lemma 2.3.13 $\{A\mathbf{N}_t\}_{t\,\in\,\mathbb{R}_+}$ fulfils condition (i) and (ii) of regularity.
(1): Consider $t>0$. As a consequence of the regularity, we have for all $\mathbf{m}\geq\mathbf{n}$ with $\mathbf{1}'\mathbf{m}-\mathbf{1}'\mathbf{n}\geq 2$ as well as for all $\mathbf{m}\ngeq\mathbf{n}$

$$ p_{\mathbf{n},\mathbf{m}}(t,t+h) \;\leq\; 1-p_{\mathbf{n},\mathbf{n}}(t,t+h)-\sum_{i=1}^{k} p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(t,t+h) $$

for all $h>0$ and

$$ p_{\mathbf{n},\mathbf{m}}(t-h,t) \;\leq\; 1-p_{\mathbf{n},\mathbf{n}}(t-h,t)-\sum_{i=1}^{k} p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(t-h,t) $$

for all $h>0$ with $t-h>0$ and hence

$$ \lim_{h\,\downarrow\,0}\frac{1}{h}\,p_{\mathbf{n},\mathbf{m}}(t,t+h) \;=\; \lim_{h\,\downarrow\,0}\frac{1}{h}\,p_{\mathbf{n},\mathbf{m}}(t-h,t) \;=\; 0 \qquad (+) $$

So Lemma 2.3.13 and regularity give

$$ \begin{aligned} &\lim_{h\,\downarrow\,0}\frac{1}{h}\;{}_A p_{\mathbf{l},\mathbf{l}+\mathbf{e}_i}(t,t+h)\\ &\quad=\; \lim_{h\,\downarrow\,0}\frac{1}{h}\sum_{\mathbf{n}\,\in\,A^{-1}(\{\mathbf{l}\})} \frac{\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}\}\right]}{\sum\limits_{\mathbf{v}\,\in\,A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_t=\mathbf{v}\}\right]} \sum_{\mathbf{m}\,\in\,A^{-1}(\{\mathbf{l}+\mathbf{e}_i\})} p_{\mathbf{n},\mathbf{m}}(t,t+h)\\ &\quad=\; \sum_{\mathbf{n}\,\in\,A^{-1}(\{\mathbf{l}\})} \frac{\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}\}\right]}{\sum\limits_{\mathbf{v}\,\in\,A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_t=\mathbf{v}\}\right]} \sum_{\mathbf{m}\,\in\,A^{-1}(\{\mathbf{l}+\mathbf{e}_i\})} \lim_{h\,\downarrow\,0}\frac{1}{h}p_{\mathbf{n},\mathbf{m}}(t,t+h)\\ &\quad=\; \sum_{\mathbf{n}\,\in\,A^{-1}(\{\mathbf{l}\})} \frac{\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}\}\right]}{\sum\limits_{\mathbf{v}\,\in\,A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_t=\mathbf{v}\}\right]} \sum_{\substack{j\,\in\,\{1,\ldots,k\}\\ \mathbf{e}_i{}'A\mathbf{e}_j\,=\,1}} \lim_{h\,\downarrow\,0}\frac{1}{h}p_{\mathbf{n},\mathbf{n}+\mathbf{e}_j}(t,t+h)\\ &\quad=\; \sum_{\mathbf{n}\,\in\,A^{-1}(\{\mathbf{l}\})} \frac{\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}\}\right]}{\sum\limits_{\mathbf{v}\,\in\,A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_t=\mathbf{v}\}\right]} \sum_{\substack{j\,\in\,\{1,\ldots,k\}\\ \mathbf{e}_i{}'A\mathbf{e}_j\,=\,1}} \lim_{h\,\downarrow\,0}\frac{1}{h}p_{\mathbf{n},\mathbf{n}+\mathbf{e}_j}(t-h,t)\\ &\quad=\; \sum_{\mathbf{n}\,\in\,A^{-1}(\{\mathbf{l}\})} \frac{\lim_{h\,\downarrow\,0}\mathrm{P}\left[\{\mathbf{N}_{t-h}=\mathbf{n}\}\right]}{\sum\limits_{\mathbf{v}\,\in\,A^{-1}(\{\mathbf{l}\})} \lim_{h\,\downarrow\,0}\mathrm{P}\left[\{\mathbf{N}_{t-h}=\mathbf{v}\}\right]} \sum_{\mathbf{m}\,\in\,A^{-1}(\{\mathbf{l}+\mathbf{e}_i\})} \lim_{h\,\downarrow\,0}\frac{1}{h}p_{\mathbf{n},\mathbf{m}}(t-h,t) \end{aligned} $$

$$
\begin{aligned}
&= \lim_{h\downarrow 0} \frac{1}{h} \sum_{\mathbf{n}\in A^{-1}(\{\mathbf{l}\})} \frac{\mathrm{P}\left[\{\mathbf{N}_{t-h}=\mathbf{n}\}\right]}{\sum\limits_{\mathbf{v}\in A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_{t-h}=\mathbf{v}\}\right]} \sum_{\mathbf{m}\in A^{-1}(\{\mathbf{l}+\mathbf{e}_i\})} p_{\mathbf{n},\mathbf{m}}(t-h,t) \\
&= \lim_{h\downarrow 0} \frac{1}{h}\ {}_Ap_{\mathbf{l},\mathbf{l}+\mathbf{e}_i}(t-h,t)
\end{aligned}
$$

for all $t>0$, $\mathbf{l}\in\mathbb{N}_0^d$ and $i\in\{1,\dots,d\}$, since there are only sums with finite many elements. Therefore, we get

$$
\begin{aligned}
{}_A\kappa_{\mathbf{l}}^{(i)}(t) &= \lim_{h\downarrow 0} \frac{1}{h}\ {}_Ap_{\mathbf{l},\mathbf{l}+\mathbf{e}_i}(t,t+h) \\
&= \sum_{\mathbf{n}\in A^{-1}(\{\mathbf{l}\})} \frac{\mathrm{P}\left[\{\mathbf{N}_{t}=\mathbf{n}\}\right]}{\sum\limits_{\mathbf{v}\in A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_{t}=\mathbf{v}\}\right]} \sum_{\substack{j\in\{1,\dots,k\}\\ \mathbf{e}_i{}'A\mathbf{e}_j=1}} \kappa_{\mathbf{n}}^{(j)}(t)
\end{aligned}
$$

and thus the continuity of ${}_A\kappa_{\mathbf{l}}^{(i)}$ on $(0,\infty)$. This identity also provides the existence and finiteness of $\lim_{t\downarrow 0}{}_A\kappa_{\mathbf{0}}^{(i)}(t)$ since we have

$$
\begin{aligned}
\lim_{t\downarrow 0}{}_A\kappa_{\mathbf{0}}^{(i)}(t) &= \lim_{t\downarrow 0} \sum_{\mathbf{n}\in A^{-1}(\{\mathbf{0}\})} \frac{\mathrm{P}\left[\{\mathbf{N}_{t}=\mathbf{n}\}\right]}{\sum\limits_{\mathbf{v}\in A^{-1}(\{\mathbf{0}\})} \mathrm{P}\left[\{\mathbf{N}_{t}=\mathbf{v}\}\right]} \sum_{\substack{j\in\{1,\dots,k\}\\ \mathbf{e}_i{}'A\mathbf{e}_j=1}} \kappa_{\mathbf{n}}^{(j)}(t) \\
&= \lim_{t\downarrow 0} \frac{\mathrm{P}\left[\{\mathbf{N}_{t}=\mathbf{0}\}\right]}{\mathrm{P}\left[\{\mathbf{N}_{t}=\mathbf{0}\}\right]} \sum_{\substack{j\in\{1,\dots,k\}\\ \mathbf{e}_i{}'A\mathbf{e}_j=1}} \kappa_{\mathbf{n}}^{(j)}(t) \\
&= \sum_{\substack{j\in\{1,\dots,k\}\\ \mathbf{e}_i{}'A\mathbf{e}_j=1}} \kappa_{\mathbf{n}}^{(j)}(0)
\end{aligned}
$$

All intensities are also finite since each sum has only finite many summands.
Again Lemma 2.3.13 and regularity, keeping also $p_{\mathbf{n},\mathbf{m}}(t,t+h)=0$ for $\mathbf{m}\not\geq\mathbf{n}$ and $h>0$ in mind, gives

$$
\begin{aligned}
\sum_{i=1}^{d}{}_A\kappa_{\mathbf{l}}^{(i)}(t) &= \sum_{i=1}^{d} \sum_{\mathbf{n}\in A^{-1}(\{\mathbf{l}\})} \frac{\mathrm{P}\left[\{\mathbf{N}_{t}=\mathbf{n}\}\right]}{\sum\limits_{\mathbf{v}\in A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_{t}=\mathbf{v}\}\right]} \sum_{\substack{j\in\{1,\dots,k\}\\ \mathbf{e}_i{}'A\mathbf{e}_j=1}} \lim_{h\downarrow 0}\frac{1}{h} p_{\mathbf{n},\mathbf{n}+\mathbf{e}_j}(t,t+h) \\
&= \sum_{\mathbf{n}\in A^{-1}(\{\mathbf{l}\})} \frac{\mathrm{P}\left[\{\mathbf{N}_{t}=\mathbf{n}\}\right]}{\sum\limits_{\mathbf{v}\in A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_{t}=\mathbf{v}\}\right]} \sum_{i=1}^{d} \sum_{\substack{j\in\{1,\dots,k\}\\ \mathbf{e}_i{}'A\mathbf{e}_j=1}} \lim_{h\downarrow 0}\frac{1}{h} p_{\mathbf{n},\mathbf{n}+\mathbf{e}_j}(t,t+h) \\
&= \sum_{\mathbf{n}\in A^{-1}(\{\mathbf{l}\})} \frac{\mathrm{P}\left[\{\mathbf{N}_{t}=\mathbf{n}\}\right]}{\sum\limits_{\mathbf{v}\in A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_{t}=\mathbf{v}\}\right]} \sum_{j=1}^{k} \lim_{h\downarrow 0}\frac{1}{h} p_{\mathbf{n},\mathbf{n}+\mathbf{e}_j}(t,t+h) \\
&= \sum_{\mathbf{n}\in A^{-1}(\{\mathbf{l}\})} \frac{\mathrm{P}\left[\{\mathbf{N}_{t}=\mathbf{n}\}\right]}{\sum\limits_{\mathbf{v}\in A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_{t}=\mathbf{v}\}\right]} \lim_{h\downarrow 0}\frac{1}{h}\Big(1-p_{\mathbf{n},\mathbf{n}}(t,t+h)\Big)
\end{aligned}
$$

$$= \sum_{\mathbf{n}\in A^{-1}(\{\mathbf{l}\})} \frac{\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}]}{\sum_{\mathbf{v}\in A^{-1}(\{\mathbf{l}\})} \mathrm{P}[\{\mathbf{N}_t = \mathbf{v}\}]} \lim_{h\downarrow 0} \frac{1}{h}\Bigl(1 - \sum_{\mathbf{m}\in A^{-1}(\{\mathbf{l}\})} p_{\mathbf{n},\mathbf{m}}(t,t+h)\Bigr)$$

$$= \lim_{h\downarrow 0} \frac{1}{h}\Biggl(1 - \sum_{\mathbf{n}\in A^{-1}(\{\mathbf{l}\})} \frac{\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}]}{\sum_{\mathbf{v}\in A^{-1}(\{\mathbf{l}\})} \mathrm{P}[\{\mathbf{N}_t = \mathbf{v}\}]} \sum_{\mathbf{m}\in A^{-1}(\{\mathbf{l}\})} p_{\mathbf{n},\mathbf{m}}(t,t+h)\Biggr)$$

$$= \lim_{h\downarrow 0} \frac{1}{h}\Bigl(1 - {}_Ap_{\mathbf{l},\mathbf{l}}(t,t+h)\Bigr)$$

as well as

$$\lim_{h\downarrow 0} \frac{1}{h}\Bigl(1 - {}_Ap_{\mathbf{l},\mathbf{l}}(t-h,t)\Bigr)$$

$$= \lim_{h\downarrow 0} \frac{1}{h}\Biggl(1 - \sum_{\mathbf{n}\in A^{-1}(\{\mathbf{l}\})} \frac{\mathrm{P}[\{\mathbf{N}_{t-h} = \mathbf{n}\}]}{\sum_{\mathbf{v}\in A^{-1}(\{\mathbf{l}\})} \mathrm{P}[\{\mathbf{N}_{t-h} = \mathbf{v}\}]} \sum_{\mathbf{m}\in A^{-1}(\{\mathbf{l}\})} p_{\mathbf{n},\mathbf{m}}(t-h,t)\Biggr)$$

$$= \sum_{\mathbf{n}\in A^{-1}(\{\mathbf{l}\})} \frac{\lim_{h\downarrow 0}\mathrm{P}[\{\mathbf{N}_{t-h} = \mathbf{n}\}]}{\sum_{\mathbf{v}\in A^{-1}(\{\mathbf{l}\})} \lim_{h\downarrow 0}\mathrm{P}[\{\mathbf{N}_{t-h} = \mathbf{v}\}]} \lim_{h\downarrow 0} \frac{1}{h}\Bigl(1 - \sum_{\mathbf{m}\in A^{-1}(\{\mathbf{l}\})} p_{\mathbf{n},\mathbf{m}}(t-h,t)\Bigr)$$

$$= \sum_{\mathbf{n}\in A^{-1}(\{\mathbf{l}\})} \frac{\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}]}{\sum_{\mathbf{v}\in A^{-1}(\{\mathbf{l}\})} \mathrm{P}[\{\mathbf{N}_t = \mathbf{v}\}]} \lim_{h\downarrow 0} \frac{1}{h}\Bigl(1 - p_{\mathbf{n},\mathbf{n}}(t-h,t)\Bigr)$$

$$= \sum_{\mathbf{n}\in A^{-1}(\{\mathbf{l}\})} \frac{\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}]}{\sum_{\mathbf{v}\in A^{-1}(\{\mathbf{l}\})} \mathrm{P}[\{\mathbf{N}_t = \mathbf{v}\}]} \lim_{h\downarrow 0} \frac{1}{h}\Bigl(1 - p_{\mathbf{n},\mathbf{n}}(t,t+h)\Bigr)$$

$$= \lim_{h\downarrow 0} \frac{1}{h}\Bigl(1 - {}_Ap_{\mathbf{l},\mathbf{l}}(t,t+h)\Bigr)$$

for all $t > 0$ and $\mathbf{l} \in \mathbb{N}_0^d$. Thus, $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ fulfils condition (iii) of regularity.

(2): Firstly, we notice that

$$\sum_{i=1}^{d} \sum_{\substack{j\in\{1,\dots,k\}\\ \mathbf{e}_i{}'A\mathbf{e}_j=1}} \lim_{h\downarrow 0} \frac{1}{h} p_{\mathbf{n},\mathbf{n}+\mathbf{e}_j}(t,t+h) = \sum_{i=1}^{d} \lim_{h\downarrow 0} \frac{1}{h} p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(t,t+h)$$

$$= \lim_{h\downarrow 0} \frac{1}{h}\bigl(1 - p_{\mathbf{n},\mathbf{n}}(t,t+h)\bigr) - \sum_{j=d+1}^{k} \lim_{h\downarrow 0} \frac{1}{h} p_{\mathbf{n},\mathbf{n}+\mathbf{e}_j}(t,t+h)$$

$$= \lim_{h\downarrow 0} \frac{1}{h}\Bigl(1 - \sum_{\mathbf{m}\in A^{-1}(\{\mathbf{l}\})} p_{\mathbf{n},\mathbf{m}}(t,t+h)\Bigr)$$

holds for all $\mathbf{n} \in A^{-1}(\{\mathbf{l}\})$ and $t > 0$ (remember (+)). Furthermore, consider $\mathbf{l} \in \mathbb{N}_0^d$

and $i \in \{1, \ldots, d\}$. By assumption

$$\sum_{\mathbf{n} \in A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \sum_{j=1}^{d} \kappa_{\mathbf{n}}^{(j)}(t)$$

and

$$\sum_{\mathbf{n} \in A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \sum_{\substack{j \in \{1,\ldots,k\} \\ \mathbf{e}_i{}' A \mathbf{e}_j = 1}} \kappa_{\mathbf{n}}^{(j)}(t) \quad = \quad \sum_{\mathbf{n} \in A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \kappa_{\mathbf{n}}^{(i)}(t)$$

converge uniformly for $t \in \mathbb{R}_+$.
Thus, the calculation in the first part of the proof, including the permutation of limits, can be transferred to the case $A \in \mathcal{A}_S$. ■

Introducing the concept of regularity immediately raises the question which kind of processes are regular. And without much surprise we find regular processes among the processes already considered.

2.3.15 Lemma. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process having the binomial property. Then there exists a family $\{\boldsymbol{\kappa}_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{N}_0^k}$ of continuous functions with $\boldsymbol{\kappa}_{\mathbf{n}} : (0, \infty) \to (\mathbf{0}, \boldsymbol{\infty})$ for all $\mathbf{n} \in \mathbb{N}_0^k$ such that for all $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$ the following is valid.*

(1) $\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] > 0$,
(2) $\lim_{h \downarrow 0} p_{\mathbf{n},\mathbf{n}}(t-h, t) = 1$, *and*
(3) *all limits used below exist and the identities*

$$\lim_{h \downarrow 0} \frac{1}{h}\Big(1 - p_{\mathbf{n},\mathbf{n}}(t-h,t)\Big) \quad = \quad \sum_{j=1}^{k} \kappa_{\mathbf{n}}^{(j)}(t)$$

$$\lim_{h \downarrow 0} \frac{1}{h}\, p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(t-h,t) \quad = \quad \kappa_{\mathbf{n}}^{(i)}(t)$$

are fulfilled for all $i \in \{1, \ldots, k\}$.

Furthermore, the family $\{\boldsymbol{\kappa}_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{N}_0^k}$ fulfils

$$\kappa_{\mathbf{n}}^{(i)}(t) \quad = \quad \frac{n^{(i)} + 1}{t}\, \frac{\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n} + \mathbf{e}_i\}\right]}{\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]}$$

Proof:
(1): The assertion is due to Lemma 2.2.3.

(2): Consider $r, t > 0$, $\mathbf{n}, \mathbf{m} \in \mathbb{N}_0^k$ with $r < t$. Using the binomial property gives for $\mathbf{n} \leq \mathbf{m}$

$$p_{\mathbf{n},\mathbf{m}}(r,t) \;=\; \left(\prod_{i=1}^{k} \binom{m^{(i)}}{n^{(i)}} \left(\frac{r}{t}\right)^{n^{(i)}} \left(1 - \frac{r}{t}\right)^{m^{(i)}-n^{(i)}} \right) \frac{\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{m}\}\right]}{\mathrm{P}\left[\{\mathbf{N}_r = \mathbf{n}\}\right]}$$

We immediately get

$$\begin{aligned} &\lim_{h \downarrow 0} p_{\mathbf{n},\mathbf{m}}(t-h,t)\, \mathrm{P}\left[\{\mathbf{N}_{t-h} = \mathbf{n}\}\right] \\ &\quad= \lim_{h \downarrow 0} \left(\prod_{i=1}^{k} \binom{m^{(i)}}{n^{(i)}} \left(\frac{t-h}{t}\right)^{n^{(i)}} \left(\frac{h}{t}\right)^{m^{(i)}-n^{(i)}} \right) \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{m}\}\right] \\ &\quad= \begin{cases} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] & \text{if } \mathbf{m} = \mathbf{n} \\ 0 & \text{else} \end{cases} \end{aligned}$$

By Fatou's lemma and the monotony of the paths of a counting process, this yields

$$\begin{aligned} \mathrm{P}\left[\{\mathbf{N}_t \geq \mathbf{n}\}\right] &= \sum_{\mathbf{m} \in [\mathbf{n},\boldsymbol{\infty})} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{m}\}\right] \\ &= \sum_{\mathbf{m} \in [\mathbf{n},\boldsymbol{\infty})} \liminf_{h \downarrow 0} p_{\mathbf{m},\mathbf{m}}(t-h,t)\, \mathrm{P}\left[\{\mathbf{N}_{t-h} = \mathbf{m}\}\right] \\ &\leq \sum_{\mathbf{m} \in [\mathbf{n},\boldsymbol{\infty})} \liminf_{h \downarrow 0} \mathrm{P}\left[\{\mathbf{N}_{t-h} = \mathbf{m}\}\right] \\ &\leq \liminf_{h \downarrow 0} \sum_{\mathbf{m} \in [\mathbf{n},\boldsymbol{\infty})} \mathrm{P}\left[\{\mathbf{N}_{t-h} = \mathbf{m}\}\right] \\ &\leq \liminf_{h \downarrow 0} \mathrm{P}\left[\{\mathbf{N}_{t-h} \geq \mathbf{n}\}\right] \\ &\leq \limsup_{h \downarrow 0} \mathrm{P}\left[\{\mathbf{N}_{t-h} \geq \mathbf{n}\}\right] \\ &\leq \limsup_{h \downarrow 0} \mathrm{P}\left[\{\mathbf{N}_{t} \geq \mathbf{n}\}\right] \\ &= \mathrm{P}\left[\{\mathbf{N}_t \geq \mathbf{n}\}\right] \end{aligned}$$

Thus

$$\mathrm{P}\left[\{\mathbf{N}_t \geq \mathbf{n}\}\right] \;=\; \lim_{h \downarrow 0} \mathrm{P}\left[\{\mathbf{N}_{t-h} \geq \mathbf{n}\}\right]$$

As $\mathbf{n}$ was arbitrary we obtain by the use of the inclusion–exclusion principle

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \;=\; \lim_{h \downarrow 0} \mathrm{P}\left[\{\mathbf{N}_{t-h} = \mathbf{n}\}\right]$$

for all $\mathbf{n} \in \mathbb{N}_0^k$. And hence with the first part of the proof

$$\lim_{h \downarrow 0} p_{\mathbf{n},\mathbf{n}}(t-h,t) \;=\; 1$$

(3): Consider $t > 0$, $\mathbf{n} \in \mathbb{N}_0^k$ and $i \in \{1, \ldots, k\}$. With (1), (2), and Corollary 2.3.3 $\mathrm{P}[\{\mathbf{N}_r = \mathbf{l}\}]$ is continuous in r on $[0, \infty)$ for all $\mathbf{l} \in \mathbb{N}_0^k$. Therewith

$$\begin{aligned}
\lim_{h \downarrow 0} \frac{1}{h} p_{\mathbf{n}, \mathbf{n}+\mathbf{e}_i}(t-h, t) &= \lim_{h \downarrow 0} \frac{1}{h} \left(\left(n^{(i)} + 1\right) \left(\frac{t-h}{t}\right)^{\mathbf{1}'\mathbf{n}} \frac{h}{t} \right) \frac{\mathrm{P}[\{\mathbf{N}_t = \mathbf{n} + \mathbf{e}_i\}]}{\mathrm{P}[\{\mathbf{N}_{t-h} = \mathbf{n}\}]} \\
&= \frac{n^{(i)} + 1}{t} \frac{\mathrm{P}[\{\mathbf{N}_t = \mathbf{n} + \mathbf{e}_i\}]}{\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}]}
\end{aligned}$$

yields the existence of a set $\{\boldsymbol{\kappa}_\mathbf{n}\}_{\mathbf{n} \in \mathbb{N}_0^k}$ of strictly positive, finite, and continuous functions on $(0, \infty)$ with

$$\kappa_\mathbf{n}^{(i)}(t) = \lim_{h \downarrow 0} \frac{1}{h} p_{\mathbf{n}, \mathbf{n}+\mathbf{e}_i}(t-h, t)$$

Taking an arbitrary $\mathbf{n} \in \mathbb{N}_0^k$, we define for $\mathbf{m} \geq \mathbf{n}$ and $h > 0$

$$\begin{aligned}
f_\mathbf{m}(h) &:= \frac{1}{h} \left(\prod_{i=1}^k \binom{m^{(i)}}{n^{(i)}} \left(\frac{t-h}{t}\right)^{n^{(i)}} \left(\frac{h}{t}\right)^{m^{(i)} - n^{(i)}} \right) \\
&= \frac{1}{t} \left(\frac{t-h}{t}\right)^{\mathbf{1}'\mathbf{n}} \left(\frac{h}{t}\right)^{\mathbf{1}'(\mathbf{m}-\mathbf{n})-1} \prod_{i=1}^k \binom{m^{(i)}}{n^{(i)}}
\end{aligned}$$

Firstly, we observe that $\lim_{h \downarrow 0} f_\mathbf{m}(h) = 0$ holds for all $\mathbf{m}$ with $\mathbf{1}'\mathbf{m} - \mathbf{1}'\mathbf{n} \geq 2$. Secondly, for every $j \in \{1, \ldots, k\}$

$$\begin{aligned}
&f_\mathbf{m}(h) - f_{\mathbf{m}+\mathbf{e}_j}(h) \\
&\quad= \left[\binom{m^{(j)}}{n^{(j)}} - \frac{h}{t} \binom{m^{(j)} + 1}{n^{(j)}} \right] \frac{1}{t} \left(\frac{t-h}{t}\right)^{\mathbf{1}'\mathbf{n}} \left(\frac{h}{t}\right)^{\mathbf{1}'(\mathbf{m}-\mathbf{n})-1} \prod_{i \neq j} \binom{m^{(i)}}{n^{(i)}} \\
&\quad= \left[\frac{m^{(j)}!\,(m^{(j)} + 1 - n^{(j)})!\, t - h\,(m^{(j)} + 1)!\,(m^{(j)} - n^{(j)})!}{n^{(j)}!\,(m^{(j)} - n^{(j)})!\,(m^{(j)} + 1 - n^{(j)})!\, t} \right] \\
&\qquad\quad \cdot \frac{1}{t} \left(\frac{t-h}{t}\right)^{\mathbf{1}'\mathbf{n}} \left(\frac{h}{t}\right)^{\mathbf{1}'(\mathbf{m}-\mathbf{n})-1} \prod_{i \neq j} \binom{m^{(i)}}{n^{(i)}} \\
&\quad\geq 0
\end{aligned}$$

if

$$h \leq t \, \frac{m^{(j)} + 1 - n^{(j)}}{m^{(j)} + 1}$$

This means together with the first argument, that there exists some $\hat{h} > 0$ such that

$$f_\mathbf{m}(h) \leq 1$$

holds for all $h \in (0,\hat{h})$ and all $\mathbf{m} \geq \mathbf{n}$ with $\mathbf{1}'\mathbf{m} - \mathbf{1}'\mathbf{n} \geq 2$. So we have

$$
\begin{aligned}
&\sum_{\substack{\mathbf{m} \in [\mathbf{n},\infty) \\ \mathbf{1}'\mathbf{m}-\mathbf{1}'\mathbf{n} \geq 2}} \frac{1}{h} p_{\mathbf{n},\mathbf{m}}(t-h,t) \\
&\quad = \sum_{\substack{\mathbf{m} \in [\mathbf{n},\infty) \\ \mathbf{1}'\mathbf{m}-\mathbf{1}'\mathbf{n} \geq 2}} \frac{1}{h} \left(\prod_{i=1}^{k} \binom{m^{(i)}}{n^{(i)}} \left(\frac{t-h}{t}\right)^{n^{(i)}} \left(\frac{h}{t}\right)^{m^{(i)}-n^{(i)}} \right) \frac{\mathrm{P}[\{\mathbf{N}_t = \mathbf{m}\}]}{\mathrm{P}[\{\mathbf{N}_{t-h} = \mathbf{n}\}]} \\
&\quad \leq \frac{1}{t\,\mathrm{P}[\{\mathbf{N}_{t-h} = \mathbf{n}\}]} \sum_{\substack{\mathbf{m} \in [\mathbf{n},\infty) \\ \mathbf{1}'\mathbf{m}-\mathbf{1}'\mathbf{n} \geq 2}} \mathrm{P}[\{\mathbf{N}_t = \mathbf{m}\}] \\
&\quad \leq \frac{1}{t\,\mathrm{P}[\{\mathbf{N}_{t-h} = \mathbf{n}\}]}
\end{aligned}
$$

for all $h \in (0,\hat{h})$, which means we can use dominated convergence for

$$
\lim_{h \downarrow 0} \sum_{\mathbf{m} \in [\mathbf{n},\infty)\setminus\{\mathbf{n}\}} \frac{1}{h} p_{\mathbf{n},\mathbf{m}}(t-h,t)
$$

Considering $\mathbf{m} \in \mathbb{N}_0^k$ with $\mathbf{m} \geq \mathbf{n}$ and $\mathbf{1}'\mathbf{m} - \mathbf{1}'\mathbf{n} \geq 2$ the binomial property also yields

$$
\lim_{h \downarrow 0} \frac{1}{h} p_{\mathbf{n},\mathbf{m}}(t-h,t) = 0
$$

Hence

$$
\begin{aligned}
\lim_{h \downarrow 0} \frac{1}{h}\big(1 - p_{\mathbf{n},\mathbf{n}}(t-h,t)\big) &= \lim_{h \downarrow 0} \sum_{\mathbf{m} \in [\mathbf{n},\infty)\setminus\{\mathbf{n}\}} \frac{1}{h} p_{\mathbf{n},\mathbf{m}}(t-h,t) \\
&= \sum_{\mathbf{m} \in [\mathbf{n},\infty)\setminus\{\mathbf{n}\}} \lim_{h \downarrow 0} \frac{1}{h} p_{\mathbf{n},\mathbf{m}}(t-h,t) \\
&= \sum_{i=1}^{k} \kappa_{\mathbf{n}}^{(i)}(t)
\end{aligned}
$$

Thus (3) is shown.

For the representation of $\kappa_{\mathbf{n}}^{(i)}(t)$ see the proof of (3). ■

2.3.16 Theorem. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process having the binomial property. Then the following are equivalent.*

(a) $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *is a regular process with intensities* $\{\boldsymbol{\kappa}_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{N}_0^k}$.

(b) $\lim_{t \downarrow 0} t^{-1}\mathrm{P}[\{\mathbf{N}_t = \mathbf{e}_i\}]$ *is finite for all* $i \in \{1,\ldots,k\}$.

If $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ satisfies one and hence all preceding items, then

$$\kappa^{(i)}_{\mathbf{n}}(t) \;=\; \frac{n^{(i)}+1}{t}\,\frac{\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}+\mathbf{e}_i\}\right]}{\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}\}\right]}$$

holds for all $t>0$, $\mathbf{n}\in\mathbb{N}_0^k$, and $i\in\{1,\ldots,k\}$.

Proof:
(a) $\Rightarrow$ (b): If $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ is regular, then the intensities are with Lemma 2.3.15 (3) of the form

$$\kappa^{(i)}_{\mathbf{n}}(t) \;=\; \frac{n^{(i)}+1}{t}\,\frac{\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}+\mathbf{e}_i\}\right]}{\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}\}\right]}$$

for all $t>0$, $\mathbf{n}\in\mathbb{N}_0^k$, and $i\in\{1,\ldots,k\}$. The right continuity of $\boldsymbol{\kappa}_{\mathbf{0}}$ at 0 together with 2.1.2 (3) yields that

$$\begin{aligned}
\lim_{t\downarrow 0} t^{-1}\mathrm{P}[\{\mathbf{N}_t=\mathbf{e}_i\}] &= \lim_{t\downarrow 0}\Big(\kappa^{(i)}_{\mathbf{0}}(t)\,\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{0}\}\right]\Big)\\
&= \lim_{t\downarrow 0}\kappa^{(i)}_{\mathbf{0}}(t)\;\cdot\;\lim_{t\downarrow 0}\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{0}\}\right]\\
&= \lim_{t\downarrow 0}\kappa^{(i)}_{\mathbf{0}}(t)\\
&= \kappa^{(i)}_{\mathbf{0}}(0)
\end{aligned}$$

is finite for all $i\in\{1,\ldots,k\}$.

(b) $\Rightarrow$ (a): If $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ has the binomial property, then it possesses the Chapman–Kolmogorov property, too (compare Lemma 2.2.8). So to show regularity we can use Theorem 2.3.4 and prove assertion (b). Lemma 2.3.15 yields all necessary properties except that $\boldsymbol{\kappa}_{\mathbf{0}}$ is only defined on $(0,\infty)$ instead of $\mathbb{R}_+$. To extend $\boldsymbol{\kappa}_{\mathbf{0}}$ continuously we need the existence and the finiteness of the limit $\lim_{t\downarrow 0}\kappa^{(i)}_{\mathbf{0}}(t)$ for all $i\in\{1,\ldots,k\}$. The assumption in connection with Lemma 2.3.15 and 2.1.2 (3) yields

$$\begin{aligned}
\lim_{t\downarrow 0}\kappa^{(i)}_{\mathbf{0}}(t) &= \lim_{t\downarrow 0}\frac{1}{t}\,\frac{\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{e}_i\}\right]}{\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{0}\}\right]}\\
&= \lim_{t\downarrow 0}\frac{\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{e}_i\}\right]}{t}\;\cdot\;\lim_{t\downarrow 0}\frac{1}{\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{0}\}\right]}\\
&= \lim_{t\downarrow 0}\frac{\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{e}_i\}\right]}{t}\\
&< \infty
\end{aligned}$$

and the proof is completed. ■

In Section 3.4 we will see that there exist multivariate counting processes which do and which do not fulfil the assumption that $\lim_{t\downarrow 0} t^{-1}\mathrm{P}[\{\mathbf{N}_t=\mathbf{e}_i\}]$ is finite for all $i\in\{1,\ldots,k\}$.

Since a multivariate counting process which has the binomial property also has the Chapman–Kolmogorov property, it fulfils the backward differential and integral equations of Theorem 2.3.5 as well as the forward differential and integral equations of Theorem 2.3.6 whenever $\lim_{t\downarrow 0} t^{-1}\mathrm{P}[\{\mathbf{N}_t = \mathbf{e}_i\}]$ is finite for all $i \in \{1,\dots,k\}$. Thus, for a multivariate counting process which has the binomial property and which fulfils the additional assumption on $\mathrm{P}[\{\mathbf{N}_t = \mathbf{e}_i\}]$, the intensities uniquely define the transition probabilities and therefore the one–dimensional distributions of the process.

This, of course, is also true for a process having the multinomial property. Furthermore, the multinomial property assures that the intensities uniquely define the finite–dimensional distributions of the process. So a regular process with the multinomial property is defined by its intensities (compare Corollary 2.3.9).

A further consequence of Theorem 2.3.16 is a straightforward recursion of the one–dimensional probabilities in terms of the intensities. So adding the binomial property to an arbitrary regular multivariate counting process changes the complicated recursion into a simple one.

Theorem 2.3.17 *Let* $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *be a multivariate counting process having the binomial property. If* $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *is regular with intensities* $\{\boldsymbol{\kappa}_\mathbf{n}\}_{\mathbf{n}\in\mathbb{N}_0^k}$*, then the recursion*

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}+\mathbf{e}_i\}\right] \;=\; \frac{t}{n^{(i)}+1}\,\kappa_\mathbf{n}^{(i)}(t)\,\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]$$

with $\mathbf{n}\in\mathbb{N}_0^k$ *and* $i\in\{1,\dots,k\}$ *and the starting value*

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{0}\}\right] \;=\; e^{-\int_0^t \sum_{i=1}^k \kappa_\mathbf{0}^{(i)}(s)\,ds}$$

holds for all $t\in\mathbb{R}_+$*.*

Proof: As every process which has the binomial property has the Chapman–Kolmogorov property (see Lemma 2.2.8), $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ fulfils the Kolmogorov systems of differential equations (Theorems 2.3.5 and 2.3.6). This yields the starting value of the recursion. For $t>0$ the recursion itself immediately follows from Theorem 2.3.16. Since the identity $\mathrm{P}\left[\{\mathbf{N}_0 = \mathbf{n}\}\right] = 0$ holds for all $\mathbf{n}\in\mathbb{N}_0^k\backslash\{\mathbf{0}\}$, the recursion is valid for $t=0$ as well. ■

Choosing $t>0$ and $\mathbf{n}\in\mathbb{N}_0^k$ and executing two steps of the recursion, we get on the one hand

$$\begin{aligned}
\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}+\mathbf{e}_i+\mathbf{e}_j\}\right] &= \frac{t}{n^{(i)}+1}\,\kappa_{\mathbf{n}+\mathbf{e}_j}^{(i)}(t)\,\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}+\mathbf{e}_j\}\right]\\
&= \frac{t^2}{(n^{(i)}+1)\,(n^{(j)}+1)}\,\kappa_{\mathbf{n}+\mathbf{e}_j}^{(i)}(t)\,\kappa_\mathbf{n}^{(j)}(t)\,\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]
\end{aligned}$$

and on the other hand

$$\begin{aligned} P[\{\mathbf{N}_t = \mathbf{n} + \mathbf{e}_i + \mathbf{e}_j\}] &= \frac{t}{n^{(j)}+1}\,\kappa^{(j)}_{\mathbf{n}+\mathbf{e}_i}(t)\,P[\{\mathbf{N}_t = \mathbf{n} + \mathbf{e}_i\}] \\ &= \frac{t^2}{(n^{(j)}+1)\,(n^{(i)}+1)}\,\kappa^{(j)}_{\mathbf{n}+\mathbf{e}_i}(t)\,\kappa^{(i)}_{\mathbf{n}}(t)\,P[\{\mathbf{N}_t = \mathbf{n}\}] \end{aligned}$$

Since $P[\{\mathbf{N}_t = \mathbf{n}\}] > 0$ holds for all $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$ for multivariate counting processes having the binomial property (see Lemma 2.2.3), we get

$$\kappa^{(i)}_{\mathbf{n}+\mathbf{e}_j}(t)\,\kappa^{(j)}_{\mathbf{n}}(t) \;=\; \kappa^{(j)}_{\mathbf{n}+\mathbf{e}_i}(t)\,\kappa^{(i)}_{\mathbf{n}}(t)$$

Illustrating this in a picture

$$\begin{tikzcd} \mathbf{n}+\mathbf{e}_j \arrow[r, "\kappa^{(i)}_{\mathbf{n}+\mathbf{e}_j}(t)"] & \mathbf{n}+\mathbf{e}_i+\mathbf{e}_j \\ \mathbf{n} \arrow[u, "\kappa^{(j)}_{\mathbf{n}}(t)"] \arrow[r, "\kappa^{(i)}_{\mathbf{n}}(t)"] & \mathbf{n}+\mathbf{e}_i \arrow[u, "\kappa^{(j)}_{\mathbf{n}+\mathbf{e}_i}(t)"'] \end{tikzcd}$$

we see that the product of the intensities along a path between two states is independent of the choice of the path. This also remains true for any two states which differ more than two steps from each other.

Furthermore, Theorem 2.3.16 yields a representation in terms of the intensities for the expectation of the process at some time t. This representation suggests a possible interpretation of the intensities of a certain class of multivariate counting processes (the details will become clearer in Section 4.4).

2.3.18 Corollary. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process which has the binomial property and which is regular. Then*

$$E\left[N^{(i)}_t\right] \;=\; t\sum_{\mathbf{n}\in\mathbb{N}_0^k} P[\{\mathbf{N}_t = \mathbf{n}\}]\;\kappa^{(i)}_{\mathbf{n}}(t)$$

holds for all $t > 0$ and $i \in \{1,\ldots,k\}$.

Proof: From Theorem 2.3.16 we get for arbitrary $t > 0$ and $i \in \{1,\ldots,k\}$

$$\begin{aligned} t\sum_{\mathbf{n}\in\mathbb{N}_0^k} P[\{\mathbf{N}_t = \mathbf{n}\}]\;\kappa^{(i)}_{\mathbf{n}}(t) &= t\sum_{\mathbf{n}\in\mathbb{N}_0^k} P[\{\mathbf{N}_t = \mathbf{n}\}]\;\frac{n^{(i)}+1}{t}\,\frac{P[\{\mathbf{N}_t = \mathbf{n}+\mathbf{e}_i\}]}{P[\{\mathbf{N}_t = \mathbf{n}\}]} \\ &= \sum_{\mathbf{n}\in\mathbb{N}_0^k} \big(n^{(i)}+1\big)\,P[\{\mathbf{N}_t = \mathbf{n}+\mathbf{e}_i\}] \\ &= E\left[N^{(i)}_t\right] \end{aligned}$$

and the assertion is shown. ■

Considering a univariate counting process with intensities which do not depend on the state (thus $\kappa_{\mathbf{n}}^{(i)}(t) = \kappa(t)$) Theorem 2.3.17 and Corollary 2.3.18 give a hint of the class of processes the binomial property is closely linked to. On one hand we get the recursion

$$\mathrm{P}\left[\{N_t = n+1\}\right] \;=\; \frac{t\,\kappa(t)}{n+1}\,\mathrm{P}\left[\{N_t = n\}\right]$$

and on the other hand the representation of the expected value

$$\mathrm{E}\left[N_t\right] \;=\; t\,\kappa(t)$$

Both formulas call to mind the Poisson process.

As a last conclusion of Theorem 2.3.16, we also obtain differential equations for the one–dimensional probabilities and the intensities were only one–dimensional probabilities and intensities, respectively, occur.

2.3.19 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process which has the binomial property and which is regular. Then*

$$\frac{d}{dt}\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \;=\; \frac{\mathbf{1}'\mathbf{n}}{t}\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] - \sum_{j=1}^{k}\frac{n^{(j)}+1}{t}\,\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}+\mathbf{e}_j\}\right]$$

and

$$\frac{d}{dt}\kappa_{\mathbf{n}}^{(i)}(t) \;=\; \kappa_{\mathbf{n}}^{(i)}(t)\left(\sum_{j=1}^{k}\kappa_{\mathbf{n}}^{(j)}(t) - \kappa_{\mathbf{n}+\mathbf{e}_i}^{(j)}(t)\right)$$

hold for all $t > 0$, $\mathbf{n} \in \mathbb{N}_0^k$, and $i \in \{1,\ldots,k\}$.

Proof: Consider $\mathbf{n} \in \mathbb{N}_0^k$ and $t > 0$.
From Theorem 2.3.16 and the differential equation from Corollary 2.3.8 we get

$$\begin{aligned}
\frac{d}{dt}\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] &= \sum_{j=1}^{k}\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}-\mathbf{e}_j\}\right]\,\kappa_{\mathbf{n}-\mathbf{e}_j}^{(j)}(t) - \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]\sum_{j=1}^{k}\kappa_{\mathbf{n}}^{(j)}(t) \\
&= \sum_{j=1}^{k}\frac{n^{(j)}}{t}\,\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] - \sum_{j=1}^{k}\frac{n^{(j)}+1}{t}\,\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}+\mathbf{e}_j\}\right] \\
&= \frac{\mathbf{1}'\mathbf{n}}{t}\,\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] - \sum_{j=1}^{k}\frac{n^{(j)}+1}{t}\,\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}+\mathbf{e}_j\}\right]
\end{aligned}$$

which proves the first identity.
Again by the differential equation from Corollary 2.3.8 and Theorem 2.3.16 we obtain

$$\begin{aligned}
\frac{d}{dt}\ln\big(\mathrm{P}[\{\mathbf{N}_t=\mathbf{n}\}]\big) &= \frac{1}{\mathrm{P}[\{\mathbf{N}_t=\mathbf{n}\}]}\frac{d}{dt}\mathrm{P}[\{\mathbf{N}_t=\mathbf{n}\}] \\
&= \sum_{j=1}^{k}\frac{\mathrm{P}[\{\mathbf{N}_t=\mathbf{n}-\mathbf{e}_i\}]}{\mathrm{P}[\{\mathbf{N}_t=\mathbf{n}\}]}\,\kappa^{(j)}_{\mathbf{n}-\mathbf{e}_i}(t)-\sum_{j=1}^{k}\kappa^{(j)}_{\mathbf{n}}(t) \\
&= \sum_{j=1}^{k}\frac{n^{(j)}}{t}-\sum_{j=1}^{k}\kappa^{(j)}_{\mathbf{n}}(t) \\
&= \frac{\mathbf{1}'\mathbf{n}}{t}-\sum_{j=1}^{k}\kappa^{(j)}_{\mathbf{n}}(t) \qquad (*)
\end{aligned}$$

Now, additionally consider $i \in \{1,\dots,k\}$. Then Theorem 2.3.16 yields for the logarithmic derivative of the intensity

$$\frac{d}{dt}\ln\big(\kappa^{(i)}_{\mathbf{n}}(t)\big) = \frac{d}{dt}\ln\big(\mathrm{P}[\{\mathbf{N}_t=\mathbf{n}+\mathbf{e}_i\}]\big)-\frac{d}{dt}\ln\big(\mathrm{P}[\{\mathbf{N}_t=\mathbf{n}\}]\big)-\frac{1}{t}$$

Inserting $(*)$ in the previous formula we get

$$\begin{aligned}
\frac{d}{dt}\ln\big(\kappa^{(i)}_{\mathbf{n}}(t)\big) &= \frac{\mathbf{1}'(\mathbf{n}+\mathbf{e}_i)}{t}-\sum_{j=1}^{k}\kappa^{(j)}_{\mathbf{n}+\mathbf{e}_i}(t)-\frac{\mathbf{1}'\mathbf{n}}{t}+\sum_{j=1}^{k}\kappa^{(j)}_{\mathbf{n}}(t)-\frac{1}{t} \\
&= \sum_{j=1}^{k}\Big(\kappa^{(j)}_{\mathbf{n}}(t)-\kappa^{(j)}_{\mathbf{n}+\mathbf{e}_i}(t)\Big)
\end{aligned}$$

which proves the second identity. ■

Chapter 3

Multivariate Mixed Poisson Processes

3.1 The Model

In this section we will introduce multivariate mixed Poisson processes within the class of counting processes and furthermore discuss some properties of these processes.

A multivariate counting process $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ is said to be a ***multivariate mixed Poisson process*** with mixing distribution $U : \mathcal{B}(\mathbb{R}^k) \to [0,1]$ if $U[(\mathbf{0}, \infty)] = 1$ and if

$$\begin{aligned} & \mathrm{P}\left[\bigcap_{j=1}^{m} \left\{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\right\}\right] \\ & \quad = \int_{\mathbb{R}^k} \prod_{j=1}^{m} e^{-\mathbf{1}'\boldsymbol{\lambda}(t_j - t_{j-1})} \frac{(\boldsymbol{\lambda}\,(t_j - t_{j-1}))^{\mathbf{n}_j}}{\mathbf{n}_j!} \, dU(\boldsymbol{\lambda}) \end{aligned}$$

holds for all $m \in \mathbb{N}$ and $t_0, t_1, \ldots, t_m \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_m$ and for all $\mathbf{n}_j \in \mathbb{N}_0^k$, $j \in \{1, \ldots, m\}$.

A short discussion about the support of the mixing distribution can be found at the end of this section.

Before we have our usual look at stability we connect multivariate mixed Poisson processes with the discussion in Section 2.2 by showing that multivariate mixed Poisson processes have the multinomial property.

3.1.1 Lemma. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process. Furthermore, let $U : \mathcal{B}(\mathbb{R}^k) \to [0,1]$ be a distribution with $U[(\mathbf{0}, \infty)] = 1$. Then the following are equivalent.*

(a) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *is a multivariate mixed Poisson process with mixing distribution* U.

(b) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *has the multinomial property and*

$$\mathrm{P}\,[\{\mathbf{N}_t = \mathbf{n}\}] \;=\; \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\,\frac{(\boldsymbol{\lambda}\, t)^{\mathbf{n}}}{\mathbf{n}!}\, dU(\boldsymbol{\lambda})$$

holds for all $t \in \mathbb{R}_+$ *and* $\mathbf{n} \in \mathbb{N}_0^k$.

Proof: Since

$$\begin{aligned}
&\int_{\mathbb{R}^k} \prod_{j=1}^{m} e^{-\mathbf{1}'\boldsymbol{\lambda}(t_j - t_{j-1})}\,\frac{(\boldsymbol{\lambda}\,(t_j - t_{j-1}))^{\mathbf{n}_j}}{\mathbf{n}_j!}\, dU(\boldsymbol{\lambda}) \\
&\qquad = \left(\prod_{i=1}^{k} \frac{n^{(i)}!}{\prod_{j=1}^{m} n_j^{(i)}!} \prod_{j=1}^{m} \left(\frac{t_j - t_{j-1}}{t_m}\right)^{n_j^{(i)}}\right) \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t_m}\,\frac{(\boldsymbol{\lambda}\, t_m)^{\mathbf{n}}}{\mathbf{n}!}\, dU(\boldsymbol{\lambda})
\end{aligned}$$

for all $m \in \mathbb{N}$ and $t_0, t_1, \ldots, t_m \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_m$ and for all $\mathbf{n}_j \in \mathbb{N}_0^k$, $j \in \{1, \ldots, m\}$ with $\sum_{j=1}^{m} \mathbf{n}_j = \mathbf{n}$ the assertion immediately is due to the definition of multivariate mixed Poisson processes. ■

In Section 3.2 we will see that the above characterization of multivariate mixed Poisson processes can be fundamentally improved. But now we use this lemma to show with the help of Section 2.2 some properties of multivariate mixed Poisson processes. Incidentally, the first two items can also be proven right from the definition.

3.1.2 Corollary. *Let* $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *be a multivariate mixed Poisson process. Then*

(1) *the inequality* $\mathrm{P}\,[\{\mathbf{N}_t = \mathbf{n}\}] > 0$ *holds for all* $t > 0$ *and* $\mathbf{n} \in \mathbb{N}_0^k$,

(2) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *has stationary increments, and*

(3) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *is a Markov process.*

3.1.3 Lemma. *Let* $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *be a multivariate counting process. Then being a mixed Poisson process is* $\mathcal{A}$*–stable. Furthermore, the mixing distribution of* $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *is given by* U_A.

Proof: As a consequence of Lemma 3.1.1 and Lemma 2.2.2 (the multinomial property is $\mathcal{A}$–stable) we just have to show the stability of the one–dimensional distributions. Thus, we consider $t \in \mathbb{R}_+$ and $\mathbf{l} \in \mathbb{N}_0^d$ and prove

$$\begin{aligned}
\mathrm{P}\,[\,\{A\mathbf{N}_t = \mathbf{l}\}] \;&=\; \int_{\mathbb{R}^d} e^{-\mathbf{1}'\boldsymbol{\lambda}^* t}\,\frac{(\boldsymbol{\lambda}^*\, t)^{\mathbf{l}}}{\mathbf{l}!}\, dU_A(\boldsymbol{\lambda}^*) \\
&=\; \int_{\mathbb{R}^k} e^{-\mathbf{1}'A\boldsymbol{\lambda} t}\,\frac{(A\boldsymbol{\lambda}\, t)^{\mathbf{l}}}{\mathbf{l}!}\, dU(\boldsymbol{\lambda}) \qquad (+)
\end{aligned}$$

for all $A \in \mathcal{A}$ with $A \in \mathbb{R}^{d\times k}$.
- Let $A \in \mathcal{A}_P$. Then (+) holds obviously.
- Let $A \in \mathcal{A}_S$. Then we have with the help of monotone convergence

$$\begin{aligned}
\mathrm{P}\left[\{A\mathbf{N}_t = \mathbf{l}\}\right] &= \sum_{\mathbf{n}\,\in\, A^{-1}(\{\mathbf{l}\})} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \\
&= \sum_{\mathbf{n}\,\in\, A^{-1}(\{\mathbf{l}\})} \int_{\mathbb{R}^k} \prod_{i=1}^{k} e^{-\lambda_i t} \frac{(\lambda_i t)^{n^{(i)}}}{n^{(i)}!}\, dU(\boldsymbol{\lambda}) \\
&= \int_{\mathbb{R}^k} \sum_{\mathbf{n}\,\in\, A^{-1}(\{\mathbf{l}\})} \prod_{i=1}^{k} e^{-\lambda_i t} \frac{(\lambda_i t)^{n^{(i)}}}{n^{(i)}!}\, dU(\boldsymbol{\lambda}) \\
&= \int_{\mathbb{R}^k} \left(\prod_{i=1}^{d} e^{-\lambda_i t} \frac{(\lambda_i t)^{l^{(i)}}}{l^{(i)}!}\right) \sum_{\mathbf{n}\,\in\, A^{-1}(\{\mathbf{l}\})} \prod_{i=d+1}^{k} e^{-\lambda_i t} \frac{(\lambda_i t)^{n^{(i)}}}{n^{(i)}!}\, dU(\boldsymbol{\lambda}) \\
&= \int_{\mathbb{R}^k} e^{-\mathbf{1}' A \boldsymbol{\lambda} t} \frac{(A\boldsymbol{\lambda} t)^{\mathbf{l}}}{\mathbf{l}!}\, dU(\boldsymbol{\lambda})
\end{aligned}$$

So (+) holds for $A \in \mathcal{A}_S$.
- Let $A \in \mathcal{A}_C$. Setting $I(i) := \{h \in \{1, \dots, k\} \,:\, \mathbf{e}_i' A \mathbf{e}_h = 1\}$ (the set of coordinates cumulated in the i–th coordinate of the transformed process) we have $\sum_{h\,\in\, I(i)} \lambda_h = \mathbf{e}_i' A \boldsymbol{\lambda}$ for all $i \in \{1, \dots, d\}$. Thus with the same formula manipulation at the beginning as before, we get

$$\begin{aligned}
\mathrm{P}\left[\{A\mathbf{N}_t = \mathbf{l}\}\right] &= \int_{\mathbb{R}^k} \sum_{\mathbf{n}\,\in\, A^{-1}(\{\mathbf{l}\})} \prod_{i=1}^{k} e^{-\lambda_i t} \frac{(\lambda_i t)^{n^{(i)}}}{n^{(i)}!}\, dU(\boldsymbol{\lambda}) \\
&= \int_{\mathbb{R}^k} \left(\prod_{i=1}^{d} e^{-\mathbf{e}_i' A \boldsymbol{\lambda} t} \frac{(\mathbf{e}_i' A \boldsymbol{\lambda} t)^{l^{(i)}}}{l^{(i)}!}\right) \\
&\qquad \cdot \sum_{\mathbf{n}\,\in\, A^{-1}(\{\mathbf{l}\})} \prod_{i=1}^{d} \frac{l^{(i)}!}{\prod_{h\,\in\, I(i)} n^{(h)}!} \prod_{h\,\in\, I(i)} \left(\frac{\lambda_h}{\mathbf{e}_i' A \boldsymbol{\lambda}}\right)^{n^{(h)}} dU(\boldsymbol{\lambda}) \\
&= \int_{\mathbb{R}^k} e^{-\mathbf{1}' A \boldsymbol{\lambda} t} \frac{(A\boldsymbol{\lambda} t)^{\mathbf{l}}}{\mathbf{l}!}\, dU(\boldsymbol{\lambda})
\end{aligned}$$

Therewith (+) holds for $A \in \mathcal{A}_C$ as well.
Furthermore, we have with $A \in \mathcal{A}$ (and hence $A \in \mathbb{R}^{d\times k}$)

$$1 \;\geq\; U_A\left[(\mathbf{0}, \boldsymbol{\infty})\right] \;=\; U\left[A^{-1}((\mathbf{0}, \boldsymbol{\infty}))\right] \;\geq\; U\left[(\mathbf{0}, \boldsymbol{\infty})\right] \;=\; 1$$

So $U_A\left[(\mathbf{0}, \boldsymbol{\infty})\right] = 1$ holds as well and the assertion is shown. ■

The product of Poisson probabilities under the integral sign raises the question of independence of the coordinates of a multivariate mixed Poisson process. An answer is given in the next theorem, which was inspired by Hofmann [1955].

3.1.4 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with mixing distribution U. Then the following are equivalent.*

(a) *The coordinates of $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ are independent.*

(b) *The identity $U = \bigotimes_{i=1}^k U_{\mathbf{e}_i{}'}$ is valid.*

Proof:
(b) $\Rightarrow$ (a): We get for all $m \in \mathbb{N}$ and $t_0, t_1, \ldots, t_m \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_m$ and for all $\mathbf{n}_j \in \mathbb{N}_0^k$, $j \in \{1, \ldots, m\}$,

$$\begin{aligned}
&\mathrm{P}\left[\bigcap_{j=1}^m \{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right] \\
&\quad= \int_{\mathbb{R}^k} \prod_{i=1}^k \prod_{j=1}^m e^{-\lambda_i (t_j - t_{j-1})} \frac{(\lambda_i (t_j - t_{j-1}))^{n_j^{(i)}}}{n_j^{(i)}!} \, dU(\boldsymbol{\lambda}) \\
&\quad= \int_{\mathbb{R}^k} \prod_{i=1}^k \prod_{j=1}^m e^{-\lambda_i (t_j - t_{j-1})} \frac{(\lambda_i (t_j - t_{j-1}))^{n_j^{(i)}}}{n_j^{(i)}!} \, d(\bigotimes_{i=1}^k U_{\mathbf{e}_i{}'})(\boldsymbol{\lambda}) \\
&\quad= \prod_{i=1}^k \int_{\mathbb{R}} \prod_{j=1}^m e^{-\lambda_i (t_j - t_{j-1})} \frac{(\lambda_i (t_j - t_{j-1}))^{n_j^{(i)}}}{n_j^{(i)}!} \, dU_{\mathbf{e}_i{}'}(\lambda_i) \\
&\quad= \prod_{i=1}^k \mathrm{P}\left[\bigcap_{j=1}^m \left\{N_{t_j}^{(i)} - N_{t_{j-1}}^{(i)} = n_j^{(i)}\right\}\right]
\end{aligned}$$

Recall that $\mathbf{e}_i{}' \in \mathcal{A}$, so the last identity is true due to Lemma 3.1.3. Thus, the coordinates are independent.

(a) $\Rightarrow$ (b): We look at certain probabilities of the process for which we treat time as a variable. For all $t_0, t_1 \ldots, t_k \in \mathbb{R}_+$ such that $0 = t_0 < t_1 < \ldots < t_k$, independence of the coordinates yields

$$\begin{aligned}
\mathrm{P}\left[\bigcap_{i=1}^k \left\{N_{t_i}^{(i)} - N_{t_{i-1}}^{(i)} = 0\right\}\right] &= \prod_{i=1}^k \mathrm{P}\left[\left\{N_{t_i}^{(i)} - N_{t_{i-1}}^{(i)} = 0\right\}\right] \\
&= \prod_{i=1}^k \int_{\mathbb{R}} e^{-\lambda_i (t_i - t_{i-1})} \, dU_{\mathbf{e}_i{}'}(\lambda_i) \\
&= \int_{\mathbb{R}^k} e^{-\sum_{j=1}^k \lambda_j (t_j - t_{j-1})} \, d(\bigotimes_{i=1}^k U_{\mathbf{e}_i{}'})(\boldsymbol{\lambda})
\end{aligned}$$

and on the other hand we obtain (where $\mathbf{e}_1{}'^{-1}(\{0\})$ indeed stands for the inverse image of the set $\{0\}$ under $\mathbf{e}_1{}'$)

$$\mathrm{P}\left[\bigcap_{i=1}^k \left\{N_{t_i}^{(i)} - N_{t_{i-1}}^{(i)} = 0\right\}\right]$$

$$
\begin{aligned}
&= \sum_{\mathbf{n}_1 \in \mathbf{e}_1{}'^{-1}(\{0\})} \cdots \sum_{\mathbf{n}_k \in \mathbf{e}_k{}'^{-1}(\{0\})} \mathrm{P}\left[\bigcap_{j=1}^{k} \{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right] \\
&= \sum_{\mathbf{n}_1 \in \mathbf{e}_1{}'^{-1}(\{0\})} \cdots \sum_{\mathbf{n}_k \in \mathbf{e}_k{}'^{-1}(\{0\})} \int_{\mathbb{R}^k} \prod_{i=1}^{k} \prod_{j=1}^{k} e^{-\lambda_i (t_j - t_{j-1})} \frac{(\lambda_i (t_j - t_{j-1}))^{n_j^{(i)}}}{n_j^{(i)}!} \, dU(\boldsymbol{\lambda}) \\
&= \int_{\mathbb{R}^k} \sum_{\mathbf{n}_1 \in \mathbf{e}_1{}'^{-1}(\{0\})} \cdots \sum_{\mathbf{n}_k \in \mathbf{e}_k{}'^{-1}(\{0\})} \prod_{i=1}^{k} \prod_{j=1}^{k} e^{-\lambda_i (t_j - t_{j-1})} \frac{(\lambda_i (t_j - t_{j-1}))^{n_j^{(i)}}}{n_j^{(i)}!} \, dU(\boldsymbol{\lambda}) \\
&= \int_{\mathbb{R}^k} \prod_{i=1}^{k} e^{-\lambda_i (t_i - t_{i-1})} \, dU(\boldsymbol{\lambda}) \\
&= \int_{\mathbb{R}^k} e^{-\sum_{j=1}^{k} \lambda_j (t_j - t_{j-1})} \, dU(\boldsymbol{\lambda})
\end{aligned}
$$

Combining these two identities we get

$$
\int_{\mathbb{R}^k} e^{-\mathbf{z}'\boldsymbol{\lambda}} \, dU(\boldsymbol{\lambda}) = \int_{\mathbb{R}^k} e^{-\mathbf{z}'\boldsymbol{\lambda}} \, d(\bigotimes_{i=1}^{k} U_{\mathbf{e}_i{}'}) \, (\boldsymbol{\lambda})
$$

for all $\mathbf{z} \in (\mathbf{0}, \boldsymbol{\infty})$. This is an identity for Laplace transforms. The uniqueness of the Laplace transform (Kallenberg [2002] Theorem 5.3) implies $U = \bigotimes_{i=1}^{k} U_{\mathbf{e}_i{}'}$. ■

The above theorem is a justification for the use of multivariate mixed Poisson processes. Only in the case of a mixing distribution being a product of its one–dimensional marginal distributions the coordinates are independent. Under such a condition the theory of one–dimensional mixed Poisson processes is sufficient. In all other situations the multivariate setting becomes necessary.

For using posterior distributions of a multivariate mixed Poisson process with mixing distribution the next theorem provides all necessary information. For univariate mixed Poisson processes the assertion has been stated in similar form for example by Willmot and Sundt [1989]. To obtain a compact notation we introduce the distribution $U_{t,\mathbf{n}} \, : \, \mathcal{B}(\mathbb{R}^k) \to [0,1]$ such that

$$
U_{t,\mathbf{n}}[B] \;\; := \;\; \frac{\int_B e^{-\mathbf{1}'\boldsymbol{\lambda} t} \, \boldsymbol{\lambda}^{\mathbf{n}} \, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \, \boldsymbol{\lambda}^{\mathbf{n}} \, dU(\boldsymbol{\lambda})}
$$

for arbitrary $\mathbf{n} \in \mathbb{N}_0^k$, $t > 0$ and a given distribution $U \, : \, \mathcal{B}(\mathbb{R}^k) \to [0,1]$ with $U[\mathbb{R}_+^k] = 1$. In accordance with the preceding definition we additionally define $U_{0,\mathbf{0}} := U$.

3.1.5 Theorem. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate mixed Poisson process with mixing distribution U. Then for all $\mathbf{n} \in \mathbb{N}_0^k$ and $t > 0$ the incremental process $\{\mathbf{K}_{t,h}\}_{h \in \mathbb{R}_+}$ is a multivariate mixed Poisson process with mixing distribution $U_{t,\mathbf{n}}$ on* *the probability space* $(\Omega, \mathcal{F}, \mathrm{P}_{t,\mathbf{n}})$.

Proof: Consider $m \in \mathbb{N}$ and $h_0, h_1, \dots, h_m \in \mathbb{R}_+$ with $0 = h_0 < h_1 < \dots < h_m$ and $\mathbf{n}_j \in \mathbb{N}_0^k$, $j \in \{1, \dots, m\}$, as well as $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$.
From the definition of multivariate mixed Poisson processes we get $\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] > 0$. Then we have

$$
\begin{aligned}
&\mathrm{P}_{t,\mathbf{n}}\left[\bigcap_{j=1}^{m} \left\{\mathbf{K}_{t,h_j} - \mathbf{K}_{t,h_{j-1}} = \mathbf{n}_j\right\}\right] \\
&= \mathrm{P}\left[\bigcap_{j=1}^{m} \left\{(\mathbf{N}_{t+h_j} - \mathbf{N}_t) - (\mathbf{N}_{t+h_{j-1}} - \mathbf{N}_t) = \mathbf{n}_j\right\} \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right] \\
&= \frac{\mathrm{P}\left[\bigcap_{j=1}^{m} \left\{\mathbf{N}_{t+h_j} - \mathbf{N}_{t+h_{j-1}} = \mathbf{n}_j\right\} \cap \{\mathbf{N}_t = \mathbf{n}\}\right]}{\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]} \\
&= \frac{\displaystyle\int_{\mathbb{R}^k} \left(\prod_{j=1}^{m} e^{-\mathbf{1}'\boldsymbol{\lambda}(h_j - h_{j-1})} \frac{(\boldsymbol{\lambda}(h_j - h_{j-1}))^{\mathbf{n}_j}}{\mathbf{n}_j!}\right) e^{-\mathbf{1}'\boldsymbol{\lambda} t} \frac{(\boldsymbol{\lambda} t)^{\mathbf{n}}}{\mathbf{n}!}\, dU(\boldsymbol{\lambda})}{\displaystyle\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \frac{(\boldsymbol{\lambda} t)^{\mathbf{n}}}{\mathbf{n}!}\, dU(\boldsymbol{\lambda})} \\
&= \int_{\mathbb{R}^k} \prod_{j=1}^{m} e^{-\mathbf{1}'\boldsymbol{\lambda}(h_j - h_{j-1})} \frac{(\boldsymbol{\lambda}(h_j - h_{j-1}))^{\mathbf{n}_j}}{\mathbf{n}_j!}\, dU_{t,\mathbf{n}}(\boldsymbol{\lambda})
\end{aligned}
$$

which yields the assertion. ■

The above theorem does not only provide a representation of the posterior probabilities, but also states that the model of multivariate mixed Poisson processes with mixing distribution remains valid despite dependent increments regardless at which time we start to observe the process.

Now, we turn towards the announced discussion concerning the support of the mixing distribution of a multivariate mixed Poisson process. To this purpose we consider stochastic processes which fulfil

$$
\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] = \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \frac{(\boldsymbol{\lambda} t)^{\mathbf{n}}}{\mathbf{n}!}\, dU(\boldsymbol{\lambda})
$$

for all $t \in \mathbb{R}_+$ and $\mathbf{n} \in \mathbb{N}_0^k$ and where $U : \mathcal{B}(\mathbb{R}^k) \to [0,1]$ is a distribution with $U\left[\mathbb{R}_+^k\right] = 1$. In comparison with mixed Poisson processes we extend the support of the mixing distribution from $(\mathbf{0}, \boldsymbol{\infty})$ to $\mathbb{R}_+^k$. The next two lemmas contain properties of the considered processes.

3.1.6 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a stochastic process in k dimensions which fulfils*

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \;=\; \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\,\frac{(\boldsymbol{\lambda}\, t)^{\mathbf{n}}}{\mathbf{n}!}\,dU(\boldsymbol{\lambda})$$

for all $t \in \mathbb{R}_+$ and $\mathbf{n} \in \mathbb{N}_0^k$ with $U\left[\mathbb{R}_+^k\right] = 1$. Then for all $A \in \mathcal{A}$

$$\mathrm{P}\left[\{A\mathbf{N}_t = \mathbf{l}\}\right] \;=\; \int_{\mathbb{R}^d} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\,\frac{(\boldsymbol{\lambda}\, t)^{\mathbf{l}}}{\mathbf{l}!}\,dU_A(\boldsymbol{\lambda})$$

holds for all $t \in \mathbb{R}_+$ and $\mathbf{l} \in \mathbb{N}_0^d$ where the mixing distribution $U_A : \mathcal{B}(\mathbb{R}^d) \to [0,1]$ fulfils $U_A\left[\mathbb{R}_+^d\right] = 1$.

Proof: The proof of the identity for the one–dimensional distributions is completely equal to the proof of Lemma 3.1.3.
Furthermore, with $A \in \mathcal{A}$ we have

$$1 \;\geq\; U_A\left[\mathbb{R}_+^d\right] \;=\; U\left[A^{-1}(\mathbb{R}_+^d)\right] \;\geq\; U\left[\mathbb{R}_+^k\right] \;=\; 1$$

Thus $U_A\left[\mathbb{R}_+^d\right] = 1$ holds as well. ■

3.1.7 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a stochastic process in k dimensions which fulfils*

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \;=\; \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\,\frac{(\boldsymbol{\lambda}\, t)^{\mathbf{n}}}{\mathbf{n}!}\,dU(\boldsymbol{\lambda})$$

for all $t \in \mathbb{R}_+$ and $\mathbf{n} \in \mathbb{N}_0^k$ with $U\left[\mathbb{R}_+^k\right] = 1$.
Then the probabilities have limits of the form

$$\lim_{t\uparrow\infty} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \;=\; \begin{cases} U\left[\{\mathbf{0}\}\right] & \text{if} \quad \mathbf{n} = \mathbf{0} \\ 0 & \text{if} \quad \mathbf{n} \in \mathbb{N}_0^k \setminus \{\mathbf{0}\} \end{cases}$$

Proof: Firstly, let us consider the function $f : (0,\infty) \to \mathbb{R}$ with $f(\lambda) := e^{-\lambda t}(\lambda\, t)^n$ with $t > 0$ and $n \geq 1$. Then we have

$$\begin{aligned} f'(\lambda) &= e^{-\lambda t}\lambda^{n-1}\, t^n\,(n - \lambda\, t) \\ f''(\lambda) &= e^{-\lambda t}\lambda^{n-2}\, t^n\left((n-\lambda\, t)^2 - n\right) \end{aligned}$$

and $\lambda^* := n/t$ is maximizer with $f(\lambda^*) = e^{-n}\, n^n$.
Therefore $e^{-\lambda t}(\lambda\, t)^n \leq e^{-n}\, n^n$ and

$$e^{-\mathbf{1}'\boldsymbol{\lambda} t}\,\frac{(\boldsymbol{\lambda}\, t)^{\mathbf{n}}}{\mathbf{n}!} \;=\; \prod_{i=1}^{k} e^{-\lambda_i t}\,\frac{(\lambda_i\, t)^{n^{(i)}}}{n^{(i)}!} \;\leq\; \prod_{i=1}^{k} e^{-n^{(i)}}\left(n^{(i)}\right)^{n^{(i)}}$$

holds for all $\boldsymbol{\lambda} \in \mathbb{R}_+^k$, $t > 0$, and $\mathbf{n} \in \mathbb{N}_0^k$. By dominated convergence we get

$$\begin{aligned}
\lim_{t\uparrow\infty} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] &= \lim_{t\uparrow\infty} \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda}\, t}\, \frac{(\boldsymbol{\lambda}\, t)^{\mathbf{n}}}{\mathbf{n}!}\, dU(\boldsymbol{\lambda}) \\
&= U\left[\{\mathbf{0}\}\right] \frac{\mathbf{0}^{\mathbf{n}}}{\mathbf{n}!} + \lim_{t\uparrow\infty} \int_{\mathbb{R}_+^k\setminus\{\mathbf{0}\}} e^{-\mathbf{1}'\boldsymbol{\lambda}\, t}\, \frac{(\boldsymbol{\lambda}\, t)^{\mathbf{n}}}{\mathbf{n}!}\, dU(\boldsymbol{\lambda}) \\
&= U\left[\{\mathbf{0}\}\right] \frac{\mathbf{0}^{\mathbf{n}}}{\mathbf{n}!} + \int_{\mathbb{R}_+^k\setminus\{\mathbf{0}\}} \lim_{t\uparrow\infty} e^{-\mathbf{1}'\boldsymbol{\lambda}\, t}\, \frac{(\boldsymbol{\lambda}\, t)^{\mathbf{n}}}{\mathbf{n}!}\, dU(\boldsymbol{\lambda}) \\
&= U\left[\{\mathbf{0}\}\right] \frac{\mathbf{0}^{\mathbf{n}}}{\mathbf{n}!}
\end{aligned}$$

which yields the assertion. ■

Adding now the properties of the paths of a multivariate counting process gives the following result.

3.1.8 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process which fulfils*

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] = \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda}\, t}\, \frac{(\boldsymbol{\lambda}\, t)^{\mathbf{n}}}{\mathbf{n}!}\, dU(\boldsymbol{\lambda})$$

for all $t \in \mathbb{R}_+$ and $\mathbf{n} \in \mathbb{N}_0^k$ with $U\left[\mathbb{R}_+^k\right] = 1$. Then $U\left[(\mathbf{0}, \boldsymbol{\infty})\right] = 1$ is valid.

Proof: The previous lemma in addition with Lemma 2.1.2 (4) yields the identity $U\left[\{\mathbf{0}\}\right] = \lim_{t\uparrow\infty} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{0}\}\right] = 0$. Since for every $A \in \mathcal{A}$ the transformed process $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ is a counting process (Lemma 2.1.1) and has mixed Poisson distributions (Lemma 3.1.6), too, we obtain $U_A\left[\{\mathbf{0}\}\right] = 0$.
For every $i \in \{1, \ldots, k\}$ the transformation $A := \mathbf{e}_i{}'$ satisfies $A \in \mathcal{A}$ and so we get

$$0 = U_{\mathbf{e}_i{}'}\left[\{0\}\right] = U\left[{\mathbf{e}_i{}'}^{-1}(\{0\})\right] = U\left[\times_{j=1}^k B_j\right] \quad \text{with } B_j := \begin{cases} \{0\} & j = i \\ \mathbb{R} & \text{else} \end{cases}$$

Therewith $U\left[\mathbb{R}_+^k \setminus (\mathbf{0}, \boldsymbol{\infty})\right] = 0$ and the assertion is shown. ■

Therefore, the restriction of the mixing distribution in the definition of multivariate mixed Poisson processes has no effect at all. The paths, which almost surely increase to infinity, do not allow the mixing distribution to have positive mass at zero in any coordinate. When omitting this property of the paths in the definition of counting processes we would have to add another property, like the paths which increase to infinity have at least strictly positive probability, to assure for example that the binomial property leads to strictly positive probabilities for all numbers of events ($\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] > 0$ for all $t > 0$ and all $\mathbf{n} \in \mathbb{N}_0^k$; compare Lemma 2.2.3).

3.2 A Characterization

In this section we will characterize multivariate mixed Poisson processes in the class of multivariate counting processes. The preliminary lemma reads easier with a new notation. For all $\mathbf{n} \in \mathbb{N}_0^k$ we set

$$\Pi_{\mathbf{n}}(\mathbf{t}) := \mathrm{P}\left[\bigcap_{i=1}^{k}\left\{N_{t_i}^{(i)} = n^{(i)}\right\}\right]$$

with $\mathbf{t} \in \mathbb{R}_+^k$.

3.2.1 Lemma. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process having the extended binomial property.*
Then

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \;=\; \frac{(-t)^{\mathbf{1}'\mathbf{n}}}{\mathbf{n}!} D^{\mathbf{n}} \Pi_{\mathbf{0}}(t\mathbf{1})$$

holds for all $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$. Furthermore, there exists a distribution with $U[(\mathbf{0}, \boldsymbol{\infty})] = 1$ such that

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \;=\; \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \frac{(\boldsymbol{\lambda} t)^{\mathbf{n}}}{\mathbf{n}!} \, dU(\boldsymbol{\lambda})$$

holds for all $t \in \mathbb{R}_+$ and $\mathbf{n} \in \mathbb{N}_0^k$.

Proof: Consider $\mathbf{t} \in \mathbb{R}_+^k$, $t \in \mathbb{R}_+$ with $\mathbf{t} \in (\mathbf{0}, t\mathbf{1})$ and $\mathbf{n} \in \mathbb{N}_0^k$. The extended binomial property yields

$$\begin{aligned}
\Pi_{\mathbf{n}}(\mathbf{t}) &= \sum_{\mathbf{l} \in [\mathbf{n}, \boldsymbol{\infty})} \mathrm{P}\left[\bigcap_{i=1}^{k}\left\{N_{t_i}^{(i)} = n^{(i)}\right\} \cap \left\{N_t^{(i)} - N_{t_i}^{(i)} = l^{(i)} - n^{(i)}\right\}\right] \\
&= \sum_{\mathbf{l} \in [\mathbf{n}, \boldsymbol{\infty})} \left(\prod_{i=1}^{k} \binom{l^{(i)}}{n^{(i)}} \left(\frac{t_i}{t}\right)^{n^{(i)}} \left(1 - \frac{t_i}{t}\right)^{l^{(i)} - n^{(i)}}\right) \Pi_{\mathbf{l}}(t\mathbf{1})
\end{aligned}$$

In particular, we have

$$\begin{aligned}
\Pi_{\mathbf{0}}(\mathbf{t}) &= \sum_{\mathbf{l} \in \mathbb{N}_0^k} \left(\prod_{i=1}^{k} \left(1 - \frac{t_i}{t}\right)^{l^{(i)}}\right) \Pi_{\mathbf{l}}(t\mathbf{1}) \\
&= \sum_{\mathbf{l} \in \mathbb{N}_0^k} \left(\prod_{i=1}^{k} (t_i - t)^{l^{(i)}}\right) \frac{\Pi_{\mathbf{l}}(t\mathbf{1})}{t^{\mathbf{1}'\mathbf{l}}}
\end{aligned}$$

The power series $\Pi_{\mathbf{0}}(\mathbf{t})$ in k coordinates is absolutely bounded for $\mathbf{t} \in (\mathbf{0}, 2t\mathbf{1})$ by $\sum_{\mathbf{l} \in \mathbb{N}_0^k} \Pi_{\mathbf{l}}(t\mathbf{1}) = 1$ and therefore absolutely convergent. Thus, $\Pi_{\mathbf{0}}$ is continuous on

$(\mathbf{0}, 2t\mathbf{1})$ and the power series can infinitely often be differentiated in this open set (see Dieudonné [1971] Chapter 9, especially Theorem 9.3.6.).

$$\begin{aligned} D^{\mathbf{n}}\Pi_{\mathbf{0}}(\mathbf{t}) &= \sum_{\mathbf{l}\in[\mathbf{n},\boldsymbol{\infty})} \left(\prod_{i=1}^{k} \frac{l^{(i)}!}{(l^{(i)}-n^{(i)})!} \left(1-\frac{t_i}{t}\right)^{l^{(i)}-n^{(i)}} \left(\frac{-1}{t}\right)^{n^{(i)}} \right) \Pi_{\mathbf{l}}(t\mathbf{1}) \\ &= \frac{\mathbf{n}!}{(-\mathbf{t})^{\mathbf{n}}} \sum_{\mathbf{l}\in[\mathbf{n},\boldsymbol{\infty})} \left(\prod_{i=1}^{k} \binom{l^{(i)}}{n^{(i)}} \left(\frac{t_i}{t}\right)^{n^{(i)}} \left(1-\frac{t_i}{t}\right)^{l^{(i)}-n^{(i)}} \right) \Pi_{\mathbf{l}}(t\mathbf{1}) \end{aligned}$$

and hence

$$\Pi_{\mathbf{n}}(\mathbf{t}) = \frac{(-\mathbf{t})^{\mathbf{n}}}{\mathbf{n}!} D^{\mathbf{n}}\Pi_{\mathbf{0}}(\mathbf{t})$$

for all $\mathbf{t} \in (\mathbf{0}, 2t\mathbf{1})$. Since t was arbitrary, the inequality

$$(-1)^{\mathbf{1}'\mathbf{n}} D^{\mathbf{n}}\Pi_{\mathbf{0}}(\mathbf{t}) \geq 0$$

holds for all $\mathbf{t} \in (\mathbf{0}, \boldsymbol{\infty})$ and $\Pi_{\mathbf{0}}$ is continuous on $(\mathbf{0}, \boldsymbol{\infty})$. From Lemma 2.1.2 (1) it follows that $\Pi_{\mathbf{0}}$ is right continuous on $\mathbb{R}_+^k$ and so $\Pi_{\mathbf{0}}$ is continuous on $\mathbb{R}_+^k$.
Last but not least, we have $\Pi_{\mathbf{0}}(\mathbf{0}) = 1$ since $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ is a multivariate counting process. Therewith $\Pi_{\mathbf{0}}$ fulfils all conditions of the Multivariate Bernstein–Widder theorem (1.3.1) which yields the existence of a distribution $U : \mathcal{B}(\mathbb{R}^k) \to [0,1]$ with $U\left[\mathbb{R}_+^k\right] = 1$ such that

$$\Pi_{\mathbf{0}}(\mathbf{t}) = \int_{\mathbb{R}^k} e^{-\mathbf{t}'\boldsymbol{\lambda}}\, dU(\boldsymbol{\lambda})$$

holds for all $\mathbf{t} \in \mathbb{R}_+^k$. So we obtain $\Pi_{\mathbf{0}}(\mathbf{t}) = M_U(-\mathbf{t})$ where M_U is the moment generating function of U. Since M_U is finite on $(-\boldsymbol{\infty}, \mathbf{0}]$ we can use Lemma 1.2.1 to differentiate $\Pi_{\mathbf{0}}$ on $(\mathbf{0}, \boldsymbol{\infty})$. Thus, we get

$$\begin{aligned} \Pi_{\mathbf{n}}(\mathbf{t}) &= \frac{(-\mathbf{t})^{\mathbf{n}}}{\mathbf{n}!} D^{\mathbf{n}}\Pi_{\mathbf{0}}(\mathbf{t}) \\ &= \frac{(-\mathbf{t})^{\mathbf{n}}}{\mathbf{n}!} \int_{\mathbb{R}^k} (-\boldsymbol{\lambda})^{\mathbf{n}}\, e^{-\mathbf{t}'\boldsymbol{\lambda}}\, dU(\boldsymbol{\lambda}) \\ &= \int_{\mathbb{R}^k} e^{-\mathbf{t}'\boldsymbol{\lambda}}\, \frac{\mathbf{t}^{\mathbf{n}}\,\boldsymbol{\lambda}^{\mathbf{n}}}{\mathbf{n}!}\, dU(\boldsymbol{\lambda}) \end{aligned}$$

Using $\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}] = \Pi_{\mathbf{n}}(t\mathbf{1})$, we immediately get that

$$\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}] = \frac{(-t)^{\mathbf{1}'\mathbf{n}}}{\mathbf{n}!} D^{\mathbf{n}}\Pi_{\mathbf{0}}(t\mathbf{1})$$

and

$$\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}] = \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, \frac{(\boldsymbol{\lambda}\, t)^{\mathbf{n}}}{\mathbf{n}!}\, dU(\boldsymbol{\lambda})$$

hold for all $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$. The last identities are also valid for $t = 0$ as all path almost surely start at $\mathbf{0}$. Furthermore, Lemma 3.1.8 gives $U[(\mathbf{0}, \boldsymbol{\infty})] = 1$ which completes the proof. ∎

Since in the preceding proof we need as many different time variables as the process has coordinates, it is not possible to replace the extended binomial property by the binomial property. However, as the binomial property carries over to the transformed process and the extended binomial property and the binomial property are identical for a univariate counting process, we have the following corollary.

3.2.2 Corollary. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process which has the binomial property. Let $A \in \mathcal{A}$ with $A \in \mathbb{R}^{1 \times k}$. Then there exists a distribution $U : \mathcal{B}(\mathbb{R}) \longrightarrow [0,1]$ with $U[(0,\infty)] = 1$ such that*

$$\mathrm{P}\left[\{A\mathbf{N}_t = n\}\right] \;=\; \int_{\mathbb{R}} e^{-\lambda t} \, \frac{(\lambda\, t)^n}{n!} \, dU(\lambda)$$

holds for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_0$.
In particular, this applies to all coordinates $\{N_t^{(i)}\}_{t \in \mathbb{R}_+}$, $i \in \{1, \ldots, k\}$, and the sum $\{\mathbf{1}'\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ of all coordinates of the multivariate counting process.

By Lemma 3.2.1, the following characterization of multivariate mixed Poisson processes is rather obvious. In the univariate case compare Schmidt and Zocher [2003] or for similar results Nawrotzki [1955], Lundberg [1964], and Albrecht [1981].

3.2.3 Theorem (Characterization). *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process. Then the following are equivalent*

(a) $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *is a multivariate mixed Poisson process.*
(b) $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *has the multinomial property.*
(c) $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *has the extended binomial property and the Markov property.*
(d) $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *has the binomial property and the Markov property.*

Proof: The equivalence of (b), (c), and (d) is already known from Lemma 2.2.7. Furthermore, Lemma 3.1.1 yields (a) ⇒ (b). Thus, only one implication remains to be shown.

(c) ⇒ (a): By Lemma 3.2.1, there exists a distribution U satisfying $U[(\mathbf{0}, \boldsymbol{\infty})] = 1$ such that

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \;=\; \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \, \frac{(\boldsymbol{\lambda}\, t)^{\mathbf{n}}}{\mathbf{n}!} \, dU(\boldsymbol{\lambda})$$

holds for all $t \in \mathbb{R}_+$ and $\mathbf{n} \in \mathbb{N}_0^k$. The multinomial property, which we know is equivalent to (c), in connection with Lemma 3.1.1 yields the assertion. ∎

So with a multivariate mixed Poisson process with mixing distribution we are in a comfortable situation. Not only that due to the multinomial property we just have to look at the one–dimensional distributions instead of the finite–dimensional distributions, but also the one–dimensional distribution can be reduced to one function which is the moment generating function of the mixing distribution.

3.2.4 Corollary. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with mixing distribution U. Then*

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \;=\; \frac{t^{\mathbf{1}'\mathbf{n}}}{\mathbf{n}!}\, D^{\mathbf{n}} M_U(-t\mathbf{1})$$

holds for all $t > 0$ and all $\mathbf{n} \in \mathbb{N}_0^k$.

Proof: Since $\Pi_{\mathbf{0}}(\mathbf{t}) = M_U(-\mathbf{t})$, we get with Lemma 3.2.1

$$\begin{aligned}
\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] &= \frac{(-t)^{\mathbf{1}'\mathbf{n}}}{\mathbf{n}!} D^{\mathbf{n}}\Pi_{\mathbf{0}}(t\mathbf{1}) \\
&= \frac{(-t)^{\mathbf{1}'\mathbf{n}}}{\mathbf{n}!} \left.\frac{\partial^{\mathbf{1}'\mathbf{n}} M_U(-\mathbf{x})}{\partial^{\mathbf{n}}\mathbf{x}}\right|_{\mathbf{x}=t\mathbf{1}} \\
&= \frac{t^{\mathbf{1}'\mathbf{n}}}{\mathbf{n}!} D^{\mathbf{n}} M_U(-t\mathbf{1})
\end{aligned}$$

and the assertion is shown. ■

Given a multivariate mixed Poisson process the function $\Pi_{\mathbf{0}}$, which can be expressed by the moment generating function of the mixing distribution, provides all necessary information for the finite–dimensional distributions. Additionally, we can also derive binomial moments of the process from this moment generating function, as can be seen in Section 3.3. Not only $\Pi_{\mathbf{0}}$ as a function of time provides all the information of a multivariate mixed Poisson process, but also the distribution of $\mathbf{N}_t$ for any $t > 0$ is sufficient to determine the finite–dimensional distributions of the process. We state the next lemma to see how this works.

3.2.5 Lemma. *Let U and V be two distributions with $U[(\mathbf{0},\infty)] = V[(\mathbf{0},\infty)] = 1$. Let $t > 0$ and let*

$$\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, \frac{(\boldsymbol{\lambda}\, t)^{\mathbf{n}}}{\mathbf{n}!}\, dU(\boldsymbol{\lambda}) \;=\; \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, \frac{(\boldsymbol{\lambda}\, t)^{\mathbf{n}}}{\mathbf{n}!}\, dV(\boldsymbol{\lambda})$$

hold for all $\mathbf{n} \in \mathbb{N}_0^k$.
Then $U = V$.

Proof: The moment generating functions M_U and M_V are both finite for all $\mathbf{s} < \mathbf{0}$. Using Lemma 1.2.1 we see that there exists for all $\mathbf{s} < \mathbf{0}$ a Taylor expansion around

$\mathbf{s}$ and thus the moment generating functions are analytic for $\mathbf{s} < \mathbf{0}$. Examining the Taylor expansion in the neighbourhood B of $-t\mathbf{1}$ we get for all $\mathbf{s} \in B$

$$\begin{aligned} M_U(\mathbf{s}) &= \sum_{\mathbf{n} \in \mathbb{N}_0^k} \frac{(\mathbf{s}+t\mathbf{1})^{\mathbf{n}}}{\mathbf{n}!} \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, \boldsymbol{\lambda}^{\mathbf{n}}\, dU(\boldsymbol{\lambda}) \\ &= \sum_{\mathbf{n} \in \mathbb{N}_0^k} \left(\frac{1}{t}(\mathbf{s}+t\mathbf{1})\right)^{\mathbf{n}} \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, \frac{(\boldsymbol{\lambda} t)^{\mathbf{n}}}{\mathbf{n}!}\, dU(\boldsymbol{\lambda}) \\ &= \sum_{\mathbf{n} \in \mathbb{N}_0^k} \left(\frac{1}{t}(\mathbf{s}+t\mathbf{1})\right)^{\mathbf{n}} \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, \frac{(\boldsymbol{\lambda} t)^{\mathbf{n}}}{\mathbf{n}!}\, dV(\boldsymbol{\lambda}) \\ &= M_V(\mathbf{s}) \end{aligned}$$

Thus, $M_U(\mathbf{s}) = M_V(\mathbf{s})$ for all $\mathbf{s} \in B$ and the principal of analytic continuation (see Dieudonné [1971] 9.4.2) yields $M_U(\mathbf{s}) = M_V(\mathbf{s})$ for all $\mathbf{s} < \mathbf{0}$. Since U and V have only mass on the positive cone of $\mathbb{R}^k$ as well as M_U and M_V are continuous on $(-\boldsymbol{\infty}, \mathbf{0}]$ the identity $M_U(\mathbf{s}) = M_V(\mathbf{s})$ holds for all $\mathbf{s} \leq \mathbf{0}$, which is equivalent to $\mathcal{L}_U = \mathcal{L}_V$ under the assumptions of the lemma, where $\mathcal{L}_U$ and $\mathcal{L}_V$ denote the Laplace transform of the measure U and V, respectively. The uniqueness of the Laplace transform (see Kallenberg [2002] Theorem 5.3) now gives $U = V$. ■

Now it is obvious that the mixing distribution U is determined by the distribution of a single random vector $\mathbf{N}_t$, regardless which $t > 0$ is taken. In the univariate setting certain proofs can be found (see e.g. Teicher [1961] or Grandell [1976] Theorem 1.1) that a mixing distribution is uniquely determined by the mixed Poisson distribution received.

Since a multivariate counting process which has the binomial property and independent increments possesses the multinomial property (see Corollary 2.2.11), it is a multivariate mixed Poisson process with mixing distribution, too. However, to be more precise it is a somewhat special multivariate mixed Poisson process. To make the meaning of the word special plain we give the following definition.

A multivariate counting process $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ is said to be a ***multivariate Poisson process*** if it is a multivariate mixed Poisson process with mixing distribution U and there exists some $\mathbf{x} \in (\mathbf{0}, \boldsymbol{\infty})$ such that $U[\{\mathbf{x}\}] = 1$.

In other words, a multivariate Poisson process is a multivariate mixed Poisson process with degenerated mixing distribution. Thus

$$\mathrm{P}\left[\bigcap_{j=1}^{m} \{\mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j\}\right] = \prod_{i=1}^{k}\prod_{j=1}^{m} e^{-x_i(t_j - t_{j-1})} \frac{(x_i(t_j - t_{j-1}))^{n_j^{(i)}}}{n_j^{(i)}!}$$

holds for all $m \in \mathbb{N}$ and $t_0, t_1, \ldots, t_m \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_m$ and for all $\mathbf{n}_j \in \mathbb{N}_0^k$, $j \in \{1, \ldots, m\}$. It is easy to see that this process has independent coordinates, which are univariate Poisson processes in the usual sense. Therefore, such a process is not really a multivariate process. However, the preceding definition shall just serve as a benchmark, for example in the next theorem.

3.2.6 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process. Then the following are equivalent.*

(a) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *is a multivariate Poisson process.*
(b) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *has the binomial property and independent increments.*
(c) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *has independent coordinates and each coordinate is a Poisson process.*

Proof:
(c) $\Leftrightarrow$ (a): The representation of the finite–dimensional distributions immediately yields the assertion.

(a) $\Rightarrow$ (b): As every multivariate Poisson process is a multivariate mixed Poisson process, it has the binomial property. The independent increments are evident from the representation of the finite–dimensional distributions.

(b) $\Rightarrow$ (a): By Corollary 2.2.11 and Theorem 3.2.3 $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ is a multivariate mixed Poisson process with mixing distribution U. Furthermore, the transformed process $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ with $A \in \mathcal{A}$ also has, as a consequence of Lemma 2.2.2 and 2.2.1, the binomial property and independent increments. In particular, this holds with $A = \mathbf{e}_i{}'$ for every coordinate $\{\mathbf{N}_t^{(i)}\}_{t\in\mathbb{R}_+}$, $i \in \{1, \dots, k\}$. By Schmidt and Zocher [2003] Theorem 3.2 every coordinate is a Poisson process then and thus for all $i \in \{1, \dots, k\}$ there exists some $x_i \in (0, \infty)$ such that $\{\mathbf{N}_t^{(i)}\}_{t\in\mathbb{R}_+}$ is a mixed Poisson process with mixing distribution δ_{x_i}. Therefore

$$U = \delta_{\mathbf{x}} = \bigotimes_{i=1}^{k} \delta_{x_i}$$

is the only distribution with $U_{\mathbf{e}_i{}'} = \delta_{x_i}$ for all $i \in \{1, \dots, k\}$. Hence, $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ is a multivariate Poisson process. ■

With the above theorem we can answer the question whether a multivariate mixed Poisson process can have independent increments.

3.2.7 Corollary. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with mixing distribution U. Then the following are equivalent.*

(a) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *has independent increments.*
(b) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *is a multivariate Poisson process.*

3.3 Moments

This section will provide some properties of binomial moments, moments around the origin, and central moments of the random vectors $\mathbf{N}_t$, $t \in \mathbb{R}_+$. And again, the moment generating function of the mixing distribution will play a leading role. Additionally, the probability generating function will also help a lot to analyze the moments of random vectors. The according theory has been stated in Section 1.1.

3.3.1 Theorem. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate mixed Poisson process with mixing distribution U. Then*

$$g_{\mathbf{N}_t}(\mathbf{r}) \;=\; M_U(t(\mathbf{r}-\mathbf{1}))$$

holds for all $\mathbf{r} \in [\mathbf{0},\mathbf{1}]$ and $t \in \mathbb{R}_+$.

Proof: Consider $\mathbf{r} \in [\mathbf{0},\mathbf{1}]$ and $t \in \mathbb{R}_+$. Then

$$\begin{aligned}
g_{\mathbf{N}_t}(\mathbf{r}) &= \sum_{\mathbf{n} \in \mathbb{N}_0^k} \mathbf{r}^{\mathbf{n}}\, P[\{\mathbf{N}_t = \mathbf{n}\}] \\
&= \sum_{\mathbf{n} \in \mathbb{N}_0^k} \mathbf{r}^{\mathbf{n}} \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \frac{(\boldsymbol{\lambda} t)^{\mathbf{n}}}{\mathbf{n}!}\, dU(\boldsymbol{\lambda}) \\
&= \int_{\mathbb{R}^k} \sum_{\mathbf{n} \in \mathbb{N}_0^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \frac{(\boldsymbol{\lambda} t)^{\mathbf{n}}\, \mathbf{r}^{\mathbf{n}}}{\mathbf{n}!}\, dU(\boldsymbol{\lambda}) \\
&= \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \prod_{i=1}^{k} \sum_{n^{(i)} \in \mathbb{N}_0} \frac{(\lambda_i\, t\, r_i)^{n^{(i)}}}{n^{(i)}!}\, dU(\boldsymbol{\lambda}) \\
&= \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, e^{\mathbf{r}'\boldsymbol{\lambda} t}\, dU(\boldsymbol{\lambda}) \\
&= M_U(t(\mathbf{r}-\mathbf{1}))
\end{aligned}$$

and the assertion is shown. ■

3.3.2 Corollary. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate mixed Poisson process with mixing distribution U and let $A \in \mathcal{A}$. Then*
(1) $g_{A\mathbf{N}_t}(\mathbf{r}) = g_{\mathbf{N}_t}(A'\mathbf{r})$ *if $A \in \mathcal{A}_P \cup \mathcal{A}_C$.*
(2) $g_{A\mathbf{N}_t}(\mathbf{r}) = g_{\mathbf{N}_t}(A'\mathbf{r} + \mathbf{1} - A'\mathbf{1})$ *if $A \in \mathcal{A}_S$.*

Proof: For arbitrary $A \in \mathcal{A}$ we get

$$\begin{aligned}
g_{A\mathbf{N}_t}(\mathbf{r}) &= M_{U_A}(t(\mathbf{r}-\mathbf{1})) \\
&= M_U(A't(\mathbf{r}-\mathbf{1})) \\
&= M_U(t(A'\mathbf{r}-A'\mathbf{1})) \\
&= M_U(t(A'\mathbf{r}-A'\mathbf{1}+\mathbf{1}-\mathbf{1})) \\
&= g_{\mathbf{N}_t}(A'\mathbf{r}+\mathbf{1}-A'\mathbf{1})
\end{aligned}$$

and thus (2). Since $A'\mathbf{1} = \mathbf{1}$ holds for $A \in \mathcal{A}_P \cup \mathcal{A}_C$ (1) follows.
Remark: Of course, in the case $A'\mathbf{1} = \mathbf{1}$ and $a_{ij} \in \{0,1\}$ the assertion can also be proven directly from the definition of the probability generating function by using power laws. ■

Theorem 3.3.1 shows that the moment generating function of the mixing distribution also contains information regarding the binomial moments of the multivariate mixed Poisson process. This enables us to formulate conditions for the finiteness of such moments we will write down in the next two theorems. But before that we state a short lemma.

3.3.3 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with mixing distribution U. Then*

$$\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] = \frac{t^{\mathbf{1}'\mathbf{l}}}{\mathbf{l}!}\int_{\mathbb{R}^k} \boldsymbol{\lambda}^{\mathbf{l}}\, dU(\boldsymbol{\lambda})$$

holds for all $\mathbf{l} \in \mathbb{N}_0^k$ and $t \in \mathbb{R}_+$.

Proof: Let $\mathbf{l} \in \mathbb{N}_0^k$ and $t > 0$. From Lemma 1.1.1, Theorem 3.3.1, Lemma 1.2.1, and the monotone convergence theorem we get

$$\begin{aligned}
\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] &= \sup_{\mathbf{r}\in[\mathbf{0},\mathbf{1})} \frac{1}{\mathbf{l}!} D^{\mathbf{l}} g_{\mathbf{N}_t}(\mathbf{r}) \\
&= \sup_{\mathbf{r}\in[\mathbf{0},\mathbf{1})} \frac{1}{\mathbf{l}!} \left.\frac{\partial^{\mathbf{1}'\mathbf{l}} M_U(t(\mathbf{x}-\mathbf{1}))}{\partial^{\mathbf{l}}\mathbf{x}}\right|_{\mathbf{x}=\mathbf{r}} \\
&= \sup_{\mathbf{s}\in[-t\mathbf{1},\mathbf{0})} \frac{t^{\mathbf{1}'\mathbf{l}}}{\mathbf{l}!} D^{\mathbf{l}} M_U(\mathbf{s}) \\
&= \frac{t^{\mathbf{1}'\mathbf{l}}}{\mathbf{l}!} \sup_{\mathbf{s}\in[-t\mathbf{1},\mathbf{0})} \int_{\mathbb{R}^k} \boldsymbol{\lambda}^{\mathbf{l}} e^{\mathbf{s}'\boldsymbol{\lambda}}\, dU(\boldsymbol{\lambda}) \\
&= \frac{t^{\mathbf{1}'\mathbf{l}}}{\mathbf{l}!} \int_{\mathbb{R}^k} \sup_{\mathbf{s}\in[-t\mathbf{1},\mathbf{0})} \boldsymbol{\lambda}^{\mathbf{l}} e^{\mathbf{s}'\boldsymbol{\lambda}}\, dU(\boldsymbol{\lambda}) \\
&= \frac{t^{\mathbf{1}'\mathbf{l}}}{\mathbf{l}!} \int_{\mathbb{R}^k} \boldsymbol{\lambda}^{\mathbf{l}}\, dU(\boldsymbol{\lambda})
\end{aligned}$$

Since $\mathbf{N}_0 = \mathbf{0}$ almost surely, the assertion holds for $t = 0$, too. ■

3.3.4 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with mixing distribution U and $\mathbf{l} \in \mathbb{N}_0^k$. Then the following are equivalent.*

(a) *There exists a some $t > 0$ such that*

$$\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] < \infty$$

holds.

(b) *The inequality*

$$\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] \;<\; \infty$$

holds for all $t \in \mathbb{R}_+$.

(c) *The mixing distribution satisfies*

$$\int_{\mathbb{R}^k} \boldsymbol{\lambda}^{\mathbf{l}} \, dU(\boldsymbol{\lambda}) \;<\; \infty$$

(d) *For all* $\mathbf{s} \in (-\boldsymbol{\infty}, \mathbf{0}]$ *the inequality*

$$\lim_{\mathbf{r} \to \mathbf{s}} D^{\mathbf{l}} M_U|_{(-\boldsymbol{\infty},\mathbf{0})}(\mathbf{r}) \;<\; \infty$$

is valid.

If $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *satisfies one and hence all preceding items, then*

$$\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] \;=\; \frac{t^{\mathbf{1}'\mathbf{l}}}{\mathbf{l}!} \lim_{\mathbf{r} \uparrow \mathbf{0}} D^{\mathbf{l}} M_U|_{(-\boldsymbol{\infty},\mathbf{0})}(\mathbf{r})$$

holds for $t \in \mathbb{R}_+$.

Proof: The equivalence of (a), (b), and (c) is due to Lemma 3.3.3.

(a) $\Leftrightarrow$ (d): With Theorem 3.3.1 $g_{\mathbf{N}_t}(\mathbf{r}) = M_U(t(\mathbf{r} - \mathbf{1}))$ holds for $\mathbf{r} \in [\mathbf{0}, \mathbf{1}]$ and $t > 0$. Thus, the assertion immediately follows from Lemma 1.1.3.

Moreover, we have

$$\begin{aligned}
\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] &= \frac{1}{\mathbf{l}!} \lim_{\mathbf{r} \uparrow \mathbf{1}} D^{\mathbf{l}} g_{\mathbf{N}_t}|_{[\mathbf{0},\mathbf{1})}(\mathbf{r}) \\
&= \frac{1}{\mathbf{l}!} \lim_{\substack{\mathbf{r} \uparrow \mathbf{1} \\ \mathbf{r} \in [\mathbf{0},\mathbf{1})}} \left. \frac{\partial^{\mathbf{1}'\mathbf{l}} M_U(t(\mathbf{x} - \mathbf{1}))}{\partial^{\mathbf{l}} \mathbf{x}} \right|_{\mathbf{x}=\mathbf{r}} \\
&= \frac{t^{\mathbf{1}'\mathbf{l}}}{\mathbf{l}!} \lim_{\mathbf{r} \uparrow \mathbf{0}} D^{\mathbf{l}} M_U|_{(-\boldsymbol{\infty},\mathbf{0})}(\mathbf{r})
\end{aligned}$$

for all $t > 0$. Since $\mathbf{N}_0 = \mathbf{0}$ almost surely, the assertion holds for $t = 0$, too. ■

3.3.5 Theorem. *Let* $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *be a multivariate mixed Poisson process with mixing distribution* U *and* $\mathbf{l} \in \mathbb{N}_0^k$. *Then the following are equivalent.*

(a) *For all* $\mathbf{m} \leq \mathbf{l}$ *there exists some* $t > 0$ *such that*

$$\mathrm{E}\left[\, (\mathbf{N}_t)^{\mathbf{m}} \,\right] \;<\; \infty$$

holds.

(b) *For all* $\mathbf{m} \le \mathbf{l}$ *the inequality*

$$\mathrm{E}\left[\,(\mathbf{N}_t)^{\mathbf{m}}\,\right] \;<\; \infty$$

holds for all $t \in \mathbb{R}_+$.

(c) *For all* $\mathbf{m} \le \mathbf{l}$ *the mixing distribution fulfils*

$$\int_{\mathbb{R}^k} \boldsymbol{\lambda}^{\mathbf{m}}\, dU(\boldsymbol{\lambda}) \;<\; \infty$$

(d) *For all* $\mathbf{m} \le \mathbf{l}$ *the* $\mathbf{m}$*–th derivative of* $M_U|_{(-\boldsymbol{\infty},\mathbf{0}]}$ *is continuous on* $(-\boldsymbol{\infty},\mathbf{0}]$.

If $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *satisfies one and hence all preceding items, then*

$$\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] \;=\; \frac{t^{\mathbf{1}'\mathbf{l}}}{\mathbf{l}!}\, D^{\mathbf{l}} M_U|_{(-\boldsymbol{\infty},\mathbf{0}]}(\mathbf{0})$$

holds for $t \in \mathbb{R}_+$.

Proof: Due to Lemma 1.1.4 the equivalence of (a), (b), and (c) follows from Theorem 3.3.4.

(b) $\Leftrightarrow$ (d): With Theorem 3.3.1 $g_{\mathbf{N}_t}(\mathbf{r}) = M_U(t(\mathbf{r}-\mathbf{1}))$ holds for all $\mathbf{r} \in [\mathbf{0},\mathbf{1}]$ and $t > 0$. Thus, Lemma 1.1.4 yields the assertion.

Since condition (c) of Theorem 3.3.4 ($\int_{\mathbb{R}^k} \boldsymbol{\lambda}^{\mathbf{m}}\, dU(\boldsymbol{\lambda}) < \infty$) is fulfilled, we additionally get with the continuity of the $\mathbf{l}$–th derivative of M_U

$$\begin{aligned}
\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] &= \frac{t^{\mathbf{1}'\mathbf{l}}}{\mathbf{l}!} \lim_{\mathbf{r}\uparrow\mathbf{0}} D^{\mathbf{l}} M_U|_{(-\boldsymbol{\infty},\mathbf{0})}(\mathbf{r}) \\
&= \frac{t^{\mathbf{1}'\mathbf{l}}}{\mathbf{l}!}\, D^{\mathbf{l}} M_U|_{(-\boldsymbol{\infty},\mathbf{0}]}(\mathbf{0})
\end{aligned}$$

which completes the proof. ∎

As a consequence of Lemma 3.3.3 and the equivalence of (c) and (d) in Theorem 3.3.4 we have

$$\lim_{\mathbf{r}\uparrow\mathbf{0}} D^{\mathbf{l}} M_U|_{(-\boldsymbol{\infty},\mathbf{0})}(\mathbf{r}) \;=\; \int_{\mathbb{R}^k} \boldsymbol{\lambda}^{\mathbf{l}}\, dU(\boldsymbol{\lambda})$$

in the case one of the terms is finite. We do not, as in Lemma 1.2.1, require the finiteness of the moment generating function M_U in a neighbourhood of $\mathbf{0}$. Under this condition all derivatives of M_U at $\mathbf{0}$ would exist and therefore all moments of $\mathbf{N}_t$ would be finite. With a detour about the probability generating function of a multivariate mixed Poisson process with mixing distribution at some time t we were able to refine results for the moment generating function.

With the properties derived so far, we can deduce conditions for the finiteness of the first and second central moment of the process at some time t, which will be illustrated in the next corollary.

3.3.6 Corollary. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with mixing distribution U and let $i, j \in \{1,\dots,k\}$ with $i \neq j$.*

(1) *If $\int_{\mathbb{R}^k} \lambda_i \, dU(\boldsymbol{\lambda}) < \infty$, then $\mathrm{E}[N_t^{(i)}] < \infty$ and*

$$\mathrm{E}\left[N_t^{(i)}\right] \;=\; t\int_{\mathbb{R}^k} \lambda_i \, dU(\boldsymbol{\lambda})$$

hold for all $t \in \mathbb{R}_+$.

(2) *If $\int_{\mathbb{R}^k} (\lambda_i)^2 \, dU(\boldsymbol{\lambda}) < \infty$, then $\mathrm{E}[(N_t^{(i)})^2] < \infty$ and*

$$\mathrm{Var}\left[N_t^{(i)}\right] \;=\; t^2 \int_{\mathbb{R}^k} \left(\lambda_i - \int_{\mathbb{R}^k} x_i \, dU(\mathbf{x})\right)^2 dU(\boldsymbol{\lambda}) \;+\; t\int_{\mathbb{R}^k} \lambda_i \, dU(\boldsymbol{\lambda})$$

hold for all $t \in \mathbb{R}_+$.

(3) *If $\max\left\{\int_{\mathbb{R}^k} \lambda_i \lambda_j \, dU(\boldsymbol{\lambda}),\ \int_{\mathbb{R}^k} \lambda_i \, dU(\boldsymbol{\lambda}),\ \int_{\mathbb{R}^k} \lambda_j \, dU(\boldsymbol{\lambda})\right\} < \infty$, then $\mathrm{E}[N_t^{(i)} N_t^{(j)}] < \infty$, $\mathrm{E}[N_t^{(i)}] < \infty$ as well as $\mathrm{E}[N_t^{(j)}] < \infty$ and*

$$\mathrm{Cov}\left[N_t^{(i)}, N_t^{(j)}\right] \;=\; t^2 \int_{\mathbb{R}^k} \left(\lambda_i - \int_{\mathbb{R}^k} x_i \, dU(\mathbf{x})\right)\left(\lambda_j - \int_{\mathbb{R}^k} x_j \, dU(\mathbf{x})\right) dU(\boldsymbol{\lambda})$$

hold for all $t \in \mathbb{R}_+$.

Proof: Since $\mathbf{N}_0 = \mathbf{0}$ almost surely, the assertion holds for $t = 0$. For $t > 0$ we can prove the assertions using Lemma 3.3.3 and transformation between binomial and central moments (compare proof of Corollary 1.1.5). ■

The above corollary shows a significant difference between multivariate Poisson processes and multivariate mixed Poisson processes with a non–degenerate mixing distribution. If U is a degenerate distribution then, and only then,

$$\mathrm{Var}\left[N_t^{(i)}\right] \;=\; t\int_{\mathbb{R}^k} \lambda_i \, dU(\boldsymbol{\lambda}) \;=\; \mathrm{E}\left[N_t^{(i)}\right]$$

for all $i \in \{1,\dots,k\}$. If U is degenerated we also have $\mathrm{Cov}[N_t^{(i)}, N_t^{(j)}] = 0$ for $i \neq j$, which is entirely clear since the coordinates of the process are independent.

To draw a conclusion of this section we remark that the moment generating function M_U of the mixing distribution does not only determine the one–dimensional distributions of the process, but also the moments of $\mathbf{N}_t$ with $t \in \mathbb{R}_+$. The transformed processes $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ with $A \in \mathcal{A}$ are multivariate mixed Poisson processes with mixing distribution U_A. As a consequence of

$$M_{U_A}(\mathbf{t}) \;=\; M_U(A'\mathbf{t})$$

(see Lemma 1.2.2) the function M_U also contains information about one–dimensional distributions and moments of the transformed processes.

3.4 Regularity

We start the section by giving a representation of the transition probabilities.

3.4.1 Lemma. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate mixed Poisson process with mixing distribution U. Then*

$$\begin{aligned} p_{\mathbf{n},\mathbf{m}}(r,t) &= \frac{(t-r)^{\mathbf{1}'(\mathbf{m}-\mathbf{n})}}{(\mathbf{m}-\mathbf{n})!} \frac{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \boldsymbol{\lambda}^{\mathbf{m}} \, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} r} \boldsymbol{\lambda}^{\mathbf{n}} \, dU(\boldsymbol{\lambda})} \\ &= \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda}(t-r)} \frac{(\boldsymbol{\lambda}(t-r))^{\mathbf{m}-\mathbf{n}}}{(\mathbf{m}-\mathbf{n})!} \, dU_{r,\mathbf{n}}(\boldsymbol{\lambda}) \end{aligned}$$

holds for all $(\mathbf{n}, r) \in Z$ and $\mathbf{m} \in \mathbb{N}_0^k$, $t \in \mathbb{R}_+$ with $\mathbf{n} \leq \mathbf{m}$ and $r \leq t$.

Proof: Recalling $p_{\mathbf{n},\mathbf{m}}(r,t) = \mathrm{P}_{r,\mathbf{n}}[\{\mathbf{K}_{r,t-r} = \mathbf{m}-\mathbf{n}\}]$, we immediately obtain from Theorem 3.1.5

$$\begin{aligned} p_{\mathbf{n},\mathbf{m}}(r,t) &= \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda}(t-r)} \frac{(\boldsymbol{\lambda}(t-r))^{\mathbf{m}-\mathbf{n}}}{(\mathbf{m}-\mathbf{n})!} \, dU_{r,\mathbf{n}}(\boldsymbol{\lambda}) \\ &= \frac{\displaystyle\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda}(t-r)} \frac{(\boldsymbol{\lambda}(t-r))^{\mathbf{m}-\mathbf{n}}}{(\mathbf{m}-\mathbf{n})!} e^{-\mathbf{1}'\boldsymbol{\lambda} r} \boldsymbol{\lambda}^{\mathbf{n}} \, dU(\boldsymbol{\lambda})}{\displaystyle\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} r} \boldsymbol{\lambda}^{\mathbf{n}} \, dU(\boldsymbol{\lambda})} \\ &= \frac{(t-r)^{\mathbf{1}'(\mathbf{m}-\mathbf{n})}}{(\mathbf{m}-\mathbf{n})!} \frac{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \boldsymbol{\lambda}^{\mathbf{m}} \, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} r} \boldsymbol{\lambda}^{\mathbf{n}} \, dU(\boldsymbol{\lambda})} \end{aligned}$$

which yields the assertion. ■

The next aim is to characterize regular processes among multivariate mixed Poisson processes in a similar way it was done with regular processes among processes having the binomial property.

3.4.2 Theorem. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate mixed Poisson process with mixing distribution U. Then the following are equivalent.*

(a) $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *is regular with intensities* $\{\boldsymbol{\kappa}_\mathbf{n}\}_{\mathbf{n} \in \mathbb{N}_0^k}$.

(b) *The inequality* $\int_{\mathbb{R}^k} \mathbf{1}'\boldsymbol{\lambda} \, dU(\boldsymbol{\lambda}) < \infty$ *is valid.*

If $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ satisfies one and hence all preceding items, then

$$\begin{aligned} \kappa_\mathbf{n}^{(i)}(t) &= \frac{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \boldsymbol{\lambda}^{\mathbf{n}+\mathbf{e}_i} \, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \boldsymbol{\lambda}^{\mathbf{n}} \, dU(\boldsymbol{\lambda})} \\ &= \int_{\mathbb{R}^k} \lambda_i \, dU_{t,\mathbf{n}}(\boldsymbol{\lambda}) \end{aligned}$$

holds for all $(\mathbf{n}, t) \in Z$ and $i \in \{1, \ldots, k\}$.

Proof: Since a multivariate mixed Poisson process has the binomial property (Theorem 3.2.3), the assumptions of Theorem 2.3.16 are fulfilled. Thus, we can prove the equivalence of (a) and (b) by showing that $\int_{\mathbb{R}^k} \mathbf{1}'\boldsymbol{\lambda}\, dU(\boldsymbol{\lambda}) < \infty$ if, and only if, $\lim_{t\downarrow 0} t^{-1}\mathrm{P}[\{\mathbf{N}_t = \mathbf{e}_i\}] < \infty$ for all $i \in \{1,\ldots,k\}$. By monotone convergence we obtain for $i \in \{1,\ldots,k\}$

$$\begin{aligned}
\lim_{t\downarrow 0} \frac{1}{t} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{e}_i\}\right] &= \lim_{t\downarrow 0} \frac{1}{t} \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \lambda_i\, t\, dU(\boldsymbol{\lambda}) \\
&= \int_{\mathbb{R}^k} \lim_{t\downarrow 0} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \lambda_i\, dU(\boldsymbol{\lambda}) \\
&= \int_{\mathbb{R}^k} \lambda_i\, dU(\boldsymbol{\lambda})
\end{aligned}$$

and with

$$\sum_{i=1}^k \int_{\mathbb{R}^k} \lambda_i\, dU(\boldsymbol{\lambda}) = \int_{\mathbb{R}^k} \mathbf{1}'\boldsymbol{\lambda}\, dU(\boldsymbol{\lambda})$$

the equivalence of (a) and (b) is proven.
Consider $t > 0$, $\mathbf{n} \in \mathbb{N}_0^k$, and $i \in \{1,\ldots,k\}$. Using again Theorem 2.3.16 we get under the assumption of regularity

$$\begin{aligned}
\kappa_{\mathbf{n}}^{(i)}(t) &= \frac{n^{(i)}+1}{t} \frac{\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}+\mathbf{e}_i\}\right]}{\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]} \\
&= \frac{n^{(i)}+1}{t} \frac{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} (\boldsymbol{\lambda}\, t)^{\mathbf{n}+\mathbf{e}_i}/(\mathbf{n}+\mathbf{e}_i)!\, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} (\boldsymbol{\lambda}\, t)^{\mathbf{n}}/\mathbf{n}!\, dU(\boldsymbol{\lambda})} \\
&= \frac{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \boldsymbol{\lambda}^{\mathbf{n}+\mathbf{e}_i}\, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \boldsymbol{\lambda}^{\mathbf{n}}\, dU(\boldsymbol{\lambda})} \\
&= \int_{\mathbb{R}^k} \lambda_i\, dU_{t,\mathbf{n}}(\boldsymbol{\lambda})
\end{aligned}$$

and additionally with monotone convergence

$$\begin{aligned}
\kappa_{\mathbf{0}}^{(i)}(0) &= \lim_{t\downarrow 0} \kappa_{\mathbf{0}}^{(i)}(t) \\
&= \lim_{t\downarrow 0} \frac{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \lambda_i\, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, dU(\boldsymbol{\lambda})} \\
&= \left(\int_{\mathbb{R}^k} \lambda_i\, dU(\boldsymbol{\lambda})\right) \left(\int_{\mathbb{R}^k} 1\, dU(\boldsymbol{\lambda})\right)^{-1} \\
&= \int_{\mathbb{R}^k} \lambda_i\, dU(\boldsymbol{\lambda})
\end{aligned}$$

and hence the representation of the intensities. ■

Thus, a multivariate mixed Poisson process with mixing distribution is regular if, and only if, the mixing distribution and therewith the random vectors $\mathbf{N}_t$, $t > 0$, (see Theorem 3.3.5) have finite moments of first order. In Schmidt [1996] regularity requires the existence of intensities on the closed positive half line and therefore a univariate mixed Poisson process $\{N_t\}_{t \in \mathbb{R}_+}$ has to have moments of any order to be regular. In this study the concept of regularity is chosen in a way that regularity is still equivalent to the Kolmogorov system of backward and forward differential equations under the assumption of the Chapman–Kolmogorov property, but a regular multivariate mixed Poisson process may have infinite moments of order higher than one.

A multivariate mixed Poisson process has the Chapman–Kolmogorov property and hence fulfils the Kolmogorov system of backward and forward differential equations whenever the process is regular.

In the univariate setting the differential equations for the intensities from Theorem 2.3.19 characterize mixed Poisson processes among regular Markov processes (see Grandell [1997] Theorem 6.1). The proof is done via the (univariate) Bernstein–Widder theorem. For a similar result considering multivariate counting processes it is necessary to use the multivariate Bernstein–Widder theorem. Consequently, we need a function depending on k time variables (remember the extended binomial property), which is not given by the introduced intensities. It is of course possible to define intensity functions for a multivariate process which depend on multiple variables. However, the interpretation and the usefulness of such a definition is doubtful so that we abstain from introducing it. Nevertheless, we are able to give a characterization of multivariate mixed Poisson processes with mixing distribution among regular Markov processes.

3.4.3 Theorem. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process which is a regular Markov process with intensities $\{\boldsymbol{\kappa}_\mathbf{n}\}_{\mathbf{n} \in \mathbb{N}_0^k}$. Furthermore, let $U : \mathcal{B}(\mathbb{R}^k) \to [0,1]$ be a distribution with $U[(\mathbf{0}, \boldsymbol{\infty})] = 1$. Then the following are equivalent.*

(a) *$\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ is a multivariate mixed Poisson process with mixing distribution U.*

(b) *The transition probabilities fulfil*

$$p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(t, t+h) \;=\; \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} h}\, \lambda_i h \, dU_{t,\mathbf{n}}(\boldsymbol{\lambda})$$

for all $t > 0$, $\mathbf{n} \in \mathbb{N}_0^k$ and all $h \in \mathbb{R}_+$.

(c) *The intensities fulfil*

$$\kappa_\mathbf{n}^{(i)}(t) \;=\; \int_{\mathbb{R}^k} \lambda_i \, dU_{t,\mathbf{n}}(\boldsymbol{\lambda})$$

for all $t > 0$ and all $\mathbf{n} \in \mathbb{N}_0^k$.

Proof: We prove the assertion according to the following scheme: (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b): Lemma 3.4.1 yields the assertion.

(b) $\Rightarrow$ (c): From the definition of the intensities we get for all $t > 0$ and all $\mathbf{n} \in \mathbb{N}_0^k$ with the help of the monotone convergence theorem

$$\begin{aligned}
\kappa_{\mathbf{n}}^{(i)}(t) &= \lim_{h\downarrow 0} \frac{1}{h}\, p_{\mathbf{n},\mathbf{n}+\mathbf{e}_i}(t, t+h) \\
&= \lim_{h\downarrow 0} \frac{1}{h} \int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} h}\, \lambda_i\, h\, dU_{t,\mathbf{n}}(\boldsymbol{\lambda}) \\
&= \int_{\mathbb{R}^k} \lim_{h\downarrow 0} e^{-\mathbf{1}'\boldsymbol{\lambda} h}\, \lambda_i\, dU_{t,\mathbf{n}}(\boldsymbol{\lambda}) \\
&= \int_{\mathbb{R}^k} \lambda_i\, dU_{t,\mathbf{n}}(\boldsymbol{\lambda})
\end{aligned}$$

(c) $\Rightarrow$ (a): The intensities of a regular Markov process do uniquely determine the finite–dimensional distributions of the process (see Corollary 2.3.9). As intensities fulfilling $\kappa_{\mathbf{n}}^{(i)}(t) = \int_{\mathbb{R}^k} \lambda_i\, dU_{t,\mathbf{n}}(\boldsymbol{\lambda})$ belong with Theorem 3.4.2 to a multivariate mixed Poisson process with mixing distribution U the assertion follows. ∎

As an outcome of the claimed continuity of the intensity $\kappa_{\mathbf{0}}^{(i)}$ and the transition probabilities $p_{\mathbf{0},\mathbf{m}}(\cdot, h)$ at zero under regularity, we do not have to require any properties for $t = 0$ in (b) and (c). Furthermore, $\int_{\mathbb{R}^k} \mathbf{1}'\boldsymbol{\lambda}\, dU(\boldsymbol{\lambda})$ is finite in any item of the above theorem.

Without surprise we are able to express the intensities of a multivariate mixed Poisson process with mixing distribution in terms of the moment generating function. This enables us to derive with ease some properties of the intensities.

3.4.4 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a regular multivariate mixed Poisson process with mixing distribution U. Then*

$$\kappa_{\mathbf{n}}^{(i)}(t) = \frac{D^{\mathbf{n}+\mathbf{e}_i} M_U(-t\mathbf{1})}{D^{\mathbf{n}} M_U(-t\mathbf{1})}$$

holds for all $t > 0$, $\mathbf{n} \in \mathbb{N}_0^k$, and $i \in \{1, \ldots, k\}$.

Proof: The assertion follows directly from Theorem 3.4.2 and Lemma 1.2.1. ∎

3.4.5 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a regular multivariate mixed Poisson process with mixing distribution U. Then the intensities $\{\boldsymbol{\kappa}_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}_0^k}$ have the following properties.*

(1) *The intensities are infinitely often differentiable on $(0, \infty)$.*

(2) *For all* $i \in \{1, \ldots, k\}$ *and all* $t > 0$ *the intensity* $\kappa_{\mathbf{n}}^{(i)}(t)$ *is increasing in* $n^{(i)}$.

(3) *Let* $i \in \{1, \ldots, k\}$, $\mathbf{n} \in \mathbb{N}_0^k$ *and let* $\int_{\mathbb{R}^k} \boldsymbol{\lambda}^{\mathbf{n}+\mathbf{e}_i} \, dU(\boldsymbol{\lambda})$ *be finite. Then the limit* $\lim_{t \downarrow 0} \kappa_{\mathbf{n}}^{(i)}(t)$ *is finite and*

$$\lim_{t \downarrow 0} \kappa_{\mathbf{n}}^{(i)}(t) \quad = \quad \frac{\int_{\mathbb{R}^k} \boldsymbol{\lambda}^{\mathbf{n}+\mathbf{e}_i} \, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} \boldsymbol{\lambda}^{\mathbf{n}} \, dU(\boldsymbol{\lambda})}$$

holds.

Proof:

(1): For $t \in (0, \infty)$ the intensities are a quotient of derivatives of the moment generating function M_U with arguments $-t\mathbf{1} < \mathbf{0}$ (see Theorem 3.4.4). As the moment generating function M_U is analytic on $(-\boldsymbol{\infty}, \mathbf{0})$, it is infinitely often differentiable thereon. And so are the intensities on $(0, \infty)$.

(2): Assume $t > 0$. Setting

$$c \quad := \quad e^{-\mathbf{1}'\boldsymbol{\lambda} t} \, \boldsymbol{\lambda}^{\mathbf{n}}$$

we get for all $i \in \{1, \ldots, k\}$ by the representation of the intensities (Theorem 3.4.2)

$$\begin{aligned}
\kappa_{\mathbf{n}+\mathbf{e}_i}^{(i)}(t) - \kappa_{\mathbf{n}}^{(i)}(t) \quad &= \quad \frac{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \, \boldsymbol{\lambda}^{\mathbf{n}+2\mathbf{e}_i} \, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \, \boldsymbol{\lambda}^{\mathbf{n}+\mathbf{e}_i} \, dU(\boldsymbol{\lambda})} - \frac{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \, \boldsymbol{\lambda}^{\mathbf{n}+\mathbf{e}_i} \, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \, \boldsymbol{\lambda}^{\mathbf{n}} \, dU(\boldsymbol{\lambda})} \\
&= \quad \frac{\int_{\mathbb{R}^k} (\lambda_i)^2 \, c \, dU(\boldsymbol{\lambda}) \, \int_{\mathbb{R}^k} c \, dU(\boldsymbol{\lambda}) - \left(\int_{\mathbb{R}^k} \lambda_i \, c \, dU(\boldsymbol{\lambda}) \right)^2}{\int_{\mathbb{R}^k} \lambda_i \, c \, dU(\boldsymbol{\lambda}) \, \int_{\mathbb{R}^k} c \, dU(\boldsymbol{\lambda})}
\end{aligned}$$

Using the Cauchy–Schwarz inequality for $\lambda_i \sqrt{c}$ and $\sqrt{c}$ we obtain

$$\int_{\mathbb{R}^k} (\lambda_i)^2 \, c \, dU(\boldsymbol{\lambda}) \int_{\mathbb{R}^k} c \, dU(\boldsymbol{\lambda}) - \left(\int_{\mathbb{R}^k} \lambda_i \, c \, dU(\boldsymbol{\lambda}) \right)^2 \quad \geq \quad 0$$

and therefore $\kappa_{\mathbf{n}}^{(i)}(t)$ increases in $n^{(i)}$ for all $t > 0$.

(3): Since $U[(\mathbf{0}, \boldsymbol{\infty})] = 1$ holds, we get for all $\mathbf{n} \in \mathbb{N}_0^k$

$$\int_{\mathbb{R}^k} \boldsymbol{\lambda}^{\mathbf{n}} \, dU(\boldsymbol{\lambda}) \quad > \quad 0$$

Now, taking the limit of the intensity yields

$$\begin{aligned}
\lim_{t \downarrow 0} \kappa_{\mathbf{n}}^{(i)}(t) \quad &= \quad \lim_{t \downarrow 0} \frac{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \, \boldsymbol{\lambda}^{\mathbf{n}+\mathbf{e}_i} \, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \, \boldsymbol{\lambda}^{\mathbf{n}} \, dU(\boldsymbol{\lambda})} \\
&= \quad \frac{\int_{\mathbb{R}^k} \boldsymbol{\lambda}^{\mathbf{n}+\mathbf{e}_i} \, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} \boldsymbol{\lambda}^{\mathbf{n}} \, dU(\boldsymbol{\lambda})}
\end{aligned}$$

which is finite by assumption. ■

Considering the monotony in t, we have from Grandell [1997] that the intensities of a univariate mixed Poisson process are decreasing in t. This assertion cannot be carried over to the multivariate setting since there exist intensities which are strictly increasing in t.

Example: We consider the bivariate case and set

$$c \ := \ e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, \boldsymbol{\lambda}^{\mathbf{n}}$$

Then we obtain from Theorem 2.3.19 and the representation of intensities

$$\begin{aligned}
\frac{d}{dt}\kappa_{\mathbf{n}}^{(1)}(t) \ &= \ \kappa_{\mathbf{n}}^{(1)}(t)\left(\sum_{j=1}^{2}\kappa_{\mathbf{n}}^{(j)}(t) - \kappa_{\mathbf{n}+\mathbf{e}_1}^{(j)}(t)\right)\\
&= \ \frac{\int_{\mathbb{R}^2}\lambda_1\, c\, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^2} c\, dU(\boldsymbol{\lambda})}\left(\sum_{j=1}^{2}\frac{\int_{\mathbb{R}^2}\lambda_j\, c\, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^2} c\, dU(\boldsymbol{\lambda})} - \frac{\int_{\mathbb{R}^2}\lambda_j\,\lambda_1\, c\, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^2}\lambda_1\, c\, dU(\boldsymbol{\lambda})}\right)\\
&= \ \frac{\int_{\mathbb{R}^2}\lambda_1\, c\, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^2} c\, dU(\boldsymbol{\lambda})}\left(\frac{\int_{\mathbb{R}^2}(\lambda_1+\lambda_2)\, c\, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^2} c\, dU(\boldsymbol{\lambda})} - \frac{\int_{\mathbb{R}^2}(\lambda_1+\lambda_2)\,\lambda_1\, c\, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^2}\lambda_1\, c\, dU(\boldsymbol{\lambda})}\right)\\
&= \ \frac{\int_{\mathbb{R}^2}\lambda_1\, c\, dU(\boldsymbol{\lambda})\ \int_{\mathbb{R}^2}(\lambda_1+\lambda_2)\, c\, dU(\boldsymbol{\lambda}) - \int_{\mathbb{R}^2}(\lambda_1+\lambda_2)\,\lambda_1\, c\, dU(\boldsymbol{\lambda})\ \int_{\mathbb{R}^2} c\, dU(\boldsymbol{\lambda})}{\left(\int_{\mathbb{R}^2} c\, dU(\boldsymbol{\lambda})\right)^2}
\end{aligned}$$

Furthermore, we consider the set $A := \{\mathbf{x} \in \mathbb{R}^2 \ : \ x_1 = 0.5\,(\sqrt{(x_2)^2+8} - x_2),\ x_2 \in (0,\infty)\}$ and a distribution U with $U(A) = 1$. This means, that the mass of U is concentrated on the set where the equation $x_1 = 2/(x_1+x_2)$ is fulfilled. Together with this assumption we get

$$\begin{aligned}
\frac{d}{dt}\kappa_{\mathbf{n}}^{(i)}(t) \ &= \ 2\left(\frac{\int_{\mathbb{R}^2}(\lambda_1+\lambda_2)^{-1}\, c\, dU(\boldsymbol{\lambda})\ \int_{\mathbb{R}^2}(\lambda_1+\lambda_2)\, c\, dU(\boldsymbol{\lambda}) - \left(\int_{\mathbb{R}^2} c\, dU(\boldsymbol{\lambda})\right)^2}{\left(\int_{\mathbb{R}^2} c\, dU(\boldsymbol{\lambda})\right)^2}\right)\\
&> \ 0
\end{aligned}$$

by the use of the Cauchy–Schwarz inequality for $\sqrt{(\lambda_1+\lambda_2)^{-1}\, c}$ and $\sqrt{(\lambda_1+\lambda_2)\, c}$ which are not almost surely linear dependent. $\square$

Our next aim is to rewrite Corollary 2.3.18 for multivariate mixed Poisson processes. This result can be used to obtain a bound for the absolute alteration of the probabilities $\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]$ in an infinitesimal time interval.

3.4.6 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a regular multivariate mixed Poisson process with mixing distribution U. Then*

$$\int_{\mathbb{R}^k}\lambda_i\, dU(\boldsymbol{\lambda}) \ = \ \sum_{\mathbf{n}\in\mathbb{N}_0^k}\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]\,\kappa_{\mathbf{n}}^{(i)}(t)$$

holds for all $t \in \mathbb{R}_+$ and $i \in \{1,\dots,k\}$.

Proof: The assertion follow from Corollary 2.3.18 and Corollary 3.3.6. ■

3.4.7 Corollary. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a regular multivariate mixed Poisson process with mixing distribution U. Then*

$$\left|\frac{d}{dt}\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}\}\right]\right| \;\leq\; \int_{\mathbb{R}^k}\mathbf{1}'\boldsymbol{\lambda}\,dU(\boldsymbol{\lambda})$$

holds for all $t>0$ and all $\mathbf{n}\in\mathbb{N}_0^k$.

Proof: Let $t>0$ and $\mathbf{n}\in\mathbb{N}_0^k$.
Under the assumption of the corollary $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ has the Chapman–Kolmogorov property and we can use Corollary 2.3.8 to obtain

$$\frac{d}{dt}\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}\}\right] \;=\; \sum_{i=1}^k \mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}-\mathbf{e}_i\}\right]\,\kappa^{(i)}_{\mathbf{n}-\mathbf{e}_i}(t) - \mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}\}\right]\sum_{i=1}^k \kappa^{(i)}_{\mathbf{n}}(t)$$

By Lemma 3.4.6, this yields

$$\begin{aligned}\frac{d}{dt}\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}\}\right] \;&\leq\; \sum_{i=1}^k \mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}-\mathbf{e}_i\}\right]\,\kappa^{(i)}_{\mathbf{n}-\mathbf{e}_i}(t)\\ &\leq\; \sum_{i=1}^k\int_{\mathbb{R}^k}\lambda_i\,dU(\boldsymbol{\lambda})\\ &=\; \int_{\mathbb{R}^k}\mathbf{1}'\boldsymbol{\lambda}\,dU(\boldsymbol{\lambda})\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}\}\right] \;&\geq\; -\sum_{i=1}^k \mathrm{P}\left[\{\mathbf{N}_t=\mathbf{n}\}\right]\kappa^{(i)}_{\mathbf{n}}(t)\\ &\geq\; -\sum_{i=1}^k\int_{\mathbb{R}^k}\lambda_i\,dU(\boldsymbol{\lambda})\\ &=\; -\int_{\mathbb{R}^k}\mathbf{1}'\boldsymbol{\lambda}\,dU(\boldsymbol{\lambda})\end{aligned}$$

Thus, the assertion holds. ■

In the next lines the concept of martingales will take the leading role. Thereto, using the word martingale we will always refer to a martingale adapted to the natural filtration of the underlying process $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$.

3.4.8 Theorem. *Let* $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *be a regular multivariate mixed Poisson process with mixing distribution* U. *Then*

(1) $\{\boldsymbol{\kappa}_{\mathbf{N}_t}(t)\}_{t \in \mathbb{R}_+}$ *is a martingale.*

(2) $\{\mathbf{N}_t - t\,\boldsymbol{\kappa}_{\mathbf{N}_t}(t)\}_{t \in \mathbb{R}_+}$ *is a martingale.*

Proof: Before we prove the two assertions we derive an equation we will need in both proofs.
Consider $s, t \in \mathbb{R}_+$ with $0 < s \le t$ and $i \in \{1, \dots, k\}$. From Theorem 3.4.2 we see that $\int_{\mathbb{R}^k} \lambda_i \, dU_{t,\mathbf{n}}(\boldsymbol{\lambda})$ is finite and as a consequence of Theorem 3.1.5 (the incremental process is a multivariate mixed Poisson process) and Corollary 3.3.6 (representation of moments) we get

$$
\begin{aligned}
\mathrm{E}\left(N_t^{(i)} - N_s^{(i)} \,\middle|\, \mathbf{N}_s\right) &= \sum_{\mathbf{n} \in \mathbb{N}_0^k} \mathrm{E}\left[K_{s,t-s}^{(i)} \,\middle|\, \{\mathbf{N}_s = \mathbf{n}\}\right] \chi_{\{\mathbf{N}_s = \mathbf{n}\}} \\
&= \sum_{\mathbf{n} \in \mathbb{N}_0^k} (t-s) \int_{\mathbb{R}^k} \lambda_i \, dU_{s,\mathbf{n}}(\boldsymbol{\lambda}) \, \chi_{\{\mathbf{N}_s = \mathbf{n}\}} \\
&= \sum_{\mathbf{n} \in \mathbb{N}_0^k} (t-s)\, \kappa_{\mathbf{n}}^{(i)}(s) \, \chi_{\{\mathbf{N}_s = \mathbf{n}\}} \\
&= (t-s)\, \kappa_{\mathbf{N}_s}^{(i)}(s) \qquad\qquad (+)
\end{aligned}
$$

By Theorem 3.4.2 and Corollary 3.3.6 the equation is valid for $s = 0$, too.

(1): Now consider $(\mathbf{n}, s) \in Z$ and $i \in \{1, \dots, k\}$. Then by the representation of transition probabilities and intensities for multivariate mixed Poisson processes we have for all $t > s$

$$
\begin{aligned}
\mathrm{E}\left[\kappa_{\mathbf{N}_t}^{(i)}(t) \,\middle|\, \{\mathbf{N}_s = \mathbf{n}\}\right] &= \sum_{\mathbf{l} \in \mathbb{N}_0^k} p_{\mathbf{n},\mathbf{n}+\mathbf{l}}(s,t)\, \kappa_{\mathbf{n}+\mathbf{l}}^{(i)}(t) \\
&= \sum_{\mathbf{l} \in \mathbb{N}_0^k} \frac{(t-s)^{\mathbf{1}'\mathbf{l}}}{\mathbf{l}!} \, \frac{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \, \boldsymbol{\lambda}^{\mathbf{n}+\mathbf{l}} \, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} s} \, \boldsymbol{\lambda}^{\mathbf{n}} \, dU(\boldsymbol{\lambda})} \, \frac{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \, \boldsymbol{\lambda}^{\mathbf{n}+\mathbf{l}+\mathbf{e}_i} \, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \, \boldsymbol{\lambda}^{\mathbf{n}+\mathbf{l}} \, dU(\boldsymbol{\lambda})} \\
&= \sum_{\mathbf{l} \in \mathbb{N}_0^k} \frac{l^{(i)}+1}{t-s} \, \frac{(t-s)^{\mathbf{1}'(\mathbf{l}+\mathbf{e}_i)}}{(\mathbf{l}+\mathbf{e}_i)!} \, \frac{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t} \, \boldsymbol{\lambda}^{\mathbf{n}+\mathbf{l}+\mathbf{e}_i} \, dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} s} \, \boldsymbol{\lambda}^{\mathbf{n}} \, dU(\boldsymbol{\lambda})} \\
&= \sum_{\mathbf{l} \in \mathbb{N}_0^k} \frac{l^{(i)}+1}{t-s} \, p_{\mathbf{n},\mathbf{n}+\mathbf{l}+\mathbf{e}_i}(s,t) \\
&= \frac{1}{t-s} \, \mathrm{E}\left[N_t^{(i)} - N_s^{(i)} \,\middle|\, \{\mathbf{N}_s = \mathbf{n}\}\right] \\
&= \kappa_{\mathbf{n}}^{(i)}(s)
\end{aligned}
$$

Thus, for all $t \ge s$

$$
\mathrm{E}\left(\kappa_{\mathbf{N}_t}^{(i)}(t) \,\middle|\, \mathbf{N}_s\right) = \kappa_{\mathbf{N}_s}^{(i)}(s)
$$

and together with the Markov property the assertion is shown.

(2): Consider $s, t \in \mathbb{R}_+$ with $s \leq t$. Then (1) and the equation shown before yield

$$\begin{aligned} \mathrm{E}\left(\mathbf{N}_t - t\,\boldsymbol{\kappa}_{\mathbf{N}_t}(t) \,\middle|\, \mathbf{N}_s\right) &= \mathrm{E}\left(\mathbf{N}_t - \mathbf{N}_s \,\middle|\, \mathbf{N}_s\right) + \mathbf{N}_s - t\,\mathrm{E}\left(\boldsymbol{\kappa}_{\mathbf{N}_t}(t) \,\middle|\, \mathbf{N}_s\right) \\ &= (t-s)\,\boldsymbol{\kappa}_{\mathbf{N}_s}(s) + \mathbf{N}_s - t\,\boldsymbol{\kappa}_{\mathbf{N}_s}(s) \\ &= \mathbf{N}_s - s\,\boldsymbol{\kappa}_{\mathbf{N}_s}(s) \end{aligned}$$

and the proof is completed. ■

The first property generalizes a well known property for univariate mixed Poisson processes (see Lundberg [1964] and Grandell [1997]) to the multivariate setting.

The second property deserves a short discussion we will restrict to the univariate case. There are various ways of deriving a centred process which is a martingale out of a mixed Poisson process. One of them is to use the compensator, that means subtracting a process which has to be predictable and increasing in t from $\{N_t\}_{t\in\mathbb{R}_+}$. It is well known that the compensator is unique (see e.g. Liptser and Shiryayev [1978]). For a Poisson process $\{N_t\}_{t\in\mathbb{R}_+}$, where the intensities are independent of the state and the time, the process $\{t\,\kappa_{N_t}(t)\}_{t\in\mathbb{R}_+}$ used in Theorem 3.4.8 (2) coincides with the compensator for mixed Poisson processes derived by Grigelionis [1998]. A general coincidence does not exist, since the process $\{t\,\kappa_{N_t}(t)\}_{t\in\mathbb{R}_+}$ does not always fulfil the requirements of a compensator which will be illustrated within the next lines.

Example: Using Theorems 2.3.19 and 3.4.2 we obtain

$$\begin{aligned} \frac{d\,t\kappa_n(t)}{dt} &= \kappa_n(t) + t\,\kappa_n(t)\,(\kappa_n(t) - \kappa_{n+1}(t)) \\ &= \kappa_n(t) + t\,(\kappa_n(t))^2 - t\,\kappa_n(t)\,\kappa_{n+1}(t) \\ &= \int_{\mathbb{R}} \lambda\,dU_{t,n}(\lambda) + t\left(\int_{\mathbb{R}} \lambda\,dU_{t,n}(\lambda)\right)^2 - t\int_{\mathbb{R}} \lambda\,dU_{t,n}(\lambda)\int_{\mathbb{R}} \lambda\,dU_{t,n+1}(\lambda) \\ &= \int_{\mathbb{R}} \lambda\,dU_{t,n}(\lambda) + t\left(\int_{\mathbb{R}} \lambda\,dU_{t,n}(\lambda)\right)^2 - t\int_{\mathbb{R}} \lambda^2\,dU_{t,n}(\lambda) \end{aligned}$$

Assuming $U[\{1\}] = U[\{5\}] = 0.5$ we get for $n = 4$ and $t = 2$

$$U_{t,n}[B] = \left(e^{-2}\cdot 1^4 \cdot 0.5 + e^{-10}\cdot 5^4\cdot 0.5\right)^{-1} \sum_{\lambda\in B\cap\{1,5\}} e^{-2\lambda}\,\lambda^4\cdot 0.5$$

and thus

$$\left.\frac{d\,t\kappa_4(t)}{dt}\right|_{t=2} \approx -2.9$$

and the process $\{t\,\kappa_{N_t}(t)\}_{t\in\mathbb{R}_+}$ is not increasing for all $t\in(0,\infty)$. □

At the end of this section we will consider the remark after Theorem 2.3.17. For a multivariate mixed Poisson process with mixing distribution we have

$$\begin{aligned}
\kappa^{(j)}_{\mathbf{n}+\mathbf{e}_i}(t)\,\kappa^{(i)}_{\mathbf{n}}(t) &= \frac{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda}t}\,\boldsymbol{\lambda}^{\mathbf{n}+\mathbf{e}_i+\mathbf{e}_j}\,dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda}t}\,\boldsymbol{\lambda}^{\mathbf{n}+\mathbf{e}_i}\,dU(\boldsymbol{\lambda})}\,\frac{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda}t}\,\boldsymbol{\lambda}^{\mathbf{n}+\mathbf{e}_i}\,dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda}t}\,\boldsymbol{\lambda}^{\mathbf{n}}\,dU(\boldsymbol{\lambda})} \\
&= \frac{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda}t}\,\boldsymbol{\lambda}^{\mathbf{n}+\mathbf{e}_i+\mathbf{e}_j}\,dU(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda}t}\,\boldsymbol{\lambda}^{\mathbf{n}}\,dU(\boldsymbol{\lambda})} \\
&= \int_{\mathbb{R}^k} \lambda_i\,\lambda_j\,dU_{t,\mathbf{n}}(\boldsymbol{\lambda})
\end{aligned}$$

Interpreting the intensity $\kappa^{(i)}_{\mathbf{n}}(t)$ as tendency to jump at time t from state $\mathbf{n}$ into state $\mathbf{n}+\mathbf{e}_i$ we see that the moments around the origin of the posterior distribution $U_{t,\mathbf{n}}$ represent the tendency to jump at time t from state $\mathbf{n}$ into the according state.

Chapter 4

Multivariate Mixed Poisson Processes with Random Parameter

4.1 The Model

A mixed Poisson process can be seen as the outcome of a two–step model. First a parameter according to the mixing distribution is chosen and then the occurrence of the events under consideration in the unit time interval is Poisson distributed where the expectation is exactly the chosen parameter. We can now specify the model of multivariate mixed Poisson processes with mixing distribution by assuming that the parameter is a realization of a random vector and additionally considering the conditional probabilities of the process with respect to the existing random vector. Altogether we are in the following setting.

A multivariate counting process $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ is said to be a ***multivariate mixed Poisson process with parameter*** $\boldsymbol{\Lambda}$ if $\boldsymbol{\Lambda}$ is a random vector with $\mathrm{P}_{\boldsymbol{\Lambda}}[(\mathbf{0}, \boldsymbol{\infty})] = 1$ such that

$$\mathrm{P}\left(\bigcap_{j=1}^{m} \left\{ \mathbf{N}_{t_j} - \mathbf{N}_{t_{j-1}} = \mathbf{n}_j \right\} \middle| \boldsymbol{\Lambda} \right) = \prod_{j=1}^{m} e^{-\mathbf{1}'\boldsymbol{\Lambda}\,(t_j - t_{j-1})} \frac{(\boldsymbol{\Lambda}\,(t_j - t_{j-1}))^{\mathbf{n}_j}}{\mathbf{n}_j!}$$

holds for all $m \in \mathbb{N}$ and $t_0, t_1, \ldots, t_m \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_m$ and for all $\mathbf{n}_j \in \mathbb{N}_0^k$, $j \in \{1, \ldots, m\}$.

This definition is dealing with conditional probabilities, in the sense of conditional expectation, of the finite–dimensional increments of the process. As in the case of a univariate mixed Poisson process (see Schmidt [1996]), there exists an equivalent definition using other properties of stochastic processes. Therefore, we introduce the concept of conditionally independent and conditionally stationary increments.

Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process and $\mathbf{\Lambda}$ a random vector.
$\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ is said to have ***conditionally independent increments*** with respect to $\mathbf{\Lambda}$ if

$$P\left(\bigcap_{j=1}^{m}\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\}\middle|\mathbf{\Lambda}\right) \;=\; \prod_{j=1}^{m} P\left(\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\}\middle|\mathbf{\Lambda}\right)$$

holds for all $m\in\mathbb{N}$ and $t_0,t_1,\dots,t_m\in\mathbb{R}_+$ with $0=t_0<t_1<\dots<t_m$ and for all $\mathbf{n}_j\in\mathbb{N}_0^k$, $j\in\{1,\dots,m\}$.
$\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ is said to have ***conditionally stationary increments*** with respect to $\mathbf{\Lambda}$ if

$$P\left(\bigcap_{j=1}^{m}\{\mathbf{N}_{t_j+h}-\mathbf{N}_{t_{j-1}+h}=\mathbf{n}_j\}\middle|\mathbf{\Lambda}\right) \;=\; P\left(\bigcap_{j=1}^{m}\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\}\middle|\mathbf{\Lambda}\right)$$

holds for all $m\in\mathbb{N}$ and $t_0,t_1,\dots,t_m,h\in\mathbb{R}_+$ with $0=t_0<t_1<\dots<t_m$ and for all $\mathbf{n}_j\in\mathbb{N}_0^k$, $j\in\{1,\dots,m\}$.

4.1.1 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate counting process and $\mathbf{\Lambda}$ a random vector. Then the following are equivalent.*

(a) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *is a multivariate mixed Poisson process with parameter $\mathbf{\Lambda}$.*

(b) $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *has conditionally independent and conditionally stationary increments with respect to $\mathbf{\Lambda}$ and*

$$P\left(\{\mathbf{N}_t=\mathbf{n}\}\middle|\mathbf{\Lambda}\right) \;=\; e^{-\mathbf{1}'\mathbf{\Lambda}\,t}\,\frac{(\mathbf{\Lambda}\,t)^{\mathbf{n}}}{\mathbf{n}!}$$

holds for all $t\in\mathbb{R}_+$ and all $\mathbf{n}\in N_0^k$.

Proof:
(a) $\Rightarrow$ (b): obvious

(b) $\Rightarrow$ (a): Let $m\in\mathbb{N}$ and $t_0,t_1,\dots,t_m\in\mathbb{R}_+$ with $0=t_0<t_1<\dots<t_m$ and $\mathbf{n}_j\in\mathbb{N}_0^k$, $j\in\{1,\dots,m\}$. Then

$$\begin{aligned} P\left(\bigcap_{j=1}^{m}\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\}\middle|\mathbf{\Lambda}\right) &= \prod_{j=1}^{m} P\left(\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\}\middle|\mathbf{\Lambda}\right)\\ &= \prod_{j=1}^{m} P\left(\{\mathbf{N}_{t_j-t_{j-1}}=\mathbf{n}_j\}\middle|\mathbf{\Lambda}\right)\\ &= \prod_{j=1}^{m} e^{-\mathbf{1}'\mathbf{\Lambda}\,(t_j-t_{j-1})}\,\frac{(\mathbf{\Lambda}\,(t_j-t_{j-1}))^{\mathbf{n}_j}}{\mathbf{n}_j!} \end{aligned}$$

which yields the assertion. ∎

In contrast with the definition of multivariate mixed Poisson processes with mixing distribution the definition of multivariate mixed Poisson processes with parameter uses conditional probabilities. Considering the unconditional probabilities, the next corollary is obvious.

4.1.2 Corollary. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$. Then $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ is a multivariate mixed Poisson process with mixing distribution $\mathrm{P}_{\boldsymbol{\Lambda}}$.*

Thus, a multivariate mixed Poisson process with parameter possesses all properties which a multivariate mixed Poisson with mixing distribution has. The opposite implication does not seem to be true, as it is in general not possible to construct the conditional probabilities from the unconditional ones. Therefore, the characterizations of multivariate mixed Poisson processes with mixing distribution in terms of the multinomial property (Theorem 3.2.3) cannot be carried over to multivariate mixed Poisson processes with parameter. First, we are not able to make sure the existence of a random vector with the distribution originating from the Bernstein–Widder theorem on the given probability space. On the other hand, assuming there exists such a random vector it is in general not possible to construct the conditional probabilities in the definition of the multivariate mixed Poisson process with parameter from the unconditional ones.

Furthermore, the characterization of multivariate mixed Poisson processes by the multinomial property and the one–dimensional distributions (Lemma 3.1.1) does not apply to multivariate mixed Poisson processes with parameter. Hence, to show that the property of being a mixed Poisson process with parameter is $\mathcal{A}$–stable, we cannot employ the proof of Lemma 3.1.3. However, by applying this time Theorem 4.1.1 we still only have to use the one–dimensional distributions. Therefore first the following lemma.

4.1.3 Lemma. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate counting process and $\boldsymbol{\Lambda}$ a random vector. Then*

(1) *The property of having conditionally independent increments with respect to $\boldsymbol{\Lambda}$ is $\mathcal{A}$–stable.*
(2) *The property of having conditionally stationary increments with respect to $\boldsymbol{\Lambda}$ is $\mathcal{A}$–stable.*

Proof: The assertion can be proven by exactly the same transformations as in the proof of Lemma 2.2.1, with the sole difference that we use conditional probabilities instead of unconditional probabilities. ∎

4.1.4 Lemma. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$ and let $A \in \mathcal{A}$. Then*

(1) *The process* $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *is a multivariate mixed Poisson process with parameter* $A\boldsymbol{\Lambda}$. *This means, being a multivariate mixed Poisson process with parameter is* $\mathcal{A}$*–stable.*

(2) *The identity*

$$\mathrm{P}\left(\{A\mathbf{N}_t=\mathbf{l}\}\,\middle|\,A\boldsymbol{\Lambda}\right) \;=\; \mathrm{P}\left(\{A\mathbf{N}_t=\mathbf{l}\}\,\middle|\,\boldsymbol{\Lambda}\right)$$

holds for all $t\in\mathbb{R}_+$ *and* $\mathbf{l}\in\mathbb{N}_0^d$.

Proof: Consider $t\in\mathbb{R}_+$ and $\mathbf{l}\in\mathbb{N}_0^d$.
Firstly, we prove that

$$\mathrm{E}\left(\chi_{\{A\mathbf{N}_t=\mathbf{l}\}}\middle|\boldsymbol{\Lambda}\right) \;=\; e^{-\mathbf{1}'A\boldsymbol{\Lambda}\,t}\,\frac{(A\boldsymbol{\Lambda}\,t)^{\mathbf{l}}}{\mathbf{l}!} \qquad (+)$$

holds for all $A\in\mathcal{A}$ with $A\in\mathbb{R}^{d\times k}$.
- Let $A\in\mathcal{A}_P$. Then (+) holds obviously.
- Let $A\in\mathcal{A}_S$. Then we have with the help of monotone convergence for conditional expectation

$$\begin{aligned}
\mathrm{E}\left(\chi_{\{A\mathbf{N}_t=\mathbf{l}\}}\middle|\boldsymbol{\Lambda}\right) &= \mathrm{E}\left(\sum_{\mathbf{n}\in A^{-1}(\{\mathbf{l}\})}\chi_{\{\mathbf{N}_t=\mathbf{n}\}}\,\middle|\,\boldsymbol{\Lambda}\right)\\
&= \sum_{\mathbf{n}\in A^{-1}(\{\mathbf{l}\})}\mathrm{P}\left(\{\mathbf{N}_t=\mathbf{n}\}\,\middle|\,\boldsymbol{\Lambda}\right)\\
&= \sum_{\mathbf{n}\in A^{-1}(\{\mathbf{l}\})}\prod_{i=1}^{k}e^{-\Lambda_i\,t}\,\frac{(\Lambda_i\,t)^{n^{(i)}}}{n^{(i)}!}\\
&= \left(\prod_{i=1}^{d}e^{-\Lambda_i\,t}\,\frac{(\Lambda_i\,t)^{l^{(i)}}}{l^{(i)}!}\right)\sum_{\mathbf{n}\in A^{-1}(\{\mathbf{l}\})}\prod_{i=d+1}^{k}e^{-\Lambda_i\,t}\,\frac{(\Lambda_i\,t)^{n^{(i)}}}{n^{(i)}!}\\
&= e^{-\mathbf{1}'A\boldsymbol{\Lambda}\,t}\,\frac{(A\boldsymbol{\Lambda}\,t)^{\mathbf{l}}}{\mathbf{l}!}
\end{aligned}$$

So (+) holds for $A\in\mathcal{A}_S$.
- Let $A\in\mathcal{A}_C$. Setting $I(i):=\{h\in\{1,\dots,k\}\;:\;\mathbf{e}_i{}'A\mathbf{e}_h=1\}$ (the set of coordinates cumulated in the i–th coordinate of the transformed process) we have $\sum_{h\in I(i)}\lambda_h=\mathbf{e}_i{}'A\boldsymbol{\lambda}$ for all $i\in\{1,\dots,d\}$. Thus, with the same formula manipulation at the beginning as before, we get

$$\begin{aligned}
\mathrm{E}\left(\chi_{\{A\mathbf{N}_t=\mathbf{l}\}}\middle|\boldsymbol{\Lambda}\right) &= \sum_{\mathbf{n}\in A^{-1}(\{\mathbf{l}\})}\prod_{i=1}^{k}e^{-\Lambda_i\,t}\,\frac{(\Lambda_i\,t)^{n^{(i)}}}{n^{(i)}!}\\
&= \left(\prod_{i=1}^{d}e^{-\mathbf{e}_i{}'A\boldsymbol{\Lambda}\,t}\,\frac{(\mathbf{e}_i{}'A\boldsymbol{\Lambda}\,t)^{l^{(i)}}}{l^{(i)}!}\right)
\end{aligned}$$

$$
\begin{aligned}
&\cdot \sum_{\mathbf{n}\,\in\, A^{-1}(\{\mathbf{l}\})} \prod_{i=1}^{d} \frac{l^{(i)}!}{\prod_{h\,\in\, I(i)} n^{(h)}!} \prod_{h\,\in\, I(i)} \left(\frac{\Lambda_h}{\mathbf{e}_i{}' A\boldsymbol{\Lambda}}\right)^{n^{(h)}} \\
= \;& e^{-\mathbf{1}' A\boldsymbol{\Lambda}\, t}\, \frac{(A\boldsymbol{\Lambda}\, t)^{\mathbf{l}}}{\mathbf{l}!}
\end{aligned}
$$

Therefore, (+) holds for all $A \in \mathcal{A}$. We are now going to prove the assertions.

(1): Equation (+) gives

$$
\begin{aligned}
\mathrm{P}\left(\{A\mathbf{N}_t = \mathbf{l}\}\,\middle|\,A\boldsymbol{\Lambda}\right) &= \mathrm{E}\left(\mathrm{E}\left(\chi_{\{A\mathbf{N}_t = \mathbf{l}\}}\middle|\boldsymbol{\Lambda}\right)\middle|A\boldsymbol{\Lambda}\right) \\
&= \mathrm{E}\left(e^{-\mathbf{1}' A\boldsymbol{\Lambda}\, t}\, \frac{(A\boldsymbol{\Lambda}\, t)^{\mathbf{l}}}{\mathbf{l}!}\,\middle|A\boldsymbol{\Lambda}\right) \\
&= e^{-\mathbf{1}' A\boldsymbol{\Lambda}\, t}\, \frac{(A\boldsymbol{\Lambda}\, t)^{\mathbf{l}}}{\mathbf{l}!}
\end{aligned}
$$

Now, Theorem 4.1.1 in connection with Lemma 4.1.3 yields the assertion.

(2): Using equation (+) and (1) we get

$$
\begin{aligned}
\mathrm{P}\left(\{A\mathbf{N}_t = \mathbf{l}\}\,\middle|\,A\boldsymbol{\Lambda}\right) &= e^{-\mathbf{1}' A\boldsymbol{\Lambda}\, t}\, \frac{(A\boldsymbol{\Lambda}\, t)^{\mathbf{l}}}{\mathbf{l}!} \\
&= \mathrm{E}\left(\chi_{\{A\mathbf{N}_t = \mathbf{l}\}}\middle|\boldsymbol{\Lambda}\right) \\
&= \mathrm{P}\left(\{A\mathbf{N}_t = \mathbf{l}\}\,\middle|\,\boldsymbol{\Lambda}\right)
\end{aligned}
$$

and the assertion is shown. ■

In short terms, the lemma states that the parameter of the transformed process is the transformed parameter. For example $\{N_t^{(i)}\}_{t\,\in\,\mathbb{R}_+}$ is a univariate mixed Poisson process with parameter Λ_i. In addition, conditioning of a transformed process with respect to the parameter is equal to conditioning with respect to transformed parameter.

Since a multivariate mixed Poisson process with parameter is a multivariate mixed Poisson process with mixing distribution we transfer some results of Chapter 3. We can of course use $P_{\boldsymbol{\Lambda}}$ instead of U and so the results will probable be easier to remember. The first assertion under consideration is Theorem 3.1.4.

4.1.5 Theorem. *Let $\{\mathbf{N}_t\}_{t\,\in\,\mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$. Then the coordinates of $\{\mathbf{N}_t\}_{t\,\in\,\mathbb{R}_+}$ are independent if, and only if, the coordinates of $\boldsymbol{\Lambda}$ are independent.*

A proof, which refers to multivariate mixed Poisson processes with parameter, of the preceding theorem can be found in Zocher [2003]. The following theorem is stated in this reference, too. It considers conditional independence with respect to the parameter which is now possible because of the parameter $\boldsymbol{\Lambda}$ introduced in the model.

4.1.6 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$. Then the coordinates of $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ are conditionally independent with respect to $\boldsymbol{\Lambda}$.*

Proof: Using the transformation $A = \mathbf{e}_i'$ with $A\boldsymbol{\Lambda} = \Lambda_i$ and Lemma 4.1.4 we get

$$\begin{aligned}
\mathrm{P}\left(\bigcap_{j=1}^{m}\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\}\,\middle|\,\boldsymbol{\Lambda}\right) &= \prod_{i=1}^{k}\prod_{j=1}^{m} e^{-\Lambda_i\,(t_j-t_{j-1})}\,\frac{(\Lambda_i\,(t_j-t_{j-1}))^{n_j^{(i)}}}{n_j^{(i)}!}\\
&= \prod_{i=1}^{k}\mathrm{P}\left(\bigcap_{j=1}^{m}\left\{N_{t_j}^{(i)}-N_{t_{j-1}}^{(i)}=n_j^{(i)}\right\}\,\middle|\,\Lambda_i\right)\\
&= \prod_{i=1}^{k}\mathrm{P}\left(\bigcap_{j=1}^{m}\left\{N_{t_j}^{(i)}-N_{t_{j-1}}^{(i)}=n_j^{(i)}\right\}\,\middle|\,\boldsymbol{\Lambda}\right)
\end{aligned}$$

for all $m \in \mathbb{N}$ and $t_0, t_1, \ldots, t_m \in \mathbb{R}_+$ with $0 = t_0 < t_1 < \ldots < t_m$ and for all $\mathbf{n}_j \in \mathbb{N}_0^k$, $j \in \{1, \ldots, m\}$. ■

As in Chapter 3, the moment generating function plays a leading role. In order to keep the notation simple we use the symbol $M_{\boldsymbol{\Lambda}}$ instead of $M_{P_{\boldsymbol{\Lambda}}}$ for the moment generating function (of the distribution) of the random vector $\boldsymbol{\Lambda}$. From the results of Chapter 3 we have

$$\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}] = \frac{t^{\mathbf{1}'\mathbf{n}}}{\mathbf{n}!}D^{\mathbf{n}}M_{\boldsymbol{\Lambda}}(-t\mathbf{1})$$

for $\mathbf{n} \in \mathbb{N}_0^k$ and $t > 0$ as well as $M_{A\boldsymbol{\Lambda}}(\mathbf{t}) = M_{\boldsymbol{\Lambda}}(A'\mathbf{t})$ for $\mathbf{t} \in \mathbb{R}^d$. So the one–dimensional probabilities, which are the relevant ones because of the multinomial property, are determined by the moment generating function of $\boldsymbol{\Lambda}$. In addition, the one–dimensional probabilities of the transformed processes are also determined by the moment generating function $M_{\boldsymbol{\Lambda}}$. The application of this function with regard to the moments of the process is discussed in the succeeding section.

4.2 Moments

In Section 3.3 we have derived necessary and sufficient conditions for the finiteness of the binomial moments, moments around the origin, and central moments of the multivariate mixed Poisson process with mixing distribution. An important tool was the probability generating function. In the case of a multivariate mixed Poisson process with parameter the results look as follows.

4.2.1 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$. Then*

$$g_{\mathbf{N}_t}(\mathbf{r}) \;=\; M_{\boldsymbol{\Lambda}}(t(\mathbf{r}-\mathbf{1}))$$

holds for all $\mathbf{r} \in [\mathbf{0},\mathbf{1}]$ and $t \in \mathbb{R}_+$. The binomial moment of $\mathbf{N}_t$ fulfils

$$\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] \;=\; \frac{t^{\mathbf{1}'\mathbf{l}}}{\mathbf{l}!}\,\mathrm{E}\left[\boldsymbol{\Lambda}^{\mathbf{l}}\right]$$

for all $\mathbf{l} \in \mathbb{N}_0^k$ and $t \in \mathbb{R}_+$.

4.2.2 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$ and $\mathbf{l} \in \mathbb{N}_0^k$. Then the following are equivalent.*

(a) *There exists some $t > 0$ such that*

$$\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] \;<\; \infty$$

holds.

(b) *The inequality*

$$\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] \;<\; \infty$$

holds for all $t \in \mathbb{R}_+$.

(c) *The parameter satisfies*

$$\mathrm{E}\left[\boldsymbol{\Lambda}^{\mathbf{l}}\right] \;<\; \infty$$

(d) *For all $\mathbf{s} \in (-\boldsymbol{\infty},\mathbf{0}]$ the inequality*

$$\lim_{\mathbf{r}\to\mathbf{s}} D^{\mathbf{l}} M_{\boldsymbol{\Lambda}}|_{(-\boldsymbol{\infty},\mathbf{0})}(\mathbf{r}) \;<\; \infty$$

is valid.

If $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ satisfies one and hence all preceding items, then

$$\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] \;=\; \frac{t^{\mathbf{1}'\mathbf{l}}}{\mathbf{l}!}\,\lim_{\mathbf{r}\uparrow\mathbf{0}} D^{\mathbf{l}} M_{\boldsymbol{\Lambda}}|_{(-\boldsymbol{\infty},\mathbf{0})}(\mathbf{r})$$

holds for $t \in \mathbb{R}_+$.

4.2.3 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$ and $\mathbf{l} \in \mathbb{N}_0^k$. Then the following are equivalent.*

(a) *For all* $\mathbf{m} \leq \mathbf{l}$ *there exists some* $t > 0$ *such that*

$$\mathrm{E}\left[\,(\mathbf{N}_t)^{\mathbf{m}}\,\right] \;<\; \infty$$

holds.

(b) *For all* $\mathbf{m} \leq \mathbf{l}$ *the inequality*

$$\mathrm{E}\left[\,(\mathbf{N}_t)^{\mathbf{m}}\,\right] \;<\; \infty$$

holds for all $t \in \mathbb{R}_+$.

(c) *For all* $\mathbf{m} \leq \mathbf{l}$ *the parameter fulfils*

$$\mathrm{E}\left[\boldsymbol{\Lambda}^{\mathbf{l}}\right] \;<\; \infty$$

(d) *For all* $\mathbf{m} \leq \mathbf{l}$ *the* $\mathbf{m}$*–th derivative of* $M_{\boldsymbol{\Lambda}}|_{(-\boldsymbol{\infty},\mathbf{0}]}$ *is continuous on* $(-\boldsymbol{\infty},\mathbf{0}]$.

If $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *satisfies one and hence all preceding items, then*

$$\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] \;=\; \frac{t^{\mathbf{1}'\mathbf{l}}}{\mathbf{l}!}\, D^{\mathbf{l}} M_{\boldsymbol{\Lambda}}|_{(-\boldsymbol{\infty},\mathbf{0}]}(\mathbf{0})$$

holds for $t \in \mathbb{R}_+$.

As an outcome of this theorems it is possible to formulate explicit formulas for the first and second central moment of $\mathbf{N}_t$ as in Corollary 3.3.6. However, this results can also be obtained in compact manner by the use of conditional expectation. This approach enables us to specify the covariance of $\mathbf{N}_t$ and $\mathbf{N}_{t+h}$ additionally (compare also Zocher [2003]).

4.2.4 Theorem. *Let* $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *be a multivariate mixed Poisson process with parameter* $\boldsymbol{\Lambda}$.

(1) *If* $\boldsymbol{\Lambda}$ *has a finite moment of first order, then*

$$\mathrm{E}\left[\mathbf{N}_t\right] \;=\; t\,\mathrm{E}\left[\boldsymbol{\Lambda}\right]$$

holds for all $t \in \mathbb{R}_+$.

(2) *If* $\boldsymbol{\Lambda}$ *has a finite moment of second order, then*

$$\mathrm{Cov}\left[\mathbf{N}_t, \mathbf{N}_{t+h}\right] \;=\; t\,\mathrm{Diag}\left(\mathrm{E}\left[\boldsymbol{\Lambda}\right]\right) + t\,(t+h)\,\mathrm{Var}\left[\boldsymbol{\Lambda}\right]$$

holds for all $t, h \in \mathbb{R}_+$.

(3) *If* $\boldsymbol{\Lambda}$ *has a finite moment of second order and* $\mathrm{Var}[\Lambda_i] > 0$ *and* $\mathrm{Var}[\Lambda_l] > 0$ *for* $i, l \in \{1, \ldots, k\}$ *with* $i \neq l$, *then*

$$\lim_{t\uparrow\infty} \varrho\left(N_t^{(i)}, N_t^{(l)}\right) \;=\; \varrho\left(\Lambda_i, \Lambda_l\right)$$

holds. The absolute value of the coefficient of correlation $\varrho\,(N_t^{(i)}, N_t^{(l)})$ *is increasing on* $(0,\infty)$ *additionally.*

Proof: Before we prove the assertions, we make some preliminary remarks. With Lemma 4.1.4 we have ($\mathbf{\Lambda}$ is almost surely finite) $\mathrm{E}(N_t^{(i)}|\mathbf{\Lambda}) = \mathrm{E}(N_t^{(i)}|\Lambda_i) = t\Lambda_i$ and $\mathrm{Var}(N_t^{(i)}|\mathbf{\Lambda}) = \mathrm{Var}(N_t^{(i)}|\Lambda_i) = t\Lambda_i$ for all $t \in \mathbb{R}_+$. Conditional independence of the coordinates with respect to $\mathbf{\Lambda}$ (Theorem 4.1.6) yields $\mathrm{Cov}(N_t^{(i)}, N_t^{(l)}|\mathbf{\Lambda}) = 0$ for $i \neq l$. Altogether, we have

$$\mathrm{E}\left(\mathbf{N}_t \middle| \mathbf{\Lambda}\right) \;=\; t\mathbf{\Lambda} \qquad\qquad \mathrm{Var}\left(\mathbf{N}_t \middle| \mathbf{\Lambda}\right) \;=\; t\,\mathrm{Diag}\,(\mathbf{\Lambda})$$

(1): With $\mathbf{\Lambda}$ also $\mathbf{N}_t$ has a finite moment of first order for all $t \in \mathbb{R}_+$ (Theorem 4.2.3), so the theory of conditional expectation yields

$$\mathrm{E}\,[\mathbf{N}_t] \;=\; \mathrm{E}\left[\mathrm{E}\left(\mathbf{N}_t \middle| \mathbf{\Lambda}\right)\right] \;=\; \mathrm{E}\,[t\mathbf{\Lambda}] \;=\; t\,\mathrm{E}\,[\mathbf{\Lambda}]$$

(2): If $\mathbf{\Lambda}$ has a finite moment of second order, then it has a finite moment of first order, too. Thus, $\mathbf{N}_t$ has finite moments of first and second order for all $t \in \mathbb{R}_+$ (Theorem 4.2.3) and so covariance decomposition yields

$$\begin{aligned}
\mathrm{Cov}\,[\mathbf{N}_t, \mathbf{N}_{t+h}] \;&=\; \mathrm{E}\left[\mathrm{Cov}\left(\mathbf{N}_t, \mathbf{N}_{t+h} \middle| \mathbf{\Lambda}\right)\right] + \mathrm{Cov}\left[\mathrm{E}\left(\mathbf{N}_t \middle| \mathbf{\Lambda}\right), \mathrm{E}\left(\mathbf{N}_{t+h} \middle| \mathbf{\Lambda}\right)\right] \\
&=\; \mathrm{E}\left[\mathrm{Cov}\left(\mathbf{N}_t, \mathbf{N}_{t+h} - \mathbf{N}_t \middle| \mathbf{\Lambda}\right)\right] + \mathrm{E}\left[\mathrm{Var}\left(\mathbf{N}_t \middle| \mathbf{\Lambda}\right)\right] + \mathrm{Cov}\,[t\mathbf{\Lambda}, (t+h)\mathbf{\Lambda}] \\
&=\; \mathrm{E}\,[\,t\,\mathrm{Diag}\,(\mathbf{\Lambda})] + t\,(t+h)\,\mathrm{Var}\,[\mathbf{\Lambda}] \\
&=\; t\,\mathrm{Diag}\,(\mathrm{E}\,[\mathbf{\Lambda}]) + t\,(t+h)\,\mathrm{Var}\,[\mathbf{\Lambda}]
\end{aligned}$$

where $\mathrm{Cov}\left(\mathbf{N}_t, \mathbf{N}_{t+h} - \mathbf{N}_t \middle| \mathbf{\Lambda}\right) = 0$ since $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ has conditionally independent increments with respect to $\mathbf{\Lambda}$.

(3): Since

$$\begin{aligned}
\varrho\left(N_t^{(i)}, N_t^{(l)}\right) \;&=\; \frac{\mathrm{Cov}\left[N_t^{(i)}, N_t^{(l)}\right]}{\sqrt{\mathrm{Var}\left[N_t^{(i)}\right]\;\mathrm{Var}\left[N_t^{(l)}\right]}} \\
&=\; \frac{t^2\,\mathrm{Cov}\,[\Lambda_i, \Lambda_l]}{\sqrt{t^4\,\mathrm{Var}\,[\Lambda_i]\;\mathrm{Var}\,[\Lambda_l] + t^3\,(\mathrm{Var}\,[\Lambda_i]\;\mathrm{E}\,[\Lambda_l] + \mathrm{E}\,[\Lambda_i]\;\mathrm{Var}\,[\Lambda_l]) + t^2\,\mathrm{E}\,[\Lambda_i]\;\mathrm{E}\,[\Lambda_l]}} \\
&=\; \frac{\mathrm{Cov}\,[\Lambda_i, \Lambda_l]}{\sqrt{\mathrm{Var}\,[\Lambda_i]\;\mathrm{Var}\,[\Lambda_l] + t^{-1}\,(\mathrm{Var}\,[\Lambda_i]\;\mathrm{E}\,[\Lambda_l] + \mathrm{E}\,[\Lambda_i]\;\mathrm{Var}\,[\Lambda_l]) + t^{-2}\,\mathrm{E}\,[\Lambda_i]\;\mathrm{E}\,[\Lambda_l]}}
\end{aligned}$$

we have

$$\lim_{t \uparrow \infty} \varrho\left(N_t^{(i)}, N_t^{(l)}\right) \;=\; \varrho\,(\Lambda_i, \Lambda_l)$$

and the absolute value of the correlation coefficient is increasing on $(0, \infty)$. ■

The above theorem gives an easy way to check the correlation of two coordinates of the process. For $i \neq l$ we have

$$\mathrm{Cov}[N_s^{(i)}\,,\, N_t^{(l)}] = s\,t\,\mathrm{Cov}[\Lambda_i\,,\,\Lambda_l]$$

for all $s, t > 0$. Therefore, it is only necessary to check the correlation between two coordinates for one time pair to get the correlation of the coordinates of the parameter, which is significant for the correlation of the coordinates for all time pairs. Let us also have a look at the correlation of two increments of the process.

4.2.5 Corollary. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\mathbf{\Lambda}$. If $\mathbf{\Lambda}$ has a finite moment of second order, then*

$$\begin{aligned}\mathrm{Cov}\,[\mathbf{N}_{t_2} - \mathbf{N}_{t_1}, \mathbf{N}_{t_4} - \mathbf{N}_{t_3}] \;&=\; (t_2 - t_1)\,(t_4 - t_3)\,\mathrm{Var}\,[\mathbf{\Lambda}] \\ &\quad+\; \big((t_2 - t_3)^+ - (t_2 - t_4)^+\big)\,\mathrm{Diag}\,(\mathrm{E}\,[\mathbf{\Lambda}])\end{aligned}$$

holds for all $t_1, t_2, t_3, t_4 \in \mathbb{R}_+$ with $t_1 < t_2$, $t_3 < t_4$ and $t_1 \leq t_3$.

Proof: Consider $t_1, t_2, t_3, t_4 \in \mathbb{R}_+$ with $t_1 < t_2$, $t_3 < t_4$ and $t_1 \leq t_3$. From Theorem 4.2.4 (2) we get

$$\begin{aligned}&\mathrm{Cov}\,[\mathbf{N}_{t_2} - \mathbf{N}_{t_1}, \mathbf{N}_{t_4} - \mathbf{N}_{t_3}] \\ &\quad= \mathrm{Cov}\,[\mathbf{N}_{t_2}, \mathbf{N}_{t_4}] - \mathrm{Cov}\,[\mathbf{N}_{t_2}, \mathbf{N}_{t_3}] - \mathrm{Cov}\,[\mathbf{N}_{t_1}, \mathbf{N}_{t_4}] + \mathrm{Cov}\,[\mathbf{N}_{t_1}, \mathbf{N}_{t_3}] \\ &\quad= \min\{t_2, t_4\}\,\mathrm{Diag}\,(\mathrm{E}\,[\mathbf{\Lambda}]) + t_2\,t_4\,\mathrm{Var}\,[\mathbf{\Lambda}] \\ &\qquad - \min\{t_2, t_3\}\,\mathrm{Diag}\,(\mathrm{E}\,[\mathbf{\Lambda}]) - t_2\,t_3\,\mathrm{Var}\,[\mathbf{\Lambda}] \\ &\qquad - \min\{t_1, t_4\}\,\mathrm{Diag}\,(\mathrm{E}\,[\mathbf{\Lambda}]) - t_1\,t_4\,\mathrm{Var}\,[\mathbf{\Lambda}] \\ &\qquad + \min\{t_1, t_3\}\,\mathrm{Diag}\,(\mathrm{E}\,[\mathbf{\Lambda}]) + t_1\,t_3\,\mathrm{Var}\,[\mathbf{\Lambda}] \\ &\quad= \mathrm{Var}\,[\mathbf{\Lambda}]\,(t_2\,t_4 - t_2\,t_3 - t_1\,t_4 + t_1\,t_3) \\ &\qquad + \mathrm{Diag}\,(\mathrm{E}\,[\mathbf{\Lambda}])\,(\min\{t_2, t_4\} - \min\{t_2, t_3\} - \min\{t_1, t_4\} + \min\{t_1, t_3\}) \\ &\quad= \mathrm{Var}\,[\mathbf{\Lambda}]\,(t_2 - t_1)\,(t_4 - t_3) \\ &\qquad + \mathrm{Diag}\,(\mathrm{E}\,[\mathbf{\Lambda}])\,(\min\{0, t_4 - t_2\} - \min\{0, t_3 - t_2\}) \\ &\quad= \mathrm{Var}\,[\mathbf{\Lambda}]\,(t_2 - t_1)\,(t_4 - t_3) \\ &\qquad + \mathrm{Diag}\,(\mathrm{E}\,[\mathbf{\Lambda}])\,\big((t_2 - t_3)^+ - (t_2 - t_4)^+\big)\end{aligned}$$

and the assertion is shown. ■

Only in the case when the two intervals are not disjoint the term $(t_2 - t_3)^+ - (t_2 - t_4)^+$ does not vanish. Then it takes the value of the length of the common interval.

4.3 Posterior Distributions

Introducing the random parameter in the model of multivariate mixed Poisson processes in this chapter offers the possibility to study the conditional distribution of the parameter with respect to the process at some time t. Since the roles of the random vectors are permuted regarding the definition of the process we can speak of posterior distributions. The consideration of posterior distributions is linked to the question of stability of the model over time, which is also answered in this section. To avoid double execution, we first determine the common distribution of the finite–dimensional increments and the parameter.

4.3.1 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\mathbf{\Lambda}$. Then*

$$\mathrm{P}\left[\bigcap_{j=1}^{m}\left\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\right\}\cap\{\mathbf{\Lambda}\in B\}\right]$$
$$= \int_\Omega \chi_{\{\mathbf{\Lambda}\in B\}} \prod_{j=1}^{m} e^{-\mathbf{1}'\mathbf{\Lambda}\,(t_j-t_{j-1})}\,\frac{(\mathbf{\Lambda}\,(t_j-t_{j-1}))^{\mathbf{n}_j}}{\mathbf{n}_j!}\,d\mathrm{P}$$

holds for all $m\in\mathbb{N}$ and $t_0,t_1,\ldots,t_m\in\mathbb{R}_+$ with $0=t_0<t_1<\ldots<t_m$ and $\mathbf{n}_j\in\mathbb{N}_0^k$, $j\in\{1,\ldots,m\}$, and all $B\in\mathcal{B}(\mathbb{R}^k)$.

Proof: Consider $m\in\mathbb{N}$ and $t_0,t_1,\ldots,t_m\in\mathbb{R}_+$ with $0=t_0<t_1<\ldots<t_m$ and $\mathbf{n}_j\in\mathbb{N}_0^k$, $j\in\{1,\ldots,m\}$, as well as $B\in\mathcal{B}(\mathbb{R}^k)$. Then we obtain

$$\begin{aligned}
&\mathrm{P}\left[\bigcap_{j=1}^{m}\left\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\right\}\cap\{\mathbf{\Lambda}\in B\}\right]\\
&\qquad= \int_\Omega \chi_{\cap_{j=1}^m\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\}\cap\{\mathbf{\Lambda}\in B\}}\,d\mathrm{P}\\
&\qquad= \int_\Omega \mathrm{E}\left(\chi_{\cap_{j=1}^m\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\}\cap\{\mathbf{\Lambda}\in B\}}\,\middle|\,\mathbf{\Lambda}\right)d\mathrm{P}\\
&\qquad= \int_\Omega \chi_{\{\mathbf{\Lambda}\in B\}}\,\mathrm{E}\left(\chi_{\cap_{j=1}^m\{\mathbf{N}_{t_j}-\mathbf{N}_{t_{j-1}}=\mathbf{n}_j\}}\,\middle|\,\mathbf{\Lambda}\right)d\mathrm{P}\\
&\qquad= \int_\Omega \chi_{\{\mathbf{\Lambda}\in B\}} \prod_{j=1}^{m} e^{-\mathbf{1}'\mathbf{\Lambda}\,(t_j-t_{j-1})}\,\frac{(\mathbf{\Lambda}\,(t_j-t_{j-1}))^{\mathbf{n}_j}}{\mathbf{n}_j!}\,d\mathrm{P}
\end{aligned}$$

which yields the assertion. ■

4.3.2 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\mathbf{\Lambda}$. Then for all $t>0$ and all $\mathbf{n}\in\mathbb{N}_0^k$ the process $\{\mathbf{K}_{t,h}\}_{h\in\mathbb{R}_+}$ is a multivariate mixed Poisson process with parameter $\mathbf{\Lambda}$ on the probability space $(\Omega,\mathcal{F},\mathrm{P}_{t,\mathbf{n}})$.*

Proof: Consider $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$.
Our first aim is to show that the identity

$$\int_\Omega (f \circ \mathbf{\Lambda})\, d\mathrm{P}_{t,\mathbf{n}} = \frac{\int_\Omega (f \circ \mathbf{\Lambda}) \left(e^{-\mathbf{1}'\mathbf{\Lambda}\, t}\, (\mathbf{\Lambda}\, t)^{\mathbf{n}}/\mathbf{n}!\right) d\mathrm{P}}{\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]}$$

holds for all measurable functions $f : \mathbb{R}^k \to \mathbb{R}_+$. Therefore, we consider $B \in \mathcal{B}(\mathbb{R}^k)$ and $f := \chi_B$. Then $f \circ \mathbf{\Lambda} = \chi_{\mathbf{\Lambda}^{-1}(B)}$ and Lemma 4.3.1 yield

$$\begin{aligned}
\int_\Omega (f \circ \mathbf{\Lambda})\, d\mathrm{P}_{t,\mathbf{n}} &= \int_\Omega \chi_{\mathbf{\Lambda}^{-1}(B)}\, d\mathrm{P}_{t,\mathbf{n}} \\
&= \mathrm{P}_{t,\mathbf{n}}\left[\mathbf{\Lambda}^{-1}(B)\right] \\
&= \frac{\mathrm{P}\left[\mathbf{\Lambda}^{-1}(B) \cap \{\mathbf{N}_t = \mathbf{n}\}\right]}{\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]} \\
&= \frac{\int_\Omega \chi_{\mathbf{\Lambda}^{-1}(B)} \left(e^{-\mathbf{1}'\mathbf{\Lambda}\, t}\, (\mathbf{\Lambda}\, t)^{\mathbf{n}}/\mathbf{n}!\right) d\mathrm{P}}{\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]} \\
&= \frac{\int_\Omega (f \circ \mathbf{\Lambda}) \left(e^{-\mathbf{1}'\mathbf{\Lambda}\, t}\, (\mathbf{\Lambda}\, t)^{\mathbf{n}}/\mathbf{n}!\right) d\mathrm{P}}{\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]}
\end{aligned}$$

Now, the representation of positive measurable functions in terms of simple functions and the monotone convergence theorem gives the desired identity.
For the rest of the proof we additionally consider $m \in \mathbb{N}$ and $h_0, h_1, \ldots, h_m \in \mathbb{R}_+$ with $0 = h_0 < h_1 < \ldots < h_m$ and $\mathbf{n}_j \in \mathbb{N}_0^k$, $j \in \{1, \ldots, m\}$ as well as an arbitrary $C \in \sigma(\mathbf{\Lambda})$. Lemma 4.3.1 and the previous identity yield

$$\begin{aligned}
&\int_C \chi_{\cap_{j=1}^m \{\mathbf{K}_{t,h_j} - \mathbf{K}_{t,h_{j-1}} = \mathbf{n}_j\}}\, d\mathrm{P}_{t,\mathbf{n}} \\
&= \int_\Omega \chi_{\cap_{j=1}^m \{\mathbf{N}_{t+h_j} - \mathbf{N}_{t+h_{j-1}} = \mathbf{n}_j\} \cap C}\, d\mathrm{P}_{t,\mathbf{n}} \\
&= \mathrm{P}_{t,\mathbf{n}}\left[\bigcap_{j=1}^m \left\{\mathbf{N}_{t+h_j} - \mathbf{N}_{t+h_{j-1}} = \mathbf{n}_j\right\} \cap C\right] \\
&= \frac{\mathrm{P}\left[\bigcap_{j=1}^m \left\{\mathbf{N}_{t+h_j} - \mathbf{N}_{t+h_{j-1}} = \mathbf{n}_j\right\} \cap \{\mathbf{N}_t = \mathbf{n}\} \cap C\right]}{\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]} \\
&= \frac{\displaystyle\int_\Omega \chi_C \left(\prod_{j=1}^m e^{-\mathbf{1}'\mathbf{\Lambda}\,(h_j - h_{j-1})}\, \frac{(\mathbf{\Lambda}\,(h_j - h_{j-1}))^{\mathbf{n}_j}}{\mathbf{n}_j!}\right) e^{-\mathbf{1}'\mathbf{\Lambda}\, t}\, \frac{(\mathbf{\Lambda}\, t)^{\mathbf{n}}}{\mathbf{n}!}\, d\mathrm{P}}{\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]} \\
&= \int_\Omega \chi_C \prod_{j=1}^m e^{-\mathbf{1}'\mathbf{\Lambda}\,(h_j - h_{j-1})}\, \frac{(\mathbf{\Lambda}\,(h_j - h_{j-1}))^{\mathbf{n}_j}}{\mathbf{n}_j!}\, d\mathrm{P}_{t,\mathbf{n}}
\end{aligned}$$

$$= \int_C \prod_{j=1}^{m} e^{-\mathbf{1}'\mathbf{\Lambda}\,(h_j-h_{j-1})}\,\frac{(\mathbf{\Lambda}\,(h_j-h_{j-1}))^{\mathbf{n}_j}}{\mathbf{n}_j!}\,d\mathrm{P}_{t,\mathbf{n}}$$

Thus, we have

$$\mathrm{P}_{t,\mathbf{n}}\left(\bigcap_{j=1}^{m}\{\mathbf{K}_{t,h_j}-\mathbf{K}_{t,h_{j-1}}=\mathbf{n}_j\}\,\middle|\,\mathbf{\Lambda}\right) = \prod_{j=1}^{m} e^{-\mathbf{1}'\mathbf{\Lambda}\,(h_j-h_{j-1})}\,\frac{(\mathbf{\Lambda}\,(h_j-h_{j-1}))^{\mathbf{n}_j}}{\mathbf{n}_j!}$$

which proves the assertion. ■

Hence, introducing the parameter in the model of multivariate mixed Poisson processes does not change the stability of the model over time in the sense that it is not important at which time we start to observe the process. While the model stays unchanged the distribution alters of course. Thus, the next theorem is concerned with the conditional distribution of the parameter with respect to the process at some time t.

4.3.3 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\mathbf{\Lambda}$. Then*

$$\mathrm{P}_{\mathbf{\Lambda}\,|\,\mathbf{N}_t}(B) = \frac{\int_B e^{-\mathbf{1}'\boldsymbol{\lambda}\,t}\,\boldsymbol{\lambda}^{\mathbf{N}_t}\,d\mathrm{P}_{\mathbf{\Lambda}}(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda}\,t}\,\boldsymbol{\lambda}^{\mathbf{N}_t}\,d\mathrm{P}_{\mathbf{\Lambda}}(\boldsymbol{\lambda})}$$

holds for all $t>0$ and all $B\in\mathcal{B}(\mathbb{R}^k)$.

Proof: We consider $t>0$ and $B\in\mathcal{B}(\mathbb{R}^k)$ as well as $\mathbf{n}\in\mathbb{N}_0^k$. By Lemma 4.3.1 we have

$$\begin{aligned}\mathrm{P}\left(\{\mathbf{\Lambda}\in B\}\cap\{\mathbf{N}_t=\mathbf{n}\}\right) &= \int_\Omega \chi_{\{\mathbf{\Lambda}\in B\}}\,e^{-\mathbf{1}'\mathbf{\Lambda}\,t}\,\frac{(\mathbf{\Lambda}\,t)^{\mathbf{n}}}{\mathbf{n}!}\,d\mathrm{P}\\ &= \int_{\mathbf{\Lambda}^{-1}(B)} e^{-\mathbf{1}'\mathbf{\Lambda}\,t}\,\frac{(\mathbf{\Lambda}\,t)^{\mathbf{n}}}{\mathbf{n}!}\,d\mathrm{P}\\ &= \int_B e^{-\mathbf{1}'\boldsymbol{\lambda}\,t}\,\frac{(\boldsymbol{\lambda}\,t)^{\mathbf{n}}}{\mathbf{n}!}\,d\mathrm{P}_{\mathbf{\Lambda}}(\boldsymbol{\lambda})\end{aligned}$$

and therefore

$$\begin{aligned}\mathrm{P}_{\mathbf{\Lambda}\,|\,\mathbf{N}_t=\mathbf{n}}[B] &= P\left[\{\mathbf{\Lambda}\in B\}\,\middle|\,\{\mathbf{N}_t=\mathbf{n}\}\right]\\ &= \frac{\int_B e^{-\mathbf{1}'\boldsymbol{\lambda}\,t}\,(\boldsymbol{\lambda}\,t)^{\mathbf{n}}/\mathbf{n}!\;d\mathrm{P}_{\mathbf{\Lambda}}(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda}\,t}\,(\boldsymbol{\lambda}\,t)^{\mathbf{n}}/\mathbf{n}!\;d\mathrm{P}_{\mathbf{\Lambda}}(\boldsymbol{\lambda})}\end{aligned}$$

and hence

$$P_{\mathbf{\Lambda}\,|\,\mathbf{N}_t}(B) = \sum_{\mathbf{n}\in\mathbb{N}_0^k} \mathrm{P}_{\mathbf{\Lambda}\,|\,\mathbf{N}_t=\mathbf{n}}[B]\,\chi_{\{\mathbf{N}_t=\mathbf{n}\}}$$

$$
\begin{aligned}
&= \sum_{\mathbf{n} \in \mathbb{N}_0^k} \frac{\int_B e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, \boldsymbol{\lambda}^{\mathbf{n}}\, d\mathrm{P}_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, \boldsymbol{\lambda}^{\mathbf{n}}\, d\mathrm{P}_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda})}\, \chi_{\{\mathbf{N}_t = \mathbf{n}\}} \\
&= \sum_{\mathbf{n} \in \mathbb{N}_0^k} \frac{\int_B e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, \boldsymbol{\lambda}^{\mathbf{N}_t}\, d\mathrm{P}_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, \boldsymbol{\lambda}^{\mathbf{N}_t}\, d\mathrm{P}_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda})}\, \chi_{\{\mathbf{N}_t = \mathbf{n}\}} \\
&= \frac{\int_B e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, \boldsymbol{\lambda}^{\mathbf{N}_t}\, d\mathrm{P}_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, \boldsymbol{\lambda}^{\mathbf{N}_t}\, d\mathrm{P}_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda})}
\end{aligned}
$$

which completes the proof. ■

4.4 Regularity

As in the sections before, we will use the fact that a multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$ is a multivariate mixed Poisson process with mixing distribution $P_{\boldsymbol{\Lambda}}$ and thus most of the results of Section 3.4 are valid for the process considered in this chapter. Nevertheless, we carry over some results where the introduction of the parameter shortens the presentation or makes the essential point of the assertion more obvious.

4.4.1 Lemma. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$. Then*

$$
\begin{aligned}
p_{\mathbf{n},\mathbf{m}}(r,t) &= \frac{(t-r)^{\mathbf{1}'(\mathbf{m}-\mathbf{n})}}{(\mathbf{m}-\mathbf{n})!}\, \frac{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, \boldsymbol{\lambda}^{\mathbf{m}}\, dP_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} r}\, \boldsymbol{\lambda}^{\mathbf{n}}\, dP_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda})} \\
&= \mathrm{E}\left[e^{-\mathbf{1}'\boldsymbol{\Lambda}(t-r)}\, \frac{(\boldsymbol{\Lambda}\,(t-r))^{\mathbf{m}-\mathbf{n}}}{(\mathbf{m}-\mathbf{n})!} \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\} \right]
\end{aligned}
$$

holds for all $(\mathbf{n}, r) \in Z$ and $\mathbf{m} \in \mathbb{N}_0^k$, $t \in \mathbb{R}_+$ with $\mathbf{n} \le \mathbf{m}$ and $r \le t$.

4.4.2 Theorem. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$. Then the following are equivalent.*

(a) *$\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ is regular with intensities $\{\boldsymbol{\kappa}_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{N}_0^k}$.*

(b) *The condition $\mathrm{E}[\boldsymbol{\Lambda}] < \infty$ is valid.*

If $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ satisfies one and hence all preceding items, then

$$
\begin{aligned}
\kappa_{\mathbf{n}}^{(i)}(t) &= \frac{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, \boldsymbol{\lambda}^{\mathbf{n}+\mathbf{e}_i}\, dP_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\mathbf{1}'\boldsymbol{\lambda} t}\, \boldsymbol{\lambda}^{\mathbf{n}}\, dP_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda})} \\
&= \mathrm{E}\left[\Lambda_i \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\} \right]
\end{aligned}
$$

holds for all $(\mathbf{n}, t) \in Z$ and $i \in \{1, \ldots, k\}$.

We can see that a multivariate mixed Poisson process with parameter is regular if, and only if, the parameter has a finite first order moment. The identity of the fraction of integrals with the conditional expected value holds due to Theorem 4.3.3. In combination with the characterization of regularity we get the following somewhat strange looking identity.

4.4.3 Corollary. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a regular multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$. Then the identity*

$$\mathrm{E}\left[e^{-\mathbf{1}'\boldsymbol{\Lambda}\, h} \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right] \;=\; e^{-\int_t^{t+h} \mathrm{E}\,\left[\mathbf{1}'\boldsymbol{\Lambda} \,\middle|\, \{\mathbf{N}_s = \mathbf{n}\}\right]\, ds}$$

holds for all $(\mathbf{n}, t) \in Z$ and $h \in \mathbb{R}_+$.

Proof: Consider $(\mathbf{n}, t) \in Z$ and $h \in \mathbb{R}_+$. Then we obtain with the help of Theorem 2.3.6

$$\begin{aligned}
\mathrm{E}\left[e^{-\mathbf{1}'\boldsymbol{\Lambda}\, h} \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right] &= p_{\mathbf{n},\mathbf{n}}(t, t+h) \\
&= e^{-\int_t^{t+h} \sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(s)\, ds} \\
&= e^{-\int_t^{t+h} \sum_{i=1}^k \mathrm{E}\left[\Lambda_i \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right]\, ds} \\
&= e^{-\int_t^{t+h} \mathrm{E}\,\left[\mathbf{1}'\boldsymbol{\Lambda} \,\middle|\, \{\mathbf{N}_s = \mathbf{n}\}\right]\, ds}
\end{aligned}$$

which yields the assertion. ■

Similar to the characterization in terms of the multinomial property (Theorem 3.2.3), the definition of multivariate mixed Poisson processes by means of conditional probabilities does not seem to allow to carry over the characterization in terms of the representation of transition probabilities and intensities (Theorem 3.4.3) from the setting with mixing distribution to the setting with parameter. Therefore, we turn to the properties of the intensities.

4.4.4 Theorem. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a regular multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$. Then the intensities $\{\boldsymbol{\kappa}_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{N}_0^k}$ have the following properties.*

(1) *The identity*

$$\kappa_{\mathbf{n}}^{(i)}(t) \;=\; \frac{D^{\mathbf{n}+\mathbf{e}_i} M_{\boldsymbol{\Lambda}}(-t\mathbf{1})}{D^{\mathbf{n}} M_{\boldsymbol{\Lambda}}(-t\mathbf{1})}$$

holds for all $t > 0$, $\mathbf{n} \in \mathbb{N}_0^k$, and $i \in \{1, \ldots, k\}$.

(2) *Let $i \in \{1, \ldots, k\}$, $\mathbf{n} \in \mathbb{N}_0^k$ and let $\mathrm{E}\left[\boldsymbol{\Lambda}^{\mathbf{n}+\mathbf{e}_i}\right]$ be finite. Then $\lim_{t \downarrow 0} \kappa_{\mathbf{n}}^{(i)}(t)$ is finite and*

$$\lim_{t \downarrow 0} \kappa_{\mathbf{n}}^{(i)}(t) \;=\; \frac{\mathrm{E}\left[\boldsymbol{\Lambda}^{\mathbf{n}+\mathbf{e}_i}\right]}{\mathrm{E}\left[\boldsymbol{\Lambda}^{\mathbf{n}}\right]}$$

is valid.

4.4.5 Corollary. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a regular multivariate mixed Poisson process with parameter $\mathbf{\Lambda}$. Then the inequality*

$$\left| \frac{d}{dt} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \right| \quad \leq \quad \mathrm{E}\left[\mathbf{1}'\mathbf{\Lambda}\right]$$

holds for all $t > 0$ and all $\mathbf{n} \in \mathbb{N}_0^k$.

The transfer of Lemma 3.4.6 into the setting of mixed Poisson processes with parameter would have brought up a meaningless equation. The intensities are conditional expected values of coordinates of the parameter and thus the product of the intensities with the probabilities is in fact the expected value of the coordinates of the parameter.

Chapter 5

Multivariate Mixed Poisson Processes with Special Parameter

5.1 The Model

As we have seen in the previous chapter, the multivariate mixed Poisson process with parameter $\mathbf{\Lambda}$ is determined by the moment generating function $M_{\mathbf{\Lambda}}$ of the parameter. So the distribution of the random vector $\mathbf{\Lambda} \in (\mathbf{0}, \boldsymbol{\infty})$ controls the process. To make the parameter more practical we introduce a new assumption. Therefore, we define two new random vectors, where one of them is indeed a random variable.

$\Lambda := \mathbf{1}'\mathbf{\Lambda}$ is the sum of all coordinates of the parameter. Since $\mathbf{1}' \in \mathcal{A}$ Lemma 4.1.4 says that Λ is the parameter of the mixed Poisson process $\{N_t\}_{t\in\mathbb{R}_+}$ with $N_t := \mathbf{1}'\mathbf{N}_t$, $t \in \mathbb{R}_+$. Furthermore we have $\Lambda \in (0, \infty)$.

$\mathbf{\Theta} := (\mathbf{1}'\mathbf{\Lambda})^{-1}\mathbf{\Lambda}$ is the vector of the proportions of the coordinates of the parameter with respect to the sum of all coordinates. Defining $\Delta_k := \{\mathbf{x} \in \mathbb{R}^k : \mathbf{x} > 0,\ \mathbf{1}'\mathbf{x} = 1\}$ as the strictly positive boundary of the k–dimensional unit simplex we have $\mathbf{\Theta} \in \Delta_k$. It is obvious that $\mathbf{\Lambda} = \Lambda\mathbf{\Theta}$ holds. Hence, the assumption we will study in this chapter is as follows.

A multivariate mixed Poisson process fulfils the ***independence assumption*** **(I)** if Λ and $\mathbf{\Theta}$ are independent.

There are various reasons for studying this assumption. First, it seems easier to handle two random vectors with support $(0, \infty)$ and Δ_k than the whole parameter with support $(\mathbf{0}, \boldsymbol{\infty})$. Second, Λ is the parameter of the process which is the sum of all coordinates. So assuming one has applied a univariate mixed Poisson process with parameter and now wants to divide the events of interest in certain sub–events one can still use the information about Λ which is already available. In this case one just has to concentrate on the random vector $\mathbf{\Theta}$ and its properties. This will become clearer later in this chapter, but a first hint is given by Theorem 5.1.2. Last but not least, the independence assumption, which also can be transferred to the setting of multivariate mixed Poisson processes with mixing distribution, is valid in most of models of multivariate mixed Poisson distributions and multivariate mixed Poisson processes discussed in literature so far, as in Bates and Neyman [1952], Picard [1976], Nelson [1984], and Walhin and Paris [2001], to name a few.

For a possible choice of distributions for Λ and $\mathbf{\Theta}$ and the therewith following consequences for the multivariate mixed Poisson process see Zocher [2002] and Zocher [2005].

Naturally, we first look at how the transformation works according to the independence assumption

5.1.1 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter Λ which fulfils* **(I)** *and let $A \in \mathcal{A}_P \cup \mathcal{A}_C$. Then $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ is a multivariate mixed Poisson process with parameter $A\mathbf{\Lambda}$ which fulfils* **(I)**.

Proof: Due to Lemma 4.1.4 (1) the transformed process is again a multivariate mixed Poisson process which has parameter $A\mathbf{\Lambda} =: \mathbf{\Lambda}^*$. Since $\mathbf{1}'A = \mathbf{1}'$ holds for $A \in \mathcal{A}_P \cup \mathcal{A}_C$ we get

$$\Lambda^* := \mathbf{1}'\mathbf{\Lambda}^* = \mathbf{1}'A\mathbf{\Lambda} = \mathbf{1}'\mathbf{\Lambda} = \Lambda$$

as well as

$$\mathbf{\Theta}^* := (\mathbf{1}'\mathbf{\Lambda}^*)^{-1}\,\mathbf{\Lambda}^* = \Lambda^{-1}\,A\Lambda\mathbf{\Theta} = A\mathbf{\Theta}$$

A is measurable and thus the independence of Λ^* and $\mathbf{\Theta}^*$ follows. ■

For $A \in \mathcal{A}_S$ the transformed process $\{A\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ need not to fulfil the independence assumption whenever $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ fulfills **(I)**.
Example: Consider a three–dimensional random vector $\mathbf{\Lambda}$ taking values according to the subsequent table. The derived variables Λ and $\mathbf{\Theta}$ are listed, too.

Λ_1	Λ_2	Λ_3	Λ	Θ_1	Θ_2	Θ_3	P
1	1	2	4	1/4	1/4	1/2	0.2
1	2	1	4	1/4	1/2	1/4	0.2
2	1	1	4	1/2	1/4	1/4	0.6

We can see that Λ is constant and thus Λ and $\mathbf{\Theta}$ are independent. Considering furthermore $A \in \mathcal{A}_S$ with

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and denoting $\mathbf{\Lambda}^* := A\mathbf{\Lambda}$ we obtain

Λ_1^*	Λ_2^*	Λ^*	Θ_1^*	Θ_2^*	P
1	1	2	1/2	1/2	0.2
1	2	3	1/3	2/3	0.2
2	1	3	2/3	1/3	0.6

and therefore

$$\mathrm{P}[\{\Lambda^* = 3\} \cap \{\Theta_1^* = 2/3\}] \;=\; 0.6 \;\neq\; 0.48 \;=\; \mathrm{P}[\{\Lambda^* = 3\}]\;\mathrm{P}[\{\Theta_1^* = 2/3\}]$$

Hence, Λ^* and $\boldsymbol{\Theta}^*$ are not independent. □

A multivariate mixed Poisson process with parameter has the multinomial property, so that just the one–dimensional distributions are relevant.

5.1.2 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter Λ which fulfils* **(I)**. *Then*

$$\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}] \;=\; \int_{\mathbb{R}} e^{-\lambda t}\,\frac{(\lambda\, t)^{\mathbf{1}'\mathbf{n}}}{(\mathbf{1}'\mathbf{n})!}\,d\mathrm{P}_\Lambda(\lambda)\;\int_{\mathbb{R}^k}\binom{\mathbf{1}'\mathbf{n}}{\mathbf{n}}\,\boldsymbol{\theta}^{\mathbf{n}}\,d\mathrm{P}_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$$

holds for all $t \in \mathbb{R}_+$ and all $\mathbf{n} \in \mathbb{N}_0^k$. Furthermore

$$\begin{aligned}\mathrm{P}[\{N_t = n\}] \;&=\; \int_{\mathbb{R}} e^{-\lambda t}\,\frac{(\lambda\, t)^n}{n!}\,d\mathrm{P}_\Lambda(\lambda)\\ \mathrm{P}\big[\{\mathbf{N}_t = \mathbf{n}\}\,\big|\,\{N_t = n\}\big] \;&=\; \int_{\mathbb{R}^k}\binom{n}{\mathbf{n}}\,\boldsymbol{\theta}^{\mathbf{n}}\,d\mathrm{P}_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\end{aligned}$$

hold for all $n \in \mathbb{N}_0$ and $\mathbf{n} \in \mathbb{N}_0^k$ with $\mathbf{1}'\mathbf{n} = n$.

Proof: Consider $t \in \mathbb{R}_+$ and $\mathbf{n} \in \mathbb{N}_0^k$. Then we have

$$\begin{aligned}\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}] \;&=\; \mathrm{E}\left[e^{-\mathbf{1}'\boldsymbol{\Lambda} t}\,\frac{(\boldsymbol{\Lambda}\, t)^{\mathbf{n}}}{\mathbf{n}!}\right]\\ &=\; \mathrm{E}\left[e^{-\mathbf{1}'\Lambda\boldsymbol{\Theta} t}\,\frac{(\Lambda\boldsymbol{\Theta}\, t)^{\mathbf{n}}}{\mathbf{n}!}\right]\\ &=\; \mathrm{E}\left[e^{-\Lambda t}\,(\Lambda\, t)^{\mathbf{1}'\mathbf{n}}\,\frac{\boldsymbol{\Theta}^{\mathbf{n}}}{\mathbf{n}!}\right]\\ &=\; \mathrm{E}\left[e^{-\Lambda t}\,\frac{(\Lambda\, t)^{\mathbf{1}'\mathbf{n}}}{(\mathbf{1}'\mathbf{n})!}\right]\;\mathrm{E}\left[\binom{\mathbf{1}'\mathbf{n}}{\mathbf{n}}\,\boldsymbol{\Theta}^{\mathbf{n}}\right]\\ &=\; \int_{\mathbb{R}} e^{-\lambda t}\,\frac{(\lambda\, t)^{\mathbf{1}'\mathbf{n}}}{(\mathbf{1}'\mathbf{n})!}\,d\mathrm{P}_\Lambda(\lambda)\;\int_{\mathbb{R}^k}\binom{\mathbf{1}'\mathbf{n}}{\mathbf{n}}\,\boldsymbol{\theta}^{\mathbf{n}}\,d\mathrm{P}_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\end{aligned}$$

Since $\mathbf{1}' \in \mathcal{A}$, the process $\{N_t\}_{t\in\mathbb{R}_+}$ is a univariate mixed Poisson process with parameter Λ (see Lemma 4.1.4) and

$$\mathrm{P}[\{N_t = n\}] \;=\; \int_{\mathbb{R}} e^{-\lambda t}\,\frac{(\lambda\, t)^n}{n!}\,d\mathrm{P}_\Lambda(\lambda)$$

is valid for all $n \in \mathbb{N}_0$. As a result of the decomposition

$$\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right] \;=\; \mathrm{P}\left[\{N_t = n\}\right] \, \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\} \,\middle|\, \{N_t = n\}\right]$$

for $n \in \mathbb{N}_0$ and $\mathbf{n} \in \mathbb{N}_0^k$ with $\mathbf{1}'\mathbf{n} = n$, the last equation follows immediately. ■

Theorem 5.1.2 shows the different influence of the two random vectors Λ and $\mathbf{\Theta}$. The parameter Λ drives the process of the sum of all coordinates and determines how many events occur in a certain time interval, whereas $P_{\mathbf{\Theta}}$ is the mixing distribution of a mixed multinomial distribution which divides the events into the given classes (coordinates). It is very remarkable that this distribution is independent of time t. So we can in fact think of a three step model. First, the mixed Poisson process $\{N_t\}_{t \in \mathbb{R}_+}$ determines the number of events that occur until a certain time t. By the mixed multinomial distribution the events are then divided onto the coordinates. And as the last step, the multinomial property provides the distribution of the events into the past periods. The first step depends on Λ, the second one is influenced by $\mathbf{\Theta}$, whereas the last step is independent of the parameter.

5.2 Moments

Keeping in mind that the moment generating function $M_{\mathbf{\Lambda}}$ does not only determine the one–dimensional distributions of the multivariate mixed Poisson process, but also the binomial moments, it would be desirable if this function can be decomposed under the independence assumption **(I)** into the moment generating functions of Λ and $\mathbf{\Theta}$. As a consequence of **(I)** we have (see also Theorem 1.2.3)

$$M_{(\Lambda,\mathbf{\Theta})}(t, \mathbf{s}) \;=\; M_\Lambda(t) \, M_{\mathbf{\Theta}}(\mathbf{s})$$

with $t \in \mathbb{R}_+$ and $\mathbf{s} \in \mathbb{R}^k$. However, to replace the moment generating function $M_{(\Lambda,\mathbf{\Theta})}$ of the common distribution of Λ and $\mathbf{\Theta}$ by the moment generating function $M_{\mathbf{\Lambda}}$ is not possible because $M_{\mathbf{\Lambda}}$ is defined on $\mathbb{R}^k$ and not on $\mathbb{R}^{k+1}$.

Nevertheless, there exists a factorization of $M_{\mathbf{\Lambda}}$ for certain arguments.

5.2.1 Lemma. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\mathbf{\Lambda}$ which fulfils* **(I)***. Then*

$$\begin{aligned} D^{\mathbf{n}} M_{\mathbf{\Lambda}}(-t\mathbf{1}) \;&=\; D^n M_\Lambda(-t) \binom{n}{\mathbf{n}}^{-1} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\} \,\middle|\, \{N_t = n\}\right] \\ &=\; D^n M_\Lambda(-t) \, \mathrm{E}\left[\mathbf{\Theta}^{\mathbf{n}}\right] \end{aligned}$$

holds for all $t > 0$ and all $\mathbf{n} \in \mathbb{N}_0^k$ where $n := \mathbf{1}'\mathbf{n}$.

Proof: Since $\{N_t\}_{t \in \mathbb{R}_+}$ is a mixed Poisson process with parameter Λ we get for all $\mathbf{n} \in \mathbb{N}_0^k$ with the help of Corollary 3.2.4

$$\begin{aligned} \frac{t^n}{\mathbf{n}!} D^{\mathbf{n}} M_{\mathbf{\Lambda}}(-t\mathbf{1}) &= \mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}] \\ &= \mathrm{P}[\{N_t = n\}]\, \mathrm{P}\big[\{\mathbf{N}_t = \mathbf{n}\} \,\big|\, \{N_t = n\}\big] \\ &= \frac{t^n}{n!} D^n M_\Lambda(-t)\, \mathrm{P}\big[\{\mathbf{N}_t = \mathbf{n}\} \,\big|\, \{N_t = n\}\big] \end{aligned}$$

which yields the first equation. By

$$\mathrm{P}\big[\{\mathbf{N}_t = \mathbf{n}\} \,\big|\, \{N_t = n\}\big] = \mathrm{E}\left[\binom{n}{\mathbf{n}} \mathbf{\Theta}^{\mathbf{n}}\right]$$

the second equation directly follows from the first one. ∎

So under the assumption **(I)** the moment generating function M_Λ and the conditional probabilities of $\mathbf{N}_t$ with respect to $\{N_t = n\}$, which are nothing else than the moments around the origin of $\mathbf{\Theta}$, take over the role of $M_{\mathbf{\Lambda}}$ in determining the one–dimensional distributions of the process. This is also true for the binomial moments of $\mathbf{N}_t$, as can be seen in the next lines.

5.2.2 Lemma. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\mathbf{\Lambda}$ which fulfils* **(I)**. *The binomial moment of $\mathbf{N}_t$ fulfils*

$$\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] = \mathrm{E}\left[\binom{N_t}{l}\right] \mathrm{P}\big[\{\mathbf{N}_t = \mathbf{l}\} \,\big|\, \{N_t = l\}\big]$$

for all $t \in \mathbb{R}_+$ and $\mathbf{l} \in \mathbb{N}_0^k$ where $l := \mathbf{1}'\mathbf{l}$.

Proof: Theorem 4.2.1 twice and **(I)** yield

$$\begin{aligned} \mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] &= \frac{t^l}{\mathbf{l}!} \mathrm{E}[\mathbf{\Lambda}^{\mathbf{l}}] \\ &= \frac{t^l}{\mathbf{l}!} \mathrm{E}\big[(\Lambda\mathbf{\Theta})^{\mathbf{l}}\big] \\ &= \frac{t^l}{l!}\mathrm{E}[\Lambda^l]\ \mathrm{E}\left[\binom{l}{\mathbf{l}} \mathbf{\Theta}^{\mathbf{l}}\right] \\ &= \mathrm{E}\left[\binom{N_t}{l}\right] \mathrm{P}\big[\{\mathbf{N}_t = \mathbf{l}\} \,\big|\, \{N_t = l\}\big] \end{aligned}$$

which proves the assertion. ∎

5.2.3 Theorem. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\mathbf{\Lambda}$ which fulfils* **(I)**. *For $l \in \mathbb{N}_0$ the following are equivalent.*

(a) *There exists some $t > 0$ and some $\mathbf{l} \in \mathbb{N}_0^k$ with $\mathbf{1}'\mathbf{l} = l$ such that*

$$\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] \;<\; \infty$$

holds.

(b) *The identity*

$$\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{m}}\right] \;<\; \infty$$

holds for all $t \in \mathbb{R}_+$ and all $\mathbf{m} \in \mathbb{N}_0^k$ with $\mathbf{1}'\mathbf{m} \le l$.

(c) *There exists some $t > 0$ and some $\mathbf{l} \in \mathbb{N}_0^k$ with $\mathbf{1}'\mathbf{l} = l$ such that*

$$\mathrm{E}\left[(\mathbf{N}_t)^{\mathbf{l}}\right] \;<\; \infty$$

holds.

(d) *The identity*

$$\mathrm{E}\left[(\mathbf{N}_t)^{\mathbf{m}}\right] \;<\; \infty$$

holds for all $t \in \mathbb{R}_+$ and all $\mathbf{m} \in \mathbb{N}_0^k$ with $\mathbf{1}'\mathbf{m} \le l$.

(e) *There exists some $t > 0$ such that*

$$\mathrm{E}\left[(N_t)^l\right] \;<\; \infty$$

holds.

(f) *The identity*

$$\mathrm{E}\left[(N_t)^m\right] \;<\; \infty$$

holds for all $t \in \mathbb{R}_+$ and all $m \in \mathbb{N}$ with $m \le l$.

(g) *The parameter of the sum process fulfils*

$$\mathrm{E}\left[\Lambda^l\right] \;<\; \infty$$

(h) *For all $m \le l$ the m–th derivative of $M_\Lambda|_{(-\infty,0]}$ is continuous on $(\infty, 0]$.*

If $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ satisfies one and hence all preceding items, then

$$\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] \;=\; \frac{t^l}{l!}\, D^l M_\Lambda|_{(-\infty,0]}(0)\; \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{l}\}\,\middle|\,\{N_t = l\}\right]$$

holds for all $t \in \mathbb{R}_+$ and all $\mathbf{l} \in \mathbb{N}_0^k$ with $\mathbf{1}'\mathbf{l} = l$.

Proof: The condition $t = 0$ in whatever assertion in this theorem is not significant since a multivariate mixed Poisson process always starts in $\mathbf{0}$ and therefore all

moments of $\mathbf{N}_0$ are finite.
As $\{N_t\}_{t\in\mathbb{R}_+}$ is a univariate mixed Poisson process with parameter Λ, the equivalence of (e), (f), (g), and (h) is due to Theorem 4.2.3. This theorem also yields in connection with Lemma 5.2.2

$$\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] \;=\; \frac{t^l}{l!}\, D^l M_\Lambda|_{(-\infty,0]}(0)\ \mathrm{P}\left[\{\mathbf{N}_t=\mathbf{l}\}\,\middle|\,\{N_t=l\}\right]$$

for all $t>0$. Therefore, to prove the theorem it is sufficient to show (g) $\Rightarrow$ (b) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (a) $\Rightarrow$ (e).

(g) $\Rightarrow$ (b): This follows from Lemma 5.2.2 and Theorem 4.2.1.

(b) $\Rightarrow$ (d): This follows from Lemma 1.1.4.

(d) $\Rightarrow$ (c): obvious

(c) $\Rightarrow$ (a): Since for all $t>0$ and all $\mathbf{l}\in\mathbb{N}_0^k$

$$\mathrm{E}\left[\binom{\mathbf{N}_t}{\mathbf{l}}\right] \;\leq\; \mathrm{E}\left[(\mathbf{N}_t)^{\mathbf{l}}\right]$$

the assumptions follows immediately.

(a) $\Rightarrow$ (e): Using Lemma 5.2.2 and the fact that for a positive one–dimensional discrete random variable the binomial moment of order l is finite if, and only if, the moment of order l is finite (see Schmidt [2002]), we obtain the assertion. ■

Under assumption **(I)** the finiteness of the moments of $\mathbf{N}_t$ depends only on the finiteness of the moments of N_t, which can be verified through the moment generating function of Λ. As before, we want to give explicit formulas for the first and second central moments of the process in the case they are finite. Although the next lemma contains only identities for two independent random variables, it is formulated in terms of the process $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ to stay in the familiar notation.

5.2.4 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$ which fulfils* **(I)**. *The moments of $\boldsymbol{\Lambda}$ can be decomposed in the succeeding manner.*

(1) $\mathrm{E}[\boldsymbol{\Lambda}] \;=\; \mathrm{E}[\Lambda]\ \mathrm{E}[\boldsymbol{\Theta}]$
(2) *If Λ has a finite first moment, then*

$$\mathrm{Var}[\boldsymbol{\Lambda}] \;=\; \mathrm{Var}[\Lambda]\ \mathrm{Var}[\boldsymbol{\Theta}] + (\mathrm{E}[\Lambda])^2\ \mathrm{Var}[\boldsymbol{\Theta}] + \mathrm{Var}[\Lambda]\ \mathrm{E}[\boldsymbol{\Theta}]\ \mathrm{E}[\boldsymbol{\Theta}]'$$

is valid.

Proof:
(1): The independence assumption **(I)** directly yields

$$\mathrm{E}\left[\boldsymbol{\Lambda}\right] \;=\; \mathrm{E}\left[\Lambda\boldsymbol{\Theta}\right] \;=\; \mathrm{E}\left[\Lambda\right]\,\mathrm{E}\left[\boldsymbol{\Theta}\right]$$

(2): If Λ a has finite first moment, then so has $\boldsymbol{\Lambda}$. Hence, we get

$$\begin{aligned}
\mathrm{Var}\left[\boldsymbol{\Lambda}\right] &= \mathrm{E}\left[\boldsymbol{\Lambda}\boldsymbol{\Lambda}'\right] - \mathrm{E}\left[\boldsymbol{\Lambda}\right]\,\mathrm{E}\left[\boldsymbol{\Lambda}\right]' \\
&= \mathrm{E}\left[\Lambda^2\boldsymbol{\Theta}\boldsymbol{\Theta}'\right] - \mathrm{E}\left[\Lambda\boldsymbol{\Theta}\right]\,\mathrm{E}\left[\Lambda\boldsymbol{\Theta}\right]' \\
&= \mathrm{E}\left[\Lambda^2\right]\,E\left[\boldsymbol{\Theta}\boldsymbol{\Theta}'\right] - \left(\mathrm{E}\left[\Lambda\right]\right)^2\,\mathrm{E}\left[\boldsymbol{\Theta}\right]\,\mathrm{E}\left[\boldsymbol{\Theta}\right]' \\
&= \mathrm{Var}\left[\Lambda\right]\,E\left[\boldsymbol{\Theta}\boldsymbol{\Theta}'\right] + \left(\mathrm{E}\left[\Lambda\right]\right)^2\,\mathrm{E}\left[\boldsymbol{\Theta}\boldsymbol{\Theta}'\right] - \left(\mathrm{E}\left[\Lambda\right]\right)^2\,\mathrm{E}\left[\boldsymbol{\Theta}\right]\,\mathrm{E}\left[\boldsymbol{\Theta}\right]' \\
&= \mathrm{Var}\left[\Lambda\right]\,\mathrm{Var}\left[\boldsymbol{\Theta}\right] + \left(\mathrm{E}\left[\Lambda\right]\right)^2\,\mathrm{Var}\left[\boldsymbol{\Theta}\right] + \mathrm{Var}\left[\Lambda\right]\,\mathrm{E}\left[\boldsymbol{\Theta}\right]\,\mathrm{E}\left[\boldsymbol{\Theta}\right]'
\end{aligned}$$

which completes the proof. ∎

5.2.5 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$ which fulfils* **(I)**. *Then the following is valid.*

(1) *If Λ has a finite moment of first order, then*

$$\mathrm{E}\left[\mathbf{N}_t\right] \;=\; t\,\mathrm{E}\left[\Lambda\right]\,\mathrm{E}\left[\boldsymbol{\Theta}\right]$$

holds for all $t \in \mathbb{R}_+$.

(2) *If Λ has a finite moment of second order, then*

$$\begin{aligned}
\mathrm{Cov}\left[\mathbf{N}_t, \mathbf{N}_{t+h}\right] &= t\,\mathrm{E}\left[\Lambda\right]\,\mathrm{Diag}\left(\mathrm{E}\left[\boldsymbol{\Theta}\right]\right) \\
&\quad + t\,(t+h)\left(\mathrm{Var}\left[\Lambda\right]\,\mathrm{Var}\left[\boldsymbol{\Theta}\right] + \left(\mathrm{E}\left[\Lambda\right]\right)^2\,\mathrm{Var}\left[\boldsymbol{\Theta}\right] + \mathrm{Var}\left[\Lambda\right]\,\mathrm{E}\left[\boldsymbol{\Theta}\right]\,\mathrm{E}\left[\boldsymbol{\Theta}\right]'\right)
\end{aligned}$$

holds for all $t, h \in \mathbb{R}_+$.

Proof: This theorem immediately follows from Theorem 4.2.4 and Lemma 5.2.4, since the finiteness of the moment of order l of Λ is equivalent to the finiteness of all moments of order $\mathbf{l}$ with $\mathbf{1}'\mathbf{l} = l$ of $\boldsymbol{\Lambda}$ (compare Theorem 5.2.3 and 4.2.3). ∎

The above theorem shows that still a wide range of correlation structures are possible under the independence assumption. For example if Λ is degenerated, then the correlation structure assumed for $\boldsymbol{\Theta}$ carries over to the correlation structure of the multivariate mixed Poisson process.

5.3 Posterior Distributions

As in Section 4.3, we will study conditional distributions and the question of stability of the model over time. We start with the conditional distribution of Λ and $\boldsymbol{\Theta}$ with respect to the process at some time t.

5.3.1 Theorem. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\mathbf{\Lambda}$ which fulfils* **(I)***. Then the identities*

$$
\begin{aligned}
\mathrm{P}_{\Lambda \,|\, \mathbf{N}_t} &= \mathrm{P}_{\Lambda \,|\, N_t} \\
\mathrm{P}_{\mathbf{\Theta} \,|\, \mathbf{N}_t}(B) &= \frac{\int_B \boldsymbol{\theta}^{\mathbf{N}_t} \, d\mathrm{P}_{\mathbf{\Theta}}(\boldsymbol{\theta})}{\int_{\mathbb{R}^k} \boldsymbol{\theta}^{\mathbf{N}_t} \, d\mathrm{P}_{\mathbf{\Theta}}(\boldsymbol{\theta})}
\end{aligned}
$$

hold for all $t > 0$ and $B \in \mathcal{B}(\mathbb{R}^k)$.

Proof: The proof uses the same ideas as the proofs of Lemma 4.3.1 and Theorem 4.3.3. Therefore, it is a bit shortened.
Starting with the second identity, we consider $t > 0$ and $B \in \mathcal{B}(\mathbb{R}^k)$ as well as $\mathbf{n} \in \mathbb{N}_0^k$ and set $n := \mathbf{1}'\mathbf{n}$. Then we have

$$
\begin{aligned}
\mathrm{P}\left(\{\mathbf{\Theta} \in B\} \cap \{\mathbf{N}_t = \mathbf{n}\}\right) &= \mathrm{E}\left[\mathrm{E}\left(\chi_{\{\mathbf{\Theta} \in B\} \cap \{\mathbf{N}_t = \mathbf{n}\}} \middle| \mathbf{\Lambda}\right)\right] \\
&= \mathrm{E}\left[\chi_{\{\mathbf{\Theta} \in B\}} \, \mathrm{E}\left(\chi_{\{\mathbf{N}_t = \mathbf{n}\}} \middle| \mathbf{\Lambda}\right)\right] \\
&= \mathrm{E}\left[\chi_{\{\mathbf{\Theta} \in B\}} \; e^{-\mathbf{1}'\mathbf{\Lambda}\, t} \, \frac{(\mathbf{\Lambda}\, t)^{\mathbf{n}}}{\mathbf{n}!}\right] \\
&= \mathrm{E}\left[e^{-\Lambda\, t} \, \frac{(\Lambda\, t)^n}{n!}\right] \mathrm{E}\left[\chi_{\{\mathbf{\Theta} \in B\}} \binom{n}{\mathbf{n}} \mathbf{\Theta}^{\mathbf{n}}\right] \\
&= \int_\Omega e^{-\Lambda\, t} \, \frac{(\Lambda\, t)^n}{n!} \, d\mathrm{P} \int_{\mathbf{\Theta}^{-1}(B)} \binom{n}{\mathbf{n}} \mathbf{\Theta}^{\mathbf{n}} \, d\mathrm{P} \\
&= \int_{\mathbb{R}} e^{-\lambda\, t} \, \frac{(\lambda\, t)^n}{n!} \, d\mathrm{P}_\Lambda(\lambda) \int_B \binom{n}{\mathbf{n}} \boldsymbol{\theta}^{\mathbf{n}} \, d\mathrm{P}_{\mathbf{\Theta}}(\boldsymbol{\theta})
\end{aligned}
$$

and thus from Theorem 5.1.2

$$
\begin{aligned}
\mathrm{P}_{\mathbf{\Theta} \,|\, \mathbf{N}_t = \mathbf{n}}[B] &= P\left[\{\mathbf{\Theta} \in B\} \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right] \\
&= \frac{\mathrm{P}\left[\{\mathbf{\Theta} \in B\} \cap \{\mathbf{N}_t = \mathbf{n}\}\right]}{\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]} \\
&= \frac{\displaystyle\int_{\mathbb{R}} e^{-\lambda\, t} \, \frac{(\lambda\, t)^n}{n!} \, d\mathrm{P}_\Lambda(\lambda) \int_B \binom{n}{\mathbf{n}} \boldsymbol{\theta}^{\mathbf{n}} \, d\mathrm{P}_{\mathbf{\Theta}}(\boldsymbol{\theta})}{\displaystyle\int_{\mathbb{R}} e^{-\lambda\, t} \, \frac{(\lambda\, t)^n}{n!} \, d\mathrm{P}_\Lambda(\lambda) \int_{\mathbb{R}^k} \binom{n}{\mathbf{n}} \boldsymbol{\theta}^{\mathbf{n}} \, d\mathrm{P}_{\mathbf{\Theta}}(\boldsymbol{\theta})} \\
&= \frac{\int_B \boldsymbol{\theta}^{\mathbf{n}} \, d\mathrm{P}_{\mathbf{\Theta}}(\boldsymbol{\theta})}{\int_{\mathbb{R}^k} \boldsymbol{\theta}^{\mathbf{n}} \, d\mathrm{P}_{\mathbf{\Theta}}(\boldsymbol{\theta})}
\end{aligned}
$$

which proves the second identity.
Now, we turn to the first identity. Let $t > 0$ and $B \in \mathcal{B}(\mathbb{R})$ as well as $\mathbf{n} \in \mathbb{N}_0^k$ and

set $n := \mathbf{1}'\mathbf{n}$. This gives

$$\begin{aligned}
\mathrm{P}\left(\{\Lambda \in B\} \cap \{\mathbf{N}_t = \mathbf{n}\}\right) &= \mathrm{E}\left[\mathrm{E}\left(\chi_{\{\Lambda \in B\} \cap \{\mathbf{N}_t = \mathbf{n}\}} \middle| \boldsymbol{\Lambda}\right)\right] \\
&= \mathrm{E}\left[\chi_{\{\Lambda \in B\}}\; e^{-\mathbf{1}'\boldsymbol{\Lambda}\, t}\, \frac{(\boldsymbol{\Lambda}\, t)^{\mathbf{n}}}{\mathbf{n}!}\right] \\
&= \mathrm{E}\left[\chi_{\{\Lambda \in B\}}\; e^{-\Lambda\, t}\, \frac{(\Lambda\, t)^{n}}{n!}\right] \mathrm{E}\left[\binom{n}{\mathbf{n}} \boldsymbol{\Theta}^{\mathbf{n}}\right] \\
&= \int_{\Lambda^{-1}(B)} e^{-\Lambda\, t}\, \frac{(\Lambda\, t)^{n}}{n!}\, d\mathrm{P} \int_{\Omega} \binom{n}{\mathbf{n}} \boldsymbol{\Theta}^{\mathbf{n}}\, d\mathrm{P} \\
&= \int_{B} e^{-\lambda\, t}\, \frac{(\lambda\, t)^{n}}{n!}\, d\mathrm{P}_{\Lambda}(\lambda) \int_{\mathbb{R}^k} \binom{n}{\mathbf{n}} \boldsymbol{\theta}^{\mathbf{n}}\, d\mathrm{P}_{\boldsymbol{\Theta}}(\boldsymbol{\theta})
\end{aligned}$$

and thus using again Theorem 5.1.2

$$\begin{aligned}
\mathrm{P}_{\Lambda \,|\, \mathbf{N}_t = \mathbf{n}}[B] &= P\left[\{\Lambda \in B\} \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right] \\
&= \frac{\displaystyle\int_{B} e^{-\lambda t}\, \frac{(\lambda\, t)^{n}}{n!}\, d\mathrm{P}_{\Lambda}(\lambda) \int_{\mathbb{R}^k} \binom{n}{\mathbf{n}} \boldsymbol{\theta}^{\mathbf{n}}\, d\mathrm{P}_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}{\displaystyle\int_{\mathbb{R}} e^{-\lambda t}\, \frac{(\lambda\, t)^{n}}{n!}\, d\mathrm{P}_{\Lambda}(\lambda) \int_{\mathbb{R}^k} \binom{n}{\mathbf{n}} \boldsymbol{\theta}^{\mathbf{n}}\, d\mathrm{P}_{\boldsymbol{\Theta}}(\boldsymbol{\theta})} \\
&= \frac{\int_{B} e^{-\lambda t}\, \lambda^{n}\, d\mathrm{P}_{\Lambda}(\lambda)}{\int_{\mathbb{R}} e^{-\lambda t}\, \lambda^{n}\, d\mathrm{P}_{\Lambda}(\lambda)}
\end{aligned}$$

Hence

$$\mathrm{P}_{\Lambda \,|\, \mathbf{N}_t}(B) = \frac{\int_{B} e^{-\lambda t}\, \lambda^{N_t}\, d\mathrm{P}_{\Lambda}(\lambda)}{\int_{\mathbb{R}} e^{-\lambda t}\, \lambda^{N_t}\, d\mathrm{P}_{\Lambda}(\lambda)}$$

Since $\{N_t\}_{t \in \mathbb{R}_+}$ is a univariate mixed Poisson process with parameter Λ, it follows from Theorem 4.3.3 that $\mathrm{P}_{\Lambda \,|\, \mathbf{N}_t} = \mathrm{P}_{\Lambda \,|\, N_t}$. ■

We have pointed out, that the distribution which divides the events into the given classes (coordinates) and which was a mixed multinomial distribution with parameter $\boldsymbol{\Theta}$ was independent of time t. This property goes over to the conditional distribution of $\boldsymbol{\Theta}$ with respect to the process at some time t, as can be seen from the preceding theorem.

A natural question to ask is whether Λ and $\boldsymbol{\Theta}$ are conditionally independent with respect to the process at some time t. To answer this question we use the results of Section 1.2.

Theorem 5.3.2 *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$. Then the following are equivalent.*

(a) Λ *and* $\boldsymbol{\Theta}$ *are independent.*
(b) Λ *and* $\boldsymbol{\Theta}$ *are conditionally independent with respect to* $\mathbf{N}_t$ *for some* $t > 0$.
(c) Λ *and* $\boldsymbol{\Theta}$ *are conditionally independent with respect to* $\mathbf{N}_t$ *for all* $t \in \mathbb{R}_+$.

Proof: We prove the assertion according to the following scheme: (a) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a).

(a) $\Rightarrow$ (c): Consider $t > 0$. By the representation of conditional expectation in terms of integrals with respect to the conditional distribution for random vectors (see Hess [1997] 4.2.1) and the conditional distributions calculated so far (see Theorems 4.3.3 and 5.3.1), we obtain for arbitrary $h > 0$ and for all $n \in \mathbb{N}_0$ and $\mathbf{l} \in \mathbb{N}_0^k$

$$
\begin{aligned}
\mathrm{E}\left(e^{-\Lambda h}\,\Lambda^n\,\boldsymbol{\Theta}^{\mathbf{l}}\middle|\mathbf{N}_t\right)
&= \int_{\mathbb{R}^k} e^{-\lambda h}\,\lambda^n\,\boldsymbol{\theta}^{\mathbf{l}}\,d\mathrm{P}_{\boldsymbol{\Lambda}\,|\,\mathbf{N}_t}(\boldsymbol{\lambda}) \\
&= \frac{\int_{\mathbb{R}^k} e^{-\lambda h}\,\lambda^n\,\boldsymbol{\theta}^{\mathbf{l}}\,e^{-\lambda t}\,\boldsymbol{\lambda}^{\mathbf{N}_t}\,d\mathrm{P}_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\lambda t}\,\boldsymbol{\lambda}^{\mathbf{N}_t}\,d\mathrm{P}_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda})} \\
&= \sum_{\mathbf{m} \in \mathbb{N}_0^k} \frac{\mathrm{E}\left[e^{-\Lambda(t+h)}\,\Lambda^{(n+\mathbf{1}'\mathbf{m})}\,\boldsymbol{\Theta}^{\mathbf{l}+\mathbf{m}}\right]}{\mathrm{E}\left[e^{-\Lambda t}\,\Lambda^{\mathbf{1}'\mathbf{m}}\,\boldsymbol{\Theta}^{\mathbf{m}}\right]}\,\chi_{\{\mathbf{N}_t = \mathbf{m}\}} \\
&= \sum_{\mathbf{m} \in \mathbb{N}_0^k} \frac{\mathrm{E}\left[e^{-\Lambda(t+h)}\,\Lambda^{(n+\mathbf{1}'\mathbf{m})}\right]\,\mathrm{E}\left[\boldsymbol{\Theta}^{\mathbf{l}+\mathbf{m}}\right]}{\mathrm{E}\left[e^{-\Lambda t}\,\Lambda^{\mathbf{1}'\mathbf{m}}\right]\,\mathrm{E}\left[\boldsymbol{\Theta}^{\mathbf{m}}\right]}\,\chi_{\{\mathbf{N}_t = \mathbf{m}\}} \\
&= \frac{\int_{\mathbb{R}} e^{-\lambda(t+h)}\,\lambda^{(n+N_t)}\,d\mathrm{P}_{\Lambda}(\lambda)\,\int_{\mathbb{R}^k} \boldsymbol{\theta}^{\mathbf{l}+\mathbf{N}_t}\,d\mathrm{P}_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}{\int_{\mathbb{R}} e^{-\lambda t}\,\lambda^{N_t}\,d\mathrm{P}_{\Lambda}(\lambda)\,\int_{\mathbb{R}^k} \boldsymbol{\theta}^{\mathbf{N}_t}\,d\mathrm{P}_{\boldsymbol{\Theta}}(\boldsymbol{\theta})} \\
&= \mathrm{E}\left(e^{-\Lambda h}\,\Lambda^n\middle|\mathbf{N}_t\right)\,\mathrm{E}\left(\boldsymbol{\Theta}^{\mathbf{l}}\middle|\mathbf{N}_t\right)
\end{aligned}
$$

Now, Corollary 1.2.6 yields the conditionally independence of Λ and $\boldsymbol{\Theta}$ with respect to $\mathbf{N}_t$ for arbitrary $t > 0$.
Consider $t = 0$. Since $\sigma(\mathbf{N_0})$ contains only sets which have either probability mass zero or one, the conditional independence of Λ and $\boldsymbol{\Theta}$ with respect to $\mathbf{N}_0$ is due to the independence of Λ and $\boldsymbol{\Theta}$. Thus, the assertion is shown.

(c) $\Rightarrow$ (b): obvious

(b) $\Rightarrow$ (a): Using again Hess [1997] 4.2.1 and Theorems 4.3.3 and 5.3.1 we get for all $n \in \mathbb{N}_0$ and $\mathbf{l} \in \mathbb{N}_0^k$

$$
\frac{\int_{\mathbb{R}^k} \lambda^n\,\boldsymbol{\theta}^{\mathbf{l}}\,e^{-\lambda t}\,\boldsymbol{\lambda}^{\mathbf{N}_t}\,d\mathrm{P}_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda})}{\int_{\mathbb{R}^k} e^{-\lambda t}\,\boldsymbol{\lambda}^{\mathbf{N}_t}\,d\mathrm{P}_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda})} = \mathrm{E}\left(\Lambda^n\,\boldsymbol{\Theta}^{\mathbf{l}}\middle|\mathbf{N}_t\right)
$$

$$
\begin{aligned}
&= \mathrm{E}\left(\Lambda^n \middle| \mathbf{N}_t\right)\ \mathrm{E}\left(\boldsymbol{\Theta}^{\mathbf{l}} \middle| \mathbf{N}_t\right) \\
&= \frac{\int_{\mathbb{R}} e^{-\lambda t}\, \lambda^{(n+N_t)}\, d\mathrm{P}_\Lambda(\lambda)\ \int_{\mathbb{R}^k} \boldsymbol{\theta}^{\mathbf{l}+\mathbf{N}_t}\, d\mathrm{P}_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}{\int_{\mathbb{R}} e^{-\lambda t}\, \lambda^{N_t}\, d\mathrm{P}_\Lambda(\lambda)\ \int_{\mathbb{R}^k} \boldsymbol{\theta}^{\mathbf{N}_t}\, d\mathrm{P}_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}
\end{aligned}
$$

Considering the event $\{\mathbf{N}_t = \mathbf{0}\}$ which has positive probability we obtain

$$
\begin{aligned}
\mathrm{E}\left[e^{-\Lambda t}\, \Lambda^n\, \boldsymbol{\Theta}^{\mathbf{l}}\right] &= \int_{\mathbb{R}^k} e^{-\lambda t}\, \lambda^n\, \boldsymbol{\theta}^{\mathbf{l}}\, d\mathrm{P}_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}) \\
&= \int_{\mathbb{R}} e^{-\lambda t}\, \lambda^n\, d\mathrm{P}_\Lambda(\lambda) \int_{\mathbb{R}^k} \boldsymbol{\theta}^{\mathbf{l}}\, d\mathrm{P}_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \\
&= \mathrm{E}\left[e^{-\Lambda t}\, \Lambda^n\right]\ \mathrm{E}\left[\boldsymbol{\Theta}^{\mathbf{l}}\right]
\end{aligned}
$$

and thus from Theorem 1.2.3 the assertion follows. ∎

Using the previous theorem and Theorem 4.3.2, the next result is rather obvious.

5.3.3 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$ which fulfils* **(I)***. Then for all $t > 0$ and all $\mathbf{n} \in \mathbb{N}_0^k$ the process $\{\mathbf{K}_{t,h}\}_{h\in\mathbb{R}_+}$ is a multivariate mixed Poisson process on the probability space $(\Omega, \mathcal{F}, \mathrm{P}_{t,\mathbf{n}})$ with parameter $\boldsymbol{\Lambda}$ which fulfils* **(I)***.*

This means, to accept the model treated in this chapter it is not crucial to know when the process started, since the model for the incremental process remains unchanged for different starting times. The change just affects the underlying probability distributions.

5.4 Regularity

After decomposing probabilities and moments of a multivariate mixed Poisson process with parameter which fulfils **(I)** we turn to transition probabilities and intensities. With the possible decomposition we will regain some properties of the intensities, which are valid in the univariate case but not have been valid in the general multivariate setting. But first, let us turn to the transition probabilities.

5.4.1 Lemma. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$ which fulfils* **(I)***. Then*

$$
p_{\mathbf{n},\mathbf{m}}(r,t) \;=\; {}_{\mathbf{1}'}p_{n,m}(r,t)\ \mathrm{E}\left[\binom{m-n}{\mathbf{m}-\mathbf{n}} \boldsymbol{\Theta}^{\mathbf{m}-\mathbf{n}} \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right]
$$

holds for all $(\mathbf{n}, r) \in Z$ and $\mathbf{m} \in \mathbb{N}_0^k$, $t \in \mathbb{R}_+$ with $\mathbf{n} \le \mathbf{m}$ and $r \le t$ where $n := \mathbf{1}'\mathbf{n}$ and $m := \mathbf{1}'\mathbf{m}$.

Proof: Consider $(\mathbf{n}, r) \in Z$ and $\mathbf{m} \in \mathbb{N}_0^k$, $t \in \mathbb{R}_+$ with $\mathbf{n} \leq \mathbf{m}$ and $r \leq t$. Using first Lemma 4.4.1 and then Theorem 5.3.3 and Theorem 5.3.1 in connection with the same formula manipulation as in the proof of the decomposition of the unconditional probabilities (Theorem 5.1.2) we get

$$\begin{aligned}
p_{\mathbf{n},\mathbf{m}}(r,t) &= \mathrm{E}\left[e^{-\mathbf{1}'\boldsymbol{\Lambda}\,(t-r)} \, \frac{(\boldsymbol{\Lambda}\,(t-r))^{\mathbf{m}-\mathbf{n}}}{(\mathbf{m}-\mathbf{n})!} \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\} \right] \\
&= \mathrm{E}\left[e^{-\Lambda\,(t-r)} \, \frac{(\Lambda\,(t-r))^{m-n}}{(m-n)!} \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\} \right] \, \mathrm{E}\left[\binom{m-n}{\mathbf{m}-\mathbf{n}} \boldsymbol{\Theta}^{\mathbf{m}-\mathbf{n}} \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\} \right] \\
&= \mathrm{E}\left[e^{-\Lambda\,(t-r)} \, \frac{(\Lambda\,(t-r))^{m-n}}{(m-n)!} \,\middle|\, \{N_t = n\} \right] \, \mathrm{E}\left[\binom{m-n}{\mathbf{m}-\mathbf{n}} \boldsymbol{\Theta}^{\mathbf{m}-\mathbf{n}} \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\} \right] \\
&= {}_{\mathbf{1}'}p_{n,m}(r,t) \; \mathrm{E}\left[\binom{m-n}{\mathbf{m}-\mathbf{n}} \boldsymbol{\Theta}^{\mathbf{m}-\mathbf{n}} \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\} \right]
\end{aligned}$$

since $\{N_t\}_{t \in \mathbb{R}_+}$ is a mixed Poisson process with parameter Λ (see Lemma 4.1.4). ■

5.4.2 Theorem. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$ which fulfils* **(I)**. *Then the following are equivalent.*

(a) $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ *is regular with intensities* $\{\boldsymbol{\kappa}_\mathbf{n}\}_{\mathbf{n} \in \mathbb{N}_0^k}$.
(b) *The condition* $\mathrm{E}[\Lambda] < \infty$ *is valid.*

If $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ satisfies one and hence all preceding items, then

$$\kappa_\mathbf{n}^{(i)}(t) \;=\; {}_{\mathbf{1}'}\kappa_n(t) \; \mathrm{E}\left[\Theta_i \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right]$$

holds for all $(\mathbf{n}, t) \in Z$ and $i \in \{1, \ldots, k\}$ where $n := \mathbf{1}'\mathbf{n}$.

Proof: The equivalence of (a) and (b) follows immediately from Theorem 4.4.2. For $(\mathbf{n}, t) \in Z$ and $i \in \{1, \ldots, k\}$ we additionally get from Theorem 5.3.2 and Theorem 5.3.1

$$\begin{aligned}
\kappa_\mathbf{n}^{(i)}(t) &= \mathrm{E}\left[\Lambda_i \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right] \\
&= \mathrm{E}\left[\Lambda \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right] \, \mathrm{E}\left[\Theta_i \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right] \\
&= \mathrm{E}\left[\Lambda \,\middle|\, \{N_t = n\}\right] \, \mathrm{E}\left[\Theta_i \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right] \\
&= {}_{\mathbf{1}'}\kappa_n(t) \; \mathrm{E}\left[\Theta_i \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right]
\end{aligned}$$

since $\{N_t\}_{t \in \mathbb{R}_+}$ is a mixed Poisson process with parameter Λ (see Lemma 4.1.4). ■

Thus, the intensities are driven by intensities generated from a univariate mixed Poisson process. This leads to the following theorem.

5.4.3 Theorem. *Let $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ be a regular multivariate mixed Poisson process with parameter $\boldsymbol{\Lambda}$ which fulfils* **(I)**. *Then the intensities $\{\boldsymbol{\kappa}_\mathbf{n}\}_{\mathbf{n} \in \mathbb{N}_0^k}$ have the following properties.*

(1) *The identity*

$$\kappa_{\mathbf{n}}^{(i)}(t) \;=\; \frac{D^{n+1} M_\Lambda(-t)}{D^n M_\Lambda(-t)}\, \mathrm{E}\left[\Theta_i \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right]$$

holds for all $t > 0$, $\mathbf{n} \in \mathbb{N}_0^k$, and $i \in \{1, \dots, k\}$ where $n := \mathbf{1}'\mathbf{n}$.

(2) *For all $\mathbf{n} \in \mathbb{N}_0^k$ and all $i \in \{1, \dots, k\}$ the intensity $\kappa_{\mathbf{n}}^{(i)}$ is decreasing.*

(3) *Let $\mathbf{n} \in \mathbb{N}_0^k$, $n := \mathbf{1}'\mathbf{n}$, $i \in \{1, \dots, k\}$, and let $\mathrm{E}[\Lambda^{n+1}]$ be finite. Then $\lim_{t\downarrow 0} \kappa_{\mathbf{n}}^{(i)}(t)$ is finite and*

$$\lim_{t\downarrow 0} \kappa_{\mathbf{n}}^{(i)}(t) \;=\; \frac{\mathrm{E}[\Lambda^{n+1}]}{\mathrm{E}[\Lambda^n]}\, \mathrm{E}\left[\Theta_i \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right]$$

is valid.

Proof:
(1): By the fact that $\{N_t\}_{t\in\mathbb{R}_+}$ is a mixed Poisson process with parameter Λ, the assertion follows from Theorem 4.4.4 and Theorem 5.4.2.

(2): As the conditional expected value $\mathrm{E}[\Theta_i \,|\, \{\mathbf{N}_t = \mathbf{n}\}]$ does not dependent on time t and the intensities of a univariate mixed Poisson process are decreasing (see Grandell [1997]), the assertion follows from Theorem 5.4.2.

(3): Since $\mathrm{E}[\Theta_i \,|\, \{\mathbf{N}_t = \mathbf{n}\}]$ is finite and independent of t for all $\mathbf{n} \in \mathbb{N}_0^k$ and $i \in \{1, \dots, k\}$, the assertion follows from 4.4.4 (2). ■

The next theorem contains a list of equivalent properties of multivariate mixed Poisson processes with parameter. These properties are fulfilled if the process possesses **(I)**. Thus, they are necessary conditions for the validity of **(I)** and can be used for rejecting the hypothesis 'The process fulfils **(I)**'.

5.4.4 Theorem. *Let $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ be a regular multivariate mixed Poisson process with parameter $\mathbf{\Lambda}$. Then the following are equivalent.*

(a) *For all $i, j \in \{1, \dots, k\}$ and $\mathbf{n} \in \mathbb{N}_0^k$ there exists a constant $c_{\mathbf{n}}^{(i,j)} \in \mathbb{R}_+$ such that*

$$\kappa_{\mathbf{n}}^{(i)}(t) \;=\; c_{\mathbf{n}}^{(i,j)}\, \kappa_{\mathbf{n}}^{(j)}(t)$$

holds for all $t > 0$.

(b) *The identity*

$$\sum_{i=1}^{k} \kappa_{\mathbf{n}}^{(i)}(t) \;=\; {}_{\mathbf{1}'}\kappa_n(t)$$

holds for all $t > 0$ and $\mathbf{n} \in \mathbb{N}_0^k$ where $n := \mathbf{1}'\mathbf{n}$.

(c) *The identity*

$$\mathrm{E}\left[\Lambda \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right] \;=\; \mathrm{E}\left[\Lambda \,\middle|\, \{N_t = n\}\right]$$

holds for all $t > 0$ *and* $\mathbf{n} \in \mathbb{N}_0^k$ *where* $n := \mathbf{1}'\mathbf{n}$.

(d) *The identity*

$$\mathrm{P}_{\Lambda\,|\{\mathbf{N}_t=\mathbf{n}\}} \;=\; \mathrm{P}_{\Lambda\,|\{N_t=n\}}$$

holds for all $t > 0$ *and* $\mathbf{n} \in \mathbb{N}_0^k$ *where* $n := \mathbf{1}'\mathbf{n}$.

(e) *The identity*

$$p_{\mathbf{n},\mathbf{n}}(t, t+h) \;=\; {}_{\mathbf{1}'}p_{n,n}(t, t+h)$$

holds for all $t > 0$, $\mathbf{n} \in \mathbb{N}_0^k$ *where* $n := \mathbf{1}'\mathbf{n}$ *and all* $h \in \mathbb{R}_+$.

(f) *The identity*

$$\mathrm{P}\left[\{\mathbf{N}_{t+h} = \mathbf{n}\} \,\middle|\, \{N_{t+h} = n\}\right] \;=\; \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\} \,\middle|\, \{N_t = n\}\right]$$

holds for all $t > 0$, $\mathbf{n} \in \mathbb{N}_0^k$ *where* $n := \mathbf{1}'\mathbf{n}$ *and all* $h \in \mathbb{R}_+$.

(g) *The identity*

$$\begin{aligned}
&\mathrm{P}\left[\{\mathbf{N}_{t+h} = \mathbf{n}\} \cap \{\mathbf{N}_t = \mathbf{n}\}\right] \\
&\qquad = \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\} \,\middle|\, \{N_t = n\}\right] \mathrm{P}\left[\{N_{t+h} = n\} \cap \{N_t = n\}\right]
\end{aligned}$$

holds for all $t > 0$, $\mathbf{n} \in \mathbb{N}_0^k$ *where* $n := \mathbf{1}'\mathbf{n}$ *and all* $h \in \mathbb{R}_+$.

If $\{\mathbf{N}_t\}_{t\in\mathbb{R}_+}$ *fulfils* **(I)**, *then the process possesses property* (a) - (g).

Proof: We prove the assertion according to the following scheme: (a) $\Leftrightarrow$ (b), (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (b), (e) $\Leftrightarrow$ (f), and (e) $\Leftrightarrow$ (g).

(a) $\Rightarrow$ (b): Consider $\mathbf{n} \in \mathbb{N}_0^k$, $t > 0$, and $i, j \in \{1, \ldots, k\}$. The assumption immediately yields

$$\frac{\frac{d}{dt}\kappa_{\mathbf{n}}^{(i)}(t)}{\kappa_{\mathbf{n}}^{(i)}(t)} \;=\; \frac{\frac{d}{dt}\kappa_{\mathbf{n}}^{(j)}(t)}{\kappa_{\mathbf{n}}^{(j)}(t)}$$

Now, using the differential equation for the intensities (see Theorem 2.3.19) we obtain

$$\begin{aligned}
\sum_{l=1}^{k} \kappa_{\mathbf{n}+\mathbf{e}_i}^{(l)}(t) &= -\frac{\frac{d}{dt}\kappa_{\mathbf{n}}^{(i)}(t)}{\kappa_{\mathbf{n}}^{(i)}(t)} + \sum_{l=1}^{k} \kappa_{\mathbf{n}}^{(l)}(t) \\
&= -\frac{\frac{d}{dt}\kappa_{\mathbf{n}}^{(j)}(t)}{\kappa_{\mathbf{n}}^{(j)}(t)} + \sum_{l=1}^{k} \kappa_{\mathbf{n}}^{(l)}(t) \\
&= \sum_{l=1}^{k} \kappa_{\mathbf{n}+\mathbf{e}_j}^{(l)}(t)
\end{aligned}$$

Since $\mathbf{n}$ was arbitrary we obtain for all $\mathbf{n}$, $\mathbf{m} \in \mathbb{N}_0^k$ with $\mathbf{1}'\mathbf{n} = \mathbf{1}'\mathbf{m}$

$$\sum_{i=1}^{k} \kappa_{\mathbf{n}}^{(i)}(t) \;=\; \sum_{i=1}^{k} \kappa_{\mathbf{m}}^{(i)}(t)$$

and thus Theorem 2.3.14 yields with $A = \mathbf{1}' \in \mathcal{A}_C$

$$\begin{aligned}
\sum_{i=1}^{k} \kappa_{\mathbf{n}}^{(i)}(t) &= \frac{\sum\limits_{\substack{\mathbf{m}\in\mathbb{N}_0^k \\ \mathbf{1}'\mathbf{m}=n}} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{m}\}\right]}{\sum\limits_{\substack{\mathbf{v}\in\mathbb{N}_0^k \\ \mathbf{1}'\mathbf{v}=n}} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{v}\}\right]} \sum_{i=1}^{k} \kappa_{\mathbf{n}}^{(i)}(t) \\
&= \sum_{\substack{\mathbf{m}\in\mathbb{N}_0^k \\ \mathbf{1}'\mathbf{m}=n}} \frac{\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{m}\}\right]}{\sum\limits_{\substack{\mathbf{v}\in\mathbb{N}_0^k \\ \mathbf{1}'\mathbf{v}=n}} \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{v}\}\right]} \sum_{i=1}^{k} \kappa_{\mathbf{m}}^{(i)}(t) \\
&= {}_{\mathbf{1}'}\kappa_n(t)
\end{aligned}$$

(b) $\Rightarrow$ (a): Consider $\mathbf{n} \in \mathbb{N}_0^k$ and $i, j \in \{1, \ldots, k\}$. Using again Theorem 2.3.19 we get

$$\begin{aligned}
\frac{\frac{d}{dt}\kappa_{\mathbf{n}}^{(i)}(t)}{\kappa_{\mathbf{n}}^{(i)}(t)} &= \sum_{j=1}^{k} \Big(\kappa_{\mathbf{n}}^{(j)}(t) - \kappa_{\mathbf{n}+\mathbf{e}_i}^{(j)}(t)\Big) \\
&= {}_{\mathbf{1}'}\kappa_n(t) - {}_{\mathbf{1}'}\kappa_{n+1}(t)
\end{aligned}$$

for all $t > 0$. Since the right hand side does not depend on i, the identity

$$\frac{\frac{d}{dt}\kappa_{\mathbf{n}}^{(i)}(t)}{\kappa_{\mathbf{n}}^{(i)}(t)} \;=\; \frac{\frac{d}{dt}\kappa_{\mathbf{n}}^{(j)}(t)}{\kappa_{\mathbf{n}}^{(j)}(t)}$$

holds for all $t > 0$. This means, that the logarithmic derivatives and thus we get the existence of a constant $c_{\mathbf{n}}^{(i,j)} \in \mathbb{R}_+$ such that

$$\kappa_{\mathbf{n}}^{(i)}(t) \;=\; c_{\mathbf{n}}^{(i,j)}\, \kappa_{\mathbf{n}}^{(j)}(t)$$

holds for all $t > 0$.

(b) $\Rightarrow$ (c): As a consequence of the representation of the intensities (Theorem 4.4.2), we get for all $\mathbf{n} \in \mathbb{N}_0^k$ and all $t > 0$

$$\begin{aligned}
\mathrm{E}\left[\Lambda \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right] &= \sum_{i=1}^{k} \mathrm{E}\left[\Lambda_i \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right] \\
&= \sum_{i=1}^{k} \kappa_{\mathbf{n}}^{(i)}(t) \\
&= {}_{\mathbf{1}'}\kappa_n(t) \\
&= \mathrm{E}\left[\Lambda \,\middle|\, \{N_t = n\}\right]
\end{aligned}$$

(c) $\Rightarrow$ (d): Consider $\mathbf{n} \in \mathbb{N}_0^k$ and $t > 0$. By the representation of the conditional distribution of the parameter with respect to the mixed Poisson process (see Theorems 4.3.3), we obtain for all $h > 0$

$$\begin{aligned}
\frac{D\, M_{\mathrm{P}_{\Lambda|\{\mathbf{N}_t = \mathbf{n}\}}}(-h)}{M_{\mathrm{P}_{\Lambda|\{\mathbf{N}_t = \mathbf{n}\}}}(-h)} &= \frac{\int_{\mathbb{R}} \lambda\, e^{-\lambda h}\, d\mathrm{P}_{\Lambda|\{\mathbf{N}_t = \mathbf{n}\}}(\lambda)}{\int_{\mathbb{R}} e^{-\lambda h}\, d\mathrm{P}_{\Lambda|\{\mathbf{N}_t = \mathbf{n}\}}(\lambda)} \\
&= \frac{\mathrm{E}\left[\Lambda\, e^{-\Lambda(t+h)}\, \mathbf{\Lambda}^{\mathbf{n}}\right]}{\mathrm{E}\left[e^{-\Lambda t}\, \mathbf{\Lambda}^{\mathbf{n}}\right]}\, \frac{\mathrm{E}\left[e^{-\Lambda t}\, \mathbf{\Lambda}^{\mathbf{n}}\right]}{\mathrm{E}\left[e^{-\Lambda(t+h)}\, \mathbf{\Lambda}^{\mathbf{n}}\right]} \\
&= \mathrm{E}\left[\Lambda \middle| \{\mathbf{N}_{t+h} = \mathbf{n}\}\right] \\
&= \mathrm{E}\left[\Lambda \middle| \{N_{t+h} = n\}\right] \\
&= \frac{\int_{\mathbb{R}} \lambda\, e^{-\lambda h}\, d\mathrm{P}_{\Lambda|\{N_t = n\}}(\lambda)}{\int_{\mathbb{R}} e^{-\lambda h}\, d\mathrm{P}_{\Lambda|\{N_t = n\}}(\lambda)} \\
&= \frac{D\, M_{\mathrm{P}_{\Lambda|\{N_t = n\}}}(-h)}{M_{\mathrm{P}_{\Lambda|\{N_t = n\}}}(-h)}
\end{aligned}$$

Since the moment generating function is positive, the equivalence of the logarithmic derivatives yields the existence of a constant $c \in \mathbb{R}_+$ such that

$$M_{\mathrm{P}_{\Lambda|\{\mathbf{N}_t = \mathbf{n}\}}}(-h) \;=\; c\, M_{\mathrm{P}_{\Lambda|\{N_t = n\}}}(-h)$$

holds for all $h > 0$. Furthermore the continuity of the moment generating functions on $(-\infty, 0]$ and

$$M_{\mathrm{P}_{\Lambda|\{\mathbf{N}_t = \mathbf{n}\}}}(0) \;=\; 1 \;=\; M_{\mathrm{P}_{\Lambda|\{N_t = n\}}}(0)$$

yield $c = 1$. Now the identity of the moment generating functions for all $h \leq 0$ gives the equivalence of the distributions. (see Billingsley [1995] Theorem 22.2).

(d) $\Rightarrow$ (e): The representation of the transition probabilities of a mixed Poisson process immediately yields

$$\begin{aligned}
p_{\mathbf{n},\mathbf{n}}(t, t+h) &= \mathrm{E}\left[e^{-\Lambda h} \middle| \{\mathbf{N}_t = \mathbf{n}\}\right] \\
&= \mathrm{E}\left[e^{-\Lambda h} \middle| \{N_t = n\}\right] \\
&= {}_{\mathbf{1}'}p_{n,n}(t, t+h)
\end{aligned}$$

for all $\mathbf{n} \in \mathbb{N}_0^k$, $t > 0$ and all $h \in \mathbb{R}_+$.

(e) $\Rightarrow$ (b): The regularity of the processes $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ and $\{N_t\}_{t \in \mathbb{R}_+}$ gives

$$\begin{aligned}
\sum_{i=1}^k \kappa_{\mathbf{n}}^{(i)}(t) &= \lim_{h \downarrow 0} \frac{1}{h}\big(1 - p_{\mathbf{n},\mathbf{n}}(t, t+h)\big) \\
&= \lim_{h \downarrow 0} \frac{1}{h}\big(1 - {}_{\mathbf{1}'}p_{n,n}(t, t+h)\big) \\
&= {}_{\mathbf{1}'}\kappa_n(t)
\end{aligned}$$

for all $\mathbf{n} \in \mathbb{N}_0^k$ and all $t > 0$.

(e) $\Leftrightarrow$ (f) : Using the binomial property we have for all $\mathbf{n} \in \mathbb{N}_0^k$, $t > 0$ and all $h \in \mathbb{R}_+$

$$\begin{aligned} p_{\mathbf{n},\mathbf{n}}(t, t+h) &= \left(\frac{t}{t+h}\right)^n \frac{\mathrm{P}[\{\mathbf{N}_{t+h} = \mathbf{n}\}]}{\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}]} \\ &= \left(\frac{t}{t+h}\right)^n \frac{\mathrm{P}[\{N_{t+h} = n\}]}{\mathrm{P}[\{N_t = n\}]} \frac{\mathrm{P}\left[\{\mathbf{N}_{t+h} = \mathbf{n}\} \,\middle|\, \{N_{t+h} = n\}\right]}{\mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\} \,\middle|\, \{N_t = n\}\right]} \end{aligned}$$

as well as

$$_{\mathbf{1}'}p_{n,n}(t, t+h) = \left(\frac{t}{t+h}\right)^n \frac{\mathrm{P}[\{N_{t+h} = n\}]}{\mathrm{P}[\{N_t = n\}]}$$

Since $\mathrm{P}[\{N_s = n\}] > 0$ for all $s > 0$ the assertion follows.

(e) $\Leftrightarrow$ (g): Consider $\mathbf{n} \in \mathbb{N}_0^k$, $t > 0$, and $h \in \mathbb{R}_+$. Multiplying

$$_{\mathbf{1}'}p_{n,n}(t, t+h)\, \mathrm{P}[\{N_t = n\}] = \mathrm{P}[\{N_{t+h} = n\} \cap \{N_t = n\}]$$

on both sides with $\mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}]\; (\mathrm{P}[\{N_t = n\}])^{-1}$ yields

$$\begin{aligned} &{}_{\mathbf{1}'}p_{n,n}(t, t+h)\, \mathrm{P}[\{\mathbf{N}_t = \mathbf{n}\}] \\ &\quad = \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\} \,\middle|\, \{N_t = n\}\right] \mathrm{P}[\{N_{t+h} = n\} \cap \{N_t = n\}] \end{aligned}$$

where we can see that $p_{\mathbf{n},\mathbf{n}}(t, t+h) = {}_{\mathbf{1}'}p_{n,n}(t, t+h)$ is a necessary and sufficient condition for

$$\begin{aligned} &\mathrm{P}[\{\mathbf{N}_{t+h} = \mathbf{n}\} \cap \{\mathbf{N}_t = \mathbf{n}\}] \\ &\quad = \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\} \,\middle|\, \{N_t = n\}\right] \mathrm{P}[\{N_{t+h} = n\} \cap \{N_t = n\}] \end{aligned}$$

to be valid.

From Theorem 5.3.1 we see that a multivariate mixed Poisson process with parameter which fulfils **(I)** satisfies (d) and hence the proof is completed. ■

The above theorem contains in fact equivalences between conditional expected values of Λ with respect to events of the multivariate mixed Poisson process $\{\mathbf{N}_t\}_{t \in \mathbb{R}_+}$ and conditional expected values of Λ with respect to events of the transformed process $\{N_t\}_{t \in \mathbb{R}_+}$. The crucial point thereto is step (c) $\Rightarrow$ (d). Since the conditional expected values are identical for all $t > 0$ the equality can be carried over to the distribution. The effect can be illustrated on the basis of the equivalence of (b) and (e). While the first identity deals with the equality of limits of the transition probabilities the second one shows that even the corresponding transition probabilities are identical.

The following lines will contain an example which shows that the conditions in Theorem 5.4.4 are not necessarily fulfilled by arbitrary multivariate mixed Poisson processes.

Example: We consider the bivariate case. Then the equation in 5.4.4 (e) is equivalent to

$$\frac{\mathrm{E}\left[e^{-\Lambda(t+h)}\,(\Lambda_1)^{n^{(1)}}\,(\Lambda_2)^{n^{(2)}}\right]}{\mathrm{E}\left[e^{-\Lambda t}\,(\Lambda_1)^{n^{(1)}}\,(\Lambda_2)^{n^{(2)}}\right]} \;=\; \frac{\mathrm{E}\left[e^{-\Lambda(t+h)}\,\Lambda^n\right]}{\mathrm{E}\left[e^{-\Lambda t}\,\Lambda^n\right]}$$

Now, Consider $\Lambda_1 = 0.5\cdot(\sqrt{(\Lambda_2)^2+8}-\Lambda_2)$ and $\Lambda_2 \in (0,\infty)$, which implies that $\Lambda_1+\Lambda_2 = \Lambda > \sqrt{2}$ as well as that Λ_1 is the positive solution of $\Lambda_1 = 2/(\Lambda_1+\Lambda_2)$. Furthermore, assume $n^{(1)} = n$, $n^{(2)} = 0$. Multiplying both side with the denominators we obtain

$$E\left[e^{-\Lambda t}\,\Lambda^n\right]\;\mathrm{E}\left[e^{-\Lambda(t+h)}\,2^n\,\Lambda^{-n}\right] \;=\; \mathrm{E}\left[e^{-\Lambda(t+h)}\,\Lambda^n\right]\;\mathrm{E}\left[e^{-\Lambda t}\,2^n\,\Lambda^{-n}\right]$$

Adding the further assumption that Λ has probability mass p_1 at l_1 and $1-p_1$ at l_2 with $l_1, l_2 > \sqrt{2}$, $l_1 \neq l_2$, and $0 < p_1 < 1$ the previous equation transforms into

$$\begin{aligned}
&\left(e^{-l_1 t}\,l_1^n\,p_1 + e^{-l_2 t}\,l_2^n\,(1-p_1)\right)\left(e^{-l_1(t+h)}\,l_1^{-n}\,p_1 + e^{-l_2(t+h)}\,l_2^{-n}\,(1-p_1)\right)\\
&\quad = \;\left(e^{-l_1(t+h)}\,l_1^n\,p_1 + e^{-l_2(t+h)}\,l_2^n\,(1-p_1)\right)\left(e^{-l_1 t}\,l_1^{-n}\,p_1 + e^{-l_2 t}\,l_2^{-n}\,(1-p_1)\right)
\end{aligned}$$

Thus

$$\begin{aligned}
0 \;&=\; e^{-l_1 t - l_2 t - l_2 h}\,l_1^n\,l_2^{-n}\,p_1\,(1-p_1) \;+\; e^{-l_1 t - l_2 t - l_1 h}\,l_1^{-n}\,l_2^n\,p_1\,(1-p_1)\\
&\quad - \left(e^{-l_1 t - l_2 t - l_1 h}\,l_1^n\,l_2^{-n}\,p_1\,(1-p_1) \;+\; e^{-l_1 t - l_2 t - l_2 h}\,l_1^{-n}\,l_2^n\,p_1\,(1-p_1)\right)\\
&=\; p_1\,(1-p_1)\,\left(e^{-l_1 t - l_2 t - l_2 h} - e^{-l_1 t - l_2 t - l_1 h}\right)\left(l_1^n\,l_2^{-n} - l_1^{-n}\,l_2^n\right)
\end{aligned}$$

This equation can only be satisfied for all $n \in \mathbb{N}$ and all $h > 0$ if $l_1 = l_2$ which is a contradiction to the assumption of a two point distribution. Therefore 5.4.4 (e) is not fulfilled by all multivariate mixed Poisson processes. □

List of Symbols

$\mathbb{N}$	the set $\{1, 2, \dots\}$
$\mathbb{N}_0$	the set $\{0, 1, 2, \dots\}$
$\mathbb{N}_0^k$	k–fold cartesian product of $\mathbb{N}_0$
$\mathbb{Z}$	the integers
$\mathbb{Z}^k$	k–fold cartesian product of $\mathbb{Z}$
$\mathbb{R}$	the real numbers
$\mathbb{R}_+$	the real numbers in the interval $[0, \infty)$
$\mathbb{R}^k$	k–dimensional Euclidian space
$\mathbb{R}_+^k$	the elements of $[\mathbf{0}, \boldsymbol{\infty})$
$\mathcal{B}(\mathbb{R}^k)$	the σ–algebra of Borel–sets in $\mathbb{R}^k$
$\mathbf{0}$	vector with all elements being equal to 0
$\mathbf{1}$	vector with all elements being equal to 1
$\boldsymbol{\infty}$	vector with all elements being equal to ∞
$\mathbf{e}_i$	the i–th unit vector
$\mathrm{Diag}(\mathbf{x})$	diagonal matrix of elements of the vector $\mathbf{x}$
$\mathbf{x} \leq \mathbf{y}$	$\mathbf{x} \leq \mathbf{y}$ iff $x^{(i)} \leq y^{(i)}$ for all i
$\mathbf{x} < \mathbf{y}$	$\mathbf{x} < \mathbf{y}$ iff $x^{(i)} < y^{(i)}$ for all i
$[\mathbf{x}, \mathbf{y}]$	$\mathbf{z} \in [\mathbf{x}, \mathbf{y}]$ iff $\mathbf{x} \leq \mathbf{z} \leq \mathbf{y}$
$(\mathbf{x}, \mathbf{y})$	$\mathbf{z} \in (\mathbf{x}, \mathbf{y})$ iff $\mathbf{x} < \mathbf{z} < \mathbf{y}$
$\mathbf{x}^{\mathbf{y}}$	$\mathbf{x}^{\mathbf{y}} := \prod_{i=1}^{k} (x^{(i)})^{y^{(i)}}$
$\mathbf{n}!$	$\mathbf{n}! := \prod_{i=1}^{k} n^{(i)}!$
$\binom{\mathbf{1}'\mathbf{n}}{\mathbf{n}}$	$\binom{\mathbf{1}'\mathbf{n}}{\mathbf{n}} := (\mathbf{1}'\mathbf{n})!/\mathbf{n}!$
$\binom{\mathbf{n}}{\mathbf{l}}$	$\binom{\mathbf{n}}{\mathbf{l}} := \prod_{i=1}^{k} \binom{n^{(i)}}{l^{(i)}}$, $\binom{\mathbf{n}}{\mathbf{l}} = \frac{\mathbf{n}!}{\mathbf{l}!\,(\mathbf{n}-\mathbf{l})!}$ if $\mathbf{l} \in [\mathbf{0}, \mathbf{n}]$
$I(\mathbf{n}, \mathbf{m})$	$I(\mathbf{n}, \mathbf{m}) := \big\{ i \in \{1, \dots, k\} \ : \ n^{(i)} < m^{(i)} \big\}$
$\mathrm{grad} f$	gradient of f
$\mathrm{Hess} f$	Hessian matrix of f
$D^{\mathbf{n}}$	differential operator
$\mathbf{N}_t$	stochastic process at time t
$\mathbf{K}_{t,h}$	incremental process with respect to t at time h
$\Pi_{\mathbf{n}}(t)$	$\Pi_{\mathbf{n}}(t) := \mathrm{P}\left[\{\mathbf{N}_t = \mathbf{n}\}\right]$
$\mathrm{P}_{t,\mathbf{n}}$	$\mathrm{P}_{t,\mathbf{n}}[B] := \mathrm{P}\left[B \,\middle|\, \{\mathbf{N}_t = \mathbf{n}\}\right]$
$p_{\,\mathbf{n},\mathbf{m}}(r, t)$	transition probability
${}_A p_{\,\mathbf{n},\mathbf{m}}(r, t)$	transition probability of transformed process
$\boldsymbol{\kappa}_{\mathbf{n}}$	transition intensity
${}_A\boldsymbol{\kappa}_{\mathbf{n}}$	transition intensity of transformed process
Z	the set of admissible pairs $Z := \{(\mathbf{0}, 0)\} \cup \big(\mathbb{N}_0^k \times (0, \infty)\big)$

U	distribution
U_A	transformed distribution of U under A
$U_{t,\mathbf{n}}$	posterior distribution of U
M_U	moment generating function of U
$\mathcal{L}_U$	Laplace–transform of U
$g_{\mathbf{X}}$	probability generating function of the random vector $\mathbf{X}$
$\boldsymbol{\Lambda}$	parameter of the multivariate mixed Poisson process
Λ	$\Lambda := \mathbf{1}'\boldsymbol{\Lambda}$
$\boldsymbol{\Theta}$	$\boldsymbol{\Theta} := (\mathbf{1}'\boldsymbol{\Lambda})^{-1}\boldsymbol{\Lambda}$
(I)	independence assumption of Λ and $\boldsymbol{\Theta}$

Literature

Albrecht, P. [1981]: *Über einige Eigenschaften des gemischten Poissonprozesses.* Mitt. VSVM, 241-249

Bates, G.E. and Neyman, J. [1952]: *Contributions to the theory of accident proneness, I: An optimistic model of the correlation between light and severe accidents.* In: University of California Publications in Statistics, University of California Press, **Vol. 1**, pp. 215-254

Berg, Ch., Christensen, J.P.R. and Ressel, P. [1984]: *Harmonic Analysis on Semigroups.* New York: Springer

Billingsley, P. [1995]: *Probability and Measure.* New York: Wiley

Bocher, S. [1955]: *Harmonic Analysis and the Theory of Probability.* Berkeley: University of California Press

Bourbaki, N. [2003]: *Functions of a Real Variable / Elements of Mathematics.* Berlin: Springer

Consael, R. [1952]: *Sur les processus composés de Poisson à deux variables aléatoires.* Académie Royale de Belgique. Mém. Cl. Sci., fasc. 6, t.27

Dieudonné, J. [1971]: *Grundzüge der modernen Analysis.* Berlin: Deutscher Verlag der Wissenschaften

Ferrari, A., Letac, G. and Tourneret, J.-Y. [2004]: *Multivariate Mixed Poisson Distributions.* Proceedings of the 12th Conf. On Sign. Proc. (Vienna), 1067-1070

Grandell, J. [1976]: *Doubly Stochastic Poisson Processes.* Berlin: Springer

Grandell, J. [1997]: *Mixed Poisson Processes.* London: Chapman & Hall

Grigelionis, B. [1998]: *On Mixed Poisson Processes and Martingales.* Scandinavian Actuarial Journal, 81-88

Heß, K.-Th. [1997]: *Bedingt unabhängige und identisch verteilte Folgen von Zufallsvariablen.* Dresden: Dissertation TU Dresden

Heuser, H. [2003a]: *Lehrbuch der Analysis.* Stuttgart: Teubner

Heuser, H. [2003b]: *Lehrbuch der Analysis Teil 2.* Stuttgart: Teubner

Hofmann, M. [1955]: *Über zusammengesetzte Poisson–Prozesse und ihre Anwendungen in der Unfallversicherung.* Mitt. SVVM **55**, 499-575

Kallenberg, O. [2002]: *Foundations of Modern Probability* Berlin: Springer

Kannan, R. and Krueger, C.K. [1996]: *Advanced Analysis on the real line.* New York: Springer

Lemaire, J. [1995]: *Bonus–Malus Systems in Automobile Insurance.* Boston: Kluwer

Liptser, R.S. and Schiryayev. A.N. [1978]: *Statistics of Random Processes II, Applications.* New York: Springer

Lundberg, O. [1964]: *On random processes and their application to sickness and accident statistics.* Uppsala: Almqvist & Wiksell

Nawrotzki, K. [1962]: *Ein Grenzwertsatz für homogene zufällige Punktwolken (Verallgemeinerung eines Satzes von A. Renyi).* Mathematische Nachrichten **24**, 201-217

Nelson, J.F. [1984]: *The Dirichlet–Gamma–Poisson Model of Repeated Events.* Sociological Methods and Research **12**, No. 4, 347-373

Partrat, C. [1994]: *Compound Model for Two Dependent Kinds of Claims.* Insurance: Mathematics and Economics **15**, 219-231

Picard, P. [1976]: *Generalisation de l'étude sur la survenance des sinistres en assurance automobile.* Bulletin Trimestriel de l'Institut des Actuaires Français **296**, 204-268

Schmidt, K.D. [1996]: *Lectures on Risk Theory.* Stuttgart: Teubner

Schmidt, K.D. [2002]: *Versicherungsmathematik.* Berlin: Springer

Schmidt, K.D. and Zocher, M. [2003]: *Claim Number Processes having the Multinomial Property.* Dresdner Schriften zur Versicherungsmathematik **1/2003**

Schmidt, K.D. and Zocher, M. [2005]: *Loss reserving and Hofmann Distributions.* preprint

Teicher, H. [1961]: *Identifiability of mixtures.* Annals of Mathematical Statistics **32**, 244-248

Walhin, J.F. and Paris J. [2001]: *The Mixed Bivariate Hofmann Distribution.* ASTIN Bulletin **31**, 123-138

Willmot, G.E. and Sundt B. [1989]: *On Posterior Probabilities and Moments in Mixed Poisson Processes.* Scandinavian Actuarial Journal, 139-146

Zocher, M. [2002]: *Multivariate gemischte Poissonprozesse.* Diploma thesis TU–Dresden

Zocher, M. [2003]: *Multivariate Mixed Poisson Processes and the Dependence of Their Coordinates.* Dresdner Schriften zur Versicherungsmathematik **2/2003**

Zocher, M. [2005]: *Statistik für bivariate gemischte Poisson-Prozesse am Beispiel der Kraftfahrthaftpflichtversicherung.* to appear in: Allgemeines Statistisches Archiv

Versicherung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

Die vorgelegte Dissertation wurde am Institut für Mathematische Stochastik der Technischen Universität Dresden unter der wissenschaftlichen Betreuung von Herrn Prof. Dr. sc. math. habil. Klaus D. Schmidt angefertigt.

Ich erkenne die Promotionsordnung der Fakultät Mathematik und Naturwissenschaften der TU Dresden vom 20. März 2000 an.

Dresden, den 20.06.2005 gez. Mathias Zocher