Preconditioners for the p-Version of the Boundary Element Galerkin Method in $\mathbb{R}^3$

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Chapter 1

Introduction

The most universal approach for approximately solving elliptic boundary value problems is by means of variational formulations over discrete function spaces, which is called Galerkin's method. There are several ways in deriving variational formulations. One can test the differential equation with suitable functions of appropriate Sobolev spaces by using the L^2-inner product over the domain of the problem (more precisely the extension of the inner product by duality) and then by integrating by parts. Its solution within spaces of functions with small support is called finite element method (FEM), see, e.g., [6, 139, 32]. This method leads to large linear systems with sparse matrices and is most often used in practise. When considering problems on unbounded domains, e.g., scattering or transmission problem, this method is not directly applicable. In the case a fundamental solution of the problem exists, e.g., when the differential operator is linear with constant coefficients, an elegant way of solving is to represent its solution via Green's formula by an integral equation on the boundary of the problem. Then, testing with functions of appropriate Sobolev trace spaces leads to a variational formulation whose discretization needs only to consider the boundary of the domain. Here again, it is advantageous to take functions with small supports and then, the method is called boundary element method (BEM). However, since we now have to deal with integral operators which are non-local the arising linear systems are not sparse but in general fully occupied. On the other hand, since only the boundary needs to be discretized, the spatial dimensionality of the problem is reduced by one. Therefore, the linear systems of the BEM are not as large as for the FEM for comparable situations.

The conversion of elliptic boundary value problems into corresponding integral equations for investigating existence and uniqueness of solutions goes back to Neumann and Hilbert [107, 76]. In potential theory integral equation methods have been widely used to solve scattering problems in acoustics and electro magnetics, see, e.g., [91, 78, 47] and the references therein. Actually, they have been used by Kleinman and Wendland to construct solutions via Neumann iterations, see [147, 88]. Classic layer approaches with unknown densities leading to Fredholm integral equations of the second kind can be treated by using Hölder spaces and the Riesz-Schauder theory of compact operators. On the other hand, adequate tools to deal with first kind integral equations, which often encounter physically meaningful unknowns, are Sobolev spaces and the calculus of pseudo-differential operators, cf., e.g., [82, 138, 83, 149, 44, 39]. For an introduction to the theory of pseudo-differential operators we refer to [122, 140]. However, on Lipschitz boundaries which are only piecewise smooth the standard theory of pseudo-differential operators is not applicable. In this case, continuity

and strong ellipticity of the most important first kind operators of the single layer and of the normal derivative of the double layer have been proved by Costabel [35]. In two dimensions the Mellin transformation can be used to investigate the mapping properties of integral operators in the scale of Sobolev spaces. The usefulness of this transformation in connection with domains with corners was shown by Kondratiev [90]. For the investigation of first kind integral operators by the Mellin transformation we refer to a series of papers by Costabel and Stephan, see, e.g., [38, 40, 39]. The Mellin transformation is even useful when considering countably normed spaces which are adequate to show exponentially fast convergence of the *hp*-version of the boundary element method for problems on polygonal domains [74], for the convergence of the *hp*-version with geometrically graded meshes see also [10, 9, 62, 73]. For theoretical results concerning the approximation on geometric meshes for problems in three dimensions see [56, 5, 71, 99, 98].

We consider, in abstract form, a variational equation

$$U \in \tilde{H}^{\alpha/2}(\Gamma): \ a(U,v) := \langle A^\alpha U, v\rangle = \langle g, v\rangle \quad \text{for any } v \in \tilde{H}^{\alpha/2}(\Gamma) \tag{1.1}$$

where Γ is a Lipschitz continuous surface and the right hand side function $g \in H^{-\alpha/2}(\Gamma)$ is given. The operator A^α maps the Sobolev space $\tilde{H}^{\alpha/2}(\Gamma)$ (see Chapter 2 for a definition) bijectively onto its dual space $H^{-\alpha/2}(\Gamma)$ and is referred to as a pseudo-differential operator of order α. However, the term pseudo-differential operator is in general used for defining a class of operators by the symbol of its Fourier transformed counterpart on smooth domains, see, e.g., [122, 140]. Here, we also consider non-smooth domains which are locally the graphs of Lipschitz continuous functions.

In general, we assume that (1.1) is uniquely solvable. Besides this, the main assumption which is necessary to ensure convergence of Galerkin's method is that A^α is strongly elliptic, i.e., it satisfies a Gårding inequality

$$\Re\langle \theta A^\alpha U, U\rangle \geq \gamma \|U\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} - |k(U,U)|$$

for all $U \in \tilde{H}^{\alpha/2}(\Gamma)$. Here, γ is a constant independent of U, $k(\cdot,\cdot)$ denotes a compact bilinear form on $\tilde{H}^{\alpha/2}(\Gamma) \times \tilde{H}^{\alpha/2}(\Gamma)$ and θ is a smooth complex function on Γ. For our examples we will have $\theta = 1$ or, when considering systems of integral equations, θ will be a constant real matrix. For $u, v \in L^2(\Gamma)$ $\langle u, v\rangle$ denotes the usual $L^2(\Gamma)$-inner product and for generalized functions $u \in \tilde{H}^{\alpha/2}(\Gamma)$, $v \in H^{-\alpha/2}(\Gamma)$ $\langle u, v\rangle$ denotes the duality pairing.

The Galerkin method for solving (1.1) is as follows. *Given an N-dimensional subspace $X_N \subset \tilde{H}^{\alpha/2}(\Gamma)$ find $u \in X_N$ such that*

$$a(u,v) = \langle g, v\rangle \quad \textit{for any } v \in X_N. \tag{1.2}$$

Due to the strong ellipticity of A^α (and the unique solvability of (1.1)) we know that there exists an integer N_0 such that (1.2) is uniquely solvable for any $N \geq N_0$ and there holds the quasi-optimal error estimate

$$\|U - u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \leq c \inf_{v \in X_N} \|U - v\|_{\tilde{H}^{\alpha/2}(\Gamma)}$$

for a constant $c > 0$ which is independent of N, see [77, 138].

By choosing a basis $\{\phi_1, \ldots, \phi_N\}$ in X_N the variational problem over X_N reduces to a linear system

$$Au = g$$

where for simplicity we use the same notations again, i.e. $u = (u_1, \ldots, u_N)^T$ is the vector of coefficients of the function $u \in X_N$ and $g = (g_1, \ldots, g_N)^T$ with $g_j = \langle g, \phi_j \rangle$. The stiffness or system matrix of the linear system is $A = (a(\phi_i, \phi_j))_{i,j=1}^N$.

To precisely define the ansatz spaces X_N that will be used let us introduce some notations. In most situations we consider smooth open surfaces $\Gamma \subset \mathbb{R}^3$ which will be identified with domains $\Gamma \subset \mathbb{R}^2$. This is more general than considering closed surfaces since $\tilde{H}^{\alpha/2}(\Gamma) = H^{\alpha/2}(\Gamma)$ when Γ is closed. (In general we only have the inclusion $\tilde{H}^{\alpha/2}(\Gamma) \subset H^{\alpha/2}(\Gamma)$.) The situation of a closed surface can be dealt with by local mappings. We restrict ourselves to domains which can be discretized by rectangular meshes. This is to allow for taking basis functions of the tensor product type for the BEM which is very convenient for the p-version we will deal with. Curvilinear elements can be included by additional smooth mappings onto reference rectangles. The extension of our methods to some extent to triangular meshes is possible and one can also consider Schur complement methods to separate functions defined on triangles from those defined on rectangles, e.g., when dealing with meshes consisting of some triangles and many rectangles. However, several of our estimates are based on rectangular elements and an extension to triangular meshes is not explicitly studied. For a more detailed discussion of this point see page 9 below.

Now let

$$\Gamma_h = \{\Gamma_1, \ldots, \Gamma_J\} \quad \text{with} \quad \bar{\Gamma} = \cup_{j=1}^J \bar{\Gamma}_j$$

denote a quasi-uniform mesh of rectangular elements Γ_j with side length of order $O(h)$. The nodes $\{x_j;\ j = 1, \ldots, J_{V'}\}$ of the mesh are numbered such that the first J_V nodes are in the interior of Γ. Depending on the regularity of the space $\tilde{H}^{\alpha/2}(\Gamma)$, which is the energy space of the pseudo-differential operator A^α under consideration, we need continuous or not necessarily continuous functions. The piecewise polynomial ansatz space of continuous functions is

$$S^1_{h,p}(\Gamma) := \{v \in C^0(\Gamma);\ v = 0 \text{ on } \partial\Gamma, v|_{\Gamma_j} \text{ is a polynomial of degree } p, j = 1, \ldots, J\}. \quad (1.3)$$

The upper index 1 in $S^1_{h,p}(\Gamma)$ indicates the continuity of the functions and refers to the Sobolev space $H^1(\Gamma)$ which is a super space of $S^1_{h,p}(\Gamma)$.

If continuous functions are not required we use the ansatz space

$$S^0_{h,p}(\Gamma) := \{v \in L^2(\Gamma);\ v|_{\Gamma_j} \text{ is a polynomial of degree } p, j = 1, \ldots, J\}. \quad (1.4)$$

Here, the index 0 in $S^0_{h,p}(\Gamma)$ refers to the space $L^2(\Gamma) = H^0(\Gamma)$. We also use the notations $S^1_p(\Gamma_h) := S^1_{h,p}(\Gamma)$ and $S^0_p(\Gamma_h) := S^0_{h,p}(\Gamma)$. Now, the standard h-version of the boundary element method consists in refining the mesh and taking piecewise polynomials of low degree. For the p-version we take a fixed mesh and increase the polynomial degree p in order to improve the Galerkin approximation to the true solution U of (1.1). The classical h-version works with low degree, $p = 0$ or 1 depending on the operator. In these cases we also use the notations $S^1_h(\Gamma) := S^1_{h,1}(\Gamma)$ and $S^0_h(\Gamma) := S^0_{h,0}(\Gamma)$. The first space simply consists of piecewise bilinear functions on the mesh Γ_h that vanish at the boundary of Γ and the space

$S_h^0(\Gamma)$ comprises the piecewise constant functions on Γ_h. Sometimes, when the mesh size h or the domain Γ is obvious from the context, we will neglect the respective variables in the notation. The spaces $S_{h,p}^1(\Gamma)$ and $S_{h,p}^0(\Gamma)$ are conforming ansatz spaces for the Galerkin method for integral operators of order one and minus one, respectively, i.e. $S_{h,p}^1(\Gamma) \subset \tilde{H}^{1/2}(\Gamma)$ and $S_{h,p}^0(\Gamma) \subset \tilde{H}^{-1/2}(\Gamma)$.

For problems when the solution U to (1.1) behaves singularly, e.g., at the edges and corners of Γ both methods the h- and the p-version converge algebraically in N, the p-version twice as fast as the h-version, see [11] for the finite element method and [133, 134] for the boundary element method. For the optimal approximation of singularities on polyhedra with the p-version BEM see [120]. The spectral condition number of the stiffness matrix A behaves like $O(h^{-|\alpha|}p^{2|\alpha|})\kappa(M)$ where $\kappa(M)$ is the condition number of the Gram (or mass) matrix for the basis functions in use. For standard basis functions in $\mathbb{R}^2$ (which corresponds to the situation of the BEM in $\mathbb{R}^3$) one has $\kappa(M) = O(1)$ if $\alpha = -1$ (scaled tensor products of Legendre polynomials, i.e., L^2-orthonormal functions) or $\kappa(M) = O(p^4)$ if $\alpha = +1$ (tensor products of scaled antiderivatives of Legendre polynomials), see [86] for details. Therefore, it is in general more expensive to solve the linear system (1.2) for the p-version than for the h-version. Since usually the dimension of the subspace X_N tends to be large we have to use iterative methods and, due to the ill-conditioned matrices especially for large polynomial degrees, preconditioners are necessary in order to keep the numbers of required iteration steps moderate.

As mentioned above, solutions to elliptic problems in polyhedral domains in general behave singularly at the corners and edges of the domain. When these problems are converted via the direct method to boundary integral equations then the solutions of the latter equations possess these edge and corner-edge singularities. These singularities diminish the rate of convergence of the h- and the p-version of the BEM (and the FEM as well). But combining the h- and the p-version in the right way by refining the mesh geometrically towards the singularities and by increasing the polynomial degrees on large elements, then one obtains exponentially fast convergence. For problems in $\mathbb{R}^2$ see [57, 58] (concerning the FEM) and [10, 62, 73] (concerning the BEM). Here, in $\mathbb{R}^2$, one only has to deal with corner singularities of the type r^α (r being the distance to the corner). However, for problems in $\mathbb{R}^3$ one has to deal with edge singularities that are best approximated by distorted elements and anisotropic polynomial degrees. The corresponding analysis can be found in [56, 99, 71].

In this work we propose and analyze preconditioning methods for linear systems (1.2) of the p-version BEM for boundary value problems in $\mathbb{R}^3$. The two important cases $\alpha = 1$ (hypersingular operators) and $\alpha = -1$ (weakly singular operators) are considered. We will mainly focus on the p-version of the Galerkin method. The aim is to define solvers for the arising linear systems whose efficiency should not deteriorate too much when p increases. However, often the dependence of efficient solution procedures on the mesh size h is important. This is especially the case for practical applications where the polynomial degree cannot be made arbitrarily large. Then, also a reduction of the mesh size is in order. In this situation the Galerkin method can be considered as an h-version for high degree polynomials. If the dependence of the efficiency of solvers on the mesh size is almost negligible the method is scalable, i.e., larger problems (h is small) can be solved as efficiently as smaller problems where h is rather large. Therefore, an important feature of efficient preconditioners for linear systems arising from the p-version of the BEM is the independence of essential parameters on

h. This will be the case with all of our methods. Only when considering indefinite operators it is necessary to choose h small enough, this depending on p, in order to have locally positive definite problems. However, this does not mean that the methods become less efficient with smaller mesh size.

Preconditioners for systems which stem from integral operators of order one are not only of interest in its own. They are integral part of some domain decomposition methods for solving linear systems arising from the FEM. This will be explained in the following. For an overview of domain decomposition methods we refer to [31, 126]. Domain decomposition methods are used to reduce a given boundary value problem, most often its discrete finite element or finite differences formulation, into a couple of smaller problems which are more easily invertible. This is to make the solution of the whole problem accessible for parallel implementations or even only to speed up its sequential solution. Even though the individual smaller problems shall be solved independently in parallel they must be coupled in order to reproduce the global solution. This can be done in several ways. One way is to use a specific amount of overlap between the subdomains in order to share unknowns by more than one subspace of the decomposition. Another way is to consider non-overlapping decompositions and to solve the individual sub-problems independently subject to some boundary conditions. The boundary data must be adapted to ensure conformity of the solution across interfaces. It then remains to iteratively solve the interface problem whose stiffness matrix is the so-called Schur complement of the block of the original stiffness matrix which belongs to the unknowns on the interfaces. Each iteration for the solution of the interface problem requires the solution of the problems on the subdomains which can be performed in parallel.

The Schur complement can be considered as the discrete form of the Poincaré-Steklov operator which maps the Dirichlet data on the interfaces onto the Neumann data. This is a pseudo-differential operator of order one mapping $\tilde{H}^{1/2}$ on the interfaces onto $H^{-1/2}$. Thus, preconditioners for the inversion of the interface operator are also preconditioners for pseudo-differential operators of order one and vice versa, subject to the availability of suitable trace and extension operators (see below). The Poincaré-Steklov operator can be given explicitly, see, e.g., [119], and has been used in domain decomposition methods to solve the individual problems on the subdomains by the BEM, see [85, 81]. In [92, 63] this explicit representation as well as its implicit form by finite element discretization is used for the efficient solution of practically relevant problems. This mixed form yields systems which couple the FEM and the BEM.

Considering the Schur complement for the unknowns of a finite element system which are associated with the boundary Γ of the domain Ω of the boundary value problem is equivalent to decoupling the functions interior to Ω and the functions on the boundary. More precisely, we then have

$$a(u, v) = 0$$

for all interior functions u and functions v on the boundary Γ. Here, $a(\cdot, \cdot)$ is the finite element bilinear form which is assumed to be equivalent to the H^1-inner product. The functions v then are called discrete harmonic. For harmonic functions u on Ω we have the equivalence of the semi-norms (which neglect the L^2-contributions)

$$|u|_{H^1(\Omega)} \simeq |u|_{\partial\Omega}|_{H^{1/2}(\partial\Omega)},$$

see [93, 105]. For discrete harmonic functions we need the trace theorem, see [93, 105], and an extension theorem to prove corresponding mutual estimates:

$$c|u|_{\partial\Omega}|_{H^{1/2}(\partial\Omega)} \leq |u|_{H^1(\Omega)} \leq C|u|_{\partial\Omega}|_{H^{1/2}(\partial\Omega)}$$

Here, $c > 0$ is a constant and depends only on the domain Ω and C might depend also on u. For piecewise polynomials of degree p on uniform rectangular meshes an upper bound for C like $O(\log p)$ is given by Theorem 2.3. A local extension theorem, i.e., for polynomials on the faces of a cube, which yields a bound C which is independent of p is given by Ben Belgacem [16]. For extensions dealing with piecewise polynomials of low order, allowing for bounded norms, we refer to [150, 23, 20], see also [31].

By the above equivalence, FEM preconditioners dealing with the H^1-inner product for discrete harmonic functions associated with interfaces can also be considered as preconditioners for the BEM for operators of order one, and vice versa. The neglect of the L^2-contribution to the norms is usually justified by a quotient space argument or by Poincaré's inequality. However, since we do not know of an extension operator for piecewise polynomials whose norm does not depend on the polynomial degree the above equivalence of norms gives only sub-optimal results. Therefore, in this work we will not deal with global extension operators to prove the efficiency of preconditioners for the boundary element method. In contrast, we usually derive local estimates which give better results than when dealing with our extension theorem (Theorem 2.3).

The investigation of domain decomposition methods or preconditioners for discretizations of problems in $\mathbb{R}^3$ using large polynomial degrees is still in progress. The p-version FEM in $\mathbb{R}^3$ has been considered in [100, 112, 113, 114]. In [100] Mandel proposed a partial orthogonalization process where a simple coarse space is considered by introducing a special bilinear form. This coarse space is necessary to obtain a method that gives spectral condition numbers not depending on the mesh size. In [112, 113] Pavarino considered overlapping decompositions which give bounded condition numbers. This method is also successful for the BEM for hypersingular operators, cf. Section 3.2.2, and is also used as a localization procedure in $\tilde{H}^{1/2}(\Gamma)$, cf. Section 2.2. Here, the coarse space of global functions consists of piecewise bilinear functions. Pavarino and Widlund [114] have taken special discrete harmonic functions as basis for the FEM which allow for direct decompositions of the ansatz space to define almost optimal preconditioners. Here, a rather larger space of functions which are associated with the nodes and edges of the mesh, the so-called wire basket space, has to be taken. We will use the traces of those basis functions on surfaces to define preconditioners for operators of order one and we will show that even the diagonal scaling (plus the global space of piecewise bilinear functions) provides an almost optimal method, see Section 3.2.3.

A preconditioning method for the hp-version BEM in $\mathbb{R}^2$ has been proposed and analyzed in [75]. Preconditioning methods for the hp-version of the FEM in $\mathbb{R}^2$ can be found in [2, 1, 109, 59]. However, all these methods for the hp-version are restricted to problems in two space dimensions. For problems in $\mathbb{R}^3$ we also refer to [61]. Here, the hp-version with non-distorted elements has been considered. As mentioned above, the hp-version for problems in $\mathbb{R}^3$ with corner-edge singularities uses distorted elements to achieve exponentially fast convergence. These elements cause difficulties in investigating preconditioners since we often use scaling arguments (scaling with respect to the size h of the elements) for norm estimates in different Sobolev spaces. But these elements cannot be scaled by a general factor since here the mesh widths in different space directions are decoupled. A second

difficulty with geometrically graded meshes concerns the so-called coarse grid space used by the preconditioners to achieve scalability of the method. Technically this coarse grid space is necessary to deal with semi-norms instead of norms for local subspaces defining the preconditioners. These semi-norms are scalable (whereas the norms usually do not scale) and allow for eliminating the dependence of the efficiency of the method on the mesh size. On locally quasi-uniform meshes, the separation of the coarse space from the local subspaces can be performed by Clément's interpolation (see [33]) but is an open problem for general non-uniform meshes.

Let us also mention the mortar finite element method which is another class of domain decomposition. This method is based on a weak coupling of the sub-problems which amounts to a non-conforming FEM for the whole problem, see, e.g., [18, 15]. This allows for using different meshes (and subspaces) on different subdomains which need not match at the interfaces. Even for the *hp*-version with geometrically graded meshes in $\mathbb{R}^2$ optimal convergence of the mortar finite element method has been proved, see [123]. Essential theoretical ingredient is an extension theorem for piecewise polynomials on non-uniform meshes mapping $H^{1/2+\varepsilon}(\partial\Omega)$ to $H^1(\Omega)$ ($\varepsilon > 0$).

In the following we do not further mention the relation between Schur complement methods in domain decomposition and the BEM. On the contrary, we consider preconditioners for linear systems purely from the point of view of boundary element methods. They are not only important in the framework of domain decomposition but more generally when pseudo-differential operators on surfaces need to be discretized. As mentioned above, typical examples appear when solving transmission or scattering problems with the BEM. Of course, often a coupling of the FEM and the BEM is advantageous, see, e.g., [27, 87, 148, 79, 41, 37], and we will show that our methods are applicable also in this context. The coupling of the FEM and the BEM can also be efficiently used to solve various nonlinear problems, see [51, 43, 131, 30, 53, 52], and even the pure BEM is applicable in several nonlinear situations, see [80] and the references therein. When performing an outer Newton iteration to handle the nonlinearities efficient iterative solvers are even more important. In these situations our methods are applicable as well, see also [102].

As theoretical tool for the investigation of preconditioners we use the additive Schwarz framework. The additive Schwarz method is based upon subspace decompositions of the ansatz space X_N of the Galerkin method. It solves variational problems over the subspaces independently and the local solutions are then assembled to give an approximation to the solution of the full problem, see Section 3.1. This method goes back to H. A. Schwarz [121] who used an iterative technique on overlapping domains to prove the existence of harmonic functions on irregular regions. The first application of Schwarz iterations for the solution of boundary element systems we know of is prescribed by Hebeker [69] where the method by Lions [94] for partial differential equations is generalized. Both algorithms are multiplicative variants of the Schwarz method which use informations about the solution as soon as they are available. The multiplicative Schwarz method can be considered as a generalization of the block Gauss-Seidel iteration to overlapping subspaces. The additive variant generalizes the block Jacobi iteration and can be more easily parallelized. For a theoretical comparison of both methods which does not make use of specific situations, we refer to [54].

As an iterative solver for linear systems arising from the *h*-version of the BEM, the additive Schwarz method has been considered first by Hahne and Stephan [66]. The application of this method as preconditioner for the BEM in $\mathbb{R}^2$ has been investigated in [70, 141, 142].

For an overview of the additive Schwarz method as a tool for solving BEM systems, also for problems in three dimensions, see [132].

In the two-dimensional situation, when dealing with integral equations on curves, additive Schwarz methods for weakly singular operators directly correspond to additive Schwarz methods for hypersingular operators, and vice versa. This is due to the existence of simple isomorphisms between $\tilde{H}^{1/2}(\Gamma)$ and

$$\tilde{H}_0^{-1/2}(\Gamma) := \{\psi \in \tilde{H}^{-1/2}(\Gamma);\ \int_\Gamma \psi\, ds = 0\}$$

which are the energy spaces of operators of orders one and minus one, respectively. The extensions of standard differentiation and integration, which preserve polynomials, onto $\tilde{H}^{1/2}(\Gamma)$ and $\tilde{H}_0^{-1/2}(\Gamma)$, respectively, can be taken. By these mappings, any subspace decomposition of an ansatz space for hypersingular operators gives a related subspace decomposition of the ansatz space of differentiated functions for weakly singular operators, and vice versa. Both decompositions then provide the same spectral properties of the corresponding additive Schwarz methods.

Such an easy isomorphism which preserves polynomials on surfaces in $\mathbb{R}^3$ is not known. For example $(-\Delta)^{1/2}$ and its inverse would be candidates but they are only pseudo-differential operators which in general do not map polynomials onto polynomials. Therefore, our tools for preconditioners for operators of order one and of order minus one are different.

Let us mention some other approaches for preconditioning linear systems arising from the h-version of the boundary element method. Oswald has proved norm equivalences for finite element multilevel splittings both in $H^{1/2}$ and $H^{-1/2}$, cf. [111, 110]. These equivalences yield estimates for multilevel additive Schwarz preconditioners, see also [104] for an application to the BEM. However, although we also present and investigate two- and multilevel methods for completeness, we do not rely on those results.

We note that in this work we do not consider wavelet methods. So far they are restricted to the h-version of the BEM and are used to reduce the density of stiffness matrices, to accelerate matrix-vector multiplications and as preconditioning methods. There are many references in this direction, we only mention [19, 46, 45, 144]. In this context we also refer to the panel clustering method for the efficient matrix vector multiplication [64]. For a first approach to reduce the density of stiffness matrices arising from the p-version of the BEM in two dimensions see [70].

Further, we mention the method by Steinbach [127] who uses operators of opposite orders to construct preconditioners. This method is especially worth being considered when one deals with systems where all the needed operators occur. Then there is no extra work to construct the needed stiffness matrices. In the framework of domain decomposition this approach has also been proposed by Xu and Zhang, see [153]. Here, the explicit representation of the inverse of the Poincaré-Steklov operator by a weakly singular operator, which is well-known in the boundary element literature, see, e.g., [119], is used to precondition the Poincaré-Steklov operator which is hypersingular.

In the following we give an overview of our work. We investigate preconditioning methods for scalar boundary integral equations of order plus one and minus one and show that they can be combined (also with preconditioners for the finite element method) to efficiently solve systems of integral operators (and differential operators). General non-selfadjoint and indefinite operators that are strongly elliptic are covered by the theory. As standard iterative

solver we use the generalized minimum residual method (GMRES) by Saad and Schultz [116] which can be applied in all cases. We focus on the p-version of the Galerkin method on rectangular meshes such that we can use piecewise tensor products of polynomials as ansatz and test functions. This restriction at hand, one can precisely define decompositions of the ansatz spaces which lead, via the additive Schwarz method, to almost optimal preconditioners. However, we note that for integral operators of order -1, where continuity of the basis functions is not required, our preconditioners provide the same results also on general quasi-uniform regular meshes. Indeed, most of the needed technical results used in this case (and collected in Chapter 2) hold for quite general decompositions into Lipschitz domains. Only the inverse property of basis functions (Lemma 2.7) is proved on rectangular meshes. On the other hand, most of the technical results needed to prove efficiency of preconditioners for integral operators of order $+1$ are heavily based on a tensor product structure of the basis functions. Due to this tensor product structure, that is associated with rectangular meshes, rigorous decompositions of the ansatz space can be investigated. The extreme case leads to an almost diagonal preconditioner. However, some of the technical results for this case concerning estimates for polynomials in Sobolev spaces (collected in Chapter 2) do not assume special basis functions and can be used in more general circumstances.

In principle, we propose two different preconditioning strategies for positive definite integral operators of order $+1$. The overlapping method for standard basis functions proposed in §3.2.2 can be expected to work also for triangular meshes. A proof of this generalization requires a different proof for the localization technique presented in §2.2. The iterative substructuring method of §3.2.3, however, is directly associated with special basis functions on rectangles. These so-called discrete harmonic polynomials are easy to calculate since their tensor product structure can be exploited. In principle, similar basis functions can be defined on triangles as well. However, their efficient calculation is not studied yet.

Let us emphasize that our strategies to generalize preconditioners for positive definite operators to preconditioners for indefinite operators or to systems of operators do not make use of any special assumption for the ansatz spaces. These methods are completely generic by making use of the existence of preconditioners for the positive definite scalar cases.

The main three parts of the work deal with Sobolev norms for general functions and for polynomials (Chapter 2), with preconditioners for pseudo-differential operators and for systems thereof (Chapter 3), and with academic as well as more practical examples the methods can be applied to (Chapter 4).

The standard mathematical formulation of the boundary element Galerkin method is given in terms of bilinear forms defined on Sobolev spaces. Theoretical estimates which describe the behavior of additive Schwarz preconditioners are also based on these bilinear forms. It is therefore necessary to deduce various norm estimates for general functions in Sobolev spaces as well as for special polynomials that will be used in the boundary element method. This is done in Chapter 2. Some estimates concern, e.g., the interaction of interpolating Sobolev norms and composing Sobolev spaces from individual components (Lemma 2.1), the separation of Sobolev spaces over a domain into individual Sobolev spaces over parts of the domain (Lemma 2.2, Corollary 2.1), embedding, trace and extension operators (Lemmas 2.8 and 2.9, Theorem 2.3). Main results of this part are an energy preserving localization operator acting on the space of continuous piecewise polynomials (Theorem 2.2) and bounds for extension and trace operators dealing with discrete harmonic polynomials in Section 2.3. The localization procedure is used to prove the efficiency of overlapping decompositions for

operators of order one and is also a main ingredient in the proof of the efficiency of almost arbitrary decompositions when using discrete harmonic basis functions.

All the preconditioners investigated in this work are of the additive Schwarz type, i.e., they are based on the superposition of independent solutions of individual problems defined on subspaces of the whole ansatz space. Therefore, a close relationship between the preconditioners and the operators under consideration is given. Boundary integral operators of different orders possess different requirements concerning the ansatz spaces and need different procedures for the construction of subspaces.

Hypersingular operators, for which we must take continuous functions, require sophisticated decompositions. The continuity of the ansatz functions can be dealt with by defining overlapping subspaces, the functions of the subspaces being continuously extendible by zero. This method is proven to be optimal, i.e., the required numbers of iterations of the GMRES method are bounded independently of the polynomial degree and the mesh size (Theorem 3.6 in Section 3.2). For the h-version using piecewise polynomials of degree one an extension of this method to hierarchical piecewise bilinear basis functions is the multilevel additive Schwarz method for which almost optimal results are obtained (Theorems 3.4 and 3.5).

For the p-version, instead of dealing with overlapping subspaces, another possibility is to take discrete harmonic polynomials for the construction of the basis functions. In that case spans of individual (or sets of) basis functions can be chosen as subspaces for the decomposition of the ansatz space. Any of these decompositions then leads to an almost optimal preconditioner (Theorem 3.7).

For weakly singular operators, which can deal with discontinuous functions, simple non-overlapping decompositions are sufficient to obtain almost optimal preconditioners (Theorem 3.8 in Section 3.3). As a corollary we also obtain almost optimal results for a two-level method for the h-version (Corollary 3.1) which improves the result by Mund, Stephan and Weiße in [104].

Most of the above methods are directly applicable to non-selfadjoint or indefinite operators. Here, the main restriction is that, besides a small subspace of global ansatz functions, the remaining subspaces have only local supports, cf. Section 3.4.1. and Theorems 3.9, 3.10. A slightly less restrictive method, called hybrid method, only requires the inversion of the full operator on a low-dimensional subspace of global ansatz functions plus the application of an arbitrary preconditioner which is known to be efficient for a positive definite operator of the same order as the original operator (Theorem 3.11 in Section 3.4.2). The abstract results of Theorems 3.9, 3.10, and 3.11 are used to show that all the presented decompositions are generally applicable to indefinite operators when only taking care of the order of the operator. This is explicitly formulated for operators of order one in Section 3.4.3 and for operators of order minus one in Section 3.4.4. Section 3.5 deals with systems of pseudo-differential operators which is important for practical applications. The main result of this section is that essentially any combination of our preconditioners for individual scalar operators provides an efficient method for systems of the operators (Theorems 3.16, 3.17). This comprises also systems which arise from the coupling of the finite element and the boundary element method. Here, for the finite element part, any preconditioner of the literature can be taken. This general case is covered by Theorem 3.17 whereas additive Schwarz type preconditioners for the finite element part are also covered by Theorem 3.16. The special case of block skew-symmetric systems, which may appear when coupling the FEM and the BEM [41], is considered in Section 3.5.2. In this situation we prove that preconditioners for the individual

diagonal blocks (belonging to different operators) can be combined in an optimal manner. That means the worst spectral bounds of the individual methods are also bounds for the combined method (Theorem 3.18).

In the experimental part Chapter 4 we describe typical examples and show how our methods apply. The numerical verification of the theoretical estimates for scalar problems is provided by solving the Neumann and Dirichlet screen problems for the Helmholtz operator, cf. Section 4.1. These problems are modeled by hypersingular and weakly singular integral equations, respectively. Let us note that we study the Helmholtz problem as a typical example that leads to a non-selfadjoint operator, i.e., we do not have the special case of a symmetric positive definite linear system. On the other hand the theoretical results for our preconditioners are based on abstract assumptions about the operators like their strong ellipticity. Therefore, these results do not yield estimates that characterize the influence of special parameters like the wave number of the Helmholtz problem. The influence of the wave number in the p-version of the BEM is subject of ongoing research and is not covered here.

A more practical example than the Helmholtz problem is the so-called electric screen problem in Section 4.2 which is a Dirichlet problem for Maxwell's equations. This problem is modeled by a non-selfadjoint system of indefinite first kind integral operators of order minus one and plus one. The numerical results for various preconditioning methods demonstrate their ability for accelerating the GMRES method for the solution of the linear systems. The corresponding Neumann problem for Maxwell's equations is considered in Section 4.3. Theoretical results for preconditioners for this so-called magnetic screen problem are given which are similar to the methods and results in Section 4.2. Therefore, numerical results can be expected to be analogous to those for the electric screen problem and are not reported. Finally, an interface problem for the Helmholtz operator is considered in Section 4.4 which can be solved by coupling the finite element and the boundary element method. We show that the abstract results for systems of pseudo-differential operators are also applicable in this case. For the coupling procedure, numerical results are given only in the two-dimensional case. In Section 4.5 comments about the implementation of the examples and with regard to the numerical expense of the preconditioners are given.

Throughout the work $c, C, \tilde{c}$ denote generic positive constants which do not depend on essential parameters like the mesh size h or the polynomial degree p if not otherwise stated.

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Chapter 2

Sobolev norms and polynomials

In this chapter we collect various technical results about Sobolev spaces both in the continuous case as well as for polynomials (Section 2.1). Motivation for most of the estimates is to replace norms over domains by norms over subdomains which are in the optimal case almost equivalent. However, the constants in the mutual estimates might depend on parameters like the number of subdomains or the type of functions under consideration, e.g., on the polynomial degree. The aim is to minimize these dependencies. In fact, a dependence on the mesh size of the final methods will not be present in any case. In Section 2.2 we define a partition of unity within $\tilde{H}^{1/2}(\Gamma)$ which will be used as a localization tool. Section 2.3 is devoted to special basis functions which are discrete harmonic. For a given polynomial degree these basis functions are easy to calculate and can therefore be used for defining ansatz spaces of the BEM. These functions allow for an almost total decoupling within $\tilde{H}^{1/2}(\Gamma)$ which thus gives an efficient additive Schwarz method for operators of order one.

First let us deal with the continuous case. We define the Sobolev spaces that will be used and prove some technical lemmas.

We take the usual L^2- and H^1-norms and the H^1-semi-norm. The Sobolev spaces of non-integral orders are introduced as interpolation spaces. Here we use Peetre's K-method, cf. [17]. For two normed spaces A_0 and A_1 the interpolation space $A_s = [A_0, A_1]_s$ $(0 < s < 1)$ is equipped with the norm

$$\|a\|_{[A_0,A_1]_s} := \left(\int_0^\infty \left(t^{-s} \inf_{a=a_0+a_1} \left(\|a_0\|_{A_0} + t \, \|a_1\|_{A_1} \right) \right)^2 \frac{dt}{t} \right)^{1/2}. \tag{2.1}$$

For $0 < s < 1$ we define

$$H^s(\Gamma) = \left[L^2(\Gamma), H^1(\Gamma) \right]_s, \quad \tilde{H}^s(\Gamma) = \left[L^2(\Gamma), H^1_0(\Gamma) \right]_s.$$

The notation $\tilde{H}^s$ is used by Grisvard and is common in the boundary element literature, whereas the notation $H^s_{00} = \tilde{H}^s$ is used by Lions and Magenes and is common in the finite element literature.

For higher $s > 1$ the interpolation is analogously defined by using Sobolev spaces of higher integral orders. The spaces $H^{-s}(\Gamma)$ (resp. $\tilde{H}^{-s}(\Gamma)$) for $s > 0$ are the dual spaces of $\tilde{H}^s(\Gamma)$ (resp. $H^s(\Gamma)$) with respect to the L^2-inner product.

For smooth Γ (or for restricted values of s if Γ is not smooth) the Sobolev spaces on Γ can also be defined as trace spaces. To this end let $\tilde{\Gamma}$ be a smooth, closed surface containing

Γ. Then, for $s \geq 0$, $H^s(\tilde{\Gamma})$ is the restriction of $H^{s+1/2}_{\rm loc}(\mathbb{R}^3)$ to $\tilde{\Gamma}$ and, as before, $H^{-s}(\tilde{\Gamma}) = (H^s(\tilde{\Gamma}))'$ and $H^0(\tilde{\Gamma}) = L^2(\tilde{\Gamma})$. Then, for real s,

$$\tilde{H}^s(\Gamma) = \{u \in H^s(\tilde{\Gamma});\ \operatorname{supp} u \subset \bar{\Gamma}\}$$

and

$$H^s(\Gamma) = \{u|_\Gamma;\ u \in H^s(\tilde{\Gamma})\},$$

see, e.g., [93]. The spaces $\tilde{H}^s(\Gamma)$ for $s = \pm 1/2$ are the energy spaces of integral operators of orders ± 1 and are, for open surfaces Γ, not equivalent to the corresponding spaces $H^s(\Gamma)$. However, for $|s| < 1/2$ and Lipschitz domains Γ we have equivalence of the norms in $\tilde{H}^s(\Gamma)$ and $H^s(\Gamma)$, see [55, Theorems 1.4.2.4, 1.4.5.2 (c)]. Indeed, this property is of central importance in our theory of preconditioners for integral operators of order minus one. A bound for the (non-trivial) equivalence constant being dependent on s is given by Lemma 2.6 below.

The next technical lemma deals with the norms of combinations of interpolation spaces and product spaces. These spaces typically appear when using the interpolation theory globally on the whole domain Γ in connection with domain decomposition methods as will be done here.

Lemma 2.1 *Let $A_0 = \Pi_{i=1}^n A_{0,i}$, $A_1 = \Pi_{i=1}^n A_{1,i}$ be product spaces of normed spaces $A_{0,i}$ and $A_{1,i}$ with $A_{0,i} \subset A_{1,i}$. Then for $a \in A_0$ with pairwise independent components a_i, $i = 1, \ldots, n$,*

$$C_1 \|a\|_{[A_0,A_1]_s} \leq \|a\|_{\Pi_i [A_{0,i},A_{1,i}]_s} \leq C_2 \|a\|_{[A_0,A_1]_s}$$

for constants C_1, C_2 independent of a and n. If the components a_i, $i = 1, \ldots, n$, are not independent only the estimate

$$\|a\|_{\Pi_i [A_{0,i},A_{1,i}]_s} \leq C_2 \|a\|_{[A_0,A_1]_s}$$

holds.

Proof. Let $a = (a_1, \ldots, a_n)^T \in A_0$ and $a^0 = (a_1^0, \ldots, a_n^0)^T \in A_0$, $a^1 = (a_1^1, \ldots, a_n^1)^T \in A_1$. Then (cf. (2.1))

$$\begin{aligned}
\|a\|^2_{[A_0,A_1]_s} &= \int_0^\infty t^{-2s} \Big(\inf_{a=a^0+a^1} \big(\|a^0\|_{A_0} + t\, \|a^1\|_{A_1} \big) \Big)^2 \frac{dt}{t} \\
&= \int_0^\infty t^{-2s} \inf_{a_i = a_i^0 + a_i^1,\ i=1,\ldots,n} \Big(\big(\sum_{i=1}^n \|a_i^0\|^2_{A_{0,i}} \big)^{1/2} + t \big(\sum_{i=1}^n \|a_i^1\|^2_{A_{1,i}} \big)^{1/2} \Big)^2 \frac{dt}{t} \\
&\simeq \int_0^\infty t^{-2s} \inf_{a_i = a_i^0 + a_i^1,\ i=1,\ldots,n} \Big(\sum_{i=1}^n \|a_i^0\|^2_{A_{0,i}} + t^2 \sum_{i=1}^n \|a_i^1\|^2_{A_{1,i}} \Big) \frac{dt}{t} \qquad (2.2)
\end{aligned}$$

since

$$(a+b)^2 \leq 2(a^2 + b^2) \leq 2(a+b)^2$$

for $a, b \geq 0$. If the components $a_i = a_i^0 + a_i^1$ of a in (2.2) can be chosen independently for $i = 1, \ldots, n$ then we can exchange the sum and infimum and obtain

$$\begin{aligned}\|a\|^2_{[A_0,A_1]_s} &\simeq \sum_{i=1}^n \int_0^\infty t^{-2s} \inf_{a_i=a_i^0+a_i^1} (\|a_i^0\|^2_{A_{0,i}} + t^2\|a_i^1\|^2_{A_{1,i}}) \frac{dt}{t} \\ &\simeq \sum_{i=1}^n \int_0^\infty t^{-2s} \inf_{a_i=a_i^0+a_i^1} (\|a_i^0\|_{A_{0,i}} + t\|a_i^1\|_{A_{1,i}})^2 \frac{dt}{t} \\ &= \sum_{i=1}^n \|a_i\|^2_{[A_{0,i},A_{1,i}]_s} = \|a\|^2_{\Pi_i[A_{0,i},A_{1,i}]_s}.\end{aligned}$$

If the components $a_i = a_i^0 + a_i^1$ of a in (2.2) cannot be chosen independently the exchange of the sum and the infimum gives only a lower bound. □

The next lemma is essential in performing domain decompositions for integral operators. It decomposes Sobolev norms of non-integral orders into norms over subdomains which is trivial for norms of integral orders.

Lemma 2.2 *Let Γ be a Lipschitz domain in $\mathbb{R}^n$ and let $\{\Gamma_j;\ j = 1, \ldots, J\}$ be a decomposition of Γ, i.e.*

$$\cup_{j=1}^J \bar{\Gamma}_j = \bar{\Gamma} \quad \textit{and} \quad \Gamma_i \cap \Gamma_j = \emptyset \quad \textit{if} \quad i \neq j.$$

Then, for $s > 0$ and $u \in \tilde{H}^s(\Gamma)$ with $u_j := u|_{\Gamma_j} \in \tilde{H}^s(\Gamma_j)$, $j = 1, \ldots, J$, there exists a constant $C > 0$ which is independent of u and J such that

$$\|u\|^2_{\tilde{H}^s(\Gamma)} \leq C \sum_{j=1}^J \|u_j\|^2_{\tilde{H}^s(\Gamma_j)}. \tag{2.3}$$

Further, for arbitrary $u \in H^s(\Gamma)$,

$$C \sum_{j=1}^J \|u_j\|^2_{H^s(\Gamma_j)} \leq \|u\|^2_{H^s(\Gamma)}. \tag{2.4}$$

Proof. We use the ideas of von Petersdorff who proved the assertions for the norms defined by the complex interpolation method, cf. [143, Lemma 3.2]. Indeed, the essential new ingredient is the relation of interpolation norms and product spaces given by Lemma 2.1.

Let $0 \leq s \leq 1$ and define the restriction operator

$$R : \left\{ \begin{array}{ccl} H^s(\Gamma) & \to & \Pi_{j=1}^J H^s(\Gamma_j) \\ u & \mapsto & (u_1, \ldots, u_J)^T := (u|_{\Gamma_1}, \ldots, u|_{\Gamma_J})^T \end{array} \right.$$

and the composition operator

$$R^{-1} : \left\{ \begin{array}{ccl} \Pi_{j=1}^J \tilde{H}^s(\Gamma_j) & \to & \tilde{H}^s(\Gamma) \\ (u_1, \ldots, u_J)^T & \mapsto & u :\ u|_{\Gamma_j} := u_j,\ j = 1, \ldots, J. \end{array} \right.$$

For $s = 0$ and $s = 1$ both operators, R and R^{-1}, are continuous with bound 1 if one endows $\Pi_{j=1}^{J} \tilde{H}^s(\Gamma_j)$ with the norm

$$\|(u_1, \ldots, u_J)^T\|_{\tilde{s}}^2 = \sum_{j=1}^{J} \|u_j\|_{\tilde{H}^s(\Gamma_j)}^2, \quad s = 0, 1,$$

and analogously for $\Pi_{j=1}^{J} H^s(\Gamma_j)$ using the notation $\|\cdot\|_s$ for the norm. Thus, using the interpolation theorem, R and R^{-1} are continuous for all $s \in [0, 1]$ that means

$$\|(u_1, \ldots, u_J)^T\|_s^2 \leq \|u\|_{H^s(\Gamma)}^2$$

and

$$\|u\|_{\tilde{H}^s(\Gamma)}^2 \leq \|(u_1, \ldots, u_J)^T\|_{\tilde{s}}^2.$$

Due to Lemma 2.1 we can interchange the composition and the interpolation (which gives a lower bound in the first case and an equivalent norm in the second case above). Therefore there exist constants $C_1, C_2 > 0$ such that for $0 \leq s \leq 1$

$$C_1 \sum_{j=1}^{J} \|u_j\|_{H^s(\Gamma_j)}^2 \leq \|(u_1, \ldots, u_J)^T\|_s^2 \leq C_2 \|u\|_{H^s(\Gamma)}^2$$

which is (2.4) and

$$\|u\|_{\tilde{H}^s(\Gamma)}^2 \leq \|(u_1, \ldots, u_J)^T\|_{\tilde{s}}^2 \simeq \sum_{j=1}^{J} \|u_j\|_{\tilde{H}^s(\Gamma_j)}^2$$

which is (2.3).

The proof for $s > 1$ is analogous by interpolating between Sobolev spaces of higher integral order. □

In general, for a function $u \in \tilde{H}^{1/2}(\Gamma)$, we do not have $u|_{\Gamma_j} \in \tilde{H}^{1/2}(\Gamma_j)$. Further, the energy norm given by integral operators of order plus one is equivalent to the $\tilde{H}^{1/2}(\Gamma)$-norm. Therefore, the assertions (2.3) and (2.4) together do not prove the efficiency of the related non-overlapping domain decomposition method. For integral operators of order minus one the energy norm is equivalent to the $\tilde{H}^{-1/2}(\Gamma)$-norm. For a piecewise polynomial $u \in \tilde{H}^{-1/2}(\Gamma)$, we have $u|_{\Gamma_j} \in \tilde{H}^{-1/2}(\Gamma_j)$ since $u|_{\Gamma_j} \in L^2(\Gamma_j)$. This does not mean that the local norms (i.e., $(\sum_j \|\cdot\|_{\tilde{H}^{-1/2}(\Gamma_j)}^2)^{1/2}$) are equivalent to the global norm. Again, in general only the estimate for the energy norm corresponding to (2.3) holds. Therefore, the efficiency of a non-overlapping domain decomposition method is not directly given by the norm estimates. The next corollary deals with Sobolev norms of negative orders which cover the cases of operators of negative orders.

Corollary 2.1 *Let $s < 0$ and $u \in \tilde{H}^s(\Gamma)$ with $u_j := u|_{\Gamma_j} \in \tilde{H}^s(\Gamma_j)$, $j = 1, \ldots, J$. There exists a constant $c > 0$ which is independent of u and J such that*

$$\|u\|_{\tilde{H}^s(\Gamma)}^2 \leq c \sum_{j=1}^{J} \|u_j\|_{\tilde{H}^s(\Gamma_j)}^2. \tag{2.5}$$

Further, for $u \in H^s(\Gamma)$, there holds

$$c \sum_{j=1}^{J} \|u_j\|^2_{H^s(\Gamma_j)} \leq \|u\|^2_{H^s(\Gamma)} \tag{2.6}$$

for a constant $c > 0$ which is independent of u and J.

Proof. By duality the assertions (2.5) and (2.6) follow from (2.4) and (2.3), respectively, see also [143, Lemma 3.2]. □

To bound the maximum eigenvalue of additive Schwarz operators one usually has to bound a global norm by norms over subdomains. This is straight forward for Sobolev spaces of integral order but requires a proof for intermediate spaces. Actually, we have to use Lemma 2.2, Corollary 2.1 and a coloring argument. In the following lemma we consider decompositions of Γ into possibly overlapping subdomains. In contrast, Lemma 2.2 only deals with non-overlapping decompositions.

Lemma 2.3 *Let $\{\Gamma_i;\ i = 1, \dots, k\}$ be a finite covering of Γ by subdomains Γ_i with Lipschitz boundary,*

$$\bar{\Gamma} = \cup_{i=1}^{k} \bar{\Gamma}_i,$$

with a covering constant K, i.e. we can color $\{\Gamma_i;\ i = 1, \dots, k\}$ using at most K colors in such a way that subdomains of the same color are disjoint. Let $v = \sum_{i=1}^{k} v_i \in \tilde{H}^s(\Gamma)$ for real s with $v_i \in \tilde{H}^s(\Gamma_i)$, $i = 1, \dots, k$. Then there holds

$$\|v\|^2_{\tilde{H}^s(\Gamma)} \leq cK \sum_{i=1}^{k} \|v_i\|^2_{\tilde{H}^s(\Gamma_i)}$$

with a constant $c > 0$ which is independent of v and the decomposition of Γ.

Proof. Due to the coloring assumption we can split the sum $\sum_{i=1}^{k} \varphi_i$ into at most K sums,

$$\varphi = \sum_{i=1}^{k} \varphi_i = \sum_{j=1}^{K} \sum_{i \in I_j} \varphi_i,$$

where each two functions have at most a subdomain's boundary as a common support:

$$\text{interior } (\operatorname{supp} \varphi_m \cap \operatorname{supp} \varphi_n) = \emptyset \quad \text{for } m, n \in I_j,\ j = 1, \dots, K.$$

Then we can apply Lemma 2.2 (estimate (2.3) if $s > 0$) or Corollary 2.1 ($s < 0$) to each of the partial sums,

$$\| \sum_{i \in I_j} \varphi_i \|^2_{\tilde{H}^s(\Gamma)} \leq c \sum_{i \in I_j} \|\varphi_i\|^2_{\tilde{H}^s(\Gamma_i)},$$

and obtain by using the triangle inequality

$$\|\varphi\|^2_{\tilde{H}^s(\Gamma)} \leq K \sum_{j=1}^{K} \| \sum_{i \in I_j} \varphi_i \|^2_{\tilde{H}^s(\Gamma)} \leq cK \sum_{j=1}^{K} \sum_{i \in I_j} \|\varphi_i\|^2_{\tilde{H}^s(\Gamma_i)} = cK \sum_{i=1}^{k} \|\varphi_i\|^2_{\tilde{H}^s(\Gamma_i)}.$$

□

To estimate the norm of a function locally on the actual mesh Γ_h we use a transformation onto a reference element. We therefore need to know how Sobolev norms behave on regions of different sizes. For the scaling of Sobolev norms of nonnegative integer order we also refer, e.g., to [32, Theorems 3.1.2, 3.1.3].

Lemma 2.4 *Let $I^n = (0,1)^n$, $I_h^n = (0,h)^n$ and let*

$$T_h^n : I_h^n \to I^n, \quad n = 1,2,3,$$

be an affine transformation. For functions u, $\tilde{u}$ such that $u = \tilde{u} \circ T_h^n$ on I_h^n there holds for $n = 1,2,3$

$$\|u\|^2_{\tilde{H}^s(I_h^n)} \simeq h^{n-2s} \|\tilde{u}\|^2_{\tilde{H}^s(I^n)} \quad (0 \le s \le 1)$$

and

$$\|u\|^2_{H^s(I_h^n)} \simeq h^{n-2s} \|\tilde{u}\|^2_{H^s(I^n)} \quad (-1 \le s \le 0).$$

Proof. For $s = 0, 1$ the first relation can be verified by substitution. For $0 < s < 1$ we use interpolation. The second relation holds for $-1 \le s \le 0$ by duality:

$$\|u\|_{H^s(I_h^n)} = \sup_{\psi \in \tilde{H}^{-s}(I_h^n)} \frac{\langle u, \psi \rangle_{L^2(I_h^n)}}{\|\psi\|_{\tilde{H}^{-s}(I_h^n)}} \simeq \sup_{\tilde{\psi} \in \tilde{H}^{-s}(I^n)} \frac{h^n \langle \tilde{u}, \tilde{\psi} \rangle_{L^2(I^n)}}{h^{(n+2s)/2} \|\tilde{\psi}\|_{\tilde{H}^{-s}(I^n)}} = h^{(n-2s)/2} \|\tilde{u}\|_{H^s(I^n)}.$$

□

The scaling property of $\tilde{H}^s$-norms for $s \ge 0$ as given by the above lemma can be extended to negative values of s if one considers only functions with integral mean zero.

Lemma 2.5 *We use the same notations as in Lemma 2.4. For a function u with integral mean zero on I_h^n there holds for $n = 1,2,3$*

$$\|u\|^2_{H^s(I_h^n)} \simeq h^{n-2s} \|\tilde{u}\|^2_{H^s(I^n)} \quad (0 \le s \le 1)$$

and

$$\|u\|^2_{\tilde{H}^s(I_h^n)} \simeq h^{n-2s} \|\tilde{u}\|^2_{\tilde{H}^s(I^n)} \quad (-1 \le s \le 0),$$

provided one of the respective norms is finite.

Proof. The first assertion holds by Poincaré's inequality for $s = 1$ and for $0 < s < 1$ this follows by interpolation. Now let us prove the second assertion. First we consider the case $s = -1$. By duality we obtain

$$\begin{aligned}
\|u\|_{\tilde{H}^{-1}(I_h^n)} &= \sup_{\psi \in H^1(I_h^n)} \frac{\langle u, \psi \rangle_{L^2(I_h^n)}}{\|\psi\|_{H^1(I_h^n)}} = \sup_{\psi \in H^1(I_h^n)} \sup_{c \in \mathbb{R}} \frac{\langle u, \psi + c \rangle_{L^2(I_h^n)}}{\|\psi + c\|_{H^1(I_h^n)}} \\
&= \sup_{\psi \in H^1(I_h^n)} \frac{\langle u, \psi \rangle_{L^2(I_h^n)}}{\inf_{c \in \mathbb{R}} \|\psi + c\|_{H^1(I_h^n)}} \simeq \sup_{\psi \in H^1(I_h^n)} \frac{\langle u, \psi \rangle_{L^2(I_h^n)}}{|\psi|_{H^1(I_h^n)}} \\
&\simeq \sup_{\tilde{\psi} \in H^1(I^n)} \frac{h^n \langle \tilde{u}, \tilde{\psi} \rangle_{L^2(I^n)}}{h^{(n-2)/2} |\tilde{\psi}|_{H^1(I^n)}} = h^{(n+2)/2} \sup_{\tilde{\psi} \in H^1(I^n)} \frac{\langle \tilde{u}, \tilde{\psi} \rangle_{L^2(I^n)}}{|\tilde{\psi}|_{H^1(I^n)}} \\
&\simeq h^{(n+2)/2} \|\tilde{u}\|_{\tilde{H}^{-1}(I^n)}.
\end{aligned}$$

The last equivalence holds by the same arguments as used in the steps before. For the range $-1 < s < 0$ the assertion then follows by interpolation. □

The next lemma gives a bound for the norm of the injection $H^s \to \tilde{H}^s$ on Lipschitz domains for $|s| < 1/2$. In fact $\tilde{H}^s = H^s$ on Lipschitz domains for $|s| < 1/2$, see [55, Theorems 1.4.2.4, 1.4.5.2 (c)]. This result will be used to bound the $\tilde{H}^{1/2}$-norm of a polynomial by its $H^{1/2}$-norm, the bound depending of the polynomial degree, see Lemma 2.8 below.

Lemma 2.6 *Let $R \subset \mathbb{R}^2$ be a Lipschitz domain. There exists a constant $c > 0$ such that for any $s \in (-1/2, 1/2)$ and any $v \in H^s(R)$ there holds*

$$\|v\|_{\tilde{H}^s(R)} \leq \frac{c}{1/2 - |s|} \|v\|_{H^s(R)}.$$

Proof. For $0 < s < 1/2$ we use the following equivalent characterization of the $\tilde{H}^s(R)$-norm,

$$\|v\|^2_{\tilde{H}^s(R)} = \|v\|^2_{H^s(R)} + \Big\| \frac{v(x)}{\operatorname{dist}(x, \partial R)^s} \Big\|^2_{L^2(R)} \tag{2.7}$$

(see, e.g., [93, Theorem 11.7] and [55, Lemma 1.3.2.6]). Following the proof of Theorem 1.4.4.4 in [55] we estimate the latter term in the above relation.

First we consider the case when $R = \mathbb{R}_+$ the positive real axis. We make use of the identity [55, (1,4,4,9)]

$$v(x) = -w(x) + \int_x^\infty \frac{w(y)}{y}\, dy \quad \text{where} \quad w(x) = \frac{1}{x} \int_0^x (v(t) - v(x))\, dt.$$

Due to Hardy's inequality [68, Theorem 330],

$$\int_0^\infty x^{-r} \left(\int_x^\infty f(t)\, dt \right)^2 dx \leq \frac{4}{(r-1)^2} \int_0^\infty x^{2-r} f^2(x)\, dx$$

for $r < 1$, there holds

$$\int_0^\infty \left(x^{-s} \int_x^\infty \frac{w(y)}{y}\, dy \right)^2 dx \leq \frac{1}{(1/2 - s)^2} \int_0^\infty w^2(x) x^{-2s}\, dx \quad \text{for } s < 1/2.$$

We estimate the latter integral above by

$$\begin{aligned}
\int_0^\infty w^2(x) x^{-2s}\, dx &= \int_0^\infty x^{-2(s+1)} \left(\int_0^x \big(v(t) - v(x)\big)\, dt \right)^2 dx \\
&\leq \int_0^\infty x^{-2(s+1)} \int_0^x 1\, dt \int_0^x \big(v(t) - v(x)\big)^2\, dt\, dx \\
&= \int_0^\infty x^{-2s-1} \int_0^x \big(v(t) - v(x)\big)^2\, dt\, dx \\
&\leq \int_0^\infty \int_0^\infty \frac{\big(v(t) - v(x)\big)^2}{|x - t|^{2s+1}}\, dt\, dx \leq \|v\|^2_{H^s(0,\infty)}
\end{aligned}$$

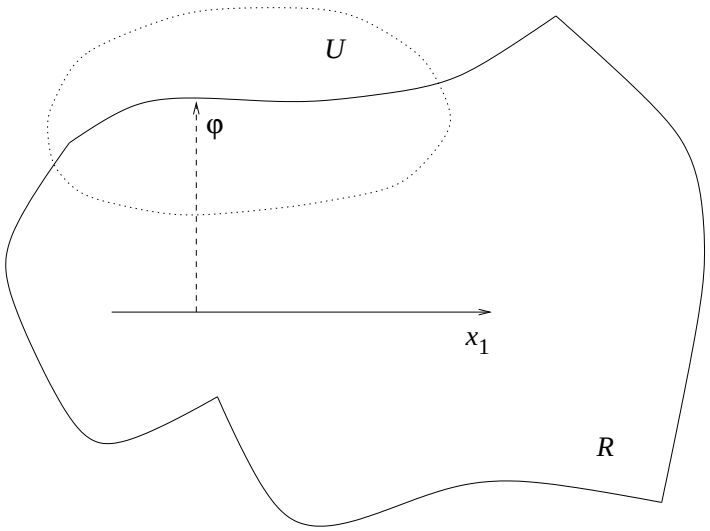


Figure 2.1: Proof of Lemma 2.6: the domain R with Lipschitz continuous boundary.

where the last inequality is due to the equivalence of norms

$$\|v\|^2_{H^s(0,\infty)} \simeq \|v\|^2_{L^2(0,\infty)} + \int_0^\infty \int_0^\infty \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,dx\,dy,$$

cf., e.g., [55, (1,3,2,2)]. Then we obtain

$$\begin{aligned}
\int_0^\infty (x^{-s}v(x))^2\,dx &= \int_0^\infty \left(-x^{-s}w(x) + x^{-s}\int_x^\infty \frac{w(y)}{y}\,dy\right)^2 dx \\
&\le 2\left(\int_0^\infty \big(x^{-s}w(x)\big)^2\,dx + \int_0^\infty \big(x^{-s}\int_x^\infty \frac{w(y)}{y}\,dy\big)^2\,dx\right) \\
&\le 2\left(\|x^{-s}w\|^2_{L^2(0,\infty)} + \frac{1}{(1/2-s)^2}\|x^{-s}w\|^2_{L^2(0,\infty)}\right) \\
&\le \frac{c}{(1/2-s)^2}\|v\|^2_{H^s(0,\infty)}.
\end{aligned}$$

By (2.7) this proves the assertion when $R = \mathbb{R}_+$ for $0 < s < 1/2$. The constant c does not depend on v and s.

This result applies to general Lipschitz domains which can be seen by using the technique described in the proof of Theorem 1.4.4.4 in [55]. More precisely the boundary of R is locally the graph of a Lipschitz continuous function φ and locally we can define

$$v_{x_1}(t) = v\big(x_1, \varphi(x_1) - t\big)$$

where $(x_1, \varphi(x_1))$ describes a boundary piece and $(x_1, \varphi(x_1) - t) \in R$ for $0 < t < t_0$ and a $t_0 > 0$, see Figure 2.1. (Here we assume that the local boundary piece under consideration is a Lipschitz continuous function of the variable x_1.)

By using a partition of unity we can assume that the support of v is contained in a local neighborhood U of the boundary. We obtain

$$\|t^{-s}v_{x_1}\|^2_{L^2(0,\infty)} \le \frac{c}{(1/2-s)^2}\|v_{x_1}\|^2_{H^s(0,\infty)}$$

for almost every x_1 such that $(x_1, \varphi(x_1)) \in U \cap \partial R$ where the constant c does not depend on x_1 and s. Integrating with respect to x_1 this yields

$$\begin{aligned}
&\int_{\mathbb{R}} \int_0^\infty t^{-2s} v^2\big(x_1, \varphi(x_1) - t\big)\, dt\, dx_1 \\
&\simeq \int_{\mathbb{R}} \int_{R \cap \{(x_1, \varphi(x_1) - t);\, t > 0\}} \big(\varphi(x_1) - x_2\big)^{-2s} v^2(x_1, x_2)\, dx_2\, dx_1 \\
&\simeq \Big\| \frac{v(x)}{\operatorname{dist}(x, \partial R)^s} \Big\|^2_{L^2(R)} \leq \frac{c}{(1/2 - s)^2} \|v\|^2_{H^s(R)}
\end{aligned}$$

and therefore

$$\|v\|^2_{\tilde{H}^s(R)} \leq \frac{c}{|s - 1/2|^2} \|v\|^2_{H^s(R)} \quad \text{for } 0 < s < 1/2.$$

For $-1/2 < s < 0$ we obtain

$$\|v\|_{\tilde{H}^s(R)} = \sup_{\varphi \in C_0^\infty(R)} \frac{\langle v, \varphi \rangle_{L^2(R)}}{\|\varphi\|_{H^{-s}(R)}} \leq \frac{c}{|-s - 1/2|} \sup_{\varphi \in C_0^\infty(R)} \frac{\langle v, \varphi \rangle_{L^2(R)}}{\|\varphi\|_{\tilde{H}^{-s}(R)}} = \frac{c}{|s + 1/2|} \|v\|_{H^s(R)}$$

and the proof is complete. □

2.1 Discrete Sobolev inequalities

We collect some results about Sobolev norms for polynomials.

The next lemma deals with the inverse property of piecewise polynomials. This property is well known for standard continuous finite element functions. Since we need this property also for higher degree polynomials and for discontinuous functions we include a proof for these cases. For simplicity we only consider rectangular meshes. The case of piecewise polynomials on curves is implicitly covered, see also [134, Remark 3.4].

Lemma 2.7 *Let Γ be a polygonal domain and let $\{\Gamma_j;\ j = 1, \ldots, J\}$ be a rectangular mesh on Γ which is assumed to be quasi-uniform, the lengths of the elements being proportional to h. Further let v be a piecewise polynomial of degree p, i.e. $v|_{\Gamma_j}$ is a polynomial of degree p for $j = 1, \ldots, J$. If $v \in \tilde{H}^r(\Gamma)$ for a real number $r \leq 1$, then for $s \leq r$ there holds*

$$\|v\|_{\tilde{H}^r(\Gamma)} \leq c h^{s-r} p^{2(r-s)} \|v\|_{\tilde{H}^s(\Gamma)}.$$

Accordingly, for $v \in H^r(\Gamma)$, there holds

$$\|v\|_{H^r(\Gamma)} \leq c h^{s-r} p^{2(r-s)} \|v\|_{H^s(\Gamma)}.$$

Proof. First we deal with continuous piecewise polynomials. Let R be a Lipschitz domain such as one of the subdomains Γ_j. Locally for polynomials v we have by Schmidt's inequality (see [13])

$$|v|_{H^1(R)} \leq c(h) p^2 \|v\|_{L^2(R)}.$$

Therefore, we conclude with the help of Lemma 2.4 that there holds

$$\begin{aligned}\|v\|^2_{\tilde{H}^1(\Gamma)} &\simeq |v|^2_{H^1(\Gamma)} = \sum_{j=1}^{J} |v|_{\Gamma_j}|^2_{H^1(\Gamma_j)} \simeq \sum_{j=1}^{J} |\widetilde{v|_{\Gamma_j}}|^2_{H^1(I^2)} \\ &\leq c \sum_{j=1}^{J} p^4 \|\widetilde{v|_{\Gamma_j}}\|^2_{L^2(I^2)} \simeq \sum_{j=1}^{J} h^{-2} p^4 \|v|_{\Gamma_j}\|^2_{L^2(\Gamma_j)} = h^{-2} p^4 \|v\|^2_{L^2(\Gamma)}.\end{aligned} \tag{2.8}$$

Here, $\widetilde{v|_{\Gamma_j}}$ means the linearly transformed function $v|_{\Gamma_j}$ as defined in Lemma 2.4. By interpolation we now obtain

$$\|v\|_{\tilde{H}^r(\Gamma)} \leq c h^{-(r-s)} p^{2(r-s)} \|v\|_{\tilde{H}^s(\Gamma)} \quad \text{for} \quad 0 \leq s \leq r \leq 1. \tag{2.9}$$

Here we make use of the fact that interpolating between polynomial spaces reproduces the polynomials, see [96] for fundamental results on intervals and [14, 16] for an extension to domains in $\mathbb{R}^2$. This fact will be used several times in the following.

We note that the use of local scaling properties of Sobolev norms is standard in proving the usual inverse property of continuous piecewise polynomials of low degree, see, e.g., [32, Theorem 3.2.6].

To prove (2.9) for negative values of r and s we use the induction method of the proof of Theorem 4.1.3 in [6]. More precisely, we proceed as follows. Let $\alpha \in [0, 1/2)$ and $\varepsilon > 0$ be given such that $\alpha + \varepsilon \leq 1$. Then from (2.9) we obtain by interpolation

$$\begin{aligned}\|v\|^2_{\tilde{H}^\alpha(\Gamma)} &= \|v\|^2_{[\tilde{H}^{\alpha-\varepsilon}(\Gamma), \tilde{H}^{\alpha+\varepsilon}(\Gamma)]_{1/2}} \leq \|v\|_{\tilde{H}^{\alpha-\varepsilon}(\Gamma)} \|v\|_{\tilde{H}^{\alpha+\varepsilon}(\Gamma)} \\ &\leq \|v\|_{\tilde{H}^{\alpha-\varepsilon}(\Gamma)} c h^{-\varepsilon} p^{2\varepsilon} \|v\|_{\tilde{H}^\alpha(\Gamma)},\end{aligned}$$

i.e.

$$\|v\|_{\tilde{H}^\alpha(\Gamma)} \leq c h^{-\varepsilon} p^{2\varepsilon} \|v\|_{\tilde{H}^{\alpha-\varepsilon}(\Gamma)}.$$

Therefore, (2.9) holds for $r \in [0, 1]$ and $s \in [-r, r]$. By induction it follows that (2.9) holds for all numbers $s \leq r$. The extension to negative numbers r can be performed again by interpolation. For real numbers $s < r < 0 \leq t$ and $0 < \theta < 1$ with $(1-\theta)s + \theta t = r$ there holds

$$\begin{aligned}\|v\|_{\tilde{H}^r(\Gamma)} &= \|v\|_{[\tilde{H}^s(\Gamma), \tilde{H}^t(\Gamma)]_\theta} \leq \|v\|^{1-\theta}_{\tilde{H}^s(\Gamma)} \|v\|^\theta_{\tilde{H}^t(\Gamma)} \\ &\leq \|v\|^{1-\theta}_{\tilde{H}^s(\Gamma)} c h^{-\theta(t-s)} p^{2\theta(t-s)} \|v\|^\theta_{\tilde{H}^s(\Gamma)} = c h^{-(r-s)} p^{2(r-s)} \|v\|_{\tilde{H}^s(R)}.\end{aligned}$$

This proves the assertion of the lemma for continuous functions.

We now prove the assertion when the piecewise polynomials are not required to be continuous. In this case we only deal with Sobolev norms of order less than 1/2. We simply make use of the inclusion $v|_{\Gamma_j} \in \tilde{H}^r(\Gamma_j)$ $(r < 1/2)$ for a piecewise polynomial v. We then

obtain by Lemmas 2.2 and 2.4 for parameters $0 \le s \le r < 1/2$

$$\begin{aligned}\|v\|^2_{\tilde{H}^r(\Gamma)} &\le c\sum_{j=1}^{J} \|v|_{\Gamma_j}\|^2_{\tilde{H}^r(\Gamma_j)} \simeq \sum_{j=1}^{J} h^{2-2r}\big\|\widetilde{v|_{\Gamma_j}}\big\|^2_{\tilde{H}^r(I^2)} \\ &\le c\sum_{j=1}^{J} h^{2-2r}p^{4(r-s)}\big\|\widetilde{v|_{\Gamma_j}}\big\|^2_{\tilde{H}^s(I^2)} \simeq \sum_{j=1}^{J} h^{2-2r}h^{-2+2s}p^{4(r-s)}\|v|_{\Gamma_j}\|^2_{\tilde{H}^s(\Gamma_j)} \\ &\le ch^{2(s-r)}p^{4(r-s)}\|v\|^2_{\tilde{H}^s(\Gamma)}. \end{aligned} \tag{2.10}$$

Here again, $\widetilde{v|_{\Gamma_j}}$ means the linearly transformed function $v|_{\Gamma_j}$ as defined in Lemma 2.4. Further, we made use of Schmidt's inequality and its extension to real numbers $0 \le s \le r \le 1$, see also [48, Lemma 4.1].

The extension of the above inverse property to negative numbers s and r can be performed inductively by interpolation as shown in the case of continuous piecewise polynomials. This finishes the proof of the first assertion of the lemma which deals with $\tilde{H}^r(\Gamma)$-norms.

The assertion for the $H^r(\Gamma)$-norms can be obtained by slight modifications of the above proof. In the case of continuous functions we note that (2.8) holds also when adding the $L^2(\Gamma)$-norm at the left hand side which gives

$$\|v\|^2_{H^1(\Gamma)} \le ch^{-2}p^4\|v\|^2_{L^2(\Gamma)}.$$

The rest of the proof then is the same by replacing $\tilde{H}^r$-norms with H^r-norms. In the case of not necessarily continuous functions we note that (2.10) holds also for non-tilde norms which can be seen by using norms with weighted L^2-term. This weighting is admissible by Poincaré's inequality since C_0^∞ is dense in H^r for $|r| < 1/2$. The remainder of the proof for functions which are not necessarily continuous is analogous to the case when dealing with $\tilde{H}^r$-norms. □

With the help of Lemmas 2.6 and 2.7 we are now able to bound the $\tilde{H}^{1/2}$ and $\tilde{H}^{-1/2}$-norms of piecewise polynomials by their $H^{1/2}$ and $H^{-1/2}$-norms, respectively, see Lemma 2.8 below. For corresponding results regarding the $\tilde{H}^{1/2}$- and $H^{1/2}$-norms of polynomials on an interval we refer to [7]. By this lemma we then can switch from local $\tilde{H}^{-1/2}$-norms to local $H^{-1/2}$-norms which can be bounded by the global $\tilde{H}^{-1/2}$-norm by Corollary 2.1. This gives an estimate for the minimum eigenvalue of a non-overlapping domain decomposition method for operators of order minus one. However, this method would depend on the mesh size. In order to avoid this dependence we need a more sophisticated decomposition whose efficiency is proved without directly using the next lemma (see Theorem 3.8). Only the technique of its proof will be used but we give the result which is of interest in its own.

Lemma 2.8 *Let v be a piecewise polynomial of degree p defined on a quasi-uniform rectangular mesh R_h on R with mesh size h. There exists a constant $c > 0$ which is independent of p but may depend on the diameter of R such that there holds*

$$\|v\|_{\tilde{H}^{-1/2}(R)} \le c(1 + \log\frac{p+1}{h})\|v\|_{H^{-1/2}(R)} \quad (h \le 1).$$

Analogously, if $v \in \tilde{H}^{1/2}(R)$ is a piecewise polynomial then there holds

$$\|v\|_{\tilde{H}^{1/2}(R)} \le c(1 + \log\frac{p}{h})\|v\|_{H^{1/2}(R)} \quad (h \le 1).$$

Proof. The first assertion is proved as follows. We have $v \in \tilde{H}^{-1/2}(R)$ since $v \in L^2(R) \subset \tilde{H}^{-1/2}(R)$. Now let $\delta \in (0, 1/2)$ be given. Using Lemma 2.6 we deduce

$$\|v\|_{\tilde{H}^{-1/2}(R)} \leq \|v\|_{\tilde{H}^{-1/2+\delta}(R)} \leq c/\delta \|v\|_{H^{-1/2+\delta}(R)},$$

and by the inverse property (Lemma 2.7) we conclude for $p > 0$

$$\|v\|_{\tilde{H}^{-1/2}(R)} \leq \frac{c}{\delta} h^{-\delta} p^{2\delta} \|v\|_{H^{-1/2}(R)}.$$

Here the constant c does not depend on p or h. Setting $\delta := 1/\log \frac{p^2}{h}$ we obtain

$$\|v\|_{\tilde{H}^{-1/2}(R)} \leq c \log \frac{p}{h} \|v\|_{H^{-1/2}(R)} \quad (p/h > 1)$$

which gives the first assertion.

Analogously, we obtain for continuous piecewise polynomials v by using the inverse property (Lemma 2.7) and Lemma 2.6

$$\begin{aligned} \|v\|_{\tilde{H}^{1/2}(R)} &\leq c h^{-\delta} p^{2\delta} \|v\|_{\tilde{H}^{1/2-\delta}(R)} \leq \frac{c}{\delta} h^{-\delta} p^{2\delta} \|v\|_{H^{1/2-\delta}(R)} \\ &\leq \frac{c}{\delta} h^{-\delta} p^{2\delta} \|v\|_{H^{1/2}(R)} \end{aligned}$$

($\delta \in (0, 1/2)$). Choosing $\delta := 1/\log \frac{p^2}{h}$ the second assertion of the lemma follows. □

The following lemma gives a bound for the trace operator for polynomials on rectangles. We note that Pavarino and Widlund [114, Lemma 5.3] proved a related result which gives for $u \in H^1(\Omega_{\text{ref}})$

$$\|u\|^2_{L^2(I)} \leq C(1 + \log p) \|u\|^2_{H^1(\Omega_{\text{ref}})}$$

where I is a line segment in $\Omega_{\text{ref}} := (-1, 1)^3$ which is parallel to a coordinate axis. Their estimate is based on an L_∞ estimate by Babuška, Craig, Mandel and Pitkäranta [7, Theorem 6.2] and by Bramble and Xu [26, Lemma 2.2] which is also used in our proof.

Lemma 2.9 (i) *Let I be a side of $\Gamma_{\text{ref}} := (-1, 1)^2$ and let u be a polynomial of degree p on Γ_{ref}. There exists a constant $C > 0$ such that for any $p \geq 1$*

$$\|u\|^2_{L^2(I)} \leq C(1 + \log p) \|u\|^2_{H^{1/2}(\Gamma_{\text{ref}})}.$$

(ii) *Let I be a line segment in Γ_{ref} which is parallel to a coordinate axis and let u be a polynomial of degree p on Γ_{ref} with $u|_{\partial\Gamma_{\text{ref}}} = 0$. There exists a constant $C > 0$ such that for any $p \geq 2$*

$$\|u\|^2_{L^2(I)} \leq C(1 + \log p) \|u\|^2_{\tilde{H}^{1/2}(\Gamma_{\text{ref}})}.$$

Proof. We will prove part (i) of the lemma. Part (ii) then follows from (i) by splitting Γ_{ref} along I and by making use of (2.4).

Assume that $I = (-1,1) \times \{-1\}$. There holds

$$\begin{aligned}\|u\|^2_{L^2(I)} &= \int_{-1}^{1} u(x, y=-1)^2 \, dx \le \int_{-1}^{1} \|u(x,\cdot)\|^2_{L^\infty(-1,1)} \, dx \\ &\le C(1+\log p) \int_{-1}^{1} \|u(x,\cdot)\|^2_{H^{1/2}(-1,1)} \, dx. \end{aligned} \tag{2.11}$$

The last estimate is due to [7, Theorem 6.2]. The integral $\int_{-1}^{1} \|u(x,\cdot)\|^2_{H^{1/2}(-1,1)} \, dx$ can be considered as an anisotropic norm $\|\cdot\|^2_{L^2(-1,1),H^{1/2}(-1,1)}$ of u. For any extension U of u onto $\mathbb{R}^2$ there holds

$$\begin{aligned}\|u\|^2_{L^2(-1,1),H^{1/2}(-1,1)} \le \|U\|^2_{L^2(\mathbb{R}),H^{1/2}(\mathbb{R})} &\simeq \int_{\mathbb{R}^2} |\mathcal{F}(U)(\xi_1,\xi_2)|^2 (1+\xi_2^2)^{1/2} \, d\xi_1 d\xi_2 \\ &\le \int_{\mathbb{R}^2} |\mathcal{F}(U)(\xi_1,\xi_2)|^2 (1+\xi_1^2+\xi_2^2)^{1/2} \, d\xi_1 d\xi_2 \\ &\simeq \|U\|^2_{H^{1/2}(\mathbb{R}^2)}. \end{aligned} \tag{2.12}$$

Here $\mathcal{F}$ denotes the Fourier transform with $(x,y) \to (\xi_1,\xi_2)$. Therefore, combining (2.11) and (2.12),

$$\|u\|^2_{L^2(I)} \le C(1+\log p) \inf_{U|_{\Gamma_{\rm ref}}=u} \|U\|^2_{H^{1/2}(\mathbb{R}^2)} \simeq (1+\log p)\|u\|^2_{H^{1/2}(\Gamma_{\rm ref})}.$$

□

2.2 Localization

Ansatz functions of the Galerkin method are usually given in a piecewise manner on a mesh consisting of a couple of elements. Most often polynomials are used as the pieces. Therefore, polynomials on elements are the appropriate objects for norm estimates to deal with. On the other hand, in the Sobolev spaces $\tilde{H}^{1/2}$ and H^1, continuity of the piecewise polynomials is required. Thus, we cannot just take restrictions of functions as local representations. We need a localization technique which yields for a given function local representations with zero traces on the boundaries of their supports. The zero traces of the functions allow for extending them by zero onto the whole domain without losing continuity.

The localization will be realized by a partition of unity which simply consists of a number of piecewise bilinear functions. However, multiplication of an ansatz function by a piecewise bilinear function increases the polynomial degree by one. In order to stay within the ansatz space under consideration we need to reduce the polynomial degree again. For this we use polynomial interpolations. We note that our partition of unity actually is the method used by Pavarino [112].

Let us recall and introduce some notations. For a domain $\Gamma \subset \mathbb{R}^2$ let

$$\Gamma_h = \{\Gamma_1, \ldots, \Gamma_J\} \quad \text{with} \quad \bar{\Gamma} = \cup_{j=1}^{J} \bar{\Gamma}_j$$

denote the mesh of rectangular elements Γ_j with side length h. The nodes $\{x_j;\ j = 1, \ldots, J_{V'}\}$ of the mesh are numbered such that the first J_V nodes are in the interior of

Γ. To define our partition of unity we need to take nodes at incoming corners into account. These so-called L-nodes shall be $x_{J_V+j},\ j = 1,\dots,J_L$. The polynomial ansatz space of continuous functions is $S^1_{h,p}(\Gamma)$, see (1.3).

We define a partition of unity $\{\theta_j;\ j = 1,\dots,J_V + J_L\}$ which consists of continuous, piecewise bilinear functions, cf. Figure 2.2:

$$\sum_j \theta_j = 1,\ \operatorname{supp}\theta_j = \bar{\Gamma}'_j,\ 0 \le \theta_j^l \le 1. \tag{2.13}$$

The domain Γ'_j is the union of the elements Γ_i which are adjacent to the node x_j. This implies that the slope of θ_j in coordinate direction is bounded like

$$|\frac{\partial}{\partial_x}\theta_j|, |\frac{\partial}{\partial_y}\theta_j| \le c/h.$$

In the interior of Γ the nodal values of θ_j are uniquely determined by the above relations:

$$\theta_j(x_i) = \delta_{ji}, \quad j,i = 1,\dots,J_V.$$

At the boundary, i.e. $x_i \in \partial\Gamma$, we define, with E_{i_1} and E_{i_2} being the edges on the boundary of Γ'_j which meet at x_i,

$$\theta_j(x_i) = \begin{cases} 1 & \text{if both } E_{j_1} \text{ and } E_{j_2} \text{ are on the boundary of } \Gamma \\ 0 & \text{otherwise} \end{cases}, \quad j = 1,\dots,J_V.$$

To satisfy the condition $\sum_j \theta_j = 1$ we need to introduce additional cut-off functions at the L-points:

$$\theta_j(x_i) = \delta_{ji} \quad j = J_V+1,\dots,J_V+J_L,\ i = 1,\dots,J_{V'}.$$

There holds $\theta_j v \in S^1_{h,p+1}(\Gamma'_j)$ for any $v \in S^1_{h,p}(\Gamma)$ and for any cut-off function θ_j of the partition of unity. Since we need a piecewise polynomial of degree p we use an interpolation operator which interpolates piecewise polynomials of degree $p+1$ by piecewise polynomials of degree p:

$$\Pi_p : S^1_{h,p+1} \to S^1_{h,p}$$

For a given $v \in S^1_{h,p+1}$, the interpolating function $\Pi_p v$ is defined element-wise, by mapping each element onto the reference element $\Gamma_{\text{ref}} := (-1,1)\times(-1,1)$, as the polynomial of degree p which interpolates v at the $(p+1)^2$ points (x_n, y_m) where the x_n's are the zeroes of the polynomial

$$\mathcal{L}_{p+1}(x) := \int_{-1}^{x} L_p(t)\,dt. \tag{2.14}$$

L_p is the Legendre polynomial of degree p.

The next lemma proves the continuity of the interpolation operator Π_p in several norms. We note that the integral cases of the L^2 and H^1_0-norms have already been considered by Pavarino [113, Lemma 5], [112, Lemma 2].

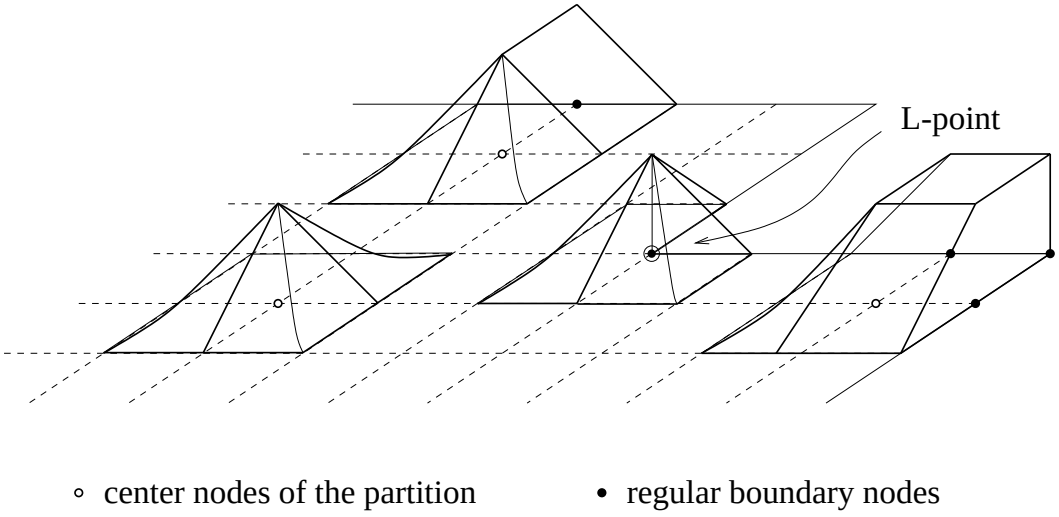


Figure 2.2: Partition of unity: four typical cut-off functions.

Lemma 2.10 *The interpolation operator*

$$\Pi_p : S^1_{h,p+1} \to S^1_{h,p}$$

is continuous with respect to the $L^2(\Gamma)$, $H^1_0(\Gamma)$ *and* $\tilde{H}^{1/2}(\Gamma)$*-norms uniformly in* h *and* p.

Proof. By definition of Π_p this operator maps onto piecewise polynomials of degree p. There are $p+1$ distinct zeroes of $\mathcal{L}_{p+1}$ in $[-1,1]$ and $\mathcal{L}_{p+1}(\pm 1) = 0$. Therefore on each edge E_j of the mesh Γ_h there are $p+1$ interpolation nodes which uniquely define a pth degree interpolation polynomial. The nodes are implicitly defined by mapping back the elements onto the reference element. If the edge E_j is an interior edge there are two adjacent elements and since the transformation is linear the resulting nodes on the edge are the same by taking either of the elements for the local definition. We therefore have continuity across element boundaries and the operator Π_p is well defined.

First we note that Pavarino [112, Lemma 2] has shown that Π_p is continuous with respect to the H^1-semi-norm on the reference element, uniformly in p. Since the operator Π_p is a local mapping we directly obtain the continuity

$$\Pi_p : \big(S^1_{h,p+1}, |\cdot|_{H^1(\Gamma)}\big) \to \big(S^1_{h,p}, |\cdot|_{H^1(\Gamma)}\big).$$

Having shown the continuity of

$$\Pi_p : \big(S^1_{h,p+1}, \|\cdot\|_{L^2(\Gamma)}\big) \to \big(S^1_{h,p}, \|\cdot\|_{L^2(\Gamma)}\big) \tag{2.15}$$

the assertion of the Lemma with respect to the $\tilde{H}^{1/2}(\Gamma)$-norm then follows by interpolation. It therefore remains to prove the $L^2(\Gamma)$-continuity. We note that Pavarino already considered the L^2-norm, see [113, Lemma 5]. However, we give a more explicit proof which shows that the bound of the operator Π_p with respect to the $L^2(\Gamma)$-norm is actually the square of the bound with respect to the $H^1(\Gamma)$-semi-norm. Indeed, our proof uses the technique presented in [112].

As mentioned above Π_p is a local operator and it suffices to prove the continuity locally on the elements. For this we take the reference element $\Gamma_{\text{ref}} = (-1,1)\times(-1,1)$ instead of an arbitrary element Γ_j.

For the space of polynomials of degree p (individually in the space variables) on Γ_{ref} we choose the basis

$$\{\mathcal{L}_i^\star(x)\mathcal{L}_j^\star(y);\ i,j=0,\cdots,p\}$$

where $\mathcal{L}_i^\star := \mathcal{L}_i/\|\mathcal{L}_i\|_{L^2(-1,1)}$ for $i\geq 2$ and $\mathcal{L}_0^\star(x) := 1/\sqrt{2}$, $\mathcal{L}_1^\star(x) := (1+x)\sqrt{3/8}$, cf. (2.14). Then there holds for $\varphi(x,y) := \sum_{i,j=0}^{p+1} c_{ij}\mathcal{L}_i^\star(x)\mathcal{L}_j^\star(y)$

$$\Pi_p(\varphi) = \sum_{i,j=0}^{p} c_{ij}\mathcal{L}_i^\star(x)\mathcal{L}_j^\star(y)$$

and

$$\|\varphi\|^2_{L^2(\Gamma_{\text{ref}})} = \vec{c}^T S\vec{c}, \quad \|\Pi_p(\varphi)\|^2_{L^2(\Gamma_{\text{ref}})} = \vec{c}^T BSB\vec{c}$$

for

$$\vec{c} = (c_{ij})_{(ij)} \in \mathbb{R}^{(p+2)^2}, \quad S = \left(\langle \mathcal{L}_i^\star\mathcal{L}_j^\star, \mathcal{L}_k^\star\mathcal{L}_l^\star\rangle\right)_{(ij,kl)} \in \mathbb{R}^{(p+2)^2\times(p+2)^2}.$$

$B = (b_{ij,kl}) \in \mathbb{R}^{(p+2)^2\times(p+2)^2}$ is the projection matrix which is diagonal for the chosen basis:

$$b_{ij,kl} = \begin{cases} 1 & \text{if } (i,j)=(k,l) \text{ and } \max\{i,j,k,l\} < p+1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\|\Pi_p\|^2_{L^2(\Gamma_{\text{ref}})\to L^2(\Gamma_{\text{ref}})} = \sup_{\varphi\in P_{p+1}} \frac{\|\Pi_p(\varphi)\|^2_{L^2(\Gamma_{\text{ref}})}}{\|\varphi\|^2_{L^2(\Gamma_{\text{ref}})}} = \sup_{\vec{c}\in\mathbb{R}^{(p+2)^2}} \frac{\vec{c}^T BSB\vec{c}}{\vec{c}^T S\vec{c}}$$

which is bounded by the largest eigenvalue of the generalized eigenvalue problem

$$BSB\vec{c} = \lambda S\vec{c}. \tag{2.16}$$

Here, P_{p+1} is the space of polynomials of degree $p+1$ in each variable on $(-1,1)^2$. To be more specific let us fix the ordering of the basis functions via the indices (i,j) by

$$(0,0),(0,1),\ldots,(0,p+1),\ldots,(p+1,0),\ldots,(p+1,p+1).$$

Using the relation

$$\mathcal{L}_j(x) = \int_{-1}^{x} L_{j-1}(t)\,dt = \frac{1}{2j-1}\left(L_j(x) - L_{j-2}(x)\right)$$

and the orthogonality of the Legendre polynomials we obtain

$$A_{p+2} := \left(\langle \mathcal{L}_i^\star, \mathcal{L}_j^\star \rangle\right)_{(i,j)} = \begin{pmatrix} 1 & c_0 & b_0 & & & & & \\ c_0 & 1 & c_1 & b_1 & & & & \\ b_0 & c_1 & 1 & 0 & \cdot & & & \\ & b_1 & 0 & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot & \cdot & b_{p-1} \\ & & & & \cdot & \cdot & \cdot & 0 \\ & & & & & b_{p-1} & 0 & 1 \end{pmatrix}$$

with

$$c_0 = \langle \mathcal{L}_0^\star, \mathcal{L}_1^\star \rangle, \quad c_1 = \langle \mathcal{L}_1^\star, \mathcal{L}_2^\star \rangle, \quad b_0 = \langle \mathcal{L}_0^\star, \mathcal{L}_2^\star \rangle, \quad b_1 = \langle \mathcal{L}_1^\star, \mathcal{L}_3^\star \rangle, \quad b_j = \langle \mathcal{L}_j^\star, \mathcal{L}_{j+2}^\star \rangle \quad (j \geq 2).$$

Now, rewriting $\langle \mathcal{L}_i^\star \mathcal{L}_j^\star, \mathcal{L}_k^\star \mathcal{L}_l^\star \rangle = \langle \mathcal{L}_i^\star, \mathcal{L}_k^\star \rangle \times \langle \mathcal{L}_j^\star, \mathcal{L}_l^\star \rangle$, we find for the whole matrix $S = \left(\langle \mathcal{L}_i^\star \mathcal{L}_j^\star, \mathcal{L}_k^\star \mathcal{L}_l^\star \rangle\right)_{(ij,kl)}$ the structure

$$\begin{aligned} S &= \begin{pmatrix} A_{p+2} & c_0 A_{p+2} & b_0 A_{p+2} & & & & & & \\ c_0 A_{p+2} & A_{p+2} & c_1 A_{p+2} & b_1 A_{p+2} & & & & & \\ b_0 A_{p+2} & c_1 A_{p+2} & A_{p+2} & \mathcal{O} & \cdot & & & & \\ & b_1 A_{p+2} & \mathcal{O} & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \cdot & b_{p-1} A_{p+2} & \\ & & & & \cdot & \cdot & \cdot & \mathcal{O} & \\ & & & & & b_{p-1} A_{p+2} & \mathcal{O} & A_{p+2} & \end{pmatrix} \\ &= \begin{pmatrix} A_{p+2} & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ & & & & & & & A_{p+2} \end{pmatrix} \begin{pmatrix} I & c_0 I & b_0 I & & & & & \\ c_0 I & I & c_1 I & b_1 I & & & & \\ b_0 I & c_1 I & I & \mathcal{O} & \cdot & & & \\ & b_1 I & \mathcal{O} & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot & \cdot & b_{p-1} I \\ & & & & \cdot & \cdot & \cdot & \mathcal{O} \\ & & & & & b_{p-1} I & \mathcal{O} & I \end{pmatrix} \\ &=: S_1 \times S_2 \end{aligned}$$

where $I, \mathcal{O} \in \mathbb{R}^{(p+2)\times(p+2)}$ are the identity matrix and zero matrix, respectively. The same way we obtain for the matrix BSB, which can be reduced from S simply by replacing the

entries by 0 which belong to degrees larger than p,

$$BSB = \begin{pmatrix} \tilde{A}_{p+1} & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \tilde{A}_{p+1} \end{pmatrix} \begin{pmatrix} I & c_0 I & b_0 I & & & & & \\ c_0 I & I & c_1 I & b_1 I & & & & \\ b_0 I & c_1 I & I & \mathcal{O} & \cdot & & & \\ & b_1 I & \mathcal{O} & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \cdot & b_{p-2} I & \\ & & & \cdot & \cdot & \cdot & \cdot & \mathcal{O} \\ & & & & b_{p-2} I & \mathcal{O} & I & \mathcal{O} \\ & & & & & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix}$$
$$= B_1 S_1 B_1 \times B_2 S_2 B_2$$

where $\tilde{A}_{p+1} = \begin{pmatrix} A_{p+1} & 0 \\ 0 & 0 \end{pmatrix}$,

$$B_1 = \begin{pmatrix} B_{1,p+1} & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & B_{1,p+1} \end{pmatrix}, \quad B_{1,p+1} = \begin{pmatrix} 1 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 1 \\ & & & & & 0 \end{pmatrix} \in \mathbb{R}^{(p+2)\times(p+2)},$$

and

$$B_2 = \begin{pmatrix} I & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & I \\ & & & & & \mathcal{O} \end{pmatrix} \in \mathbb{R}^{(p+2)^2\times(p+2)^2}.$$

Thus we can estimate the largest eigenvalue of the generalized eigenvalue problem (2.16), which is now rewritten by

$$BSB\vec{c} = B_1 S_1 B_1 \; B_2 S_2 B_2 \vec{c} = \lambda S_1 S_2 \vec{c},$$

by successively estimating the eigenvalues of the two generalized eigenvalue problems

$$BS_2 B\vec{c} = \mu_2 S_2 \vec{c} \tag{2.17}$$

and

$$BS_1 B\vec{c} = \mu_1 S_1 \vec{c}. \tag{2.18}$$

The eigenvalue problem for S_1 is exactly that treated in [112, Proof of Lemma 2] where a bound for μ_1 is given,

$$\mu_1 < \frac{4}{1-\varepsilon} =: C_\varepsilon \quad (\varepsilon > 0)$$

if p is large enough. The eigenvalue problem for S_2 has the same structure as that for the diagonal blocks A_{p+2} of S_1. The only difference is that all the entries are multiplied by an identity matrix which enlarges the dimensions of the eigenspaces but does not change the eigenvalues. Therefore we conclude the boundedness

$$\mu_2 < C_\varepsilon \quad (\varepsilon > 0)$$

for p large enough. Finally, we obtain

$$\lambda \leq \sup\{\mu_1;\ \mu_1 \text{ is eigenvalue of (2.18)}\}\ \sup\{\mu_2;\ \mu_2 \text{ is eigenvalue of (2.17)}\} \leq C_\varepsilon^2$$

for p large enough. Thus we have shown the continuity of the interpolation operator with respect to the $L^2(\Gamma_{\text{ref}})$-norm. As explained above this local continuity proves (2.15) and the proof of the lemma is finished by interpolating between $L^2(\Gamma)$ and $H_0^1(\Gamma)$. □

In the previous proof we obtained a bound $\sqrt{C_\varepsilon}$ for the norm $\|\Pi_p\|_{H_0^1(\Gamma)\to H_0^1(\Gamma)}$ and a bound C_ε for the norm $\|\Pi_p\|_{L^2(\Gamma)\to L^2(\Gamma)}$. However, these bounds are just valid if the degree p is large enough. By interpolation we obtained the continuity of Π_p as a map from $\tilde{H}^{1/2}(\Gamma)$ onto itself. We use the real K-method of the interpolation theory which provides a so-called exact interpolation functor (see [17, Definition 2.4.3, Theorem 3.1.2]) which means that there holds

$$\|\Pi_p\|_{\tilde{H}^{1/2}(\Gamma)\to\tilde{H}^{1/2}(\Gamma)} \leq C\|\Pi_p\|^{1/2}_{L^2(\Gamma)\to L^2(\Gamma)}\|\Pi_p\|^{1/2}_{H_0^1(\Gamma)\to H_0^1(\Gamma)}$$

with $C = 1$. Therefore from the above proof we obtain

$$\|\Pi_p\|_{\tilde{H}^{1/2}(\Gamma)\to\tilde{H}^{1/2}(\Gamma)} \leq C_\varepsilon^{1/2}C_\varepsilon^{1/4} \to 4^{3/4} \approx 2.83 \quad \text{for } \varepsilon \to 0.$$

This is valid only for large p and does not give an exact bound for general p. However, in the special case $p = 1$ smaller bounds for the norm of Π_p can be obtained. We consider the case of interpolating a function $v \in S^1_{h,2}(\Gamma)$ which is given by $\theta_j w$ for $w \in S^1_{h,1}(\Gamma)$ where θ_j is a cut-off function of the partition of unity (2.13). This is the typical situation we have to deal with when considering the pure h-version of the Galerkin method.

Lemma 2.11 *For any $w \in S^1_{h,1}$ there holds*

$$\|\Pi_1\theta_j w\|_{\tilde{H}^{1/2}(\Gamma)} \leq C\|\theta_j w\|_{\tilde{H}^{1/2}(\Gamma)}$$

uniformly in h and for all cut-off functions θ_j in (2.13)*. Here, C is an arbitrary constant larger than* $(125/36)^{1/4} \approx 1.37$*.*

Proof. As in the proof of Lemma 2.10 we first prove the continuity of Π_1 with respect to the L^2- and H_0^1-norms and then interpolate these results.

Let us consider the reference rectangle $(0, 2h)^2$ and the part $\Gamma_1 := (0, h)^2$ therein. By θ we denote the hat function concentrated at (h, h), i.e. $\theta(x, y) = 1$ for $(x, y) = (h, h)$ and $\theta(x, y) = 0$ at the remaining eight nodes. On Γ_1 the space of continuous, piecewise bilinear functions is spanned by the hat functions φ_1, φ_2, φ_3, φ_4 which are 1 at the nodes $(0, 0)$, $(h, 0)$, (h, h), $(0, h)$, respectively. Then, on Γ_1, any $w \in S^1_h$ can be represented by

$$w = \chi_1\varphi_1 + \cdots + \chi_4\varphi_4$$

with $\chi_i \in \mathbb{R}$, $i = 1, \dots, 4$. Further we have

$$\theta w = \varphi_3 w \quad \text{on } \Gamma_1$$

and

$$\Pi_1 \theta w = \chi_3 \varphi_3 \quad \text{on } \Gamma_1.$$

Defining $\tilde{\varphi}_1(x) := 1 - x/h$ and $\tilde{\varphi}_2(x) := x/h$ we rewrite the basis functions by

$$\begin{aligned} \varphi_1(x,y) &= \tilde{\varphi}_1(x)\,\tilde{\varphi}_1(y), \quad \varphi_2(x,y) = \tilde{\varphi}_2(x)\,\tilde{\varphi}_1(y), \\ \varphi_3(x,y) &= \tilde{\varphi}_2(x)\,\tilde{\varphi}_2(y), \quad \varphi_4(x,y) = \tilde{\varphi}_1(x)\,\tilde{\varphi}_2(y), \end{aligned}$$

and compute

$$\|\Pi_1 \theta w\|^2_{L^2(\Gamma_1)} = \|\chi_3 \varphi_3\|^2_{L^2(\Gamma_1)} = \|\chi_3 \tilde{\varphi}_2(x) \tilde{\varphi}_2(y)\|^2_{L^2(\Gamma_1)} = \chi_3^2 \frac{h^2}{9}$$

and

$$\begin{aligned} &\|\theta w\|^2_{L^2(\Gamma_1)} = \|\chi_1\varphi_1\varphi_3 + \chi_2\varphi_2\varphi_3 + \chi_3\varphi_3^2 + \chi_4\varphi_3\varphi_4\|^2_{L^2(\Gamma_1)} \\ &= \frac{h^2}{1800}\big(2\chi_1^2 + 12\chi_2^2 + 72\chi_3^2 + 12\chi_4^2 + 6\chi_1\chi_2 + 9\chi_1\chi_3 + 6\chi_1\chi_4 + 36\chi_2\chi_3 + 9\chi_2\chi_4 + 36\chi_3\chi_4\big). \end{aligned}$$

We show that there exists a constant $C_1 > 0$ such that

$$C_1 \|\Pi_1 \theta w\|^2_{L^2(\Gamma_1)} \le \|\theta w\|^2_{L^2(\Gamma_1)}. \tag{2.19}$$

We define

$$\begin{aligned} f_C(\chi_1,\chi_2,\chi_3,\chi_4) &:= \frac{1800}{h^2}\left(\|\theta w\|^2_{L^2(\Gamma_1)} - C\,\|\Pi_1\theta w\|^2_{L^2(\Gamma_1)}\right) \\ &= 2\chi_1^2 + 12\chi_2^2 + (72 - 200C)\chi_3^2 + 12\chi_4^2 + 6\chi_1\chi_2 + 9\chi_1\chi_3 \\ &\qquad + 6\chi_1\chi_4 + 36\chi_2\chi_3 + 9\chi_2\chi_4 + 36\chi_3\chi_4 \end{aligned}$$

and find

$$\nabla f_C(\chi_1,\chi_2,\chi_3,\chi_4) = \begin{pmatrix} 4 & 6 & 9 & 6 \\ 6 & 24 & 36 & 9 \\ 9 & 36 & 16(9-25C) & 36 \\ 6 & 9 & 36 & 24 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{pmatrix}.$$

Obviously, $f_C(\chi)$ tends to $+\infty$ for $|\chi| \to \infty$ if $C < 72/200 = 9/25$ and since ∇f_C is a multi-linear function the only extreme of f_C in $\mathbb{R}^4$ is a minimum at 0. Thus

$$f_C(\chi) \ge \inf_{\chi \in \mathbb{R}^4} f_C(\chi) = f_C(0) = 0 \quad (C < 9/25)$$

and therefore (2.19) holds for $C_1 < 9/25$.

Now we show that there exists a constant $C_2 > 0$ such that

$$C_2 \|\Pi_1 \theta w\|^2_{H^1_0(\Gamma_1)} \le \|\theta w\|^2_{H^1_0(\Gamma_1)}. \tag{2.20}$$

Analogously as before we find

$$\|\Pi_1\theta w\|^2_{H^1_0(\Gamma_1)} = |\Pi_1\theta w|^2_{H^1(\Gamma_1)} = \frac{2}{3}\chi_3^2$$

and

$$\|\theta w\|^2_{H^1_0(\Gamma_1)} = \frac{1}{90}\left(2\chi_1^2 + 10\chi_2^2 + 48\chi_3^2 + 10\chi_4^2 + \chi_1\chi_2 - 6\chi_1\chi_3 + \chi_1\chi_4 - 6\chi_2\chi_4\right).$$

Defining

$$\begin{aligned} f_C(\chi_1,\chi_2,\chi_3,\chi_4) &:= 90\left(\|\theta w\|^2_{H^1_0(\Gamma_1)} - C\|\Pi_1\theta w\|^2_{H^1_0(\Gamma_1)}\right) \\ &= 2\chi_1^2 + 10\chi_2^2 + (48-60C)\chi_3^2 + 10\chi_4^2 + \chi_1\chi_2 - 6\chi_1\chi_3 + \chi_1\chi_4 - 6\chi_2\chi_4 \end{aligned}$$

we find the multi-linear gradient

$$\nabla f_C(\chi_1,\chi_2,\chi_3,\chi_4) = \begin{pmatrix} 4 & 1 & -6 & 1 \\ 1 & 20 & 0 & -6 \\ -6 & 0 & 24(4-5C) & 0 \\ 1 & -6 & 0 & 20 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{pmatrix}.$$

Again, $f_C(\chi)$ tends to $+\infty$ for $|\chi| \to \infty$ as long as $C < 48/60 = 4/5$. Thus

$$f_C(\chi) \geq \inf_{\chi\in\mathbb{R}^4} f_C(\chi) = f_C(0) = 0 \quad (C < 4/5)$$

and therefore (2.20) holds for $C_2 < 4/5$.

Obviously the estimates (2.19) and (2.20) are valid on the whole reference rectangle $(0,2h)^2$ and we obtain via interpolation

$$\|\Pi_1\theta_j w\|_{\tilde{H}^{1/2}(\Gamma)} \leq C\|\theta_j w\|_{\tilde{H}^{1/2}(\Gamma)}, \quad j = 1,\dots,J_V,$$

for a constant

$$C \geq (C_1C_2)^{-1/4} > (36/125)^{1/4} \approx 1.37$$

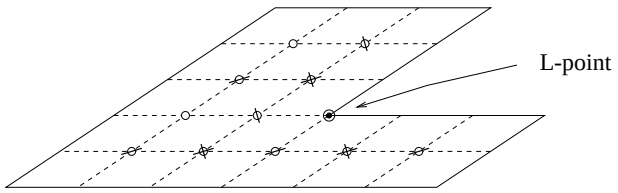
which is independent of h. □

Next we define the localization operator by

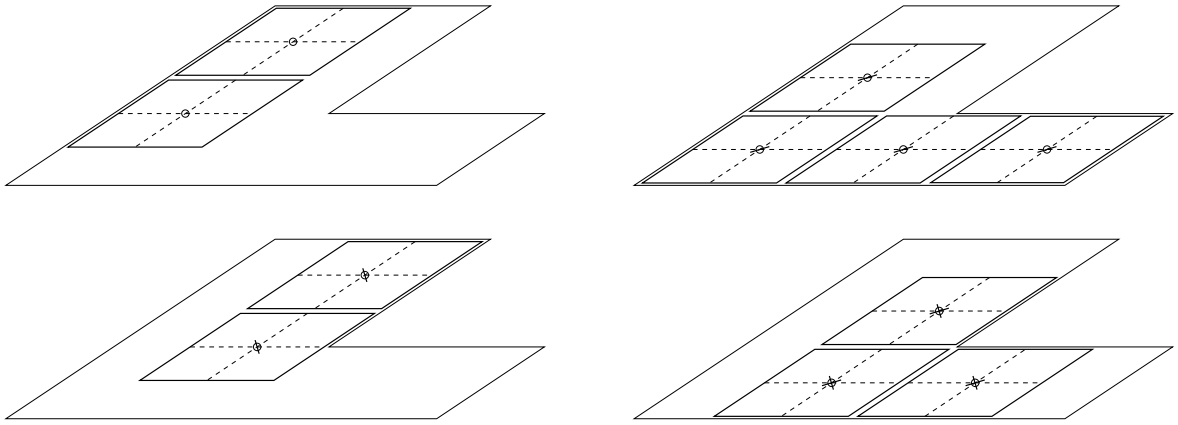
$$\Lambda : \left\{ \begin{array}{ccc} S^1_{h,p}(\Gamma) & \to & S^1_{h,1}(\Gamma) \times S^1_{h,p}(\Gamma_1') \times \cdots \times S^1_{h,p}(\Gamma'_{J_V+J_L}) \\ v & \mapsto & \left((\Lambda v)_j\right)_{j=0}^{J_V+J_L} := \left(Q_h v, \Pi_p\theta_1(v - Q_h v), \dots, \Pi_p\theta_{J_V+J_L}(v - Q_h v)\right) \end{array} \right. .$$

Here, $Q_h v$ is the $L^2(\Gamma)$-projection of v onto $S^1_{h,1}(\Gamma)$, the ansatz space of piecewise bilinear functions.

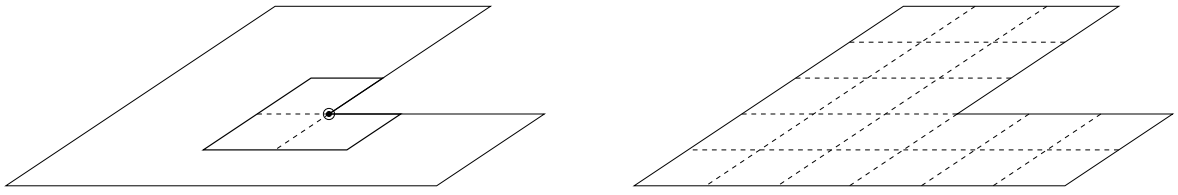
The operator Λ provides, with the exception of a piecewise bilinear contribution, an energy conserving localization of the ansatz space $S^1_{h,p}(\Gamma)$:



The mesh with marked interior nodes



Local subspaces defined by the overlapping decomposition



Additional subspace for each L-point (temporary)

Additional subspace of piecewise bilinear functions

Figure 2.3: The decomposition of $S^1_{h,p}(\Gamma)$ induced by the localization operator Λ.

Theorem 2.1 (Localization, preliminary version) *The operator Λ provides a localization of the space $S^1_{h,p}(\Gamma)$, i.e. for any $v \in S^1_{h,p}(\Gamma)$ there holds $v = \sum_{j=0}^{J_V+J_L}(\Lambda v)_j$. Further there exist positive constants C_1 and C_2 which are independent of h and p such that*

$$C_1\|v\|^2_{\tilde{H}^{1/2}(\Gamma)} \le \sum_{j=0}^{J_V+J_L} \|(\Lambda v)_j\|^2_{\tilde{H}^{1/2}(\Gamma)} \le C_2\|v\|^2_{\tilde{H}^{1/2}(\Gamma)}. \tag{2.21}$$

Proof. Let $v \in S^1_{h,p}(\Gamma)$ be given. Then $(\Lambda v)_0 = Q_h v \in S^1_{h,1}(\Gamma)$ by definition of the projection operator Q_h. Further, since $\theta_j(v - Q_h v) = 0$ in $\Gamma \setminus \Gamma'_j$ and since Π_p interpolates polynomials of degree less than $p+1$ exactly (i.e., the function 0 remains 0 on elements), there holds $(\Lambda v)_j = \Pi_p\theta_j(v - Q_h v) \in S^1_{h,p}(\Gamma'_j)$ for $j = 1, \dots, J_V + J_L$. Thus, the operator Λ is well defined. Due to the linearity of Π_p and since $\sum_j \theta_j = 1$, there holds

$$\sum_{j=0}^{J_V+J_L}(\Lambda v)_j = Q_h v + \sum_{j=1}^{J_V+J_L}\Pi_p\theta_j(v - Q_h v) = Q_h v + \Pi_p(v - Q_h v) = v.$$

Using the identity above we now prove the left inequality in (2.21). We start with the triangle inequality

$$\|v\|^2_{\tilde{H}^{1/2}(\Gamma)} \le 2\big(\|(\Lambda v)_0\|^2_{\tilde{H}^{1/2}(\Gamma)} + \|\sum_{j=1}^{J_V+J_L}(\Lambda v)_j\|^2_{\tilde{H}^{1/2}(\Gamma)}\big).$$

The second term on the right hand side can be estimated with the help of Lemma 2.3. Since the supports of the components $(\Lambda v)_j$ of Λv consist of at most four rectangles the covering constant N_c of that lemma is bounded by 4 and we obtain

$$\|\sum_{j=1}^{J_V+J_L}(\Lambda v)_j\|^2_{\tilde{H}^{1/2}(\Gamma)} \le c\sum_{j=1}^{J_V+J_L}\|(\Lambda v)_j\|^2_{\tilde{H}^{1/2}(\Gamma'_j)} = c\sum_{j=1}^{J_V+J_L}\|(\Lambda v)_j\|^2_{\tilde{H}^{1/2}(\Gamma)}.$$

The last two estimates yield

$$\|v\|^2_{\tilde{H}^{1/2}(\Gamma)} \le c\sum_{j=0}^{J_V+J_L}\|(\Lambda v)_j\|^2_{\tilde{H}^{1/2}(\Gamma)}. \tag{2.22}$$

It remains to determine the constant C_2 in (2.21). We show preliminary estimates for the $L^2(\Gamma)$ and $H^1_0(\Gamma)$-norms separately and then use interpolation to switch to the $\tilde{H}^{1/2}(\Gamma)$-norm.

Using the continuity of Π_p with respect to the L^2-norm on an element Γ_k (see Lemma 2.10) we obtain for $v \in S^1_{h,p}(\Gamma)$ and $w := v - Q_h v$

$$\|(\Lambda v)_j\|^2_{L^2(\Gamma_k)} = \|\Pi_p\theta_j w\|^2_{L^2(\Gamma_k)} \le c\|\theta_j w\|^2_{L^2(\Gamma_k)} \le c\|w\|^2_{L^2(\Gamma_k)}, \quad j = 1, \cdots, J_V + J_L.$$

Since at most four functions $(\Lambda v)_j$ are nonzero on Γ_k we have by summing over j

$$\sum_{j=1}^{J_V+J_L}\|(\Lambda v)_j\|^2_{L^2(\Gamma_k)} \le c\|w\|^2_{L^2(\Gamma_k)}$$

and, summing over k,

$$\sum_{j=1}^{J_V+J_L} \|(\Lambda v)_j\|^2_{L^2(\Gamma'_j)} \leq c\|w\|^2_{L^2(\Gamma)} \leq c\|v\|^2_{L^2(\Gamma)}. \tag{2.23}$$

The last estimate is due to the triangle inequality and due to the boundedness of the L^2-projection operator Q_h.

Now, again by Lemma 2.10, we obtain for the H^1-semi-norm analogously as before

$$|(\Lambda v)_j|^2_{H^1(\Gamma_k)} = |\Pi_p \theta_j w|^2_{H^1(\Gamma_k)} \leq c|\theta_j w|^2_{H^1(\Gamma_k)} \leq c\Big(2/h^2\|w\|^2_{L^2(\Gamma_k)} + |w|^2_{H^1(\Gamma_k)}\Big),$$

$j = 1, \cdots, J_V + J_L$. Again, by summing over j,

$$\sum_{j=1}^{J_V+J_L} |(\Lambda v)_j|^2_{H^1(\Gamma_k)} \leq c\Big(2/h^2\|w\|^2_{L^2(\Gamma_k)} + |w|^2_{H^1(\Gamma_k)}\Big)$$

and by summing over k

$$\sum_{j=1}^{J_V+J_L} |(\Lambda v)_j|^2_{H^1(\Gamma'_j)} \leq c\Big(2/h^2\|w\|^2_{L^2(\Gamma)} + |w|^2_{H^1(\Gamma)}\Big).$$

Estimating

$$\|w\|^2_{L^2(\Gamma)} = \|v - Q_h v\|^2_{L^2(\Gamma)} \leq ch^2|v|^2_{H^1(\Gamma)} \tag{2.24}$$

and using the boundedness of the L^2-projection with respect to the H^1-semi-norm (see, e.g., [26] for both results) we thus obtain

$$\sum_{j=1}^{J_V+J_L} |(\Lambda v)_j|^2_{H^1(\Gamma'_j)} \leq c|v|^2_{H^1(\Gamma)}. \tag{2.25}$$

By a coloring argument we can split the sums in (2.23) and (2.25) into a finite number N_c of sums such that any two functions in a sum have disjoint supports:

$$\sum_{j\in I_i} \|(\Lambda v)_j\|^2_{L^2(\Gamma'_j)} \leq c\|v\|^2_{L^2(\Gamma)}, \quad i = 1, \ldots, N_c \tag{2.26}$$

and

$$\sum_{j\in I_i} |(\Lambda v)_j|^2_{H^1(\Gamma'_j)} \leq c|v|^2_{H^1(\Gamma)}, \quad i = 1, \ldots, N_c \tag{2.27}$$

where $I_1 \cup \cdots \cup I_{N_c} = \{1, \ldots, J_V + J_L\}$. With the help of Lemma 2.1 we obtain by interpolating (2.26) and (2.27)

$$\begin{aligned}
\sum_{j\in I_i} \|(\Lambda v)_j\|^2_{\tilde{H}^{1/2}(\Gamma'_j)} &= \|(\Lambda v)_{j\in I_i}\|^2_{\Pi_{j\in I_i}[L^2(\Gamma'_j), H^1_0(\Gamma'_j)]_{1/2}} \\
&\leq c\,\|(\Lambda v)_{j\in I_i}\|^2_{[\Pi_{j\in I_i} L^2(\Gamma'_j), \Pi_{j\in I_i} H^1_0(\Gamma'_j)]_{1/2}} \\
&\leq c\,\|v\|^2_{[L^2(\Gamma), H^1_0(\Gamma)]_{1/2}} = c\,\|v\|^2_{\tilde{H}^{1/2}(\Gamma)}.
\end{aligned}$$

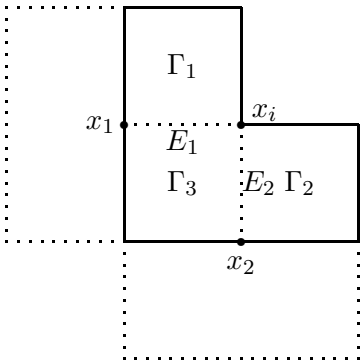


Figure 2.4: Decomposition of $S^1_{h,p}(\Gamma'_i)$ for an L-point x_i.

Finally, summing the above estimate over $i = 1, \dots, N_c$ and using the boundedness of the L^2-projection with respect to the $\tilde{H}^{1/2}$-norm,

$$\|(\Lambda v)_0\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{j=1}^{J_V+J_L} \|(\Lambda v)_j\|^2_{\tilde{H}^{1/2}(\Gamma'_j)} \le c\|v\|^2_{\tilde{H}^{1/2}(\Gamma)}$$

which is the right inequality in (2.21). □

The next theorem shows that the additional components at the L-points in the localized representation of $S^1_{h,p}(\Gamma)$ are actually not necessary. They are just due to the definition of the partition of unity.

Theorem 2.2 (Localization, final version) *For any $v \in S^1_{h,p}(\Gamma)$ there exist $v_0 \in S^1_{h,1}(\Gamma)$ and $v_j \in S^1_{h,p}(\Gamma'_j)$, $j = 1, \dots, J_V$, such that $v = \sum_{j=0}^{J_V} v_j$ and*

$$C_1\|v\|^2_{\tilde{H}^{1/2}(\Gamma)} \le \sum_{j=0}^{J_V} \|v_j\|^2_{\tilde{H}^{1/2}(\Gamma)} \le C_2\|v\|^2_{\tilde{H}^{1/2}(\Gamma)}. \tag{2.28}$$

Here, C_1 and C_2 are positive constants which are independent of h and p.

Proof. We modify the proof of Theorem 2.1 by showing that the components of a function $v \in S^1_{h,p}(\Gamma)$ which belong to L-points can be represented by components which belong to interior nodes. A typical situation is given in Figure 2.4: The L-point x_i is a node of the elements Γ_1, Γ_2 and Γ_3 and x_1 and x_2 are interior nodes.

For a given $v \in S^1_{h,p}(\Gamma)$ we define $w_1 \in S^1_{h,p}(\Gamma'_1)$ and $w_2 \in S^1_{h,p}(\Gamma'_2)$ such that $v_i = \Pi_p\theta_i v = w_1 + w_2$ and

$$\|w_1\|^2_{\tilde{H}^{1/2}(\Gamma)} + \|w_2\|^2_{\tilde{H}^{1/2}(\Gamma)} \le C\|v_i\|^2_{\tilde{H}^{1/2}(\Gamma)}. \tag{2.29}$$

If x_i is the only L-point, i.e. $J_L = 1$, we then deduce from (2.29) by using the results of

Theorem 2.1 that

$$\begin{aligned}\|(\Lambda v)_0\|^2_{\tilde H^{1/2}(\Gamma)} + \|(\Lambda v)_1 + w_1\|^2_{\tilde H^{1/2}(\Gamma)} + \|(\Lambda v)_2 + w_2\|^2_{\tilde H^{1/2}(\Gamma)} + \sum_{j=3}^{J_V} \|(\Lambda v)_j\|^2_{\tilde H^{1/2}(\Gamma)}\\ \le c\Big(\sum_{j=0}^{J_V} \|(\Lambda v)_j\|^2_{\tilde H^{1/2}(\Gamma)} + \|w_1\|^2_{\tilde H^{1/2}(\Gamma)} + \|w_2\|^2_{\tilde H^{1/2}(\Gamma)}\Big)\\ \le c\sum_{j=0}^{J_V+1} \|(\Lambda v)_j\|^2_{\tilde H^{1/2}(\Gamma)} \le c\|v\|^2_{\tilde H^{1/2}(\Gamma)}.\end{aligned}$$

We also obtain, by applying the triangle inequality and Lemma 2.3 just as in the proof of Theorem 2.1,

$$\begin{aligned}\|v\|^2_{\tilde H^{1/2}(\Gamma)} \le c\|(\Lambda v)_0\|^2_{\tilde H^{1/2}(\Gamma)} + \|(\Lambda v)_1 + w_1\|^2_{\tilde H^{1/2}(\Gamma)}\\ + \|(\Lambda v)_2 + w_2\|^2_{\tilde H^{1/2}(\Gamma)} + \sum_{j=3}^{J_V} \|(\Lambda v)_j\|^2_{\tilde H^{1/2}(\Gamma)}.\end{aligned}$$

More than one L-point can be handled the same way. It therefore remains to prove an estimate of the form (2.29). This can be done by taking the proof of [112, Lemma 3] by Pavarino who showed that there exist $w_1 \in S^1_{h,p}(\Gamma_1)$ and $w_2 \in S^1_{h,p}(\Gamma_2)$ such that $v_i = w_1 + w_2$ and

$$|w_1|^2_{H^1(\Gamma_1')} + |w_2|^2_{H^1(\Gamma_2')} \le C|v_i|^2_{H^1(\Gamma_i')}.$$

For the convenience of the reader we recall that proof and we will see that the estimates also hold for the L^2-norm instead of the H^1-semi-norm. The estimate (2.29) then follows by interpolation.

We define

$$w_1 = \begin{cases} v_i & \text{in } \Gamma_1 \cup E_1\\ \text{reflection of } v_i \text{ across } E_1 & \text{in } \Gamma_3\\ 0 & \text{elsewhere.}\end{cases}$$

Since w_1 vanishes at the boundary of $\Gamma_1 \cup \Gamma_3 \cup E_1$ we have $w_1 \in S^1_{h,p}(\Gamma_1')$ and

$$|w_1|^2_{H^1(\Gamma_1')} = 2|w_1|^2_{H^1(\Gamma_1)} = 2|v_i|^2_{H^1(\Gamma_1)} \le 2|v_i|^2_{H^1(\Gamma_i')}$$

and

$$\|w_1\|^2_{L^2(\Gamma_1')} = 2\|w_1\|^2_{L^2(\Gamma_1)} = 2\|v_i\|^2_{L^2(\Gamma_1)} \le 2\|v_i\|^2_{L^2(\Gamma_i')}.$$

$v_i - w_1$ vanishes on E_1 and by the triangle inequality

$$|v_i - w_1|_{H^1(\Gamma_i')} \le 2|v_i|_{H^1(\Gamma_i')}$$

and

$$\|v_i - w_1\|_{L^2(\Gamma_i')} \le 2\|v_i\|_{L^2(\Gamma_i')}.$$

We can therefore define

$$w_2 = \begin{cases} v_i - w_1 & \text{in } \Gamma_2 \cup \Gamma_3 \cup E_2 \\ 0 & \text{elsewhere} \end{cases}$$

with the required properties. □

2.3 Discrete harmonic polynomials

Now we introduce a special type of polynomials that will be used to construct the ansatz space $S^1_{h,p}(\Gamma)$ when dealing with iterative substructuring preconditioners for operators of order one. The used basis functions which are tensor products of these polynomials are only weakly coupled, i.e., neglecting the mixed terms in stiffness matrices then yield preconditioners which are almost independent of the polynomial degree. In the FEM, the strong coupling of standard basis functions (e.g., of tensor products of anti-derivatives of Legendre polynomials) with respect to the H^1-inner product, especially of functions associated with edges and with the interior of elements, is well-known, see, e.g., [8]. In order to overcome this problem in the construction of preconditioners often a Schur complement step is used to decouple the strongly coupled functions (which amounts to a basis transformation). This step is equivalent to eliminate one set of the strongly coupled functions. For more details see, e.g., [7, 100, 8]. Here, we construct basis functions which are a-priori only weakly coupled. These basis functions have previously been used by Canuto and Funaro [29] for the spectral method and by Pavarino and Widlund for the FEM [114]. In fact, our methods are based on the work of Pavarino and Widlund. However, we propose a more rigorous decomposition and eventually we have to deal with estimates in $\tilde{H}^{1/2}$ exclusively.

First, let us define polynomials of one variable which are the discrete counterparts of tensor product solutions of the Laplace equation.

Let P^p be the space of polynomials of degree p on $(-1,1)$ and let P^p_0 be the subspace of polynomials that vanish at the endpoints of the interval.

Definition 2.1 *Let φ_0 be the polynomial of degree p satisfying*

$$\|\varphi_0\|_{L^2(-1,1)} = \min_{\varphi} \|\varphi\|_{L^2(-1,1)}, \quad \varphi_0(1) = 1,\ \varphi_0(-1) = 0.$$

We also define $\varphi_0^-(x) \equiv \varphi_0(-x)$.

Definition 2.2 *Let $\Phi_i \in P^p_0$ and $\lambda_i^{(p)}$, $i = 1, \ldots, p-1$, be the eigenfunctions and eigenvalues defined by*

$$\int_{-1}^{1} \Phi_i'(x) v'(x)\, dx = \lambda_i^{(p)} \int_{-1}^{1} \Phi_i(x) v(x)\, dx \quad \forall v \in P^p_0$$

and $|\Phi_i|_{H^1(-1,1)} = 1$, and denote $\Phi_i^-(x) \equiv \Phi_i(-x)$.

Definition 2.3 *Let* $\{\lambda_i^{(p)}\}_{i=1,\dots,p-1}$ *be the eigenvalues of Definition* 2.2. *Define a set* $\{\varphi_i\}_{i=1,\dots,p-1}$ *of polynomials of degree* p *by*

$$\int_{-1}^{1} \varphi_i'(x)v'(x)\,dx + \frac{\lambda_i^{(p)}}{2}\int_{-1}^{1} \varphi_i(x)v(x)\,dx = 0 \quad \forall v \in P_0^p$$

and $\varphi_i(-1) = 0,\ \varphi_i(1) = 1$. *We use also the notation* $\varphi_i^-(x) \equiv \varphi_i(-x)$.

The next type of polynomials will be used only for technical reasons to extend boundary element functions to finite element functions as are used in [114].

Definition 2.4 *Let* $\{\lambda_i^{(p)}\}_{i=1,\dots,p-1}$ *be the eigenvalues of Definition* 2.2. *Define a set* $\{\varphi_{ij}\}_{i,j=1,\dots,p-1}$ *of polynomials of degree* p *by*

$$\int_{-1}^{1} \varphi_{ij}'(x)v'(x)\,dx + (\lambda_i^{(p)} + \lambda_j^{(p)})\int_{-1}^{1} \varphi_{ij}(x)v(x)\,dx = 0 \quad \forall v \in P_0^p$$

and $\varphi_{ij}(-1) = 0,\ \varphi_{ij}(1) = 1$. *We use the notation* $\varphi_{ij}^-(x) \equiv \varphi_{ij}(-x)$.

We now describe our basis functions on the reference element $\Gamma_{\text{ref}} = (-1,1)^2$. They are tensor products of the polynomials defined above.

- The *interior basis functions* are defined by

$$f_{ij}(x,y) = \Phi_i(x)\Phi_j(y),\ i,j = 1,\dots,p-1.$$

- One of the sets of *edge basis functions*, for the edge $E_1 = \{(x,y);\ y = -1\}$, is given by

$$e_i^{(1)}(x,y) = \Phi_i(x)\varphi_i^-(y),\ i = 1,\dots,p-1.$$

  (The remaining edges are $E_2 = \{(x,y);\ x = 1\}$, $E_3 = \{(x,y);\ y = 1\}$, and $E_4 = \{(x,y);\ x = -1\}$ and the corresponding edge basis functions are defined accordingly.)

- One of the four *vertex basis functions*, the one for vertex $V_1 = (-1,-1)$, is given by

$$v^{(1)}(x,y) = \varphi_0^-(x)\varphi_0^-(y).$$

  (The remaining vertices are $V_2 = (1,-1)$, $V_3 = (1,1)$, and $V_4 = (-1,1)$ and the respective vertex basis functions are defined accordingly.)

One of the interior basis functions for $p = 8$ is shown in Figure 2.5. Of course, it looks much alike a standard bubble basis function which are tensor products of anti-derivatives of Legendre polynomials. An edge basis function $e_i^{(2)}$ (the part on Γ_{ref}) for the edge $E_2 = \{(x,y);\ x = 1\}$ can be seen in Figure 2.6 and the vertex basis function $v^{(3)}$ is given in Figure 2.7, both for the case $p = 8$. Note that the nodal and the edge basis functions behave like a pth degree polynomial in both x and y directions. In the standard case nodal basis functions are piecewise bilinear and edge basis functions behave linearly in the direction perpendicular to the edge they are associated with. Indeed, for $p = 8$ there are 49 interior basis functions and 7 edge basis functions for each edge.

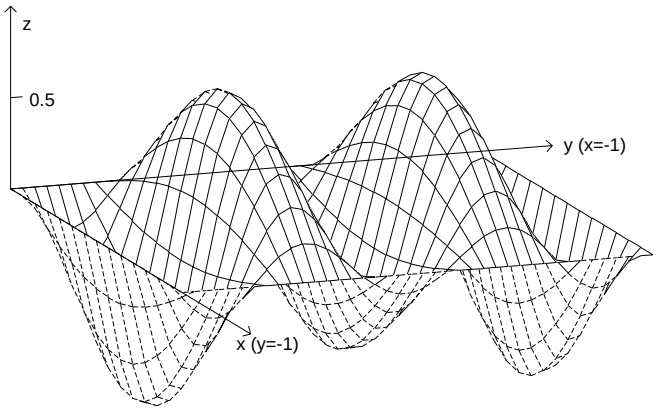


Figure 2.5: Discrete harmonic functions: an interior basis function for $p = 8$.

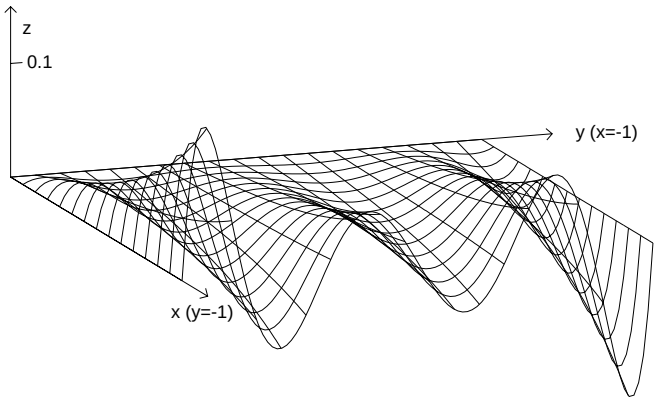


Figure 2.6: Discrete harmonic functions: an edge basis function for $p = 8$.

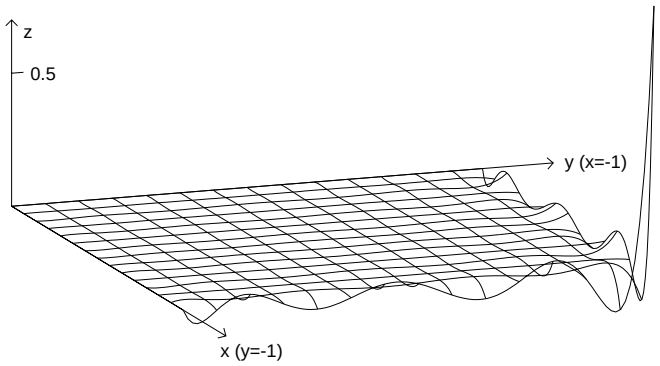


Figure 2.7: Discrete harmonic functions: vertex basis function for $p = 8$.

2.3.1 Representation of boundary element functions

In this section we specify for any boundary element function $u \in S^1_{h,p}(\Gamma)$ its representation in terms of the special basis functions defined above. This representation is used to decompose any $u \in S^1_{h,p}(\Gamma)$ according to the decomposition of $S^1_{h,p}(\Gamma)$ into subspaces for the iterative substructuring method (cf. Section 3.2.3). Let $u \in S^1_{h,p}(\Gamma)$ be given on the mesh $\bar{\Gamma}_h = \cup_{j=1}^J \bar{\Gamma}_j$. u is defined by mapping each element Γ_j back to the reference element Γ_{ref}. On Γ_{ref} we use the same function name u again. We use the decomposition

$$u = u_W + u_I \quad \text{on } \Gamma_{\text{ref}}$$

which is the sum of a vertex-edge component u_W, which is called the wire basket component, and an interior component which vanishes at the boundary of Γ_{ref}.

- *Vertex components:* For the vertices V_k of Γ_{ref} let

$$\begin{aligned} u_{V_1} &= u(-1,-1)\, v^{(1)} \\ u_{V_2} &= u(1,-1)\, v^{(2)} \\ u_{V_3} &= u(1,1)\, v^{(3)} \\ u_{V_4} &= u(-1,1)\, v^{(4)} \end{aligned}$$

  The vertex component of u then is

$$u_V = \sum_{k=1}^{4} u_{V_k}.$$

- *Edge components:* Since $u - u_V$ vanishes at the vertices of Γ_{ref} it can be expanded, on each edge E_k, in the $\{\Phi_i\}$ basis: $u - u_V = \sum_{i=1}^{p-1} \alpha_i^{(k)} \Phi_i$ with

$$\alpha_i^{(k)} = \lambda_i^{(p)} \int_{E_k} (u - u_V)\Phi_i \, ds.$$

Let

$$u_{E_1} = \sum_{i=1}^{p-1} \alpha_i^{(1)} e_i^{(1)}, \quad \text{for } E_1 = \{(x,y);\ y = -1\}$$

$$u_{E_2} = \sum_{i=1}^{p-1} \alpha_i^{(2)} e_i^{(2)}, \quad \text{for } E_2 = \{(x,y);\ x = 1\}$$

$$u_{E_3} = \sum_{i=1}^{p-1} \alpha_i^{(3)} e_i^{(3)}, \quad \text{for } E_3 = \{(x,y);\ y = 1\}$$

$$u_{E_4} = \sum_{i=1}^{p-1} \alpha_i^{(4)} e_i^{(4)}, \quad \text{for } E_4 = \{(x,y);\ x = -1\}.$$

The edge component of u then is

$$u_E = \sum_{k=1}^{4} u_{E_k}.$$

- *Wire basket component:* The wire basket component of u is defined as

$$u_W = u_V + u_E.$$

- *Interior component:* The interior component of u is given by

$$u_I = u - u_W$$

and vanishes on the boundary of Γ_{ref} by construction. Therefore it can be expanded in the $\{\Phi_i \Phi_j\}$ basis: $u_I = \sum_{i,j=1}^{p-1} \beta_{ij} f_{ij}$ with

$$\beta_{ij} = \lambda_i^{(p)} \lambda_j^{(p)} \int_{\Gamma_{\text{ref}}} (u - u_W) f_{ij}(x,y) \, d(x,y).$$

2.3.2 Extensions of boundary element functions

We will use some of the technical lemmas given by Pavarino and Widlund in [114]. In order to do so we need to extend the boundary element functions given in Section 2.3.1 to the functions used in [114]. This is done in the following and, thereafter, we derive all the local estimates which are used to prove the extension theorem (Theorem 2.3) and to prove the efficiency of iterative substructuring methods in Section 3.2.3.

As described in Section 2.3.1 we represent a boundary element function $u \in S^1_{h,p}(\Gamma)$ as

$$u = u_V + u_E + u_I$$

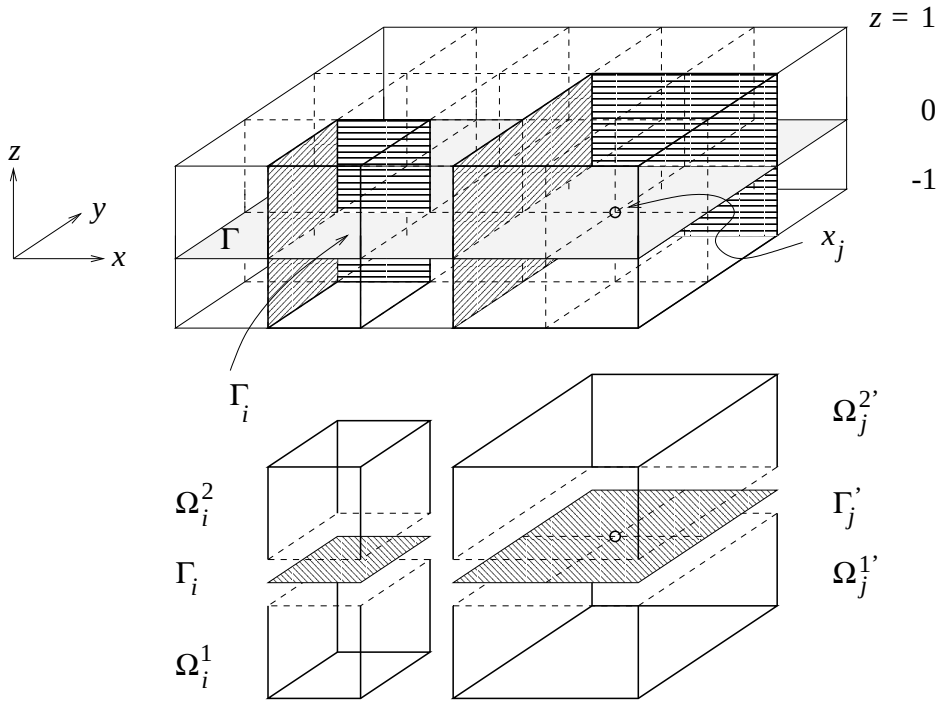


Figure 2.8: Extension of $S^1_{h,p}(\Gamma)$: extended domain Ω with mesh Ω_h.

where the vertex, edge and interior components u_V, u_E and u_I are defined locally by mapping each element Γ_j of Γ_h to the reference element $\Gamma_{\text{ref}} = (-1,1)^2$. Analogously we extend the components separately by extending them locally on Γ_{ref} onto $\Omega_{\text{ref}} = \Gamma_{\text{ref}} \times (-1,1)$. The wire basket of Ω_{ref} will be denoted by W_{ref}. Using the same notation we identify Γ_{ref} and $\Omega_{\text{ref}}|_{z=1}$. By mapping Ω_{ref} back to the elements $\Gamma_j \times (-h,0)$ and $\Gamma_j \times (0,h)$ such that Γ_{ref} falls onto $\Gamma_j \times \{0\}$ we implicitly get an extended mesh Ω_h of Γ_h with $\Gamma_h = \Omega_h|_{\{z=0\}}$. The extended mesh consists of $2J$ elements

$$\bar{\Omega}_h = \cup_{j=1}^J \bar{\Omega}_j = \cup_{j=1}^J \left(\bar{\Omega}_j^1 \cup \bar{\Omega}_j^2\right)$$

where Ω_j^1 and Ω_j^2 are the two brick elements adjacent to the two dimensional element Γ_j, see Figure 2.8.

The extensions of the boundary element functions of $S^1_{h,p}(\Gamma)$ onto Ω_h are given locally by defining the extensions on Γ_{ref} onto functions on Ω_{ref} as follows.

- *Extensions of vertex components:* For the vertex basis functions we define their extensions by

  $$\hat{v}^{(k)}(x,y,z) = v^{(k)}(x,y)\,\varphi_0(z), \quad k = 1,\ldots,4,$$

  where the function φ_0 is given by Definition 2.1. The extensions of the vertex components are

  $$\hat{u}_{V_k} = u(V_k)\,\hat{v}^{(k)}, \quad k = 1,\ldots,4.$$

The extension of the vertex component of u then is

$$\hat{u}_V = \sum_{k=1}^{4} \hat{u}_{V_k}.$$

- *Extensions of edge components:* For the edge basis functions we define their extensions by

$$\hat{e}_i^{(k)}(x,y,z) = e_i^{(k)}(x,y)\,\varphi_i(z), \quad k = 1,\dots,4,\ i = 1,\dots,p-1,$$

where the functions φ_i are given by Definition 2.3. We obtain extensions of the edge components

$$\hat{u}_{E_k} = \sum_{i=1}^{p-1} \alpha_i^{(k)} \hat{e}_i^{(k)}, \quad k = 1,\dots,4.$$

The coefficients $\alpha_i^{(k)}$ are the same as in Section 2.3.1. The extension of the edge component of u then is

$$\hat{u}_E = \sum_{k=1}^{4} \hat{u}_{E_k}.$$

- *Extension of the wire basket component:* The extension of the wire basket component of u is defined as

$$\hat{u}_W = \hat{u}_V + \hat{u}_E.$$

- *Extension of the interior component:* For the interior basis functions we define their extensions by

$$\hat{f}_{ij}(x,y,z) = f_{ij}(x,y)\,\varphi_{ij}(z), \quad i,j = 1,\dots,p-1,$$

where the functions φ_{ij} are that of Definition 2.4. The extension of the interior component is then given by

$$\hat{u}_I = \sum_{i,j=1}^{p-1} \beta_{ij} \hat{f}_{ij}.$$

The coefficients β_{ij} are the same as in Section 2.3.1.

Definition 2.5 *For defining the extension $\mathcal{E}(u)$ of a boundary element function $u \in S^1_{h,p}(\Gamma)$ we represent u via*

$$u = u_0 + \sum_{j=1}^{J_V} u_j$$

according to the construction in the proof of Theorem 2.2. The global piecewise bilinear component u_0 of u is extended onto Ω by extending the bilinear basis functions on Γ to trilinear functions on Ω that vanish on $\partial\Omega$. The components u_j of u are extended locally on the reference element Γ_{ref} by

$$\mathcal{E}(u_j) := \hat{u}_j := \hat{u}_{j,W} + \hat{u}_{j,I}.$$

We note that we have $\hat{u}|_{\partial\Omega} = 0$ due to the boundary conditions of the extension functions $\varphi_0(-1) = \phi_i(-1) = \varphi_{ij}(-1) = 0$ and since $u|_{\partial\Gamma} \equiv 0$. Therefore,

$$\mathcal{E}(u) \in H_0^1(\Omega) \quad \text{for all} \quad u \in S_{h,p}^1(\Gamma).$$

Further, since $\varphi_0(1) = \phi_i(1) = \varphi_{ij}(1) = 1$, we have

$$\mathcal{E}(u)|_\Gamma = u \quad \text{for all} \quad u \in S_{h,p}^1(\Gamma),$$

i.e. $\mathcal{E}$ is an extension operator.

The next theorem which gives a bound for the extension operator can be used, together with the trace theorem, to prove the efficiency of domain decomposition methods for the BEM which are given by the trace of domain decomposition methods for the FEM. Indeed, all the results by Pavarino and Widlund [114] can be extended to the BEM. However, these results then comprise an additional term $\log^2 p$. Therefore, we will not use the extension theorem for the proof of the efficiency of iterative substructuring methods (using discrete harmonic basis functions) in Section 3.2.3 but we will invoke sharper estimates of local nature for that proof. Nevertheless, we give the result for the extension operator which is of interest in itself and which can also be used to study the convergence of the p-version of the mortar element method in three dimensions, cf. [123] for the hp-version in two dimensions.

Theorem 2.3 (Extension) *For the extension $\mathcal{E}(u)$ of a piecewise polynomial $u \in S_{h,p}^1(\Gamma)$ from Γ onto $\Omega = \Gamma \times (-h, h)$ with (using the same notation) $\Gamma = \{(x, y, 0)^T;\ (x, y, z)^T \in \Omega\}$ there holds*

$$|\mathcal{E}(u)|_{H^1(\Omega)} \leq C(1 + \log p)\|u\|_{\tilde{H}^{1/2}(\Gamma)}.$$

The constant C does not depend on p and the mesh size h.

The proof of this theorem is based on the localization procedure of Section 2.2 (Theorem 2.1) and on local traces and extensions. First we collect the required local estimates.

Lemma 2.12 (Corollary 5.5, Lemma 5.6, Lemma 5.4 in [114]) *Let $u = u_W + u_I$ be a polynomial of degree p on Γ_{ref} where u_W and u_I are its wire basket component and interior component, respectively. Further, let u_{E_k} denote a single edge component of u. By $\hat{u}$, $\hat{u}_W$, $\hat{u}_I$, and $\hat{u}_{E_k}$ we denote the extensions onto Ω_{ref} of u, u_W, u_I, and u_{E_k}, respectively, as defined above. Then there holds*

$$|\hat{u}_I|^2_{H^1(\Omega_{\text{ref}})} \leq C\|u_I\|^2_{\tilde{H}^{1/2}(\Gamma_{\text{ref}})}, \tag{2.30}$$

$$|\hat{u}_W|^2_{H^1(\Omega_{\text{ref}})} \leq C\|u\|^2_{L^2(W_{\text{ref}})}, \tag{2.31}$$

$$|\hat{u}_{E_k}|^2_{H^1(\Omega_{\text{ref}})} \leq C\|u_{E_k}\|^2_{L^2(E_k)}. \tag{2.32}$$

Further, for the nodal basis functions $v^{(k)}$ and their extensions $\hat{v}^{(k)}$ there holds

$$|\hat{v}^{(k)}|^2_{H^1(\Omega_{\text{ref}})} \leq C\|v^{(k)}\|^2_{L^2(W_{\text{ref}})}. \tag{2.33}$$

The next three lemmas present bounds of the different components of piecewise polynomials locally on a reference element $\tilde{\Gamma}_{\rm ref} := (-2,2)^2$ which consists of four translations of the reference element $\Gamma_{\rm ref}$,

$$\text{closure}(\tilde{\Gamma}_{\rm ref}) = \text{closure}(\cup_{j=1}^4 \Gamma_{{\rm ref},j}).$$

We denote the interior edges of $\tilde{\Gamma}_{\rm ref}$ by $\tilde{E}_k$, $k = 1, \ldots, 4$.

The following lemma gives bounds for norms of the nodal component of a boundary element function.

Lemma 2.13 *Let u_V denote the vertex component of a function $u \in S_p^1(\tilde{\Gamma}_{\rm ref})$. There exists a constant $C > 0$ such that*

$$\sum_{k=1}^4 \|u_V\|^2_{L^2(\tilde{E}_k)} \le C(1+\log p)\|u\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\rm ref})} \tag{2.34}$$

and

$$\|u_V\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\rm ref})} \le C(1+\log p)\|u\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\rm ref})}. \tag{2.35}$$

Proof. Define a polynomial $\tilde{v} \in S_p^1(\tilde{\Gamma}_{\rm ref})$ by

$$\|\tilde{v}\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\rm ref})} = \min_{\varphi \in S_p^1(\tilde{\Gamma}_{\rm ref})} \|\varphi\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\rm ref})}, \quad \tilde{v}(0,0) = 1. \tag{2.36}$$

This new vertex basis function may not coincide with the vertex basis function v concentrated at the interior vertex $(0,0)$ as defined in Section 2.3. By using Definition 2.1, Lemma 2.9 (i), property (2.4) and the condition (2.36) we obtain

$$\begin{aligned}
\sum_{k=1}^4 \|u_V\|^2_{L^2(\tilde{E}_k)} &= \sum_{k=1}^4 \|u(0,0)v\|^2_{L^2(\tilde{E}_k)} \le \sum_{k=1}^4 \|u(0,0)\tilde{v}|_{\tilde{E}_k}\|^2_{L^2(\tilde{E}_k)} \\
&\le C(1+\log p)\sum_{k=1}^4 \|u(0,0)\tilde{v}\|^2_{H^{1/2}(\Gamma_{{\rm ref},k})} \\
&\le C(1+\log p)\|u(0,0)\tilde{v}\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\rm ref})} \le C(1+\log p)\|u\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\rm ref})}.
\end{aligned}$$

This proves (2.34). To prove (2.35) we take the three dimensional extension $\tilde{\Omega}_{\rm ref} = (-2,2)^3$ of $\tilde{\Gamma}_{\rm ref}$ which consists of eight elements $\Omega_{{\rm ref},k}$, $k = 1, \ldots, 8$. The eight elements are translations of the reference cube $\Omega_{\rm ref}$. Due to the trace theorem we have

$$\|u_V\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\rm ref})} \le C\|\hat{u}_V\|^2_{H_0^1(\tilde{\Omega}_{\rm ref})} = C\sum_{k=1}^8 |\hat{u}_V|^2_{H^1(\Omega_{{\rm ref},k})}.$$

Then we use (2.33) which gives for the nodal component u_V

$$|\hat{u}_V|^2_{H^1(\Omega_{{\rm ref},k})} \le C\|u_V\|^2_{L^2(W_{{\rm ref},k})}$$

where $W_{\mathrm{ref},k}$ is the wire basket of $\Omega_{\mathrm{ref},k}$ and obtain, by noting $\hat{u}_V|_{\partial\tilde{\Omega}_{\mathrm{ref}}} \equiv 0$, that

$$\|u_V\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\mathrm{ref}})} \leq C \sum_{k=1}^{4} \|u_V\|^2_{L^2(\tilde{E}_k)}.$$

Then (2.34) completes the proof. □

The next lemma presents L^2-estimates for a boundary element function and its edge component.

Lemma 2.14 *Let u_E denote the edge component of a function $u \in S^1_p(\tilde{\Gamma}_{\mathrm{ref}})$. There exists a constant $C > 0$ such that*

$$\sum_{k=1}^{4} \|u\|^2_{L^2(\tilde{E}_k)} \leq C(1+\log p)\|u\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\mathrm{ref}})} \tag{2.37}$$

and

$$\sum_{k=1}^{4} \|u_E\|^2_{L^2(\tilde{E}_k)} \leq C(1+\log p)\|u\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\mathrm{ref}})}. \tag{2.38}$$

Proof. Due to Lemma 2.9 (i) there holds

$$\sum_{k=1}^{4} \|u\|^2_{L^2(\tilde{E}_k)} \leq C(1+\log p) \sum_{k=1}^{4} \|u\|^2_{H^{1/2}(\Gamma_{\mathrm{ref},k})}.$$

Since

$$\sum_{k=1}^{4} \|u\|^2_{H^{1/2}(\Gamma_{\mathrm{ref},k})} \leq C\|u\|^2_{H^{1/2}(\tilde{\Gamma}_{\mathrm{ref}})} \leq C\|u\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\mathrm{ref}})},$$

cf. (2.4), the proof of (2.37) is complete.

Since $u = u_V + u_E$ on the edges of $\tilde{\Gamma}_{\mathrm{ref}}$ we obtain, by applying the triangle inequality, (2.34) and (2.37),

$$\begin{aligned}\sum_{k=1}^{4} \|u_E\|^2_{L^2(\tilde{E}_k)} &\leq C\left(\sum_{k=1}^{4} \|u_V\|^2_{L^2(\tilde{E}_k)} + \sum_{k=1}^{4} \|u\|^2_{L^2(\tilde{E}_k)}\right) \\ &\leq C(1+\log p)\|u\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\mathrm{ref}})}.\end{aligned}$$

This proves (2.38). □

Now we prove the required local energy estimates for the components of a boundary element function.

Lemma 2.15 (i) *Let u_W denote the wire basket component and $u_I^{(k)}$ denote the component which is interior to $\Gamma_{\mathrm{ref},k}$ ($k = 1, \ldots, 4$) of a function $u \in S_p^1(\tilde{\Gamma}_{\mathrm{ref}})$. There exists a constant $C > 0$ such that*

$$\|u_W\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\mathrm{ref}})} \leq C(1 + \log p)\|u\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\mathrm{ref}})} \tag{2.39}$$

and

$$\|u_I^{(k)}\|^2_{\tilde{H}^{1/2}(\Gamma_{\mathrm{ref},k})} \leq C(1 + \log p)^2\|u\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\mathrm{ref}})}, \quad k = 1, \ldots, 4. \tag{2.40}$$

(ii) *Let $u_{\tilde{E}_k} = \sum_{i=1}^{p-1} \alpha_i \tilde{e}_i^{(k)}$ be the edge component on a single edge $\tilde{E}_k$ of $\tilde{\Gamma}_{\mathrm{ref}}$. Here, $\tilde{e}_i^{(k)}$ is an edge basis function transformed onto $\tilde{E}_k$. There exists a constant $C > 0$ such that*

$$\|u_{\tilde{E}_k}\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\mathrm{ref}})} \leq C(1 + \log p)\|u\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\mathrm{ref}})}, \quad k = 1, \ldots, 4, \tag{2.41}$$

and

$$\sum_{i=1}^{p-1} \|\alpha_i \tilde{e}_i^{(k)}\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\mathrm{ref}})} \leq C(1 + \log p)\|u\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\mathrm{ref}})}, \quad k = 1, \ldots, 4. \tag{2.42}$$

(iii) *Let $u_{\Gamma_{\mathrm{ref},k}} = \sum_{i,j=1}^{p-1} \alpha_{ij} f_{ij}^{(k)}$ be the interior component on the element $\Gamma_{\mathrm{ref},k}$ of $\tilde{\Gamma}_{\mathrm{ref}}$. Here, $f_{ij}^{(k)}$ is the interior basis function f_{ij} transformed onto $\Gamma_{\mathrm{ref},k}$. There exists a constant $C > 0$ such that*

$$\sum_{i,j=1}^{p-1} \|\alpha_{ij} f_{ij}^{(k)}\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\mathrm{ref}})} \leq C(1 + \log p)^2\|u\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\mathrm{ref}})}, \quad k = 1, \ldots, 4. \tag{2.43}$$

Proof. By successively using the trace theorem and the estimate (2.31) we obtain

$$\|u_W\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\mathrm{ref}})} \leq C\|\hat{u}_W\|^2_{H_0^1(\tilde{\Omega}_{\mathrm{ref}})} = \sum_{j=1}^{8} |\hat{u}_W|^2_{H^1(\Omega_{\mathrm{ref},j})} \leq C \sum_{j=1}^{8} \|\hat{u}_W\|^2_{L^2(W_{\mathrm{ref},j})}.$$

By checking the proof of (2.31) (cf. [114, Lemma 5.6]) one can see that it is actually sufficient to take the L^2-norm on the edges of $\tilde{\Gamma}_{\mathrm{ref}}$ instead of the L^2-norm on the whole wire basket of $\tilde{\Omega}_{\mathrm{ref}}$. Therefore, since $u = u_W$ on the edges of $\tilde{\Gamma}_{\mathrm{ref}}$, by applying (2.37)

$$\|u_W\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\mathrm{ref}})} \leq C \sum_{j=1}^{4} \|u_W\|^2_{L^2(\tilde{E}_j)} \leq C(1 + \log p)\|u\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\mathrm{ref}})}.$$

This proves (2.39).

To prove (2.40) we note that from the proof of Lemma 5.7 in [114] we obtain

$$\begin{aligned} \|u_I^{(k)}\|^2_{\tilde{H}^{1/2}(\Gamma_{\mathrm{ref},k})} &= \|u - u_W\|^2_{\tilde{H}^{1/2}(\Gamma_{\mathrm{ref},k})} \\ &\leq c(1 + \log p)(\max_x \|u(x,\cdot)\|^2_{L^2(I_y)} + \max_y \|u(\cdot,y)\|^2_{L^2(I_x)}). \end{aligned}$$

Here, I_x and I_y are intervals of the x-arguments and y-arguments, respectively, used for the rectangle $\Gamma_{\mathrm{ref},k}$. By Lemma 2.9 we bound the last term by

$$\begin{aligned}\max_x \|u(x,\cdot)\|^2_{L^2(I_y)} + \max_y \|u(\cdot,y)\|^2_{L^2(I_x)} &\le c(1+\log p)\|u\|^2_{H^{1/2}(\Gamma_{\mathrm{ref},k})}\\ &\le c(1+\log p)\|u\|^2_{\tilde H^{1/2}(\tilde\Gamma_{\mathrm{ref}})}\end{aligned}$$

which yields, together with the estimate before, the assertion (2.40).

To bound the norm of a single edge component $u_{\tilde E_k}$ we use the trace theorem, the estimate (2.32)

$$|\hat u_{\tilde E_k}|^2_{H^1(\tilde\Omega_{\mathrm{ref}})} \le C\|u_{\tilde E_k}\|^2_{L^2(\tilde E_k)}$$

for the extension $\hat u_{\tilde E_k}$ of $u_{\tilde E_k}$ and the estimate (2.38) to conclude

$$\|u_{\tilde E_k}\|^2_{\tilde H^{1/2}(\tilde\Gamma_{\mathrm{ref}})} \le |\hat u_{\tilde E_k}|^2_{H^1(\tilde\Omega_{\mathrm{ref}})} \le C\|u_{\tilde E_k}\|^2_{L^2(\tilde E_k)} \le C(1+\log p)\|u\|^2_{\tilde H^{1/2}(\tilde\Gamma_{\mathrm{ref}})}.$$

This is the required estimate (2.41). For the proof of (2.42) we use the $H^1(\Omega_{\mathrm{ref}})$-orthogonality of the extensions of the edge basis functions $\hat e_j^{(k)}$, $j=1,\dots,p-1$, on a specific edge (cf. [114]) and obtain analogously as above

$$\sum_{i=1}^{p-1}\|\alpha_i\tilde e_i^{(k)}\|^2_{\tilde H^{1/2}(\tilde\Gamma_{\mathrm{ref}})} \le \sum_{i=1}^{p-1}|\alpha_i\hat{\tilde e}_i^{(k)}|^2_{H^1(\tilde\Omega_{\mathrm{ref}})} = |\hat u_{\tilde E_k}|^2_{H^1(\tilde\Omega_{\mathrm{ref}})} \le C(1+\log p)\|u\|^2_{\tilde H^{1/2}(\tilde\Gamma_{\mathrm{ref}})}.$$

Analogously, the estimate (2.43) is obtained by the $H^1(\Omega_{\mathrm{ref}})$-orthogonality of the extensions of the interior basis functions (cf. [114]) and by making use of (2.30) and (2.40). □

Proof of Theorem 2.3. As defined above the mesh Γ_h is extended to a mesh Ω_h in $\mathbb{R}^3$ such that the elements Γ_j are faces of the cubes Ω_j^1 and Ω_j^2. We introduce the notations $\Omega_j^{1\prime}$ and $\Omega_j^{2\prime}$ for extensions of the domains Γ_j'. The domain Γ_j' is the union of the elements $\Gamma_{j,k}$ adjacent to the node x_j. We use the triangle inequality and a coloring argument to localize the H^1-semi-norm of an extended function to the family $\{\Omega_j';\ j=1,\dots,J_V\}$ of overlapping patches where

$$\Omega_j' := \mathrm{interior}\big(\Omega_j^{1\prime}\cup\Gamma_j'\cup\Omega_j^{2\prime}\big).$$

Since the extension operator is defined by extending the projection onto the piecewise bilinears and the basis functions on Γ to functions on Ω it is a linear operator. We obtain, by using the representation $u = u_0 + \sum_{j=1}^{J_V} u_j$ for a given function $u\in S^1_{h,p}(\Gamma)$,

$$|\mathcal{E}(u)|^2_{H^1(\Omega)} = \Big|\mathcal{E}\big(\sum_{j=0}^{J_V}u_j\big)\Big|^2_{H^1(\Omega)} = \Big|\sum_{j=0}^{J_V}\mathcal{E}\big(u_j\big)\Big|^2_{H^1(\Omega)} \le c\sum_{j=0}^{J_V}\Big|\mathcal{E}\big(u_j\big)\Big|^2_{H^1(\Omega)}. \tag{2.44}$$

First we note that by the discrete harmonic extension theorem, see [150, 23, 20],

$$\Big|\mathcal{E}\big(u_0\big)\Big|^2_{H^1(\Omega)} \le c\|u_0\|^2_{\tilde H^{1/2}(\Gamma)}. \tag{2.45}$$

(This is trivial due to the $H_0^1(\Omega)$-orthogonality of $\mathcal{E}\big(u_0\big)$ to the piecewise trilinear functions on Ω that vanish on Γ and on $\partial\Omega$, which is an empty set of functions.) Now we consider the remaining components u_j, $j = 1, \ldots, J_V$. We use the abbreviations $u_{j,W}$ and $u_{j,I}$ for the wire basket and interior components of u_j, respectively. For the wire basket components we apply (2.31) and (2.37) and obtain, by considering the transformation rules by Lemma 2.4,

$$\begin{aligned}|\mathcal{E}(u_{j,W})|^2_{H^1(\Omega)} = |\mathcal{E}(u_{j,W})|^2_{H^1(\Omega'_j)} &\simeq h|\mathcal{E}(u_{j,W}) \circ (T_h^3)^{-1}|^2_{H^1(\tilde{\Omega}_{\rm ref})} \le ch\|u_j \circ (T_h^3)^{-1}\|^2_{L^2(\tilde{W}_{\rm ref})} \\ &\le ch(1+\log p)\|u_j \circ (T_h^2)^{-1}\|^2_{\tilde{H}^{1/2}(\tilde{\Gamma}_{\rm ref})} \simeq (1+\log p)\|u_j\|^2_{\tilde{H}^{1/2}(\Gamma'_j)}. \qquad (2.46)\end{aligned}$$

More precisely in the above estimate we first must decompose the domain Ω'_j into its cubic elements and then take the affine map of the elements individually onto the reference cube. After that we can collect the L^2-norm contributions from the individual edges obtained by (2.31) and apply the estimate (2.37).

Analogously we obtain from (2.30) and (2.40) for the interior components

$$|\mathcal{E}(u_{j,I})|^2_{H^1(\Omega)} = |\mathcal{E}(u_{j,I})|^2_{H^1(\Omega'_j)} \le c\sum_{k=1}^{4} \|u_{j,I}\|^2_{\tilde{H}^{1/2}(\Gamma_{j,k})} \le c(1+\log p)^2\|u_j\|^2_{\tilde{H}^{1/2}(\Gamma'_j)}. \qquad (2.47)$$

Combining (2.44), (2.45), (2.46), and (2.47) and making use of the Localization Theorem 2.1 we conclude

$$\begin{aligned}|\mathcal{E}(u)|^2_{H^1(\Omega)} &\le c\sum_{j=0}^{J_V}\Big|\mathcal{E}\big(u_j\big)\Big|^2_{H^1(\Omega)} = c\Big|\mathcal{E}\big(u_0\big)\Big|^2_{H^1(\Omega)} + c\sum_{j=1}^{J_V}\Big|\mathcal{E}\big(u_{j,W}+u_{j,I}\big)\Big|^2_{H^1(\Omega)} \\ &\le c\|u_0\|^2_{\tilde{H}^{1/2}(\Gamma)} + c(1+\log p)^2\sum_{j=1}^{J_V}\|u_j\|^2_{\tilde{H}^{1/2}(\Gamma'_j)} \le c(1+\log p)^2\|u\|^2_{\tilde{H}^{1/2}(\Gamma)}\end{aligned}$$

which is the assertion of Theorem 2.3. □

Chapter 3

Preconditioners

We propose methods for the solution of linear systems arising from the boundary element method for solving first kind integral equations (or systems of integral equations). Since we focus on the solutions of large linear systems our solvers will be iterative methods. Among these there are stationary (e.g., Jacobi, Gauss-Seidel, successive overrelaxation) and non-stationary types (e.g., conjugate gradient, minimum residual, generalized minimum residual). The former types use the same matrix throughout the iteration process whereas the latter types adapt the iteration matrix during the iteration. Here, we focus on non-stationary methods which are usually more effective than stationary ones.

The rate at which an iterative method converges depends greatly on the spectrum of the coefficient matrix A. For operators of order α the boundary element method leads to linear systems with spectral condition numbers κ behaving like

$$\kappa(A) = O(h^{-|\alpha|}p^{2|\alpha|})\kappa(M),$$

see, e.g., [70, 86]. Here, h and p are the sizes of the elements and the polynomial degree of the ansatz functions, respectively, of the ansatz space for the Galerkin method, and $\kappa(M)$ is the spectral condition number of the mass matrix. For the dependence of the condition number on h see, e.g., Hsiao and Wendland [83]. Their proof is based on the order of the operator and on the inverse property of the basis functions of the ansatz space. Therefore, the dependence on p follows in the same manner by making use of the inverse inequality of Lemma 2.7 which deals with piecewise polynomials of arbitrary degree, see also [70]. The condition number of the mass matrix depends on the basis functions. In the case $\alpha = -1$ one can choose L^2-orthonormal functions and then $\kappa(M)$ is constant. In the case $\alpha = 1$ continuity of the basis functions is needed. For standard tensor product shape functions based on antiderivatives of scaled Legendre polynomials one has $\kappa(M) = O(p^8)$, see [86]. However, taking a closer look, one finds $\kappa(A) = O(h^{-1}p^6)$ in this case. By the above relations especially the p-version of the boundary element method may yield stiffness matrices with large spectral condition numbers and therefore, preconditioners are important for the fast solution of the arising linear systems. Here we note that, even though the condition number increases faster for the p-version than for the h-version, it is nevertheless worth while to use the p-version since it converges twice as fast as the h-version (see [134]) and even better for problems with smooth solutions. This will also be underlined by our numerical experiments, see Section 4.2 and Figure 4.12.

The method of choice for solving symmetric positive definite linear systems is the conjugate gradient method (CG). However, only some special problems give rise to symmetric

positive definite stiffness matrices. In general, we have to deal with non-symmetric, indefinite matrices. There are various methods for solving these types of linear systems. A classical idea is to take the normal equations in order to derive a symmetric positive definite system and to apply the CG method. However, in that case the spectral condition number will be squared and the resulting iterative method may become slow. We will use the generalized minimum residual method (GMRES) as test solver for our model problems. The GMRES method is generally applicable to non-symmetric systems and leads to the smallest residual for a fixed number of iterations. This method was introduced by Saad and Schultz [116] and is mathematically equivalent to the generalized conjugate residual method (GCR) [49]. For an overview of iterative solvers and their relations to one another we refer to [12] and the references therein.

Let us consider the linear system and some properties of the GMRES method in more detail. We have to solve a linear system $Au = g$ where $A \in \mathbb{R}^{N\times N}$ is the stiffness matrix of a Galerkin system and $g \in \mathbb{R}^N$ is the right hand side vector. (The complex case $A \in \mathbb{C}^{N\times N}$ can be dealt with analogously.) When considering the boundary element method for an integral operator A^α (or a system thereof) of order α this matrix stems from testing all the basis functions of an ansatz space X_N of dimension N against themselves with respect to the bilinear form $\langle A^\alpha \cdot, \cdot\rangle = \langle A^\alpha \cdot, \cdot\rangle_{L^2(\Gamma)}$ where Γ is the domain of definition of A^α. For non-selfadjoint and/or indefinite operators the matrix A can be non-symmetric or indefinite.

Given an initial guess u_0 with initial residual $r_0 = g - Au_0$, the jth iterate u_j of the GMRES method is computed such that

$$\|g - Au_j\|_c = \min\{\|g - Av\|_c;\ v \in K_j(r_0)\}$$

where $K_j(r_0)$ is the Krylov subspace

$$K_j(r_0) = \operatorname{span}\left\{r_0, Ar_0, \dots, A^{j-1}r_0\right\}.$$

Here $\|\cdot\|_c$ is the norm induced by an appropriate inner product $c(\cdot,\cdot)$ in $\mathbb{R}^N$.

The rate of convergence of the GMRES method depends on the two numbers

$$\Lambda_0 = \inf_{v\in\mathbb{R}^N} \frac{c(v, Av)}{c(v, v)}, \quad \text{and} \quad \Lambda_1 = \sup_{v\in\mathbb{R}^N} \frac{\|Av\|_c}{\|v\|_c}. \tag{3.1}$$

The quantity Λ_0 is the minimum eigenvalue of the symmetric part of A and Λ_1 is the matrix norm of A which is induced by the norm $\|\cdot\|_c$. When A is a complex matrix Λ_0 is defined to be the minimum eigenvalue of the Hermitian part of A (with respect to $c(\cdot,\cdot)$), i.e.,

$$\Lambda_0 = \inf_{v\in\mathbb{C}^N} \Re\frac{c(v, Av)}{c(v, v)}.$$

Theorem 3.1 ([49]) *If $\Lambda_0 > 0$ then the GMRES method converges and for the jth iterate residual $r_j = g - Au_j$ there holds*

$$\|r_j\|_c \le \left(1 - \frac{\Lambda_0^2}{\Lambda_1^2}\right)^{j/2} \|r_0\|_c.$$

In order to reduce the number of iterations of the GMRES method we need to find a preconditioner for the matrix A which bounds the ratio Λ_0/Λ_1 from below. In general, for non-selfadjoint integral operators which lead to non-symmetric matrices, the parameters Λ_0 and Λ_1 depend on the inner product $c(\cdot,\cdot)$ that is taken for the GMRES method. In most cases we will take the discrete counter part of the inner product of the energy space which is given by the order of the integral operator. In practise, the stiffness matrix being obtained by considering only the main part of the operator can be taken.

In the special case of block skew-symmetric systems we will also consider the inner product which is given by a preconditioner, see Section 3.5.2. This will be explained in the following.

Given a symmetric positive definite preconditioning matrix B^{-1} there are basically two approaches to define a preconditioned GMRES method. Either one applies the GMRES method directly to the system

$$B^{-1}Au = B^{-1}g$$

or one uses the symmetric formulation

$$B^{-1/2}AB^{-1/2}z = B^{-1/2}g, \quad z = B^{1/2}u.$$

Mund and Stephan have shown that both formulations are equivalent when using the appropriate inner products.

Theorem 3.2 (Theorem 2.1 in [103]) *Let u_j be the jth iterate of the GMRES method for the solution of $Au = g$ when using an inner product $(\cdot,\cdot)$ and initial guess u_0. Let $\hat{u}_j$ be the jth iterate of the GMRES method for the solution of $\big(S^{-1}AS\big)\hat{u} = S^{-1}b$ when using the inner product $(\cdot,\cdot)_{S^TS} := (S\cdot, S\cdot)$ and initial guess $\hat{u}_0 = S^{-1}u_0$. Then $\hat{u}_j = S^{-1}u_j$.*

This theorem can be used as follows. A symmetric positive definite preconditioner B^{-1} can be written as $B^{-1} = EE^T$. When using the GMRES method for the preconditioned system $B^{-1}Au = B^{-1}g$ and taking the inner product $(\cdot,\cdot)_B = (\cdot, B\cdot)_{l_2}$ this is equivalent to solving $E^TAE\hat{u} = E^Tg$ and using the l_2-inner product $(\cdot,\cdot)_{l_2}$. It is therefore possible to implicitly solve the symmetrically preconditioned system with the GMRES method based on the l_2-inner product.

In the next section we give a short introduction to the additive Schwarz method which will be used to define and analyze preconditioners for the linear systems arising from the boundary element method. In Sections 3.2 and 3.3 we exclusively deal with selfadjoint positive definite integral operators of orders one and minus one, respectively. For operators of order one we analyze two- and multilevel methods for the pure h-version which give bounded or almost bounded condition numbers (Theorems 3.4, 3.5). For the p-version we propose an overlapping method which leads to bounded condition numbers (Theorem 3.6). By using discrete harmonic basis functions as defined in Section 2.3 any decomposition of the ansatz space plus a coarse space of bilinear functions yields almost optimal results (Theorem 3.7). These methods are called iterative substructuring methods. For operators of order minus one we propose a two-level method which is nearly optimal for both the p-version and the pure h-version (lowest polynomial degrees), cf. Theorem 3.8 and Corollary 3.1. In Section 3.4 we show that the methods for positive definite operators lead also to efficient methods when considering non-selfadjoint or indefinite operators which satisfy a Gårding inequality. For this

we use abstract results for the additive Schwarz method applied to indefinite systems and for a hybrid method which is more general than the additive Schwarz method (Theorems 3.9, 3.10, and 3.11). We then consider in detail the overlapping method and the iterative substructuring methods for the p-version dealing with operators of order one (Theorems 3.12, 3.13, and 3.14) and the two-level method for operators of order minus one (Theorem 3.15). Eventually, in Section 3.5, we consider systems of integral operators (or more generally pseudo-differential operators) of different orders. We give abstract results for the additive Schwarz method as well as for the hybrid method mentioned above (Theorems 3.16 and 3.17). Concrete results for combinations of specific preconditioners then follow directly by using the abstract theorems. For this we refer to the electric screen problem (Section 4.2, Theorems 4.2, 4.3, and 4.4), the magnetic screen problem (Section 4.3, Theorems 4.6, 4.7, and 4.8), and the Helmholtz transmission problem (Section 4.4, Theorem 4.9). A specific situation appears when considering block skew-symmetric positive definite systems. In that case the results for the positive definite situations (independently for the diagonal blocks of the system) can be directly combined to obtain optimal results for the whole system (Theorem 3.18). This situation occurs, e.g., for a special coupling procedure of the BEM and the FEM for solving a Helmholtz transmission problem (Section 4.4, Theorems 4.10 and 4.11).

For most situations where we have theoretical results at hand as mentioned above numerical results are reported in Section 4.

3.1 Additive Schwarz method

Our aim is to find preconditioners for the GMRES method for solving linear systems arising form the boundary element Galerkin method. For the definition and the theoretical investigation of the preconditioners we use the additive Schwarz framework which will be introduced in the following.

The additive Schwarz method was originally used as an iterative solver for linear systems. However, it also leads to very efficient preconditioners. The main advantage of this method is that it reduces the original boundary value problem to a couple of local subproblems which are independent of each other and can therefore be solved in parallel. Here we show that this is also the case when considering integral equations for which, in contrast to standard boundary value problems, we have to consider non-local operators. The independence of the individual problems makes an implementation on a parallel architecture straight forward. But the algorithms are also efficient in their sequential forms. An important feature of appropriately designed additive Schwarz algorithms is their scalability. In most cases the local subproblems can be chosen in such a way that the iteration numbers which are necessary to solve the global problem do not depend on the number of subproblems. That is, a large problem with a large number of subproblems can be solved as efficiently as a smaller problem with a smaller number of subproblems if the sizes of the subproblems in both cases are comparable.

To introduce the additive Schwarz method we decompose the ansatz space X_N:

$$X_N = H_0 \cup \cdots \cup H_n \tag{3.2}$$

In general there is no requirement concerning the intersections of the subspaces H_j. The additive Schwarz method (ASM) consists in solving, by an iterative method, the equation

$$Pu := (P_0 + \cdots + P_n)u = f, \tag{3.3}$$

where the projections $P_j : X_N \to H_j$, $j = 0, \ldots, n$, are defined for any $v \in X_N$ by

$$a(P_j v, \varphi) = a(v, \varphi) \quad \text{for any } \varphi \in H_j. \tag{3.4}$$

In general we will directly invert these local problems. However, in practical applications, it is often desirable to reduce the amount of work by replacing some or all of the local solvers with iterative or approximate solvers.

Having chosen a basis $\{\phi_j;\ j = 0, \ldots, N\}$ of X_N and bases $\{\phi_{j,i};\ i = 1, \ldots, \dim H_j\}$ of H_j where, only for simplicity in the formulation, we require

$$\{\phi_{j,i};\ i = 1, \ldots, \dim H_j\} \subset \{\phi_j;\ j = 0, \ldots, N\}, \tag{3.5}$$

then the equation (3.4) can be written in matrix form as

$$A_j(P_j v) = \tilde{A}_j v \tag{3.6}$$

where $P_j v$ and v are the coefficient vectors of the respective functions. The matrix A_j is the diagonal block of A which belongs to the basis functions of the subspace H_j and $\tilde{A}_j$ is built up by $\dim H_j$ rows of A:

$$\tilde{A}_j = \big(a(\phi_{j,i}, \phi_k)\big)_{i=1,\ldots,\dim H_j; k=1,\ldots,N}.$$

By padding A_j with zeroes to fit to the dimension $N \times N$ of A and denoting the resulting matrix by $\widehat{A_j}$ the solution of the relation (3.6), embedded in $\mathbb{R}^N$, can be written as

$$\widehat{P_j v} = \widehat{A_j^{-1}}\,\widehat{\tilde{A}_j} v = \widehat{A_j^{-1}} A v.$$

The last equality holds since all the rows of A which are zero in $\widehat{\tilde{A}_j}$ are cancelled by the multiplication with $\widehat{A_j^{-1}}$. In the following we will neglect the padding notation "$\widehat{\ }$" by assuming that all vectors and matrices are filled up with zeroes in the right way if necessary. We note that if the condition (3.5) is not fulfilled some basis transformations in the matrix formulations are required to map between the subspaces H_j and the ansatz space X_N.

The right hand side of (3.3), $f = \sum_{j=0}^{n} P_j u$, can be computed without knowing the solution u of the original Galerkin system (1.2) by

$$a(P_j u, \varphi) = \langle g, \varphi \rangle \quad \text{for any } \varphi \in H_j,\ j = 0, \ldots, n,$$

i.e., in matrix form,

$$f = \sum_{j=0}^{n} P_j u = \sum_{j=0}^{n} A_j^{-1} A u = \sum_{j=0}^{n} A_j^{-1} g.$$

Now we can rewrite (3.3) in matrix form as

$$\sum_{j=0}^{n} A_j^{-1} A u = \sum_{j=0}^{n} A_j^{-1} g.$$

Therefore (3.3) is the preconditioned linear system and the preconditioning matrix is

$$B^{-1} = \sum_{j=0}^{n} A_j^{-1}.$$

If direct sum decompositions (3.2) are used fulfilling the condition (3.5) then no basis transformations are required and the preconditioner B^{-1} is a simple block Jacobi preconditioner. Further, if the extreme case of total decomposition of X_N, i.e. $n = N$ and $\dim H_j = 1$, is considered then B^{-1} is the Jacobi preconditioner which gives a diagonal scaling of the stiffness matrix.

In the following we will use the additive Schwarz method to define preconditioners which can be used, e.g., for the conjugate gradient method when solving linear systems with symmetric positive definite stiffness matrices or for the generalized minimum residual method for non-symmetric or indefinite systems.

The main task will be to find appropriate decompositions (3.2) which yield relatively easily solvable subproblems and which substantially reduce the amount of work which is necessary to solve the linear systems. In the symmetric case it is essential to bound the spectral condition number of the linear system. Ideally this condition number does not depend on the degree p and the mesh width h of the ansatz space X_N of the Galerkin method. In that case the number of iterations, e.g. of the conjugate gradient method, which are necessary to solve the system (3.3) up to a given accuracy would not depend on the dimension N of X_N. The next theorem and lemmas give theoretical bounds for the extreme eigenvalues of the additive Schwarz operator which is algebraically the preconditioned matrix. These estimates will also enter the theory of non-symmetric systems.

Theorem 3.3 (Theorem 3.1 in [154]) *For the additive Schwarz operator P there holds*

$$\lambda_{\min}(P) = \min_{u \in X_N} \frac{a(u,u)}{a(P^{-1}u,u)} = \min_{u \in X_N} \frac{a(u,u)}{\min_{\sum u_j = u} \sum_j a(u_j,u_j)}$$

and

$$\lambda_{\max}(P) = \max_{u \in X_N} \frac{a(u,u)}{a(P^{-1}u,u)} = \max_{u \in X_N} \frac{a(u,u)}{\min_{\sum u_j = u} \sum_j a(u_j,u_j)}.$$

The notation $\sum u_j = u$ for $u \in X_N$ means a representation of u with $u_j \in H_j$.

The next two lemmas are direct consequences of the above theorem, the first one known as Lions' lemma, see also, e.g., [106, 151, 112].

Lemma 3.1 *If there exists a constant C_1 such that for any $u \in X_N$ there exist $u_j \in H_j$, $j = 0, \ldots, n$, satisfying $u = \sum_{j=0}^{n} u_j$ and*

$$\sum_{j=0}^{n} a(u_j,u_j) \leq C_1^{-1} a(u,u)$$

then

$$\lambda_{\min}(P) \geq C_1.$$

The above condition is also necessary.

Lemma 3.2 *If there exists a constant C_2 such that for any $u \in X_N$ and $u_j \in H_j$, $j = 0, \ldots, n$, satisfying $u = \sum_{j=0}^{n} u_j$ and*

$$a(u,u) \leq C_2 \sum_{j=0}^{n} a(u_j, u_j)$$

then

$$\lambda_{\max}(P) \leq C_2.$$

The above condition is also necessary.

We note that additive Schwarz methods for the boundary element method dealing with boundary value problems in two dimensions are rather well understood, both the case of the h-version and the p-version, see [66, 141, 70, 142, 136, 135]. Even the hp-version with geometric meshes (which yields exponentially fast convergence of the Galerkin solution against the true solution) has been considered, see [75].

In two dimensions, where one has to deal with operators on curves, it is straight forward to show that if one has an additive Schwarz method for an operator of order one then one obtains a related method for an operator of order minus one which gives the same behavior for the condition number. One just has to take the derivatives w.r.t. the arc length of the basis functions for the operator of positive order to define new basis functions for the operator of negative order. The appropriate subspace decomposition is given accordingly. This corresponding method works due to the fact that standard integration and differentiation can be extended to mappings which are inverses of each other and which are isomorphisms between the spaces $\tilde{H}^{1/2}$ and $\tilde{H}^{-1/2}$ (for the latter identifying functions which differ only by constants), see [66]. By this isomorphism estimates of the form appearing in Lemmas 3.1 and 3.2 then are equivalent in $\tilde{H}^{1/2}$ and $\tilde{H}^{-1/2}/\{\text{constants}\}$ for corresponding ansatz spaces and decompositions.

In the three-dimensional Euclidean space, where one has to deal with operators on surfaces, we do not know of such an easy mechanism. Following an idea of Bramble et al. in [24] one can use the positive square root $(-\Delta)^{1/2}$ to define a norm in $\tilde{H}^{1/2}$ and one has the inverse operator $(-\Delta)^{-1/2}$. However, these operators in general do not map piecewise polynomials to piecewise polynomials which are needed as basis functions for the usual boundary element method. Therefore, our methods for operators of different orders will be introduced independently and for the proofs we use different ideas.

3.2 Pseudo-differential operators of order one

In this section we consider selfadjoint positive definite pseudo-differential operators A^α of order $\alpha = 1$ mapping $\tilde{H}^{1/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$. Without restriction we identify the norms induced by A^α and the norm in $\tilde{H}^{1/2}(\Gamma)$:

$$\|v\|^2_{\tilde{H}^{1/2}(\Gamma)} = \langle A^\alpha v, v \rangle_{L^2(\Gamma)} \quad \text{for any } v \in \tilde{H}^{1/2}(\Gamma).$$

For solving (1.1) $a(U,v) := \langle A^\alpha U, v\rangle = g(v)$ ($v \in X := \tilde{H}^{1/2}(\Gamma)$) by the Galerkin method we need to take continuous piecewise polynomials as basis functions. These ansatz spaces

are denoted by $S^1_{h,p}(\Gamma)$. A typical problem which can be modeled by a first kind integral equation with an operator of order one is the Neumann problem for the Helmholtz operator, see Section 4.1 and Lemma 4.4. More practical examples dealing with operators of order one can be found in Sections 4.2 and 4.3.

Even though we are mainly interested in the p-version of the BEM we first present some results for the pure h-version (lowest polynomial degrees), namely for the multilevel method in Section 3.2.1. Preconditioners for systems which stem form the pure h-version of the Galerkin method are of practical interest also for the p-version. This is the case, e.g., when dealing with fine meshes when the subspace of lowest order piecewise polynomials is large and, thus, the inversion of the corresponding block in the stiffness matrix is expensive. In that case preconditioners for the p-version and the h-version can be combined in order to avoid large subspaces in the overall decomposition of the ansatz space.

3.2.1 Multilevel additive Schwarz method for the h-version

We investigate a multilevel additive Schwarz method for the pure h-version Galerkin method. We therefore solve the Galerkin system (1.2) for ansatz spaces $X_N := S^1_h := S^1_1(\Gamma_h) := S^1_{h,1}(\Gamma)$ of piecewise polynomials of degree 1 on a sequence of successively refined uniform meshes Γ_h. As basis functions we use the continuous, piecewise bilinear functions $\phi_{h,j}$ which take the value 1 at a single node x_j and which are zero at the remaining mesh points. If Γ is an open surface piece the nodes on the boundary of Γ are not considered for this construction since the conformity condition $X_N \subset \tilde{H}^{1/2}(\Gamma)$ requires $\varphi|_{\partial\Gamma} = 0$ for $\varphi \in X_N$. The interior of the support of $\phi_{h,j}$, which consists of at most four rectangles, is denoted by Γ'_j. The dimension N of the space X_N is equal to the number J_V of nodes of the mesh which are not on the boundary of Γ.

In this section we follow the lines of Bramble and Pasciak [22] (who deal with the FEM) and of Stephan and Tran [141] (who deal with the BEM for problems in $\mathbb{R}^2$). We not only verify that the results by Stephan and Tran are also true when dealing with the BEM in $\mathbb{R}^3$ but also improve their result for the multilevel method by avoiding the unspecific term $c(\varepsilon)h^{-\varepsilon}$ in their bound of the condition number.

For the general multilevel method we decompose the ansatz space like

$$S^1_h = S^1_{h_1} \cup \left(\cup_{l=2}^{L_h} \cup_{j=1}^{N_l} S^1_{h_l,j} \right) \tag{3.7}$$

where l denotes the level. h_l is the mesh width of level l and we always choose $h_{l-1} = 2\,h_l$, i.e. L_h denotes the finest level with mesh width $h_{L_h} = h$. $S^1_{h_1} = S^1_1(\Gamma_{h_1})$ is the space of continuous, piecewise bilinear functions on the coarsest mesh and is not further decomposed. The space of the lth level $S^1_{h_l} := \cup_{j=1}^{N_l} S^1_{h_l,j}$ is decomposed into the one dimensional subspaces $S^1_{h_l,j} = \operatorname{span}\{\phi_{h_l,j}\}$, $j = 1, \ldots, N_l$. The support $\bar{\Gamma}'_{l,j}$ of $\phi_{h_l,j}$ consists of the elements of the lth level mesh which are adjacent to the interior node with number j of the lth level mesh. At the finest level $l = L_h$ the union of these elements have been denoted by $\Gamma'_{L_h,j} = \Gamma'_j$.

There exist two extreme cases of the multilevel method, namely the 2-level method where we choose $L_h = 2$ independently of the mesh width h and the fully multilevel method where we choose L_h such that h_1 is independent of the actual mesh width h. In the latter case the number of levels is increasing for decreasing h due to the relation

$$h_1 = 2^{L_h - 1} h_{L_h} = 2^{L_h - 1} h.$$

The very extreme case of $L_h = 1$ would result in directly inverting the problem.

The 2-level method. We consider the case $L_h = 2$ in (3.7). Therefore, we have to deal with the decomposition

$$S_h^1 = S_H^1 \cup S_{h,1}^1 \cup \cdots \cup S_{h,J_V}^1 \tag{3.8}$$

where S_H^1 is the space of continuous, piecewise bilinear functions on a coarser mesh with size $H = 2h$ and where $S_{h,j}^1 = \text{span}\{\phi_{h,j}\}$, $j = 1, \ldots, J_V$.

Theorem 3.4 *The condition number of the two-level additive Schwarz operator corresponding to the decomposition* (3.8) *is bounded, i.e.*

$$\kappa(P) \leq C$$

for a constant C which is independent of h.

Proof. For estimating the minimum eigenvalue of the additive Schwarz operator we will use Lemma 3.1, i.e. for an arbitrary function $v \in S_h^1$ we have to define a representation

$$v = v_H + v_{h,1} + \cdots + v_{h,J_V}$$

according to (3.8) such that

$$\|v_H\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{j=1}^{J_V} \|v_{h,j}\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq c\|v\|^2_{\tilde{H}^{1/2}(\Gamma)}.$$

This will be done by using cut-off functions θ_j and the interpolation operator $\Pi_p = \Pi_1$ of the localization (see Section 2.2). We note that it is sufficient to take the standard hat functions for the interior nodes x_j, $j = 1, \ldots, J_V$, rather than the cut-off functions of (2.13) which are modified near the boundary of Γ. In Section 2.2 we took the L^2-projection $Q_h v$ onto the space of piecewise bilinear functions as the global component. Here we use a coarser mesh for this projection and define $v_H := Q_H v$, the L^2-projection of v onto $S_1^1(\Gamma_H)$. Further, $v_{h,j} := \Pi_1(\theta_j w_h)$ $(j = 1, \ldots, J_V)$ with $w_h := v - v_H$. As before, the support of θ_j is denoted by $\bar{\Gamma}'_j$ (the union of several elements of the mesh at the second level). We have $\text{supp}\,\Pi_1(\theta_j w_h) = \bar{\Gamma}'_j$ and $\Pi_1(\theta_j w_h)$ is continuous, piecewise bilinear by definition. Therefore $v_{h,j} = \Pi_1(\theta_j w_h) \in S_{h,j}^1$. Further we have

$$\sum_{j=1}^{J_V} v_{h,j} = \Pi_1(\sum_{j=1}^{J_V} \theta_j w_h) = w_h \tag{3.9}$$

due to the linearity of Π_1 and since $w_h = 0$ at the boundary of Γ (because $w_h \in S_h^1 \subset \tilde{H}^{1/2}(\Gamma)$). Therefore, $\sum_{j=1}^{J_V} \theta_j w_h$ has the same nodal values as w_h itself on $\bar{\Gamma}$ and the relation (3.9) follows. We note that $\{\theta_j;\ j = 1, \ldots, J_V\}$ is not a partition of unity since $\sum_{j=1}^{J_V} \theta_j$ vanishes at the boundary of Γ. However, in the special situation of the h-version the values of $\sum_{j=1}^{J_V} \theta_j$ at the boundary of Γ do not influence the partition since the construction $v_{h,j} = \Pi_1 \theta_j v$ is

based on linear interpolation in the nodes of the mesh and $v = 0$ at boundary nodes. This is different for the p-version of the Galerkin method where nodal interpolation is not sufficient in the localization procedure.

By a slight modification of the proof of the right estimate in (2.21) we obtain the desired relation

$$\|v_H\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{j=1}^{J_V} \|v_{h,j}\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq c\|v\|^2_{\tilde{H}^{1/2}(\Gamma)}.$$

In fact, we just have to use a slightly different projection for which one obtains

$$\|w_h\|_{L^2(\Gamma)} = \|v - Q_H v\|_{L^2(\Gamma)} \leq cH|v|_{H^1(\Gamma)} = 2ch|v|_{H^1(\Gamma)},$$

and one can use the improved bound by Lemma 2.11 for the operator Π_1 instead of Lemma 2.10 which is used in the proof of Theorem 2.1. The boundedness of the minimum eigenvalue of P from below by a positive constant which is independent of h then follows by Lemma 3.1. In order to prove the boundedness of the maximum eigenvalue of P we use Lemma 3.2, i.e. we have to show that there holds

$$\|v\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq c\big(\|v_H\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{j=1}^{J_V} \|v_{h,j}\|^2_{\tilde{H}^{1/2}(\Gamma)}\big)$$

for any $v \in S_1^1(\Gamma_h)$ and any representation $v = v_H + v_{h,1} + \cdots + v_{h,J_V}$ according to (3.8) where c is a constant independent of h. The required estimate corresponds to the left inequality in the Localization Theorem 2.2. Even though we consider a special representation in that theorem the proof works identically for an arbitrary representation which fits to the decomposition (3.8). We therefore refer to the essential estimate (2.22) in the proof of Theorem 2.2 and to the argumentation before. Thus, the maximum eigenvalue of the two-level additive Schwarz operator P is bounded and we finally obtain the boundedness of the condition number of P. □

The fully multilevel method. Now we consider the additive Schwarz operator which belongs to the multilevel decomposition (3.7) of the ansatz space where the coarsest mesh of size h_1 is fixed. This implies an increase of the number of levels if the mesh size h of the underlying original mesh is decreased to improve the boundary element solution of the problem under consideration.

Lemma 3.3 *In the case Γ is a closed surface, the minimum eigenvalue of the multilevel additive Schwarz operator P is bounded from below by a constant which is independent of h and the number of levels L,*

$$\lambda_{\min}(P) \geq c.$$

In the case Γ is an open surface piece,

$$\lambda_{\min}(P) \geq c(1 + \log 1/h)^{-1/2}.$$

Proof. Applying Lemma 3.1 we have to find, for an arbitrary function $v \in S_h^1$, a representation

$$v = v_{h_1} + \sum_{l=2}^{L_h} \sum_{j=1}^{N_l} v_{h_l,j}$$

such that

$$\|v_{h_1}\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{l=2}^{L_h} \sum_{j=1}^{N_l} \|v_{h_l,j}\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq c\|v\|^2_{\tilde{H}^{1/2}(\Gamma)}$$

(in the case Γ is a closed surface) or

$$\|v_{h_1}\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{l=2}^{L_h} \sum_{j=1}^{N_l} \|v_{h_l,j}\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq c(1+\log 1/h)^{1/2}\|v\|^2_{\tilde{H}^{1/2}(\Gamma)}$$

(in the case Γ is an open surface piece). To this end we define

$$v_{h_l} := (G_{h_l} - G_{h_{l-1}})v, \quad l = 1, \ldots, L_h,$$

where G_{h_l} is the Galerkin projector (i.e. the projection with respect to the bilinear form $a(\cdot,\cdot) = \langle A^\alpha \cdot, \cdot \rangle$ which defines the norm in $\tilde{H}^{1/2}(\Gamma)$) onto the space $S^1_{h_l}$ of continuous, piecewise bilinear functions on the mesh of size h_l, and $G_{h_0} := 0$. Obviously, $\sum_{l=1}^{L_h} v_{h_l} = v$.

By construction the components v_{h_l} for different levels l of v are orthogonal with respect to the $\tilde{H}^{1/2}(\Gamma)$-inner product. At each level we proceed similarly as in the proof of Theorem 3.4 by making use of the localization procedure derived in Section 2.2. We define $v_{h_l,j} := \Pi_{h_l}(\theta^l_j v_{h_l})$ where Π_{h_l} is the interpolation operator onto $S^1_{h_l}$ and θ^l_j is a piecewise bilinear basis function of the lth level mesh. As in the proof of Theorem 2.1 we conclude that at each level $l > 1$ there holds

$$\sum_{j=1}^{N_l} \|v_{h_l,j}\|^2_{L^2(\Gamma)} \leq c\|v_{h_l}\|^2_{L^2(\Gamma)} \tag{3.10}$$

and

$$\sum_{j=1}^{N_l} |v_{h_l,j}|^2_{H^1(\Gamma)} \leq c\big(\frac{1}{h_l^2}\|v_{h_l}\|^2_{L^2(\Gamma)} + |v_{h_l}|^2_{H^1(\Gamma)}\big). \tag{3.11}$$

In the proof of Theorem 2.1 we used properties of the $L^2(\Gamma)$-projection onto a coarser grid to bound the $L^2(\Gamma)$-norm in (3.11) by a multiple of the mesh size times the $H^1(\Gamma)$-semi-norm, cf. (2.24). Here we cannot take the $L^2(\Gamma)$-projection of v_{h_l} onto an ansatz space with a coarser mesh. This would disturb the orthogonality with respect to the $\tilde{H}^{1/2}(\Gamma)$-inner product of the components v_{h_l} for different levels l. However, we note that there holds

$$v_{h_l} = v_{h_l} - G_{h_{l-1}} v_{h_l} = (1 - G_{h_{l-1}})v_{h_l}$$

by construction and therefore

$$\|v_{h_l}\|^2_{L^2(\Gamma)} \leq c h_{l-1}^{2-\varepsilon} |v_{h_l}|^2_{H^1(\Gamma)} \tag{3.12}$$

for any $\varepsilon > 0$. This estimate follows by combining the standard error estimate

$$\|v_{h_l}\|_{\tilde{H}^{1/2}(\Gamma)} = \|v_{h_l} - G_{h_{l-1}} v_{h_l}\|_{\tilde{H}^{1/2}(\Gamma)} \leq c h_{l-1}^{1/2} |v_{h_l}|_{H^1(\Gamma)} \tag{3.13}$$

(see [138]) with the Aubin-Nitsche trick to obtain the $L^2(\Gamma)$-error estimate. For the Aubin-Nitsche trick we refer to [4], [108] and to [84] for the generalization to Sobolev norms of non-integer and negative orders. The appearing parameter $\varepsilon > 0$ is due to the reduced regularity of the solutions of pseudo-differential operator equations of order one on the screen near the boundary. Even if the boundary is smooth the solutions behave like $O(\rho^{1/2})$ where ρ is the distance to the boundary of the screen, cf. [130]. Due to this behavior we only have $H^{1-\varepsilon}(\Gamma)$-regularity with positive ε. For closed Lipschitz surfaces we do have $H^1(\Gamma)$-regularity and the parameter ε in the above estimates can be replaced with 0.

However, in the case of open surfaces with polygonal boundary, we can replace the dependence on the parameter ε by a logarithmic dependence on the mesh size h. In the following this will be done by prescribing the dependence of the constant c in (3.12) on the parameter ε and by letting ε tend to zero for $h_l \to 0$.

To this end we take a closer look at the regularity and at the Aubin-Nitsche trick where the regularity enters. For simplicity let us consider a rectangular screen $\Gamma = (0,1)^2$. From [145] we know that the solution of the pseudo-differential operator equation

$$A^\alpha v = f \quad (f \in H^1(\Gamma))$$

can be decomposed, near the origin, into edge and corner-edge singularities and a smoother remainder like

$$v = v_0 + \chi(r) a_1 r^\gamma \omega(\theta) + \tilde{\chi}(\theta) e_1(r) r^{1/2} \sin^{1/2}\theta + \tilde{\chi}\left(\frac{\pi}{2} - \theta\right) e_2(r) r^{1/2} \cos^{1/2}\theta$$

with $v_0 \in \tilde{H}^{3/2-\delta}(G)$, $a_1 \in \mathbb{R}$, $\omega \in H^{3/2-\delta}[0,\pi/2]$, $e_i(r) = b_i r^{\gamma+1/2} + c_i(r)$, $c_i \in H^1_0(\mathbb{R}_+)$, $b_i \in \mathbb{R}$, $i = 1,2$, and arbitrary $\delta > 0$. Corresponding decompositions hold at the other vertices of Γ. Here, (r,θ) are plane polar coordinates concentrated at the origin. χ and $\tilde{\chi}$ are C^∞ cut-off-functions with $\chi \equiv 1$ for $r < 1/4$, $\tilde{\chi} \equiv 1$ for $\theta < \pi/4$. The exponents $\gamma \approx 0.2966$ and $\tilde{\gamma} \approx 1.426$ are obtained in [101]. The only components of v which are not $H^1_0(\Gamma)$-regular are the corner-edge singularities, e.g.

$$e_1(r) r^{1/2} \sin^{1/2}\theta \quad \text{near the edge} \quad \{(r,\theta);\ 0 \leq r < 1, \theta = 0\} = \{(x,y);\ 0 \leq x < 1, y = 0\}.$$

For this singularity we have the behavior

$$e_1(r) r^{1/2} \sin^{1/2}\theta \sim y^{1/2} \quad \text{for} \quad y \to 0$$

which disturbs the $H^1_0(\Gamma)$-regularity of v. Defining the function $\phi(x,y) := y^{1/2}$ we obtain the asymptotic behavior $\|\phi\|^2_{\tilde{H}^{1-\varepsilon}(\Gamma)} \sim 1/\varepsilon$ $(\varepsilon \to 0)$. This follows from the characterization

$$\|\phi\|^2_{\tilde{H}^\sigma(\Gamma)} \simeq \|\phi\|^2_{H^\sigma(\Gamma)} + \int_\Gamma |\phi(x)|^2 \operatorname{dist}(x, \partial\Gamma)^{-2\sigma}\, dx$$

for $\sigma \in (0,1)$, cf., e.g., [55, (1,3,2,11)], where the first term on the right hand side can be estimated via Fourier transformation and the second term can be calculated directly. Therefore, by the triangle inequality,

$$\|v\|^2_{\tilde{H}^{1-\varepsilon}(\Gamma)} \leq c/\varepsilon \tag{3.14}$$

for the solution v of $A^\alpha v = f$ with $f \in H^1(\Gamma)$. Having this estimate at hand we can take a closer look at the Aubin-Nitsche trick to derive a more explicit $L^2(\Gamma)$-estimate than (3.12). Define $z \in \tilde{H}^{1/2}(\Gamma)$ and $z_{h_{l-1}} \in S^1_{h_{l-1}}$ via

$$\begin{aligned} a(z,\psi) &= \langle v_{h_l}, \psi\rangle \quad \text{for all} \quad \psi \in \tilde{H}^{1/2}(\Gamma), \\ a(z_{h_{l-1}},\psi) &= \langle v_{h_l}, \psi\rangle \quad \text{for all} \quad \psi \in S^1_{h_{l-1}}. \end{aligned}$$

Since $v_{h_l} \in H^1(\Gamma)$ we have $z \in \tilde{H}^{1-\varepsilon}(\Gamma)$ with the above representation by corner and edge-corner singularities and a smoother remainder. This gives (cf. [130])

$$\|z - z_{h_{l-1}}\|_{\tilde{H}^{1/2}(\Gamma)} \leq c h_{l-1}^{1/2-\varepsilon} \|z\|_{\tilde{H}^{1-\varepsilon}(\Gamma)}$$

and therefore

$$\begin{aligned} \|v_{h_l}\|^2_{L^2(\Gamma)} &= \langle v_{h_l}, v_{h_l}\rangle = a(z, v_{h_l}) = a(z - z_{h_{l-1}}, v_{h_l}) \\ &\leq \|A^\alpha (z - z_{h_{l-1}})\|_{H^{-1/2}(\Gamma)} \|v_{h_l}\|_{\tilde{H}^{1/2}(\Gamma)} \leq \|z - z_{h_{l-1}}\|_{\tilde{H}^{1/2}(\Gamma)} \|v_{h_l}\|_{\tilde{H}^{1/2}(\Gamma)} \\ &\leq c h_{l-1}^{1/2-\varepsilon} \|z\|_{\tilde{H}^{1-\varepsilon}(\Gamma)} h_{l-1}^{1/2} |v_{h_l}|_{H^1(\Gamma)} = c h_{l-1}^{1-\varepsilon} \|z\|_{\tilde{H}^{1-\varepsilon}(\Gamma)} |v_{h_l}|_{H^1(\Gamma)} \end{aligned} \tag{3.15}$$

with a constant c which is independent of ε and h_{l-1}. In the above relations we used

$$a(z_{h_{l-1}}, v_{h_l}) = a(z_{h_{l-1}}, v_{h_l} - G_{h_{l-1}} v_{h_l}) = 0$$

(note that $G_{h_{l-1}} v_{h_l} = 0$), the continuity of the operator A^α, and the error estimate (3.13). Let us bound the term $\|z\|_{\tilde{H}^{1-\varepsilon}(\Gamma)}$ in the above estimate. By the definition of z and the mapping properties of A^α we have

$$\|z\|_{\tilde{H}^{1-\varepsilon}(\Gamma)} = \|(A^\alpha)^{-1} v_{h_l}\|_{\tilde{H}^{1-\varepsilon}(\Gamma)} \simeq \|v_{h_l}\|_{H^{-\varepsilon}(\Gamma)}.$$

Since $v_{h_l} \in H^1_0(\Gamma) \subset L^2(\Gamma)$ the norm $\|v_{h_l}\|_{H^{-\varepsilon}(\Gamma)}$ is bounded if $\varepsilon \to 0$. On the other hand, due to (3.14), $\|z\|_{\tilde{H}^{1-\varepsilon}(\Gamma)}$ blows up like $O\big(\varepsilon^{-1/2}\big)$. We therefore conclude that there holds

$$\|z\|_{\tilde{H}^{1-\varepsilon}(\Gamma)} \leq c\varepsilon^{-1/2} \|v_{h_l}\|_{L^2(\Gamma)} \quad (\varepsilon \to 0).$$

From (3.15) we thus obtain

$$\|v_{h_l}\|^2_{L^2(\Gamma)} \leq c h_{l-1}^{1-\varepsilon} \|z\|_{\tilde{H}^{1-\varepsilon}(\Gamma)} |v_{h_l}|_{H^1(\Gamma)} \leq c\varepsilon^{-1/2} h_{l-1}^{1-\varepsilon} \|v_{h_l}\|_{L^2(\Gamma)} |v_{h_l}|_{H^1(\Gamma)},$$

i.e.

$$\|v_{h_l}\|^2_{L^2(\Gamma)} \leq c\varepsilon^{-1} h_{l-1}^{2-2\varepsilon} |v_{h_l}|^2_{H^1(\Gamma)} \tag{3.16}$$

for a constant c which is independent of ε and h_{l-1}.

Now we return to the estimates (3.10) and (3.11). Using (3.16) we obtain from (3.11)

$$\sum_{j=1}^{N_l} |v_{h_l,j}|^2_{H^1(\Gamma)} \leq c\varepsilon^{-1} h_{l-1}^{-2\varepsilon} |v_{h_l}|^2_{H^1(\Gamma)}$$

and, together with the $L^2(\Gamma)$-estimate (3.10), by using interpolation theory analogously as in the proof of Theorem 2.1 (cf. (2.26), (2.27) and the arguments thereafter) we conclude that there holds

$$\sum_{j=1}^{N_l} \|v_{h_l,j}\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq c\varepsilon^{-1/2} h_{l-1}^{-\varepsilon} \|v_{h_l}\|^2_{\tilde{H}^{1/2}(\Gamma)}. \tag{3.17}$$

Now, since the functions v_{h_l} for different levels l are orthogonal with respect to $a(\cdot,\cdot)$ by definition, we obtain by summing up (3.17)

$$\|v_{h_1}\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{l=2}^{L_h} \sum_{j=1}^{N_l} \|v_{h_l,j}\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq c\varepsilon^{-1/2} \sum_{l=1}^{L_h} h_{l-1}^{-\varepsilon} \|v_{h_l}\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq c\varepsilon^{-1/2} h^{-\varepsilon} \|v\|^2_{\tilde{H}^{1/2}(\Gamma)} \tag{3.18}$$

with $h_0 := 1$. Choosing $\varepsilon := -1/\log h$ we obtain

$$\|v_{h_1}\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{l=2}^{L_h} \sum_{j=1}^{N_l} \|v_{h_l,j}\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq c(1+\log 1/h)^{1/2} \|v\|^2_{\tilde{H}^{1/2}(\Gamma)}$$

which gives the assertion if Γ is an open surface piece. As explained above, in the case Γ is a closed surface, the parameter ε in (3.12) can be replaced with 0 or, synonymously, the term $\varepsilon^{-1/2} h^{-\varepsilon}$ in (3.18) can be replaced with 1 which gives a bounded minimum eigenvalue of P from below in this case. □

To obtain a bound for the maximum eigenvalue of the multilevel additive Schwarz operator we use the technique by Bramble and Pasciak [22]. An essential assumption in [22] is that there holds a strengthened Cauchy-Schwarz inequality, which is given by the next lemma. This lemma is essentially that of Stephan and Tran [141, Lemma 2.7] where the multilevel additive Schwarz method for hypersingular integral equations has been investigated for problems in the two dimensional Euclidean space. For completeness we present their proof which is also valid in the higher dimensional case.

Lemma 3.4 (Stephan, Tran [141, Lemma 2.7]) *There exist constants $c > 0$ and $\gamma \in (0,1)$ such that for any $v \in S^1_{h_k}$ there holds*

$$a(P_l v, v) \leq c\gamma^{l-k} \|v\|^2_{\tilde{H}^{1/2}(\Gamma)} \quad (1 \leq k \leq l \leq L_h)$$

where $P_l := \sum_{j=1}^{N_l} P_{l,j}$ and $P_{l,j}$ is the additive Schwarz projector onto the one dimensional subspace $S^1_{h_l,j}$.

Proof. As noted above this proof is a modification to the higher dimensional case of the proof of Lemma 2.7 in [141]. Since $S^1_{h_l,j} = \mathrm{span}\{\phi_{h_l,j}\}$, $\phi_{h_l,j}$ being the piecewise bilinear basis function for the jth node of the mesh of size h_l, we have for any $v \in S^1_h$

$$P_{l,j}v = \frac{a(v,\phi_{h_l,j})}{a(\phi_{h_l,j},\phi_{h_l,j})}\phi_{h_l,j}$$

and therefore

$$a(P_l v, v) = \sum_{j=1}^{N_l} a(P_{l,j}v, v) = \sum_{j=1}^{N_l} \frac{a(v,\phi_{h_l,j})^2}{a(\phi_{h_l,j},\phi_{h_l,j})}.$$

In order to estimate $a(v,\phi_{h_l,j})$ we define $A_l:\ S^1_{h_l} \to S^1_{h_l}$ by

$$\langle A_l\phi,\psi\rangle = a(\phi,\psi) = \langle A^\alpha\phi,\psi\rangle \quad \text{for any} \quad \phi,\psi \in S^1_{h_l}, \quad l = 1,\dots,L_h.$$

Then we obtain

$$\begin{aligned} a(P_l v, v) &= \sum_{j=1}^{N_l} \frac{\langle A_l v,\phi_{h_l,j}\rangle^2}{\|\phi_{h_l,j}\|^2_{\tilde H^{1/2}(\Gamma)}} \le c\sum_{j=1}^{N_l} \frac{\|A_l v\|^2_{L^2(\Gamma'_{l,j})}\|\phi_{h_l,j}\|^2_{L^2(\Gamma'_{l,j})}}{\|\phi_{h_l,j}\|^2_{\tilde H^{1/2}(\Gamma)}} \\ &\le c\sum_{j=1}^{N_l} \frac{\|A_l v\|^2_{L^2(\Gamma'_{l,j})} h_l\|\phi_{h_l,j}\|^2_{\tilde H^{1/2}(\Gamma'_{l,j})}}{\|\phi_{h_l,j}\|^2_{\tilde H^{1/2}(\Gamma)}} \le c h_l \|A_l v\|^2_{L^2(\Gamma)}. \end{aligned} \tag{3.19}$$

Here we used the notation $\Gamma'_{l,j} = \mathrm{supp}\,\phi_{h_l,j}$ and Lemma 2.4 which gives

$$\|\phi_{h_l,j}\|^2_{L^2(\Gamma'_{l,j})} \le c h_l \|\phi_{h_l,j}\|^2_{\tilde H^{1/2}(\Gamma'_{l,j})}.$$

Due to the continuity of A^α and the inverse property (Lemma 2.7) we obtain

$$\begin{aligned} \|A_l v\|^2_{L^2(\Gamma)} &= \langle A_l v, A_l v\rangle = \langle A^\alpha v, A_l v\rangle \le \|A^\alpha v\|_{L^2(\Gamma)}\|A_l v\|_{L^2(\Gamma)} \\ &\le |v|_{H^1(\Gamma)}\|A_l v\|_{L^2(\Gamma)} \le c h_k^{-1/2}\|v\|_{\tilde H^{1/2}(\Gamma)}\|A_l v\|_{L^2(\Gamma)} \end{aligned}$$

that is

$$\|A_l v\|^2_{L^2(\Gamma)} \le c h_k^{-1}\|v\|^2_{\tilde H^{1/2}(\Gamma)}.$$

Together with (3.19) this gives

$$a(P_l v, v) \le c h_l/h_k a(v,v) = c\, 2^{-(l-k)} a(v,v).$$

□

Using the above lemma the boundedness of the maximum eigenvalue of the multilevel additive Schwarz operator follows by abstract estimates which do not use the order of the operator A^α or the dimension of the Euclidean space under consideration (Theorem 3.1 in [22]), cf. also [141, Lemma 2.8].

Lemma 3.5 *The maximum eigenvalue of the multilevel additive Schwarz operator P is bounded by a constant which is independent of h and the number of levels L,*

$$\lambda_{\max}(P) \le c.$$

Proof. We use the technique by Bramble and Pasciak [22], see also [141, Lemma 2.8]. We prove that there holds

$$a(Pv,v) \le ca(v,v) \quad \text{for any} \quad v \in S_h^1 \tag{3.20}$$

which is equivalent to $\lambda_{\max}(P) \le c$ since P is selfadjoint with respect to $a(\cdot,\cdot)$. For $l = 1,\dots,L_h$ let $G_{h_l} : S_h^1 \to S_{h_l}^1$ be the Galerkin projection, i.e.,

$$a(G_{h_l}v,w) = a(v,w) \quad \text{for any} \quad w \in S_{h_l}^1.$$

For any $v \in S_h^1$ there holds

$$G_{h_l}v = \sum_{k=1}^{l} \big(G_{h_k} - G_{h_{k-1}}\big)v \quad \text{with} \quad G_{h_0} := 0.$$

Therefore,

$$a(Pv,v) = \sum_{l=1}^{L_h} a(P_l v,v) = \sum_{l=1}^{L_h} a(P_l v, G_{h_l}v) = \sum_{l=1}^{L_h}\sum_{k=1}^{l} a\big((P_l v,(G_{h_k} - G_{h_{k-1}})v\big).$$

Since $a(P_l\cdot,\cdot)$ is symmetric and positive definite we can use the Cauchy-Schwarz inequality with respect to that inner product. Using Lemma 3.4 we therefore obtain for any $\eta > 0$ and for $S := \sum_{i=0}^{\infty}\gamma^i$

$$\begin{aligned}
a(Pv,v) &\le \sum_{l=1}^{L_h}\sum_{k=1}^{l} a\big(P_l v,v\big)^{1/2} a\big(P_l(G_{h_k} - G_{h_{k-1}})v,(G_{h_k} - G_{h_{k-1}})v\big)^{1/2} \\
&\le c\sum_{l=1}^{L_h}\sum_{k=1}^{l} \gamma^{l-k} a\big(P_l v,v\big)^{1/2} a\big((G_{h_k} - G_{h_{k-1}})v,(G_{h_k} - G_{h_{k-1}})v\big)^{1/2} \\
&= c\sum_{l=1}^{L_h}\sum_{k=1}^{l} \gamma^{l-k} a\big(P_l v,v\big)^{1/2} a\big((G_{h_k} - G_{h_{k-1}})v,v\big)^{1/2} \\
&\le c\Big(\frac{\eta}{S}\sum_{l=1}^{L_h}\sum_{k=1}^{l} \gamma^{l-k} a(P_l v,v) + \frac{S}{4\eta}\sum_{l=1}^{L_h}\sum_{k=1}^{l} \gamma^{l-k} a\big((G_{h_k} - G_{h_{k-1}})v,v\Big) \\
&\le c\Big(\eta a(Pv,v) + \frac{S^2}{4\eta}a(v,v)\Big).
\end{aligned}$$

Choosing η sufficiently small this proves (3.20) and therefore the assertion. □

Combining Lemmas 3.3 and 3.5 we have proved

Theorem 3.5 *In the case Γ is a closed surface, the multilevel additive Schwarz operator P corresponding to the decomposition* (3.7) *has a condition number which is bounded independently of h and the number of levels L, i.e.*

$$\kappa(P) \leq c$$

for a constant c which is independent of h and L. In the case Γ is an open surface piece, there holds

$$\kappa(P) \leq c(1 + \log 1/h)^{1/2}.$$

Here, $h = h_L$ is the mesh size of the boundary element space X_N.

3.2.2 Overlapping additive Schwarz method for the p-version

Now we consider the p-version of the boundary element method for solving first kind integral equations with operators of order one. The ansatz space X_N of the Galerkin method is denoted by $X_N = S_p^1 := S_p^1(\Gamma_h)$ and consists of continuous piecewise polynomials of degree p on a rectangular mesh of size h. As for the h-version the number of interior nodes of the mesh is denoted by J_V.

We decompose the ansatz space into overlapping subspaces

$$S_p^1 = S_1^1 \cup S_{p,1}^1 \cup \cdots \cup S_{p,J_V}^1. \tag{3.21}$$

Here, S_1^1 denotes the space of continuous, piecewise bilinear functions on the actual mesh and coincides with S_h^1 as defined in Section 3.2.1. $S_{p,j}^1$ is the space of piecewise polynomials on Γ_j' of degree p that vanish at the boundary of Γ_j'. As in the case of the h-version Γ_j' comprises x_j as an interior node and is the union of four elements of the mesh, cf. Figure 3.1. The decomposition (3.21) is identical to that of Pavarino [112] used for the finite element method in two dimensions.

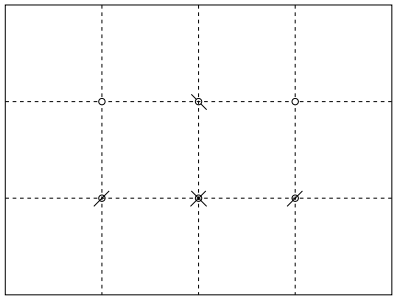
We define the additive Schwarz operator P by the abstract formulation (3.3), (3.4) and by using the decomposition (3.21). This additive Schwarz operator has bounded condition number.

Theorem 3.6 *The condition number of the additive Schwarz operator corresponding to the decomposition* (3.21) *is bounded, i.e. there exists a constant C which is independent of p and h such that there holds*

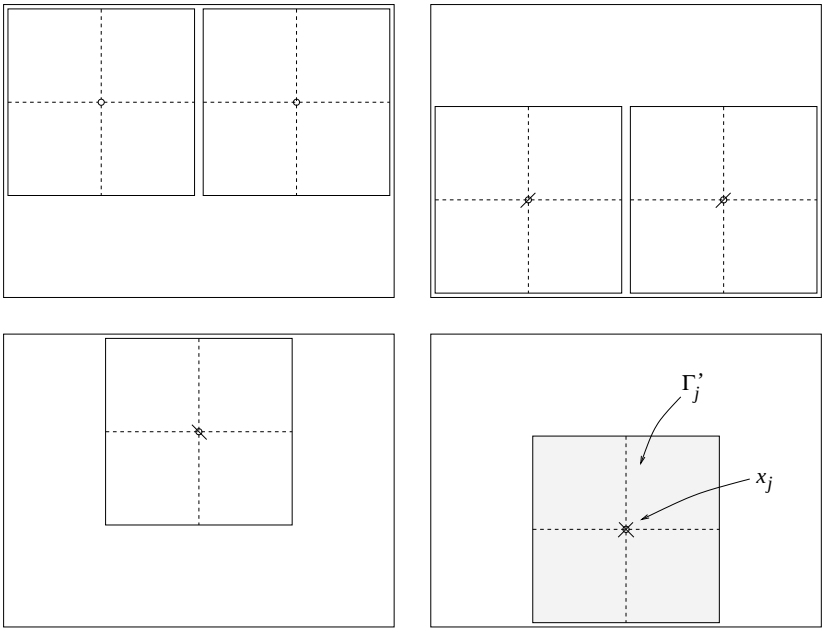
$$\kappa(P) \leq C.$$

Proof. The main part of the proof is given by Theorem 2.2 which gives an energy conserving partition of unity within the space $S_p^1(\Gamma_h)$. More precisely, for any $\varphi_p \in S_p^1$ this theorem proves the existence of a representation

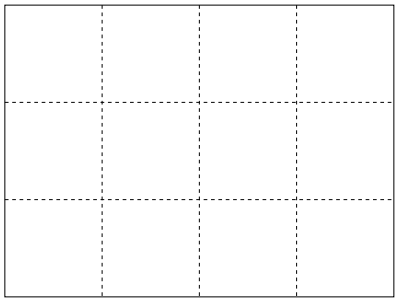
$$\varphi_p = \varphi_1 + \sum_{j=1}^{J_V} \varphi_{p,j}$$



The mesh with marked interior nodes



Local subspaces defined by the overlapping decomposition



Additional subspace of piecewise bilinear functions

Figure 3.1: Operators of order one: the mesh and its partition for the overlapping additive Schwarz method.

according to the decomposition (3.21) for which we obtain

$$\|\varphi_1\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{j=1}^{J_V} \|\varphi_{p,j}\|^2_{\tilde{H}^{1/2}(\Gamma'_j)} \leq C_2 \|\varphi_p\|^2_{\tilde{H}^{1/2}(\Gamma)}.$$

Due to Lemma 3.1 this proves the boundedness of the minimum eigenvalue of P from below by C_2^{-1}. It remains to bound the maximum eigenvalue of P. Using Lemma 3.2 it suffices to find a constant $C > 0$ such that

$$\|\varphi_p\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C\Big(\|\varphi_1\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{j=1}^{J_V} \|\varphi_{p,j}\|^2_{\tilde{H}^{1/2}(\Gamma'_j)}\Big)$$

for any $\varphi_p \in S_p^1$ and any representation $\varphi_p = \varphi_1 + \sum_{j=1}^{J_V} \varphi_{p,j}$ according to the decomposition (3.21). This estimate is obtained by first using the triangle inequality,

$$\|\varphi_p\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq 2\Big(\|\varphi_p - \varphi_1\|^2_{\tilde{H}^{1/2}(\Gamma)} + \|\varphi_1\|^2_{\tilde{H}^{1/2}(\Gamma)}\Big)$$

and then splitting the norm $\|\varphi_p - \varphi_1\|_{\tilde{H}^{1/2}(\Gamma)}$ with the help of Lemma 2.3. Here we use that the coloring constant of the decomposition is bounded by 4. This gives a bound for the maximum eigenvalue of P which does not depend on h and p and, thus, the theorem is proved. □

Numerical experiments for this preconditioner which confirm the boundedness of the condition numbers are reported in Section 4.1. Here, a Neumann screen problem for the Helmholtz operator is considered and by setting the wave number $k = 0$ this is the positive definite case being dealt with by the above theorem. See Table 4.2 on page 123.

3.2.3 Iterative substructuring method for the p-version

The overlapping additive Schwarz method of the previous section has the drawback that the local problems can become rather large for large values of p and that, due to the overlapping, many unknowns are taken more than once in the solution process. In this section we further decompose the ansatz space into subspaces which do not overlap. The natural decomposition would be to take subspaces of functions associated with individual nodes, edges, and elements. For this decomposition we cannot use standard basis functions. It is well-known that the $H^1((0,1)^3)$-semi-norm of the nodal components (the trilinear interpolants) of a polynomial u of degree p is in general not bounded by a polylogarithmic term $\log^\alpha p$ times the H^1-norm of u, see, e.g., [7]. By the trace theorem and the extension theorem (Theorem 2.3) it then follows that the bilinear interpolation in two dimensions it also not sufficiently bounded with respect to the $\tilde{H}^{1/2}$-norm. Another difficulty in this natural decomposition is the strong coupling of the edge components and the interior components belonging to the elements adjacent to the respective edges. This has been investigated numerically for the finite element method in two dimensions in [8]. By considering also the L^2-norm and then interpolating between H_0^1 and L^2 it is clear that this strong coupling is also present when taking the $\tilde{H}^{1/2}$-norm (it would only be reduced by the square root). Thus one cannot expect to improve the condition number of the stiffness matrix (for hypersingular integral operators) to behave as $\log^\alpha p$ when using

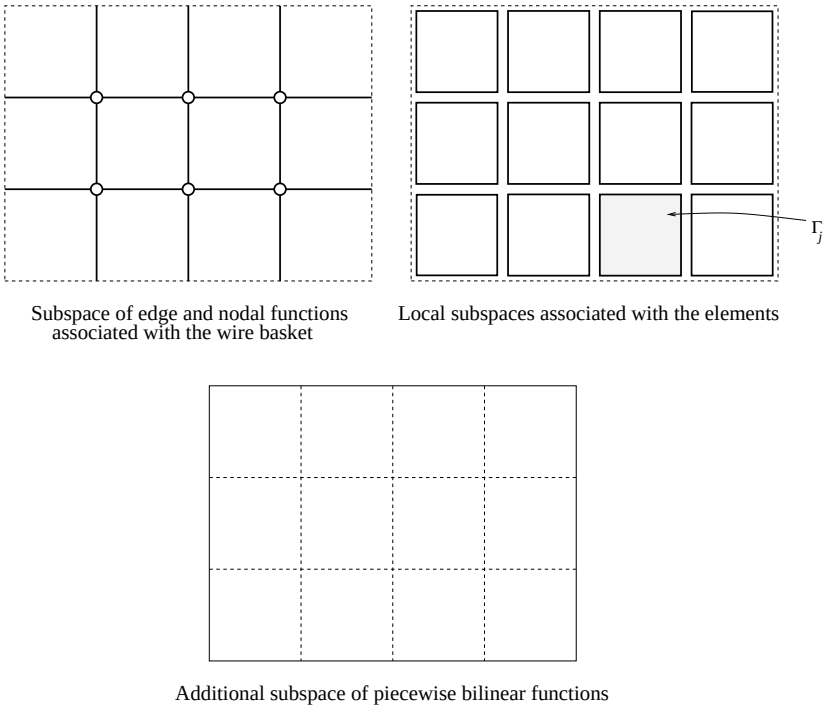


Figure 3.2: Operators of order one: the partition of the mesh for the wire basket preconditioner.

the natural decomposition for standard basis functions (decomposing into nodal, edge, and interior functions). For that reason we now consider the discrete harmonic basis functions introduced in Section 2.3 (see page 39 and Figures 2.5, 2.6, and 2.7).

According to our representation of boundary element functions described in Section 2.3.1 we decompose

$$S_p^1(\Gamma_h) = \tilde{H}_0 \cup (H_0 \oplus H_1 \oplus \cdots \oplus H_J) \tag{3.22}$$

Here, $\tilde{H}_0$ is the subspace of piecewise bilinear functions and H_0 consists of the wire basket functions on Γ_h and $H_j = S_p^1(\Gamma_h) \cap \tilde{H}^{1/2}(\Gamma_j)$, $j = 1, \ldots, J$, see Figure 3.2.

That means there exist two global spaces $\tilde{H}_0$ and H_0 of functions on the whole mesh and for each element Γ_j we have a space of polynomials vanishing at the boundary of the element. The direct sum part of the decomposition (3.22) is similar to the trace of the decomposition of a finite element space presented in [114]. The preconditioner implicitly defined by (3.22) is called the wire basket preconditioner and the related additive Schwarz operator will be denoted by P_W.

We note that the dimension of the subspace H_0 can be rather large compared to the dimensions of the remaining subspaces. Therefore, in a second step, we further decompose H_0 into subspaces for each nodal function and each edge and obtain

$$S_p^1(\Gamma_h) = \tilde{H}_0 \cup \left(\cup_{j=1}^{J_V} H_{V_j} \ \oplus \ \cup_{j=1}^{J_E} H_{E_j} \ \oplus \ \cup_{j=1}^{J} H_{\Gamma_j} \right). \tag{3.23}$$

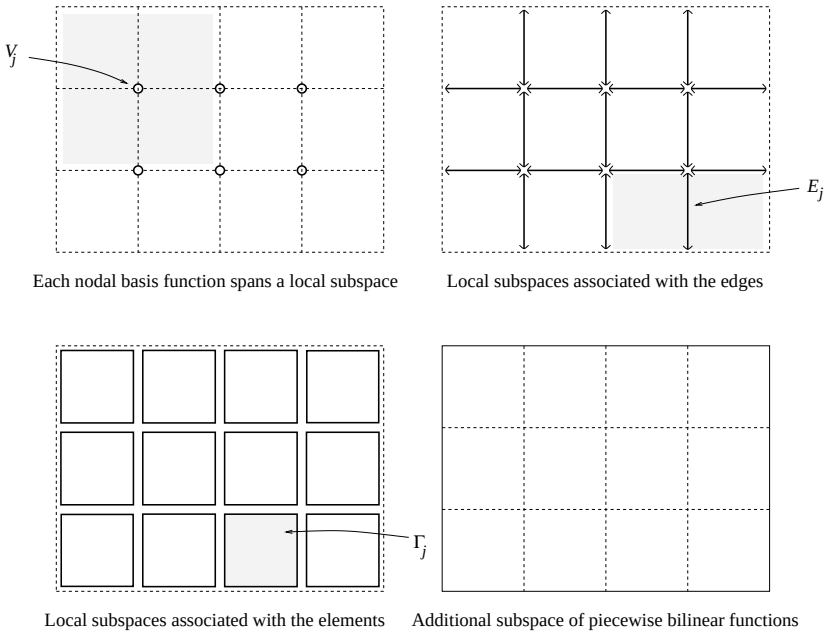


Figure 3.3: Operators of order one: the partition of the mesh for the non-overlapping additive Schwarz preconditioner.

Here, J_V and J_E are the numbers of interior vertices and edges on Γ, respectively. The space H_{V_j} is spanned by the vertex basis function concentrated at the vertex V_j. Analogously, H_{E_j} is spanned by the edge basis functions which belong to the edge E_j, cf. Figure 3.3.

This decomposition leads to a three dimensional generalization of non-overlapping additive Schwarz preconditioners in two dimensions, see, e.g., [7, 70, 142]. The resulting additive Schwarz operator is denoted by P_{ASM}. Even though the space $\tilde{H}_0$ and the direct sum part in (3.23) are not disjoint we call this method non-overlapping since, in contrast to the usual overlapping methods, all the remaining subspaces of higher polynomial degrees are disjoint.

As an extreme case we also study the diagonal preconditioner which is modified by adding the block belonging to the subspace $\tilde{H}_0$ from above. We obtain the additive Schwarz preconditioner P_D defined by the decomposition

$$S_p^1(\Gamma_h) = \tilde{H}_0 \cup \left(\cup_{j=1}^{J_V} H_{V_j} \ \oplus \ \cup_{j=1}^{J_E} \cup_{k=1}^{p-1} H_{E_j,k} \ \oplus \ \cup_{j=1}^{J} \cup_{k,l=1}^{p-1} H_{\Gamma_j,k,l} \right). \tag{3.24}$$

Here, $H_{E_j,k}$ is spanned by an edge basis function concentrated at the edge E_j. This edge basis function is piecewise a linear transformation of the basis functions $e_k^{(1)}$, $e_k^{(2)}$, $e_k^{(3)}$, $e_k^{(4)}$. $H_{\Gamma_j,k,l}$ is the span of the interior basis function f_{kl} transformed onto the element Γ_j. (For the definition of the basis functions see page 39.)

The following theorem essentially says that, if we use the special discrete harmonic basis functions, any decomposition of the ansatz space leads to an almost optimal additive Schwarz preconditioner if we add the space of piecewise bilinear functions to the decomposition.

Theorem 3.7 *For the condition number κ of the additive Schwarz operators P_W, P_{ASM} and*

P_D implicitly defined by the decompositions (3.22), (3.23) *and* (3.24)*, respectively, there holds*

$$\kappa \leq C(1+\log p)^2.$$

The constant C is independent of the mesh size h and the polynomial degree p.

Proof. To estimate the extreme eigenvalues of the additive Schwarz operators P_W, P_{ASM} and P_D we make use of the abstract Lemmas 3.1 and 3.2.

To prove the theorem we first derive bounds for the minimum and maximum eigenvalues of the wire basket and non-overlapping additive Schwarz preconditioners. After that we use orthogonal properties of the edge and interior basis functions to carry over the obtained results to the modified diagonal preconditioner.

Let us start estimating the maximum eigenvalues of P_W and P_{ASM}. We take an arbitrary function $u = \tilde{u}_0 + w \in S_p^1(\Gamma_h)$ with $\tilde{u}_0 \in \tilde{H}_0$ and $w \in S_p^1(\Gamma_h)$. The subspace $\tilde{H}_0$ consists of the piecewise bilinear functions, cf. (3.22) and (3.23). The component w of u can be uniquely represented by discrete harmonic basis functions, according to the direct sum parts of the decompositions (3.22) and (3.23). The interior component w_I of $w \in S_p^1(\Gamma_h)$ satisfies $w_I|_{\Gamma_j} = w_{\Gamma_j} \in \tilde{H}^{1/2}(\Gamma_j), j = 1, \ldots, J$, by construction. Therefore, due to Lemma 2.3 we obtain

$$\|w_I\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C \sum_{j=1}^{J} \|w_{\Gamma_j}\|^2_{\tilde{H}^{1/2}(\Gamma)}.$$

By splitting w into its wire basket component and the interior remainder, and then again using Lemma 2.3 to split the wire basket component into the edge and nodal components, we obtain the relations

$$\begin{aligned}
\|u\|^2_{\tilde{H}^{1/2}(\Gamma)} &\leq 2\left(\|\tilde{u}_0\|^2_{\tilde{H}^{1/2}(\Gamma)} + \|w\|^2_{\tilde{H}^{1/2}(\Gamma)}\right) = 2\left(\|\tilde{u}_0\|^2_{\tilde{H}^{1/2}(\Gamma)} + \|w_W + w_I\|^2_{\tilde{H}^{1/2}(\Gamma)}\right) \\
&\leq C\left(\|\tilde{u}_0\|^2_{\tilde{H}^{1/2}(\Gamma)} + \|w_W\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{j=1}^{J} \|w_{\Gamma_j}\|^2_{\tilde{H}^{1/2}(\Gamma)}\right) \qquad (3.25) \\
&\leq C\left(\|\tilde{u}_0\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{j=1}^{J_V} \|w_{V_j}\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{j=1}^{J_E} \|w_{E_j}\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{j=1}^{J} \|w_{\Gamma_j}\|^2_{\tilde{H}^{1/2}(\Gamma)}\right). \qquad (3.26)
\end{aligned}$$

The estimates (3.25) and (3.26) prove the boundedness of the maximum eigenvalues of the wire basket preconditioner and of the non-overlapping additive Schwarz preconditioner, respectively.

To obtain bounds for the minimum eigenvalues we need to find for a given ansatz function $u \in S_p^1(\Gamma_h)$ representations which correspond to the decompositions (3.22), (3.23) and which give small bounds for the energy norm of the single components, cf. Lemma 3.1.

Since all of the above decompositions are almost direct ones, i.e. they are direct with exception of the global piecewise bilinear part, the needed representations are unique for the terms of the higher polynomial degrees. However, since we need to use local estimates which deal with $\tilde{H}^{1/2}$-norms, even in the case of global components in the representations,

we have to reduce our estimates to local versions. This makes it necessary to find also localized representations of the given ansatz function u. For this step we use the localization procedure from Section 2.2. The local components are then represented via discrete harmonic basis functions for which we can apply local estimates. Since the localization procedure is linear the assembly of appropriate basis functions of different local components again gives a representation of the global function u. And, vice versa, due to Theorem 2.2 also the local estimates can be put together to provide global estimates. This will be performed in detail in the following.

By the localization procedure of Section 2.2 we represent a function $u \in S_p^1(\Gamma_h)$ by a global piecewise bilinear part and a sum of local components on overlapping subdomains of Γ. We use a partition of unity by piecewise bilinear functions θ_j. Further we need an interpolation operator Π_p which interpolates piecewise polynomials of degree $p+1$ by piecewise polynomials of degree p. The cut-off functions are chosen such that $\theta_j = 1$ at the vertex V_j and $\theta_j = 0$ at the remaining internal vertices. The support of θ_j which is denoted by the closure of $\tilde{\Gamma}_j$ consists of the elements adjacent to V_j. We obtain the representation

$$u = \tilde{u}_0 + \sum_{j=1}^{J_V} \tilde{u}_j \quad \text{with} \quad \tilde{u}_0 := Q_h(u) \quad \text{and} \quad \tilde{u}_j := \Pi_p \theta_j w, \quad w := u - \tilde{u}_0$$

(plus some modifications near L-points) where Q_h is the $L^2(\Gamma)$-projection onto $\tilde{H}_0 = S_1^1(\Gamma_h)$. Due to Theorem 2.2 there holds

$$\|u\|^2_{\tilde{H}^{1/2}(\Gamma)} \simeq \|\tilde{u}_0\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{j=1}^{J_V} \|\tilde{u}_j\|^2_{\tilde{H}^{1/2}(\Gamma'_j)}. \tag{3.27}$$

We continue by representing each component $\tilde{u}_j$ of u by discrete harmonic basis functions, this representation being unique. More precisely we split $\tilde{u}_j$ into its vertex, edge and interior components as done in Section 2.3.1. Let $\Gamma_{j,k}$ and $E_{j,k}$, $k = 1, \ldots, 4$, denote the elements and internal edges of Γ'_j, respectively. Note that the components of $\tilde{u}_j$ may not coincide with the respective components of $w_{\Gamma'_j} = (u - \tilde{u}_0)|_{\Gamma'_j}$. Since constructing the vertex, edge and interior components is a linear operation and since

$$\sum_{j=1}^{J_V} \tilde{u}_j = \sum_{j=1}^{J_V} \Pi_p \theta_j (u - \tilde{u}_0) = w$$

(with possible modifications of the middle term near L-points) we also get a representation for w by collecting the components of the $\tilde{u}_j$s for each vertex, edge, and element:

$$\begin{aligned} u = \sum_{j=0}^{J_V} \tilde{u}_j &= \tilde{u}_0 + \sum_{j=1}^{J_V} \left(\tilde{u}_{V_j} + \sum_{k=1}^{4} \tilde{u}_{E_{j,k}} + \sum_{k=1}^{4} \tilde{u}_{\Gamma_{j,k}} \right) \\ &= \tilde{u}_0 + \sum_{j=1}^{J_V} \tilde{u}_{V_j} + \sum_{j=1}^{J_E} \left(\sum_{(k,l):\, E_{k,l}=E_j} \tilde{u}_{E_{k,l}} \right) + \sum_{j=1}^{J} \left(\sum_{(k,l):\, \Gamma_{k,l}=\Gamma_j} \tilde{u}_{\Gamma_{k,l}} \right) \\ &= \tilde{u}_0 + \sum_{j=1}^{J_V} w_{V_j} + \sum_{j=1}^{J_E} \qquad w_{E_j} \qquad + \sum_{j=1}^{J} \qquad w_{\Gamma_j} \end{aligned} \tag{3.28}$$

By Lemma 2.3 we conclude that there exists a constant $C > 0$ such that

$$\|w_{E_j}\|^2_{\tilde{H}^{1/2}(\Gamma)} = \|\sum_{(k,l):\, E_{k,l}=E_j} \tilde{u}_{E_{k,l}}\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C \sum_{(k,l):\, E_{k,l}=E_j} \|\tilde{u}_{E_{k,l}}\|^2_{\tilde{H}^{1/2}(\Gamma)}, \quad j = 1, \ldots, J_E, \tag{3.29}$$

and

$$\|w_{\Gamma_j}\|^2_{\tilde{H}^{1/2}(\Gamma)} = \|\sum_{(k,l):\, \Gamma_{k,l}=\Gamma_j} \tilde{u}_{\Gamma_{k,l}}\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C \sum_{(k,l):\, \Gamma_{k,l}=\Gamma_j} \|\tilde{u}_{\Gamma_{k,l}}\|^2_{\tilde{H}^{1/2}(\Gamma)}, \quad j = 1, \ldots, J. \tag{3.30}$$

In the next three steps we bound the norms of the nodal, edge and interior components globally on Γ. This is done by using the corresponding local estimates (2.35), (2.41), (2.40), by taking into account the mesh size dependencies of the norms as given by Lemma 2.4 and by using the norm equivalence (3.27). For the nodal components we obtain

$$\sum_{j=1}^{J_V} \|w_{V_j}\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C(1+\log p) \sum_{j=1}^{J_V} \|\tilde{u}_j\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C(1+\log p)\|u\|^2_{\tilde{H}^{1/2}(\Gamma)}. \tag{3.31}$$

For the edge components we also take (3.29) into account to conclude

$$\begin{aligned}\sum_{j=1}^{J_E} \|w_{E_j}\|^2_{\tilde{H}^{1/2}(\Gamma)} &\leq C \sum_{j=1}^{J_E} \sum_{(k,l):\, E_{k,l}=E_j} \|\tilde{u}_{E_{k,l}}\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C \sum_{k=1}^{J_V} \sum_{l=1}^{4} \|\tilde{u}_{E_{k,l}}\|^2_{\tilde{H}^{1/2}(\Gamma)} \\ &\leq C(1+\log p) \sum_{k=1}^{J_V} \|\tilde{u}_k\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C(1+\log p)\|u\|^2_{\tilde{H}^{1/2}(\Gamma)}. \end{aligned} \tag{3.32}$$

Correspondingly, we use (3.30) for the interior components and obtain

$$\begin{aligned}\sum_{j=1}^{J} \|w_{\Gamma_j}\|^2_{\tilde{H}^{1/2}(\Gamma)} &\leq C \sum_{j=1}^{J} \sum_{(k,l):\, \Gamma_{k,l}=\Gamma_j} \|\tilde{u}_{\Gamma_{k,l}}\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C \sum_{k=1}^{J_V} \sum_{l=1}^{4} \|\tilde{u}_{\Gamma_{k,l}}\|^2_{\tilde{H}^{1/2}(\Gamma)} \\ &\leq C(1+\log p)^2 \sum_{k=1}^{J_V} \|\tilde{u}_k\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C(1+\log p)^2\|u\|^2_{\tilde{H}^{1/2}(\Gamma)}. \end{aligned} \tag{3.33}$$

By noting that the energy norm of $\tilde{u}_0$ is bounded by the energy norm of u due to (3.27) the estimates (3.31), (3.32) and (3.33) prove the bound of the minimum eigenvalue of the non-overlapping additive Schwarz operator P_{ASM},

$$\lambda_{\min} \geq C(1+\log p)^{-2}.$$

The same bound holds also for the minimum eigenvalue of the wire basket preconditioner as can be seen by combining (3.31), (3.32) and (3.33) with the estimate from (3.25) to (3.26).

Now we bound the eigenvalues of the modified diagonal preconditioner. The edge basis functions $e_i^{(k)}$, $i = 1, \ldots, p-1$, for fixed $k \in \{1, \ldots, 4\}$ are orthogonal with respect to the

$L^2(\Gamma_{\rm ref})$- and $H^1_0(\Gamma_{\rm ref})$-inner products, cf. [114]. Therefore, by interpolation theory, we obtain for a single edge component $w_{E_j} = \sum_{i=1}^{p-1} \alpha_i e_i^{(j)}$ the relation

$$\|w_{E_j}\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C \sum_{i=1}^{p-1} \|\alpha_i e_i^{(j)}\|^2_{\tilde{H}^{1/2}(\Gamma)}.$$

The constant C does not depend on p. Here we use that the interpolation norm does not depend on the polynomial degree (cf. [96, Theorem 1]) and that we can interchange interpolation and composition, cf. Lemma 2.1. An analogous relation holds for the interior components. Thus, by (3.26), the maximum eigenvalue of P_D is bounded. The lower bound $C(1+\log p)^{-2}$ for the minimum eigenvalue of P_D is obtained analogously to that of P_W by further splitting the edge and the interior components of w in the representation (3.28) of a function $u = \tilde{u}_0 + w \in S^1_p(\Gamma_h)$ and by using (2.42) and (2.43) instead of (2.41) and (2.40), respectively. Therefore, the proof of Theorem 3.7 is finished. □

As for the overlapping method we refer to Section 4.1 for numerical results which confirm the above theorem, cf. Table 4.3 on page 124 and Figure 4.2 on page 125.

3.3 Pseudo-differential operators of order minus one

Now we consider selfadjoint positive definite pseudo-differential operators A^α of order $\alpha = -1$ mapping $\tilde{H}^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$. Again, we identify the norms induced by A^α and the Sobolev norm in the energy space $\tilde{H}^{-1/2}(\Gamma)$:

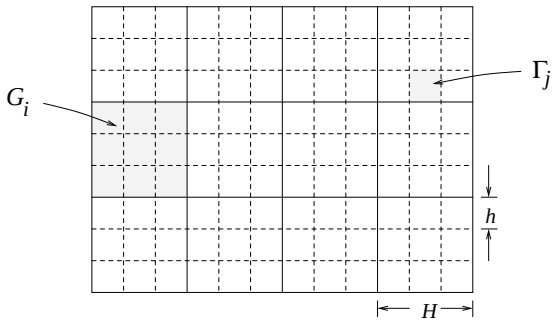
$$\|v\|^2_{\tilde{H}^{-1/2}(\Gamma)} = \langle A^\alpha v, v\rangle_{L^2(\Gamma)} \quad \text{for any } v \in \tilde{H}^{-1/2}(\Gamma).$$

For solving (1.1) $a(U,v) := \langle A^\alpha U, v\rangle = g(v)$ $(v \in \tilde{H}^{-1/2}(\Gamma))$ by the Galerkin method the basis functions need not to be continuous. The ansatz space under consideration is $S^0_{h,p}(\Gamma)$.

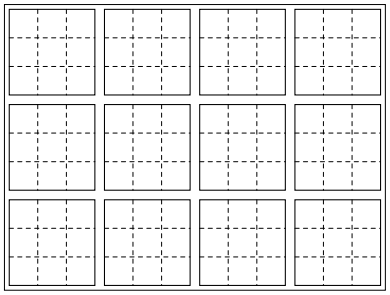
A typical example of this situation occurs, e.g., when treating the Dirichlet problem for the Helmholtz operator, see Section 4.1 and Lemma 4.3. Also the electric screen problem in Section 4.2 and the magnetic screen problem in Section 4.3 comprise weakly singular operators which are of order minus one.

We concentrate on the p-version of the BEM for which we introduce a non-overlapping method which will be proved to be almost optimal. This result can be adapted to propose an efficient 2-level preconditioner for the h-version (Corollary 3.1). We note that this is the same method investigated by Mund et al. in [104]. However, in that reference only a bound $O(H^2/h^2)$ is proved where H is the mesh size of the coarse mesh and h is the size of the fine mesh. Here, we prove a bound which behaves polylogarithmically in H/h. An extension of the two-level method to more than two levels is possible. For this we refer to [104] where norm equivalences in multilevel splittings by Oswald [110] have been used.

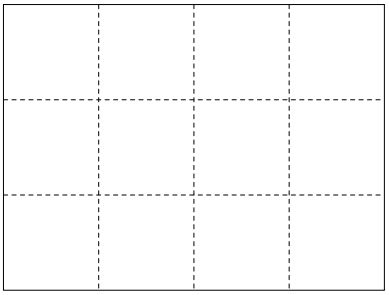
To define the preconditioner for our model problem let $\bar{\Gamma}_h = \cup_{j=1}^J \bar{\Gamma}_j$ be a given mesh of J rectangles which implicitly define the space X_N of piecewise polynomials on Γ_h by specifying the polynomial degrees. We simply use the same degree p everywhere but our method works for non-uniform degree distributions as well. For the decomposition of X_N we choose a coarse mesh $\bar{\Gamma}_H = \cup_{j=1}^n \bar{G}_j$ of size $H \geq h$. We assume that the coarse mesh Γ_H is compatible with the boundary element mesh Γ_h. The extreme case $H = h$ of identical meshes is included. The



The mesh decomposed into subdomains of size H



Local subspaces associated with the subdomains
(without constants on the subdomains)



Subspace of piecewise constants
on the coarse mesh

Figure 3.4: Operators of order minus one: the partition of the mesh for the non-overlapping additive Schwarz preconditioner.

subdomains G_j are implicitly meshed by the restrictions $\Gamma_h|_{G_j}$ and the respective elements will be denoted by $\Gamma_{j,i}$, $i = 1, \ldots, J_j$. We decompose

$$X_N = H_0 \oplus H_1 \oplus \cdots \oplus H_n \tag{3.34}$$

where H_0 is the space of piecewise constant functions on the mesh Γ_H (see Figure 3.4) and

$$H_j := \{v|_{G_j};\ v \in X_N \setminus H_0,\ \langle v, 1\rangle_{L^2(G_j)} = 0\}, \quad j = 1, \ldots, n.$$

Note that (3.34) is a direct sum decomposition which gives a unique representation for any $v \in X_N$. Further, in the case $H > h$, the subspaces H_j $(j > 0)$ also comprise piecewise constant functions, the pieces understood with respect to the local fine meshes $\Gamma_h|_{G_j}$. The functions being constant on the whole pieces G_j are excluded from these subspaces. They are covered by H_0.

Theorem 3.8 *There exists a constant $c > 0$ independent of h, H, and p such that for the condition number of the additive Schwarz operator P implicitly defined by the decomposition* (3.34) *there holds*

$$\kappa(P) \leq c\Big(1 + \log\big(\frac{H}{h}(p+1)\big)\Big)^2.$$

Proof. Due to Lemma 3.2 the boundedness of the maximum eigenvalue of P is a direct consequence of Lemma 2.3 (setting $s = -1/2$) and the triangle inequality. In order to bound the minimum eigenvalue we proceed as follows.

First we note that, for $s \geq 0$, the L^2-projection

$$Q_{0,j} : \begin{cases} H^s(G_j) & \to & H^s(G_j) \\ \psi & \mapsto & \frac{\langle \psi, 1 \rangle_{L^2(G_j)}}{|G_j|} \end{cases} \tag{3.35}$$

is continuous. This follows by interpolating $\|Q_{0,j}\psi\|_{L^2(G_j)} \leq \|\psi\|_{L^2(G_j)}$ and

$$\|Q_{0,j}\psi\|_{H^k(G_j)} = \|Q_{0,j}\psi\|_{L^2(G_j)} \leq \|\psi\|_{H^k(G_j)}$$

where k is an integer greater than or equal to s.

Now let $v = v_0 + \sum_{j=1}^n w_j$ be the representation of a function $v \in H_N$ which corresponds to the decomposition (3.34). We have to bound the constant C in

$$\langle A^\alpha v_0, v_0 \rangle + \sum_{j=1}^n \langle A^\alpha w_j, w_j \rangle \leq C \langle A^\alpha v, v \rangle$$

which is an upper bound for the inverse of the minimum eigenvalue of P. First we bound the sum of the local terms $\sum_{j=1}^n \langle A^\alpha w_j, w_j \rangle$. Using the above projection operator $Q_{0,j}$ and noting that

$$w_j = \big(v - v_0\big)|_{G_j} = v|_{G_j} - Q_{0,j}\big(v|_{G_j}\big)$$

we obtain, for any $0 < \varepsilon < 1/2$,

$$\begin{aligned} \|w_j\|_{\tilde{H}^{-1/2+\varepsilon}(G_j)} &= \sup_{\psi \in H^{1/2-\varepsilon}(G_j)} \frac{\langle v - Q_{0,j}v, \psi \rangle_{L^2(G_j)}}{\|\psi\|_{H^{1/2-\varepsilon}(G_j)}} = \sup_{\psi \in H^{1/2-\varepsilon}(G_j)} \frac{\langle v, \psi - Q_{0,j}\psi \rangle_{L^2(G_j)}}{\|\psi\|_{H^{1/2-\varepsilon}(G_j)}} \\ &\leq \|v\|_{H^{-1/2+\varepsilon}(G_j)} \sup_{\psi \in H^{1/2-\varepsilon}(G_j)} \frac{\|\psi - Q_{0,j}\psi\|_{\tilde{H}^{1/2-\varepsilon}(G_j)}}{\|\psi\|_{H^{1/2-\varepsilon}(G_j)}}. \end{aligned}$$

With Lemma 2.6 we estimate the $\tilde{H}^{1/2-\varepsilon}(G_j)$-norm by the $H^{1/2-\varepsilon}(G_j)$-norm. Together with the continuity of $Q_{0,j}$ given by (3.35) this gives for the last term the bound

$$\frac{\|\psi - Q_{0,j}\psi\|_{\tilde{H}^{1/2-\varepsilon}(G_j)}}{\|\psi\|_{H^{1/2-\varepsilon}(G_j)}} \leq \frac{c}{\varepsilon} \frac{\|\psi - Q_{0,j}\psi\|_{H^{1/2-\varepsilon}(G_j)}}{\|\psi\|_{H^{1/2-\varepsilon}(G_j)}} \leq \frac{c}{\varepsilon},$$

i.e.,

$$\|w_j\|_{\tilde{H}^{-1/2+\varepsilon}(G_j)} \leq \frac{c}{\varepsilon} \|v\|_{H^{-1/2+\varepsilon}(G_j)}.$$

The constant c does not depend on the size H of G_j since the $\tilde{H}^{-1/2+\varepsilon}(G_j)$ and the $H^{-1/2+\varepsilon}(G_j)$-norms scale with the same order of H since w_j has integral mean zero, cf. Lemmas 2.4 and 2.5. Therefore, by (2.6), we obtain

$$\sum_{j=1}^n \|w_j\|^2_{\tilde{H}^{-1/2+\varepsilon}(G_j)} \leq \frac{c}{\varepsilon^2} \sum_{j=1}^n \|v\|^2_{H^{-1/2+\varepsilon}(G_j)} \leq \frac{c}{\varepsilon^2} \|v\|^2_{H^{-1/2+\varepsilon}(\Gamma)}. \tag{3.36}$$

By noting that $\langle w_j, 1\rangle_{L^2(G_j)} = 0$, we use the scaling property given by Lemma 2.5 and obtain

$$\|w_j\|^2_{\tilde{H}^{-1/2}(G_j)} \leq cH^{2\varepsilon}\|w_j\|^2_{\tilde{H}^{-1/2+\varepsilon}(G_j)}.$$

Together with the inverse property (Lemma 2.7) and (3.36) we therefore conclude

$$\sum_{j=1}^{n} \|w_j\|^2_{\tilde{H}^{-1/2}(G_j)} \leq c\frac{H^{2\varepsilon}}{\varepsilon^2}\|v\|^2_{H^{-1/2+\varepsilon}(G)} \leq c\Big(\frac{H}{h}\Big)^{2\varepsilon}\frac{p^{4\varepsilon}}{\varepsilon^2}\|v\|^2_{\tilde{H}^{-1/2}(G)}. \tag{3.37}$$

Setting

$$\varepsilon := \frac{1}{2\log(\frac{H}{h}p^2)} \quad \text{for } \frac{H}{h}p^2 > 1$$

this gives

$$\sum_{j=1}^{n} \|w_j\|^2_{\tilde{H}^{-1/2}(G_j)} \leq c(1+\log\frac{H}{h}(p+1))^2\|v\|^2_{\tilde{H}^{-1/2}(G)} \tag{3.38}$$

for $p \geq 0$ where the constant c does not depend on p, H, and h.

The bound for the component v_0 of v then follows by the triangle inequality and by (2.5):

$$\begin{aligned}\|v_0\|^2_{\tilde{H}^{-1/2}(\Gamma)} &\leq 2\big(\|v\|^2_{\tilde{H}^{-1/2}(\Gamma)} + \|v-v_0\|^2_{\tilde{H}^{-1/2}(\Gamma)}\big) \leq c\big(\|v\|^2_{\tilde{H}^{-1/2}(\Gamma)} + \sum_{j=1}^{n}\|w_j\|^2_{\tilde{H}^{-1/2}(G_j)}\big)\\ &\leq c(1+\log\frac{H}{h}(p+1))^2\|v\|^2_{\tilde{H}^{-1/2}(G)}.\end{aligned}$$

Adding (3.38) this yields

$$\|v_0\|^2_{\tilde{H}^{-1/2}(\Gamma)} + \sum_{j=1}^{n}\|w_j\|^2_{\tilde{H}^{-1/2}(G_j)} \leq c(1+\log\frac{H}{h}(p+1))^2\|v\|^2_{\tilde{H}^{-1/2}(G)}$$

which proves that the minimum eigenvalue of P is bounded from below by a term which behaves like $(1+\log\frac{H}{h}(p+1))^{-2}$. □

The above theorem can be directly applied to the h-version (using piecewise constant ansatz functions). For completeness we give the decomposition and the result for the corresponding additive Schwarz preconditioner.

Given two mesh sizes H and h ($H > h$, H/h integer) we decompose (using the notation analogous to that used for the two-level decomposition for hypersingular operators (3.8))

$$X_N = S^0_h = S^0_H \cup S^0_{h,1} \cup \cdots \cup S^0_{h,n} \tag{3.39}$$

where S^0_H (= H_0 from above) is the space of piecewise constant functions on the coarse mesh Γ_H of domains G_j, $j = 1, \ldots, n$, and $S^0_{h,j}$ is the space of piecewise constant functions on the mesh $\Gamma_h|_{G_j}$ which are L^2-orthogonal to 1. This decomposition defines the two-level additive Schwarz operator P and the above theorem gives the following result.

Corollary 3.1 *There exists a constant $c > 0$ such that for any two mesh sizes $H > h$ (H/h integer) there holds for the additive Schwarz operator P implicitly defined by the decomposition* (3.39)

$$\kappa(P) \leq c(1 + \log \frac{H}{h})^2.$$

Numerical experiments for this preconditioner (only the p-version) are reported in Section 4.1. There, a Dirichlet screen problem for the Helmholtz operator is modeled by a first kind integral equation with operator of order $\alpha = -1$. Setting the wave number $k = 0$ this is the positive definite case of the above theorem. See Table 4.5 on page 126 and Figure 4.3 on page 127.

3.4 Indefinite or non-selfadjoint pseudo-differential operators

In Sections 3.2 and 3.3 we considered pseudo-differential operators of orders one and minus one which are selfadjoint and positive definite. The theorems of those sections provide estimates for the spectral condition numbers of various additive Schwarz operators. These estimates can be used to bound the rates of convergence of the conjugate gradient method for the preconditioned systems. We now deal with indefinite or non-selfadjoint operators which are strongly elliptic and whose main parts are selfadjoint and positive definite. Then we cannot use the conjugate gradient method as iterative solver. Our standard solver will be the GMRES method. The methods of the previous sections then are applicable as well and provide estimates which are almost as optimal as in the positive definite cases. In Section 3.4.1 we define the additive Schwarz method for these kind of operators and formulate assumptions which lead to abstract estimates for the efficiency of these preconditioners, cf. Theorems 3.9 and 3.10. A hybrid method for defining preconditioners for indefinite systems is proposed in Section 3.4.2. The requirements for this method are less restrictive. On the other hand, the theoretical efficiency is not as good as that of the additive Schwarz method which is also covered by the hybrid theory, see Theorem 3.11. In Sections 3.4.3 and 3.4.4 we explicitly consider the methods proposed for the positive definite case (in Sections 3.2 and 3.3) when applied to indefinite systems.

The weak formulation of the abstract problem under consideration is

$$\langle \mathcal{A}^\alpha U, v \rangle = \langle g, v \rangle \quad \text{for any} \quad v \in \tilde{H}^{\alpha/2}(\Gamma) \tag{3.40}$$

where $\mathcal{A}^\alpha = A^\alpha + \mathcal{K}^\alpha$ is the sum of a selfadjoint positive definite operator

$$A^\alpha : \tilde{H}^{\alpha/2}(\Gamma) \to H^{-\alpha/2}(\Gamma)$$

and a bounded operator $\mathcal{K}^\alpha : \tilde{H}^{\alpha/2}(\Gamma) \to H^{-\alpha/2+\delta}(\Gamma)$ of lower order $\alpha - \delta$ ($\delta > 0$). We always assume that (3.40) is uniquely solvable. The positive definiteness of A^α allows for identifying the norms

$$\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} = \langle A^\alpha u, u \rangle^{1/2} \quad \text{for any} \quad u \in \tilde{H}^{\alpha/2}(\Gamma).$$

We do not assume that $\mathcal{K}^\alpha$ is selfadjoint or positive definite. For sufficiently smooth Γ the operator $\mathcal{K}^\alpha : \tilde{H}^{\alpha/2}(\Gamma) \to H^{-\alpha/2}(\Gamma)$ is compact by Rellich's theorem:

$$\mathcal{K}^\alpha : \tilde{H}^{\alpha/2}(\Gamma) \to H^{-\alpha/2+\delta}(\Gamma) \xrightarrow{\text{compact}} H^{-\alpha/2}(\Gamma).$$

If pseudo-differential operator theory applies then A^α is the principal part of $\mathcal{A}^\alpha$ and $\mathcal{K}^\alpha$ is of order one less than the order of A^α, i.e. $\mathcal{K}^\alpha$ is of order $\alpha - 1$ and $\delta = 1$ holds. The compactness of $\mathcal{K}^\alpha$ yields the strong ellipticity of $\mathcal{A}^\alpha$, i.e. there exists a compact operator $\mathcal{C}$ and $\gamma > 0$ such that

$$\Re\langle(\mathcal{A}^\alpha + \mathcal{C})u, u\rangle \geq \gamma \|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}$$

(of course, $\mathcal{C} := -\mathcal{K}^\alpha$ can be chosen). By the above argument the strong ellipticity is equivalent to the existence of a Gårding inequality

$$\Re\langle\mathcal{A}^\alpha u, u\rangle \geq \gamma_1 \|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} - \gamma_2 \|u\|^2_{\tilde{H}^{\alpha/2-\delta}(\Gamma)} \tag{3.41}$$

for constants $\gamma_1, \gamma_2 > 0$. For a precise definition of strong ellipticity in the framework of pseudo-differential operators and the proof of a Gårding inequality for strongle elliptic operators see, e.g., [89].

Concerning the investigation of preconditioners for indefinite operators it is therefore legitimate to assume that there exist constants $c_0, \delta > 0$ such that there holds

$$\left|\langle\mathcal{K}^\alpha u, v\rangle\right| \leq c_0 \|u\|_{\tilde{H}^{\alpha/2-\delta}(\Gamma)} \|v\|_{\tilde{H}^{\alpha/2}(\Gamma)} \quad \text{for any} \quad u, v \in \tilde{H}^{\alpha/2}(\Gamma). \tag{3.42}$$

The same argumentation could have been started with the adjoint operator of $\mathcal{K}^\alpha$. Therefore, we also have

$$\left|\langle\mathcal{K}^\alpha u, v\rangle\right| \leq c_0 \|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \|v\|_{\tilde{H}^{\alpha/2-\delta}(\Gamma)} \quad \text{for any} \quad u, v \in \tilde{H}^{\alpha/2}(\Gamma). \tag{3.43}$$

This assumption on the continuity of $\mathcal{K}^\alpha$ is essential for proving that preconditioners for positive definite operators work also for indefinite operators. The operator $\mathcal{K}^\alpha$ can be treated by estimating the $\tilde{H}^{\alpha/2-\delta}(\Gamma)$-norm at appropriate places which is, when considering functions with local support of size h, of lower order in h than the energy norm. This means that the operator $\mathcal{A}^\alpha$ is in fact a perturbation of A^α when considering preconditioners. The essential estimates in this context will be given by Lemma 2.4 and Lemma 2.5.

The Galerkin approximation $u \in X_N$ to the exact solution U of (3.40) is given by

$$\langle\mathcal{A}^\alpha u, v\rangle = \langle g, v\rangle \quad \text{for any} \quad v \in X_N$$

where X_N is a finite dimensional subspace of $X = \tilde{H}^{\alpha/2}(\Gamma)$. Since there holds a Gårding inequality for $\mathcal{A}^\alpha$, and since we assumed unique solvability of (3.40), this system is uniquely solvable if N is large enough, see [138]. When considering finite dimensional spaces we use a subscript to denote the discrete representations of operators, e.g. $\mathcal{A}^\alpha_N$, A^α_N, $\mathcal{K}^\alpha_N$ instead of $\mathcal{A}^\alpha$, A^α, $\mathcal{K}^\alpha$, respectively, when considering X_N. For instance, $\mathcal{A}^\alpha_N$ is defined by

$$\langle\mathcal{A}^\alpha_N u, v\rangle = \langle\mathcal{A}^\alpha u, v\rangle \quad \text{for any} \quad u, v \in X_N.$$

3.4.1 Additive Schwarz method

In this section we show how the abstract additive Schwarz method can be used for systems arising form discretizing indefinite or non-selfadjoint boundary integral operators. The aim is to derive bounds for the parameters Λ_0 and Λ_1 in (3.1) which determine the rate of convergence of the GMRES method.

Our theory extends the method of Cai and Widlund [28] who address indefinite systems of the finite element method for second order differential operators. Stephan and Tran [136], [135] have shown that this method provides efficient preconditioners also for indefinite boundary element systems which arise when solving boundary value problems in $\mathbb{R}^2$. Here we deal with the boundary element method for problems in the three dimensional Euclidean space. Our theory relies on the works of Stephan, Tran and Cai, Widlund who make use of specific situations to prove the efficiency of the method. Cai and Widlund deal with the finite element method which allows to rely on Sobolev norms with integral orders. Stephan and Tran consider the boundary element method and have to deal with Sobolev norms of non-integral orders. However, they restricted their theory to boundary value problems in the plane where the proofs of some of the technical results, which have been derived in Section 2 for problems in $\mathbb{R}^3$, are quite different.

For this reason we rewrite the theory of Cai, Widlund and Stephan, Tran in a more abstract way. More precisely, we clearly formulate assumptions, which are also implicitly used in the works of Cai, Widlund and Stephan, Tran, which allow for applying the essential technical results of Section 2 to prove the efficiency of the method. Therefore, in more specific situations, it is only necessary to verify the needed assumptions and the convergence properties of the method follow from our abstract results.

For indefinite operators there are two variants in choosing the individual projections of the additive Schwarz operator. In both cases the coarse grid projection is defined with respect to the original, indefinite, operator $\mathcal{A}^\alpha$. The remaining projections are performed either also with respect to $\mathcal{A}^\alpha$ or with respect to the positive definite part A^α. More precisely we proceed as follows.

As defined in Section 3.1 the additive Schwarz method uses a decomposition of the ansatz space X_N,

$$X_N = H_0 \cup \cdots \cup H_n,$$

see (3.2). As usual, H_0 will comprise piecewise polynomials of low degree on the whole mesh Γ_h whereas H_j, $j > 0$, consists of functions which live only on a local part of Γ. The additive Schwarz operator is the sum of several projection operators and represents the preconditioned linear system. We define the projection operators $\mathcal{P}_j : X_N \to H_j$, $j = 0, \ldots, n$, and $P_j : X_N \to H_j$, $j = 1, \ldots, n$, such that, for given $u \in X_N$,

$$\langle \mathcal{A}^\alpha \mathcal{P}_j u, v\rangle = \langle \mathcal{A}^\alpha u, v\rangle \quad \text{for any} \quad v \in H_j \tag{3.44}$$

and

$$\langle A^\alpha P_j u, v\rangle = \langle \mathcal{A}^\alpha u, v\rangle \quad \text{for any} \quad v \in H_j. \tag{3.45}$$

Two types of the additive Schwarz operator are now defined by

$$\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_1 + \cdots + \mathcal{P}_n \quad \text{and} \quad P = \mathcal{P}_0 + P_1 + \cdots + P_n.$$

For the implementation of the operator P only positive definite local problems need to be solved. This might be advantageous, e.g., when replacing the exact solvers by iterative solvers. The theoretical results for both operators are asymptotically the same.

The main assumptions which are necessary when dealing with indefinite systems are the following:

Assumptions:

(A1) The subspace H_0 is rich enough, e.g. by choosing the mesh size h small enough, such that the Galerkin projection $\tilde{H}^{\alpha/2}(\Gamma) \to \tilde{H}_0$ with respect to the operator $\mathcal{A}^\alpha$ exists and is bounded for all subspaces $\tilde{H}_0$ with $H_0 \subset \tilde{H}_0 \subset \tilde{H}^{\alpha/2}(\Gamma)$.

(A2) The subspaces H_j, $j = 1, \dots, n$, in the decomposition of X_N are locally supported. That means the union $\bar{G}_j$ of the supports of the functions in a single subspace H_j is covered by a finite number of neighboring elements of the boundary element mesh. For simplicity we further assume that G_j is rectangular.
In the case $\alpha = -1$ all the functions in any of the subspaces $H_1, \dots, H_n$ have integral mean zero.

(A3) The set $\{G_j;\ j = 1, \dots, n\}$ is a finite covering of Γ. That means we can color $\{G_j;\ j = 1, \dots, n\}$ using a finite number of colors (let's say K) in such a way that subdomains G_j of the same color are disjoint.

(A4) There exists a constant $\lambda_0 > 0$ such that for any $u \in X_N$ there exists a representation $u = \sum_{j=0}^{n} u_j$ according to the decomposition of X_N such that

$$\sum_{j=0}^{n} \|u_j\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \le \lambda_0^{-1} \|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}.$$

Remark 3.1 *Assumptions* (A1), (A2), *and* (A3) *directly influence the decomposition of the boundary element space X_N. They ensure that additive Schwarz preconditioners for indefinite bilinear forms eventually behave like preconditioners for positive definite bilinear forms if the mesh size h is small enough. Assumption* (A4) *is independent of the indefinite operator $\mathcal{A}^\alpha$. It requires that the minimum eigenvalue λ_0 of the additive Schwarz preconditioner for the positive definite part A^α can be bounded from below. In applications this bound might slightly depend on the mesh size h or on the polynomial degree p. This has to be taken into account when applying the main theorems of this section (Theorems* 3.9 *and* 3.10*).*

We first derive some conclusions of the above assumptions. After that we formulate and prove the main theorems.

Lemma 3.6 (i) *There exists a constant $c > 0$ such that*

$$\|\mathcal{P}_0 u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \le c \|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \quad \textit{for any} \quad u \in X_N. \tag{3.46}$$

(ii) *There exist constants $c, \delta_1 > 0$ such that*

$$\|u - \mathcal{P}_0 u\|_{\tilde{H}^{\alpha/2-\delta}(\Gamma)} \le c h^{\delta_1} \|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \quad \textit{for any} \quad u \in X_N. \tag{3.47}$$

(iii) *Let $u \in H_j$, $j > 0$, be given. There holds*

$$\|u\|_{\tilde{H}^{\alpha/2-\delta}(\Gamma)} \le c h^{\delta_2} \|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \tag{3.48}$$

and

$$\Re\langle \mathcal{A}^\alpha u, u\rangle \geq \big(\gamma_1 - c\gamma_2 h^{2\delta_2}\big)\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \tag{3.49}$$

where

$$\delta_2 = \begin{cases} \delta & \text{if } \alpha = 1 \text{ and } (\delta \leq 1/2 \text{ or } \delta > 1/2 \text{ and } \langle u, 1\rangle = 0) \\ 1/2 & \text{if } \alpha = 1 \text{ and } (\delta > 1/2 \text{ and } \langle u, 1\rangle \neq 0) \\ \delta & \text{if } \alpha = -1 \end{cases} .$$

Proof. The assertions (i) and (ii) are direct consequences of Assumption (A1). Gårding's inequality (3.41) and the assumption on uniqueness of the solution of (3.40) imply that there exists a number $h_0 > 0$ such that, for any mesh size $h < h_0$, the Galerkin projection onto H_0 exists and is uniformly bounded in h. Further the Galerkin projection $G_h u$ of u converges against u, the convergence rate being dependent on the norm under consideration and on the mapping properties of $(\mathcal{A}^\alpha)^{-1}$. For details we refer to Stephan and Wendland [138]. By noting that the operator $\mathcal{P}_0$ is the Galerkin projection $X_N \to H_0$ with respect to $\mathcal{A}^\alpha$ the assertions (i) and (ii) of the lemma then follow.

To prove (iii) we need Assumption (A2) and a scaling argument. For $\alpha = 1$ ($\delta \leq 1/2$) we use Lemma 2.4 and for $\alpha/2 - \delta < 0$ or $\alpha/2 < 0$ we use Lemma 2.5 where the condition $\langle u, 1\rangle = 0$ enters. Since we formulated Lemmas 2.4 and 2.5 only for rectangular domains we also made this assumption for the domains G_j, $j > 0$. In the case $\alpha = 1$ and $\delta > 1/2$ and $\langle u, 1\rangle \neq 0$ we simply estimate the Sobolev norm with negative order by the L^2-norm, and then use the scaling argument. To prove the second assertion of (iii) we use Gårding's inequality (3.41) and the first assertion in (iii). □

Lemma 3.7 *For any $u \in X_N$ there holds*

$$\|\sum_{j=1}^{n} \mathcal{P}_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \leq cK \sum_{j=1}^{n} \|\mathcal{P}_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \tag{3.50}$$

and

$$\|\sum_{j=0}^{n} \mathcal{P}_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \leq 2cK \sum_{j=0}^{n} \|\mathcal{P}_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \tag{3.51}$$

for a constant $c > 0$. The same estimates hold if one replaces $\mathcal{P}_j$ by P_j for $j > 0$.

Proof. Estimate (3.50) is obtained from Assumption (A3) and Lemma 2.3. Inequality (3.51) follows from (3.50) by the triangle inequality. □

Lemma 3.8 *There exist constants $c, h_0 > 0$ such that, if $h \leq h_0$, there holds*

$$\sum_{j=0}^{n} \|\mathcal{P}_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \geq c(\lambda_0^{-1} + \gamma_2^2)^{-1} \|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \quad \text{for any} \quad u \in X_N.$$

The same estimate holds if one replaces $\mathcal{P}_j$ by P_j for $j > 0$.

Proof. Let $u \in X_N$ be given. By (3.46), (3.47)

$$\begin{aligned}\|u\|^2_{\tilde{H}^{\alpha/2-\delta}(\Gamma)} &\leq 2\|u-\mathcal{P}_0 u\|^2_{\tilde{H}^{\alpha/2-\delta}(\Gamma)} + 2\|\mathcal{P}_0 u\|^2_{\tilde{H}^{\alpha/2-\delta}(\Gamma)} \\ &\leq ch^{2\delta_1}\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} + c\|\mathcal{P}_0 u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\end{aligned}$$

Therefore, together with (3.41), we obtain

$$\begin{aligned}\Re\langle \mathcal{A}^\alpha u, u\rangle &\geq \gamma_1\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} - \gamma_2\|u\|^2_{\tilde{H}^{\alpha/2-\delta}(\Gamma)} \\ &\geq \gamma_1\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} - \gamma_2\Big(ch^{2\delta_1}\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} + c\|\mathcal{P}_0 u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\Big) \\ &= \big(\gamma_1 - c\gamma_2 h^{2\delta_1}\big)\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} - c\gamma_2\|\mathcal{P}_0 u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)}. \qquad (3.52)\end{aligned}$$

Now let $u = \sum_{j=0}^n u_j$ be a representation of u which satisfies Assumption (A4). Using the definition of the projection operators $\mathcal{P}_j$, the boundedness of $\mathcal{A}^\alpha$, the Cauchy-Schwarz inequality, and (A4) we conclude that there holds

$$\begin{aligned}\Re\langle \mathcal{A}^\alpha u, u\rangle &= \sum_{j=0}^n \Re\langle \mathcal{A}^\alpha u, u_j\rangle = \sum_{j=0}^n \Re\langle \mathcal{A}^\alpha \mathcal{P}_j u, u_j\rangle \\ &\leq c\sum_{j=0}^n \|\mathcal{P}_j u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\|u_j\|_{\tilde{H}^{\alpha/2}(\Gamma)} \leq c\Big(\sum_{j=0}^n \|\mathcal{P}_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}\Big)^{1/2}\Big(\sum_{j=0}^n \|u_j\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}\Big)^{1/2} \\ &\leq c\Big(\sum_{j=0}^n \|\mathcal{P}_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}\Big)^{1/2}\lambda_0^{-1/2}\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)}. \qquad (3.53)\end{aligned}$$

Combining (3.52) and (3.53) we find

$$\begin{aligned}\big(\gamma_1 - c\gamma_2 h^{2\delta_1}\big)\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} &\leq c\lambda_0^{-1/2}\Big(\sum_{j=0}^n \|\mathcal{P}_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}\Big)^{1/2}\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \\ &\quad + \tilde{c}\gamma_2\|\mathcal{P}_0 u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \\ &\leq c(\lambda_0^{-1/2} + \gamma_2)\Big(\sum_{j=0}^n \|\mathcal{P}_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}\Big)^{1/2}\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)}.\end{aligned}$$

That means, for h small enough such that $\gamma_1 - c\gamma_2 h^{2\delta_1} > 0$,

$$\begin{aligned}\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} &\leq c\big(\gamma_1 - C\gamma_2 h^{2\delta_1}\big)^{-2}(\lambda_0^{-1/2} + \gamma_2)^2 \sum_{j=0}^n \|\mathcal{P}_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \\ &\leq c(\lambda_0^{-1} + \gamma_2^2)\sum_{j=0}^n \|\mathcal{P}_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}.\end{aligned}$$

This proves the assertion for the operator $\mathcal{P}$. To prove the assertion for P we replace (3.53). To this end let us denote $P_0 := \mathcal{P}_0$. Then, analogously to (3.53), we obtain

$$\begin{aligned}\Re\langle \mathcal{A}^\alpha u, u\rangle &= \sum_{j=0}^n \Re\langle \mathcal{A}^\alpha u, u_j\rangle = \Re\langle \mathcal{A}^\alpha \mathcal{P}_0 u, u_0\rangle + \sum_{j=1}^n \langle A^\alpha P_j u, u_j\rangle \\ &\leq c\sum_{j=0}^n \|P_j u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\|u_j\|_{\tilde{H}^{\alpha/2}(\Gamma)} \leq c\Big(\sum_{j=0}^n \|P_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}\Big)^{1/2}\lambda_0^{-1/2}\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)}.\end{aligned}$$

The remainder of the proof is the same as for $\mathcal{P}$. □

The next theorems present abstract bounds for the minimum eigenvalues of the Hermitian parts of the additive Schwarz operators $\mathcal{P}$ and P and for the norms of $\mathcal{P}$ and P. These bounds determine the convergence of the GMRES method (see (3.1)) for solving the preconditioned indefinite or non-Hermitian linear boundary element systems

$$\mathcal{P}u = f_1 := \sum_{j=0}^{n} \mathcal{P}_j u \quad \text{and} \quad Pu = f_2 := \mathcal{P}_0 u + \sum_{j=1}^{n} P_j u$$

where the right hand sides f_1 and f_2 can be computed without knowing the solution u, cf. Section 3.1.

Theorem 3.9 *There exist constants $c, C, h_0, \delta_3 > 0$ such that, if $0 < h \le h_0$, there holds*

$$\Re\langle A^\alpha \mathcal{P}u, u\rangle \ge c\big(C(\lambda_0^{-1} + \gamma_2^2)^{-1} - h^{\delta_3}\big)\langle A^\alpha u, u\rangle \quad \textit{for any} \quad u \in X_N$$

and

$$\|\mathcal{P}u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \le c\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \quad \textit{for any} \quad u \in X_N.$$

Here, any $\delta_3 \le \min\{\delta_1, \delta_2\}$ can be chosen. The parameters δ_1 and δ_2 are those in Lemma 3.6.

Proof. We recall the steps by Cai, Widlund [28, Theorem 1] and Stephan, Tran [135, Theorem 2.1] to demonstrate where the assumptions and their conclusions enter.

First let us bound the norm of $\mathcal{P}$. By (3.49), the continuity of $\mathcal{A}^\alpha$, and (3.50)

$$\begin{aligned}
\sum_{j=1}^{n} \|\mathcal{P}_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} &\le c\big(C - h^{2\delta_2}\big)^{-1} \sum_{j=1}^{n} \Re\langle \mathcal{A}^\alpha \mathcal{P}_j u, \mathcal{P}_j u\rangle \\
&= c\big(C - h^{2\delta_2}\big)^{-1} \sum_{j=1}^{n} \Re\langle \mathcal{A}^\alpha u, \mathcal{P}_j u\rangle = c\big(C - h^{2\delta_2}\big)^{-1} \Re\langle \mathcal{A}^\alpha u, \sum_{j=1}^{n} \mathcal{P}_j u\rangle \\
&\le c\big(C - h^{2\delta_2}\big)^{-1} \|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \| \sum_{j=1}^{n} \mathcal{P}_j u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \\
&\le c\sqrt{K}\big(C - h^{2\delta_2}\big)^{-1} \|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \Big(\sum_{j=1}^{n} \|\mathcal{P}_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}\Big)^{1/2},
\end{aligned}$$

that is

$$\sum_{j=1}^{n} \|\mathcal{P}_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \le cK\big(C - h^{2\delta_2}\big)^{-2} \|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}. \tag{3.54}$$

Here, we have to assume that h is small enough such that $C - h^{2\delta_2} > 0$. Combining this with (3.51) and (3.46) we conclude

$$\| \sum_{j=0}^{n} \mathcal{P}_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \le \tilde{c}K\Big(c + K\big(C - h^{2\delta_2}\big)^{-2}\Big) \|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)},$$

i.e. the additive Schwarz operator $\mathcal{P}$ remains bounded if the mesh size h decreases. This is the second assertion of the theorem.

Now we bound the minimum eigenvalue of the Hermitian part of $\mathcal{P}$. Rewriting the relevant term by

$$\Re\langle A^\alpha \mathcal{P}u, u\rangle = \sum_{j=0}^{n} \Re\langle A^\alpha \mathcal{P}_j u, u\rangle \geq \sum_{j=0}^{n} \langle A^\alpha \mathcal{P}_j u, \mathcal{P}_j u\rangle - \Big|\sum_{j=0}^{n} \langle A^\alpha \mathcal{P}_j u, u\rangle - \langle A^\alpha \mathcal{P}_j u, \mathcal{P}_j u\rangle\Big|$$

and proving the estimate

$$\Big|\sum_{j=0}^{n} \langle A^\alpha \mathcal{P}_j u, u\rangle - \langle A^\alpha \mathcal{P}_j u, \mathcal{P}_j u\rangle\Big| \leq c\Big(h^{\delta_1} + h^{\delta_2}\big(C - h^{2\delta_2}\big)^{-1} + h^{\delta_2}\big(C - h^{2\delta_2}\big)^{-2}\Big)\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \tag{3.55}$$

we obtain by Lemma 3.8

$$\Re\langle A^\alpha \mathcal{P}u, u\rangle \geq \big(C(\lambda_0^{-1} + \gamma_2^2)^{-1} - c\big(h^{\delta_1} + h^{\delta_2}\big)\big)\langle A^\alpha u, u\rangle$$

if h is small enough. This is the first assertion of the theorem. It therefore remains to prove (3.55).

$$\begin{aligned}\Big|\sum_{j=0}^{n} \langle A^\alpha \mathcal{P}_j u, u\rangle - \langle A^\alpha \mathcal{P}_j u, \mathcal{P}_j u\rangle\Big| &= \Big|\sum_{j=0}^{n} \langle A^\alpha \mathcal{P}_j u, u - \mathcal{P}_j u\rangle\Big| = \Big|\sum_{j=0}^{n} \langle \mathcal{P}_j u, A^\alpha(u - \mathcal{P}_j u)\rangle\Big| \\ &= \Big|\sum_{j=0}^{n} \underbrace{\langle \mathcal{P}_j u, \mathcal{A}^\alpha(u - \mathcal{P}_j u)\rangle}_{=0} - \langle \mathcal{P}_j u, \mathcal{K}^\alpha(u - \mathcal{P}_j u)\rangle\Big| \\ &\leq \underbrace{\Big|\langle \mathcal{P}_0 u, \mathcal{K}^\alpha(u - \mathcal{P}_0 u)\rangle\Big|}_{=:I_1} + \underbrace{\Big|\sum_{j=1}^{n} \langle \mathcal{P}_j u, \mathcal{K}^\alpha u\rangle\Big|}_{=:I_2} + \underbrace{\Big|\sum_{j=1}^{n} \langle \mathcal{P}_j u, \mathcal{K}^\alpha \mathcal{P}_j u\rangle\Big|}_{=:I_3}\end{aligned} \tag{3.56}$$

We estimate the terms I_1, I_2, and I_3 separately. By the continuity of $\mathcal{K}^\alpha$ (3.42), and by (3.46), (3.47)

$$\begin{aligned}I_1 &= \big|\langle \mathcal{P}_0 u, \mathcal{K}^\alpha(u - \mathcal{P}_0 u)\rangle\big| \leq c_0 \|\mathcal{P}_0 u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \|u - \mathcal{P}_0 u\|_{\tilde{H}^{\alpha/2-\delta}(\Gamma)} \\ &\leq c c_0 \|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} h^{\delta_1} \|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} = c c_0 h^{\delta_1} \|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}.\end{aligned} \tag{3.57}$$

By (3.43), (3.50) (which holds also for the $\tilde{H}^{\alpha/2-\delta}$-norm instead of the $\tilde{H}^{\alpha/2}$-norm), (3.48),

and (3.54)

$$
\begin{aligned}
I_2 &= \Big|\sum_{j=1}^{n}\langle \mathcal{P}_j u, \mathcal{K}^\alpha u\rangle\Big| = \Big|\langle\sum_{j=1}^{n}\mathcal{P}_j u, \mathcal{K}^\alpha u\rangle\Big| \\
&\le c_0\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\|\sum_{j=1}^{n}\mathcal{P}_j u\|_{\tilde{H}^{\alpha/2-\delta}(\Gamma)} \le c_0\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} c\sqrt{K}\Big(\sum_{j=1}^{n}\|\mathcal{P}_j u\|^2_{\tilde{H}^{\alpha/2-\delta}(\Gamma)}\Big)^{1/2} \\
&\le c_0\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\sqrt{K}ch^{\delta_2}\Big(\sum_{j=1}^{n}\|\mathcal{P}_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}\Big)^{1/2} \\
&\le c_0\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\sqrt{K}ch^{\delta_2}\sqrt{K}\big(C-h^{2\delta_2}\big)^{-1}\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \\
&= c_0 Kch^{\delta_2}\big(C-h^{2\delta_2}\big)^{-1}\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}. \qquad (3.58)
\end{aligned}
$$

By (3.42), (3.48), (3.54)

$$
\begin{aligned}
I_3 &= \Big|\sum_{j=1}^{n}\langle \mathcal{P}_j u, \mathcal{K}^\alpha \mathcal{P}_j u\rangle\Big| \le \sum_{j=1}^{n}\Big|\langle \mathcal{P}_j u, \mathcal{K}^\alpha \mathcal{P}_j u\rangle\Big| \\
&\le \sum_{j=1}^{n} c_0\|\mathcal{P}_j u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\|\mathcal{P}_j u\|_{\tilde{H}^{\alpha/2-\delta}(\Gamma)} \le \sum_{j=1}^{n} c_0\|\mathcal{P}_j u\|_{\tilde{H}^{\alpha/2}(\Gamma)} ch^{\delta_2}\|\mathcal{P}_j u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \\
&= c_0 ch^{\delta_2}\sum_{j=1}^{n}\|\mathcal{P}_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \le c_0 ch^{\delta_2}K\big(C-h^{2\delta_2}\big)^{-2}\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}. \qquad (3.59)
\end{aligned}
$$

Combining (3.56), (3.57), (3.58), and (3.59) we obtain (3.55) which was left to be proved. □

The result for the operator P is asymptotically the same as for $\mathcal{P}$.

Theorem 3.10 *There exist constants $c, C, h_0, \delta_3 > 0$ such that, if $0 < h \le h_0$, there holds*

$$\Re\langle A^\alpha P u, u\rangle \ge c\big(C(\lambda_0^{-1}+\gamma_2^2)^{-1} - h^{\delta_3}\big)\langle A^\alpha u, u\rangle \quad \textit{for any} \quad u \in X_N$$

and

$$\|Pu\|_{\tilde{H}^{\alpha/2}(\Gamma)} \le c\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \quad \textit{for any} \quad u \in X_N.$$

Here, any $\delta_3 \le \min\{\delta_1, \delta_2\}$ can be chosen. The parameters δ_1 and δ_2 are those in Lemma 3.6.

Proof. The proof consists of parts of the proof of Theorem 3.9. First we bound the norm of P. By the continuity of $\mathcal{A}^\alpha$ and (3.50) (by replacing $\mathcal{P}_j$ by P_j) we obtain

$$
\begin{aligned}
\sum_{j=1}^{n}\|P_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} &= \sum_{j=1}^{n}\langle A^\alpha P_j u, P_j u\rangle = \langle \mathcal{A}^\alpha u, \sum_{j=1}^{n} P_j u\rangle \\
&\le c\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\|\sum_{j=1}^{n} P_j u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \\
&\le c\sqrt{K}\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\Big(\sum_{j=1}^{n}\|P_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}\Big)^{1/2},
\end{aligned}
$$

that is

$$\sum_{j=1}^{n} \|P_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \leq cK \|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}.$$

Combining this with (3.51) (and again replacing $\mathcal{P}_j$ by P_j) and (3.46) we conclude

$$\|Pu\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \leq 2\big(\|\mathcal{P}_0 u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} + \|\sum_{j=1}^{n} P_j u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}\big) \leq c\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)},$$

i.e. the additive Schwarz operator P is bounded. This is the second assertion of the theorem.

To bound the minimum eigenvalue of the Hermitian part of P we write, by denoting $P_0 := \mathcal{P}_0$,

$$\begin{aligned}
\Re\langle A^\alpha Pu, u\rangle &= \sum_{j=0}^{n} \Re\langle A^\alpha P_j u, u\rangle \\
&\geq \sum_{j=0}^{n} \langle A^\alpha P_j u, P_j u\rangle - \Big|\sum_{j=0}^{n} \langle A^\alpha P_j u, u\rangle - \langle A^\alpha P_j u, P_j u\rangle\Big| \\
&\geq \sum_{j=0}^{n} \langle A^\alpha P_j u, P_j u\rangle - \big|\langle A^\alpha \mathcal{P}_0 u, u - \mathcal{P}_0 u\rangle\big| - \big|\sum_{j=1}^{n} \langle A^\alpha P_j u, u - P_j u\rangle\big| \\
&= \sum_{j=0}^{n} \langle A^\alpha P_j u, P_j u\rangle - \underbrace{\big|\langle \mathcal{P}_0 u, \mathcal{K}^\alpha(u - \mathcal{P}_0 u)\rangle\big|}_{=I_1} - \underbrace{\big|\sum_{j=1}^{n} \langle P_j u, \mathcal{K}^\alpha u\rangle\big|}_{=:I_2'}.
\end{aligned}$$

By (3.57) there holds

$$I_1 = \big|\langle \mathcal{P}_0 u, \mathcal{K}^\alpha(u - \mathcal{P}_0 u)\rangle\big| \leq cc_0 h^{\delta_1} \|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}$$

and, analogously to (3.58), we obtain

$$I_2' = \big|\sum_{j=1}^{n} \langle P_j u, \mathcal{K}^\alpha u\rangle\big| \leq ch^{\delta_2}\big(C - h^{2\delta_2}\big)^{-1} \|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}.$$

Therefore, together with Lemma 3.8,

$$\Re\langle A^\alpha Pu, u\rangle \geq \big(C(\lambda_0^{-1} + \gamma_2^2)^{-1} - c\big(h^{\delta_1} + h^{\delta_2}\big)\big)\langle A^\alpha u, u\rangle$$

if h is small enough. This is the first assertion of the theorem which was left to be proved. □

Remark 3.2 *In general the parameter λ_0 which is the minimum eigenvalue of the corresponding additive Schwarz operator for the positive definite case may slightly depend on h or p. For instance, for the p-version of the boundary element method, we have a polylogarithmic dependence on p of λ_0^{-1} for all but the overlapping method. In order to obtain positive definiteness of the Hermitian part of $\mathcal{P}$ or P we then have to choose the mesh size h small enough, this depending on λ_0.*

3.4.2 Hybrid method

In this section we present a hybrid method for constructing preconditioners for indefinite boundary element systems. It consists in merging the solution of a small global problem for the original operator and an arbitrary preconditioner which is applicable for the positive definite part of the original operator. This hybrid method goes back to Xu and Cai [152] where it has been introduced as a preconditioner for the finite element method. This section relies on their theory in [152, Section 2].

In this work the hybrid method is used to prove the efficiency of the wire basket preconditioner for indefinite integral operators of order one. The decomposition (3.22) defining this preconditioner has an additional subspace which is not local. Therefore, Assumption (A2) (see page 82) does not hold and the theoretical results of the previous section are not applicable. However, the hybrid method is not restricted to the analysis of additive Schwarz type preconditioners.

We choose a subspace $H_0 \subset X_N$ which must satisfy Assumption (A1) on page 82, i.e., $\mathcal{A}^\alpha$ is invertible on H_0 and on finer spaces and the inverse operator is uniformly bounded. As before we define a coarse solver $\mathcal{P}_0 : X_N \to H_0$ such that, for given $u \in X_N$,

$$\langle \mathcal{A}^\alpha \mathcal{P}_0 u, v\rangle = \langle \mathcal{A}^\alpha u, v\rangle \quad \text{for any} \quad v \in H_0.$$

Further we define the restricted operator $\mathcal{A}_0^\alpha : H_0 \to H_0$ of $\mathcal{A}^\alpha$ by

$$\langle \mathcal{A}_0^\alpha u, v\rangle = \langle \mathcal{A}^\alpha u, v\rangle \quad \text{for any} \quad u, v \in H_0$$

and the L^2-projection $Q_0 : X_N \to H_0$ by

$$\langle Q_0 u, v\rangle = \langle u, v\rangle \quad \text{for any} \quad v \in H_0.$$

where $u \in X_N$ is given. There holds

$$\mathcal{A}_0^\alpha \mathcal{P}_0 = Q_0 \mathcal{A}_N^\alpha.$$

Now, let B^{-1} be a preconditioner for A_N^α, the Hermitian positive definite part of the discrete counterpart $\mathcal{A}_N^\alpha : H_N \to H_N$ of $\mathcal{A}^\alpha$. We specify the minimum and maximum eigenvalues of $B^{-1}A_N^\alpha : X_N \to X_N$ by λ_0 and λ_1, respectively. The final preconditioner for $\mathcal{A}_N^\alpha$ is defined by

$$\mathcal{B}^{-1} := (\mathcal{A}_0^\alpha)^{-1} Q_0 + \beta B^{-1}$$

which gives

$$\mathcal{B}^{-1} \mathcal{A}_N^\alpha = \mathcal{P}_0 + \beta B^{-1} \mathcal{A}_N^\alpha.$$

Here, $\beta > 0$ is a real parameter which balances the coarse grid contribution and the positive definite preconditioner. Theoretically the parameter β has to be chosen appropriately such that the hybrid method gives optimal results. In practise this choice can hardly be realized and β is replaced by a constant, e.g., $\beta = 1$.

The coarse grid solver $\mathcal{P}_0$ must be precise enough. This is characterized by the parameter δ_0 which is defined by

$$\delta_0 = \sup_{v \in X_N} \frac{\|v - \mathcal{P}_0 v\|_{\tilde{H}^{\alpha/2-\delta}(\Gamma)}}{\|v\|_{\tilde{H}^{\alpha/2}(\Gamma)}}.$$

From (3.47) we know that there holds $\delta_0 = O(h^{\delta_1})$. This parameter can be used to bound the norm of $\mathcal{P}_0$ (of course, the boundedness of $\mathcal{P}_0$ is already known from Lemma 3.6(i)).

Lemma 3.9 *There holds*

$$\|\mathcal{P}_0 u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \leq (1 + c_0\delta_0)\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \quad \textit{for any} \quad u \in X_N$$

and

$$\|\mathcal{P}_0 u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \leq 2\Re\langle A^\alpha \mathcal{P}_0 u, u\rangle + c_0^2\delta_0^2\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \quad \textit{for any} \quad u \in X_N.$$

The constant c_0 in the above estimates is that in (3.42).

Proof. This lemma is an application of Lemma 1 in [152] to the present situation. The proof given by Xu and Cai works in this case as well and is given for completeness.

Using (3.42) and the definition of δ_0 we obtain

$$\begin{aligned}
\|\mathcal{P}_0 u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} &= \langle A^\alpha \mathcal{P}_0 u, \mathcal{P}_0 u\rangle = \langle \mathcal{A}^\alpha u, \mathcal{P}_0 u\rangle - \langle \mathcal{K}^\alpha \mathcal{P}_0 u, \mathcal{P}_0 u\rangle \\
&= \langle A^\alpha u, \mathcal{P}_0 u\rangle + \langle \mathcal{K}^\alpha \big(I - \mathcal{P}_0\big) u, \mathcal{P}_0 u\rangle \\
&\leq \|u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\|\mathcal{P}_0 u\|_{\tilde{H}^{\alpha/2}(\Gamma)} + c_0\delta_0\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\|\mathcal{P}_0 u\|_{\tilde{H}^{\alpha/2}(\Gamma)}
\end{aligned}$$

which proves the first assertion. To prove the second assertion we note that there holds

$$\begin{aligned}
\langle A^\alpha \mathcal{P}_0 u, \mathcal{P}_0 u - u\rangle &= \langle \mathcal{A}^\alpha \mathcal{P}_0 u, \mathcal{P}_0 u - u\rangle - \langle \mathcal{K}^\alpha \mathcal{P}_0 u, \mathcal{P}_0 u - u\rangle \\
&= \langle \mathcal{A}^\alpha u, \mathcal{P}_0 u\rangle - \langle \mathcal{A}^\alpha \mathcal{P}_0 u, u\rangle - \langle \mathcal{K}^\alpha \mathcal{P}_0 u, \mathcal{P}_0 u - u\rangle \\
&= \langle \mathcal{K}^\alpha u, \mathcal{P}_0 u\rangle - \langle \mathcal{K}^\alpha \mathcal{P}_0 u, u\rangle - \langle \mathcal{K}^\alpha \mathcal{P}_0 u, \mathcal{P}_0 u - u\rangle + 2i\Im\langle A^\alpha u, \mathcal{P}_0 u\rangle \\
&= \langle \mathcal{K}^\alpha \big(I - \mathcal{P}_0\big) u, \mathcal{P}_0 u\rangle + 2i\Im\langle A^\alpha u, \mathcal{P}_0 u\rangle.
\end{aligned}$$

Here we used that A^α is selfadjoint with respect to $\langle\cdot,\cdot\rangle$. This relation gives, by using (3.42) and the definition of δ_0,

$$\begin{aligned}
\|\mathcal{P}_0 u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} &= \Re\langle A^\alpha \mathcal{P}_0 u, u\rangle + \Re\langle A^\alpha \mathcal{P}_0 u, \mathcal{P}_0 u - u\rangle = \Re\langle A^\alpha \mathcal{P}_0 u, u\rangle + \Re\langle \mathcal{K}^\alpha \big(I - \mathcal{P}_0\big) u, \mathcal{P}_0 u\rangle \\
&\leq \Re\langle A^\alpha \mathcal{P}_0 u, u\rangle + c_0\delta_0\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\|\mathcal{P}_0 u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \\
&\leq \Re\langle A^\alpha \mathcal{P}_0 u, u\rangle + \frac{1}{2}c_0^2\delta_0^2\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} + \frac{1}{2}\|\mathcal{P}_0 u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}
\end{aligned}$$

which gives the second assertion. □

The next theorem presents abstract bounds for the minimum eigenvalue of the Hermitian part of $\mathcal{B}^{-1}\mathcal{A}_N^\alpha$ and for the norm of $\mathcal{B}^{-1}\mathcal{A}_N^\alpha$. These bounds determine the convergence of the GMRES method for solving the preconditioned indefinite or non-Hermitian linear boundary element system $\mathcal{B}^{-1}\mathcal{A}_N^\alpha u = \mathcal{B}^{-1} g$, cf. (3.1).

Theorem 3.11 *There exist positive numbers $\varepsilon, \Lambda_0, \Lambda_1$ and β which depend on the extreme eigenvalues λ_0, λ_1 of $B^{-1}A_N^\alpha$ such that, if $\delta_0 \leq \varepsilon$,*

$$\Re\langle A^\alpha \mathcal{B}^{-1}\mathcal{A}_N^\alpha u, u\rangle \geq \Lambda_0\langle A^\alpha u, u\rangle \quad \textit{for any} \quad u \in X_N$$

and

$$\|\mathcal{B}^{-1}\mathcal{A}_N^\alpha u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \le \Lambda_1 \|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \quad \textit{for any} \quad u \in X_N.$$

More precisely, when choosing

$$\beta = \frac{\lambda_0}{2c_0^2\lambda_1^2} \quad \textit{and} \quad \delta_0 \le \varepsilon = \Big(\frac{\lambda_0^2}{4\lambda_1^2 c_0^2(c_0^2+1)}\Big)^{1/2}$$

the above estimates hold with

$$\Lambda_0 \ge \frac{\lambda_0^2}{8c_0^2\lambda_1^2} \quad \textit{and} \quad \Lambda_1 \le 1 + c_0\delta_0 + \beta\lambda_1(1+c_0).$$

Proof. This theorem is analogous to Theorem 1 in [152]. Here we explicitly take care of the order of the operator under consideration. However, the proof given by Xu and Cai works in this case as well and is recalled in order to precisely describe the constants Λ_0 and Λ_1 which influence the convergence of the GMRES method.

By denoting the minimum and maximum eigenvalues of $B^{-1}A_N^\alpha$ by λ_0 and λ_1, respectively, and by using the continuity of $\mathcal{K}^\alpha$ (3.42) there holds for $u \in X_N$

$$\begin{aligned}
\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} &= \langle A^\alpha u, u\rangle \le \frac{1}{\lambda_0}\langle A^\alpha B^{-1} A_N^\alpha u, u\rangle \\
&= \frac{1}{\lambda_0}\Re\langle A^\alpha B^{-1}\mathcal{A}_N^\alpha u, u\rangle - \frac{1}{\lambda_0}\Re\langle A^\alpha B^{-1}\mathcal{K}_N^\alpha u, u\rangle \\
&= \frac{1}{\lambda_0}\Re\langle A^\alpha B^{-1}\mathcal{A}_N^\alpha u, u\rangle - \frac{1}{\lambda_0}\Re\langle \mathcal{K}_N^\alpha u, B^{-1}A_N^\alpha u\rangle \\
&\le \frac{1}{\lambda_0}\Re\langle A^\alpha B^{-1}\mathcal{A}_N^\alpha u, u\rangle + \frac{c_0}{\lambda_0}\|u\|_{\tilde{H}^{\alpha/2-\delta}(\Gamma)}\|B^{-1}A_N^\alpha u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \\
&\le \frac{1}{\lambda_0}\Re\langle A^\alpha B^{-1}\mathcal{A}_N^\alpha u, u\rangle + c_0\frac{\lambda_1}{\lambda_0}\|u\|_{\tilde{H}^{\alpha/2-\delta}(\Gamma)}\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \\
&\le \frac{1}{\lambda_0}\Re\langle A^\alpha B^{-1}\mathcal{A}_N^\alpha u, u\rangle + \frac{1}{2}c_0^2\frac{\lambda_1^2}{\lambda_0^2}\|u\|^2_{\tilde{H}^{\alpha/2-\delta}(\Gamma)} + \frac{1}{2}\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}.
\end{aligned}$$

Here we used that $B^{-1}A_N^\alpha$ is symmetric with respect to the inner product given by A^α. Thus

$$\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \le \frac{2}{\lambda_0}\Re\langle A^\alpha B^{-1}\mathcal{A}_N^\alpha u, u\rangle + c_0^2\frac{\lambda_1^2}{\lambda_0^2}\|u\|^2_{\tilde{H}^{\alpha/2-\delta}(\Gamma)}.$$

Using the second assertion of Lemma 3.9 we obtain

$$\begin{aligned}
\|u\|^2_{\tilde{H}^{\alpha/2-\delta}(\Gamma)} &\le 2\|\mathcal{P}_0 u\|^2_{\tilde{H}^{\alpha/2-\delta}(\Gamma)} + 2\|u-\mathcal{P}_0 u\|^2_{\tilde{H}^{\alpha/2-\delta}(\Gamma)} \\
&\le 2\|\mathcal{P}_0 u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} + 2\delta_0^2\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \\
&\le 2\big(2\Re\langle A^\alpha\mathcal{P}_0 u, u\rangle + c_0^2\delta_0^2\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}\big) + 2\delta_0^2\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \\
&= 4\Re\langle A^\alpha\mathcal{P}_0 u, u\rangle + 2(c_0^2+1)\delta_0^2\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)},
\end{aligned}$$

and therefore, using the previous estimate,

$$\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \le \frac{2}{\lambda_0}\Re\langle A^\alpha B^{-1}\mathcal{A}_N^\alpha u, u\rangle + c_0^2\frac{\lambda_1^2}{\lambda_0^2}4\Re\langle A^\alpha\mathcal{P}_0 u, u\rangle + c_0^2\frac{\lambda_1^2}{\lambda_0^2}2(c_0^2+1)\delta_0^2\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)}.$$

Choosing $\varepsilon > 0$ such that

$$2c_0^2\frac{\lambda_1^2}{\lambda_0^2}(c_0^2+1)\,\varepsilon^2 = \frac{1}{2} \tag{3.60}$$

we obtain

$$\|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} \leq \frac{4}{\lambda_0}\Re\langle A^\alpha B^{-1}\mathcal{A}_N^\alpha u, u\rangle + 8c_0^2\frac{\lambda_1^2}{\lambda_0^2}\Re\langle A^\alpha \mathcal{P}_0 u, u\rangle$$

as long as $\delta_0 \leq \varepsilon$. Therefore the first assertion of the theorem holds with

$$\Lambda_0 = \frac{\lambda_0^2}{8c_0^2\lambda_1^2} \quad \text{and} \quad \beta = \frac{\lambda_0}{2c_0^2\lambda_1^2}.$$

Now we prove the second assertion of the theorem. Using the symmetry of B^{-1} and A_N^α, the continuity of $\mathcal{K}_N^\alpha$ (3.42), and again the symmetry of $B^{-1}A_N^\alpha$ with respect to A^α we obtain

$$\begin{aligned}\|B^{-1}\mathcal{K}_N^\alpha u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} &= \langle A^\alpha B^{-1}\mathcal{K}_N^\alpha u, B^{-1}\mathcal{K}_N^\alpha u\rangle = \langle \mathcal{K}_N^\alpha u, B^{-1}A_N^\alpha B^{-1}\mathcal{K}_N^\alpha u\rangle \\ &\leq c_0\|u\|_{\tilde{H}^{\alpha/2-\delta}(\Gamma)}\|B^{-1}A_N^\alpha B^{-1}\mathcal{K}_N^\alpha u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \\ &\leq c_0\lambda_1\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\|B^{-1}\mathcal{K}_N^\alpha u\|_{\tilde{H}^{\alpha/2}(\Gamma)},\end{aligned}$$

i.e.

$$\|B^{-1}\mathcal{K}_N^\alpha u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \leq c_0\lambda_1\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)}.$$

Using Lemma 3.9 we conclude that there holds

$$\begin{aligned}\|\mathcal{B}^{-1}\mathcal{A}_N^\alpha u\|_{\tilde{H}^{\alpha/2}(\Gamma)} &\leq \|\mathcal{P}_0 u\|_{\tilde{H}^{\alpha/2}(\Gamma)} + \beta\|B^{-1}\mathcal{A}_N^\alpha u\|_{\tilde{H}^{\alpha/2}(\Gamma)} \\ &\leq (1+c_0\delta_0)\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} + \beta\big(\|B^{-1}A_N^\alpha u\|_{\tilde{H}^{\alpha/2}(\Gamma)} + \|B^{-1}\mathcal{K}_N^\alpha u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\big) \\ &\leq (1+c_0\delta_0)\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} + \beta\big(\lambda_1\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)} + c_0\lambda_1\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)}\big) \\ &= \big(1+c_0\delta_0+\beta\lambda_1(1+c_0)\big)\|u\|_{\tilde{H}^{\alpha/2}(\Gamma)}.\end{aligned}$$

This proves the second assertion of the theorem with

$$\Lambda_1 \leq 1 + c_0\delta_0 + \beta\lambda_1(1+c_0).$$

□

Remark 3.3 *This remark addresses the same point as Remark* 3.2*. In general the extreme eigenvalues λ_0 and λ_1 of the preconditioned positive definite part will slightly depend on h or p. If λ_0 depends on p we then conclude from the condition on δ_0 that the mesh size h has to decrease for increasing p. Assuming that λ_1 is bounded, which is the case in most applications, we obtain the following asymptotic conditions and related estimates: When choosing $\delta_0 = O(h^{\delta_1})$ (cf. Lemma* 3.6*) and β such that*

$$\delta_0 \leq \frac{\lambda_0}{2\lambda_1 c_0\sqrt{c_0^2+1}} = O(\lambda_0) \quad \textit{and} \quad \beta = \frac{\lambda_0}{2c_0^2\lambda_1^2} = O(\lambda_0)$$

then

$$\Lambda_0 \geq \frac{\lambda_0^2}{8c_0^2\lambda_1^2} = O(\lambda_0^2) \quad \textit{and} \quad \Lambda_1 \leq 1 + c_0\delta_0 + \beta\lambda_1(1+c_0) = O(1).$$

3.4.3 Operators of order one

In this section we consider indefinite or non-selfadjoint operators of order one. Using the additive Schwarz method of Section 3.4.1, and in one case the hybrid method of Section 3.4.2, we show that the overlapping method of Section 3.2.2 as well as the iterative substructuring method of Section 3.2.3 provide efficient preconditioners also in the more general case of indefinite or non-selfadjoint operators.

Overlapping additive Schwarz method

Let us recall the method. By $X_N = S_p^1(\Gamma_h)$ we denote the space of continuous, piecewise polynomials of degree p on a rectangular mesh of size h. The number of interior nodes of the mesh is denoted by J_V. We take the decomposition

$$S_p^1(\Gamma_h) = S_1^1(\Gamma_h) \cup S_{p,1}^1 \cup \cdots \cup S_{p,J_V}^1 = H_0 \cup H_1 \cup \cdots \cup H_{J_V}$$

where $H_0 = S_1^1(\Gamma_h)$ denotes the space of continuous, piecewise bilinears on the actual mesh and $H_j = S_{p,j}^1$ is the space of piecewise polynomials on Γ_j' of degree p which vanish at the boundary of Γ_j' (the union of four elements around the interior node with number j).

As in Section 3.4.1 we consider the two operators

$$\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_1 + \cdots + \mathcal{P}_{J_V} \quad \text{and} \quad P = \mathcal{P}_0 + P_1 + \cdots + P_{J_V}.$$

The operators $\mathcal{P}_j$ ($j > 0$) are computed by locally inverting the indefinite operator $\mathcal{A}^\alpha$ on H_j whereas for P_j we need to solve local problems for the positive definite part A^α.

The overlapping method is optimal also for indefinite operators.

Theorem 3.12 *There exist positive constants c, C, h_0 such that, if $0 < h \le h_0$, there holds*

$$\Re\langle A^\alpha \mathcal{P} u, u\rangle \ge c\langle A^\alpha u, u\rangle \quad \textit{for any} \quad u \in X_N$$

and

$$\|\mathcal{P}u\|_{\tilde{H}^{1/2}(\Gamma)} \le C\|u\|_{\tilde{H}^{1/2}(\Gamma)} \quad \textit{for any} \quad u \in X_N.$$

Analogous estimates hold when replacing the operator $\mathcal{P}$ with P.

Proof. For an abstract proof of the theorem we refer to Theorem 3.9. We therefore only need to check the validity of Assumptions (A1)–(A4) of Section 3.4.1:
(A1) *The subspace H_0 is rich enough such that the Galerkin projection $\tilde{H}^{1/2}(\Gamma) \to \tilde{H}_0$ with respect to the operator $\mathcal{A}^\alpha$ exists and is bounded for all subspaces $\tilde{H}_0$ with $H_0 \subset \tilde{H}_0 \subset \tilde{H}^{1/2}(\Gamma)$.*
This condition can be fulfilled by choosing the mesh size h small enough since $\mathcal{A}^\alpha$ satisfy a Gårding inequality and since uniqueness is assumed, cf. [138].
(A2) *The subspaces H_j, $j = 1, \ldots, J_V$, in the decomposition of X_N are locally supported.*
This assumption holds by definition.
(A3) *The set $\{G_j;\ j = 1, \ldots, J_V\}$ is a finite covering of Γ.*
The coloring assumption is fulfilled since the union of the subdomains covers the domain Γ and since each subdomain consists of at most four elements.

(A4) *There exists a constant* $\lambda_0 > 0$ *such that for any* $u \in X_N$ *there exists a representation* $u = \sum_{j=0}^{J_V} u_j$ *according to the decomposition of* X_N *such that*

$$\sum_{j=0}^{J_V} \|u_j\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq \lambda_0^{-1} \|u\|^2_{\tilde{H}^{1/2}(\Gamma)}.$$

This is proved by Theorem 3.6. The number λ_0 can be bounded from below independently of h and p.
By Theorem 3.9 we conclude that there exist constants $c, C, h_0, \delta_3 > 0$ such that, if $0 < h \leq h_0$, there holds

$$\Re\langle A^\alpha \mathcal{P} u, u\rangle \geq c\big(C - h^{\delta_3}\big)\langle A^\alpha u, u\rangle \quad \text{for any} \quad u \in X_N$$

and

$$\|\mathcal{P} u\|_{\tilde{H}^{1/2}(\Gamma)} \leq c\|u\|_{\tilde{H}^{1/2}(\Gamma)} \quad \text{for any} \quad u \in X_N.$$

By choosing h small enough this proves the assertions of the theorem for the operator $\mathcal{P}$. In order to prove the assertions for the operator P we refer to Theorem 3.10 which is based on the same assumptions as Theorem 3.9. □

Numerical results for this preconditioner which underline the boundedness of the minimum eigenvalues Λ_0 of the Hermitian parts of $\mathcal{P}$ and P and of the norms Λ_1 of $\mathcal{P}$ and P are given in Section 4.1, see Figure 4.4 on page 129.

Iterative substructuring method

In this section we take the iterative substructuring method of Section 3.2.3 to define preconditioners for indefinite or non-selfadjoint operators of order one. Some variants of the iterative substructuring method do not directly fit into the framework presented in Section 3.4.1 which is based on standard additive Schwarz decompositions of the ansatz space. This is reflected by Assumptions (A2) and (A3), see page 82. Assumption (A2) is violated by the wire basket decomposition which comprises an additional subspace of global functions in the direct part of the decomposition. Assumption (A3) is not fulfilled when considering fine decompositions like the diagonal one. In that case, when increasing p, an increasing number of subspaces are defined over the same subdomains G_j which then are no more finite coverings of the domain Γ. However, Assumption (A3) only affects the estimate of the parameter Λ_1 (the norm of the preconditioned matrix) by Lemma 3.7 which can be proved for the diagonal decomposition without assuming (A3).

Let us consider the additive Schwarz operator obtained by the diagonal decomposition. The non-overlapping decomposition can be treated analogously and the corresponding preconditioner is not investigated in detail. We take the decomposition

$$X_N = S_p^1(\Gamma_h) = \tilde{X}_0 \cup \left(\cup_{j=1}^{J_V} H_{V_j} \ \oplus \ \cup_{j=1}^{J_E} \cup_{k=1}^{p-1} H_{E_j,k} \ \oplus \ \cup_{j=1}^{J} \cup_{k,l=1}^{p-1} H_{\Gamma_j,k,l} \right). \tag{3.61}$$

Here, $\tilde{X}_0 = S_1^1(\Gamma_h)$ denotes the space of continuous, piecewise bilinears on the actual mesh. J_V and J_E are the numbers of interior vertices and edges on Γ, respectively. The subspace

H_{V_j} is spanned by the vertex basis function concentrated at the vertex V_j. The subspace $H_{E_j,k}$ is spanned by an edge basis function concentrated at the edge E_j. This edge basis function is piecewise a linear transformation of the basis functions $e_k^{(1)}$, $e_k^{(2)}$, $e_k^{(3)}$, $e_k^{(4)}$. $H_{\Gamma_j,k,l}$ is the span of the interior basis function f_{kl} transformed onto the element Γ_j.

As before we consider two additive Schwarz operators

$$\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_1 + \cdots + \mathcal{P}_N \quad \text{and} \quad P = \mathcal{P}_0 + P_1 + \cdots + P_N.$$

The operators $\mathcal{P}_j$ ($j > 0$) are computed by locally inverting the indefinite operator $\mathcal{A}^\alpha$ on H_j whereas for P_j we need to solve positive definite problems involving A^α.

The estimates for the minimum eigenvalues Λ_0 of the Hermitian parts of $\mathcal{P}$ and P and for the norms of $\mathcal{P}$ and P for the non-overlapping decomposition and the modified diagonal decomposition are as follows.

Theorem 3.13 *There exist positive constants c, C, h_0, δ_3 such that, if $0 < h \le h_0$, there holds*

$$\Re\langle A^\alpha \mathcal{P}u, u\rangle \ge c\Big(C\big(1+\log(p+1)\big)^{-2} - h^{\delta_3}\Big)\langle A^\alpha u, u\rangle \quad \textit{for any} \quad u \in X_N$$

and

$$\|\mathcal{P}u\|_{\tilde{H}^{1/2}(\Gamma)} \le C\|u\|_{\tilde{H}^{1/2}(\Gamma)} \quad \textit{for any} \quad u \in X_N.$$

Here, any $\delta_3 \le \min\{\delta_1, \delta\}$ can be chosen. The parameter δ_1 is that in Lemma 3.6 *and δ is the parameter mentioned in* (3.42).

Analogous estimates hold when considering the non-overlapping decomposition (3.23) *instead of* (3.61) *and when replacing the operator $\mathcal{P}$ with P.*

Proof. An abstract proof of the theorem for the operator $\mathcal{P}$ is given by Theorem 3.9. We check Assumptions (A1), (A2), and (A4) of Section 3.4.1 and verify Lemma 3.7 without assuming (A3).

(A1) *The subspace H_0 is rich enough such that the Galerkin projection $\tilde{H}^{1/2}(\Gamma) \to \tilde{H}_0$ with respect to the operator $\mathcal{A}^\alpha$ exists and is bounded for all subspaces $\tilde{H}_0$ with $H_0 \subset \tilde{H}_0 \subset \tilde{H}^{1/2}(\Gamma)$.*

This can be fulfilled by choosing h small enough.

(A2) *The subspaces H_j, $j = 1, \ldots, J_V$, in the decomposition of X_N are locally supported.*

This assumption holds by definition.

(A4) *There exists a constant $\lambda_0 > 0$ such that for any $u \in X_N$ there exists a representation $u = \sum_{j=0}^N u_j$ according to the decomposition of X_N such that*

$$\sum_{j=0}^N \|u_j\|^2_{\tilde{H}^{1/2}(\Gamma)} \le \lambda_0^{-1}\|u\|^2_{\tilde{H}^{1/2}(\Gamma)}.$$

This is proved by Theorem 3.7 with

$$\lambda_0 \ge C\big(1+\log(p+1)\big)^{-2}$$

for a constant $C > 0$ which is independent of h and p.
Finally, we note that for the present decomposition Lemma 3.7, which was proved by assuming (A3), follows directly from the proof of Theorem 3.7. Indeed, the boundedness of the maximum eigenvalue of the additive Schwarz operator in the symmetric positive definite case for the present decomposition gives the required estimates. Thus, by Theorem 3.9 we conclude that there exist constants $c, C, h_0, \delta_3 > 0$ such that, if $0 < h \leq h_0$, there holds

$$\Re\langle A^\alpha \mathcal{P} u, u\rangle \geq c\big(C\big(1+\log(p+1)\big)^{-2} - h^{\delta_3}\big)\langle A^\alpha u, u\rangle \quad \text{for any} \quad u \in X_N$$

and

$$\|\mathcal{P}u\|_{\tilde{H}^{1/2}(\Gamma)} \leq c\|u\|_{\tilde{H}^{1/2}(\Gamma)} \quad \text{for any} \quad u \in X_N.$$

By choosing h small enough this proves the assertions of the theorem for $\mathcal{P}$ and the modified diagonal decomposition.

When considering the non-overlapping decomposition (3.23) instead of the full decomposition (3.61) the proof is applicable as well by noting that, in this case, Assumption (A3) is satisfied and Lemma 3.7 can be directly used.

The proof for the operator P is analogous by using Theorem 3.10 instead of Theorem 3.9. □

For completeness we now consider the preconditioner which is obtained by the hybrid method and the wire basket decomposition. We need a coarse subspace $H_0 \subset X_N$ on which the indefinite operator $\mathcal{A}^\alpha$ has to be inverted (yielding the projection operator $\mathcal{P}_0$) and a preconditioner B^{-1} which is known to be efficient for positive definite operators of order one. Denoting the L^2-projection operator $X_N \to H_0$ by Q_0 the final preconditioner then is given by

$$\mathcal{B}^{-1} := (\mathcal{A}_0^\alpha)^{-1} Q_0 + \beta B^{-1} \quad \text{and} \quad \mathcal{B}^{-1}\mathcal{A}_N^\alpha = \mathcal{P}_0 + \beta B^{-1}\mathcal{A}_N^\alpha.$$

Here, $\mathcal{A}_N^\alpha$ is the finite dimensional representation of $\mathcal{A}^\alpha$ on the ansatz space X_N and the parameter $\beta > 0$ has to chosen appropriately.

The operator $\mathcal{P}_0$ is already completely defined. For B^{-1} we take the additive Schwarz preconditioner defined by the wire basket decomposition (3.22) of X_N:

$$S_p^1(\Gamma_h) = \tilde{H}_0 \cup (H_0 \oplus H_1 \oplus \cdots \oplus H_J)$$

Here, $\tilde{H}_0$ is the subspace of piecewise bilinear functions and H_0 consists of the wire basket functions on Γ_h and $H_j = S_p^1(\Gamma_h) \cap \tilde{H}^{1/2}(\Gamma_j)$, $j = 1, \ldots, J$, see Figure 3.2. Then the preconditioner B^{-1} is the sum (over the subspaces) of the inverses of the positive definite part A^α of $\mathcal{A}^\alpha$ restricted to the individual subspaces of the above decomposition.

In both theoretical sections dealing with preconditioners for indefinite operators, Sections 3.4.1 and 3.4.2, the assumption that the coarse space is fine enough is present. For the hybrid method in Section 3.4.2 we introduced the parameter

$$\delta_0 = \sup_{v \in X_N} \frac{\|v - \mathcal{P}_0 v\|_{\tilde{H}^{1/2-\delta}(\Gamma)}}{\|v\|_{\tilde{H}^{1/2}(\Gamma)}} \leq c h^{\delta_1} \tag{3.62}$$

(the bound is due to Lemma 3.6).

The next theorem is the translation of Theorem 3.11 to the present particular situation. It presents abstract bounds for the minimum eigenvalue of the Hermitian part of $\mathcal{B}^{-1}\mathcal{A}_N^\alpha$ and for the norm of $\mathcal{B}^{-1}\mathcal{A}_N^\alpha$. These bounds determine the convergence of the GMRES method for solving the preconditioned indefinite or non-Hermitian linear boundary element system $\mathcal{B}^{-1}\mathcal{A}_N^\alpha u = \mathcal{B}^{-1}g$, cf. (3.1).

Theorem 3.14 *There exist constants $c, C > 0$ such that, for any integer $p_0 > 0$, there exists $h_0 > 0$ such that, if $h \le h_0$ and $p \le p_0$ there holds*

$$\Re\langle A^\alpha \mathcal{B}^{-1}\mathcal{A}_N^\alpha u, u\rangle \ge c(1+\log p)^{-4}\langle A^\alpha u, u\rangle \quad \text{for any} \quad u \in X_N$$

and

$$\|\mathcal{B}^{-1}\mathcal{A}_N^\alpha u\|_{\tilde{H}^{1/2}(\Gamma)} \le C\|u\|_{\tilde{H}^{1/2}(\Gamma)} \quad \text{for any} \quad u \in X_N.$$

The parameter β in the definition of $\mathcal{B}^{-1}$ has to decrease like $(1+\log p)^{-2}$.

Proof. We use Theorem 3.11 and recall sufficient conditions on the parameters and their conclusions. First, the coarse subspace H_0 must be fine enough such that $\delta_0 \le \varepsilon$ with $\varepsilon > 0$ satisfying

$$2c_0^2\frac{\lambda_1^2}{\lambda_0^2}(c_0^2+1)\ \varepsilon^2 = \frac{1}{2},$$

cf. (3.60). By (3.62) and the above relation it suffices to require that h is small enough such that there holds

$$ch^{\delta_1} \le \varepsilon = \Big(\frac{\lambda_0^2}{4\lambda_1^2 c_0^2(c_0^2+1)}\Big)^{1/2}.$$

Here, λ_0 and λ_1 are the minimum and maximum eigenvalues of the additive Schwarz operator P_W which corresponds to the preconditioner B^{-1}. From the proof of Theorem 3.7 we know that λ_1 is bounded and that the minimum eigenvalue can be bounded from below like

$$\lambda_0 \ge c(1+\log p)^{-2}.$$

We therefore have to ensure that there holds

$$h^{\delta_1} \le c(1+\log p)^{-2}.$$

This is an asymptotic condition which couples h and p. For any integer $p_0 > 0$ there exists a real number $h_0 > 0$ such that, if $h \le h_0$, this condition is satisfied for $p \le p_0$.

When choosing

$$\beta = \frac{\lambda_0}{2c_0^2\lambda_1^2} = O\big((1+\log p)^{-2}\big)$$

we conclude from Theorem 3.11 that the minimum eigenvalue Λ_0 of the Hermitian part of $\mathcal{B}^{-1}\mathcal{A}_N^\alpha$ is bounded from below like

$$\Lambda_0 \ge \frac{\lambda_0^2}{8c_0^2\lambda_1^2} \ge c(1+\log p)^{-4}.$$

Finally, by Theorem 3.11 the norm Λ_1 of $\mathcal{B}^{-1}\mathcal{A}_N^\alpha$ satisfies

$$\Lambda_1 \le 1 + c_0\delta_0 + \beta\lambda_1(1+c_0),$$

i.e. Λ_1 is bounded. This concludes the proof of the theorem. □

Numerical experiments with the iterative substructuring method are reported in Section 4.1. The results for the different decompositions and using the discrete harmonic basis functions are almost the same (cf. the numbers for the positive definite case in Table 4.3 on page 124) and therefore, only the results for the modified diagonal preconditioner are given in the indefinite case, see Figure 4.5 on page 131.

3.4.4 Operators of order minus one

We now deal with indefinite or non-selfadjoint operators $\mathcal{A}^\alpha : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, i.e. $\alpha = -1$. As prescribed in Section 3.4.1 the additive Schwarz method requires a decomposition of the ansatz space which has to fulfill several assumptions. We take the method of Section 3.3 which deals with additive Schwarz operators for positive definite operators of order minus one and verify these assumptions. Let us recall the decomposition of the ansatz space X_N.

The boundary element mesh is, as usual, $\bar{\Gamma}_h = \cup_{j=1}^J \bar{\Gamma}_j$ with rectangles Γ_j and X_N consists of all piecewise polynomials of degree p. We take a coarse mesh $\bar{\Gamma}_H = \cup_{j=1}^n \bar{G}_j$ of size $H \ge h$ and decompose

$$X_N = H_0 \cup H_1 \cup \cdots \cup H_n$$

where H_0 is the space of piecewise constant functions on the mesh Γ_H and

$$H_j := \{v|_{G_j};\ v \in X_N \setminus H_0,\ \langle v, 1\rangle_{L^2(G_j)} = 0\}, \quad j = 1, \dots, n.$$

Two versions of the preconditioner are defined,

$$\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_1 + \cdots + \mathcal{P}_n \quad \text{and} \quad P = \mathcal{P}_0 + P_1 + \cdots + P_n.$$

The operators $\mathcal{P}_j$ ($j > 0$) are computed by locally inverting the indefinite operator $\mathcal{A}^\alpha$ whereas for P_j we need to solve positive definite problems for A^α.

The efficiency of the additive Schwarz operators $\mathcal{P}$ and P is proved by the following theorem.

Theorem 3.15 *There exist positive constants c, C, H_0, δ_3 such that, if $0 < H \le H_0$, there holds*

$$\Re\langle A^\alpha \mathcal{P}u, u\rangle \ge c\Big(C\big(1 + \log(\frac{H}{h}(p+1))\big)^{-2} - H^{\delta_3}\Big)\langle A^\alpha u, u\rangle \quad \textit{for any} \quad u \in X_N$$

and

$$\|\mathcal{P}u\|_{\tilde{H}^{-1/2}(\Gamma)} \le c\|u\|_{\tilde{H}^{-1/2}(\Gamma)} \quad \textit{for any} \quad u \in X_N.$$

Here, any $\delta_3 \le \min\{\delta_1, \delta\}$ can be chosen. The parameter δ_1 is that in Lemma 3.6 and δ is the parameter mentioned in (3.42).

Analogous estimates hold for the operator P.

Proof. The abstract form of the theorem for the operator $\mathcal{P}$ has been proved with Theorem 3.9. We therefore only need to check the validity of Assumptions (A1)–(A4).
(A1) *The subspace H_0 is rich enough such that the Galerkin projection $\tilde{H}^{-1/2}(\Gamma) \to \tilde{H}_0$ with respect to the operator $\mathcal{A}^\alpha$ exists and is bounded for all subspaces $\tilde{H}_0$ with $H_0 \subset \tilde{H}_0 \subset \tilde{H}^{-1/2}(\Gamma)$.*
This assumption can be satisfied by choosing the mesh size H of the coarse mesh fine enough since $H_0 = S_0^0(\Gamma_H)$.
(A2) *The subspaces H_j, $j = 1, \ldots, n$, in the decomposition of X_N are locally supported. In the case $\alpha = -1$ all the functions in any of the subspaces $H_1, \ldots, H_n$ have integral mean zero.*
This assumption holds by construction since the subspaces H_j $(j > 0)$ are confined to single elements of the coarse mesh Γ_H which can be made small enough. The condition that functions of the subspaces H_j $(j > 0)$ must have integral mean zero holds by construction.
(A3) *The set $\{G_j;\ j = 1, \ldots, n\}$ is a finite covering of Γ.*
The coloring assumption is trivially fulfilled since the subdomains are non-overlapping.
(A4) *There exists a constant $\lambda_0 > 0$ such that for any $u \in X_N$ there exists a representation $u = \sum_{j=0}^n u_j$ according to the decomposition of X_N such that*

$$\sum_{j=0}^{n} \|u_j\|^2_{\tilde{H}^{-1/2}(\Gamma)} \leq \lambda_0^{-1} \|u\|^2_{\tilde{H}^{-1/2}(\Gamma)}.$$

This is the contents of Theorem 3.8. Actually, its proof shows that the maximum eigenvalue of the additive Schwarz operator in the positive definite case is bounded and that its minimum eigenvalue λ_0 can be bounded from below by $c(1 + \log \frac{H}{h}(p+1))^{-2}$. The constant c does not depend on p and h. By Theorem 3.9 and by noting that the constant γ_2 which appears in that theorem (actually, it is due to the Gårding's inequality, cf. (3.41)) does not depend on p this concludes the proof.

The proof for the operator P is analogous by using Theorem 3.10 instead of Theorem 3.9. □

3.5 Systems of pseudo-differential operators

In this section we consider preconditioners for linear systems arising from the Galerkin method for systems of pseudo-differential equations. We first consider symmetric positive definite systems which are of no practical relevance but are used in some proofs. In Section 3.5.1 we show that the standard additive Schwarz method and the hybrid method are applicable to non-positive definite systems in an efficient way. Numerical results are given in Section 4.2. The main theoretical results are generalizations of the results in the scalar case given in Section 3.4.1 and 3.4.2. Finally, in Section 3.5.2 we consider skew-symmetric systems which may arise in the coupling of the FEM and the BEM. For numerical experiments we refer to Section 4.4. We note that there are efficient procedures for saddle point problems which are not considered here, see, e.g., [21, 115, 146, 124, 137, 65]

As a first step, we consider symmetric positive definite systems $\underline{A}^\alpha$ of continuous pseudo-differential operators

$$\underline{A}^\alpha :\ X = \tilde{H}^{\alpha_1/2}(D_1) \times \cdots \times \tilde{H}^{\alpha_n/2}(D_n) \to X' = H^{-\alpha_1/2}(D_1) \times \cdots \times H^{-\alpha_n/2}(D_n),$$

$$\underline{A}^{\alpha}(u,u) := \langle \underline{A}^{\alpha} u, u\rangle \geq c \sum_{j=1}^{n} \|u_j\|^2_{\tilde{H}^{\alpha_j/2}(D_j)} =: c\|u\|^2_{\tilde{H}^{\alpha/2}}. \tag{3.63}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the duality between X and X' and $\alpha = (\alpha_1, \ldots, \alpha_n)$. The domains D_j need not to be the same. For example, when coupling the finite element and the boundary element method, some of the domains may be part of the three-dimensional space whereas other domains may be surfaces. In order to solve a system

$$\underline{A}^{\alpha} U = g$$

for given $g \in X'$ we use the Galerkin method: For a given finite dimensional subspace

$$X_N = X_{1,N_1} \times \cdots \times X_{n,N_n} \subset X$$

find $u = (u_1, \ldots, u_n) \in X_N$ such that

$$\underline{A}^{\alpha}(u,v) = g(v) := \langle g, v\rangle \quad \text{for any} \quad v = (v_1, \ldots, v_n) \in X_N. \tag{3.64}$$

The abstract additive Schwarz method can also be applied to systems of pseudo-differential operators. This will be explained in the following.

To this end we implicitly decompose the ansatz space X_N by decomposing its components X_{j,N_j}:

$$X_{j,N_j} = H_{j,0} \cup \cdots \cup H_{j,n_j}.$$

In this way we obtain a decomposition for X_N,

$$X_N = \hat{H}_{1,0} \cup \cdots \cup \hat{H}_{1,n_1} \cup \cdots \cup \hat{H}_{n,0} \cup \cdots \cup \hat{H}_{n,n_n}, \tag{3.65}$$

where

$$\hat{H}_{j,i} := \underbrace{\{0\} \times \cdots \times \{0\}}_{j-1 \text{ times}} \times H_{j,i} \times \underbrace{\{0\} \times \cdots \times \{0\}}_{n-j \text{ times}}$$

is the embedded space of $H_{j,i}$ in X_N. The additive Schwarz method consists in solving, by an iterative method, e.g. the GMRES method, the equation

$$\underline{P}(u) := \Big(\sum_{j=1}^{n} \sum_{i=0}^{n_j} \underline{P}_{j,i}\Big)(u) = F, \tag{3.66}$$

where the projections $\underline{P}_{j,i} : X_N \to \hat{H}_{j,i}$, $j = 1, \ldots, n$, $i = 0, \ldots, n_j$, are defined for any $v \in X_N$ by

$$\underline{A}^{\alpha}(\underline{P}_{j,i} v, w) = \underline{A}^{\alpha}(v,w) \quad \text{for any } w \in \hat{H}_{j,i}.$$

As in the scalar case, the right hand side of (3.66), $F = \sum_{j=1}^{n} \sum_{i=0}^{n_j} \underline{P}_{j,i}(u)$, can be computed without knowing the solution u of (3.64).

Let us define individual additive Schwarz operators P_j on the individual components of X_N by $P_j := \sum_{i=0}^{n_j} P_{j,i}$ where $P_{j,i} : X_{j,N_j} \to H_{j,i}$ is given by

$$\underline{A}^{\alpha}(\widehat{P_{j,i} v}, \hat{w}) = \underline{A}^{\alpha}(\hat{v}, \hat{w}) \quad \text{for any} \quad w \in H_{j,i}.$$

Here, $\hat{\varphi}$ denotes the embedding of $\varphi \in H_{j,i}$ or $\varphi \in X_{j,N_j}$ within X_N, as appropriate. By estimating the extreme eigenvalues of the additive Schwarz operators $P_j := \sum_{i=0}^{n_j} P_{j,i}$ we obtain bounds for the extreme eigenvalues of the whole additive Schwarz operator $\underline{P}$.

Lemma 3.10 *Let $\lambda_{j,0}$ and $\lambda_{j,1}$ denote the minimum and maximum eigenvalues, respectively, of the additive Schwarz operators $P_j := \sum_{i=0}^{n_j} P_{j,i}$, $j = 1, \dots, n$. There exist constants $c, C > 0$ which are independent of the extreme eigenvalues of P_j, $j = 1 \dots, n$, such that there holds*

$$\lambda_{\min}(\underline{P}) \geq c \min\{\lambda_{j,0};\ j = 1, \dots, n\} \quad \textit{and} \quad \lambda_{\max}(\underline{P}) \leq C \max\{\lambda_{j,1};\ j = 1, \dots, n\}.$$

Proof. Since

$$X_N = \hat{X}_{1,N_1} \cup \cdots \cup \hat{X}_{n,N_n}$$

is a direct sum decomposition the lemma is consequence of Lemmas 3.1 and 3.2. Let $u = (u_1, \dots, u_n) \in X_N$ be given and let $u_j = \sum_{i=0}^{n_j} u_{j,i}$ be any representation of the component u_j, $j = 1, \dots, n$. By Lemma 3.2 there holds

$$\|u_j\|^2_{\tilde{H}^{\alpha_j/2}(D_j)} \leq \lambda_{j,1} \sum_{i=0}^{n_j} \|u_{j,i}\|^2_{\tilde{H}^{\alpha_j/2}(D_j)},$$

and therefore, by the continuity and the positive definiteness of $\underline{A}^\alpha$,

$$\begin{aligned}\underline{A}^\alpha(u,u) = \underline{A}^\alpha(\sum_{j=1}^{n} \hat{u}_j, \sum_{j=1}^{n} \hat{u}_j) &\leq C \sum_{j=1}^{n} \|u_j\|^2_{\tilde{H}^{\alpha_j/2}(D_j)} \leq C \sum_{j=1}^{n} \lambda_{j,1} \sum_{i=0}^{n_j} \|u_{j,i}\|^2_{\tilde{H}^{\alpha_j/2}(D_j)} \\ &\leq C \max\{\lambda_{j,1};\ j = 1, \dots, n\} \sum_{j=1}^{n} \sum_{i=0}^{n_j} \underline{A}^\alpha(\hat{u}_{j,i}, \hat{u}_{j,i}).\end{aligned}$$

This gives the bound for the maximum eigenvalue of $\underline{P}$. The bound for the minimum eigenvalue is obtained as follows. By Lemma 3.1 there exist for all components u_j of u a representation $u_j = \sum_{i=0}^{n_j} u_{j,i}$ such that

$$\lambda_{j,0} \sum_{i=0}^{n_j} \|u_{j,i}\|^2_{\tilde{H}^{\alpha_j/2}(D_j)} \leq \|u_j\|^2_{\tilde{H}^{\alpha_j/2}(D_j)}$$

and therefore, by the positive definiteness and the continuity of $\underline{A}^\alpha$,

$$\begin{aligned}\underline{A}^\alpha(u,u) = \underline{A}^\alpha(\sum_{j=1}^{n} \hat{u}_j, \sum_{j=1}^{n} \hat{u}_j) &\geq c \sum_{j=1}^{n} \|u_j\|^2_{\tilde{H}^{\alpha_j/2}(D_j)} \geq c \sum_{j=1}^{n} \lambda_{j,0} \sum_{i=0}^{n_j} \|u_{j,i}\|^2_{\tilde{H}^{\alpha_j/2}(D_j)} \\ &\geq c \min\{\lambda_{j,0};\ j = 1, \dots, n\} \sum_{j=1}^{n} \sum_{i=0}^{n_j} \underline{A}^\alpha(\hat{u}_{j,i}, \hat{u}_{j,i}),\end{aligned}$$

which yields the bound for the minimum eigenvalue of $\underline{A}^\alpha$. □

3.5.1 Indefinite systems

Now, we consider indefinite or non-Hermitian systems in more detail. This section is a generalization to systems of Section 3.4 which deals with scalar operators.

The weak formulation of the problem is: For given $g \in H^{-\alpha/2}$ find $U \in X = \tilde{H}^{\alpha/2}$ such that

$$\langle \underline{\mathcal{A}}^{\alpha} U, v\rangle = \langle g, v\rangle \quad \text{for any} \quad v \in X \tag{3.67}$$

where

$$\underline{\mathcal{A}}^{\alpha} = \underline{A}^{\alpha} + \underline{\mathcal{K}}^{\alpha}$$

is the sum of a selfadjoint, positive definite operator $\underline{A}^{\alpha} : X \to X'$ and a bounded operator $\underline{\mathcal{K}}^{\alpha} : \tilde{H}^{\alpha/2} \to H^{-\alpha/2+\delta}$ of lower order $\alpha - \delta$, $\delta = (\delta_1, \dots, \delta_n)$. We assume that $\underline{\mathcal{A}}^{\alpha}$ satisfies a Gårding inequality

$$\Re\langle \theta \underline{\mathcal{A}}^{\alpha} u, u\rangle \geq \gamma_1 \|u\|^2_{\tilde{H}^{\alpha/2}} - \gamma_2 \|u\|^2_{\tilde{H}^{\alpha/2-\delta}} \tag{3.68}$$

for constants $\gamma_1, \gamma_2 > 0$ and a smooth complex valued matrix $\theta(x) \in \mathbb{C}^{n\times n}$, where, as defined previously,

$$\|u\|^2_{\tilde{H}^{\alpha/2}} = \sum_{j=1}^{n} \|u_j\|^2_{\tilde{H}^{\alpha_j/2}(D_j)}, \quad \alpha = (\alpha_1, \dots, \alpha_n),$$

and analogously for $\alpha/2 - \delta$. Further, we assume that there exist constants $c_0, \delta_j > 0$, $j = 1, \dots, n$, such that there holds

$$\left|\langle \underline{\mathcal{K}}^{\alpha} u, v\rangle\right| \leq c_0 \|u\|_{\tilde{H}^{\alpha/2-\delta}} \|v\|_{\tilde{H}^{\alpha/2}} \quad \text{for any} \quad u, v \in \tilde{H}^{\alpha/2} \tag{3.69}$$

and

$$\left|\langle \underline{\mathcal{K}}^{\alpha} u, v\rangle\right| \leq c_0 \|u\|_{\tilde{H}^{\alpha/2}} \|v\|_{\tilde{H}^{\alpha/2-\delta}} \quad \text{for any} \quad u, v \in \tilde{H}^{\alpha/2}. \tag{3.70}$$

In general we require that (3.67) has a unique solution. As for the positive definite case we use the Galerkin method to approximately solve (3.67): For a given finite dimensional subspace $X_N = X_{1,N_1} \times \cdots \times X_{n,N_n} \subset X$ find $u = (u_1, \dots, u_n) \in X_N$ such that

$$\underline{\mathcal{A}}^{\alpha}(u, v) = g(v) := \langle g, v\rangle \quad \text{for any} \quad v = (v_1, \dots, v_n) \in X_N. \tag{3.71}$$

The aim is to find preconditioners for the above linear system. As is the scalar case we investigate two variants of the usual additive Schwarz method as well as the abstract hybrid method, cf. Sections 3.4.1 and 3.4.2.

Additive Schwarz method

We use the general decomposition (3.65) and define the additive Schwarz operators

$$\underline{\mathcal{P}} := \underline{\mathcal{P}}_0 + \sum_{j=1}^{n} \sum_{i=1}^{n_j} \underline{\mathcal{P}}_{j,i} \quad \text{and} \quad \underline{P} := \underline{\mathcal{P}}_0 + \sum_{j=1}^{n} \sum_{i=1}^{n_j} \underline{P}_{j,i}. \tag{3.72}$$

Here, $\underline{\mathcal{P}}_0 : X_N \to H_0 := \cup_{j=1}^n \hat{H}_{j,0}$ is the Galerkin projection onto H_0, i.e., for $v \in X_N$

$$\underline{\mathcal{A}}^\alpha(\underline{\mathcal{P}}_0 v, w) = \underline{\mathcal{A}}^\alpha(v, w) \quad \text{for any } w \in H_0,$$

and the operators $\underline{\mathcal{P}}_{j,i}$, $j = 1, \ldots, n$, $i = 1, \ldots, n_j$, project onto $\hat{H}_{j,i}$, i.e., for $v \in X_N$,

$$\underline{\mathcal{A}}^\alpha(\underline{\mathcal{P}}_{j,i} v, w) = \underline{\mathcal{A}}^\alpha(v, w) \quad \text{for any } w \in \hat{H}_{j,i}.$$

Finally, the operators $\underline{P}_{j,i} : X_N \to \hat{H}_{j,i}$, $j = 1, \ldots, n$, $i = 1, \ldots, n_j$, are defined for any $v \in X_N$ by

$$\underline{A}^\alpha(\underline{P}_{j,i} v, w) = \underline{\mathcal{A}}^\alpha(v, w) \quad \text{for any } w \in \hat{H}_{j,i}.$$

Therefore, only the inversions of local positive definite problems are required in these cases.

As for scalar indefinite operators we need to make some assumptions concerning the coarse grid space H_0 and the decompositions of the individual components of the ansatz space X_N.

Assumptions:

(A1) The subspace H_0 is rich enough, e.g. by choosing the individual mesh sizes h_j small enough, such that the Galerkin projection $\tilde{H}^{\alpha/2} \to \tilde{H}_0$ with respect to the operator $\underline{\mathcal{A}}^\alpha$ exists and is bounded for all subspaces $\tilde{H}_0$ with $H_0 \subset \tilde{H}_0 \subset \tilde{H}^{\alpha/2}$.

For $j = 1, \ldots, n$ we require

(A2) The subspaces $H_{j,i}$, $i = 1, \ldots, n_j$, in the decomposition of X_{j,N_j} are locally supported. That means the union $\bar{G}_{j,i}$ of the supports of the functions in a single subspace $H_{j,i}$ is covered by a finite number of neighboring elements of the boundary element mesh on D_j. For simplicity we further assume that $G_{j,i}$ is rectangular.
In the case $\alpha_j = -1$ all the functions in any of the subspaces $H_{j,1}, \ldots, H_{j,n_j}$ have integral mean zero.

(A3) The set $\{G_{j,i};\ i = 1, \ldots, n_j\}$ is a finite covering of D_j. That means we can color $\{G_{j,i};\ i = 1, \ldots, n_j\}$ using a finite number of colors (let's say K_j) in such a way that subdomains $G_{j,i}$ of the same color are disjoint.

(A4) There exists a constant $\lambda_{j,0} > 0$ such that for any $u_j \in X_{j,N_j}$ there exists a representation $u_j = \sum_{i=0}^{n_j} u_{j,i}$ according to the decomposition

$$X_{j,N_j} = H_{j,0} \cup \ldots \cup H_{j,n_j}$$

such that

$$\sum_{i=0}^{n_j} \|u_{j,i}\|^2_{\tilde{H}^{\alpha_j/2}(D_j)} \le \lambda_{j,0}^{-1} \|u_j\|^2_{\tilde{H}^{\alpha_j/2}(D_j)}.$$

We need to prove a result which corresponds to Lemma 3.8 which deals with the scalar case.

Lemma 3.11 *There exist constants $c, h_0 > 0$ such that, if $h_j \le h_0$, $j = 1, \dots, n$, there holds*

$$\|\underline{\mathcal{P}}_0 u\|^2_{\tilde{H}^{\alpha/2}} + \sum_{j=1}^{n}\sum_{i=1}^{n_j} \|\underline{\mathcal{P}}_{j,i} u\|^2_{\tilde{H}^{\alpha/2}} \ge c(\max_j \lambda_{j,0}^{-1} + \gamma_2^2)^{-1} \|u\|^2_{\tilde{H}^{\alpha/2}} \quad \textit{for any} \quad u \in X_N.$$

The same estimate holds if one replaces $\underline{\mathcal{P}}_{j,i}$ by $\underline{P}_{j,i}$.

Proof. Let $u = (u_1, \dots, u_n)^T \in X_N$ be given. For the trivially embedded function $u_j \to X_N$ we again use the abbreviation $\hat{u}_j$. Analogously to (3.46) Assumption (A1) means that there exists a constant $c > 0$ such that

$$\|\underline{\mathcal{P}}_0 u\|_{\tilde{H}^{\alpha/2}} \le c\|u\|_{\tilde{H}^{\alpha/2}} \quad \text{for any} \quad u \in X_N, \tag{3.73}$$

and, depending on the regularity of $(\underline{\mathcal{A}}^\alpha)^{-1}$, there exist constants $c, \delta_{j,1} > 0$, $j = 1 \dots, n$, such that

$$\|u - \underline{\mathcal{P}}_0 u\|^2_{\tilde{H}^{\alpha/2-\delta}} \le c \sum_{j=1}^{n} h_j^{2\delta_{j,1}} \|u_j\|^2_{\tilde{H}^{\alpha_j/2}(D_j)} \quad \text{for any} \quad u \in X_N.$$

Thus we obtain

$$\begin{aligned}
\|u\|^2_{\tilde{H}^{\alpha/2-\delta}} &\le 2\|u - \underline{\mathcal{P}}_0 u\|^2_{\tilde{H}^{\alpha/2-\delta}} + 2\|\underline{\mathcal{P}}_0 u\|^2_{\tilde{H}^{\alpha/2-\delta}} \\
&\le c \sum_{j=1}^{n} h_j^{2\delta_{j,1}} \|u_j\|^2_{\tilde{H}^{\alpha_j/2}(D_j)} + c\|\underline{\mathcal{P}}_0 u\|_{\tilde{H}^{\alpha/2}} \|u\|_{\tilde{H}^{\alpha/2}}
\end{aligned}$$

and, together with Gårding's inequality, this gives

$$\begin{aligned}
\Re\langle \underline{\mathcal{A}}^\alpha u, u\rangle &\ge \gamma_1 \|u\|^2_{\tilde{H}^{\alpha/2}} - \gamma_2 \|u\|^2_{\tilde{H}^{\alpha/2-\delta}} \\
&\ge \gamma_1 \|u\|^2_{\tilde{H}^{\alpha/2}} - \gamma_2 \Big(c \sum_{j=1}^{n} h_j^{2\delta_{j,1}} \|u_j\|^2_{\tilde{H}^{\alpha_j/2}(D_j)} + c\|\underline{\mathcal{P}}_0 u\|_{\tilde{H}^{\alpha/2}} \|u\|_{\tilde{H}^{\alpha/2}} \Big) \\
&= \sum_{j=1}^{n} \big(\gamma_1 - c\gamma_2 h_j^{2\delta_{j,1}}\big) \|u_j\|^2_{\tilde{H}^{\alpha_j/2}(D_j)} - c\gamma_2 \|\underline{\mathcal{P}}_0 u\|_{\tilde{H}^{\alpha/2}} \|u\|_{\tilde{H}^{\alpha/2}}.
\end{aligned} \tag{3.74}$$

Now let $u_j = \sum_{i=0}^{n_j} u_{j,i}$, $j = 1, \dots, n$, be representations of the components u_j of u which satisfy Assumption (A4) and let $u_0 := \sum_{j=1}^{n} \hat{u}_{j,0}$. Using the definition of the projection operators $\underline{\mathcal{P}}_0$, $\underline{\mathcal{P}}_{j,i}$, the boundedness of $\underline{\mathcal{A}}^\alpha$, the Cauchy-Schwarz inequality, and (A4) we

conclude that there holds

$$\begin{aligned}
\Re\langle\underline{\mathcal{A}}^{\alpha}u,u\rangle &= \Re\langle\underline{\mathcal{A}}^{\alpha}u,u_0\rangle + \sum_{j=1}^{n}\sum_{i=1}^{n_j}\Re\langle\underline{\mathcal{A}}^{\alpha}u,\hat{u}_{j,i}\rangle \\
&= \Re\langle\underline{\mathcal{A}}^{\alpha}\underline{\mathcal{P}}_0u,u_0\rangle + \sum_{j=1}^{n}\sum_{i=1}^{n_j}\Re\langle\underline{\mathcal{A}}^{\alpha}\underline{\mathcal{P}}_{j,i}u,\hat{u}_{j,i}\rangle \\
&\leq c\|\underline{\mathcal{P}}_0u\|_{\tilde{H}^{\alpha/2}}\|u_0\|_{\tilde{H}^{\alpha/2}} + c\sum_{j=1}^{n}\sum_{i=1}^{n_j}\|\underline{\mathcal{P}}_{j,i}u\|_{\tilde{H}^{\alpha/2}}\|\hat{u}_{j,i}\|_{\tilde{H}^{\alpha/2}} \\
&= c\|\underline{\mathcal{P}}_0u\|_{\tilde{H}^{\alpha/2}}\|u_0\|_{\tilde{H}^{\alpha/2}} + c\sum_{j=1}^{n}\sum_{i=1}^{n_j}\|\underline{\mathcal{P}}_{j,i}u\|_{\tilde{H}^{\alpha/2}}\|u_{j,i}\|_{\tilde{H}^{\alpha_j/2}(D_j)} \\
&\leq c\Big(\|\underline{\mathcal{P}}_0u\|^2_{\tilde{H}^{\alpha/2}} + \sum_{j=1}^{n}\sum_{i=1}^{n_j}\|\underline{\mathcal{P}}_{j,i}u\|^2_{\tilde{H}^{\alpha/2}}\Big)^{1/2}\Big(\|u_0\|^2_{\tilde{H}^{\alpha/2}} + \sum_{j=1}^{n}\sum_{i=1}^{n_j}\|u_{j,i}\|^2_{\tilde{H}^{\alpha_j/2}(D_j)}\Big)^{1/2} \\
&\leq c\Big(\|\underline{\mathcal{P}}_0u\|^2_{\tilde{H}^{\alpha/2}} + \sum_{j=1}^{n}\sum_{i=1}^{n_j}\|\underline{\mathcal{P}}_{j,i}u\|^2_{\tilde{H}^{\alpha/2}}\Big)^{1/2}\Big(\sum_{j=1}^{n}\lambda_{j,0}^{-1}\|u_j\|^2_{\tilde{H}^{\alpha_j/2}(D_j)}\Big)^{1/2}. \qquad (3.75)
\end{aligned}$$

In the last step we made use of the identity $\|u_0\|^2_{\tilde{H}^{\alpha/2}} = \sum_{j=1}^{n}\|u_{j,0}\|^2_{\tilde{H}^{\alpha_j/2}(D_j)}$. Combining (3.74) and (3.75) we find

$$\begin{aligned}
&\sum_{j=1}^{n}\big(\gamma_1 - c\gamma_2 h_j^{2\delta_{j,1}}\big)\|u_j\|^2_{\tilde{H}^{\alpha_j/2}(D_j)} \leq \\
&c\Big(\|\underline{\mathcal{P}}_0u\|^2_{\tilde{H}^{\alpha/2}} + \sum_{j=1}^{n}\sum_{i=1}^{n_j}\|\underline{\mathcal{P}}_{j,i}u\|^2_{\tilde{H}^{\alpha/2}}\Big)^{1/2}\Big(\sum_{j=1}^{n}\lambda_{j,0}^{-1}\|u_j\|^2_{\tilde{H}^{\alpha_j/2}(D_j)}\Big)^{1/2} + c\gamma_2\|\underline{\mathcal{P}}_0u\|_{\tilde{H}^{\alpha/2}}\|u\|_{\tilde{H}^{\alpha/2}} \\
&\qquad\leq c(\max_j\lambda_{j,0}^{-1/2} + \gamma_2)\Big(\|\underline{\mathcal{P}}_0u\|^2_{\tilde{H}^{\alpha/2}} + \sum_{j=1}^{n}\sum_{i=1}^{n_j}\|\underline{\mathcal{P}}_{j,i}u\|^2_{\tilde{H}^{\alpha/2}}\Big)^{1/2}\|u\|_{\tilde{H}^{\alpha/2}}.
\end{aligned}$$

That means, for h_j, $j = 1,\ldots,n$, small enough such that $\min_j\big(\gamma_1 - c\gamma_2 h_j^{2\delta_{j,1}}\big) > 0$,

$$\begin{aligned}
\|u\|^2_{\tilde{H}^{\alpha/2}} &\leq c\max_j\big(\gamma_1 - c\gamma_2 h_j^{2\delta_{j,1}}\big)^{-2}(\max_j\lambda_{j,0}^{-1/2} + \gamma_2)^2\Big(\|\underline{\mathcal{P}}_0u\|^2_{\tilde{H}^{\alpha/2}} + \sum_{j=1}^{n}\sum_{i=1}^{n_j}\|\underline{\mathcal{P}}_{j,i}u\|^2_{\tilde{H}^{\alpha/2}}\Big) \\
&\leq c(\max_j\lambda_{j,0}^{-1} + \gamma_2^2)\Big(\|\underline{\mathcal{P}}_0u\|^2_{\tilde{H}^{\alpha/2}} + \sum_{j=1}^{n}\sum_{i=1}^{n_j}\|\underline{\mathcal{P}}_{j,i}u\|^2_{\tilde{H}^{\alpha/2}}\Big)
\end{aligned}$$

This proves the assertion for the operator $\underline{\mathcal{P}}$. As in the scalar case the assertion for $\underline{P}$ can be obtained by replacing (3.75) by the analogous estimate involving the operators $\underline{P}_{j,i}$ instead of $\underline{\mathcal{P}}_{j,i}$. The remainder of the proof for $\underline{P}$ is the same as for $\underline{\mathcal{P}}$. □

Now we are able to prove the main results of this section which bound the minimum eigenvalues of the Hermitian parts and the norms of the additive Schwarz operators $\underline{\mathcal{P}}$ and $\underline{P}$ for indefinite or non-Hermitian systems.

Theorem 3.16 *There exist constants $c, C, h_0, \delta_{j,3} > 0$, $j = 1, \dots, n$, such that, if $0 < h_j \le h_0$, $j = 1, \dots, n$, there holds*

$$\Re\langle \underline{A}^\alpha \underline{\mathcal{P}} u, u\rangle \ge c\Big(C(\max_j \lambda_{j,0}^{-1} + \gamma_2^2)^{-1} - \max_j h_j^{\delta_{j,3}}\Big)\|u\|^2_{\tilde{H}^{\alpha/2}} \quad \text{for any} \quad u \in X_N$$

and

$$\|\underline{\mathcal{P}} u\|_{\tilde{H}^{\alpha/2}} \le c\|u\|_{\tilde{H}^{\alpha/2}} \quad \text{for any} \quad u \in X_N.$$

Here, any $\delta_{j,3} \le \min\{\delta_{j,1}, \delta_{j,2}\}$, $j = 1, \dots, n$, can be chosen. The parameters $\delta_{j,1}$ are the orders of convergence when projecting $u \in X_N$ with respect to $\underline{\mathcal{A}}^\alpha$ onto H_0, cf. the proof of Lemma 3.11. *The parameters $\delta_{j,2}$ are defined by* (3.78) *below.*

Analogous estimates hold for the operator $\underline{P}$.

Proof. As in the scalar case, cf. Lemma 3.6(iii), Assumption (A2) ensures that the real part of the operator $\underline{\mathcal{A}}^\alpha$ is locally positive definite if the mesh sizes are small enough. More precisely, for $u \in H_{j,i}$, $i > 0$, there holds

$$\|\hat{u}\|_{\tilde{H}^{\alpha/2-\delta}} = \|u\|_{\tilde{H}^{\alpha_j/2-\delta_j}(D_j)} \le c h_j^{\delta_{j,2}} \|u\|_{\tilde{H}^{\alpha_j/2}(D_j)} \tag{3.76}$$

and

$$\Re\langle \underline{\mathcal{A}}^\alpha u, u\rangle \ge \big(\gamma_1 - c\gamma_2 h_j^{2\delta_{j,2}}\big)\|u\|^2_{\tilde{H}^{\alpha_j/2}(D_j)} \tag{3.77}$$

where

$$\delta_{j,2} = \begin{cases} \delta_j & \text{if } \alpha_j = 1 \text{ and } (\delta_j \le 1/2 \text{ or } \delta_j > 1/2 \text{ and } \langle u, 1\rangle_{L^2(D_j)} = 0) \\ 1/2 & \text{if } \alpha_j = 1 \text{ and } (\delta_j > 1/2 \text{ and } \langle u, 1\rangle_{L^2(D_j)} \ne 0) \\ \delta_j & \text{if } \alpha_j = -1 \end{cases} . \tag{3.78}$$

Now we bound the norm of $\underline{\mathcal{P}}$. Let $u \in X_N$ be given. By (3.77), the continuity of $\underline{\mathcal{A}}^\alpha$, and a coloring estimate like (3.50) (using the coloring constants K_j instead of K) which applies

due to Assumption (A3) we conclude

$$\begin{aligned}
\sum_{j=1}^{n}\sum_{i=1}^{n_j}\|\underline{\mathcal{P}}_{j,i}u\|^2_{\tilde{H}^{\alpha/2}} &\le \sum_{j=1}^{n} c\big(C-h_j^{2\delta_{j,2}}\big)^{-1}\sum_{i=1}^{n_j}\Re\langle\underline{\mathcal{A}}^{\alpha}\underline{\mathcal{P}}_{j,i}u,\underline{\mathcal{P}}_{j,i}u\rangle\\
&= \sum_{j=1}^{n} c\big(C-h_j^{2\delta_{j,2}}\big)^{-1}\sum_{i=1}^{n_j}\Re\langle\underline{\mathcal{A}}^{\alpha}u,\underline{\mathcal{P}}_{j,i}u\rangle\\
&\le c\max_j\big(C-h_j^{2\delta_{j,2}}\big)^{-1}\Re\langle\underline{\mathcal{A}}^{\alpha}u,\sum_{j=1}^{n}\sum_{i=1}^{n_j}\underline{\mathcal{P}}_{j,i}u\rangle\\
&\le c\max_j\big(C-h_j^{2\delta_{j,2}}\big)^{-1}\|u\|_{\tilde{H}^{\alpha/2}}\|\sum_{j=1}^{n}\sum_{i=1}^{n_j}\underline{\mathcal{P}}_{j,i}u\|_{\tilde{H}^{\alpha/2}}\\
&= c\max_j\big(C-h_j^{2\delta_{j,2}}\big)^{-1}\|u\|_{\tilde{H}^{\alpha/2}}\Big(\sum_{j=1}^{n}\|\Big(\sum_{i=1}^{n_j}\underline{\mathcal{P}}_{j,i}u\Big)_j\|^2_{\tilde{H}^{\alpha_j/2}(D_j)}\Big)^{1/2}\\
&\le c\max_j\big(C-h_j^{2\delta_{j,2}}\big)^{-1}\|u\|_{\tilde{H}^{\alpha/2}}\Big(\sum_{j=1}^{n}K_j\sum_{i=1}^{n_j}\|\big(\underline{\mathcal{P}}_{j,i}u\big)_j\|^2_{\tilde{H}^{\alpha_j/2}(D_j)}\Big)^{1/2}\\
&\le c\max_j K_j^{1/2}\max_j\big(C-h_j^{2\delta_{j,2}}\big)^{-1}\|u\|_{\tilde{H}^{\alpha/2}}\Big(\sum_{j=1}^{n}\sum_{i=1}^{n_j}\|\big(\underline{\mathcal{P}}_{j,i}u\big)_j\|^2_{\tilde{H}^{\alpha_j/2}(D_j)}\Big)^{1/2}\\
&= c\max_j K_j^{1/2}\max_j\big(C-h_j^{2\delta_{j,2}}\big)^{-1}\|u\|_{\tilde{H}^{\alpha/2}}\Big(\sum_{j=1}^{n}\sum_{i=1}^{n_j}\|\underline{\mathcal{P}}_{j,i}u\|^2_{\tilde{H}^{\alpha}}\Big)^{1/2}.
\end{aligned}\tag{3.79}$$

Here, $(\cdot)_j$ means the restriction onto the jth component. Further, we have to assume that the mesh sizes h_j are small enough such that $\min_j(C-h_j^{2\delta_{j,2}})>0$. Therefore, we obtain

$$\sum_{j=1}^{n}\sum_{i=1}^{n_j}\|\underline{\mathcal{P}}_{j,i}u\|^2_{\tilde{H}^{\alpha_j/2}(D_j)} \le c\max_j K_j\max_j\big(C-h_j^{2\delta_{j,2}}\big)^{-2}\|u\|^2_{\tilde{H}^{\alpha/2}}.\tag{3.80}$$

Note that in the above inequality chain an intermediate step proves

$$\|\sum_{j=1}^{n}\sum_{i=1}^{n_j}\underline{\mathcal{P}}_{j,i}u\|^2_{\tilde{H}^{\alpha/2}} \le c\max_j K_j\sum_{j=1}^{n}\sum_{i=1}^{n_j}\|\underline{\mathcal{P}}_{j,i}u\|^2_{\tilde{H}^{\alpha/2}}.$$

This estimate, together with (3.80), the boundedness (3.73) and the triangle inequality, gives

$$\|\Big(\underline{\mathcal{P}}_0+\sum_{j=1}^{n}\sum_{i=1}^{n_j}\underline{\mathcal{P}}_{j,i}\Big)u\|^2_{\tilde{H}^{\alpha/2}} \le c\max_j K_j^2\max_j\big(C-h_j^{2\delta_{j,2}}\big)^{-2}\|u\|^2_{\tilde{H}^{\alpha/2}},$$

i.e. the additive Schwarz operator $\underline{\mathcal{P}}$ remains bounded if the mesh sizes h_j decrease. This is the second assertion of the theorem.

Now we bound the minimum eigenvalue of the Hermitian part of $\underline{\mathcal{P}}$. As in the scalar case we rewrite

$$\begin{aligned}\Re\langle \underline{A}^\alpha \underline{\mathcal{P}} u, u\rangle &= \Re\langle \underline{A}^\alpha \underline{\mathcal{P}}_0 u, u\rangle + \sum_{j=1}^{n}\sum_{i=1}^{n_j} \Re\langle \underline{A}^\alpha \underline{\mathcal{P}}_{j,i} u, u\rangle \\ &\geq \langle \underline{A}^\alpha \underline{\mathcal{P}}_0 u, \underline{\mathcal{P}}_0 u\rangle + \sum_{j=1}^{n}\sum_{i=1}^{n_j} \langle \underline{A}^\alpha \underline{\mathcal{P}}_{j,i} u, \underline{\mathcal{P}}_{j,i} u\rangle - \Big|\langle \underline{A}^\alpha \underline{\mathcal{P}}_0 u, u\rangle - \langle \underline{A}^\alpha \underline{\mathcal{P}}_0 u, \underline{\mathcal{P}}_0 u\rangle\Big| \\ &\qquad - \sum_{j=1}^{n}\Big|\sum_{i=1}^{n_j} \langle \underline{A}^\alpha \underline{\mathcal{P}}_{j,i} u, u\rangle - \langle \underline{A}^\alpha \underline{\mathcal{P}}_{j,i} u, \underline{\mathcal{P}}_{j,i} u\rangle\Big|.\end{aligned}$$

Proving the estimates

$$\Big|\langle \underline{A}^\alpha \underline{\mathcal{P}}_0 u, u\rangle - \langle \underline{A}^\alpha \underline{\mathcal{P}}_0 u, \underline{\mathcal{P}}_0 u\rangle\Big| \leq c \max_j h_j^{\delta_{j,1}} \|u\|^2_{\tilde{H}^{\alpha/2}} \tag{3.81}$$

and

$$\sum_{j=1}^{n}\Big|\sum_{i=1}^{n_j} \langle \underline{A}^\alpha \underline{\mathcal{P}}_{j,i} u, u\rangle - \langle \underline{A}^\alpha \underline{\mathcal{P}}_{j,i} u, \underline{\mathcal{P}}_{j,i} u\rangle\Big| \leq c \max_j \big(K_j h_j^{\delta_{j,2}}\big) \|u\|^2_{\tilde{H}^{\alpha/2}} \tag{3.82}$$

we obtain by Lemma 3.11

$$\Re\langle \underline{A}^\alpha \underline{\mathcal{P}} u, u\rangle \geq \Big(C(\max_j \lambda_{j,0}^{-1} + \gamma_2^2)^{-1} - c \max_j h_j^{\delta_{j,1}} - c \max_j \big(K_j h_j^{\delta_{j,2}}\big)\Big) \|u\|^2_{\tilde{H}^{\alpha/2}}$$

if the mesh sizes h_j, $j = 1, \ldots, n$, are small enough. This gives the first assertion of the theorem. It therefore remains to prove (3.81) and (3.82). The estimate (3.81) can be obtained analogously to (3.56) and (3.57):

$$\Big|\langle \underline{A}^\alpha \underline{\mathcal{P}}_0 u, u\rangle - \langle \underline{A}^\alpha \underline{\mathcal{P}}_0 u, \underline{\mathcal{P}}_0 u\rangle\Big| = \Big|\langle \underline{\mathcal{P}}_0 u, \underline{\mathcal{K}}^\alpha \big(u - \underline{\mathcal{P}}_0 u\big)\rangle\Big| \leq c \max_j h_j^{\delta_{j,1}} \|u\|^2_{\tilde{H}^{\alpha/2}}.$$

We also obtain, like (3.56), (3.58), and (3.59),

$$\begin{aligned}\sum_{j=1}^{n}\Big|\sum_{i=1}^{n_j} \langle \underline{A}^\alpha \underline{\mathcal{P}}_{j,i} u, u\rangle - \langle \underline{A}^\alpha \underline{\mathcal{P}}_{j,i} u, \underline{\mathcal{P}}_{j,i} u\rangle\Big| &\leq \sum_{j=1}^{n}\Big|\sum_{i=1}^{n_j} \langle \underline{\mathcal{P}}_{j,i} u, \underline{\mathcal{K}}^\alpha u\rangle\Big| + \sum_{j=1}^{n}\Big|\sum_{i=1}^{n_j} \langle \underline{\mathcal{P}}_{j,i} u, \underline{\mathcal{K}}^\alpha \underline{\mathcal{P}}_{j,i} u\rangle\Big| \\ &\leq \sum_{j=1}^{n}\Big|\langle \sum_{i=1}^{n_j} \underline{\mathcal{P}}_{j,i} u, \underline{\mathcal{K}}^\alpha u\rangle\Big| + \sum_{j=1}^{n}\sum_{i=1}^{n_j}\Big|\langle \underline{\mathcal{P}}_{j,i} u, \underline{\mathcal{K}}^\alpha \underline{\mathcal{P}}_{j,i} u\rangle\Big| \\ &\leq \sum_{j=1}^{n} c K_j h_j^{\delta_{j,2}} \big(C - h_j^{2\delta_{j,2}}\big)^{-1} \|u\|^2_{\tilde{H}^{\alpha/2}} + \sum_{j=1}^{n} c h_j^{\delta_{j,2}} K_j \big(C - h_j^{2\delta_{j,2}}\big)^{-2} \|u\|^2_{\tilde{H}^{\alpha/2}} \\ &\leq cn \max_j \big(K_j h_j^{\delta_{j,2}}\big) \|u\|^2_{\tilde{H}^{\alpha/2}}.\end{aligned}$$

This gives (3.82) which was left to be proved for the operator $\underline{\mathcal{P}}$.

The respective estimates for the additive Schwarz operator $\underline{P} = \underline{\mathcal{P}}_0 + \sum_{j=1}^{n} \sum_{i=1}^{n_j} \underline{P}_{j,i}$ are obtained analogously as in the scalar case, cf. the proof of Theorem 3.10. □

The above theorem will be applied in Sections 4.2, 4.3, and 4.4 to analyze specific preconditioners for the electric and the magnetic screen problems and a Helmholtz transmission problem (being discretized by a coupled FEM/BEM method), cf. Theorems 4.2, 4.3, 4.6, 4.7, and 4.9. Numerical results will be given only for the electric screen problem.

Hybrid method

The hybrid method of Section 3.4.2 is formulated in a rather abstract way and can be applied to indefinite systems without modifying the proofs. We therefore only recall the setting of the method and present the main theorem without proof.

We choose a subspace $H_0 \subset X_N$ such that $\underline{\mathcal{A}}^\alpha$ restricted to H_0 and to finer spaces is invertible and such that its inverse is bounded. Then there exists a linear operator $\underline{\mathcal{P}}_0 : X_N \to H_0$ such that, for given $u \in X_N$,

$$\langle \underline{\mathcal{A}}^\alpha \underline{\mathcal{P}}_0 u, v \rangle = \langle \underline{\mathcal{A}}^\alpha u, v \rangle \quad \text{for any} \quad v \in H_0.$$

Further we define the restricted operator $\underline{\mathcal{A}}_0^\alpha : H_0 \to H_0$ of $\underline{\mathcal{A}}^\alpha$ by

$$\langle \underline{\mathcal{A}}_0^\alpha u, v \rangle = \langle \underline{\mathcal{A}}^\alpha u, v \rangle \quad \text{for any} \quad u, v \in H_0,$$

and the L^2-projection $\underline{Q}_0 : X_N \to H_0$ by

$$\langle \underline{Q}_0 u, v \rangle = \langle u, v \rangle \quad \text{for any} \quad v \in H_0.$$

where $u \in X_N$ is given. Now, let $\underline{B}^{-1}$ be a symmetric positive definite preconditioner for $\underline{A}_N^\alpha$, the symmetric positive definite part of the discrete analogue $\underline{\mathcal{A}}_N^\alpha$ of the continuous operator $\underline{\mathcal{A}}^\alpha$. We specify the minimum and maximum eigenvalues of $\underline{B}^{-1}\underline{A}_N^\alpha : X_N \to X_N$ by λ_0 and λ_1, respectively. Eventually, we define a preconditioner for $\underline{\mathcal{A}}_N^\alpha$ by

$$\underline{\mathcal{B}}^{-1} := (\underline{\mathcal{A}}_0^\alpha)^{-1} \underline{Q}_0 + \beta \underline{B}^{-1}$$

which gives

$$\underline{\mathcal{B}}^{-1} \underline{\mathcal{A}}_N^\alpha = \underline{\mathcal{P}}_0 + \beta \underline{B}^{-1} \underline{\mathcal{A}}_N^\alpha.$$

Here, $\beta > 0$ is a real parameter which balances the coarse grid contribution and the positive definite preconditioner.

The approximation properties of the coarse grid solver $\mathcal{P}_0$ are characterized by the parameter

$$\delta_0 = \sup_{v \in X_N} \frac{\|v - \underline{\mathcal{P}}_0 v\|_{\tilde{H}^{\alpha/2-\delta}}}{\|v\|_{\tilde{H}^{\alpha/2}}}.$$

The next theorem presents abstract bounds for the minimum eigenvalue of the symmetric part of $\underline{\mathcal{B}}^{-1}\underline{\mathcal{A}}_N^\alpha$ and for the norm of $\underline{\mathcal{B}}^{-1}\underline{\mathcal{A}}_N^\alpha$. It is a translation of Theorem 3.11 to the present situation.

Theorem 3.17 *There exist positive numbers $\varepsilon, \Lambda_0, \Lambda_1$ and β which depend on the extreme eigenvalues λ_0, λ_1 of $\underline{B}^{-1}\underline{A}_N^\alpha$ such that, if $\delta_0 \le \varepsilon$,*

$$\Re\langle \underline{A}^\alpha \underline{B}^{-1} \underline{A}_N^\alpha u, u\rangle \ge \Lambda_0 \langle \underline{A}^\alpha u, u\rangle \quad \text{for any} \quad u \in X_N$$

and

$$\|\underline{B}^{-1}\underline{A}_N^\alpha u\|_{\tilde{H}^{\alpha/2}} \le \Lambda_1 \|u\|_{\tilde{H}^{\alpha/2}} \quad \text{for any} \quad u \in X_N.$$

More precisely, when choosing

$$\beta = \frac{\lambda_0}{2c_0^2\lambda_1^2} \quad \text{and} \quad \delta_0 \le \varepsilon = \Big(\frac{\lambda_0^2}{4\lambda_1^2 c_0^2(c_0^2+1)}\Big)^{1/2}$$

the above estimates hold with

$$\Lambda_0 \ge \frac{\lambda_0^2}{8c_0^2\lambda_1^2} \quad \text{and} \quad \Lambda_1 \le 1 + c_0\delta_0 + \beta\lambda_1(1+c_0).$$

The constant c_0 is the bound of $\underline{K}^\alpha$ in (3.69).

This theorem will be used to analyze the efficiency of some preconditioners for the electric and the magnetic screen problems when a component of the energy spaces is decomposed like the wire basket decomposition, i.e. when the assumptions for the standard additive Schwarz method are not fulfilled. See Theorem 4.4 in Section 4.2 (where also numerical results are given) and Theorem 4.8 in Section 4.3.

3.5.2 Block skew-symmetric systems

There are special cases when the stiffness matrix of the Galerkin system is block skew-symmetric with positive definite symmetric part, i.e. when A is a real matrix of the form

$$A = \begin{pmatrix} A_1 & K_{12} & K_{13} & \dots & K_{1n} \\ -K_{12}^T & A_2 & K_{23} & \dots & K_{2n} \\ -K_{13}^T & -K_{23}^T & A_3 & \dots & K_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ -K_{1n}^T & -K_{2n}^T & -K_{3n}^T & \dots & A_n \end{pmatrix} \tag{3.83}$$

with symmetric positive definite diagonal blocks A_j. This situation occurs when coupling the finite element and the boundary element method for a special transmission problem, see Section 4.4.

For a fixed basis of the ansatz space X_N the diagonal blocks A_j of the above matrix correspond to the bilinear form $\underline{A}(\cdot,\cdot)$ (induced by A) restricted to subspaces $H_j \subset X_N$. This corresponds to a direct sum decomposition (3.2) of X_N.

It turns out that the properties of individual additive Schwarz preconditioners for the blocks A_j of the matrix carry over to an accumulated preconditioner for the GMRES method.

To be specific let the subspaces H_j be further decomposed:

$$H_j = H_{j,0} \cup \dots \cup H_{j,n_j}, \quad j = 1, \dots, n.$$

For simplicity we assume that also these decompositions are direct ones. We define projection operators $\underline{P}_{j,i}$ such that for $v \in X_N$

$$\underline{A}(\underline{P}_{j,i}v, \varphi) = \underline{A}(v, \varphi) \quad \text{for any } \varphi \in H_{j,i}.$$

The resulting additive Schwarz operator $\underline{P} = \sum_{j,i} \underline{P}_{j,i}$ then belongs to the decomposition

$$X_N = \cup_{j=1}^{n} \Big(H_{j,0} \cup \ldots \cup H_{j,n_j} \Big). \tag{3.84}$$

The resulting preconditioning form is

$$\underline{B}(v, w) = \sum_{j=1}^{n} \sum_{i=0}^{n_j} \underline{A}(v_{j,i}, w_{j,i})$$

for $v, w \in X_N$ with $v = \sum_{j=1}^{n} \sum_{i=0}^{n_j} v_{j,i}$ and $w = \sum_{j=1}^{n} \sum_{i=0}^{n_j} w_{j,i}$ according to the decomposition (3.84).

Since we want to gather from properties of individual additive Schwarz operators on H_j we explicitly specify them: $P_j = \sum_{i=0}^{n_j} P_{j,i}$ where for $v \in H_j$

$$\underline{A}(P_{j,i}v, \varphi) = \underline{A}(v, \varphi) \quad \text{for any } \varphi \in H_{j,i}.$$

The projection operators $P_{j,i}$ act on the subspaces H_j whereas the operators $\underline{P}_{j,i}$ act on the full space X_N.

We obtain the following result.

Theorem 3.18 *Assume that there holds*

$$\lambda_{\min}(P_j) \geq c_j \quad \textit{and} \quad \lambda_{\max}(P_j) \leq C_j, \quad j = 1 \ldots, n.$$

Then

$$\inf_{v \in X_n} \frac{\underline{B}(v, \underline{P}v)}{\underline{B}(v, v)} \geq c \quad \textit{and} \quad \sup_{v \in X_n} \frac{\underline{B}(\underline{P}v, \underline{P}v)}{\underline{B}(v, v)} \leq C^2$$

where $c = \min\{c_1, \ldots, c_n\}$ *and* $C = \max\{C_1, \ldots, C_n\}$.

Proof. First let us derive a lower bound for the minimum eigenvalue of the additive Schwarz operator $\underline{P}$. For a given $v \in X_N$ we use the representation $v = \sum_{j=1}^{n} v_j = \sum_{j=1}^{n} \sum_{i=0}^{n_j} v_{j,i}$ with $v_j \in H_j$ and $v_{j,i} \in H_{j,i}$. By the definition of the bilinear form $\underline{B}(\cdot, \cdot)$ and the projection properties of the operators $\underline{P}_{j,i}$ we obtain

$$\begin{aligned} \underline{B}(v, \underline{P}v) &= \sum_{j=1}^{n} \sum_{i=0}^{n_j} \underline{A}(v_{j,i}, \underline{P}_{j,i}v) = \sum_{j=1}^{n} \sum_{i=0}^{n_j} \underline{A}(v_{j,i}, v) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} \underline{A}(v_j, v_k) = \sum_{j=1}^{n} \underline{A}(v_j, v_j) \end{aligned}$$

since

$$\underline{A}(v_j, v_k) + \underline{A}(v_k, v_j) = 0 \quad \text{if } j \neq k$$

due to the skew-symmetry of the bilinear form $\underline{A}(\cdot,\cdot)$ which is given by the matrix (3.83). The assumption $\lambda_{\min}(P_j) \geq c_j$ is equivalent to

$$c_j \sum_{i=0}^{n_j} \underline{A}(v_{j,i}, v_{j,i}) \leq \underline{A}(v_j, v_j) \quad \text{for any } v_j \in H_j.$$

We therefore conclude

$$\underline{B}(v, \underline{P}v) \geq \sum_{j=1}^{n} c_j \sum_{i=0}^{n_j} \underline{A}(v_{j,i}, v_{j,i}) \geq c \sum_{j=1}^{n} \sum_{i=0}^{n_j} \underline{A}(v_{j,i}, v_{j,i}) = c\underline{B}(v, v)$$

with $c = \min\{c_1, \ldots, c_n\}$. This proves the first assertion of the lemma. It remains to derive an upper bound for the norm of $\underline{P}$. Similarly as above and by using the Cauchy-Schwarz inequality we get

$$\begin{aligned} \underline{B}(\underline{P}v, \underline{P}v) &= \sum_{j=1}^{n} \sum_{i=0}^{n_j} \underline{A}(\underline{P}_{j,i}v, \underline{P}_{j,i}v) = \sum_{j=1}^{n} \sum_{i=0}^{n_j} \underline{A}(v, \underline{P}_{j,i}v) \\ &= \underline{A}(v, \underline{P}v) \leq \Big(\underline{A}(v,v)\Big)^{1/2} \Big(\underline{A}(\underline{P}v, \underline{P}v)\Big)^{1/2}. \end{aligned} \tag{3.85}$$

Now we use the assumption $\lambda_{\max}(P_j) \leq C_j$ which means that there holds

$$\underline{A}(v_j, v_j) \leq C_j \sum_{i=0}^{n_j} \underline{A}(v_{j,i}, v_{j,i}) \quad \text{for any } v_j \in H_j \tag{3.86}$$

and we obtain by first using the skew-symmetry of $\underline{A}(\cdot,\cdot)$

$$\begin{aligned} \underline{A}(\underline{P}v, \underline{P}v) &= \sum_{j=1}^{n} \underline{A}(\underline{P}_j v, \underline{P}_j v) \leq \sum_{j=1}^{n} C_j \sum_{i=0}^{n_j} \underline{A}(\underline{P}_{j,i}v, \underline{P}_{j,i}v) \\ &\leq C \sum_{j=1}^{n} \sum_{i=0}^{n_j} \underline{A}(\underline{P}_{j,i}v, \underline{P}_{j,i}v) = C\underline{B}(\underline{P}v, \underline{P}v) \end{aligned} \tag{3.87}$$

with $C = \max\{C_1, \ldots, C_n\}$. Combining (3.85) and (3.87) and again using (3.86) we finally obtain

$$\begin{aligned} \underline{B}(\underline{P}v, \underline{P}v) \leq C\underline{A}(v,v) &= C \sum_{j=1}^{n} \underline{A}(v_j, v_j) \\ &\leq C \max\{C_1, \ldots, C_n\} \sum_{j=1}^{n} \sum_{i=0}^{n_j} \underline{A}(v_{j,i}, v_{j,i}) = C^2 \underline{B}(v,v) \end{aligned}$$

which is the required estimate of the norm of $\underline{P}$ with respect to the bilinear form $\underline{B}(\cdot,\cdot)$. $\square$

Remark 3.4 *The choice of an appropriate inner product for the GMRES method plays an important role in the above proof. Before, this idea has been used in* [81]. *Let* $B^{-1} = EE^T \in \mathbb{R}^{N\times N}$ *be symmetric and positive definite. Then the GMRES method applied to*

$$B^{-1}Av = B^{-1}rhs$$

using the inner product defined by B *is equivalent to solving*

$$E^TAEw = E^Trhs$$

using the l_2*-inner product, cf. Theorem* 3.2 *and the notes thereafter.*

The above theorem will be used to prove the efficiency of an additive Schwarz method for a coupled FEM/BEM method for solving a Helmholtz transmission problem, see Theorem 4.10 in Section 4.4. Numerical results are given for the respective transmission problem in the two dimensional Euclidean space.

Chapter 4

Examples

We consider several typical elliptic problems which can be efficiently modeled by the boundary element method. We demonstrate that our preconditioning methods lead to efficient solution procedures by confirming the theoretical estimates and by measuring execution times. The Helmholtz screen problems in Section 4.1, which are modeled by positive definite or indefinite scalar first kind integral equations, serve as academic model problems where the theoretical behaviors of the preconditioners are more easily observed in the numerical experiments. Screen problems for time-harmonic Maxwell's equations which can be modeled by indefinite systems of integral operators of different orders are being considered in Sections 4.2 and 4.3. Numerical results are presented for the so-called electric screen problem in Section 4.2. They underline that not only in theory the presented preconditioners are almost optimal but also in practise they are very efficient in view of CPU-times. Here one also realizes that even though the stiffness matrices are in general fully occupied (which means that their assembly step is rather time consuming) their is a need to accelerate their solution processes for non-academic problems. In Section 4.4 we consider simple transmission problems. It is shown that our methods are also applicable to the coupled finite element/boundary element method which is of great importance in practise since by this method problems with non-homogeneities in bounded areas and which are linear in the exterior, i.e., in unbounded domains, can be efficiently treated. In Section 4.5 we comment on the implementation and the numerical expense of the algorithms.

4.1 Helmholtz screen problems

In this section we consider the solutions of Neumann and Dirichlet problems for the scalar Helmholtz equation via boundary integral equations on a screen Γ. For the theoretical justification of the boundary element method for these problems we refer to [130]. The problems read as follows. *For given* $g_N \in H^{-1/2}(\Gamma)$ *or* $g_D \in H^{1/2}(\Gamma)$ *find* $u \in H^1_{\mathrm{loc}}(\Omega)$ *with* $\Omega := \mathbb{R}^3 \setminus \bar{\Gamma}$

satisfying

$$(\Delta + k^2)u = 0 \quad in \quad \Omega \tag{4.1}$$

$$\begin{aligned} \frac{\partial u}{\partial n} &= g_N \quad on \quad \Gamma \quad (Neumann) \quad or \\ u &= g_D \quad on \quad \Gamma \quad (Dirichlet) \quad and \end{aligned} \tag{4.2}$$

$$\left.\begin{aligned} \frac{\partial u}{\partial r} - iku &= o(r^{-1}) \quad for \quad k \neq 0 \quad or \\ u &= O(r^{-1}) \quad for \quad k = 0 \end{aligned}\right\} \quad as \quad r := |x| \to \infty. \tag{4.3}$$

Here, n is the normal vector on Γ and the wave number k is assumed to be small. These problems appear in the scattering theory of acoustic fields u by obstacles. The Neumann problem represents a hard screen and the Dirichlet problem represents a soft screen (see [78]). For $k = 0$ the above Dirichlet problem describes the electrostatic field u of an electrified screen and one looks for the charge density (see [34]). In [130] Stephan made the general assumption

> Γ is a bounded, simply connected, orientable smooth, open surface in $\mathbb{R}^3$ with a smooth boundary curve which does not intersect itself.

Since we want to consider surfaces which have polygonal boundary curves (and which can be meshed by rectangular meshes) we only have Lipschitz continuity of the boundary curves. This problem can be overcome by extending the surface Γ to a surface S with smooth boundary curve and by extending functions on Γ by zero. This extension can be performed since both important energy spaces, $\tilde{H}^{1/2}(\Gamma)$ and $\tilde{H}^{-1/2}(\Gamma)$, can be equivalently defined, besides interpolation, by the set of functions whose extension onto the smooth surface S by zero within $H^{1/2}(S)$ and $H^{-1/2}(S)$ is continuous. Besides this consideration, one needs to extend the surface Γ to a closed smooth surface in order to derive the needed boundary integral equations, as has been done in [130]. More precisely, we extend Γ to an arbitrary smooth, simply connected, closed, orientable manifold $\partial\Omega_1$ enclosing a bounded domain Ω_1 with boundary $\partial\Omega_1$. Let $\frac{\partial}{\partial n}$ denote the derivative operator with respect to the exterior normal to $\partial\Omega_1$ and let $[v]$ denote the jump $v_- - v_+$ where the subscripts $+$ and $-$ mean the limit from $\mathbb{R}^3 \setminus \bar{\Omega}_1$ and from Ω_1, respectively. Furthermore, let B denote a sufficiently large ball including $\bar{\Omega}_1$ and let $\Omega_2 := B \cap (\mathbb{R}^3 \setminus \bar{\Omega}_1)$.

The next lemma says that the jump across Γ of the Cauchy data $u|_\Gamma$ and $\frac{\partial u}{\partial n}\big|_\Gamma$ are in the energy spaces of the weakly singular and the hypersingular operators, respectively (see below).

Lemma 4.1 (Lemma 2.2 in [130]) *Let u be a weak solution of the screen problem* (4.1)–(4.3)*. Then for the Neumann problem we have*

$$[u]\big|_\Gamma \in \tilde{H}^{1/2}(\Gamma),$$

and for the Dirichlet problem we have

$$\left[\frac{\partial u}{\partial n}\right]\Big|_\Gamma \in \tilde{H}^{-1/2}(\Gamma).$$

The above result enables one to introduce boundary integral operators for the solution of the screen problem. First we introduce the fundamental solution ϕ of the homogeneous Helmholtz equation $(\Delta + k^2)u = 0$ in Ω_j by

$$\phi(z,\zeta) := -\frac{e^{ik|z-\zeta|}}{4\pi|z-\zeta|}.$$

The potentials of the single layer and of the double layer then are

$$V_{\Omega_j} u(z) := -2\int_{\partial\Omega_j} u(\zeta)\phi(z,\zeta)\, dS_\zeta \quad \text{for} \quad z \in \Omega_j$$

$$K_{\Omega_j} u(z) := -2\int_{\partial\Omega_j} u(\zeta)\frac{\partial}{\partial n_\zeta}\phi(z,\zeta)\, dS_\zeta \quad \text{for} \quad z \in \Omega_j,$$

respectively. These integral operators at hand we can present the representation formula for the Helmholtz equation.

Lemma 4.2 (Lemma 2.4 in [130]) *For $u \in H^1_{\rm loc}(\Omega_j)$ with $(\Delta + k^2)u = 0$ in Ω_j $(j = 1, 2)$ and with Cauchy data $v := u|_{\partial\Omega_j}$, $\psi := \frac{\partial u}{\partial n}\big|_{\partial\Omega_j}$ we have*

$$u(z) = (-1)^j \frac{1}{2}\big(K_{\Omega_j} v(z) - V_{\Omega_j}\psi(z)\big) \quad \textit{for} \quad z \in \Omega_j.$$

Using the above representation formula one can derive first kind integral equations for the solutions of the Neumann and the Dirichlet screen problems (see [130] for details). For the Dirichlet problem we take the representation formula which gives

$$\begin{aligned} u(x) &= -\frac{1}{2}\big(K_{\Omega_1} u(x) - V_{\Omega_1}\frac{\partial u}{\partial n}(x)\big), \\ 0 &= -\frac{1}{2}\big(K_{\Omega_2} u(x) - V_{\Omega_2}\frac{\partial u}{\partial n}(x)\big) \end{aligned} \tag{4.4}$$

for $x \in \Omega_1$. Denoting the outer boundary by $\partial B = \{y \in \mathbb{R}^3;\ |y| = R\}$ addition of the equations (4.4) gives

$$u(x) = \int_{|y|=R} u(y)\frac{\partial}{\partial n_y}\phi(x,y)\, dS_y - \int_{|y|=R}\frac{\partial u}{\partial n}(y)\phi(x,y)\, dS_y - \int_\Gamma [\frac{\partial u}{\partial n}](y)\phi(x,y)\, dS_y,$$

where we made use of $[\frac{\partial u}{\partial n}]\big|_{\partial\Omega_j\setminus\bar{\Gamma}} = 0$, see Lemma 4.1. Now we let x tend to the screen Γ and use the continuity of the single layer potential $V_{\Omega_j}[\frac{\partial u}{\partial n}]$ and the boundary condition $u|_\Gamma = g_D$. This gives

$$g_D(x) = \int_{|y|=R}\Big(u(y)\frac{\partial}{\partial n_y}\phi(x,y) - \frac{\partial u}{\partial n}(y)\phi(x,y)\Big)\, dS_y - \int_\Gamma [\frac{\partial u}{\partial n}](y)\phi(x,y)\, dS_y.$$

By the radiation condition (4.3), which holds for u and ϕ, the integral over $|y| = R$ vanishes for $R \to \infty$ and therefore, for $x \in \Gamma$,

$$g_D(x) = -\int_\Gamma [\frac{\partial u}{\partial n}](y)\phi(x,y)\, dS_y.$$

The above integral equation is an equivalent formulation of the Dirichlet screen problem:

Lemma 4.3 (Lemma 2.1 and Theorems 2.5, 2.7 in [130]) *For $\Im(k) \geq 0$ the Dirichlet screen problem (4.1)–(4.3) is uniquely solvable. $u \in H^1_{\mathrm{loc}}(\Omega)$ is the solution of the Dirichlet screen problem if and only if the jump $[\frac{\partial u}{\partial n}]\big|_\Gamma$ is the solution of the weakly singular integral equation*

$$V_k\Big[\frac{\partial u}{\partial n}\Big](x) := -2\int_\Gamma \Big[\frac{\partial u}{\partial n}\Big](y)\phi(x,y)\,dS_y = 2g_D(x) \tag{4.5}$$

for $x \in \Gamma$.

Now let us consider the Neumann screen problem. Taking in (4.4) the normal derivative this gives

$$\begin{aligned}\frac{\partial u}{\partial n}(x) = &\int_{|y|=R}\Big(u(y)\frac{\partial}{\partial n_x}\frac{\partial}{\partial n_y}\phi(x,y) - \frac{\partial u}{\partial n}(y)\frac{\partial}{\partial n_x}\phi(x,y)\Big)\,dS_y\\ &- \frac{\partial}{\partial n_x}\int_\Gamma [u](y)\frac{\partial}{\partial n_y}\phi(x,y)\,dS_y\end{aligned}$$

for $x \in \Omega_1$ since $[u]|_{\partial\Omega_j\setminus\bar\Gamma} = 0$. ϕ and its derivatives satisfy the radiation condition (4.3). Therefore, we obtain for $x \to \Gamma$ and $R \to \infty$ the equation

$$g_N(x) = -\frac{\partial}{\partial n_x}\int_\Gamma [u](y)\frac{\partial}{\partial n_y}\phi(x,y)\,dS_y.$$

This provides an equivalent formulation of the Neumann screen problem:

Lemma 4.4 (Lemma 2.1 and Theorems 2.6, 2.7 in [130]) *For $\Im(k) \geq 0$ the Neumann screen problem (4.1)–(4.3) is uniquely solvable. $u \in H^1_{\mathrm{loc}}(\Omega)$ is the solution of the Neumann screen problem if and only if the jump $[u]|_\Gamma$ is the solution of the hypersingular integral equation*

$$D_k[u](x) := 2\frac{\partial}{\partial n_x}\int_\Gamma [u](y)\frac{\partial}{\partial n_y}\phi(x,y)\,dS_y = -2g_N(x) \tag{4.6}$$

for $x \in \Gamma$.

An essential property of the operators V_k and D_k which guarantees convergence of the Galerkin method is their strong ellipticity, i.e., they satisfy a Gårding inequality:

Lemma 4.5 (Lemma 2.8 in [130], Theorems 1 and 2 in [35])
(i) *For all $s \in [-1/2, 1/2]$ the following mappings are continuous:*

$$\begin{aligned}V_k &: \tilde H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma),\\ D_k &: \tilde H^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma).\end{aligned}$$

(ii) *There exist constants $\gamma_{V,i}, \gamma_{D,i} > 0$ $(i = 1, 2)$ such that for all $\psi \in \tilde H^{-1/2}(\Gamma)$ and $v \in \tilde H^{1/2}(\Gamma)$*

$$\Re\langle V_k\psi, \psi\rangle_{L^2(\Gamma)} \geq \gamma_{V,1}\|\psi\|^2_{\tilde H^{-1/2}(\Gamma)} - \gamma_{V,2}\|\psi\|^2_{\tilde H^{-3/2}(\Gamma)}$$

and

$$\Re\langle D_k v, v\rangle_{L^2(\Gamma)} \geq \gamma_{D,1}\|v\|^2_{\tilde H^{1/2}(\Gamma)} - \gamma_{D,2}\|v\|^2_{\tilde H^{-1/2}(\Gamma)}.$$

The Galerkin scheme for the numerical solution of the integral equations (4.5) and (4.6) is as follows. Given finite dimensional subspaces $X_M \subset \tilde{H}^{-1/2}(\Gamma)$ and $Y_N \subset \tilde{H}^{1/2}(\Gamma)$ (with $X_M \to \tilde{H}^{-1/2}(\Gamma)$ and $Y_N \to \tilde{H}^{1/2}(\Gamma)$ with respect to the resp. norms for $M, N \to \infty$) find $\psi \in X_M$ and $v \in Y_N$ such that

$$\langle V_k \psi, \eta \rangle_{L^2(\Gamma)} = 2\langle g_D, \eta \rangle_{L^2(\Gamma)} \quad \text{for any} \quad \eta \in X_M \tag{4.7}$$

and

$$\langle D_k v, w \rangle_{L^2(\Gamma)} = -2\langle g_N, w \rangle_{L^2(\Gamma)} \quad \text{for any} \quad w \in Y_N. \tag{4.8}$$

The next lemma follows from the uniqueness of the solutions of (4.5) and (4.6) and from the strong ellipticity of V_k and D_k by standard arguments [77, 138, 130].

Lemma 4.6 *There exist integers $M_0, N_0 > 0$ such that for any $M \geq M_0$ and $N \geq N_0$ the systems* (4.7) *and* (4.8) *are uniquely solvable. There hold the quasi-optimal error estimates*

$$\left\| \left[\frac{\partial u}{\partial n}\right] - \psi \right\|_{\tilde{H}^{-1/2}(\Gamma)} \leq c \inf\{ \left\| \left[\frac{\partial u}{\partial n}\right] - \eta \right\|_{\tilde{H}^{-1/2}(\Gamma)};\ \eta \in X_M \}$$

and

$$\|[u] - v\|_{\tilde{H}^{1/2}(\Gamma)} \leq c \inf\{ \|[u] - w\|_{\tilde{H}^{1/2}(\Gamma)};\ w \in Y_N \}$$

for a constant c which is independent of $[\frac{\partial u}{\partial n}]$, $[u]$, v, ψ, M and N.

This lemma reduces the problem of estimating the error of the Galerkin solution to an approximation problem for the exact solution by functions of the ansatz space. The approximation error in general depends on the norm (which is equivalent to the energy norm given by the main part of the integral operator), the approximating functions (i.e., the functions of the ansatz space for the boundary element method) and on the smoothness of the exact solution which has to be approximated. These topics are well understood in many situations, i.e., for different geometries, different types of meshes and different types of ansatz spaces. Subject of this work is the investigation of preconditioners for the linear systems (4.7) and (4.8). In the next section we demonstrate how the preconditioners of Sections 3.2, 3.3, 3.4 apply to efficiently solve these systems by the GMRES method.

Numerical experiments

We consider the Neumann and Dirichlet screen problems (4.1)–(4.3) for the homogeneous Helmholtz equation. We take the screen $\Gamma = (0,1)^2 \times \{0\} \subset \mathbb{R}^3$ which is identified with $\Gamma = (0,1)^2 \subset \mathbb{R}^2$, see Figure 4.1. As right hand side in (4.2) we simply take the function 1 in both cases. This choice does not influence the behavior of the resulting stiffness matrix. The meshes Γ_h are uniform and rectangular as indicated in Figure 4.1. The elements of Γ_h will be denoted by Γ_j: $\bar{\Gamma} = \cup_{j=1}^{J} \bar{\Gamma}_j$.

We now use the notation X_N for ansatz spaces of dimension N for both types of operators.

To assemble the linear system (4.8), which gives an approximate solution for the Neumann problem, we take the ansatz space X_N of piecewise polynomials of degree p (in both variables)

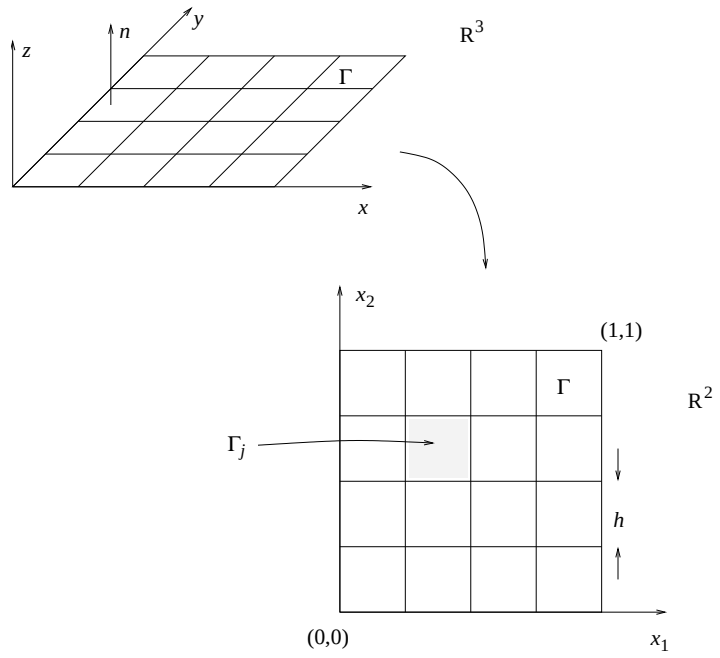


Figure 4.1: Example for the Dirichlet and Neumann problems for the Helmholtz operator: the screen Γ with uniform rectangular mesh.

on Γ_h which are continuous on Γ and which vanish at the boundary $\partial\Gamma$ of Γ. The two latter restrictions are required to ensure conformity:

$$X_N := S_p^1(\Gamma_h) := \{f \in C^0(\Gamma);\ f|_{\Gamma_j} \in P_p(\Gamma_j), j = 1, \ldots, J, f|_{\partial\Gamma} = 0\} \subset \tilde{H}^{1/2}(\Gamma) \qquad (4.9)$$

Here, $P_p(\Gamma_j)$ denotes the set of polynomials on Γ_j which are at most of degree p in each of the variables.

We consider two types of basis functions for the construction of X_N. Note that, when using exact arithmetic, the choice of the basis functions does not influence the Galerkin solution. But it affects the speed of iterative methods for the solution of the linear systems and, moreover, the choice of basis functions is strongly related with the construction of efficient preconditioners.

Our standard basis functions (abbreviated by the symbol "S") are images onto the elements of tensor products of anti-derivatives of Legendre polynomials which are normed with respect to the H^1-semi-norm. On $(-1, 1)$ the components of the tensor products are

$$\psi_0(t) = \frac{1-t}{2}, \quad \psi_1(t) = \frac{1+t}{2},$$
$$\psi_r(t) = \frac{2}{2r-1}\int_{-1}^{t} l_{r-1}(s)\,ds = \frac{1}{\sqrt{2(2r-3)}}\big(l_{r-1}(t) - l_{r-3}(t)\big)\ (2 \le r \le p)$$

where l_r is the Legendre polynomial of degree r. There holds

$$\psi_0(1) = \psi_1(-1) = \psi_r(\pm 1) = 0 \quad (r \ge 2)$$

and their tensor products are being assembled such that the resulting functions can be continuously extended by 0 onto Γ. Tensor products of ψ_0 and/or ψ_1 form the usual piecewise bilinear functions whose supports consist of at most four elements. Here, for the p-version, they are also called **nodal** or **vertex basis functions** since they can be associated with the nodes which are in the interior of Γ. Tensor products of ψ_0 or ψ_1 in one direction and of ψ_r ($r \geq 2$) in the other direction are so-called **edge basis functions** whose supports consist of two elements. Finally, tensor products of ψ_r ($2 \leq r \leq p$) form the **interior basis functions** whose supports are individual elements. This set of basis functions is also standard for the p-version of the finite element method on rectangular meshes (solving problems in the two-dimensional space). The main disadvantage of these functions, in view of preconditioning methods, is the fact that they are rather strongly coupled. That means when neglecting couplings between different functions in the stiffness matrix to build a block Jacobi preconditioner this would result in large condition numbers. In the case of the finite element method, dealing with the H^1-semi-norm, this behavior has been studied in [8] and is also explained in [7]. By taking traces and extensions this poor behavior is also expected in the case of the boundary element method when considering hypersingular operators, i.e., when dealing with the trace space of H^1. A test showing that the diagonal preconditioner, when using the standard basis functions, does not work satisfactorily will be presented below.

In order to reduce the effect of the coupling between different basis functions we consider the special discrete harmonic polynomials ("H") which have been defined in Section 2.3, see especially page 39. The **nodal basis functions** are now full polynomials in each direction. Instead of using ψ_0 and ψ_1 as the components we take the polynomials ϕ_0 and ϕ_0^-, see Definition 2.1 and Figure 2.7. The **edge basis functions** are tensor products of ϕ_i ($i = 1, \ldots, p-1$) and Φ_i ($i = 1, \ldots, p-1$) where the ϕ_is play the role of the components which are perpendicular to the edges, see Definitions 2.3, 2.2 and Figure 2.6. Tensor products of the functions Φ_i ($i = 1, \ldots, p-1$) are used to construct the **interior basis functions**, see Figure 2.5. For more details we refer to Section 2.3.

Our theoretical results of the previous sections cover asymptotic estimates for combinations of various parameters in both, the setup of the ansatz space for the boundary element method and the choice of the preconditioner for the linear system.

When using the standard basis functions (S) one can take the overlapping additive Schwarz method (OASM) as preconditioner. This method is subject of Section 3.2.2 ($k = 0$) and Section 3.4.3 ($k \neq 0$). The respective decomposition of the ansatz space X_N is given in (3.21), see also Figure 3.1. The small global subspace consists of the piecewise bilinear functions, and for each interior node we have the subspace of functions whose support consists of the elements which are adjacent to that node. In particular, the functions vanish at the boundary of that patch of elements. In order to calculate the additive Schwarz operator we have to solve a boundary integral equation on all of these subspaces. The abstract form of these problems is given in (3.4). In the case $k = 0$ this integral equation is the original equation, i.e. the bilinear form $\langle D\cdot, \cdot\rangle$ has to be inverted on the respective subspace. In the case $k \neq 0$ there are two possibilities. Either we again solve the full problem (indicated by the symbol "I" for Indefinite), i.e., we invert D_k (the abstract form is given in (3.44)) or we only solve the related positive definite problem (indicated by the symbol "P" for Positive definite), i.e., we invert $D = D_0$ (the abstract form is given in (3.45)). In the practical implementation we proceed as follows. To each of the subspaces belongs a block of the full stiffness matrix whose positive definite part is kept in the process of calculation (if $k \neq 0$). In the case $k \neq 0$ and

when solving only positive definite problems (P) we take this positive definite matrix being kept for the calculation of the blocks. All of these blocks have to be inverted separately, then they have to be filled up with zeroes in order to become of the same size as the full stiffness matrix and then they are summed up. In this way we obtain a matrix which serves as a preconditioning matrix for the original linear system.

For the discrete harmonic basis functions (H) the choice of the decomposition of the ansatz space is almost free. Due to Section 3.2.3 ($k = 0$) and Section 3.4.3 ($k \neq 0$) any decomposition then yields an almost optimal preconditioner. The only restriction is the use of an additional global subspace of piecewise bilinear functions. In Section 3.2.3 we explicitly considered the so-called wire basket preconditioner (it will be referred to by the symbol "WIRE"), a non-overlapping additive Schwarz method (NASM), and the almost diagonal preconditioner (DIAG). The decomposition of the ansatz space for the wire basket preconditioner is defined in (3.22), see also Figure 3.2. For the non-overlapping additive Schwarz method we refer to (3.23) and Figure 3.3. The decomposition for the almost diagonal preconditioner is given in (3.24). In the case of wavenumber $k = 0$, when the hypersingular Operator $D := D_0$ is selfadjoint and positive definite, all these preconditioners are directly applicable, see Section 3.2.3. When $k \neq 0$, D_k is non-selfadjoint (still real symmetric) and indefinite. In that case we use the results of Section 3.4 where the standard additive Schwarz method and a hybrid method have been presented. For the former method we made the assumption that the decomposition of the ansatz space is (besides the global space of piecewise bilinear functions) of a local nature (Assumption (A2) on page 82). This is necessary to ensure positive definiteness of the, in general indefinite, real part of the integral operator by making the support of the local polynomial space small enough. The preconditioners NASM and DIAG exclusively use (besides the mentioned global space) local subspaces for their decompositions and therefore fulfill this assumption. In these cases, there are the two variants of locally inverting the full operator (I) or inverting only its positive definite part (P). On the other hand, the wire basket preconditioner possesses another global component in the decomposition of the ansatz space, namely the wire basket component. Therefore, Assumption (A2) is not fulfilled. In order to apply this method we have to refer to the theory of the hybrid method. In this case also the global space of piecewise bilinear functions is used (where the full operator D_k has to be inverted) and an arbitrary preconditioner that works in the positive definite case can be taken. These two parts, the coarse space part and the preconditioner for the positive definite part, are added where the parameter β enters. Actually, when taking $\beta = 1$ (which is done in all our experiments), the hybrid method almost coincides with the additive Schwarz method with local positive definite problems (P). The only difference is that for the hybrid method we then have two solvers for the subspace of piecewise bilinear functions, one with respect to the full operator and one with respect to the positive definite part whereas the latter inversion is not considered for the usual additive Schwarz method. This near coincidence is true not only of the wire basket decomposition but also in the other cases (actually any decomposition then gives almost the same method). Therefore, we will not make a difference between the additive Schwarz method (with (P)) and the hybrid method. The hybrid method (with $\beta = 1$) is only used for the theoretical justification of additive Schwarz preconditioners for indefinite operators when, besides the standard global subspace, further subspaces of non-local nature are used as it is the case with the wire basket decomposition. Indeed, the hybrid method allows for considering other preconditioners than of additive Schwarz type for the positive definite part which then would give new methods. This is not considered in this work.

basis	$k = 0$	$k \neq 0$	
S	OASM	OASM	I, P
H	WIRE	WIRE	P
H	NASM	NASM	I, P
H	DIAG	DIAG	I, P

Table 4.1: Theoretically justified preconditioners for the hypersingular operator D_k.

Table 4.1 gives an overview of the most important preconditioners that are covered by our theory. The symbols are those introduced above.

Now we switch to the Dirichlet problem. The ansatz space X_N for (4.7), which gives an approximate solution for the Dirichlet problem, consists of piecewise polynomials on Γ_h which are at most of degree p (in both variables) and which need not to be continuous on Γ:

$$X_N := S_p^0(\Gamma_h) := \{f : \Gamma \to \mathbb{R};\ f|_{\Gamma_j} \in P_p(\Gamma_j), j = 1, \dots, J\} \subset \tilde{H}^{-1/2}(\Gamma) \tag{4.10}$$

As basis functions we take affine mappings onto the elements of tensor products of Legendre polynomials which are normed with respect to the L^2-norm.

Our preconditioner is the non-overlapping additive Schwarz method (NASM). It uses a direct decomposition of the ansatz space (3.34) which ensures that each unknown in the linear system is only treated once in the preconditioning step, see also Figure 3.4. In that sense, in order to use direct decompositions, we do not need special basis functions as for the hypersingular operator. On the other hand, a total decoupling of the basis functions giving the diagonal preconditioner is not recommended. This is justified by a numerical experiment presented below. The question what type of basis functions would allow for further decomposing the ansatz space (finer than for NASM) without losing efficiency is unknown so far. For the numerical experiments with NASM we choose $H = h$ (see Figure 3.4) which means that for each element of the mesh we have a single subspace:

$$X_N = S_p^0(\Gamma_h) = H_0 \cup \{v|_{\Gamma_1};\ v \in X_N \setminus H_0\} \cup \dots \cup \{v|_{\Gamma_J};\ v \in X_N \setminus H_0\}$$

Here, H_0 is the set of piecewise constant functions on Γ_h. Note that the condition $\langle v, 1\rangle_{L^2}$ in the definition of the decomposition (3.34) is automatically fulfilled since our subdomains consist only of single elements and since we are using tensor products of Legendre polynomials as basis functions.

If $k \neq 0$ we also make the distinction of locally full inversions (I) and local inversions of the main part of the weakly singular operator V_k (P).

In all the cases above for both the Neumann and the Dirichlet problem the h-version of the boundary element method with arbitrary polynomial degree and the p-version with arbitrary mesh size h are applicable.

The symmetric positive definite case ($k = 0$)

For linear systems with symmetric positive definite matrix the conjugate gradient (CG) method is the method of choice for the solution. However, in order to study the behavior of different preconditioners for different wave numbers, when the CG method is not applicable, we always use the GMRES method as solver. Then we can directly compare iteration

			S, without prec.			S, OASM		
N	h^{-1}	p	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$
4	3	1	0.126E+1	0.264	0.333	1.12	1.858	2.08
25	3	2	0.869E+2	0.502E-2	0.437	4.95	0.827	4.10
64	3	3	0.384E+3	0.117E-2	0.450	4.73	0.866	4.10
121	3	4	0.164E+4	0.274E-3	0.451	4.63	0.889	4.12
196	3	5	0.419E+4	0.107E-3	0.451	4.54	0.908	4.12
289	3	6	0.111E+5	0.405E-4	0.451	4.48	0.921	4.13
400	3	7	0.228E+5	0.197E-4	0.451	5.13	0.805	4.13
49	2	4	0.109E+4	0.575E-3	0.629	2.00	1.000	2.00
121	3	4	0.164E+4	0.274E-3	0.451	4.63	0.889	4.12
225	4	4	0.196E+4	0.180E-3	0.354	5.18	0.855	4.43
361	5	4	0.218E+4	0.134E-3	0.294	5.33	0.838	4.47
529	6	4	0.228E+4	0.108E-3	0.247	5.41	0.827	4.48

Table 4.2: Hypersingular operator: condition numbers and extreme eigenvalues (standard basis functions (S), without and with overlapping additive Schwarz preconditioner (OASM)).

numbers for different situations. In the symmetric case (indeed, symmetry of the operator with respect to the used inner product is necessary which is the case for additive Schwarz operators and the energy inner product) the essential parameters entering the error estimates of the conjugate gradient method and the GMRES method coincide: The parameter Λ_0 in (3.1) then is identical to the minimum eigenvalue $\lambda_{\min}$ of the stiffness matrix and the parameter Λ_1 in (3.1) is the maximum eigenvalue $\lambda_{\max}$ of the stiffness matrix. In either case the ratio $\kappa = \Lambda_1/\Lambda_0$ is the essential number in the standard error estimates for the iteration procedures.

Let us consider preconditioners for the hypersingular operator. Using standard basis functions and the overlapping additive Schwarz method we expect by Theorem 3.6 bounded condition numbers. Table 4.2 lists the results in this situation as well as the numbers for the original system. In the upper part of the table the mesh size $h = 1/3$ is fixed (giving 9 elements) and the maximum polynomial degree of the ansatz functions is increased from 1 to 7. This corresponds to 4 to 400 unknowns N. In the lower part of the table the polynomial degree is fixed at $p = 4$ and the mesh size h is reduced from 1/2 to 1/6. This corresponds to 4 up to 36 elements and to 49 up to 529 unknowns. We therefore can check the dependence of the essential parameters on p in the upper part and the dependence on h in the lower part. In either case the maximum and minimum eigenvalues appear to be bounded when using the overlapping additive Schwarz method. It also becomes clear that the condition number then is much smaller than without preconditioner.

Table 4.3 presents the eigenvalues and condition numbers which have been obtained by using the special basis functions (H) and the three preconditioners (wire basket, non-overlapping additive Schwarz, and modified diagonal preconditioner). The theoretical estimates are provided by Theorem 3.7 and its proof. In all three cases a bound of the condition numbers which behaves like $(1 + \log p)^2$ is expected. More precisely, by considering the proof, this behavior is that of the inverse of the minimum eigenvalues and the maximum eigenvalues should

			H, WIRE			H, NASM			H, DIAG		
N	h^{-1}	p	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$
4	3	1	1.00	2.000	2.00	1.12	1.858	2.08	1.12	1.858	2.08
25	3	2	4.93	0.455	2.24	4.21	0.483	2.04	4.21	0.483	2.04
64	3	3	6.11	0.381	2.33	5.50	0.408	2.25	5.50	0.408	2.25
121	3	4	8.52	0.276	2.35	7.78	0.294	2.29	7.96	0.283	2.26
196	3	5	9.30	0.256	2.38	8.48	0.272	2.31	8.61	0.261	2.25
289	3	6	10.95	0.217	2.38	10.02	0.231	2.32	10.43	0.215	2.25
400	3	7	11.48	0.208	2.39	10.50	0.221	2.32	10.97	0.204	2.24
49	2	4	7.38	0.314	2.32	6.72	0.324	2.18	6.85	0.315	2.16
121	3	4	8.52	0.276	2.35	7.78	0.294	2.29	7.96	0.283	2.26
225	4	4	9.04	0.261	2.36	8.19	0.282	2.32	8.38	0.271	2.27
361	5	4	9.34	0.254	2.37	8.47	0.277	2.35	8.65	0.265	2.30
529	6	4	9.51	0.250	2.38	8.62	0.274	2.36	8.80	0.262	2.31

Table 4.3: Hypersingular operator: condition numbers and extreme eigenvalues (discrete harmonic basis functions (H), with preconditioners: wire basket prec. (WIRE), non-overlapping additive Schwarz method (NASM), diagonal prec. (DIAG)).

be bounded. Indeed, the numbers of the table show almost constant maximum eigenvalues and the minimum eigenvalues are only slightly decreasing for increasing p. The last rows of the table indicate almost constant minimum eigenvalues which supports their independence on the mesh size h. Of course, the theoretical estimates are only of asymptotic nature comprising unknown constants. Nevertheless, we graphically check the logarithmic dependence of the minimum eigenvalues on the polynomial degree p. This is done in Figure 4.2 where the minimum eigenvalues obtained by the preconditioner (H, NASM) as well as the functions $(1+\log p)^{-1}$ and $(1+\log p)^{-2}$ are plotted in a double logarithmic scale. The slope of the obtained curve of the minimum eigenvalues in this range of polynomial degrees is between that of the two functions and this confirms the theoretical behavior.

Finally, the practically more important numbers of iterations for the solution of the linear systems are given in Table 4.4. As explained above, we use the GMRES method also in the positive definite case and the iteration is stopped when the initial residual is reduced by a factor of 10^{-6} where we use the vector 0 as initial guess. All the theoretically justified preconditioners substantially reduce the numbers of iterations. The overlapping additive Schwarz method for standard basis functions is a bit superior to the remaining preconditioners using discrete harmonic basis functions, at least when considering the pure p-version which corresponds to the upper part of the table. Here we also report on our numerical experiment, already mentioned above, which indicates that direct decompositions of the ansatz space do not yield efficient preconditioners when the standard basis functions are taken. The fifth row of Table 4.4 gives the numbers of iterations when using the diagonal preconditioner (including the global block which belongs to the piecewise bilinear functions) for the standard basis functions. Obviously this method does not accelerate the iterative method at all.

Now we consider the non-overlapping additive Schwarz preconditioner for the weakly singular operator, i.e., for the linear system (4.7). An estimate of the condition numbers for this

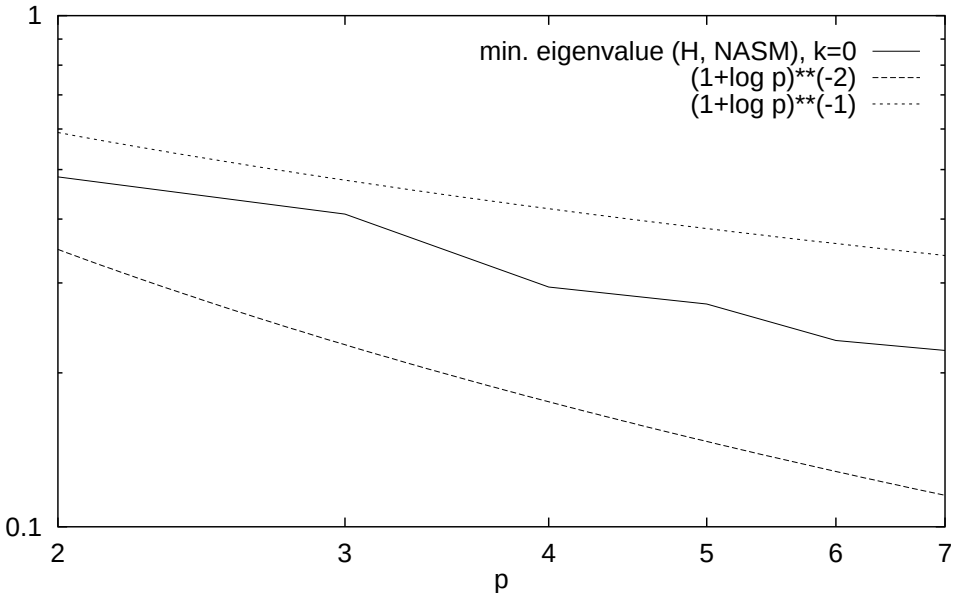


Figure 4.2: Hypersingular operator (p-version, $h^{-1} = 3$): minimum eigenvalues obtained by the non-overlapping additive Schwarz preconditioner (H, NASM).

	basis:		S	S		H		
N	h^{-1}	p	w/o prec	DIAG	OASM	WIRE	NASM	DIAG
4	3	1	2	2	2	2	4	4
25	3	2	7	7	7	7	7	7
64	3	3	12	11	9	10	10	10
121	3	4	22	24	10	14	14	14
196	3	5	32	35	10	15	15	15
289	3	6	40	38	10	16	16	17
400	3	7	47	49	9	17	17	18
49	2	4	12	11	3	10	9	10
121	3	4	22	24	10	14	14	14
225	4	4	34	41	14	16	16	16
361	5	4	47	51	15	16	17	17
529	6	4	57	55	16	17	18	18

Table 4.4: Hypersingular operator: numbers of GMRES iterations (all cases).

			without prec.			NASM		
N	h^{-1}	p	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$
4	2	0	0.397E+1	0.745E-2	0.0295	1.00	1.000	1.00
16	2	1	0.113E+2	0.266E-2	0.0301	2.95	0.564	1.66
36	2	2	0.215E+2	0.140E-2	0.0301	4.42	0.410	1.81
64	2	3	0.350E+2	0.860E-3	0.0301	5.36	0.363	1.95
100	2	4	0.521E+2	0.578E-3	0.0301	6.51	0.312	2.03
144	2	5	0.728E+2	0.414E-3	0.0301	7.33	0.286	2.10
196	2	6	0.971E+2	0.310E-3	0.0301	8.26	0.260	2.15
256	2	7	0.125E+3	0.241E-3	0.0301	8.98	0.244	2.19
324	2	8	0.156E+3	0.192E-3	0.0301	9.77	0.228	2.23
36	2	2	0.215E+2	0.140E-2	0.301E-1	4.42	0.410	1.81
144	4	2	0.491E+2	0.153E-3	0.754E-2	4.88	0.389	1.90
324	6	2	0.759E+2	0.441E-4	0.335E-2	5.01	0.382	1.92
576	8	2	0.102E+3	0.184E-4	0.188E-2	5.09	0.380	1.93
900	10	2	0.128E+3	0.937E-5	0.120E-2	5.13	0.378	1.94

Table 4.5: Weakly singular operator: condition numbers and extreme eigenvalues (without and with non-overlapping additive Schwarz preconditioner (NASM)).

method is provided by Theorem 3.8: $\kappa \leq c(1+\log(p+1))^2$ since we chose $H/h = 1$. Table 4.5 lists the extremum eigenvalues and the condition numbers of the stiffness matrix without and with preconditioner. The preconditioner efficiently reduces the condition numbers and, as it is shown theoretically in the proof of Theorem 3.8, the maximum eigenvalues are likely to be bounded. When considering the p dependence of $\lambda_{\min}$ the numbers are slightly decreasing what is expected due to the given estimate. Indeed, Figure 4.3 confirms the lower bound $c(1+\log(1+p))^{-2}$ for the minimum eigenvalues where these eigenvalues and the functions $(1+\log(p+1))^{-1}$ and $(1+\log(p+1))^{-2}$ are plotted. The lower part of Table 4.5 shows that the minimum eigenvalue is bounded when reducing the mesh size and fixing p.

Also for the weakly singular operator we use the GMRES method as iterative solver and we take the same stopping criterion as above (residual reduction by 10^{-6}). Table 4.6 lists the required numbers of iterations which are almost bounded when using the non-overlapping additive Schwarz preconditioner whereas the iteration numbers for the un-preconditioned systems are increasing. We note that a further decomposition (finer than for the NASM preconditioner) does not produce good preconditioners. To see this we test the diagonal preconditioner (still direct decomposition of the ansatz space including the global subspace of piecewise constant functions). The required iteration numbers for this preconditioner are given in the fifth row of Table 4.6. For the p-version these number are increasing nearly as fast as without preconditioner whereas for the h-version they are constant. The constant behavior for the h-version is not surprising when noting that this modified diagonal decomposition is only a modification of the NASM decomposition which is local to the elements. And since all basis functions are local to single elements (their supports are only individual elements) the influence of this modification is constant for fixed polynomial degree p (when the local subspaces on the elements are only scaled by h and not further modified).

N	h^{-1}	p	w/o prec	DIAG	NASM
4	2	0	2	2	2
16	2	1	4	4	4
36	2	2	7	7	7
64	2	3	10	9	8
100	2	4	12	11	9
144	2	5	13	12	9
196	2	6	14	13	9
256	2	7	15	14	9
324	2	8	17	14	10
36	2	2	7	7	7
144	4	2	12	13	13
324	6	2	14	13	13
576	8	2	15	13	13
900	10	2	17	13	13

Table 4.6: Weakly singular operator: numbers of GMRES iterations (without and with diagonal preconditioner (DIAG) and with non-overlapping additive Schwarz preconditioner (NASM)).

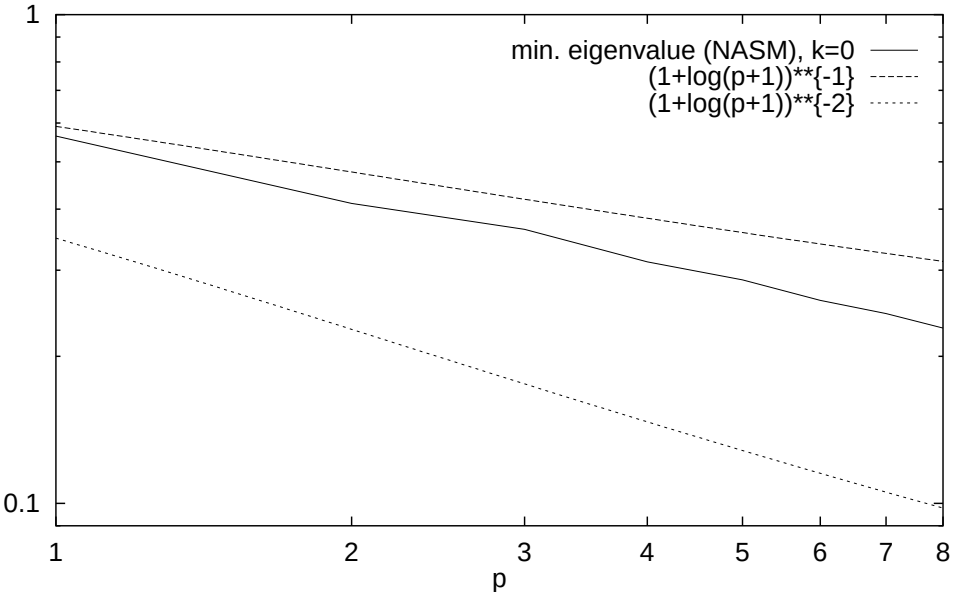


Figure 4.3: Weakly singular operator (p-version, $h^{-1} = 2$): minimum eigenvalues obtained by the non-overlapping additive Schwarz preconditioner (NASM).

The indefinite case ($k \neq 0$)

In this section we demonstrate that the preconditioners of the previous section for selfadjoint positive definite operators are also applicable when the boundary element systems (4.7) and (4.8) are indefinite or non-Hermitian.

We begin with the hypersingular operator and consider the overlapping additive Schwarz preconditioner for the standard basis functions. This case is covered by Theorem 3.12 (full local solvers (I) and positive definite local solvers (P)). In both cases the parameters Λ_0 (minimum eigenvalue of the Hermitian part of the preconditioned stiffness matrix) and Λ_1 (norm of the preconditioned stiffness matrix) are bounded for increasing polynomial degree p as long as the mesh size h is small enough. We note that the condition that h must be small enough is to ensure the positive definiteness of the, in general indefinite, real part of the integral operator on local subspaces. The abstract versions (Theorems 3.9 and 3.10) of the specific Theorem 3.12 shows that the requirement on the mesh size is steered by an expression of the type

$$c - h^{\delta_3} \quad \big(\delta_3 \leq \min\{\delta_1, \delta_2\}\big)$$

which has to be positive. The parameters δ_1 and δ_2 are prescribed in Lemma 3.6. δ_1 is the order of convergence of the h-version of the boundary element method when taking piecewise bilinear basis functions. For our specific example Γ being a screen, where strong edge singularities in the solution may occur, we expect an order of convergence $\delta_1 = 1/2 - \varepsilon$ ($\varepsilon > 0$), cf. [130]. To determine the second parameter δ_2 we note that for the hypersingular operator a Gårding's inequality (3.41) holds with $\delta = 1$, cf. Lemma 4.5. By Lemma 3.6 we then find for $\alpha = 1$ (the order of the hypersingular operator) the value $\delta_2 = 1/2$. Therefore, we can choose $\delta_3 = \min\{\delta_1, \delta_2\} = 1/2 - \varepsilon$.

Figure 4.4 shows that the parameters Λ_0 and Λ_1 are bounded as expected and that the numbers do not depend very strongly on the wavenumber k and on the type of the local solvers (I or P). We note that we do not have theoretical estimates for the dependence of the condition number on the wave number. This is a topic of ongoing research. The variations in the parameter Λ_0 from $p = 6$ to $p = 7$ are due to the finite computer's arithmetic. The analytical recurrence formulae for the computation of the entries in the stiffness matrix become unstable for higher polynomial degrees. (Of course, there are no methods which allow for taking arbitrary large degrees for the ansatz spaces without numerical problems.) We note that the numerical instabilities are not due to violation of the condition on the mesh size which, for this preconditioner, is a condition that does not depend on p.

Table 4.7 presents the needed numbers of iterations of the GMRES method for different values of k. In any case the numbers are substantially reduced but the effect of the type of the local solver becomes larger for high polynomial degrees and for larger wave numbers.

Figure 4.5 shows the parameters Λ_0 and Λ_1 in the case of discrete harmonic basis functions (H) and modified diagonal preconditioner (DIAG). This is the situation of Theorem 3.13 which proves an asymptotic bound like $(1 + \log p)^{-2}$ for the minimum eigenvalue Λ_0 of the Hermitian part of the preconditioned stiffness matrix. The norm Λ_1 of the preconditioned stiffness matrix is proven to be bounded. The requirement of Theorem 3.13 that the mesh size h must be small enough does depend on the taken polynomial degree. Especially, to ensure convergence of the GMRES method, we must have $\Lambda_0 > 0$, i.e., h must be small enough such

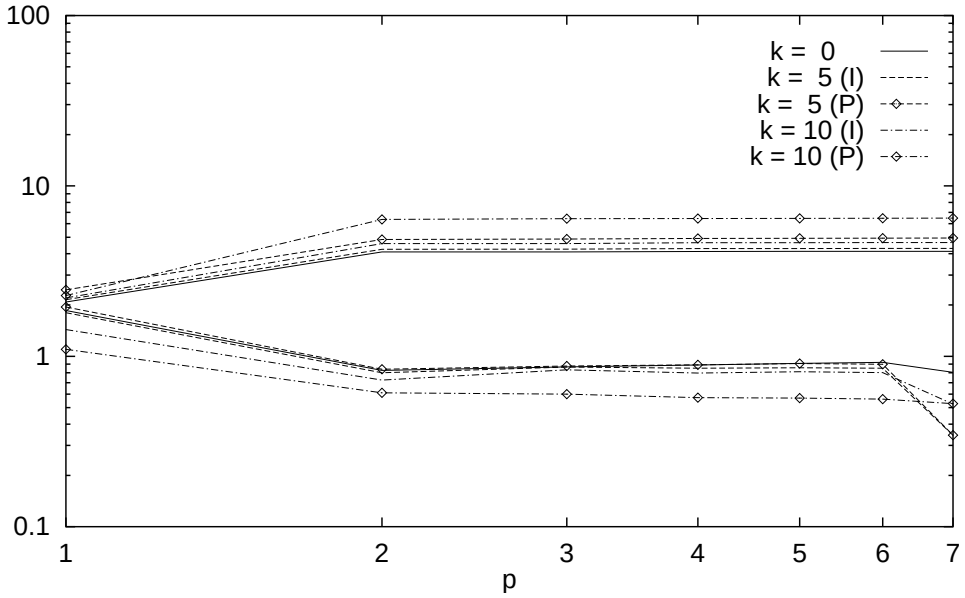


Figure 4.4: Hypersingular operator (p-version, $h^{-1} = 3$): values of Λ_0 and Λ_1 obtained by the overlapping additive Schwarz preconditioner (S, OASM) with full local solvers (I) and positive definite local solvers (P).

		k:	0		5			10		
N	h^{-1}	p	–	(P)	–	(I)	(P)	–	(I)	(P)
4	3	1	2	2	2	2	2	2	2	2
25	3	2	7	7	7	7	7	7	7	7
64	3	3	12	9	13	9	9	14	9	10
121	3	4	22	10	26	11	12	28	11	14
196	3	5	32	10	35	10	11	39	11	13
289	3	6	40	10	43	10	13	46	11	15
400	3	7	47	9	50	11	13	55	11	16
49	2	4	12	3	13	3	2	13	3	8
121	3	4	22	10	26	11	12	28	11	14
225	4	4	34	14	38	14	15	40	15	17
361	5	4	47	15	50	16	16	57	16	17
529	6	4	57	16	61	16	16	75	17	18

Table 4.7: Hypersingular operator: numbers of GMRES iterations (without prec. ("–") and with overlapping additive Schwarz preconditioner (S, OASM), different wave numbers, full local solvers and positive definite local solvers (I, P)).

that

$$C(1+\log p)^{-2} - h^{\delta_3} > 0.$$

Here, C is an unknown constant and $\delta_3 \leq \min\{\delta_1, \delta_2\}$ As already mentioned above, $\delta_3 = \delta_1 = 1/2 - \varepsilon$ for $\varepsilon > 0$ can be chosen. Nevertheless, there is no problem to satisfy these conditions since the dependence on the polynomial degree is only logarithmic. Our numerical experiments do not reflect the theoretically needed dependence of h on p. We take the same rather crude grid for all ansatz spaces being defined by different polynomial degrees and the preconditioner works as in the positive definite case. The condition on the mesh size is of asymptotic nature and may become a restriction for higher polynomial degrees or if one considers a different integral operator of order one.

We only present numerical results for the diagonal preconditioner and we note that the non-overlapping additive Schwarz preconditioner (Theorem 3.13 applies as well) and the hybrid method with wire basket decomposition (Theorem 3.14) produce almost the same results as the diagonal preconditioner. Here we note that, with regard to the local solvers, the hybrid method with wire basket decomposition coincides with the additive Schwarz preconditioner for the same decomposition if $\beta = 1$. The only difference lies in the global component H_0 where we have a full solver $\mathcal{P}_0$ plus a solution for the positive definite part P_0 for the hybrid method whereas the operator P_0 is avoided for the standard additive Schwarz method.

Referring to Figure 4.5, we observe that the diagonal additive Schwarz preconditioner (and the other two, (H, WIRE) and (H, NASM) which give nearly identical results) behaves for moderate wavenumbers k similar as in the positive definite case ($k = 0$). Here, it does not matter what type of local solver (I or P) is taken.

Table 4.8 demonstrates that the dependence of the iteration numbers of the GMRES method on the wave number is not very strong. Indeed, the numbers increase only slightly with k and p and they are in general a bit larger than when using the overlapping additive Schwarz method and standard basis functions (a bit in comparison with the un-preconditioned GMRES method).

Now we consider the system (4.7) discretizing the weakly singular integral equation when $k \neq 0$, i.e., when the single layer operator V is non-selfadjoint. Also in this case the non-overlapping additive Schwarz method is applicable. As for the hypersingular operator we consider full local solvers and positive definite local solvers. The corresponding theoretical estimates are given by Theorem 3.15. More precisely, Λ_1 is bounded in either case and the minimum eigenvalue of the Hermitian part of the stiffness matrix is bounded from below like

$$\Lambda_0 \geq C\big(1+\log(\frac{H}{h}(p+1))\big)^{-2} - cH^{\delta_3} = C(1+\log(1+p))^{-2} - ch^{\delta_3} \quad \big(\delta_3 \leq \min\{\delta_1, \delta\}\big)$$

in the case of both full local solvers and positive definite local solvers. Here, H is the mesh size of the coarse mesh Γ_H (which is used for the decomposition) and h is the mesh size of the boundary element mesh Γ_h. For our example we chose $H = h$. δ_1 is the order of convergence of the pure h-version of the boundary element method (piecewise constant ansatz functions) and δ is the parameter appearing in Gårding's inequality (3.41). As for the hypersingular operator, the solution of the weakly singular integral equation (4.5) behaves singularly at the edges of Γ which gives only a convergence h^{δ_1} of the h-version with $\delta_1 = 1/2 - \varepsilon$ ($\varepsilon > 0$), cf. [130]. The parameter $\delta = 1$ can be chosen by Lemma 4.5. Therefore, since $H/h = 1$ for our example, in both cases (I) and (P) we expect the bound $C(1+\log(p+1))^{-2} - ch^{1/2-\varepsilon}$ for

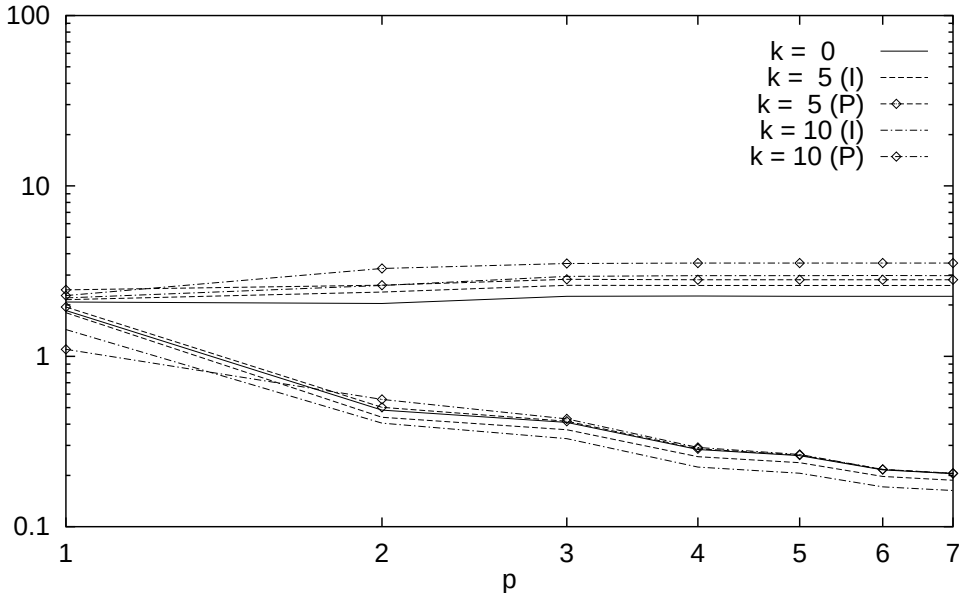


Figure 4.5: Hypersingular operator (p-version, $h^{-1} = 3$): values of Λ_0 and Λ_1 obtained by the modified diagonal preconditioner (H, DIAG) with full local solvers (I) and positive definite local solvers (P).

		k:	0		5		10	
N	h^{-1}	p	OASM	DIAG	OASM	DIAG	OASM	DIAG
4	3	1	2	4	2	4	2	4
25	3	2	7	7	7	7	7	7
64	3	3	9	10	9	10	9	11
121	3	4	10	14	11	15	11	16
196	3	5	10	15	10	16	11	17
289	3	6	10	17	10	18	11	20
400	3	7	9	18	11	20	11	22
49	2	4	3	10	3	11	3	11
121	3	4	10	14	11	15	11	16
225	4	4	14	16	14	17	15	19
361	5	4	15	17	16	18	16	21
529	6	4	16	18	16	19	17	22

Table 4.8: Hypersingular operator: numbers of GMRES iterations (with overlapping additive Schwarz preconditioner (S, OASM) and with modified diagonal preconditioner (H, DIAG), different wave numbers, full local solvers (I)).

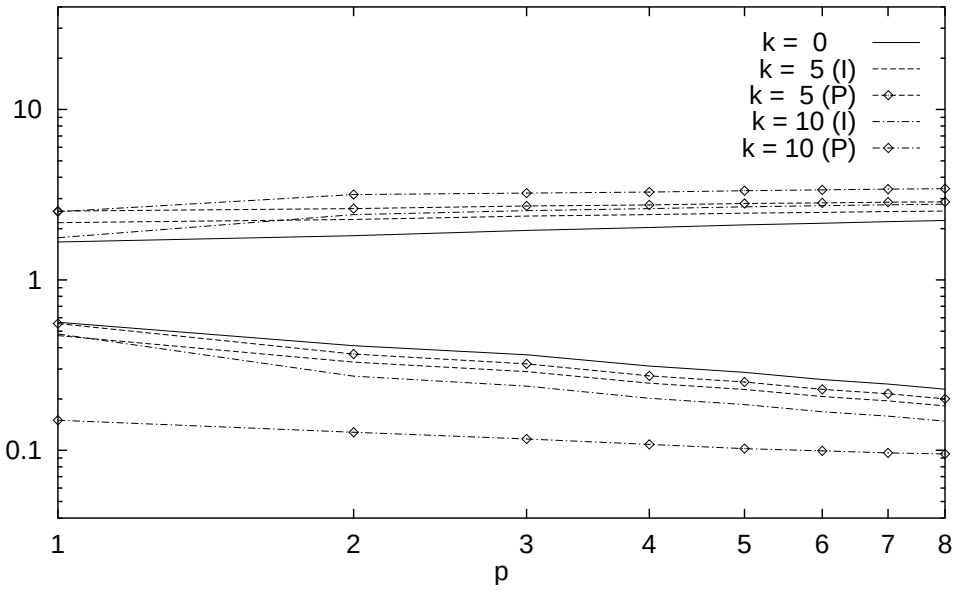


Figure 4.6: Weakly singular operator (p-version, $h^{-1} = 2$): values of Λ_0 and Λ_1 obtained by the non-overlapping additive Schwarz preconditioner (NASM) with full local solvers (I) and positive definite local solvers (P).

Λ_0 where h has to be chosen small enough such that $\mathcal{P}_0$ exists and is bounded (Assumption (A1) on page 82) and such that $C(1 + \log(p+1))^{-2} - ch^{1/2-\varepsilon} > 0$.

Figure 4.6 demonstrates that this preconditioner produce similar results as in the positive definite case. Only in the case $k = 10$ (P) the curve for the parameter Λ_0 shows smaller values but the dependence on p is even better than in the other cases. The numbers of GMRES iterations are stable and do not depend very strongly on k, cf. Table 4.9.

4.2 Electric screen problem

We consider an exterior boundary value problem for time harmonic Maxwell's equations where the Dirichlet data is given. This problem is represented by a system of first kind weakly singular and hypersingular integral operators. We assume that the unknown electric and magnetic fields are H^1-regular such that standard Sobolev spaces can be used in the following.

Let Γ denote an open plane screen in $\mathbb{R}^3$ and let $\Omega := \mathbb{R}^3 \setminus \bar{\Gamma}$. The electric problem for a perfectly conducting screen is: For a given tangential field $\mathbf{g}$ on Γ find two vector fields $\mathbf{E}$ and $\mathbf{H}$ in Ω such that

$$\operatorname{curl} \mathbf{E} = \mathbf{H}, \quad \operatorname{curl} \mathbf{H} = k^2 \mathbf{E} \quad \text{in } \Omega, \tag{4.11}$$

$$\mathbf{n} \times \mathbf{E} = \mathbf{g} \quad \text{on } \Gamma, \tag{4.12}$$

$$\mathbf{H} \times \frac{x}{|x|} - \mathbf{E} = \mathrm{o}(\frac{1}{|x|}) \quad \text{for } |x| \to \infty. \tag{4.13}$$

		k:	0		5			10		
N	h^{-1}	p	–	(P)	–	(I)	(P)	–	(I)	(P)
4	2	0	2	2	2	2	2	2	2	2
16	2	1	4	4	4	4	4	4	4	4
36	2	2	7	7	7	7	7	7	7	7
64	2	3	10	8	10	8	9	11	9	9
100	2	4	12	9	12	9	9	13	9	11
144	2	5	13	9	14	9	9	15	10	11
196	2	6	14	9	16	10	9	16	10	11
256	2	7	15	9	18	10	10	18	10	11
324	2	8	17	10	20	10	10	22	10	11
36	2	2	7	7	7	7	7	7	7	7
144	4	2	12	13	14	14	14	15	15	15
324	6	2	14	13	16	15	15	18	17	17
576	8	2	15	13	19	14	14	19	17	17
900	10	2	17	13	21	14	14	23	16	16

Table 4.9: Weakly singular operator: numbers of GMRES iterations (without prec. ("–") and with non-overlapping additive Schwarz preconditioner (NASM), different wave numbers, full local solvers and positive definite local solvers (I, P)).

Here, $\mathbf{E}$ and $\mathbf{H}$ denote the scattered electro-magnetic field and $\mathbf{n}$ is the normal vector to the screen Γ. We neglect conduction currents in Ω which means that the parameter k is real and positive and the uniqueness of the solution of (4.11)–(4.13) follows. For the treatment of more general electro-magnetic transmission problems we refer to [42].

Following [95] we represent the solution $\mathbf{E}$, $\mathbf{H}$ of (4.11)–(4.13) by

$$\mathbf{E} = V_k(\mathbf{J}) + \operatorname{grad} V_k(M), \quad \mathbf{H} = \operatorname{curl} V_k(\mathbf{J}) \tag{4.14}$$

for unknown tangential and scalar fields $\mathbf{J}$ and M, respectively. Here, V_k is the single layer potential which is defined by

$$V_k(M)(x) = \frac{1}{4\pi} \int_\Gamma M(y) \frac{e^{ik|x-y|}}{|x-y|}\, dS_y \quad (x \in \Gamma)$$

and analogously for vector fields $\mathbf{J}$.

This ansatz leads to the following system of integral equations:

$$\mathcal{A}_k \begin{pmatrix} \mathbf{J} \\ M \end{pmatrix} := \begin{pmatrix} \mathbf{n} \times V_k(\mathbf{J}) + \mathbf{n} \times \operatorname{grad}_\Gamma V_k(M) \\ \operatorname{div}(\mathbf{n}(\mathbf{n} \cdot V_k(\mathbf{J}))) - (\Delta_\Gamma + k^2) V_k(M) \end{pmatrix} = \begin{pmatrix} \mathbf{g} \\ \operatorname{div}_\Gamma \mathbf{n} \times \mathbf{g} \end{pmatrix} \tag{4.15}$$

The operator $\operatorname{div}_\Gamma$ denotes the surface divergence which is, for a tangential field, the trace onto Γ of the divergence, and $\Delta_\Gamma := \operatorname{div}_\Gamma \operatorname{grad}_\Gamma$ with the tangential component $\operatorname{grad}_\Gamma$ of the gradient.

Before recalling the equivalence of problems (4.11)–(4.13) and (4.15) we define the needed Sobolev spaces. For $s \in \mathbb{R}$ we denote by $\mathbf{H}^s(\Omega)$ the space of vector fields whose components belong to $H^s(\Omega)$. For any fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we identify $\mathbf{H}^s(\Omega)$ with

$(H^s(\Omega))^3$ by $\mathbf{u} = \sum_{i=1}^3 u_i \mathbf{e}_i,\ u_i \in H^s(\Omega)$ and use the norm

$$\|\mathbf{u}\|_{\mathbf{H}^s(\Omega)} = \left(\sum_{i=1}^3 \|u_i\|^2_{H^s(\Omega)} \right)^{1/2}.$$

Analogously we define $\mathbf{H}^s(\Gamma)$ and $\tilde{\mathbf{H}}^s(\Gamma)$. Here, $H^s(\Omega)$, $H^s(\Gamma)$ and $\tilde{H}^s(\Gamma)$ denote the usual Sobolev spaces. Since we are considering plane surfaces the decomposition of $\mathbf{H}^s(\Gamma)$ into two subspaces consisting of tangential and normal fields to Γ is straight forward:

$$\begin{aligned} \mathbf{u} = \mathbf{u}_{\mathrm{T}} + (\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \quad \mathbf{u}_{\mathrm{T}} &= (\mathbf{n}\times\mathbf{u})\times\mathbf{n} \\ N H^s(\Gamma) &= \{\mathbf{u}\in\mathbf{H}^s(\Gamma);\ \mathbf{u} = v\mathbf{n},\ v\in H^s(\Gamma)\} \\ T H^s(\Gamma) &= \{\mathbf{u}\in\mathbf{H}^s(\Gamma);\ \mathbf{u}\cdot\mathbf{n} = 0\} \\ \mathbf{H}^s(\Gamma) &= N H^s(\Gamma) \oplus T H^s(\Gamma) \end{aligned}$$

Here, $\mathbf{u}\cdot\mathbf{n}$ denotes the inner product of the three dimensional Euclidean space for the components of $\mathbf{u}$ and $\mathbf{n}$.

From [128, Corollary 2.3.6, Theorem 2.3.7] we cite the following results.

Theorem 4.1 (i) *If* $\mathbf{E}, \mathbf{H}$ *belonging to* $(H^1_{\mathrm{loc}}(\Omega))^3$ *solve* (4.11)–(4.13) *for given* $\mathbf{g} \in T H^{1/2}(\Gamma)$ *then* $\mathbf{E}, \mathbf{H}$ *are given by* (4.14) *with*

$$\mathbf{J} = [\mathbf{n}\times \operatorname{curl}\mathbf{E}]_\Gamma, \quad M = [\mathbf{n}\cdot\mathbf{E}]_\Gamma$$

satisfying (4.15). *Here,* $[\cdot]_\Gamma$ *denotes the jump across* Γ. *Conversely let* $\mathbf{J} \in T\tilde{H}^{-1/2}(\Gamma)$, $M \in \tilde{H}^{1/2}(\Gamma)$ *solve* (4.15). *Then* $\mathbf{E}$ *and* $\mathbf{H}$ *given by* (4.14) *solve* (4.11)–(4.13) *with* $\mathbf{E} \in (H^1_{\mathrm{loc}}(\Omega))^3$ *and* $\mathbf{H} \in (H^1_{\mathrm{loc}}(\Omega))^3$.

(ii) *The operator* $\mathcal{A}_k$ *in* (4.15) *is strongly elliptic, more precisely there exist* $\gamma_1, \gamma_2 > 0$ *and a smooth complex valued matrix* $\theta(x) \in \mathbb{C}^{3\times 3}$ *such that*

$$\Re\langle \theta\mathcal{A}_k\mathbf{U}, \mathbf{U}\rangle_{\mathbf{L}^2(\Gamma)} \geq \gamma_1 \left(\|\mathbf{J}\|^2_{T\tilde{H}^{-1/2}(\Gamma)} + \|M\|^2_{\tilde{H}^{1/2}(\Gamma)} \right) - \gamma_2 \left(\|\mathbf{J}\|^2_{T\tilde{H}^{-3/2}(\Gamma)} + \|M\|^2_{\tilde{H}^{-1/2}(\Gamma)} \right) \tag{4.16}$$

for all $\mathbf{U} = (\mathbf{J}, M) \in \tilde{X} := T\tilde{H}^{-1/2}(\Gamma) \times \tilde{H}^{1/2}(\Gamma)$.

(iii) *For any* k *with* $\Re(k) > 0$ $\mathcal{A}_k$ *is bijective form* $\tilde{X}$ *onto its dual* $\tilde{X}' = T H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.

We use the Galerkin method to solve the system (4.15): Let $X_N = X^{-1/2}_{N_1} \times X^{-1/2}_{N_1} \times X^{1/2}_{N_2}$ be finite dimensional spaces such that $X^{-1/2}_{N_1} \subset \tilde{H}^{-1/2}(\Gamma)$ and $X^{1/2}_{N_2} \subset \tilde{H}^{1/2}(\Gamma)$. Find $(\mathbf{J}_{N_1}, M_{N_2}) = (J_{1,N_1}, J_{2,N_1}, M_{N_2}) \in X_N$ such that

$$\langle \mathcal{A}_k(\mathbf{J}_{N_1}, M_{N_2}), (\mathbf{v}, v)\rangle_{\mathbf{L}^2(\Gamma)} = \langle \mathbf{g}, \mathbf{v}\rangle_{L^2(\Gamma)\times L^2(\Gamma)} + \langle \operatorname{div}_{\mathrm{T}} \mathbf{n}\times\mathbf{g}, v\rangle_{L^2(\Gamma)} \tag{4.17}$$

for all $(\mathbf{v}, v) \in X_N$. Since $\mathcal{A}_k$ is bijective and strongly elliptic the linear system (4.17) is uniquely solvable for sufficiently large dimensions N_1 and N_2 and due to [138] the Galerkin solution $(\mathbf{J}_{N_1}, M_{N_2})$ converges quasi-optimally in $\tilde{X}$ against the solution $(\mathbf{J}, M)$ of (4.15).

Preconditioners and numerical experiments

We consider the electric screen problem (4.11)–(4.13) for the screen $\Gamma = (0,1)^2 \times \{0\} \subset \mathbb{R}^3$ which is the same geometry as in the last section, cf. Figure 4.1. As given data we simply take $\mathbf{g} = (1,1,0)^T$ which corresponds to a given incident electric field $\mathbf{E}^0 = (1,-1,0)$ on Γ. However, the right hand side does not influence the stiffness matrix. It is only used in the experiments to check the correct implementation of the boundary element method. We use uniform rectangular meshes $\bar{\Gamma}_h = \cup_{j=1}^J \bar{\Gamma}_j$, cf. Figure 4.1. The ansatz spaces of the boundary element method consist of three components, two of them being subspaces of $\tilde{H}^{-1/2}(\Gamma)$ and one of them being a subspace of $\tilde{H}^{1/2}(\Gamma)$. The former ones will be chosen identically and are denoted by $X_{N_1}^{-1/2}$ and the latter subspace is denoted by $X_{N_2}^{1/2}$. For $X_{N_2}^{1/2}$ we take the space $S_p^1(\Gamma_h)$ of continuous piecewise polynomials defined by (4.9) in the previous section, i.e.,

$$X_{N_2}^{1/2} := S_p^1(\Gamma_h) := \{f \in C^0(\Gamma);\ f|_{\Gamma_j} \in P_p(\Gamma_j), j = 1, \ldots, J, f|_{\partial\Gamma} = 0\} \subset \tilde{H}^{1/2}(\Gamma)$$

and for $X_{N_1}^{-1/2}$ we take the space $S_q^0(\Gamma_h)$ of piecewise polynomials of degree q on the mesh Γ_h, cf. (4.10) for the weakly singular operator,

$$X_{N_1}^{-1/2} := S_q^0(\Gamma_h) := \{f :\ \Gamma \to \mathbb{R};\ f|_{\Gamma_j} \in P_q(\Gamma_j), j = 1, \ldots, J\} \subset \tilde{H}^{-1/2}(\Gamma).$$

To have a comparable number of unknowns we choose for both spaces the same mesh and take $q = p - 1$ for $X_{N_1}^{-1/2}$. Then the space of piecewise bilinear functions $S_1^1(\Gamma_h)$ for the hypersingular part of the system corresponds to the space of piecewise constant functions for the weakly singular parts. We take the same basis functions as in the previous section, i.e. affine images of tensor products of anti-derivatives of Legendre polynomials (the so-called standard basis functions (S)) or discrete harmonic polynomials (H) for $S_p^1(\Gamma_h)$ and affine images of tensor products of Legendre polynomials for $S_{p-1}^0(\Gamma_h)$. The resulting ansatz space for the Galerkin method for the electric screen problem is

$$X_N = S_{p-1}^0(\Gamma_h) \times S_{p-1}^0(\Gamma_h) \times S_p^1(\Gamma_h) \quad (p \geq 1).$$

We end up with a linear system (4.17) whose stiffness matrix is of the form

$$A_{k,N_1,N_2} = \begin{pmatrix} 0 & \langle -V_k\psi_i, \psi_j\rangle_{ij} & \langle -\partial_{x_2} V_k\psi_i, v_j\rangle_{ij} \\ \langle V_k\psi_i, \psi_j\rangle_{ij} & 0 & \langle \partial_{x_1} V_k\psi_i, v_j\rangle_{ij} \\ 0 & 0 & \langle -(\Delta_\Gamma + k^2)V_k v_i, v_j\rangle_{ij} \end{pmatrix}$$

since the operator $\mathrm{div}(\mathbf{n}(\mathbf{n}\cdot V_k(\mathbf{J})))$ vanishes for tangential fields $\mathbf{J}$ on plane screens. The functions ψ_i, $i = 1, \ldots, N_1$, are the basis functions of $X_{N_1}^{-1/2}$ and the functions v_i, $i = 1, \ldots, N_2$, span the subspace $X_{N_2}^{1/2}$. The notation $\langle \cdot, \cdot \rangle_{ij}$ means a matrix block for all combinations of indices (i,j) and the inner product is that of $L^2(\Gamma)$.

From the above structure of the matrix we also observe that the matrix function θ in Gårding's inequality in Theorem 4.16 can be chosen to be the constant matrix

$$\theta = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We always assume this multiplication implicitly.

We will demonstrate that a combination of preconditioners for the weakly singular and the hypersingular blocks yield preconditioners which allow for efficiently solving the whole system by the GMRES method.

As in the scalar case we make use of the theoretical results for the additive Schwarz method and for the hybrid method, cf. Section 3.5.1. The system of boundary integral operators for the electric screen problem maps the space $\tilde{H}^{-1/2}(\Gamma) \times \tilde{H}^{-1/2}(\Gamma) \times \tilde{H}^{1/2}(\Gamma)$ onto its dual. With regard to the notation of Section 3.5, cf. (3.63), this means $\alpha = (-1, -1, 1)$ and the domains of definition of the integral operators are $D_1 = D_2 = D_3 = \Gamma$. Gårding's inequality (3.68) holds with $\delta = (1, 1, 1)$, see (4.16).

First we consider a combination of the non-overlapping additive Schwarz method (NASM) for the weakly singular parts and the overlapping additive Schwarz method using standard basis functions (S, OASM) for the hypersingular part. This corresponds to decomposing X_N into

$$H_0 \cup_{j=1}^{J} \left(H_j^{-1/2} \times \{0\} \times \{0\} \right) \cup_{j=1}^{J} \left(\{0\} \times H_j^{-1/2} \times \{0\} \right) \cup_{j=1}^{J_V} \left(\{0\} \times \{0\} \times H_j^{1/2} \right) \quad (4.18)$$

where

$$H_0 = S_0^0(\Gamma_h) \times S_0^0(\Gamma_h) \times S_1^1(\Gamma_h), \quad (4.19)$$

$$H_j^{-1/2} = \{ v \in S_{p-1}^0(\Gamma_j);\ \langle v, 1 \rangle_{L^2(\Gamma_j)} = 0 \} \quad (4.20)$$

and

$$H_j^{1/2} = S_p^1(\Gamma_j'). \quad (4.21)$$

Here, J denotes the number of elements of the mesh and J_V is the number of nodes of the mesh Γ_h which are not on the boundary of Γ. The components of the subspace H_0 are the subspaces of global functions of the weakly singular parts and the hypersingular part. These global components are not decoupled and play the role of the overall subspace of global functions for the whole indefinite system. The remaining parts of the decompositions of the individual components correspond to the decompositions (3.34) and (3.21) (neglecting their subspaces of global components). For the decomposition of $X_{N_1}^{-1/2}$ we simply use the elements Γ_j as the domains G_j, cf. Section 3.3. In the decomposition of $X_{N_2}^{1/2}$ the domain Γ_j' consists of the elements of the mesh Γ_h which are adjacent to the interior node with number j.

Using this decomposition of X_N we define as in (3.72) the additive Schwarz operators $\underline{\mathcal{P}}_S$ (using full local solvers) and $\underline{P}_S$ (using positive definite local solvers). There holds the following result.

Theorem 4.2 *There exist constants $c, C, h_0 > 0$ such that, if $0 < h < h_0$, there holds*

$$\Re \langle \underline{A}^\alpha \underline{\mathcal{P}}_S u, u \rangle \geq \Lambda_0 \|u\|^2_{\tilde{H}^{\alpha/2}} \quad \textit{for any } u \in X_N,$$

$$\Re \langle \underline{A}^\alpha \underline{P}_S u, u \rangle \geq \Lambda_0 \|u\|^2_{\tilde{H}^{\alpha/2}} \quad \textit{for any } u \in X_N$$

where

$$\Lambda_0 \geq c\big(1+\log(\frac{H}{h}p)\big)^{-2} - CH^{1/2-\varepsilon} = c(1+\log p)^{-2} - Ch^{1/2-\varepsilon} \quad (\varepsilon > 0),$$

and there exists $\Lambda_1 > 0$ *such that*

$$\|\underline{\mathcal{P}}_S u\|_{\tilde{H}^{\alpha/2}} \leq \Lambda_1 \|u\|_{\tilde{H}^{\alpha/2}} \quad \textit{for any } u \in X_N,$$

$$\|\underline{P}_S u\|_{\tilde{H}^{\alpha/2}} \leq \Lambda_1 \|u\|_{\tilde{H}^{\alpha/2}} \quad \textit{for any } u \in X_N.$$

Proof. The proof is a combination of Theorems 3.16, 3.6, and 3.8. In order to make use of the abstract Theorem 3.16 we need to check the Assumptions (A1)–(A4) on page 103. Assumption (A1) can be fulfilled by making h small enough since $\underline{\mathcal{A}}^{\alpha}$ is bijective and strongly elliptic by Theorem 4.1. Assumptions (A2) and (A3) are satisfied by construction of the decomposition. The parameters $\lambda_{j,0}$, $j = 1,2,3$, appearing in (A4) are obtained by Theorems 3.6 and 3.8. More precisely, $\lambda_{3,0}$, which is the minimum eigenvalue of the overlapping additive Schwarz operator for standard basis functions in the scalar, positive definite case of the hypersingular operator, is bounded from below. The parameters $\lambda_{1,0}$ and $\lambda_{2,0}$, which are the minimum eigenvalue of the non-overlapping additive Schwarz operator in the scalar, positive definite case of the weakly singular operator, are bounded from below by

$$\lambda_{i,0} \geq c\big(1+\log(\frac{H}{h}p)\big)^{-2} = c(1+\log p)^{-2} \quad (i = 1,2)$$

for a constant $c > 0$ which is independent of h and p. Here we used that $H = h$ for our example. The parameters $\delta_{j,1}$, $\delta_{j,2}$, and $\delta_{j,3}$ ($j = 1,2,3$) appearing in Theorem 3.16 are as follows. $\delta_{j,1}$ ($j = 1,2,3$) are the orders of convergence of the h-version of the boundary elements method for $p = 1$ in the individual components. Here we obtain $\delta_{j,1} = 1/2 - \varepsilon$ ($j = 1,2,3$) for any $\varepsilon > 0$, cf. [128]. By (3.78), $\delta_{1,2} = \delta_{2,2} = \delta_j$ since $\alpha_1 = \alpha_2 = -1$ and $\delta = (\delta_1, \delta_2, \delta_3)$ is the parameter in the Gårding inequality (3.68). As already mentioned above, $\delta = (1,1,1)$ by (4.16). Further, by (3.78), $\delta_{3,2} = 1/2$. Therefore, $\delta_{j,3} = \min\{\delta_{j,1}, \delta_{j,2}\} = 1/2 - \varepsilon$ for any $\varepsilon > 0$. Therefore,

$$\big(\max_j \lambda_{j,0}^{-1} + \gamma_2^2\big)^{-1} - \max_j h_j^{\delta_{j,3}} = \big(\max_j \lambda_{j,0}^{-1} + \gamma_2^2\big)^{-1} - \max_j h_j^{\delta_{j,1}} \geq c(1+\log p)^{-2} - h^{1/2-\varepsilon}$$

(γ_2 is the parameter in Gårding's inequality (4.16) and is a constant) which proves the assertions of the theorem by Theorem 3.16. □

As in the scalar case we also consider decompositions of the subspace $X_{N_2}^{1/2}$ when using discrete harmonic basis function. The decomposition of the components $X_{N_1}^{-1/2}$ will remain the same as before. If the resulting decompositions of the ansatz space X_N fulfill Assumptions (A1)–(A4) then the theory of the additive Schwarz method for indefinite systems (as used above for $\underline{\mathcal{P}}_S$ and $\underline{P}_S$) is applicable. These assumptions are not satisfied in all cases. However, the combination of the decompositions NASM (non-overlapping) for $X_{N_2}^{1/2}$ and NASM (non-overlapping) for $X_{N_1}^{-1/2}$ does fulfill the assumptions and the combination of DIAG (full decomposition) for $X_{N_2}^{1/2}$ and NASM (non-overlapping decomposition) for $X_{N_1}^{-1/2}$

can be proved to be efficient without assuming (A3). For the combination of WIRE (wire basket decomposition) for $X_{N_2}^{1/2}$ and NASM for $X_{N_1}^{-1/2}$ we have to refer to the hybrid method for theoretical results. We note that the more general applicability of the hybrid method is paid off by a less sharp theoretical estimate of the essential parameter Λ_0 (minimum eigenvalue of the Hermitian part of the preconditioned stiffness matrix). Indeed, for the hybrid method we have $\Lambda_0 = O(\lambda_0^2)$ and for the additive Schwarz method we have $\Lambda_0 = O(\lambda_0)$ where λ_0 is the minimum eigenvalue of the additive Schwarz operator for a corresponding decomposition in the positive definite case (cf. Theorems 3.9, 3.10, and 3.11 and Remark 3.3).

First, let us consider the standard additive Schwarz method using the non-overlapping decomposition for $X_{N_1}^{-1/2}$ and the full decomposition for $X_{N_2}^{1/2}$. The non-overlapping decomposition for $X_{N_2}^{1/2}$ can be treated analogously and the corresponding preconditioner is not investigated in detail. We take the decomposition

$$H_0 \cup_{j=1}^{J} \Big(H_j^{-1/2} \times \{0\} \times \{0\} \Big) \cup_{j=1}^{J} \Big(\{0\} \times H_j^{-1/2} \times \{0\} \Big) \cup_{j=1}^{N_2} \Big(\{0\} \times \{0\} \times H_j^{1/2} \Big) \quad (4.22)$$

for X_N, where H_0 and $H_j^{-1/2}$ are as in (4.18), and

$$H_j^{1/2} = H_{V_j}, \quad j = 1, \dots, J_V \quad \text{(subspaces of nodal functions)},$$

$$H_j^{1/2} = H_{E_j,k}, \quad j = J_V + 1, \dots, J_V + (p-1)J_E \quad \text{(subspaces for individual edge functions)},$$

and

$$H_j^{1/2} = H_{\Gamma_j,k,l}, \quad j = J_V + (p-1)J_E + 1, N_2 \quad \text{(subspaces for individual interior functions)}.$$

Note that N_2 is the dimension of the component $X_{N_2}^{1/2}$ of the ansatz space X_N. For details we refer to (3.24). Using this decomposition of X_N we define as in (3.72) the additive Schwarz operators $\underline{\mathcal{P}}_D$ (using full local solvers) and $\underline{P}_D$ (using positive definite local solvers). There holds the following result.

Theorem 4.3 *There exist constants $c, C, h_0 > 0$ such that, if $0 < h < h_0$, there holds*

$$\Re\langle \underline{A}^{\alpha} \underline{\mathcal{P}}_D u, u\rangle \geq \Lambda_0 \|u\|^2_{\tilde{H}^{\alpha/2}} \quad \textit{for any } u \in X_N,$$

$$\Re\langle \underline{A}^{\alpha} \underline{P}_D u, u\rangle \geq \Lambda_0 \|u\|^2_{\tilde{H}^{\alpha/2}} \quad \textit{for any } u \in X_N$$

where

$$\Lambda_0 \geq c\big(1 + \log(\frac{H}{h}p)\big)^{-2} - CH^{1/2-\varepsilon} = c(1+\log p)^{-2} - Ch^{1/2-\varepsilon} \quad (\varepsilon > 0),$$

and there exists $\Lambda_1 > 0$ such that

$$\|\underline{\mathcal{P}}_D u\|_{\tilde{H}^{\alpha/2}} \leq \Lambda_1 \|u\|_{\tilde{H}^{\alpha/2}} \quad \textit{for any } u \in X_N,$$

$$\|\underline{P}_D u\|_{\tilde{H}^{\alpha/2}} \leq \Lambda_1 \|u\|_{\tilde{H}^{\alpha/2}} \quad \textit{for any } u \in X_N.$$

Analogous estimates hold for the additive Schwarz operators $\underline{\mathcal{P}}_{ASM}$ and $\underline{P}_{ASM}$ which are obtained by taking the non-overlapping decomposition (3.23) *in* (4.22) *instead of the full decomposition* (3.24).

Proof. As for the proof of Theorem 4.2 we combine Theorem 3.16 (which provides abstract results for indefinite systems) with the respective results for the scalar operators. These are Theorem 3.7 for the full decomposition of $X_{N_2}^{1/2}$ and Theorem 3.8 for the non-overlapping decomposition of $X_{N_1}^{-1/2}$. In order to make use of the abstract Theorem 3.16 we need to check Assumptions (A1)–(A4) on page 103. With the exception of (A3) they are verified as in the proof of Theorem 4.2. Assumption (A3) enters in (3.79) and for the full decomposition it can be avoided by using the boundedness of the maximum eigenvalue of the additive Schwarz operator which belongs to the full decomposition as proved in Theorem 3.7. Now the assertions of the theorem follow from Theorem 3.16 by noting that the minimum eigenvalue of the latter additive Schwarz preconditioner is bounded from below by $c(1+\log p)^{-2}$ which is asymptotically the same bound as for the minimum eigenvalue of the additive Schwarz operator which is used for the weakly singular parts. □

Next we consider the hybrid method using the non-overlapping decomposition for $X_{N_1}^{-1/2}$ and the wire basket decomposition for $X_{N_2}^{1/2}$. We define the preconditioner $\underline{\mathcal{B}}^{-1}$ by combining the coarse grid solver with respect to $\mathcal{A}_k$ on the subspace $H_0 \subset X_N$ (as defined in (4.19)) and an additive Schwarz preconditioner for the positive definite part $\underline{A}^\alpha$ of $\mathcal{A}_k$. This additive Schwarz preconditioner belongs to the decomposition of X_N into

$$H_0 \cup_{j=1}^J \Big(H_j^{-1/2} \times \{0\} \times \{0\}\Big) \cup_{j=1}^J \Big(\{0\} \times H_j^{-1/2} \times \{0\}\Big) \cup_{j=1}^{J+1} \Big(\{0\} \times \{0\} \times H_j^{1/2}\Big). \quad (4.23)$$

Here, H_0 is as above (4.19), the subspaces $H_j^{-1/2}$ are the same as in (4.20), and the subspaces $H_j^{1/2}$ are those in the wire basket decomposition (3.22) by neglecting the subspace of piecewise bilinear functions on Γ_h. We obtain the following result.

Theorem 4.4 *There exist positive numbers Λ_0, Λ_1 and β such that, if the mesh size h is small enough, then*

$$\Re\langle \underline{A}^\alpha \underline{P}_W u, u\rangle \geq \Lambda_0 \langle \underline{A}^\alpha u, u\rangle \quad \textit{for any } u \in X_N$$

and

$$\|\underline{P}_W u\|_{\tilde{H}^{\alpha/2}} \leq \Lambda_1 \|u\|_{\tilde{H}^{\alpha/2}} \quad \textit{for any } u \in X_N.$$

More precisely, when choosing

$$\beta = O\big(\log^{-2}(\frac{H}{h}p)\big) = O(\log^{-2} p)$$

and h small enough such that

$$\|u - u_h\|_{\tilde{H}^{\alpha/2-1}} \leq \frac{\lambda_0}{2\lambda_1 c_0 (c_0^2+1)^{1/2}} \|u\|_{\tilde{H}^{\alpha/2}} \quad \textit{for any } u \in X_N,$$

where $\lambda_0 = O\big(\log^{-2}(\frac{H}{h}p)\big) = O(\log^{-2} p)$, the above estimates hold with

$$\Lambda_0 \geq O\big(\log^{-4}(\frac{H}{h}p)\big) = O(\log^{-4} p)$$

and a constant $\Lambda_1 > 0$. Here, u_h is the Galerkin projection of u in H_0, λ_0 and λ_1 are the minimum and maximum eigenvalues of the symmetric positive definite part $\underline{A}^\alpha$ being preconditioned by the additive Schwarz preconditioner belonging to the decomposition of X_N, and c_0 is the constant appearing in (3.69)*.*

Proof. The abstract version of this theorem is Theorem 3.17 and the assertions follow by estimating the extreme eigenvalues λ_0 and λ_1 of the additive Schwarz operator belonging to the positive definite part of the indefinite system and decomposition (4.23). This estimate is obtained by using Lemma 3.10: There holds

$$\lambda_0 \geq c \min\{\lambda_{1,0}, \lambda_{2,0}, \lambda_{3,0}\} \quad \text{and} \quad \lambda_1 \leq C \max\{\lambda_{1,1}, \lambda_{2,1}, \lambda_{3,1}\}$$

where $\lambda_{j,0}$ and $\lambda_{j,1}$ are the minimum and maximum eigenvalues of the additive Schwarz operator which belongs to the decomposition of the jth component of X_N within the Sobolev space $\tilde{H}^{\alpha_j/2}(D_j) = \tilde{H}^{\alpha_j/2}(\Gamma)$, $j = 1, 2, 3$. The eigenvalues

$$\lambda_{1,0} = \lambda_{2,0} \geq c\big(1 + \log(\frac{H}{h}p)\big)^{-2} = c(1 + \log p)^{-2}$$

and $\lambda_{1,1} = \lambda_{2,1} \leq C$ are obtained from the proof of Theorem 3.8 and $\lambda_{3,0} \geq c(1 + \log p)^{-2}$, $\lambda_{3,1} \leq C$ by Theorem 3.7. Here, the constants $c, C > 0$ do not depend on h and p. Therefore,

$$\lambda_0 \geq c(1 + \log p)^{-2} \quad \text{and} \quad \lambda_1 \leq C$$

and the proof is finished by Theorem 3.17. □

Let us take a look at the numerical results for the different preconditioners. Figure 4.7 shows the ratios of the essential parameters Λ_0 and Λ_1 for moderate values of p in the case $k = 5$. Theoretically, we expect from Theorems 4.2 and 4.3 an asymptotic increase of $\kappa = \Lambda_1/\Lambda_0$ like $O(\log^2 p)$ for all but the (NASM)/(H,WIRE) preconditioner. By Theorem 4.4 this preconditioner is only be proved to produce ratios κ behaving like $O(\log^4 p)$. Of course, this result is not optimal and we consider this method just for completeness. The numerical results for all the preconditioners are approximately 100 to 1000 times better than without preconditioner. The curves increase at most very slowly with p but an asymptotic behavior can hardly be observed within this range of p. This is expectable for systems discretizing more practical problems which comprise different operators of different orders.

The numbers of iterations of the GMRES method which are necessary to solve the systems up to an residual reduction of 10^{-6} are given in Table 4.10. In all cases the iteration numbers are nearly constant whereas without preconditioner the numbers increase fast. It also becomes clear that, in this range of number of unknowns, there is nearly no difference between choosing full local solvers (I) or positive definite local solvers (P). Table 4.11, which shows the numbers of iterations in the case $k = 10$, underlines that the efficiency of the preconditioners is not greatly reduced by larger, moderate, wave numbers. The dependence of the iteration numbers on k becomes a bit clearer by looking at Table 4.12 where the cases $k = 0, 5, 10$ are listed for solving the original system and using the (NASM)/(S,OASM) preconditioner. However, we have no theoretical results concerning this dependence and we only demonstrate that our methods work efficiently for indefinite systems. High frequency problems are not considered in this work.

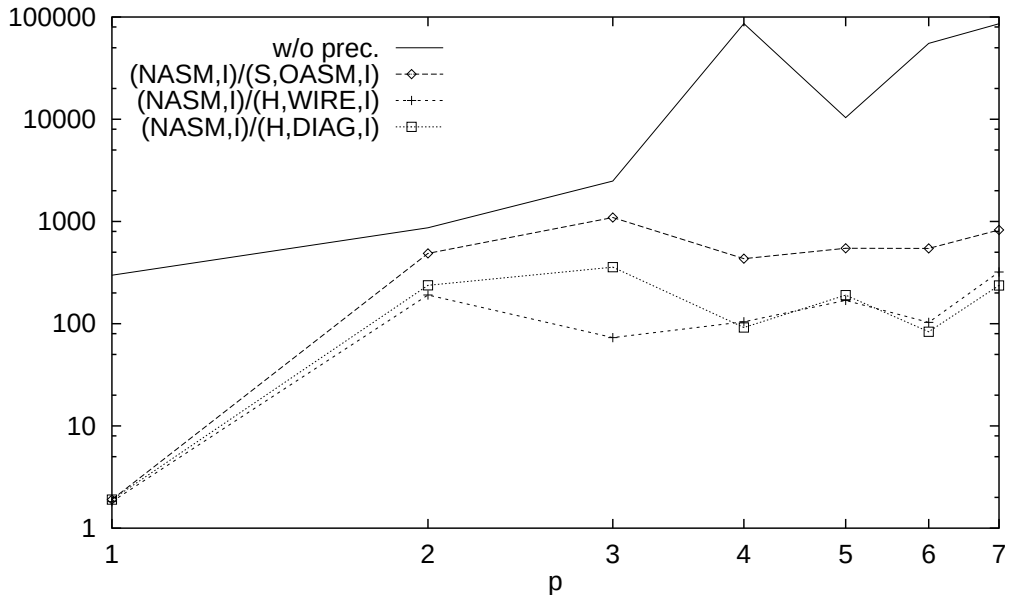


Figure 4.7: Electric screen problem ($k = 5$, p-version, $h^{-1} = 3$): the ratios $\kappa = \Lambda_1/\Lambda_0$ obtained without and with different preconditioners.

				(NASM)/(S,OASM)		(NASM)/(H,WIRE)		(NASM)/(H,DIAG)	
N	h^{-1}	p	–	(I)	(P)	(I)	(P)	(I)	(P)
22	3	1	4	2	2	2	2	2	2
97	3	2	9	7	7	7	7	7	7
226	3	3	28	13	12	13	12	13	12
409	3	4	40	13	13	13	13	13	13
646	3	5	52	15	14	15	14	15	14
937	3	6	62	15	14	15	14	15	14
1282	3	7	71	15	15	15	15	15	15
97	2	3	8	7	7	7	7	7	7
409	4	3	47	13	13	13	13	13	13
937	6	3	105	14	14	14	14	14	14
1681	8	3	159	14	14	14	14	14	14

Table 4.10: Electric screen problem ($k = 5$): numbers of GMRES iterations (without prec. ("–") and with several preconditioners using full local solvers or positive definite local solvers (I, P)).

				(NASM)/(S,OASM)		(NASM)/(H,DIAG)	
N	h^{-1}	p	–	(I)	(P)	(I)	(P)
22	3	1	4	2	2	2	2
97	3	2	8	7	7	7	7
226	3	3	34	13	13	13	13
409	3	4	43	14	13	14	13
646	3	5	55	15	15	15	15
937	3	6	65	15	16	15	16
1282	3	7	79	17	17	17	17
97	2	3	9	7	7	7	7
409	4	3	53	14	14	14	14
937	6	3	114	16	16	16	16
1681	8	3	169	16	16	16	16

Table 4.11: Electric screen problem ($k = 10$): numbers of GMRES iterations (without prec. ("–") and with several preconditioners using full local solvers or positive definite local solvers (I, P)).

			w/o prec.			with prec.		
N	h^{-1}	p	$k = 0$	5	10	0	5	10
22	3	1	4	4	4	2	2	2
97	3	2	9	9	8	7	7	7
226	3	3	26	28	34	11	13	13
409	3	4	34	40	43	13	13	14
646	3	5	46	52	55	13	15	15
937	3	6	53	62	65	14	15	15
1282	3	7	59	71	79	14	15	17
97	2	3	9	8	9	7	7	7
409	4	3	42	47	53	12	13	14
937	6	3	93	105	114	13	14	16
1681	8	3	137	159	169	13	14	16

Table 4.12: Electric screen problem: numbers of GMRES iterations (without preconditioner and with (NASM)/(S,OASM) preconditioner (full local solvers) for different wave numbers).

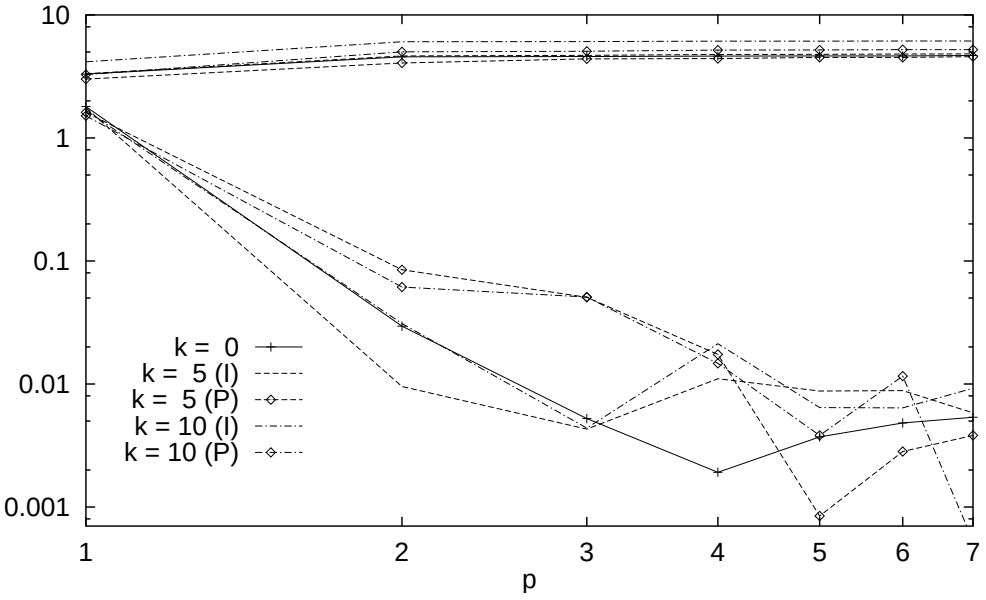


Figure 4.8: Electric screen problem (p-version, $h^{-1} = 3$): values of Λ_0 and Λ_1 obtained by combining (NASM) and the overlapping additive Schwarz preconditioner (OASM,S) with full local solvers (I) and positive definite local solvers (P).

To get an insight in the behavior of the values of Λ_0 and Λ_1 we take a look at Figures 4.8 and 4.9 where the results for the (NASM)/(S,OASM) and (NASM)/(H,DIAG) preconditioners are given (using different wave numbers and full or positive definite local solvers). In either case, the norms Λ_1 of the matrices are bounded and are nearly independent of small differences in k, of the type of the local solvers and, more importantly, of the polynomial degree of the ansatz functions. The minimum eigenvalues Λ_0 of the Hermitian parts of the matrices are a bit less stable. They are slightly decreasing with p and this is expected due to the theoretical results of Theorems 4.2 and 4.3.

In practise, more important than iteration numbers are the actual CPU-times that are used by the algorithms. Besides some initializations, the basic steps of the implementation are the assembly of the stiffness matrix (mat) and the right hand side (rhs), the calculation of the preconditioner (pre, if used), and the iterative solution of the linear system by the GMRES method (sol). In Figure 4.10 the CPU-times versus the polynomial degree are plotted for all these steps without preconditioner and with the (NASM,I)/(H,DIAG,I) preconditioner. Obviously, the solution of the system without preconditioner is the most expensive step. Even though the step of assembling the linear system is not a cheap step when considering the boundary element method, this task does not play a great role for our example. This, of course, depends on the type of implementation and on the accuracy the system has to be solved to. But, although we used analytical recurrence formulae to calculate the positive definite parts of the integral operators, our implementation of the stiffness entries is not very efficient since we took Taylor series expansions of the kernel functions in order to calculate the integral operators for $k \neq 0$. Relying on the fact that the solution of the linear system is an expensive step it becomes obvious that preconditioners play an important role.

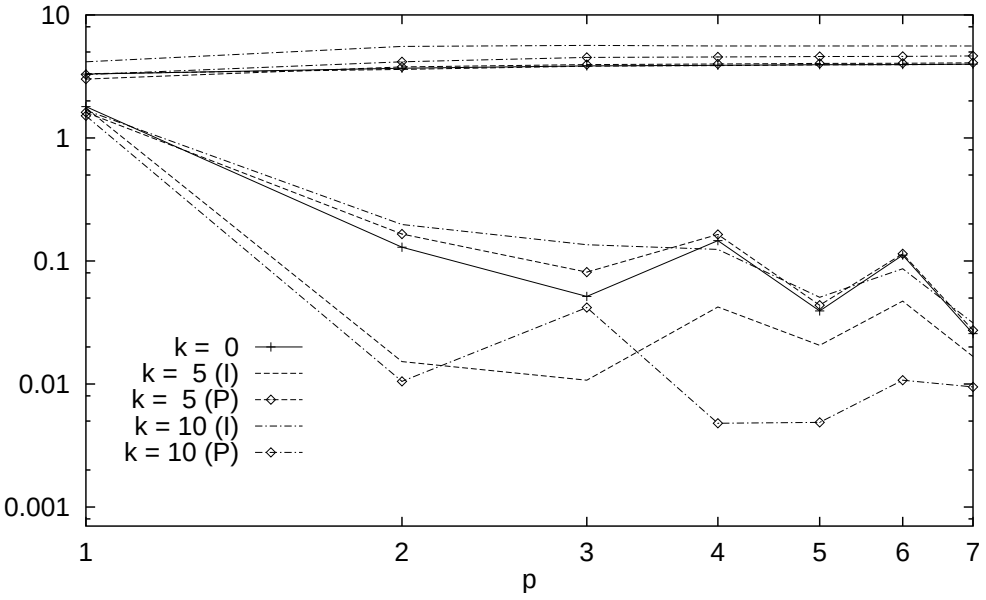


Figure 4.9: Electric screen problem (p-version, $h^{-1} = 3$): values of Λ_0 and Λ_1 obtained by combining (NASM) and the modified diagonal preconditioner (DIAG,H) with full local solvers (I) and positive definite local solvers (P).

Figure 4.10 demonstrates that the iterative solution is indeed very fast when one takes the (NASM,I)/(H,DIAG,I) preconditioner. We did not plot the results for the other methods because they all give almost the same results. Up to now we considered the dependence of the CPU-times on p. Figure 4.11 shows the corresponding CPU-times when reducing the mesh size and taking the fixed degree $p = 3$. The curves behave similarly but the difference in time between the assembly step and the remaining steps is even larger. Therefore, we took a logarithmic scale for this figure.

Remark 4.1 *Our experiments have been performed on an Intel system with PentiumPro processor* (200 Mhz) *and* 512 MB *memory under Solaris* 2.6 *using the Fortran90 compiler from Edinburgh Portable Compilers.*

The CPU-time is not the only important parameter in practise. The question arises how accurate a given problem can be solved in a fixed time. This depends not only on the assembly of the linear system and on the iterative procedure for its solution but also on the ansatz space that is in use. A question is whether it is more efficient to reduce the mesh size or to increase the polynomial degree in order to increase the accuracy when the given time is fixed. Since we do not know the exact solution for our model problem we compute an approximation to the energy norm of the true solution by extrapolating the results for different mesh sizes h. When dealing with a selfadjoint, positive definite operator A^α one obtains by the orthogonality of the Galerkin solution

$$\|\mathbf{U} - u\|^2_{\tilde{H}^{\alpha/2}} = \langle A^\alpha(\mathbf{U} - u), \mathbf{U} - u\rangle_{L^2(\Gamma)} = \|\mathbf{U}\|^2_{\tilde{H}^{\alpha/2}} - \|u\|^2_{\tilde{H}^{\alpha/2}}. \tag{4.24}$$

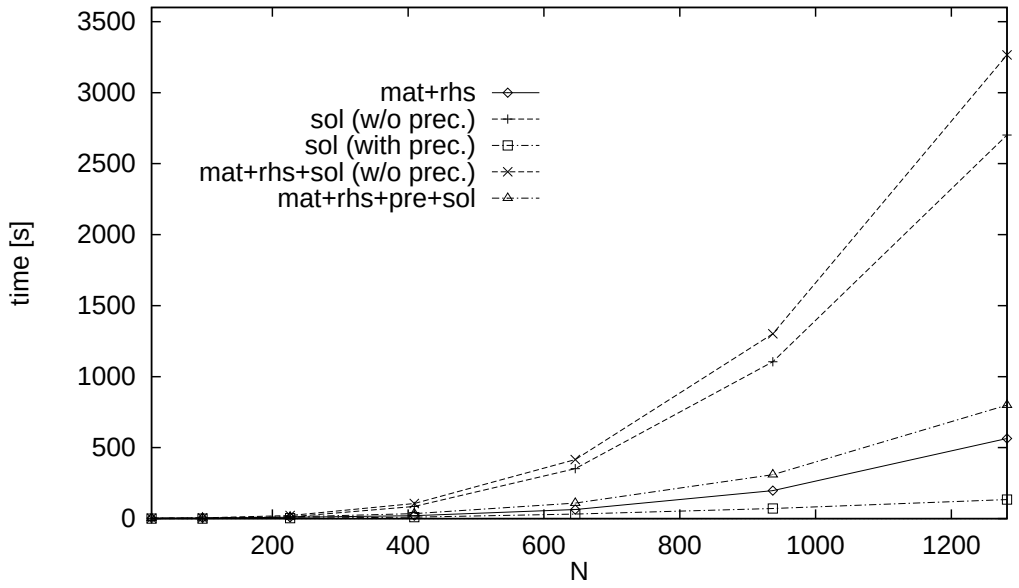


Figure 4.10: Electric screen problem ($k = 5$, p-version, $h^{-1} = 3$): CPU-times for assembling the stiffness matrix (mat), the right hand side vector (rhs), for the iterative solution (sol) and for calculating the preconditioner (pre). The (NASM,I)/(H,DIAG,I)-preconditioner is used.

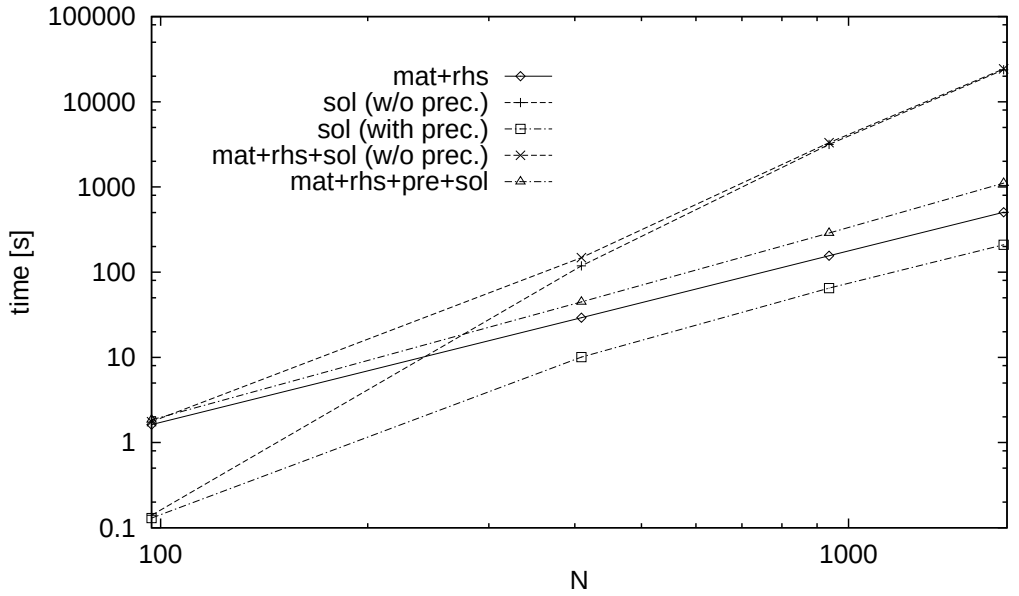


Figure 4.11: Electric screen problem ($k = 5$, h-version, $p = 3$): CPU-times for assembling the stiffness matrix (mat), the right hand side vector (rhs), for the iterative solution (sol) and for calculating the preconditioner (pre). The (NASM,I)/(H,DIAG,I)-preconditioner is used.

version	preconditioner	CPU-time	N	#iter
h, $p = 3$	–	2h 26min	1200	125
h, $p = 3$	(NASM)/(H,DIAG)	9min	1200	14
p, $1/h = 3$	–	5min	560	48
p, $1/h = 3$	(NASM)/(H,DIAG)	1min	560	15

Table 4.13: CPU-times for solving the electric screen problem ($k = 5$) up to an accuracy of 0.1 relative error in energy norm.

Here, $\mathbf{U}$ is the exact solution of the equation $A^\alpha \mathbf{U} = f$ (in the moment f is an arbitrary right hand side) and u is the Galerkin projection of $\mathbf{U}$ onto the ansatz space in use. This equality is disturbed when considering indefinite or non-selfadjoint operators. However, for our numerical experiments we simply assume that the term on the right side is a good approximation of the Galerkin error on the left side. We already know that the pure h-version converges like $O(h^{1/2-\varepsilon})$ for $\varepsilon > 0$. By the above assumption we then obtain

$$\|\mathbf{U}\|^2_{\tilde{H}^{\alpha/2}} \approx \|u\|^2_{\tilde{H}^{\alpha/2}} + ch + o(h).$$

Here, $\mathbf{U}$ is the exact solution of the system of integral equations (4.15) and u is the Galerkin solution for a fixed degree p and given mesh size h. By calculating the Galerkin solutions for several mesh sizes we obtain a good approximation for $\|\mathbf{U}\|_{\tilde{H}^{\alpha/2}}$ by extrapolation. This approximation is then used to compute approximations to the Galerkin error in the energy norm via (4.24).

Figure 4.12 plots the relative error in energy norm of the Galerkin solution versus the CPU-time that is used for the whole calculation (assembly, calculation of the preconditioner, solution of the system). One can see that, for our model problem, it is more efficient to increase the polynomial degree (even without preconditioner) than refining the mesh (even with preconditioner) if the needed accuracy is not too low. For the p-version the use of a preconditioner further reduces the required CPU-time by almost one order of magnitude.

Table 4.13 lists the resources that are required to obtain a Galerkin solution which gives 10% accuracy with respect to the energy norm. The given results (CPU-time, dimension N, and numbers of GMRES iterations #iter) are obtained by linear interpolation in the logarithmic scale (where the curves are approximately straight lines). Using piecewise polynomials of degree 3 one needs 1200 numbers of unknowns (which corresponds to 46 elements (interpolation)) where the solution of the linear system needs 125 numbers of GMRES iterations without preconditioner. The whole method then lasts nearly 2.5 hours. The (NASM)/(H,DIAG,I) preconditioner reduces the iteration steps to 14 and the whole time to 9 minutes. Even the p-version without preconditioner (using 9 elements) is faster and takes only 5 minutes, although then 48 iterations are necessary. This is due to the fact that only 560 unknowns are taken which corresponds to $p = 5$ (interpolation). The (NASM)/(H,DIAG,I) preconditioner further reduces the overall time by 80%. In this case only 15 GMRES iterations are used. Of course, the differences in efficiency would be increased by taking lower degrees for the h-version, for instance $p = 1$ which is the usual situation for the basic h-version.

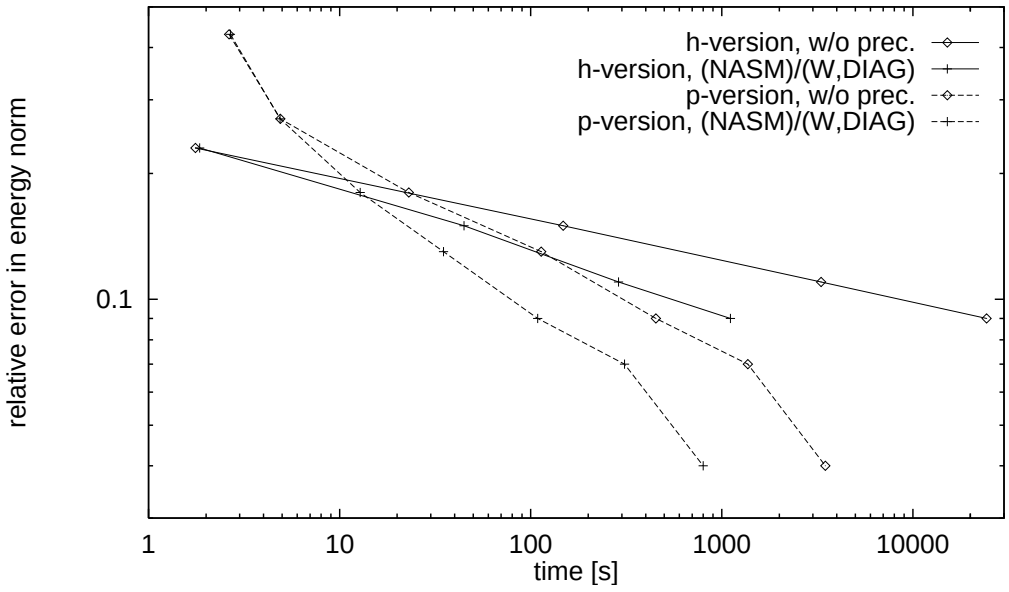


Figure 4.12: Electric screen problem ($k = 5$): relative error in energy norm versus total CPU-time (assembling stiffness matrix and right hand side, calculating the preconditioner (if present), and solving the linear system).

4.3 Magnetic screen problem

Now we consider the so-called magnetic screen problem which is the system (4.11), (4.13) with different boundary condition (we use the same notation as in Section 4.2):

For given tangential field $\mathbf{d}$ on Γ and scalar function δ on Γ find two vector fields $\mathbf{E}$ and $\mathbf{H}$ in Ω such that

$$\operatorname{curl}\mathbf{E} = \mathbf{H}, \quad \operatorname{curl}\mathbf{H} = k^2\mathbf{E} \quad \text{in } \Omega, \tag{4.25}$$

$$\mathbf{n} \times \big(\mathbf{n} \times \mathbf{H}\big) = \mathbf{d}, \quad \mathbf{n} \cdot \mathbf{H} = \delta \quad \text{on } \Gamma, \tag{4.26}$$

$$\mathbf{H} \times \frac{x}{|x|} - \mathbf{E} = \mathrm{o}\Big(\frac{1}{|x|}\Big) \quad \text{for } |x| \to \infty. \tag{4.27}$$

The wave number k is assumed to have non-negative imaginary part. As for the electric screen problem we assume also here that E, $H \in (H^1_{\mathrm{loc}}(\Omega))^3$.

Following [129] we represent the magnetic field $\mathbf{H}$ by

$$\mathbf{H} = \frac{1}{2}\Big(-\operatorname{curl} V_k\mathbf{J} + V_k(\mathbf{n}M)\Big), \quad x \in \Omega \tag{4.28}$$

for a tangential field $\mathbf{J}$ and a scalar function M, both belonging to the Cauchy data of $\mathbf{H}$ (see below). Here

$$\operatorname{curl} V_k(\mathbf{J})(x) := -2\int_\Gamma \mathbf{J}(y)\operatorname{curl}_x \phi(x,y)\, dS_y$$

and

$$V_k(\mathbf{n}M)(x) := -2\int_\Gamma (\mathbf{n}M)\phi(x,y)\,dS_y$$

with the fundamental solution

$$\phi(x,y) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}$$

of the vector Helmholtz equation $(\Delta + k^2)\mathbf{u} = 0$ in Ω. Applying the boundary conditions (4.26) to the ansatz (4.28) and substituting M by $M' = M + \operatorname{div}_{\mathrm{T}}\big(\mathbf{n}\times\mathbf{J}\big)$ we obtain the following system of integral equations:

$$\mathcal{A}_k\begin{pmatrix}\mathbf{J}\\ M'\end{pmatrix} := \begin{pmatrix} D_k(\mathbf{J}) + \mathbf{n}\times\mathbf{K}_k\big(\mathbf{n}\operatorname{div}_{\mathrm{T}}(\mathbf{n}\times\mathbf{J})\big) - \mathbf{n}\times\mathbf{K}_k(\mathbf{n}M') \\ \mathbf{n}\cdot\operatorname{curl} V_k(\mathbf{J}) - \mathbf{n}\cdot V_k(\mathbf{n}M')\end{pmatrix} = \begin{pmatrix}2\mathbf{d}\\ -2\delta\end{pmatrix} \quad \text{on } \Gamma \tag{4.29}$$

where for $x\in\Gamma$

$$D_k(\mathbf{J})(x) := \lim_{z\to x} 2\mathbf{n}\times\Big(\mathbf{n}\times\operatorname{curl}\operatorname{curl}_z\int_\Gamma \phi(z,y)\mathbf{J}(y)\,dS_y\Big)$$

and

$$\mathbf{K}_k(\mathbf{n}M')(x) := 2\int_\Gamma \mathbf{n}(x)\times\operatorname{curl}_x\phi(x,y)\mathbf{n}(y)M'(y)\,dS_y = -\mathbf{n}\times\operatorname{curl} V_k(\mathbf{n}M')(x).$$

From [129, Theorems 3.1, 3.2] we cite the following results.

Theorem 4.5 (i) $\mathbf{H}\in\big(H^1_{\mathrm{loc}}(\Omega)\big)^3$ *is the solution of the magnetic screen problem* (4.25)–(4.27) *if and only if* $\mathbf{J} = [\mathbf{n}\times\mathbf{H}]_\Gamma \in T\tilde{H}^{1/2}(\Gamma)$ *and* $M' = [\operatorname{div}\mathbf{H}]_\Gamma + \operatorname{div}_{\mathrm{T}}\big(\mathbf{n}\times[\mathbf{n}\times\mathbf{H}]_\Gamma\big)\in\tilde{H}^{-1/2}(\Gamma)$ *are the solutions of the system of integral equations* (4.29)*, and* $\mathbf{H}$ *satisfies* (4.28) *with* $M = [\operatorname{div}\mathbf{H}]_\Gamma = M' - \operatorname{div}_{\mathrm{T}}(\mathbf{n}\times\mathbf{J})$*. Here,* $[\cdot]_\Gamma$ *denotes the jump across* Γ*.*

(ii) *The operator* $\mathcal{A}_k$ *in* (4.29) *is strongly elliptic, more precisely there exist* $\gamma_1,\gamma_2>0$ *and a complex matrix* $\theta(x)\in\mathbb{C}^{3\times3}$ *such that*

$$\Re\langle\theta\mathcal{A}_k\mathbf{U},\mathbf{U}\rangle_{\mathbf{L}^2(\Gamma)} \geq \gamma_1\left(\|\mathbf{J}\|^2_{T\tilde{H}^{1/2}(\Gamma)} + \|M'\|^2_{\tilde{H}^{-1/2}(\Gamma)}\right) - \gamma_2\left(\|\mathbf{J}\|^2_{T\tilde{H}^{-1/2}(\Gamma)} + \|M'\|^2_{\tilde{H}^{-3/2}(\Gamma)}\right) \tag{4.30}$$

for all $\mathbf{U} = (\mathbf{J},M')\in\tilde{X} := T\tilde{H}^{1/2}(\Gamma)\times\tilde{H}^{-1/2}(\Gamma)$.

(iii) *For any* k *with* $\Im(k)\geq 0$ $\mathcal{A}_k$ *is bijective form* $\tilde{X}$ *onto its dual* $\tilde{X}' = TH^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)$.

We use the Galerkin method to solve the system (4.29): Let $X_N = X^{1/2}_{N_1}\times X^{1/2}_{N_1}\times X^{-1/2}_{N_2}$ be finite dimensional spaces such that $X^{1/2}_{N_1}\subset\tilde{H}^{1/2}(\Gamma)$ and $X^{-1/2}_{N_2}\subset\tilde{H}^{-1/2}(\Gamma)$. Find $(\mathbf{J}_{N_1},M'_{N_2}) = (J_{1,N_1},J_{2,N_1},M'_{N_2})\in X_N$ such that

$$\langle\mathcal{A}_k(\mathbf{J}_{N_1},M'_{N_2}),(\mathbf{v},v)\rangle_{\mathbf{L}^2(\Gamma)} = 2\langle\mathbf{d},\mathbf{v}\rangle_{L^2(\Gamma)\times L^2(\Gamma)} - 2\langle\delta,v\rangle_{L^2(\Gamma)} \tag{4.31}$$

for all $(\mathbf{v},v)\in X_N$. Since $\mathcal{A}_k$ is bijective and strongly elliptic the linear system (4.31) is uniquely solvable for sufficiently large dimensions N_1, N_2 and due to [138] the Galerkin solution $(\mathbf{J}_{N_1},M'_{N_2})$ converges quasi-optimally in $\tilde{X}$ against the solution $(\mathbf{J},M')$ of (4.29).

Preconditioners

In this section we propose preconditioners for the linear boundary element system (4.31) discretizing the magnetic screen problem. Since our methods are given in an abstract form based on Sobolev norms and Gårding's inequality the results for the electric screen problem in Section 4.2 are valid in an analogous form in this case as well. Here we deal with the space $\tilde{H}^{1/2}(\Gamma) \times \tilde{H}^{1/2}(\Gamma) \times \tilde{H}^{-1/2}(\Gamma)$ instead of $\tilde{H}^{-1/2}(\Gamma) \times \tilde{H}^{-1/2}(\Gamma) \times \tilde{H}^{1/2}(\Gamma)$ as for the electric screen problem. All the methods can be transferred to this case by decomposing subspaces of $\tilde{H}^{1/2}(\Gamma)$ and of $\tilde{H}^{-1/2}(\Gamma)$ as before. More precisely, we take the same types of ansatz spaces as in Section 4.2,

$$X_{N_1}^{1/2} := S_p^1(\Gamma_h) := \{f \in C^0(\Gamma);\ f|_{\Gamma_j} \in P_p(\Gamma_j), j = 1, \dots, J, f|_{\partial\Gamma} = 0\} \subset \tilde{H}^{1/2}(\Gamma)$$

and

$$X_{N_2}^{-1/2} := S_q^0(\Gamma_h) := \{f :\ \Gamma \to \mathbb{R};\ f|_{\Gamma_j} \in P_q(\Gamma_j), j = 1, \dots, J\} \subset \tilde{H}^{-1/2}(\Gamma).$$

Again, we choose $q = p-1$ for $X_{N_2}^{-1/2}$ and we take the same basis functions as for the electric screen problem to span $X_{N_2}^{-1/2}$ and $X_{N_1}^{1/2}$. The final ansatz space for the Galerkin method for the magnetic screen problem is

$$X_N = S_p^1(\Gamma_h) \times S_p^1(\Gamma_h) \times S_{p-1}^0(\Gamma_h) \quad (p \geq 1).$$

The system of boundary integral operators for the magnetic screen problem maps the space $\tilde{H}^{1/2}(\Gamma) \times \tilde{H}^{1/2}(\Gamma) \times \tilde{H}^{-1/2}(\Gamma)$ onto its dual. With regard to the notation of Section 3.5, cf. (3.63), this means $\alpha = (1, 1, -1)$ and the domains of definition of the integral operators are $D_1 = D_2 = D_3 = \Gamma$. By (4.30) Gårding's inequality (3.68) holds with $\delta = (1, 1, 1)$.

Now, the (NASM)/(S,OASM) preconditioner (non-overlapping decomposition for the weakly singular part and overlapping decomposition for the hypersingular parts) corresponds to decomposing X_N into

$$H_0 \cup_{j=1}^{J_V} \left(H_j^{1/2} \times \{0\} \times \{0\}\right) \cup_{j=1}^{J_V} \left(\{0\} \times H_j^{1/2} \times \{0\}\right) \cup_{j=1}^{J} \left(\{0\} \times \{0\} \times H_j^{-1/2}\right) \quad (4.32)$$

where

$$H_0 = S_1^1(\Gamma_h) \times S_1^1(\Gamma_h) \times S_0^0(\Gamma_h), \quad (4.33)$$

and $H_j^{1/2}$ and $H_j^{-1/2}$ are defined as in (4.21) and (4.20), respectively. Using this decomposition of X_N we define as in (3.72) the additive Schwarz operators $\underline{\mathcal{P}}_S$ (using full local solvers) and $\underline{P}_S$ (using positive definite local solvers). There holds the following result.

Theorem 4.6 *There exist constants $c, C, h_0 > 0$ such that, if $0 < h < h_0$, there holds*

$$\Re\langle \underline{A}^\alpha \underline{\mathcal{P}}_S u, u\rangle \geq \Lambda_0 \|u\|^2_{\tilde{H}^{\alpha/2}} \quad \textit{for any } u \in X_N,$$

$$\Re\langle \underline{A}^\alpha \underline{P}_S u, u\rangle \geq \Lambda_0 \|u\|^2_{\tilde{H}^{\alpha/2}} \quad \textit{for any } u \in X_N$$

where

$$\Lambda_0 \geq c\big(1+\log(\frac{H}{h}p)\big)^{-2} - CH^{1/2-\varepsilon} = c(1+\log p)^{-2} - Ch^{1/2-\varepsilon} \quad (\varepsilon > 0),$$

and there exists $\Lambda_1 > 0$ *such that*

$$\|\underline{P}_S u\|_{\tilde{H}^{\alpha/2}} \leq \Lambda_1 \|u\|_{\tilde{H}^{\alpha/2}} \quad \textit{for any } u \in X_N,$$

$$\|\underline{P}_S u\|_{\tilde{H}^{\alpha/2}} \leq \Lambda_1 \|u\|_{\tilde{H}^{\alpha/2}} \quad \textit{for any } u \in X_N.$$

Proof. This proof is analogous to the proof of Theorem 4.2 and we refer to there for more details. The assertions follow by combining Theorems 3.16, 3.6, and 3.8. Assumptions (A1)–(A4) on page 103 are fulfilled as before where the strong ellipticity of the system of integral equations by Theorem 4.5 enters. The parameters $\lambda_{j,0}$, $j = 1, 2, 3$, appearing in (A4) are obtained by Theorems 3.6 and 3.8. $\lambda_{1,0}$ and $\lambda_{2,0}$ are the minimum eigenvalue of the overlapping additive Schwarz operator for standard basis functions in the scalar, positive definite case of the hypersingular operator, and are bounded. $\lambda_{3,0}$ is the minimum eigenvalue of the non-overlapping additive Schwarz operator in the scalar, positive definite case of the weakly singular operator, and is bounded from below by

$$\lambda_{3,0} \geq c\big(1+\log(\frac{H}{h}p)\big)^{-2} = c(1+\log p)^{-2}.$$

Note, that H is the mesh size of the coarse mesh defining the decomposition of $X_{N_2}^{-1/2}$ and is equal to h for the decomposition being considered. Further, $\delta_{j,3} = \min\{\delta_{j,1}, \delta_{j,2}\} = 1/2 - \varepsilon$ for any $\varepsilon > 0$. Therefore,

$$\big(\max_j \lambda_{j,0}^{-1} + \gamma_2^2\big)^{-1} - \max_j h_j^{\delta_{j,3}} = \big(\max_j \lambda_{j,0}^{-1} + \gamma_2^2\big)^{-1} - \max_j h_j^{\delta_{j,1}} \geq c(1+\log p)^{-2} - h^{1/2-\varepsilon}$$

(γ_2 is the parameter in Gårding's inequality (4.30) and is a constant) which proves the assertions of the theorem by Theorem 3.16. □

Now we consider decompositions of the subspace $X_{N_1}^{1/2}$ when using discrete harmonic basis function. The decomposition of the component $X_{N_2}^{-1/2}$ will be the same as before. The full decomposition of $X_{N_1}^{1/2}$ and the non-overlapping decomposition for $X_{N_2}^{-1/2}$ gives for X_N the decomposition into

$$H_0 \cup_{j=1}^{N_1} \Big(H_j^{1/2} \times \{0\} \times \{0\}\Big) \cup_{j=1}^{N_1} \Big(\{0\} \times H_j^{1/2} \times \{0\}\Big) \cup_{j=1}^{J} \Big(\{0\} \times \{0\} \times H_j^{-1/2}\Big). \quad (4.34)$$

Here, H_0 and $H_j^{-1/2}$ are as in (4.32), and

$$H_j^{1/2} = H_{V_j}, \quad j = 1, \ldots, J_V \quad \text{(subspaces of nodal functions)},$$

$$H_j^{1/2} = H_{E_j,k}, \quad j = J_V + 1, \ldots, J_V + (p-1)J_E \quad \text{(subspaces for individual edge functions)},$$

and

$$H_j^{1/2} = H_{\Gamma_j,k,l}, \quad j = J_V + (p-1)J_E + 1, N_2 \quad \text{(subspaces for individual interior functions)}.$$

Using this decomposition of X_N we define as in (3.72) the additive Schwarz operators $\underline{\mathcal{P}}_D$ (using full local solvers) and $\underline{P}_D$ (using positive definite local solvers). We obtain the following result.

Theorem 4.7 *There exist constants $c, C, h_0 > 0$ such that, if $0 < h < h_0$, there holds*

$$\Re\langle \underline{A}^\alpha \underline{\mathcal{P}}_D u, u\rangle \geq \Lambda_0 \|u\|^2_{\tilde{H}^{\alpha/2}} \quad \textit{for any } u \in X_N,$$

$$\Re\langle \underline{A}^\alpha \underline{P}_D u, u\rangle \geq \Lambda_0 \|u\|^2_{\tilde{H}^{\alpha/2}} \quad \textit{for any } u \in X_N$$

where

$$\Lambda_0 \geq c\big(1 + \log(\frac{H}{h}p)\big)^{-2} - CH^{1/2-\varepsilon} = c(1+\log p)^{-2} - Ch^{1/2-\varepsilon} \quad (\varepsilon > 0),$$

and there exists $\Lambda_1 > 0$ such that

$$\|\underline{\mathcal{P}}_D u\|_{\tilde{H}^{\alpha/2}} \leq \Lambda_1 \|u\|_{\tilde{H}^{\alpha/2}} \quad \textit{for any } u \in X_N,$$

$$\|\underline{P}_D u\|_{\tilde{H}^{\alpha/2}} \leq \Lambda_1 \|u\|_{\tilde{H}^{\alpha/2}} \quad \textit{for any } u \in X_N.$$

Analogous estimates hold for the additive Schwarz operators $\underline{\mathcal{P}}_{ASM}$ and $\underline{P}_{ASM}$ which are obtained by taking the non-overlapping decomposition (3.23) *in* (4.34) *instead of the full decomposition* (3.24)*.*

Proof. This theorem is the counterpart of Theorem 4.3 and the proof can be very short. We combine Theorem 3.16 (providing abstract results for indefinite systems) with the respective results for the scalar operators. These are Theorem 3.7 for the full decomposition of $X_{N_1}^{1/2}$ and Theorem 3.8 for the non-overlapping decomposition of $X_{N_2}^{-1/2}$. Assumptions (<u>A1</u>), (<u>A2</u>), and (<u>A4</u>) on page 103 are fulfilled and Assumption (<u>A3</u>) can be avoided as in the proof of Theorem 4.3. □

Finally, we consider the hybrid method using the the wire basket decomposition for $X_{N_1}^{1/2}$ and the non-overlapping decomposition for $X_{N_2}^{-1/2}$. The preconditioner $\underline{\mathcal{B}}^{-1}$ is defined by combining the coarse grid solver with respect to $\mathcal{A}_k$ on the subspace $H_0 \subset X_N$ (as defined in (4.33)) and an additive Schwarz preconditioner for the symmetric positive definite part $\underline{A}^\alpha$ of $\mathcal{A}_k$. (As for the electric screen problem we assume that the operator $\mathcal{A}_k$ has been multiplied by the matrix θ appearing in Gårding's inequality (4.30) before taking the symmetric positive definite part.) To define this additive Schwarz preconditioner we decompose X_N into

$$H_0 \cup_{j=1}^{J+1} \Big(H_j^{1/2} \times \{0\} \times \{0\}\Big) \cup_{j=1}^{J+1} \Big(\{0\} \times H_j^{1/2} \times \{0\}\Big) \cup_{j=1}^{J} \Big(\{0\} \times \{0\} \times H_j^{-1/2}\Big). \quad (4.35)$$

Here, H_0 is as above (4.33), the subspaces $H_j^{1/2}$ are those in the wire basket decomposition (3.22) by neglecting the subspace of piecewise bilinear functions and the subspaces $H_j^{-1/2}$ are the same as in (4.32). The following result holds.

Theorem 4.8 *There exist positive numbers Λ_0, Λ_1 and β such that, if the mesh size h is small enough, then*

$$\Re\langle \underline{A}^\alpha \underline{P}_W u, u\rangle \geq \Lambda_0 \langle \underline{A}^\alpha u, u\rangle \quad \text{for any } u \in X_N$$

and

$$\|\underline{P}_W u\|_{\tilde{H}^{\alpha/2}} \leq \Lambda_1 \|u\|_{\tilde{H}^{\alpha/2}} \quad \text{for any } u \in X_N.$$

More precisely, when choosing

$$\beta = O\big(\log^{-2}(\frac{H}{h}p)\big) = O(\log^{-2} p)$$

and h small enough such that

$$\|u - u_h\|_{\tilde{H}^{\alpha/2-1}} \leq \frac{\lambda_0}{2\lambda_1 c_0 (c_0^2+1)^{1/2}} \|u\|_{\tilde{H}^{\alpha/2}} \quad \text{for any } u \in X_N,$$

where $\lambda_0 = O\big(\log^{-2}(\frac{H}{h}p)\big) = O(\log^{-2} p)$, the above estimates hold with

$$\Lambda_0 \geq O\big(\log^{-4}(\frac{H}{h}p)\big) = O(\log^{-4} p)$$

and a constant $\Lambda_1 > 0$. Here, u_h is the Galerkin projection of u in H_0, λ_0 and λ_1 are the minimum and maximum eigenvalues of the positive definite part $\underline{A}^\alpha$ being preconditioned by the additive Schwarz preconditioner belonging to the decomposition of X_N, and c_0 is the constant appearing in (3.69).

Proof. As in the proof of Theorem 4.4 the assertions follow from Theorem 3.17 by estimating the extreme eigenvalues λ_0 and λ_1 of the additive Schwarz operator belonging to the positive definite part of the indefinite system and decomposition (4.35). Due to Lemma 3.10 estimates for λ_0 and λ_1 are obtained by estimating the additive Schwarz operators which belong to the individual components. For this we use Theorem 3.7 (for the first two components) and Theorem 3.8 (for the third component). □

4.4 Helmholtz transmission problem

As shown in [37, 43, 41, 72] the coupling of finite element and boundary element techniques can provide an accurate and efficient solution procedure for transmission problems in partial differential equations. It is especially useful for problems with unbounded regions. Also for the coupling method the p-version is efficient since its convergence rate is at least twice that of the h-version, see [11, 134, 72]. Whereas various preconditioners for the h-version have been considered in [137, 65, 50] we show that our methods are efficient when using the p-version.

Here, we consider transmission problems for linear differential operators in both the interior and the exterior domains. Even though this type of problems can be solved by the pure boundary element method (see, e.g., [40, 42]) we focus on the coupling of the finite element and the boundary element method. Important applications of coupled procedures are problems which possess nonlinear behavior in a bounded region which can be efficiently discretized

by finite elements, see, e.g., [51, 43, 131, 30, 67, 52]. In the approximation procedure often an outer Newton iteration is used to obtain a series of linearized interface problems which can be efficiently solved by our methods.

Let Ω_1 denote a bounded simply connected domain in $\mathbb{R}^n$, $n = 2, 3$, and $\Omega_2 = \mathbb{R}^n \setminus \bar{\Omega}_1$, $\Gamma = \partial\Omega_1 = \partial\Omega_2$. The interface Γ is assumed to be sufficiently smooth. $\partial/\partial n$ denotes the normal derivative on Γ the normal pointing from Ω_1 to Ω_2. The Helmholtz transmission problem is as follows.

For given functions $f \in H^{-1}(\Omega_1)$, $v_0 \in H^{1/2}(\Gamma)$, $\psi_0 \in H^{-1/2}(\Gamma)$ and given numbers k_1, k_2, $\mu \in \mathbb{C}$ find U_1 and U_2 such that

$$\begin{aligned} -(\Delta + k_1^2)U_1 &= f \text{ in } \Omega_1 \\ -(\Delta + k_2^2)U_2 &= 0 \text{ in } \Omega_2 \end{aligned} \tag{4.36}$$

$$\left.\begin{aligned} U_1 &= U_2 + v_0 \\ \mu\frac{\partial U_1}{\partial n} &= \frac{\partial U_2}{\partial n} + \psi_0 \end{aligned}\right\} \text{ on } \Gamma \tag{4.37}$$

where U_2 has to satisfy a decay condition for $|x| \to \infty$:

$$\left.\begin{aligned} U_2(x) &= O\big(|x|^{-(n-1)/2}\big) \\ \frac{\partial U_2(x)}{\partial |x|} - ik_2U_2(x) &= o\big(|x|^{-(n-1)/2}\big) \end{aligned}\right\} \quad \text{if} \quad k_2 \neq 0,$$

$$\left.\begin{aligned} U_2(x) &= O(\log|x|) + o(1) \quad (n=2) \\ U_2(x) &= O\big(|x|^{-1}\big) \quad (n=3) \end{aligned}\right\} \quad \text{if} \quad k_2 = 0. \tag{4.38}$$

We use the following weak formulation (see [36, 37, 41]): *Find* $U \in H^1(\Omega)$ *and* $\Phi \in H^{-1/2}(\Gamma)$ *such that for all* $v \in H^1(\Omega)$ *and* $\psi \in H^{-1/2}(\Gamma)$

$$\begin{aligned} 2a(U,v) - \langle \Phi, v\rangle + \langle K'_{k_2}\Phi, v\rangle + \frac{1}{\mu}\langle D_{k_2}U, v\rangle = \\ \frac{1}{\mu}\langle \psi_0, v\rangle + \frac{1}{\mu}\langle K'_{k_2}\psi_0, v\rangle + \frac{1}{\mu}\langle D_{k_2}v_0, v\rangle + 2\langle f, v\rangle_{L^2(\Omega)}, \\ \langle U, \psi\rangle + \mu\langle V_{k_2}\Phi, \psi\rangle - \langle K_{k_2}U, \psi\rangle = \langle v_0, \psi\rangle + \langle V_{k_2}\psi_0, \psi\rangle - \langle K_{k_2}v_0, \psi\rangle. \end{aligned} \tag{4.39}$$

Here $\langle \cdot, \cdot \rangle$ means the $L^2(\Gamma)$-inner product and $a(\cdot, \cdot)$ is the bilinear form

$$a(u,v) = \int_\Omega \nabla u \nabla v + k_1^2 uv \, dx \tag{4.40}$$

for $u, v \in H^1(\Omega)$. The operators V_k, K_k, K'_k and D_k are the single layer potential, the double layer potential, the adjoint of the double layer potential and the normal derivative of the double layer potential, respectively,

$$\begin{aligned} V_k\phi(x) &= -2\int_\Gamma \phi(y)\gamma_k(x,y)\, ds_y \quad (x \in \Gamma), \\ K_k\phi(x) &= -2\int_\Gamma \phi(y)\frac{\partial}{\partial n_y}\gamma_k(x,y)\, ds_y \quad (x \in \Gamma), \\ K'_k\phi(x) &= -2\int_\Gamma \phi(y)\frac{\partial}{\partial n_x}\gamma_k(x,y)\, ds_y \quad (x \in \Gamma), \\ D_k\phi(x) &= -\frac{\partial}{\partial n_x}K_k\phi(x) \quad (x \in \Gamma). \end{aligned}$$

Here, γ_k is a fundamental solution of the Helmholtz operator $-(\Delta + k^2)$

$$\gamma_k(x,y) = \begin{cases} \frac{1}{2\pi}\log|x-y| & (k=0, n=2) \\ -\frac{i}{4}H_0^{(1)}(k|x-y|) & (k\neq 0, n=2) \\ -\frac{e^{ik|x-y|}}{4\pi|x-y|} & (n=3) \end{cases}$$

with the Hankel function $H_0^{(1)}$ of the first kind and of order 0.

The bilinear form defined by the left hand side of (4.39) will be denoted by $\underline{\mathcal{A}}(U,\Phi;v,\psi)$,

$$\underline{\mathcal{A}}(U,\Phi;v,\psi) := 2a(U,v) - \langle\Phi, v\rangle + \langle K'_{k_2}\Phi, v\rangle + \frac{1}{\mu}\langle D_{k_2}U, v\rangle + \langle U,\psi\rangle + \mu\langle V_{k_2}\Phi,\psi\rangle - \langle K_{k_2}U,\psi\rangle.$$

We make the following assumption:

> The transmission problem (4.36)–(4.38) and the Dirichlet problem for $-(\Delta + k_2^2)$ on $\mathbb{R}^3 \setminus \bar{\Omega}_2$ have no eigensolutions.

Under this assumption Costabel [36, 37] proved the equivalence of (4.36)–(4.38) and (4.39):

Let $U_1 \in H^1(\Omega)$ and $U_2 \in H^1_{\mathrm{loc}}(\bar{\Omega}_2)$ be a solution of the interface problem (4.36)–(4.38). If we set $U = U_1$ and $\Phi = \frac{\partial U_1}{\partial n}|_\Gamma$, then U and Φ satisfy (4.39). Conversely, let $U \in H^1(\Omega)$ and $\Phi \in H^{-1/2}(\Gamma)$ satisfy (4.39). Define $U_1 = U$ in Ω and U_2 in Ω_2 by the representation formula

$$U_2 = -\frac{1}{2}V_{k_2}\frac{\partial U_2}{\partial n} + \frac{1}{2}K_{k_2}U_2$$

with U_2 on Γ and $\frac{\partial U_2}{\partial n}$ on Γ obtained from the interface conditions (4.37) and by setting $\frac{\partial U_1}{\partial n} := \Phi$. Then U_1 and U_2 solve the interface problem (4.36)–(4.38). Further, $\underline{\mathcal{A}}(\cdot;\cdot)$ is strongly elliptic: There exist constants $\gamma_1, \gamma_2 > 0$ such that

$$\Re\underline{\mathcal{A}}(v,\psi;v,\psi) \geq \gamma_1\big(\|v\|^2_{H^1(\Omega)} + \|\psi\|^2_{H^{-1/2}(\Gamma)}\big) - \gamma_2\big(\|v\|^2_{L^2(\Omega)} + \|\psi\|^2_{H^{-3/2}(\Gamma)}\big). \tag{4.41}$$

For the Galerkin scheme we choose finite dimensional subspaces $X_M \subset H^1(\Omega)$ and $Y_N \subset H^{-1/2}(\Gamma)$ and define the Galerkin solution $(u,\phi) \in X_M \times Y_N$ by

$$\begin{aligned} 2a(u,v) - \langle\phi,v\rangle + \langle K'_{k_2}\phi, v\rangle + \frac{1}{\mu}\langle D_{k_2}u, v\rangle = \\ \frac{1}{\mu}\langle\psi_0, v\rangle + \frac{1}{\mu}\langle K'_{k_2}\psi_0, v\rangle + \frac{1}{\mu}\langle D_{k_2}v_0, v\rangle + 2\langle f, v\rangle_{L^2(\Omega)} \\ \langle u,\psi\rangle + \mu\langle V_{k_2}\phi,\psi\rangle - \langle K_{k_2}u,\psi\rangle = \langle v_0,\psi\rangle + \langle V_{k_2}\psi_0,\psi\rangle - \langle K_{k_2}v_0,\psi\rangle \end{aligned}$$

for all $v \in X_M$ and $\psi \in Y_N$. In the following we take $X_M := S^1_p(\Omega_{h_1})$ and $Y_N := S^0_q(\Gamma_{h_2})$ where Ω_{h_1} and Γ_{h_2} are uniform meshes of size h_1 and h_2 on Ω and Γ, respectively. For simplicity we again assume that the elements of the meshes are simplices. Both spaces consist of piecewise polynomials of the degree which is used in the notation. As usual, the upper index 1 indicates continuity of the ansatz functions whereas the index 0 refers to piecewise polynomials which need not to be continuous. We note that the two individual ansatz spaces

are in general not related, they can be chosen independently. However, for our numerical experiments we take $h_1 = h_2$ and $q = p - 1$. To be more specific let us introduce bases

$$\operatorname{span}\{v_1, \ldots, v_M\} = X_M \quad \text{and} \quad \operatorname{span}\{\psi_1, \ldots, \psi_N\} = Y_N.$$

The basis functions of X_M are supposed to be ordered such that

$$\operatorname{span}\{v_1, \ldots, v_{M_\Omega}\} = X_M \cap H_0^1(\Omega).$$

The basis functions that do not vanish on Γ then are $v_{M_\Omega+1}, \ldots, v_{M_\Omega+M_\Gamma}$. Let us denote the coefficients of u and ϕ again by u and ϕ, respectively. Further, by u_Ω and u_Γ we denote the coefficients belonging to the components of u which are interior (i.e., they vanish at the boundary) and not interior to Ω, respectively. We obtain a linear system of the form

$$\begin{bmatrix} A & B^T & 0 \\ B & C+D & K^T - I \\ 0 & I-K & V \end{bmatrix} \begin{bmatrix} u_\Omega \\ u_\Gamma \\ \phi \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}. \tag{4.42}$$

The blocks $\begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$, D, V, K, and K^T stem from the bilinear form $a(\cdot,\cdot)$ and the operators D_{k_2}, V_{k_2}, K_{k_2}, and K'_{k_2}, respectively. The finite element block A corresponds to testing finite element functions which are interior to Ω against themselves and C belongs to the remaining functions. The blocks B and B^T are due to testing interior basis functions against the remaining finite element functions and vice versa.

There are of course several ways of solving (4.42). For example taking $-\phi$ instead of ϕ we obtain a saddle point problem and a hybrid minimum residual method (HMCR) can be used as an efficient solver (see [137, 65] for the two-dimensional case ($n = 2$) and the h-version). In [103] both cases the system for ϕ and for $-\phi$ have been studied when solving with the GMRES method (also two dimensions and h-version). For a method based on a special inner product under which the originally indefinite system becomes positive definite we refer to [21].

Our aim is to show that our methods can be extended to coupled systems of the finite element and the boundary element method. In order to be able to apply the standard convergence estimate of the GMRES method Theorem 3.1 which assumes $\Lambda_0 > 0$ (cf. (3.1)) we restrict ourselves to positive definite bilinear forms for the interior problem, i.e., we assume

$$a(u,u) \geq c\|u\|^2_{H^1(\Omega)} \quad \text{for any } u \in H^1(\Omega).$$

The energy space of the bilinear form $\underline{\mathcal{A}}(\cdot;\cdot)$ is $H^1(\Omega) \times \tilde{H}^{-1/2}(\Gamma)$. With regard to the notation of Section 3.5 this means $\alpha = (2, -1)$ and the domains of definition of the operators are $D_1 = \Omega$ and $D_2 = \Gamma$. By (4.41) Gårding's inequality (3.68) holds with $\delta = (1, 1)$.

In the following we consider two situations. First, we deal with the system (4.42) when $n = 3$ and $k_2 \neq 0$ in which case the theory of Section 3.5.1 is applicable. This is done in a rather abstract way. Second, we look at the linear system when $k_2 = 0$, i.e., when the stiffness matrix is block-skew symmetric and when the symmetric part of the stiffness matrix is positive definite. This is the special situation of Theorem 3.18. Numerical results are given for the latter case in two dimensions.

Preconditioners for the indefinite system

Since we do not want to go into the details of additive Schwarz methods for finite elements we merely consider the indefinite case of the coupling procedure in an abstract way. Actually, we show how the additive Schwarz method presented in Section 3.5.1 applies in this case.

We decompose the individual components of the ansatz space $X_M \times Y_N$

$$X_M = H_0^1 \cup \cdots \cup H_{n_1}^1 \quad \text{and} \quad Y_N = H_0^{-1/2} \cup \cdots \cup H_{n_2}^{-1/2}$$

which leads to a decomposition of the whole ansatz space $X_M \times Y_N$ into

$$H_0 \cup \Big(H_1^1 \times \{0\} \cup \cdots \cup H_{n_1}^1 \times \{0\} \Big) \cup \Big(\{0\} \times H_1^{-1/2} \cup \cdots \cup \{0\} \times H_{n_2}^{-1/2} \Big). \tag{4.43}$$

The small subspace of global functions is $H_0 = H_0^1 \times H_0^{-1/2}$ (the use of the notation H_0^1 should not confuse). For H_0^1 we simply take the piecewise polynomials of lowest degree on the mesh Ω_{h_1} ($S_1^1(\Omega_{h_1})$) and $H_0^{-1/2}$ is the subspace of piecewise constant functions on Γ_{h_2} ($S_0^0(\Gamma_{h_2})$).

We assume that the decomposition (4.43) fulfills Assumptions (A1)–(A4) on page 103, i.e., the mesh sizes h_1 and h_2 are small enough such that $\underline{\mathcal{A}}$ can be inverted on H_0 and the subspaces H_j^1 and $H_j^{-1/2}$ ($j > 0$) are locally supported and are finite coverings of the domains Ω and Γ, respectively. Further, for any $v \in X_M$ and $\psi \in Y_N$ with $v = \sum_{j=0}^{n_1} v_j$ and $\psi = \sum_{j=0}^{n_2} \psi_j$ (according to (4.43)) we assume that there holds (A4):

$$\sum_{j=0}^{n_1} \|v_j\|_{H^1(\Omega)}^2 \leq \lambda_{1,0}^{-1} \|v\|_{H^1(\Omega)}^2$$

and

$$\sum_{j=0}^{n_2} \|\psi_j\|_{\tilde{H}^{-1/2}(\Gamma)}^2 \leq \lambda_{2,0}^{-1} \|\psi\|_{\tilde{H}^{-1/2}(\Gamma)}^2.$$

Note that we take the $H^1(\Omega)$-norm instead of the $\tilde{H}^1(\Omega) = H_0^1(\Omega)$-norm in the first estimate which is due to the transmission conditions across Γ (4.37). However, locally, if $\bar{\Omega}_j := \operatorname{supp} \psi_j \subset \Omega$, there holds the equivalence of norms $\|\psi_j\|_{H^1(\Omega)} \simeq \|\psi_j\|_{H_0^1(\Omega_j)}$ which is essential for obtaining preconditioners which are independent of the mesh size. Actually, that is the reason for assuming (A2) which ensures the scaling property claimed by Lemma 2.4.

Using this decomposition we define as in (3.72) the additive Schwarz operators $\underline{\mathcal{P}}_S$ (using full local solvers) and $\underline{P}_S$ (using positive definite local solvers). There holds the following result.

Theorem 4.9 *There exist constants $c, C, h_0, \delta_{1,3}, \delta_{2,3} > 0$ such that, if $0 < h_1, h_2 \leq h_0$ there holds for any $(v, \psi) \in X_M \times Y_N$*

$$\Re \langle \underline{A}^\alpha \underline{\mathcal{P}}(v,\psi), (v,\psi) \rangle \geq c \Big(C \min\{\lambda_{1,0}, \lambda_{2,0}\} - (\max\{h_1, h_2\})^{1/2} \Big) \|(v,\psi)\|_{H^1(\Omega) \times \tilde{H}^{-1/2}(\Gamma)}^2$$

and

$$\|\underline{\mathcal{P}}(v,\psi)\|_{H^1(\Omega) \times \tilde{H}^{-1/2}(\Gamma)} \leq c \|(v,\psi)\|_{H^1(\Omega) \times \tilde{H}^{-1/2}(\Gamma)}.$$

The same estimates hold for the operator $\underline{P}$.

Proof. The proof is an application of Theorem 3.16 which holds also in the finite element case, i.e., when finite element bilinear forms are included corresponding to a parameter $\alpha_j = 2$. In this case we have $\alpha = (2, -1)$, i.e., $\alpha_1 = 2$. The assumptions (A1)–(A4) are reasonable also in this case and they lead to the same conclusions, and eventually to Theorem 3.16, as in the standard situations of this work, $\alpha_j = \pm 1$, cf. Lemma 2.4 and Lemma 2.3 which are valid when $s = 1$. Now, by Theorem 3.16 we obtain for sufficiently small mesh sizes h_1 and h_2

$$\Re\langle \underline{A}^{\alpha}\underline{\mathcal{P}}(v,\psi),(v,\psi)\rangle \geq c\Big(C\min_j \lambda_{j,0} - \max_j h_j^{\delta_{j,3}}\Big)\|(v,\psi)\|^2_{H^1(\Omega)\times\tilde{H}^{-1/2}(\Gamma)}$$

and

$$\|\underline{\mathcal{P}}(v,\psi)\|_{H^1(\Omega)\times\tilde{H}^{-1/2}(\Gamma)} \leq c\|(v,\psi)\|_{H^1(\Omega)\times\tilde{H}^{-1/2}(\Gamma)}$$

for any $(v,\psi) \in X_M \times Y_N$. The parameter γ_2 is that in Gårding's inequality (4.41) and is a constant. Further, $\delta_{j,3} = \min\{\delta_{j,1}, \delta_{j,2}\}$, $j = 1, 2$, can be chosen. The parameters $\delta_{j,1}$ are the orders of convergence when projecting $(v,\psi) \in X_M \times Y_N$ with respect to $\underline{\mathcal{A}}$ onto H_0. Since Ω is a Lipschitz domain the finite element part converges linearly in h_1 and the boundary element part converges like $h_2^{1/2}$. The parameters $\delta_{j,2}$ stem from Lemma 3.6. There holds $\delta_{1,2} \geq 1$ and $\delta_{2,2} \geq 1$. Therefore, $\delta_{1,3} = \min\{1,1\} = 1$ and $\delta_{2,3} = \min\{1/2, 1\} = 1/2$ can be chosen and we conclude

$$\max_j h_j^{\delta_{j,3}} \leq \max\{h_1, h_2^{1/2}\} \leq h^{1/2} \quad \text{with } h := \max\{h_1, h_2\}.$$

This concludes the proof for the operator $\underline{\mathcal{P}}$. The proof for the operator $\underline{P}$ is analogously obtained by Theorem 3.16. □

Preconditioners for the block skew-symmetric system

Now we consider the special situation when when $a(\cdot,\cdot)$ is symmetric positive definite and $k_2 = 0$ which is the special situation of Theorem 3.18 (block skew-symmetric system with positive definite symmetric part). The symmetric part of the bilinear form is

$$\underline{\mathcal{A}}_0(u,\phi;v,\psi) := 2a(u,v) + \langle Du, v\rangle + \langle V\phi, \psi\rangle.$$

In fact, for any $u \in H^1(\Omega)$ and $\phi \in H^{-1/2}(\Gamma)$ there holds

$$\underline{\mathcal{A}}(u,\phi;u,\phi) = \underline{\mathcal{A}}_0(u,\phi;u,\phi) \simeq \left(\|u\|^2_{H^1(\Omega)} + \|\phi\|^2_{H^{-1/2}(\Gamma)}\right).$$

For the latter equivalence we note that there holds

$$0 \leq \langle Du, u\rangle \simeq \|u\|^2_{H^{1/2}(\Gamma)} \leq c\|u\|^2_{H^1(\Omega)} \quad \text{and} \quad \langle V\phi,\phi\rangle \simeq \|\phi\|^2_{H^{-1/2}(\Gamma)}.$$

Here, in the case $n = 2$, we have to assume that the interface Γ has conformal radius different from 1 such that the single layer potential is injective, cf. [125]. This can always be achieved by an appropriate scaling of Ω_1 and Ω_2.

We emphasize that in this situation we use a special bilinear form for the GMRES method, namely the bilinear form which is given by the preconditioner. For this we assume that the decomposition of all the components of the ansatz space (here X_M and Y_N) are direct ones. Of course, this is a restriction but might be avoided by modifications of the method. We note that in two dimensions direct decompositions of the ansatz spaces can be used without problems [7, 142, 75] but in three dimensions this is more sophisticated, see [114] for the finite element method. The preconditioner for weakly singular operators which is proposed in this work is already defined by a direct decomposition. Thus, there are efficient direct decompositions for the coupling procedure also in three dimensions.

According to Section 3.5.2 X_M and Y_N are directly decomposed like

$$X_M = H_0^1 \cup \cdots \cup H_{n_1}^1 \quad \text{and} \quad Y_N = H_0^{-1/2} \cup \cdots \cup H_{n_2}^{-1/2}.$$

The additive Schwarz operator $\underline{P} = \sum_{j,i} \underline{P}_{j,i}$ for the coupled system then belongs to the decomposition

$$X_M \times Y_N = \Big(H_0^1 \times \{0\} \cup \cdots \cup H_{n_1}^1 \times \{0\}\Big) \cup \Big(H_0^{-1/2} \times \{0\} \cup \cdots \cup H_{n_2}^{-1/2} \times \{0\}\Big). \quad (4.44)$$

The resulting preconditioning form is

$$\underline{B}\big((v,\phi),(w,\psi)\big) = \sum_{j=0}^{n_1} \underline{A}\big((v_j,0),(w_j,0)\big) + \sum_{j=0}^{n_2} \underline{A}\big((0,\phi_j),(0,\psi_j)\big)$$

for $v, w \in X_M$ and $\phi, \psi \in Y_N$ with $v = \sum_{j=0}^{n_1} v_j$ and $\phi = \sum_{j=0}^{n_2} \phi_j$ according to the decomposition (4.44) and analogously for w and ψ. Let $\lambda_{1,0}$, $\lambda_{1,1}$ and $\lambda_{2,0}$, $\lambda_{2,1}$ denote the minimum and maximum eigenvalues of the additive Schwarz operators defined within the components X_M and Y_N, respectively, i.e.

$$\lambda_{1,0} \sum_{j=0}^{n_1} \|v_j\|_{H^1(\Omega)}^2 \le \|v\|_{H^1(\Omega)}^2 \le \lambda_{1,1} \sum_{j=0}^{n_1} \|v_j\|_{H^1(\Omega)}^2 \quad \text{for any } v = \sum_{j=0}^{n_1} v_j \in X_M$$

and

$$\lambda_{2,0} \sum_{j=0}^{n_2} \|\psi_j\|_{\tilde{H}^{-1/2}(\Gamma)}^2 \le \|\psi\|_{\tilde{H}^{-1/2}(\Gamma)}^2 \le \lambda_{2,1} \sum_{j=0}^{n_2} \|\psi_j\|_{\tilde{H}^{-1/2}(\Gamma)}^2 \quad \text{for any } \psi = \sum_{j=0}^{n_2} \psi_j \in Y_N.$$

Then we can estimate the minimum eigenvalue of the symmetric part of $\underline{P}$ and the norm of $\underline{P}$.

Theorem 4.10 *There holds*

$$\inf_{(v,\psi)\in X_M\times Y_N} \frac{\underline{B}\big((v,\psi),\underline{P}(v,\psi)\big)}{\underline{B}\big((v,\psi),(v,\psi)\big)} \ge \min\{\lambda_{1,0},\lambda_{2,0}\}$$

and

$$\sup_{(v,\psi)\in X_M\times Y_N} \frac{\underline{B}\big(\underline{P}(v,\psi),\underline{P}(v,\psi)\big)}{\underline{B}\big((v,\psi),(v,\psi)\big)} \le \max\{\lambda_{1,1},\lambda_{2,1}\}^2.$$

Proof. The assertions are directly obtained by Theorem 3.18. □

The above theorem is still an abstract result. It allows for combining additive Schwarz preconditioners (based on direct decompositions) for the finite element method and the boundary element method for weakly singular integral operators which then gives an efficient method for the coupling procedure. In the following we consider an additive Schwarz method for this problem in two dimensions since for this case we are able to present some numerical results.

More precisely, we combine the preconditioner for the finite element method proposed in [7] and the preconditioner for the boundary element method given in [142]. In order to define the decompositions of the ansatz spaces $X_M = S_p^1(\Omega_{h_1})$ and $Y_N = S_{p-1}^0(\Gamma_{h_2})$ we classify the basis functions that will be used.

Let $\{E_j;\ j = 1, \ldots, J_E\}$ denote the set of edges of the mesh Ω_{h_1}. We assume that the basis functions can be divided into the following four sets:

- **The set H_0^1 of the nodal functions.** For each node of the mesh Ω_{h_1} there is a function which has the value one at the node and is zero at the remaining nodes.
- **The sets $H_{E_j}^1$ of the edge functions.** For each edge E_j of the mesh Ω_{h_1} there are functions vanishing at all other edges and which are nonzero only at the elements adjacent to E_j (and on E_j).
- **The sets $H_{\Omega_j}^1$ of the interior functions.** For each element Ω_j of the mesh Ω_{h_1} there are functions being nonzero only in the interior of Ω_j.
- **The sets of boundary element functions.** For each element Γ_j of the boundary element mesh Γ_{h_2} there are functions whose supports are contained in Γ_j. Note that the boundary element functions need not to be continuous since $Y_N \subset H^{-1/2}(\Gamma)$.

Using these types of basis functions we decompose the components of the ansatz space like

$$X_M = H_0^1 \cup H_{E_1}^1 \cup \cdots \cup H_{E_{J_E}}^1 \cup H_{\Omega_1}^1 \cup \cdots \cup H_{\Omega_{J_\Omega}}^1 \tag{4.45}$$

and

$$Y_N = H_0^{-1/2} \cup H_{\Gamma_1}^{-1/2} \cup \cdots \cup H_{\Gamma_{J_\Gamma}}^{-1/2}. \tag{4.46}$$

The space H_0^1 is the space of the nodal functions, $H_{E_j}^1$ is the space of the edge functions related with the edge E_j and $H_{\Omega_j}^1$ is spanned by the interior functions on the element Ω_j. For the boundary element functions we assume that $H_0^{-1/2}$ consists of the piecewise constant functions on the mesh Γ_{h_2} and $H_{\Gamma_j}^{-1/2}$ is spanned by the Legendre polynomials l_i, $i = 1, \ldots, p-1$, mapped onto the element Γ_j.

For $(u, \phi) \in X_M \times Y_N$ we define according to (4.45) and (4.46) the representations

$$u = u_1 + \sum_{j=1}^{J_E} u_{E_j} + \sum_{j=1}^{J_\Omega} u_{\Omega_j} \quad \text{and} \quad \phi = \phi_0 + \sum_{j=1}^{J_\Gamma} \phi_{\Gamma_j}. \tag{4.47}$$

The additive Schwarz preconditioning form belonging to the decomposition (4.45) and (4.46) is

$$\underline{B}(u,\phi;v,\psi) := \underline{\mathcal{A}}(u_1,0;v_1,0) + \sum_{j=1}^{J_E} \underline{\mathcal{A}}(u_{E_j},0;v_{E_j},0) + \sum_{j=1}^{J_\Omega} \underline{\mathcal{A}}(u_{\Omega_j},0;v_{\Omega_j},0)$$
$$+ \underline{\mathcal{A}}(0,\phi_0;0,\psi_0) + \sum_{j=1}^{J_\Gamma} \underline{\mathcal{A}}(0,\phi_{\Gamma_j};0,\psi_{\Gamma_j}). \tag{4.48}$$

Theorem 4.11 *Let the set of nodal functions of X_M be spanned by the standard piecewise bilinear functions and assume the edge functions to be discrete harmonic, i.e. for $i = 1,\dots,J_E$ and $j = 1,\dots,J_\Omega$*

$$a(u,v) = 0 \quad \textit{for all} \quad u \in H^1_{E_i}, v \in H^1_{\Omega_j},$$

where $a(\cdot,\cdot)$ is the finite element bilinear form (4.40)*. Then there exist constants $C_1, C_2 > 0$ such that for the additive Schwarz operator $\underline{P}$ defined by the decompositions* (4.45)*,* (4.46) *there holds*

$$\underline{B}\big((v,\psi);(v,\psi)\big) \le C_1(1+\log p)^2 \underline{B}\big((v,\psi);\underline{P}(v,\psi)\big) \quad \textit{for all } (v,\psi) \in X_M \times Y_N$$

and

$$\underline{B}\big(\underline{P}(v,\psi);\underline{P}(v,\psi)\big) \le C_2 \underline{B}\big((v,\psi);(v,\psi)\big) \quad \textit{for all } (v,\psi) \in X_M \times Y_N.$$

Proof. We use Theorem 4.10 and it is therefore left to estimate the eigenvalues of the additive Schwarz operator within the two components X_M and Y_N of the ansatz space.

For the first decomposition of the finite element space X_M we refer to Babuška et al. [7, Lemmas 3.1–3.3]: There exist constants $c_1, c_2, C_1, C_2 > 0$ such that for any $u \in X_M$ with representation $u = \sum_{j=1}^{\mathcal{J}} u_j := u_1 + \sum_{j=1}^{J_E} u_{E_j} + \sum_{j=1}^{J_\Omega} u_{\Omega_j}$ according to (4.47) there holds

$$c_1(1+\log p)^{-2} \sum_{j=1}^{\mathcal{J}} |u_j|^2_{H^1(\Omega)} \le |u|^2_{H^1(\Omega)} \le C_1 \sum_{j=1}^{\mathcal{J}} |u_j|^2_{H^1(\Omega)} \tag{4.49}$$

This result can be extended to the $H^1(\Omega)$-norm by a quotient space argument for the functions $u \in X_M$ and by using the Poincaré inequality for functions with local supports, i.e., supports which are of size proportional to h_1. For a nodal function we have to take the union of at most four elements, whereas for edge functions we have to take at most two elements and a single element for interior functions.

Referring to the work of Stephan and Tran [142], who considered for Y_N the trace onto Γ of the decomposition of X_M, we know that there exist constants $c_2, C_2 > 0$ such that for any $\phi \in Y_N$ with representation $\phi = \sum_{j=0}^{J_\Gamma} \phi_j := \phi_0 + \sum_{j=1}^{J_\Gamma} \phi_{\Gamma_j}$ according to (4.47) there holds

$$c_2(1+\log p)^{-2} \sum_{j=0}^{J_\Gamma} \|\phi_j\|^2_{H^{-1/2}(\Gamma)} \le \|\phi\|^2_{H^{-1/2}(\Gamma)} \le C_2 \sum_{j=0}^{J_\Gamma} \|\phi_j\|^2_{H^{-1/2}(\Gamma)}. \tag{4.50}$$

Now, by the estimates (4.49) and (4.50), Theorem 4.10 proves the assertion of this theorem. □

The assumption on the edge functions to be discrete harmonic can be satisfied in two ways. It is possible to choose edge basis functions which are a-priori discrete harmonic, e.g., by taking the polynomials defined by (2.2) in Section 2.3, see also [100, 114]. Or one eliminates the interior functions after having assembled the stiffness matrix as it is done in [7]. The latter case is known as a Schur complement step and this requires a transformation only of the edge functions. This transformation is performed in our numerical experiments in the next section.

Numerical experiments

We now present numerical experiments for solving the linear system (4.42) of the coupling procedure in two dimensions with the GMRES method. As preconditioner we take the bilinear form $\underline{B}$ (4.48) and the preconditioned system corresponds to an additive Schwarz operator which is denoted by $\underline{P}$. We consider the interface problem (4.36)–(4.38) with $k = i$ (imaginary unit) for the L-shaped domain Ω (see Figure 4.13) and its complement. The functions v_0, ψ_0 and f are chosen such that

$$u_1(x,y) = \Im(z^{2/3}) \text{ for } z = x + iy \qquad \text{and} \qquad u_2(x,y) = \log|(x,y) + (0.3, 0.3)|.$$

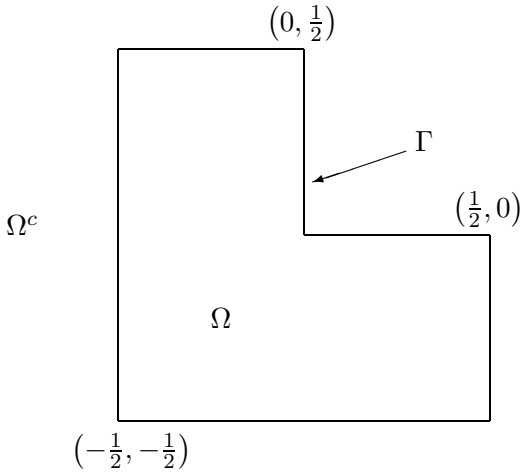


Figure 4.13: Example for the Helmholtz transmission problem in $\mathbb{R}^2$: the L-shaped domain.

For the sequence of finite dimensional subspaces X_M and Y_N we use a fixed, rather crude decomposition of the L-shaped domain into three rectangles (see Figure 4.14) and take the induced decomposition of the boundary into eight elements for the boundary element part. That means, in this case we have $h_1 = h_2 = 1/2$. For the spaces Y_N we use discontinuous piecewise Legendre polynomials on the decomposition of Γ and for the spaces X_M we use tensor products of anti-derivatives of Legendre polynomials on each of the rectangles.

We solve the linear system (4.42) via the GMRES method where we consider the unpreconditioned version and the preconditioner $\underline{B}$. The theoretical results for the preconditioner require discrete harmonic edge functions, i.e. they have to be orthogonal with respect

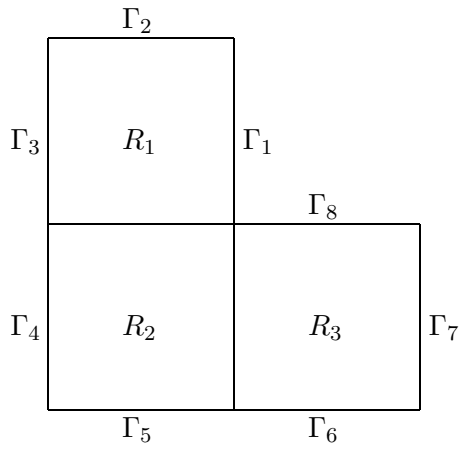


Figure 4.14: The grid used for the L-shaped domain.

to the H^1-inner product to the basis functions which are interior to the elements. This is fulfilled by performing a Schur complement with respect to the interior basis functions resulting in a basis transformation of the edge functions. The inversions of the blocks of the interior functions are performed directly, as it is done for all the blocks of the preconditioner. For practical applications these direct inversions may be replaced by indirect solvers which are not covered by our theory. Further we note that the action of performing the Schur complements is a local operation which can be parallelized on the level of the elements and is therefore not a too time-consuming task. For a more detailed discussion of a-posteriori realizing the edge functions to be discrete harmonic we refer to [7].

Table 4.14 presents the minimum eigenvalues Λ_0 of the symmetric parts of the unpreconditioned and preconditioned stiffness matrices as well as the norms Λ_1 of the unpreconditioned and preconditioned matrices up to degree $p = 14$. As proved by Theorem 4.11 the norm of the preconditioned system is bounded whereas the minimum eigenvalue of the symmetric part decreases slowly. To check the asymptotic behavior of the minimum eigenvalue we plot the values versus the polynomial degree p in a log–log scale, see Figure 4.15. As predicted by Theorem 4.11 the curve for the preconditioner $\underline{B}$ seems to decrease like $(1 + \log p)^{-2}$. Further, Table 4.14 underlines the improvements in the convergence of the GMRES method to solve the linear system when using the preconditioner $\underline{B}$. The columns entitled *#iter* list the numbers of iterations which are necessary to reduce the initial residual by a factor of 10^{-6}, and they are substantially smaller than without preconditioner.

4.5 Implementation issues

First we give an overview of the main steps of the boundary element method. We note that our aim was to write modular programs where different steps are clearly separated in order to reduce the time for the implementation and in order to make individual modules easily exchangeable. In this way it is less difficult to test different preconditioners for different operators.

		$\underline{\mathcal{A}}$			$\underline{P}$		
p	$N+M$	Λ_0	Λ_1	#iter	Λ_0	Λ_1	#iter
1	16	0.05640	31.87	16	1.00000	14.62	14
2	37	0.00840	46.01	35	0.27702	15.17	26
3	64	0.00340	46.68	59	0.24253	15.18	27
4	97	0.00170	46.72	86	0.17917	15.20	30
5	136	0.00099	46.72	113	0.16074	15.20	31
6	181	0.00062	46.73	137	0.13681	15.20	32
7	232	0.00042	46.73	159	0.12657	15.20	33
8	289	0.00029	46.73	183	0.11374	15.20	34
9	352	0.00021	46.73	206	0.10738	15.20	35
10	421	0.00016	46.73	231	0.09923	15.20	35
11	496	0.00012	46.73	253	0.09489	15.20	36
12	577	0.00010	46.73	275	0.08917	15.20	36
13	664	0.00008	46.73	297	0.08601	15.20	37
14	757	0.00006	46.73	319	0.08174	15.20	37

Table 4.14: Coupled FEM/BEM method for the Helmholtz transmission problem in $\mathbb{R}^2$: the minimum eigenvalues Λ_0 of the symmetric parts, the norms Λ_1 of the un-preconditioned and the preconditioned systems and the numbers #iter of iterations of the GMRES method.

Setup of the geometry and the ansatz space. The geometry is completely defined by the ansatz space which has been denoted by $S_p^1(\Gamma_h)$ (continuous functions) or $S_p^0(\Gamma_h)$ (contains discontinuous functions). This is done by defining the mesh Γ_h (consisting of nodes, sides, and elements in the case of continuous functions and consisting only of elements otherwise) and by specifying the polynomial degree which is associated with each of the objects. Here we use a uniform mesh and we have the same degree everywhere. But the components of the tensor products which define the basis functions may have different degrees, depending on what type of basis functions is in use.

Assembly of the stiffness matrix and the right hand side. For the construction of the stiffness matrix we use analytical recurrence formulae to compute stiffness entries for monomial terms

$$\langle A^\alpha T_r^{-1} x^i y^j, T_s^{-1} x^k y^l \rangle_{L^2(\Gamma_s)}, \quad i,j,k,l = 0, \dots, p. \tag{4.51}$$

Here T_s is an affine transformation of the element Γ_s onto the reference element $(-1,1)^2$. The operator A^α is the integral operator under consideration and in the indefinite case (wave number $k \neq 0$) an additional Taylor series expansion of the kernel function has to be performed. For instance, when $k = 5$ we took 20 terms of the expansion. The entries for the final stiffness matrix are computed by combinations of these terms. The recurrence formulae for calculating the monomial terms (4.51) and the Taylor series expansions have been derived and implemented by M. Maischak [97]. Since we only considered right hand side functions which are polynomials (even constants) these formulae are also used for assembling the right hand side vectors. We note that in the case of curved surfaces or when the elements have curved boundaries the analytical formulae must be replaced by numerical integration formulae, cf., e.g., [118, 117].

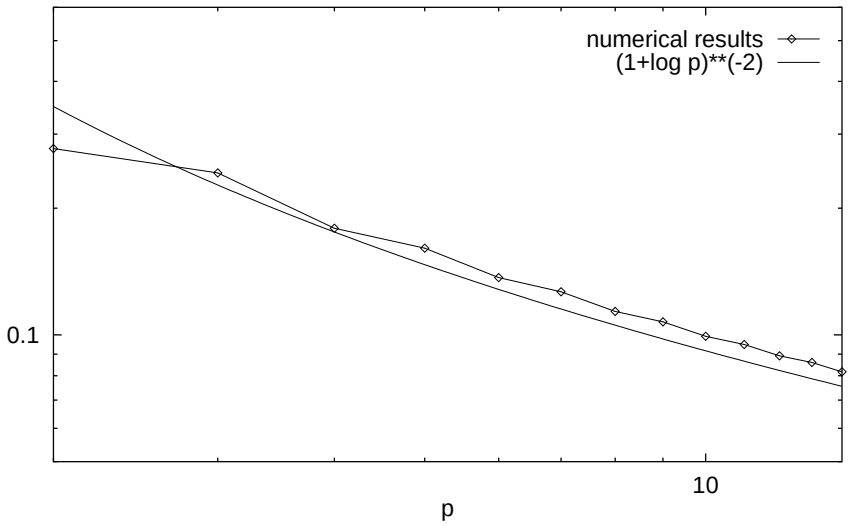


Figure 4.15: Coupled FEM/BEM method for the Helmholtz transmission problem in $\mathbb{R}^2$: the minimum eigenvalues of the symmetric parts of the system preconditioned by $\underline{B}$.

In the case of indefinite systems the matrix which belongs to the main part of the operator (i.e., $k = 0$ in the scalar case) is stored. This matrix is symmetric and positive definite and defines the bilinear form at the discrete level which has to be used for the GMRES method. However, no extra computations are necessary for this step since the main part is calculated in either way when using series expansions of the kernel functions.

Calculation of the preconditioner. The calculation of additive Schwarz type preconditioners requires the solution of a couple of independent sub-problems. In our formulation these are again boundary integral equations which must be solved in a subspace of the whole ansatz space. In the finite element method the according linear systems are easily assembled by using the local stiffness matrices. In the boundary element method, where we do not have local stiffness matrices in this abstract form, we transform the basis functions of the subspace such that we can use a block of the whole stiffness matrix as stiffness matrix for the sub-problem. In some cases this transformation is only a restriction, i.e., when the basis functions of the subspace are also basis functions of the ansatz space. For example, when using standard basis functions (S) for the stiffness matrix and taking discrete harmonic basis functions (H) for the preconditioner (cf. Section 4.1) these transformations do appear.

Now, the solution of the sub-problems corresponds to solving the individual linear systems. In the case the local problems are large enough an iterative solver or even a fixed number of iterations of an iterative solver can be used. However, when considering small individual problems, what is the case when dealing with fine decompositions as in this work, often direct inversions are reasonable. They are taken in our procedures.

Related to this aspect is the following. When taking preconditioners for iterative solvers it is usually not necessary to calculate the preconditioner in a separate step before. In contrast, most often a more efficient way is to invert the local systems within the iterative solver when necessary (directly or iteratively). However, since we want to calculate eigenvalues of the preconditioned and un-preconditioned systems in order to check theoretical estimates

we need the whole matrices as they are. Therefore, in our algorithm, the calculation of the preconditioner is a separate step.

Solution of the linear system. We use the GMRES method without restart for the solution of the linear systems. A step of the GMRES method is an orthogonalization step where orthogonality is defined with respect to a bilinear form which enters the estimate by Theorem 3.1 via (3.1). In our case this bilinear form is most often the bilinear form of the main part of the operator under consideration. At the discrete level this bilinear form is given by the main part of the stiffness matrix and is used in the algorithm. Only for the block skew-symmetric coupling system in Section 4.4 (Theorem 4.10) we use the bilinear form of the preconditioner.

Restart. When considering several steps of the boundary element method, e.g., different polynomial degrees, a restart of the program can be performed much more efficiently than a new start (e.g., by re-using old parts of the stiffness matrix). However, since we want to compare individual steps of the method independently, no optimization is used.

Check of theoretical estimates. Our theoretical estimates are checked by calculating the parameters Λ_0 and Λ_1 in (3.1),

$$\Lambda_0 = \inf_{v\in\mathbb{R}^N} \frac{c(v,Av)}{c(v,v)} \quad \text{or} \quad \Lambda_0 = \inf_{v\in\mathbb{C}^N} \Re\frac{c(v,Av)}{c(v,v)}, \quad \text{and} \quad \Lambda_1 = \sup_{v\in\mathbb{R}^N} \frac{\|Av\|_c}{\|v\|_c}.$$

Here, A is the, possibly preconditioned, stiffness matrix under consideration and $c(\cdot,\cdot)$ is a given bilinear form. Preconditioned matrix means that the stiffness matrix has been multiplied with the matrix of the preconditioner which is, in the additive Schwarz framework, the sum of the inverses of some stiffness matrices belonging to sub-problems, cf. also Remark 3.4 on page 113. Here, some transformations are necessary to map basis functions of the whole ansatz space to basis functions of the subspaces and the stiffness matrices of the sub-problems have to be embedded in $N \times N$-matrices by filling up with zeroes. (N is the dimension of the whole ansatz space.)

Let C denote the matrix which defines this bilinear form at the discrete level. Then we have to consider the generalized eigenvalue problem for the Hermitian part of CA

$$\frac{1}{2}\big((CA)^\star + CA\big)v = \lambda Cv, \quad v \in \mathbb{R}^N,$$

to determine Λ_0 and the generalized eigenvalue problem

$$CAv = \lambda Cv, \quad v \in \mathbb{R}^N,$$

to determine Λ_1. We simply reduce both generalized eigenvalue problems to standard eigenvalue problems by multiplying with the inverse of C. Those problems are solved by reducing the matrices to upper Hessenberg form and then to quasi-upper tridiagonal form (so-called Schur factorization) where the pairs of conjugate complex eigenvalues and the real eigenvalues can be observed. For all these transformations we use standard LAPACK-routines [3], indeed ready-to-use macros are present there.

Numerical expenses.

In the following we comment on the computational costs of our methods.

Assembly of the stiffness matrix. For the boundary element method it is usually necessary to assemble the whole system before solving it since the matrix is dense by the non-locality of the integral operators and since in general the matrix cannot be assembled by scaling local stiffness matrices on reference elements. However, in special situations, there are some methods for dealing with sparse matrices and for very efficiently assembling the matrices. We only mention the wavelet theory applicable to the pure h-version, cf., e.g., [19, 45], the efficient use of circulant matrices, see, e.g., [155], and an algorithm for the p-version which considers sparse matrices instead of the full ones [70]. Here, we deal with the full matrix and the calculation of $O(N^2) = O(h^{-4}p^4)$ entries is required. As before, h denotes the width of the elements and p is the maximum polynomial degree of the ansatz functions. We note that the numerical complexity of the matrix vector multiplication for the pure h-version can be reduced by the panel clustering technique of Hackbusch and Novak, see [64]. However, to our knowledge there are no corresponding results when the p-version of the BEM is used.

Calculation of the preconditioner.

- **Overlapping method for $S_p^1(\Gamma_h)$.** This method is subject of Section 3.2.2. We have to solve a couple of linear systems each of them corresponding to a subspace in the decomposition (3.21). The linear system which belongs to the subspace of global functions is of size $O(h^{-2} \times h^{-2})$ and the remaining systems are of size $O(p^2 \times p^2)$. Therefore, when the product hp is very large or very small we have to deal with linear systems which are of substantially different sizes. In a sequential algorithm, as is used here, this is not a problem but reduces the efficiency of parallel algorithms.

- **Iterative substructuring methods for $S_p^1(\Gamma_h)$.** These methods are based on decompositions of the ansatz space where discrete harmonic basis functions are used, cf. Section 3.2.3. Either one takes a-priori these special basis functions for the assembly of the stiffness matrix or one has to use transformations between the sets of basis functions in order to setup the stiffness matrices which correspond to the subspaces of the decomposition. The coefficients of the components of these basis functions are obtained by the solutions of the three problems given by Definitions 2.1, 2.2 and 2.3. We note that these problems are just one-dimensional and of the size $O(p)$, whereas the whole boundary element problem is of the size $O(h^{-2}p^2)$. Further, the computation of the coefficients of the basis functions has to be performed only once for a specific polynomial degree p. Therefore, this step is negligible in the overall process in view of its time consumption.

  Since we take the monomial terms (4.51) to assemble the entries in the stiffness matrix there is a difference in the amount of work when using standard or discrete harmonic basis functions. In the latter case, except for the subspace $S_1^1(\Gamma_h)$, the nodal functions and the components of the edge functions which are vertical to the edges are, locally on the elements, polynomials of degree p whereas in the standard case they are polynomials just of degree 1. However, all the terms (4.51) have to be computed in either way to construct the stiffness entries for the interior components and these computations are in general the most expensive step in the construction of the stiffness matrix. Therefore the larger number of monomial terms which are needed to construct the nodal and the edge components does not increase substantially the overall cost.

Let us consider the preconditioners themselves. They are defined by decompositions into a low dimensional subspace of piecewise bilinear functions plus, after transformation to discrete harmonic basis functions, natural decompositions of the ansatz space. That means we split with respect to nodal, edge and interior components, which can be directly represented by blocks of the stiffness matrix which do not overlap each other. Since the transformation to the new basis functions can be performed locally once to obtain the global transformation the costs to construct our preconditioners are essentially the costs to invert the blocks independently of each other. The block belonging to the subspace $S_1^1(\Gamma_h)$, which appears in all the three cases (3.22), (3.23), and (3.24), is of size $O(h^{-2} \times h^{-2})$.

For the wire basket preconditioner based on (3.22) the remaining blocks are of the sizes $O(h^{-2}p \times h^{-2}p)$ and $O(p^2 \times p^2)$ for the wire basket functions and interior functions, respectively. For the non-overlapping additive Schwarz preconditioner belonging to the decomposition (3.23) the sizes are $O(1)$, $O(p \times p)$ and $O(p^2 \times p^2)$ for the nodal functions, edge functions and interior functions, respectively. The implementation of the modified diagonal preconditioner requires the inversions just of the diagonal entries and of the block belonging to the subspace $S_1^1(\Gamma_h)$ and is therefore very cheap.

In view of parallelization of this method load balancing is not a problem since any decompositions of the subspaces using discrete harmonic basis functions can be used. However, the overhead for computing the transformations between the basis functions is present in any case. On the other hand even the full decomposition is as efficient as others and therefore, a sequential algorithms for this part seems to be reasonable.

- **Non-overlapping decomposition of $S_p^0(\Gamma_h)$.** This method takes the non-overlapping decomposition (3.34) which is direct and natural for the functions of degree larger than 0. This means no unknown is taken twice and no transformation of the basis functions of degree larger than 0 is necessary. Transformations of the piecewise constant functions (which are cheap) are only necessary if the domains G_j of the local subspaces differ from individual elements Γ_j. These two properties (direct and natural decomposition) reduce the overhead in the implementation of the preconditioner which is larger for the methods dealing with continuous functions. The sizes of the blocks, which have to be inverted in this method, corresponding to the decomposition (3.34) are $O(h^{-2} \times h^{-2})$ and $O(p^2 \times p^2)$. However, for a fixed polynomial degree, a load balancing can be performed (the sizes of the blocks can be equalized) by choosing appropriate domains G_j.

- **Solution of the local problems.** To apply the preconditioners we have to solve a couple of local problems associated with the local subspaces of the respective decomposition of the ansatz space. For the modified diagonal preconditioner the local problems are only one-dimensional. For the remaining decompositions the sizes of the local problems are at most $O(p^2)$ and, even for more practical applications where h is very small and p is only moderate, these problems can be solved directly. The main important feature of our preconditioners is their scalability. That means the sizes of the local problems and the resulting condition numbers are independent of h. We emphasize that away from the boundary, edges and corners of the domain the stiffness matrices of the local problems are identical on uniform meshes. This fact is due to the depen-

dence of the Greens' functions (being the kernels of the integral operators) only on the distance $|x - y|$. Here, y is the point of integration and x is the point of observation. Therefore, in practise this small stiffness matrix whose size depends only on p can be inverted once for all. We also note that the exact inversions of the local problems can be replaced by inexact methods. For instance, in the case of indefinite operators the use of an additive Schwarz operator with local positive definite problems is such a strategy, cf. the discussion in §4.1 on pages 128 ff. Concerning inexact solvers for the FEM we also refer to [60, 25].

Besides the local problems one also has to solve a relatively small problem associated with the coarse space of global functions. Most often these functions are of the lowest degree (piecewise constant or multilinear) and, therefore, standard preconditioning techniques for the h-version can be used for this problem instead of inverting it directly.

Solution of the linear system. In order to confirm the theoretical estimates of the preconditioners which enter the estimates of the needed numbers of iterations of the GMRES method via Theorem 3.1 we have to use the GMRES method without restart which requires an amount of memory that depends on the actual numbers of iterations, cf. [116]. This does not cause problems for our examples which are of relatively small size. The most expensive steps in the GMRES method are matrix vector multiplications where the stiffness matrix and the matrix defining the inner product are taken. The number of inner product calculations grows quadratically with the number of iterations of the GMRES method. As already seen in our numerical experiments the proposed preconditioners substantially reduce these numbers and in some cases they are almost constant. Thus, having computed the preconditioners, the solution procedure then is very cheap.

Bibliography

[1] M. Ainsworth, *A hierarchical domain decomposition preconditioner for h-p finite element approximation on locally refined meshes*, SIAM J. Sci. Comput., 17 (1996), pp. 1395–1413.

[2] ——, *A preconditioner based on domain decomposition for h-p finite element approximation on quasi-uniform meshes*, SIAM J. Numer. Anal., 33 (1996), pp. 1358–1376.

[3] E. Anderson, Z. Bai, C. Bischof, J. Demmel, J. Dongarra, J. Du Croz, A. Greenbaum, S. Hammarling, A. McKenney, S. Ostrouchov, and D. Sorensen, *LAPACK Users' Guide*, SIAM, Philadelphia, PA, 1992.

[4] J. P. Aubin, *Behavior of the error of the approximate solution of boundary value problems for linear elliptic operators by Galerkin's and finite difference methods*, Ann. Scuola Norm. Sup. Pisa, 21 (1967), pp. 599–637.

[5] I. Babuška, , and B. Guo, *Approximation properties of the h-p version of the finite element method*, Comput. Methods Appl. Mech. Engrg., 133 (1996), pp. 319–346.

[6] I. Babuška and A. K. Aziz, *Survey lectures on the mathematical foundations of the finite element method*, in The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, A. K. Aziz, ed., New York, 1972, Academic Press, pp. 5–359.

[7] I. Babuška, A. Craig, J. Mandel, and J. Pitkäranta, *Efficient preconditioning for the p-version finite element method in two dimensions*, SIAM J. Numer. Anal., 28 (1991), pp. 624–661.

[8] I. Babuška, M. Griebel, and J. Pitkäranta, *The problem of selecting the shape functions for a p-type finite element*, Internat. J. Numer. Methods Engrg., 28 (1989), pp. 1891–1908.

[9] I. Babuška, B. Guo, and E. P. Stephan, *The h-p version of the boundary element method with geometric mesh on polygonal domains*, Comput. Methods Appl. Mech. Engrg., 80 (1990), pp. 319–325.

[10] ——, *On the exponential convergence of the h-p version for boundary element Galerkin methods on polygons*, Math. Methods Appl. Sci., 12 (1990), pp. 413–427.

[11] I. Babuška and M. Suri, *The h-p version of the finite element method with quasiuniform meshes*, RAIRO Modél. Math. Anal. Numér., 21 (1987), pp. 199–238.

[12] R. Barrett, M. Berry, T. F. Chan, J. Demmel, J. Donato, J. Dongarra, V. Eijkhout, R. Pozo, C. Romine, and H. Van der Horst, *Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods*, SIAM, Philadelphia, PA, 1994.

[13] R. Bellman, *A note on an inequality of E. Schmidt*, Bull. Amer. Math. Soc., 50 (1944), pp. 734–737.

[14] F. Ben Belgacem, *Discrétisations 3D non conformes par la méthode de décomposition de domaine des éléments avec joints: analyse mathématique et mise en œuvre pour le problème de Poisson*, PhD thesis, Université Pierre et Marie Curie, PARIS VI, 1993.

[15] ——, *The mortar finite element method with Lagrange multipliers*, (1993). Preprint.

[16] ——, *Polynomial extensions of compatible polynomial traces in three dimensions*, Comput. Methods Appl. Mech. Engrg., 116 (1994), pp. 235–241. Proceedings of the International Conference On Spectral and High Order Methods 1992 (C. Bernardi and Y. Maday, eds.).

[17] J. Bergh and J. Löfström, *Interpolation Spaces*, no. 223 in Grundlehren der mathematischen Wissenschaften, Springer-Verlag, Berlin, 1976.

[18] C. Bernardi, Y. Maday, and A. T. Patera, *Domain decomposition by the mortar element method*, in Asymptotic and Numerical Methods for PDEs with Critical Parameters, H. G. Kaper and M. Garbey, eds., 1993, pp. 269–286.

[19] G. Beylkin, R. Coifman, and V. Rokhlin, *Fast wavelet transforms and numerical algorithms I*, Comm. Pure Appl. Math., 44 (1991), pp. 141–183.

[20] P. Bjørstad and O. B. Widlund, *Iterative methods for the solution of elliptic problems on regions partitioned into substructures*, SIAM J. Numer. Anal., 23 (1986), pp. 1093–1120.

[21] J. H. Bramble and J. E. Pasciak, *A preconditioning technique for indefinite systems resulting from mixed approximations of elliptic problems*, Math. Comp., 50 (1988), pp. 1–17.

[22] ——, *New estimates for multilevel algorithms including the V-cycle*, Math. Comp., 60 (1993), pp. 447–471.

[23] J. H. Bramble, J. E. Pasciak, and A. H. Schatz, *An iterative method for elliptic problems on regions partitioned into substructures*, Math. Comp., 46 (1986), pp. 361–369.

[24] ——, *The construction of preconditioners for elliptic problems by substructuring, IV*, Math. Comp., 53 (1989), pp. 1–24.

[25] J. H. Bramble, J. E. Pasciak, and A. T. Vassilev, *Analysis of non-overlapping domain decomposition algorithms with inexact solves*, Math. Comp., 67 (1998), pp. 1–19.

[26] J. H. Bramble and J. Xu, *Some estimates for a weighted L^2 projection*, Math. Comp., 56 (1991), pp. 463–476.

[27] F. Brezzi and C. Johnson, *On the coupling of boundary integral and finite element methods*, Calcolo, 16 (1979), pp. 189–201.

[28] X.-C. Cai and O. B. Widlund, *Domain decomposition algorithms for indefinite elliptic problems*, SIAM J. Sci. Stat. Comput., 13 (1992), pp. 243–258.

[29] C. Canuto and D. Funaro, *The Schwarz algorithm for spectral methods*, SIAM J. Numer. Anal., 25 (1988), pp. 24–40.

[30] C. Carstensen and E. P. Stephan, *Coupling of FEM and BEM for a nonlinear interface problem: The h-p version*, Numer. Methods Partial Differential Eq., 11 (1995), pp. 539–554.

[31] T. F. Chan and T. P. Mathew, *Domain decomposition algorithms*, Acta Numerica, (1994), pp. 61–143.

[32] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[33] P. Clément, *Approximation by finite element functions using local regularization*, RAIRO Numer. Anal., 9 (1975), pp. 77–84.

[34] E. T. Copson, *On the problem of the electrified disc*, Proc. Edin. Math. Soc., 8 (1947).

[35] M. Costabel, *Boundary integral operators on Lipschitz domains: Elementary results*, SIAM J. Math. Anal., 19 (1988), pp. 613–626.

[36] ——, *A symmetric method for the coupling of finite elements and boundary elements*, in The Mathematics of Finite Elements and Applications VI, J. R. Whiteman, ed., London, 1988, Academic Press, pp. 281–288.

[37] ——, *On the coupling of finite and boundary element methods*, in Proc. Intern. Symp. on Numerical Analysis, B. Karasözun, ed., Ankara, 1989, METU, pp. 1–14.

[38] M. Costabel and E. P. Stephan, *The method of Mellin transformation for boundary integral equations on curves with corners*, in Numerical Solution of Singular Integral Equations, A. Gerasoulis and R. Vichnevetsky, eds., IMACS, 1984, pp. 95–100.

[39] ——, *Boundary integral equations for mixed boundary value problems in polygonal domains and Galerkin approximations*, in Mathematical Models and Methods in Mechanics, W. Fiszdon and K. Wilmanski, eds., Warsaw, 1985, Banach Centre Publ. **15**, pp. 175–251.

[40] ——, *A direct boundary integral equation method for transmission problems*, J. Math. Anal. Appl., 106 (1985), pp. 367–413.

[41] ——, *Coupling of finite elements and boundary elements for inhomogeneous transmission problems in $\mathbb{R}^3$*, in The Mathematics of Finite Elements and Applications VI, J. R. Whiteman, ed., London, 1988, Academic Press, pp. 289–296.

[42] ——, *Strongly elliptic boundary integral equations for electromagnetic transmission problems*, Proc. Roy. Soc. Edinburgh, 109A (1988), pp. 271–296.

[43] ——, *Coupling of finite element and boundary element methods for an elasto-plastic interface problem*, SIAM J. Numer. Anal., 27 (1990), pp. 1212–1226.

[44] M. Costabel, E. P. Stephan, and W. L. Wendland, *On boundary integral equations of the first kind for the bi-Laplacian in a polygonal domain*, Annali Scuola Normale Superiore Pisa, Ser. IV, 10 (1983), pp. 197–241.

[45] W. Dahmen, S. Prössdorf, and R. Schneider, *Wavelet approximation methods for pseudodifferential equations. II: Matrix compression and fast solution*, Adv. Comput. Math., 1 (1993), pp. 259–335.

[46] ——, *Wavelet approximation methods for pseudodifferential equations. I: Stability and convergence*, Math. Z., 215 (1994), pp. 583–620.

[47] C. L. Dolph, *The integral equation method in scattering theory*, in Problems in Analysis, R. C. Gunning, ed., Princeton, N. J., 1970, Princeton University Press, pp. 201–277.

[48] M. R. Dorr, *The approximation theory for the p-version of the finite element method*, SIAM J. Numer. Anal., 21 (1984), pp. 1180–1207.

[49] S. C. Eisenstat, H. C. Elman, and M. H. Schultz, *Variational iterative methods for non-symmetric systems of linear equations*, SIAM J. Numer. Anal., 20 (1983), pp. 345–357.

[50] S. A. Funken and E. P. Stephan, *Hierarchical basis preconditioners for coupled FEM–BEM equations*, in Notes on Numerical Fluid Mechanics, W. Hackbusch and G. Wittum, eds., New York, 1996, Springer-Verlag.

[51] G. N. Gatica and G. C. Hsiao, *The coupling of boundary element and finite element methods for a nonlinear exterior boundary value problem*, Z. Anal. Anwendungen, 8 (1989), pp. 377–387.

[52] G. N. Gatica and W. L. Wendland, *Coupling of mixed finite elements and boundary elements for linear and nonlinear elliptic problems*, Appl. Anal., 63 (1996), pp. 39–75.

[53] ——, *Coupling of mixed finite elements and boundary elements for a hyperelastic interface problem*, SIAM J. Numer. Anal., 34 (1997), pp. 2335–2356.

[54] M. Griebel and P. Oswald, *On the abstract theory of additive and multiplicative Schwarz algorithms*, Numer. Math., 70 (1995), pp. 163–180.

[55] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman Publishing Inc., Boston, 1985.

[56] B. Guo, *The h-p version of the finite element method for solving boundary value problems in polyhedral domains*, in Boundary value problems and integral equations in nonsmooth domains (Luminy, 1993), vol. 167 of Lecture Notes in Pure and Appl. Math., Dekker, New York, 1995, pp. 101–120.

[57] B. Guo and I. Babuška, *The h-p version of the finite element method, Part 1: The basic approximation results*, Comp. Mech., 1 (1986), pp. 21–41.

[58] ——, *The h-p version of the finite element method, Part 2: General results and applications*, Comp. Mech., 1 (1986), pp. 203–220.

[59] B. Guo and W. Cao, *A preconditioner for the h-p version of the finite element method in two dimensions*, Numer. Math., 75 (1996), pp. 59–77.

[60] ——, *Inexact solvers on element interfaces for the p and h-p finite element method*, Comput. Methods Appl. Mech. Engrg., 150 (1997), pp. 173–189.

[61] ——, *An additive Schwarz method for the h-p version of the finite element method in three dimensions*, SIAM J. Numer. Anal., 35 (1998), pp. 632–654.

[62] B. Q. Guo, N. Heuer, and E. P. Stephan, *The h-p version of the boundary element method for transmission problems with piecewise analytic data*, SIAM J. Numer. Anal., 33 (1996), pp. 789–808.

[63] G. Haase, B. Heise, M. Kuhn, and U. Langer, *Adaptive domain decomposition methods for finite and boundary element equations*, in Reports from the Final Conference of the Priority Research Programme Boundary Element Methods 1989–1995, W. L. Wendland, ed., Berlin, 1995, Springer-Verlag.

[64] W. Hackbusch and Z. P. Nowak, *On the fast matrix multiplication in the boundary element method by panel clustering*, Numer. Math., 54 (1989), pp. 463–491.

[65] M. Hahne, M. Maischak, E. P. Stephan, and A. J. Wathen, *Efficient preconditioners for coupled FEM–BEM equations*, Numer. Methods Partial Differential Eq. To appear.

[66] M. Hahne and E. P. Stephan, *Schwarz iterations for the efficient solution of screen problems with boundary elements*, Computing, 55 (1996), pp. 61–85.

[67] M. Hahne, E. P. Stephan, and W. Thies, *Fast solvers for coupled FEM–BEM equations I*, in Fast Solvers for Flow Problems, W. Hackbusch and G. Wittum, eds., vol. 49 of Notes on Numerical Fluid Mechanics, Braunschweig, Wiesbaden, 1994, Vieweg, pp. 121–130.

[68] G. Hardy, D. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, 1952.

[69] F. K. Hebeker, *On a parallel Schwarz algorithm for symmetric strongly elliptic integral equations*, in Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, Y. A. Kuznetsov, G. A. Meurant, J. Périaux, and O. B. Widlund, eds., Philadelphia, 1991, SIAM, pp. 382–393.

[70] N. Heuer, *Efficient algorithms for the p-version of the boundary element method*, J. Integral Equations Appl., 8 (1996), pp. 337–361.

[71] N. Heuer, M. Maischak, and E. P. Stephan, *Exponential convergence of the hp-version for the boundary element method on open surfaces*, Numer. Math., 83 (1999), pp. 641–666.

[72] N. Heuer and E. P. Stephan, *Coupling of the finite element and boundary element method – the h-p version*, Z. Angew. Math. Mech., 71 (1991), pp. T 584–T 586.

[73] ——, *The hp-version of the boundary element method on polygons*, J. Integral Equations Appl., 8 (1996), pp. 173–212.

[74] ——, *Boundary integral operators in countably normed spaces*, Math. Nachr., 191 (1998), pp. 123–151.

[75] N. Heuer, E. P. Stephan, and T. Tran, *Multilevel additive Schwarz method for the h-p version of the Galerkin boundary element method*, Math. Comp., 67 (1998), pp. 501–518.

[76] D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, no. 3 in Fortschritte der Mathematischen Wissenschaften in Monographien, Teubner-Verlag, Leipzig, Berlin, 1924.

[77] S. Hildebrandt and E. Wienholtz, *Constructive proofs of representation theorems in separable Hilbert space*, Comm. Pure Appl. Math., XVII (1964), pp. 369–373.

[78] H. Hönl, A. W. Maue, and K. Westpfahl, *Theorie der Beugung*, in Handbuch der Physik, S. Flügge, ed., vol. 25/1, Berlin, 1961, Springer-Verlag.

[79] G. C. Hsiao, *The coupling of BEM and FEM – a brief review*, in Boundary Elements X, Vol. 1, C. A. Brebbia et al., ed., Berlin, 1988, Springer-Verlag, pp. 431–445.

[80] ——, *Nonlinear problems via boundary element methods*, in IABEM Symposium on Boundary Integral Methods for Nonlinear Problems, L. Morino and W. L. Wendland, eds., Dordrecht, 1997, Kluwer Academic Publishers, pp. 115–120.

[81] G. C. Hsiao, B. N. Khoromskij, and W. L. Wendland, *Boundary integral operators and domain decomposition*, (1994). DFG Priority Research Boundary Element Methods, Preprint 94-11, Universität Stuttgart, Germany.

[82] G. C. Hsiao and R. C. MacCamy, *Solution of boundary value problems by integral equations of the first kind*, SIAM Rev., 15 (1973), pp. 687–705.

[83] G. C. Hsiao and W. L. Wendland, *A finite element method for some integral equations of the first kind*, J. Math. Anal. Appl., 58 (1977), pp. 449–481.

[84] ——, *The Aubin–Nitsche lemma for integral equations*, J. Integral Equations, 3 (1981), pp. 299–315.

[85] ——, *Domain decomposition in boundary element methods*, in Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, Y. A. Kuznetsov, G. A. Meurant, J. Périaux, and O. B. Widlund, eds., Philadelphia, 1991, SIAM, pp. 41–49.

[86] N. Hu, X.-Z. Guo, and I. N. Katz, *Bounds for eigenvalues and condition numbers in the p-version of the finite element method*, Math. Comp., 67 (1998), pp. 1423–1450.

[87] C. Johnson and J. C. Nedelec, *On the coupling of boundary integral and finite element methods*, Math. Comp., 35 (1980), pp. 1063–1079.

[88] R. E. Kleinman and W. L. Wendland, *On Neumann's method for the exterior Neumann problem for the Helmholtz equation*, J. Math. Anal. Appl., 57 (1977), pp. 170–202.

[89] J. J. Kohn and L. Nirenberg, *An algebra of pseudo-differential operators*, Comm. Pure Appl. Math., XVIII (1965), pp. 269–305.

[90] V. A. Kondratiev, *Boundary problems for elliptic equations in domains with conical or angular points*, Trans. Moscow Math. Soc., 16 (1967), pp. 227–313.

[91] W. D. Kupradze, *Randwertaufgaben der Schwingungstheorie und Integralgleichungen*, Deutscher Verlag der Wissenschaften, Berlin, 1956.

[92] U. Langer, *Parallel iterative solution of symmetric coupled FE/BE-equations via domain decomposition*, Contemp. Math., 157 (1994), pp. 335–344.

[93] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications I*, Springer-Verlag, New York, 1972.

[94] P. L. Lions, *On the Schwarz alternating method I*, in Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., Philadelphia, 1988, SIAM.

[95] R. C. MacCamy and E. P. Stephan, *A boundary element method for an exterior problem for three-dimensional Maxwell's equations*, Appl. Anal., 16 (1983), pp. 141–163.

[96] Y. Maday, *Relèvements de traces polynomiales et interpolations Hilbertiennes entre espaces de polynômes*, C. R. Acad. Sci. Paris Sér. I. Math., 309 (1989), pp. 463–468.

[97] M. Maischak, *The analytical computation of the Galerkin elements for the Laplace, Lamé and Helmholtz equation in 3D-BEM*, Tech. Rep. 95-16, DFG-Schwerpunkt Randelementmethoden, 1995.

[98] M. Maischak and E. P. Stephan, *The hp-version of the BEM with geometric meshes in 3D*, in Boundary Element Topics 1989–1995, Springer-Verlag, 1997, pp. 351–362.

[99] M. Maischak and E. P. Stephan, *The hp-version of the boundary element method in $\mathbb{R}^3$. The basic approximation results*, Math. Meth. Appl. Sci., 20 (1997), pp. 461–476.

[100] J. Mandel, *Two-level domain decomposition preconditioning for the p-version finite element method in three dimensions*, Internat. J. Numer. Methods Engrg., 29 (1990), pp. 1095–1108.

[101] J. Morrison and J. Lewis, *Singularity at the corner of a flat plate*, SIAM J. Appl. Math., 31 (1976), pp. 233–250.

[102] P. Mund and E. P. Stephan, *An adaptive two-level method for the coupling of nonlinear FEM-BEM equations*, SIAM J. Numer. Anal. To appear.

[103] ——, *The preconditioned GMRES method for systems of coupled FEM-BEM equations*, (1997). Preprint Institut für Angewandte Mathematik, Universität Hannover, Germany.

[104] P. Mund, E. P. Stephan, and J. Weisse, *Two level methods for the single layer potential in $\mathbb{R}^3$*, Computing, 60 (1998), pp. 243–266.

[105] J. Nečas, *Les Méthodes Directes en Théorie des Équations Elliptiques*, Academia, Prague, 1967.

[106] S. V. Nepomnyaschikh, *Domain Decomposition and Schwarz Methods in a Subspace for Approximate Solution of Elliptic Boundary Value Problems*, PhD thesis, Computing Center of the Siberian Branch of the USSR Academy of Sciences, Novosibirsk, USSR, 1986.

[107] C. Neumann, *Untersuchungen über das logarithmische und Newtonsche Potential*, Teubner-Verlag, Leipzig, 1877.

[108] J. A. Nitsche, *Ein Kriterium für die Quasi-Optimalität des Ritzschen Verfahrens*, Numer. Math., 11 (1968), pp. 346–348.

[109] J. T. Oden, A. Patra, and Y. Feng, *Parallel domain decomposition solver for adaptive hp finite element methods*, (1994). Preprint TICAM, Austin, Texas.

[110] P. Oswald, *Multilevel-splitted norms for $H^{-1/2}$*. Manuscript.

[111] ——, *Multilevel Finite Element Approximation. Theory and Applications*, Teubner Skripten zur Numerik, Teubner, Stuttgart, 1994.

[112] L. F. Pavarino, *Additive Schwarz methods for the p-version finite element method*, Numer. Math., 66 (1994), pp. 493–515.

[113] ——, *Schwarz methods with local refinement for the p-version finite element method*, Numer. Math., 69 (1994), pp. 185–211.

[114] L. F. Pavarino and O. B. Widlund, *A polylogarithmic bound for an iterative substructuring method for spectral elements in three dimensions*, SIAM J. Numer. Anal., 33 (1996), pp. 1303–1335.

[115] T. Rusten and R. Winther, *A preconditioned iterative method for saddlepoint problems*, SIAM J. Matrix Anal. Appl., 13 (1992), pp. 887–904.

[116] Y. Saad and M. H. Schultz, *GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems*, SIAM J. Sci. Stat. Comput., 7 (1986), pp. 856–869.

[117] S. A. Sauter, *Cubature techniques for 3-D Galerkin BEM*, in Boundary Elements: Implementation and Analysis of Advanced Algorithms, W. Hackbusch and G. Wittum, eds., vol. 54 of Notes on Numerical Fluid Mechanics, Braunschweig, Wiesbaden, 1996, Vieweg, pp. 29–44.

[118] S. A. Sauter and C. Schwab, *Quadrature for hp-Galerkin BEM in* 3-*d*, Tech. Rep. 96-02, ETH Zürich, Seminar for Appl. Math., 1996.

[119] A. H. Schatz, V. Thomée, and W. L. Wendland, *Mathematical Theory of Finite and Boundary Element Methods*, Birkhäuser, Basel, 1990.

[120] C. Schwab and M. Suri, *The optimal p-version approximation of singularities on polyhedra in the boundary element method*, SIAM J. Numer. Anal., 33 (1996), pp. 729–759.

[121] H. A. Schwarz, *Gesammelte Mathematische Abhandlungen*, vol. 2, Springer-Verlag, Berlin, 1890, pp. 133–143. First published in *Vierteljahrsschrift der Naturforschenden Gesellschaft* in Zürich, 15 (1870), pp. 272–286.

[122] R. Seeley, *Topics in pseudo-differential operators*, in Pseudo-Differential Operators, L. Nirenberg, ed., C.I.M.E., Roma, Cremonese, 1969, pp. 168–305.

[123] P. Seshaiyer and M. Suri, *Uniform hp convergence results for the mortar finite element method*, (1993). Preprint Department of Mathematics and Statistics, University of Maryland, Baltimore.

[124] D. J. Silvester and A. J. Wathen, *Fast iterative solution of stabilised Stokes systems, Part II: Using general block preconditioners*, SIAM J. Numer. Anal., 31 (1994), pp. 1352–1367.

[125] I. H. Sloan and A. Spence, *The Galerkin method for integral equations of the first kind with logarithmic kernel: Theory*, IMA J. Numer. Anal., 8 (1988), pp. 105–122.

[126] B. Smith, P. Bjørstad, and W. Gropp, *Domain Decomposition*, Cambridge University Press, 1996.

[127] O. Steinbach, *Fast solvers for the symmetric boundary element method*, in Boundary Elements: Implementation and Analysis of Advanced Algorithms, W. Hackbusch and G. Wittum, eds., vol. 54 of Notes on Numerical Fluid Mechanics, Braunschweig, Wiesbaden, 1996, Vieweg, pp. 232–242.

[128] E. P. Stephan, *Boundary integral equations for mixed boundary value problems, screen and transmission problems in* $\mathbb{R}^3$, Tech. Rep. 848, Technische Hochschule Darmstadt, Germany, 1984. Habilitation Thesis.

[129] ——, *Boundary integral equations for magnetic screens in* $\mathbb{R}^3$, Proc. Roy. Soc. Edinburgh, 102A (1986), pp. 189–210.

[130] ——, *Boundary integral equations for screen problems in* $\mathbb{R}^3$, Integral Equations Operator Theory, 10 (1987), pp. 257–263.

[131] ——, *Coupling of finite elements and boundary elements for some nonlinear interface problems*, Comput. Methods Appl. Mech. Engrg., 101 (1992), pp. 61–72.

[132] ——, *Additive Schwarz methods for integral equations of the first kind*, in The Mathematics of Finite Elements and Applications, J. R. Whiteman, ed., Chichester, 1997, Wiley, pp. 123–144.

[133] E. P. Stephan and M. Suri, *On the convergence of the p-version of the boundary element method*, Math. Comp., 52 (1989), pp. 31–48.

[134] ——, *The h-p version of the boundary element method on polygonal domains with quasiuniform meshes*, RAIRO Modél. Math. Anal. Numér., 25 (1991), pp. 783–807.

[135] E. P. Stephan and T. Tran, *Domain decomposition algorithms for indefinite weakly singular integral equations – the h- and p-versions.* Preprint Institut für Angewandte Mathematik, Universität Hannover.

[136] ——, *Domain decomposition algorithms for indefinite hypersingular integral equations – the h- and p-versions*, SIAM J. Sci. Comput., 19 (1998), pp. 1139–1153.

[137] E. P. Stephan and A. J. Wathen, *Convergence of preconditioned minimum residual iteration for coupled finite element / boundary element computations*, University of Bristol, Mathematics Department Report No. AM-94-03, (1994). To appear in SIAM J. Numer. Anal.

[138] E. P. Stephan and W. L. Wendland, *Remarks to Galerkin and least squares methods with finite elements for general elliptic problems*, Manuscripta Geodaetica, 1 (1976), pp. 93–123.

[139] G. Strang and G. J. Fix, *An analysis of the finite-element method*, Prentice-Hall, Englewood Cliffs, N.J., 1973.

[140] M. Taylor, *Pseudodifferential Operators*, Princeton University Press, Princeton, N. J., 1981.

[141] T. Tran and E. P. Stephan, *Additive Schwarz method for the h-version boundary element method*, Appl. Anal., 60 (1996), pp. 63–84.

[142] ——, *Additive Schwarz algorithms for the p-version of the Galerkin boundary element method.* Preprint Institut für Angewandte Mathematik, Universität Hannover, Germany, 1997.

[143] T. von Petersdorff, *Randwertprobleme der Elastizitätstheorie für Polyeder – Singularitäten und Approximation mit Randelementmethoden*, PhD thesis, Technische Hochschule Darmstadt, Germany, 1989.

[144] T. von Petersdorff, C. Schwab, and R. Schneider, *Multiwavelets for second kind integral equations*, SIAM J. Numer. Anal., 34 (1997), pp. 2212–2227.

[145] T. von Petersdorff and E. P. Stephan, *Regularity of mixed boundary value problems in $\mathbb{R}^3$ and boundary element methods on graded meshes*, Math. Methods Appl. Sci., 12 (1990), pp. 229–249.

[146] A. J. Wathen and D. J. Silvester, *Fast iterative solution of stabilised Stokes systems, Part I: Using simple diagonal preconditioners*, SIAM J. Numer. Anal., 30 (1993), pp. 630–649.

[147] W. L. Wendland, *Die Behandlung von Randwertproblemen im $\mathbb{R}_3$ mit Hilfe von Einfach- und Doppelschichtpotentialen*, Numer. Math., 11 (1968), pp. 380–404.

[148] ——, *On asymptotic error estimates for the combined BEM and FEM*, in Innovative Numerical Methods in Engineering, R. P. Shaw et al., ed., Berlin, 1986, Springer-Verlag, pp. 55–70.

[149] W. L. Wendland, E. P. Stephan, and G. C. Hsiao, *On the integral equation method for the plane mixed boundary value problem of the Laplacian*, Math. Methods Appl. Sci., 1 (1979), pp. 265–321.

[150] O. B. Widlund, *An extension theorem for finite element spaces with three applications*, in Numerical Techniques in Continuum Mechanics: Notes on Numerical Fluid Mechanics, W. Hackbusch and K. Witsch, eds., vol. 16, Braunschweig, Wiesbaden, 1986, Vieweg, pp. 110–122.

[151] ——, *Optimal iterative refinement methods*, in Domain Decomposition Methods for Partial Differential Equations, T. F. Chan, R. Glowinski, J. Périaux, and O. B. Widlund, eds., Philadelphia, 1989, SIAM, pp. 114–125.

[152] J. Xu and X.-C. Cai, *A preconditioned GMRES method for nonsymmetric or indefinite problems*, Math. Comp., 59 (1992), pp. 311–319.

[153] J. Xu and S. Zhang, *Preconditioning the Poincaré-Steklov operator by using Green's function*, Math. Comp., 66 (1997), pp. 125–138.

[154] X. Zhang, *Multilevel Schwarz methods*, Numer. Math., 63 (1992), pp. 521–539.

[155] E. P. Zhidkov and B. N. Khoromskij, *Boundary integral equations on special surfaces and their applications*, Sov. J. Numer. Anal. Math. Model., 2 (1988), pp. 463–488.