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Ramification theory of the p-adic open disc and the lifting problem

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Introduction

In this thesis we study Galois covers of algebraic curves. In characteristic 0, the Galois theory of curves is well understood. This is in strong contrast to the situation in characteristic p where less is known. One technique for studying Galois covers in characteristic p is to relate them to the Galois theory of curves in characteristic 0. Reducing Galois covers of curves in characteristic 0 to characteristic p is one way to prove existence of Galois covers in characteristic p with a given Galois group. In the opposite direction, the lifting problem asks which covers of curves in characteristic p *lift* to characteristic 0. It is known that not all Galois covers in characteristic p lift to characteristic 0. In our work we prove results regarding the liftability of Galois covers of algebraic curves. For some Galois covers in characteristic 2, we explicitly construct lifts to characteristic 0. We also introduce a new necessary condition for the liftability of Galois covers. We then use our new necessary condition to show that certain Galois covers do not lift to characteristic 0.

Background

It is known that the set of Galois covers of a punctured Riemann surface can be described in terms of its topological fundamental group. Indeed, if S is a Riemann surface of genus g with n punctures, then the quotients of its topological fundamental group $\Gamma_{g,n}$ correspond to the Galois covers of S. An explicit description of the fundamental group $\Gamma_{g,n}$ is known in terms of the genus g and the number of punctures n, namely it is the free profinite group on $2g+n-1$ generators for $n>0$.

This theory can also be used to describe the Galois theory of curves defined over the complex numbers. This is accomplished by using algebraization techniques, which essentially state that there is an equivalence between the category of compact Riemann surfaces and the category of smooth projective algebraic curves defined over the complex numbers. This correspondence respects finite branched covers. Furthermore, one knows that inertia groups of a characteristic-0 Galois cover $D \to C$, i.e. the stabilizers of the fixed points of D, are always cyclic.

The Galois theory of curves becomes more difficult in characteristic p. One problem is that the classical analytic techniques used in the study of Riemann surfaces cannot be used in characteristic p. However, there exists an algebraic definition of the fundamental group of a punctured curve which works over any base field. In the case of a genus-g curve over the complex numbers with n punctures, this algebraic fundamental group is the profinite completion of the topological fundamental group $\Gamma_{g,n}$. For more details on this see Grothendieck et al [13].

One knows that the Galois theory of a curve in characteristic p can be described completely via the algebraic fundamental group. A serious problem is that in general an explicit description of the fundamental group is not known. Abhyankar's conjecture, proved by Raynaud [34] and Harbater [17], explicitly states which groups occur as Galois groups of curves in characteristic p. However, simply knowing which groups can occur as Galois groups is not enough to determine the structure of the algebraic fundamental group. As an example of the subtleties in characteristic p, one knows that the affine line $\mathbb{A}^1$ is no longer simply connected. Another subtlety is that the inertia groups of a cover in characteristic p need not be cyclic. In general they are extensions of cyclic by p-groups (see Serre [37]).

Grothendieck has studied the situation when one restricts to *tame* covers of curves. These are covers of which the inertia groups have order relatively prime to p. In this case the inertia groups are always cyclic and the theory is completely described in terms of the *tame* fundamental group. Grothendieck proved that the tame fundamental group is a quotient of the topological fundamental group $\Gamma_{g,n}$, where g is the genus of the curve and n is the number of branch points.

The lifting problem

One technique for obtaining information about the Galois theory in characteristic p is by *reduction*. Let k be a field of characteristic p and let R be a discrete valuation ring of characteristic 0 with residue field k. It is helpful to think of $k = \mathbb{F}_p$ and $R = \mathbb{Z}_p$, the ring of p-adic integers. Let $\mathcal{D} \to \mathcal{C}$ be a Galois cover of curves defined over R. The curves $\mathcal{D}$ and $\mathcal{C}$ as well as the finite map between them are given by polynomial equations. One may now consider the k-varieties which result when one reduces the defining polynomials modulo the prime of R.

However, these are in general not smooth. If the singularities are as mild as possible, i.e. ordinary double points, then we say that the cover has *semistable reduction*. It is known that after extending R, there exists a suitable set of defining polynomials such that the reduction is semistable (this can be made more precise). If the reduction happens to be a separable cover $D \to C$ of smooth irreducible curves, then we say that the cover $\mathcal{D} \to \mathcal{C}$ has *good* reduction. One knows that in this case the reduction $D \to C$ is also Galois. For more information on this see Liu [24].

One can also ask to go in the opposite direction, namely given a Galois cover $D \to C$ of curves over k, does there exist a Galois cover $\mathcal{D} \to \mathcal{C}$ of curves over R which reduces to $D \to C$? If this is the case then we say that the cover $D \to C$ *lifts* to characteristic 0, and the cover $\mathcal{D} \to \mathcal{C}$ is a lift of $D \to C$.

Lifting and reduction is one way of relating the Galois theory in characteristic 0 to that in characteristic p. Several interesting questions arise in this context. The first more general but very difficult question is to ask whether or not there is a necessary and sufficient condition for a Galois cover to be liftable.

Question A *Do there exist necessary and sufficient conditions for a cover in characteristic p to be liftable?*

The Oort conjecture states that all cyclic covers lift from characteristic p to characteristic 0 (see [30]). A weaker question is to ask, given a group G, does there exist some G-Galois cover of curves in characteristic p which lifts to characteristic 0. Matignon [27] asked what the situation in this context is for nonabelian p-groups. More precisely, he asked the following question.

Question B *Does there exist a G-Galois cover in characteristic p which lifts to characteristic 0, where G is a nonabelian group of order p^3?*

A slightly different question was posed by Chinburg, Guralnick and Harbater [8] for the generalized quaternion groups.

Question C *Do all G-Galois covers in characteristic 2 lift to characteristic 0, where G is a generalized quaternion group of order exceeding 8?*

Grothendieck's result on the tame fundamental group proves that all G-covers lift if the order of G is relatively prime to p, the characteristic of k. However, the lifting problem becomes very difficult if the order of G is a multiple of p. Several results are known for the lifting problem. Sekiguchi, Oort and Suwa [36] proved the Oort conjecture for the group $G = \mathbb{Z}/p\mathbb{Z}$, and this result was later extended by Green and Matignon [15] for the case $G = \mathbb{Z}/m\mathbb{Z}$, where p^2 strictly divides m. Pagot [32] also proved that all Klein four Galois covers lift to characteristic 0. Later, Bouw and Wewers [4] proved that all D_p-covers lift, where p is an odd prime.

In our work we shall give partial answers to the question of Matignon and the question of Chinburg, Guralnick and Harbater. We shall identify a family of D_4-actions in characteristic 2, where D_4 is the dihedral group of order 8, which lift to characteristic 0 (work taken from Brewis [5]). We shall also give examples of generalized quaternion actions which do not lift to characteristic 0 (work taken from Brewis–Wewers [6]). This is done by introducing a new necessary condition for liftability, namely the *Hurwitz-tree* condition. Thereafter we show that our necessary condition always holds for cyclic actions in characteristic p. This provides some new evidence for the validity of the Oort conjecture.

Our contributions

Let C_k/k be a curve over a field k of characteristic p and let $G \hookrightarrow \mathrm{Aut}_k(C_k)$ be a G-action on C_k. One says this action *lifts* to characteristic 0 if there exists a local ring R of characteristic 0 with residue field k, a smooth R-curve C_R/R together with a map $G \hookrightarrow \mathrm{Aut}_R(C_R)$ which reduces to the given G-action on C_k. This is the *global lifting problem* for the group G.

Similarly one has the *local lifting problem*: let $G \hookrightarrow \mathrm{Aut}_k(k[\![t]\!])$, a so-called *local* G-action. We ask when one can find an embedding $G \hookrightarrow \mathrm{Aut}_R(R[\![T]\!])$ reducing to the given one. By considering inertia subgroups one sees that each global lifting problem induces, by localisation and completion at each ramification point, several local lifting problems. In fact, the *local-global* principle of Green–Matignon [15] states that these two types of problems are equivalent. For more information on this see also Bertin–Mézard [2] or Henrio [19].

In Henrio [19], following ideas of Green and Matignon [16], a new understanding of the lifting problem in the case of $\mathbb{Z}/p\mathbb{Z}$-actions was given in terms of the so-called *Hurwitz trees*. Let $G := \mathbb{Z}/p\mathbb{Z}$ act on the p-adic open disc $\mathsf{Y} := \mathrm{Spf}(R[\![z]\!])$, where R is an extension of $W(k)$. Then Henrio associated a combinatorial object, the Hurwitz tree, with the action. The Hurwitz tree is an object that is defined purely in characteristic p by a semistable k-curve and differentials of the individual components of the semistable curve. It reflects the relative positions of the geometric branch points as well as the ramification theory locally around each branch point. Henrio also proved that each

Hurwitz tree is induced by a $\mathbb{Z}/p\mathbb{Z}$-action on the open disc. This technique was later exploited by Bouw and Wewers [4], who generalized the Hurwitz-tree theory to the group D_p where $p \neq 2$, and they used this to prove that all local D_p-actions lift from characteristic p to characteristic 0.

In this thesis we study the ramification theory of the p-adic open disc in terms of representation theory and we combine this with the techniques of Kato [21], [22], Kato–Saito [23] and Huber [20]. These techniques were developed to study the higher local class field theory of two-dimensional local fields (the boundary of the p-adic open disc for instance). This provides new insight into the local lifting problem.

Using Kato's theory, in particular his *differential Swan conductor*, one can associate differential forms with an action on the p-adic open disc, and many properties are known. We also associate ramification groups with such an action. These can be related to Huber's *Artin* and *depth* characters. In our work we shall prove some relations between the representations of the ramification groups, Kato's differential Swan conductor and Huber's Artin and depth characters (Chapter two).

We partially succeeded in generalizing the Hurwitz-tree concept to arbitrary p-groups (Chapter three). The crucial new ingredient in our construction is the systematic use of the Artin and depth characters. This leads to a new necessary condition for the liftability of a local action in characteristic p, called the *Hurwitz-tree* obstruction. If our new necessary condition holds, then we say that the Hurwitz-tree obstruction *vanishes*.

Theorem 3.6.2 *Let $G \hookrightarrow Aut_k(k[\![z]\!])$ be a local G-action, where G is a p-group. Let a_G be the classical Artin character of the action. If the action lifts to characteristic 0, then there exists a Hurwitz tree $\mathcal{T}$ with depth 0 and Artin character a_G.*

This condition is of a representation-theoretic flavour, and by studying the representation theory of the generalized quaternion group more closely, we are able to answer Question 1.3 of Chinburg–Guralnick–Harbater [8] negatively.

Theorem 3.6.6 *There exist local generalized-quaternion actions in characteristic 2 for which the Hurwitz-tree condition does not vanish. Hence, there exist local generalized-quaternion actions in characteristic 2 which do not lift to characteristic 0.*

We also show that our new necessary condition holds for all $\mathbb{Z}/p^n\mathbb{Z}$-actions.

Theorem 4.3.4 *Let $G \hookrightarrow Aut_k(k[\![z]\!])$ be a local G-action, where $G = \mathbb{Z}/p^n\mathbb{Z}$. Let a_G be the Artin character of this action. Then there exists a Hurwitz tree $\mathcal{T}$ for the group $\mathbb{Z}/p^n\mathbb{Z}$ with depth 0 and Artin character a_G. Hence the Hurwitz-tree obstruction vanishes for cyclic actions in characteristic p.*

This provides some new evidence for the validity of the Oort conjecture.

A crucial question that remains is whether each Hurwitz tree is induced by an action on the p-adic open disc. We highlight some of the problems that one encounters when one attempts to give a full generalization of Henrio's theory using Kato's differential Swan conductor (Chapter five). We explore the differential Swan conductor more thoroughly with examples in the case of $\mathbb{Z}/p^2\mathbb{Z}$-Galois extensions and we prove some properties of the differentials that occur as differential Swan conductors.

Finally, we study liftability of D_4-actions in characteristic 2 (Chapter six). More precisely, we study the so-called *supersimple* local D_4-actions. These are actions which can be compactified, using the Katz–Gabber compactification, into covers $D \to \mathbb{P}^1_k$, where D/H is a projective line and H is the centralizer of D_4. Our theorem is then that all supersimple actions lift to characteristic 0.

Theorem 6.5.1 *All supersimple local D_4-actions in characteristic 2 lift to characteristic 0.*

Furthermore, the genus of D is not bounded from above.

Overview

In the first chapter, we recall the definition of Kato's differential Swan conductor and its relation to the differentials that Henrio uses to study $\mathbb{Z}/p\mathbb{Z}$-extensions of two-dimensional local fields (Lemma 1.4.2 and Lemma 1.4.3). This is done in Sections 1.3.1 – 1.6. We also gather some results on the differential Swan conductor from Kato [22]. Lastly in Section 1.7 we study the differential Swan conductor explicitly in the cases of $(\mathbb{Z}/p\mathbb{Z})^2$ and generalized quaternion Galois extensions.

In Chapter two we study ramification filtrations of a Galois cover of p-adic open discs. We recall the filtration that Kato and Saito [23] introduced. Furthermore, in Section 2.3 we simplify their filtration into what we call the *simplified* ramification filtration. Following the classical approach (Serre [37]), we prove some structural results on the quotients of the simplified ramification filtration (Theorem 2.3.16).

In Section 2.4 we shall also introduce Huber's Artin and depth characters. We then prove a powerful relation between these, Kato's differential Swan conductor, and the ramification filtration of Kato and Saito (Theorem 2.4.5 and Theorem 2.4.11).

Still restricting to Galois covers of p-adic open discs, we build the theory of Hurwitz trees in Chapter three. The crucial new ingredient in this construction is the use of the Artin and depth characters introduced in Chapter two. We introduce a new necessary condition for the liftability of a local action in characteristic p (see Theorem 3.6.2). We call this the *Hurwitz-tree obstruction*, and we say that if the necessary condition holds, then the Hurwitz-tree obstruction vanishes. We also apply our new theory in Section 3.6.6 to study actions of generalized quaternion groups. Lastly, in Section 3.7 we generalize an old theorem of Green and Matignon [15] on the branch points of $(\mathbb{Z}/p\mathbb{Z})^2$-covers of p-adic open discs.

Chapter four deals exclusively with local cyclic actions in characteristic p. We prove that the Hurwitz-tree obstruction vanishes for all local $\mathbb{Z}/p^n\mathbb{Z}$-actions in characteristic p (Theorem 4.3.4).

The fifth chapter serves as an illustration of the problems encountered when one attempts to generalize Henrio's work for arbitrary p-groups. We sketch three problems relating to the values of Kato's differential Swan conductors (Section 5.2), to which extend they classify Galois extensions up to conjugation (Section 5.3), and what the role of simplified ramification filtration is in the theory of Hurwitz trees (Section 5.4). We hope that these ideas will serve as guidelines for future studies of Hurwitz trees.

Lastly in the sixth chapter we consider the problem of Matignon. We study a particular class of D_4-actions in characteristic 2, namely the *supersimple* D_4-actions. Here D_4 denotes the dihedral group of order 8. The chapter culminates in a theorem stating that all supersimple D_4-actions lift from characteristic 2 to characteristic 0 (Theorem 6.5.1).

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Chapter 1

Swan conductors I : Kato's differential character

Consider a G-Galois extension E/F of complete discrete valuation fields. In this chapter we shall introduce Kato's differential Swan conductor. In order to do this we first define the type of field extensions we shall be interested in, namely the so-called *Case-II* type extensions (Definition 1.2.1).

In Section 1.3.1 we then define the group $\tilde{S_E}$ in which the differential Swan conductor will take its values. This group is an extension of the group $\overline{E}^*$, where $\overline{E}$ denotes the residue field of E. We also define an order function on this group, which extends the order function on $\overline{E}$ in the case that $\overline{E}$ is also a discrete valuation field.

We then turn to Kato's differential Swan conductor in Section 1.3.3. Following this, we discuss and compute the differential Swan conductor explicitly in the case of $\mathbb{Z}/p\mathbb{Z}$-Galois extensions in Section 1.4. In this case the differential Swan conductor corresponds to the differentials that Henrio [19] associates to $\mathbb{Z}/p\mathbb{Z}$-Galois extensions. Thereafter we state Kato's very important version of the *Hasse–Arf* theorem (see Theorem 1.5.1). We conclude with examples, where the differential Swan conductor is related to those of intermediate cyclic extensions in the case that $G = (\mathbb{Z}/p\mathbb{Z})^2$ or G is the quaternion group.

1.1 Notation

In this chapter we let K be a *complete* discrete valuation field of characteristic 0 with *perfect* residue field k of characteristic p. We shall denote the ring of integers of K by R. We fix a p^{th}-root of unity $\zeta_p \in K$, and we define $\lambda := \zeta_p - 1 \in R$. We shall also denote by v_K the valuation on the field K, and we denote by π_K a parameter of K. We assume that v_K has been normalized such that $v_K(\pi_K) = 1$.

We let F be a complete discrete valuation field with residue field $\overline{F}$ and with parameter π_F. We denote by $\mathcal{O}_F$ the ring of integers of F. We shall assume that F contains the discrete valuation field K. Furthermore, the valuation $v_F : F^* \to \mathbb{Z}$ on F is assumed to extend the valuation $v_K : K^* \to \mathbb{Z}$ on K, i.e. that $v_F|_{K^*} = v_K$. We shall assume that $\pi_F = \pi_K$, i.e. we assume that the parameter of K is also a parameter of F.

Notice that by definition we have an embedding of characteristic- p fields

$$k \subset \overline{F}.$$

When we refer to a finite Galois extension $F \subset E$, we shall write $\mathcal{O}_E$ for the ring of integers of the (complete) discrete valuation field E. We shall write $\overline{E}$ for the residue field of E and $\overline{e} \in \overline{E}$ for the reduction of an element $e \in \mathcal{O}_E$.

1.2 Assumption and setting

In [21], Kato developed his theory for two cases of Galois extensions of the field F. In our work we shall primarily be concerned with the second case that he considered, namely, the so-called Galois extensions of *Case-II* type.

1.2.1 Definition. Let $F \subset E$ be a Galois extension. We say that the extension is of *Case-II* type if the induced extension $\overline{F} \subset \overline{E}$ is purely inseparable, has the same degree as $F \subset E$, i.e.

$$[\overline{E} : \overline{F}] = [E : F],$$

and furthermore, that the field extension $\overline{F} \subset \overline{E}$ is generated by one element, i.e. there exists at least one element $y \in \mathcal{O}_E$ such that

$$\overline{E} = \overline{F}(\overline{y}),$$

where $\overline{y}$ denotes the reduction of y. Such an element is called a *generator* of the Case-II type extension $F \subset E$.

1.2.2 Remark. Notice that if $F \subset E$ is a Galois extension of Case-II type with generator $y \in E$, then it follows that $y \in \mathcal{O}_E^*$.

1.2.3 Remark. For a Galois extension $F \subset E$ of Case-II type, the local parameter π_F of F is also a local parameter of E.

In this chapter, we shall restrict to discrete valuation fields with residue fields which satisfy the following condition.

1.2.4 Assumption. The residue field $\overline{F}$ of F is either the function field of a smooth k-curve, or a local power series field over k, i.e.

$$\overline{F} \simeq k((t)).$$

1.2.5 Remark. Let $F \subset E$ be a finite extension of discrete valuation fields. Assuming that F satisfies Assumption 1.2.4, one sees that so does E.

1.2.6 Remark. Notice that the $\overline{F}$-module of absolute differentials $\Omega_{\overline{F}}$ is generated by one element, i.e. is a one-dimensional vector space over $\overline{F}$.

In the case that $\overline{F}$ is the function field of a smooth k-curve, the following lemma will be useful later on. To make it clear we also give its proof.

1.2.7 Lemma. *Let $k(x) \subset L$ be a purely inseparable extension of fields of degree p^n. Then there exists a $y \in L$ such that $L = k(y)$, and furthermore, such that $y^{p^n} = x$.*

PROOF. We start with the case $n = 1$. In this case, there exists a $z \in L$, and a $f = \frac{\sum a_i x^i}{\sum b_i x^i} \in k(x)$ such that $L = k(x)[z]$, where z satisfies

$$z^p = f = \frac{\sum a_i x^i}{\sum b_i x^i}.$$

Consider the field $M := k(x)[y]$, where y satisfies

$$y^p = x.$$

Notice that M contains an element g, where $g^p = f$.

Therefore, M contains the field L, and since both are of degree p, we see that $L = M = k(y)$. The general case follows by induction on n, and writing $k(x) \subset L$ as a tower of degree-p inseparable extensions

$$k(x) \subset L_1 \subset \ldots\ldots \subset L. \qquad \square$$

1.2.8 Remark. A slightly more general result can be found in Liu [24] Proposition 7.4.21.

Sometimes, we shall also deal with the case that $\overline{F}$ is local power series field. We leave the proof of the following proposition for the reader.

1.2.9 Proposition. *Let $k((x)) \subset L$ be a purely inseparable extension of degree p^n. Then there exists a $y \in L$ such that $L := k((y))$, and furthermore, such that $x := y^{p^n}$.*

1.3 Kato's Swan conductor

1.3.1 The value group of Kato's Swan conductor

In this section we shall work only with Case-II type extensions of F, where F is a complete discrete valuation field assumed only to satisfy Assumption 1.2.4. Following Kato [21], for a p-extension E/F of Case-II type we define the abelian group S_E to be the group of units of the $\overline{E}$-algebra

$$A_E := \bigoplus_{i,j \in \mathbb{Z}} \mathrm{m}_E^i \,/\, \mathrm{m}_E^{i+1} \otimes_{\overline{E}} \Omega_{\overline{E}}^{\otimes j},$$

where $\overline{E}$ denotes the residue field of the discrete valuation field E, and m_E the maximal ideal of the discrete valuation field E.

We shall write $[\mathrm{d}y]$ for the class of the differential $\mathrm{d}y \in \Omega_{\overline{E}}$ inside the group S_E. Furthermore, if $f \in \mathcal{O}_E$, then there exists an unique $n \in \mathbb{Z}$ such that $f \in \mathrm{m}_E^n$ but $f \notin \mathrm{m}_E^{n+1}$. We shall then simply write $[f] \in S_E$ for the class of $f \mod \mathrm{m}_E^{n+1}$.

For a tower $F \subset M \subset E$ of Case-II type extensions of F, we may define an injection $S_M \hookrightarrow S_E$ as follows. First of all, since π_K is a parameter of both M and E, there exists a canonical mapping of $\mathrm{m}_M^i \,/\, \mathrm{m}_M^{i+1} \to \mathrm{m}_E^i \,/\, \mathrm{m}_E^{i+1}$ for each $i \in \mathbb{Z}$. Next we define a map $\Omega_{\overline{M}} \to \Omega_{\overline{E}}^{\otimes [E:M]}$. Namely, for an element $x \in \overline{M}$, we map

$$[\mathrm{d}x] \mapsto [(\mathrm{d}y)^{\otimes [E:M]}]$$

where $y \in \overline{E}$ is the unique element such that $y^{[E:M]} = x$ inside $\overline{E}$.

Assume that E/F is a Case-II type G-Galois where G is a finite p-group. Let $y \in \mathcal{O}_E$ be a generator (see Definition 1.2.1) for the extension. We define Kato's Swan character $\mathrm{sw}_{E/F}$ with values in S_E as the function

$$\mathrm{sw}_{E/F}(\sigma) = [\mathrm{d}y] - [y - \sigma y]$$

for $\sigma \neq 1$ and

$$\mathrm{sw}_{E/F}(1) = -\sum_{\sigma \neq 1} \mathrm{sw}_{E/F}(\sigma).$$

1.3.1 Remark. The definition of $\mathrm{sw}_{E/F}$ is independent of the choice of y, see Kato [21] p.319 for details.

1.3.2 Definition. We define the *different* of $F \subset E$ as the element

$$\mathcal{D}_{E/F} := -\sum_{\sigma \neq 1} \mathrm{sw}_{E/F}(\sigma) \in S_E.$$

1.3.2 Order functions

For this section only we assume that the residue field $\overline{F}$ is a discrete valuation field, i.e. a local power series field over k. Therefore the space of differentials $\Omega_{\overline{E}}^{\otimes j}$ has a natural order function $\mathrm{ord}_{\overline{E}}$. We can now define three order functions $\mathrm{ord}_{E,\pi}$, $\mathrm{ord}_{E,\Omega}$ and $\mathrm{ord}_{E,\overline{E}}$ on the group S_E.

1.3.3 Definition. Namely, for an element $u := [\pi^n] - [\omega] \in S_E$ with $\omega \in \Omega_{\overline{E}}^{\otimes j}$, we define

$$\mathrm{ord}_{E,\pi}(u) := n, \quad \mathrm{ord}_{E,\Omega}(u) := j, \quad \mathrm{ord}_{E,\overline{E}}(u) := \mathrm{ord}_{\overline{E}}(\omega).$$

Consider the group $\tilde{S_E} := S_E \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}$. We may extend the three order functions above to order functions of $\tilde{S_E}$ by considering the bilinear functions $S_E \times \tilde{\mathbb{Z}} \to \tilde{\mathbb{Z}}$

$$(u, \alpha) \mapsto \mathrm{ord}_{E,\pi}(u) \cdot \alpha, \quad (u, \alpha) \mapsto \mathrm{ord}_{E,\Omega}(u) \cdot \alpha, \quad (u, \alpha) \mapsto \mathrm{ord}_{E,\overline{E}}(u) \cdot \alpha$$

where $u \in S_E$ and $\alpha \in \mathbb{Z}$. In this way we obtain three order functions, also denoted by $\mathrm{ord}_{E,\pi}$, $\mathrm{ord}_{E,\Omega}$ and $\mathrm{ord}_{E,\overline{E}}$, on the group $\tilde{S_E}$.

1.3.4 Remark. Consider a sum $u := \sum_i u_i \otimes \alpha_i \in \tilde{S_E}$. Assume that $u \in S_E$, i.e. that $u = v \otimes 1$ with $v \in S_E$. If

$$\mathrm{ord}_{E,\pi}(u_i) = \mathrm{ord}_{E,\Omega}(u_i) = \mathrm{ord}_{E,\overline{E}}(u_i) = 0$$

for each i, then

$$\mathrm{ord}_{E,\pi}(v) = \mathrm{ord}_{E,\Omega}(v) = \mathrm{ord}_{E,\overline{E}}(v) = 0.$$

1.3.5 Remark. It seems that the group $\tilde{S_E} := S_E \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}$ is the value group of some mysterious arithmetic object E' which contains the elements of the form x^α, where $x \in E$ and $\alpha \in \tilde{\mathbb{Z}}$.

1.3.3 Kato's Swan conductor

In order to define Kato's Swan conductor, we now introduce a normalization term $\epsilon(\zeta_p)$. Let $r \in \mathbb{Z}$ be relatively prime to p. For an extension M/F of Case-II type we define

$$\epsilon_M(\zeta_p^r) := \sum_{a \in \mathbb{F}_p^*} [a] \otimes \zeta_p^{ar} \in \tilde{S_M}.$$

One checks that under the embedding $\tilde{S_F} \hookrightarrow \tilde{S_M}$ the element $\epsilon_F(\zeta_p^r) \in \tilde{S_F}$ maps to $\epsilon_M(\zeta_p^r) \in \tilde{S_M}$, and therefore we shall make no further use of the subscript emphasizing the field F.

1.3.6 Lemma. *For any $r \in \mathbb{Z}$ relatively prime to p, we have that $\epsilon(\zeta_p) + [r] \otimes 1 = \epsilon(\zeta_p^r)$ inside the group $\tilde{S_F}$.*

Finally we can define *Kato's Swan conductor*.

1.3.7 Definition. Let χ be a character of the group G. We define the *Swan conductor* $\mathrm{Sw}_{E/F}(\chi)$ of χ as

$$\mathrm{Sw}_{E/F}(\chi) := \sum_{\sigma \in G} \mathrm{sw}_{E/F}(\sigma) \otimes \chi(\sigma) + \chi(1) \cdot \epsilon(\zeta_p) \in \tilde{S_E}.$$

Let us state some important properties of Kato's Swan conductor.

1.3.8 Theorem (Kato [21] Proposition 3.3). *Let $H \triangleleft G$ be a normal subgroup and denote by $M := E^H$ the fixed field of E under H. Then $M \subset E$ is a Galois extension which is also of Case-II type. Let χ be a character of H. Then the following identity holds.*

$$\mathrm{Sw}_{E/F}(\mathrm{Ind}_H^G \chi) = [G : H] \cdot \Big(\mathrm{Sw}_{E/M}(\chi) + \chi(1) \cdot \mathcal{D}_{M/F}\Big).$$

1.3.9 Theorem (Kato [21] Proposition 3.3). *Assume the notation of Theorem 1.3.8. Let χ be a character of G/H and denote by $\chi|_G$ the restriction of χ to $G \to G/H$. Then we have the following identity.*

$$\mathrm{Sw}_{E/F}(\chi|_G) = \mathrm{Sw}_{M/F}(\chi).$$

1.3.4 Functorial properties

Let $F \subset F'$ be an extension of complete discrete valuation fields, not necessarily assumed to be finite, and both satisfying Assumption 1.2.4. Assume that the parameter π_F of F is also a parameter of F', and furthermore, that the extension $F \subset F'$ induces an embedding of $\overline{F}$-vector spaces

$$\Omega_{\overline{F}} \hookrightarrow \Omega_{\overline{F'}}.$$

1.3.10 Remark. The condition that $\Omega_{\overline{F}}$ embeds into $\Omega_{\overline{F'}}$ is not automatic, since the extension $\overline{F} \subset \overline{F'}$ may be purely inseparable and nontrivial, in which case the induced map $\Omega_{\overline{F}} \hookrightarrow \Omega_{\overline{F'}}$ is simply the zero map, i.e. not injective.

Let $F \subset E$ denote a Galois extension of Case-II type. We define $E' := F'E$ and our aim is to study the extension $F' \subset E'$.

Notice that we have an embedding

$$\Omega_{\overline{E}} \hookrightarrow \Omega_{\overline{E'}}.$$

The assumption that π_F is a parameter for both F and F' implies that we can define embeddings $i_F^{F'} : S_F \hookrightarrow S_{F'}$ and $i_E^{E'} : S_E \hookrightarrow S_{E'}$. Furthermore, one checks that the extension $F' \subset E'$ is Galois and still of Case-II type, and that the following diagram is commutative

$$\begin{array}{ccc} S_F & \xrightarrow{i_F^{F'}} & S_{F'} \\ \Big\downarrow{\scriptstyle i_F^E} & & \Big\downarrow{\scriptstyle i_{F'}^{E'}} \\ S_E & \xrightarrow{i_E^{E'}} & S_{E'}. \end{array}$$

Let χ be a virtual character of the group $G := \mathrm{Gal}(E/F)$. Since the field extensions $F \subset E$ and $F \subset F'$ are linearly disjoint, one sees that the restriction morphism

$$\mathrm{Gal}(E'/F') \to \mathrm{Gal}(E/F)$$

is an isomorphism, and hence we denote by χ' the character of $\mathrm{Gal}(E'/F')$ obtained by the composition

$$\chi' : \mathrm{Gal}(E'/F') \simeq \mathrm{Gal}(E/F) \xrightarrow{\chi} \tilde{\mathbb{Z}}.$$

1.3.11 Proposition. *The following compatibility holds inside the group* $\tilde{S_{F'}}$.

$$i_F^{F'}(\mathrm{Sw}_{E/F}(\chi)) = \mathrm{Sw}_{E'/F'}(\chi').$$

PROOF. Let $y \in \mathcal{O}_E$ be a generator (in the sense of Definition 1.2.1) of the Case-II type extension $F \subset E$. One checks that y is also a generator of $F' \subset E'$ (in the sense of Definition 1.2.1). Therefore, one reduces to checking the following:

$$i_E^{E'}(\mathrm{sw}_{E/F}(\sigma)) = \mathrm{sw}_{E'/F'}(\sigma).$$

But this follows from the definition of $\mathrm{sw}_{E/F}$ and the fact that π_F is a parameter for all of F, F', E and E'. $\square$

1.4 $\mathbb{Z}/p\mathbb{Z}$-extensions

Following the unpublished Bouw [3] we do the example where $G \simeq \mathbb{Z}/p\mathbb{Z}$. We shall often return to this example in the future, and therefore it is useful to include it here. In the following we let F be a complete discrete valuation field satisfying Assumption 1.2.4.

Let us start with the following theorem found in Henrio [19] (Proposition 5.1.6, Corollaire 5.1.8) which classifies Case-II type $\mathbb{Z}/p\mathbb{Z}$-Galois extensions of F (see also Kato [22] Proposition 4.1(6)).

1.4.1 Theorem. *Let $F \subset E$ be a $\mathbb{Z}/p\mathbb{Z}$-Galois extension and assume that $\pi_K \in \mathcal{O}_K$ is a parameter for both F and E. Then there exists an integer $n \in \mathbb{Z}$ with $0 \leq n \leq v_F(\lambda_1)$, and a unit $u \in \mathcal{O}_F^*$ with $\overline{u} \notin \overline{F}^{*p}$ such that $E = F(y)$, where y satisfies*

$$y^p = 1 + \pi^{np}u.$$

Our aim is to calculate Kato's Swan conductor using this explicit generating equation. We distinguish two cases, namely $n = 0$ or the so-called *logarithmic* case (see Lemma 1.4.2), and $n > 0$, or the so-called *exact* case (see Lemma 1.4.3).

1.4.2 Lemma (Logarithmic case). *Let $x \in \mathcal{O}_F^*$ such that $\overline{x} \notin \overline{F}^p$. Let $E := F(y)$ be the $\mathbb{Z}/p\mathbb{Z}$-Galois extension generated by y, where y satisfies*

$$y^p = 1 + x.$$

Let $\sigma \in G := \mathrm{Gal}(E/F)$ be the automorphism

$$\sigma : y \mapsto \zeta_p y.$$

We let χ_b be the irreducible character of G defined by

$$\chi_b : \sigma \mapsto \zeta_p^b,$$

where $b \in \mathbb{F}_p^*$. Then

$$\mathrm{Sw}_{E/F}(\chi_b) := [\lambda_1^p] - [b\frac{\mathrm{d}x}{1+x}].$$

PROOF. We follow Bouw [3] exactly. One checks that $\overline{E} = \overline{F}(y)$, and hence

$$\mathrm{sw}_{E/F}(\sigma^r) = [\mathrm{d}y] - [y] - [\lambda_1] - [r] - [-1] = [\frac{\mathrm{d}y}{y}] - [\lambda_1] - [r] - [-1],$$

where $r \in \mathbb{F}_p^*$. Then by definition we have

$$\begin{aligned}
\mathrm{Sw}_{E/F}(\chi) : &= \sum_{r\in\mathbb{F}_p} \mathrm{sw}_{E/F}(\sigma^r) \otimes \zeta_p^{br} + \epsilon(\zeta_p) \\
&= \sum_{r\in\mathbb{F}_p^*} \mathrm{sw}_{E/F}(\sigma^r) \otimes \zeta_p^{br} + \mathrm{sw}_{E/F}(1_G) \otimes 1 + \epsilon(\zeta_p) \\
&= \sum_{r\in\mathbb{F}_p^*} ([\frac{\mathrm{d}y}{y}] - [\lambda_1] - [r] - [-1]) \otimes \zeta_p^{br} - \sum_{r\in\mathbb{F}_p^*} ([\frac{\mathrm{d}y}{y}] - [\lambda_1] - [r] - [-1]) + \epsilon(\zeta_p) \\
&= \left([\frac{\mathrm{d}y}{y}] - [\lambda_1] - [-1]\right) \otimes \left(\sum_{r\in\mathbb{F}_p^*} \zeta_p^{br} - (p-1)\right) - \epsilon(\zeta_p^b) + \epsilon(\zeta_p) + [-1],
\end{aligned}$$

from which the result follows. □

Still following Bouw [3] word-for-word, we also state the following important variation of Lemma 1.4.2.

1.4.3 Lemma (Exact case). *Let $u \in \mathcal{O}_F^*$ such that $\overline{u} \notin \overline{F}^p$, and let $0 < n < v_K(\lambda_1)$ be a positive integer. We let $E := F(y)$, where y satisfies*

$$y^p = 1 + \pi_K^{np} u.$$

Let $\sigma \in G := \mathrm{Gal}(E/F)$ be the automorphism

$$y \mapsto \zeta_p y,$$

and let χ_b be the irreducible character

$$\sigma \mapsto \zeta_p^b,$$

where $b \in \mathbb{F}_p^*$. Then we have that

$$\mathrm{Sw}_{E/F}(\chi_b) = [\lambda_1^p \pi_K^{-pn}] - [b\,\mathrm{d}u].$$

PROOF. Again we calculate. We define an element $v \in \mathcal{O}_E$ by the substitution

$$y = 1 + \pi_K^n v.$$

One checks that $v \in \mathcal{O}_E$ is a generator for the Case-II type extension $F \subset E$, and furthermore, that

$$\overline{v}^p = \overline{u}$$

inside the purely inseparable extension of residue fields $\overline{F} \subset \overline{E}$.

Notice that $\sigma : y \mapsto \zeta_p y$ acts on v as

$$\sigma^r : v \mapsto (\lambda_r)\pi_K^{-n} + \zeta_p^r v.$$

where we have defined $\lambda_r := \zeta_p^r - 1$.

Therefore, for any $r \in \mathbb{F}_p^*$ we obtain

$$\begin{aligned}
\mathrm{sw}_{E/F}(\sigma^r) :&= [\mathrm{d}\overline{v}] - [v - \sigma^r v] \\
&= [\mathrm{d}\overline{v}] - [-\lambda_r \pi_K^{-n} - \lambda_r v] \\
&= [\mathrm{d}\overline{v}] - [\lambda_r \pi_K^{-n}] - [-1] \\
&= [\mathrm{d}\overline{v}] - [r] - [\lambda_1 \pi_K^{-n}] - [-1].
\end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{Sw}_{E/F}(\chi) :&= \sum_{r \in \mathbb{F}_p} \mathrm{sw}_{E/F}(\sigma^r) \otimes \zeta_p^{br} + \epsilon(\zeta_p) \\
&= \sum_{r \in \mathbb{F}_p^*} \mathrm{sw}_{E/F}(\sigma^r) \otimes \zeta_p^{br} + \mathrm{sw}_{E/F}(1_G) \otimes 1 + \epsilon(\zeta_p) \\
&= \sum_{r \in \mathbb{F}_p^*} ([\mathrm{d}\overline{v}] - [\lambda_1] - [r] - [-1]) \otimes \zeta_p^{br} - \sum_{r \in \mathbb{F}_p^*} ([\mathrm{d}\overline{v}] - [\lambda_1] - [r] - [-1]) + \epsilon(\zeta_p) \\
&= \Big([\mathrm{d}\overline{v}] - [\lambda_1] - [-1]\Big) \otimes \Big(\sum_{r \in \mathbb{F}_p^*} \zeta_p^{br} - (p-1)\Big) - \epsilon(\zeta_p^b) + \epsilon(\zeta_p) + [-1],
\end{aligned}$$

and the result follows once again. $\square$

1.4.4 Remark. Let E/F be $\mathbb{Z}/p\mathbb{Z}$-Galois of Case-II type and let χ be a nontrivial character of the Galois group $\mathrm{Gal}(E/F)$. We write $\mathrm{Sw}_{E/F}(\chi) = [\pi^\delta] - [\omega]$. Then $p|\delta$.

The following is also an useful result.

1.4.5 Lemma. *Let $F \subset E$, G and χ_b be as in Lemma 1.4.2 or Lemma 1.4.3. Then we have the following relation between the different $\mathcal{D}_{E/F}$ and $\mathrm{Sw}_{E/F}(\chi_b)$.*

$$p \cdot \mathcal{D}_{E/F} = (p-1) \cdot \mathrm{Sw}_{E/F}(\chi_b) + [-1]$$

inside the group S_E.

PROOF. This follows directly from computation. We shall prove the result in the logarithmic case and leave the exact case to the reader. By definition

$$\begin{aligned} p \cdot \mathcal{D}_{E/F} : &= -p \cdot \sum_{r \in \mathbb{F}_p^*} \mathrm{sw}_{E/F}(\sigma^r) \\ &= -p \cdot \sum_{r \in \mathbb{F}_p^*} [[\frac{\mathrm{d}y}{y}] - [\lambda_1] - [r] - [-1]] \\ &= \sum_{r \in \mathbb{F}_p^*} [r] + p(p-1) \cdot [\lambda_1] + p(p-1) \cdot [-1] - p(p-1) \cdot [\frac{\mathrm{d}y}{y}] \\ &= p(p-1) \cdot [\lambda_1] + [-1] - (p-1) \cdot [\frac{\mathrm{d}(1+x)}{1+x}] \\ &= (p-1) \cdot \mathrm{Sw}_{E/F}(\chi) + [-1]. \end{aligned}$$

□

We can now state a preliminary version of Kato's Hasse–Arf Theorem (see Theorem 1.5.1 for the full version).

1.4.6 Theorem (Kato [21] Theorem 3.4). *Assume that $F \subset E$ are complete discrete valuation fields satisfying Assumption 1.2.4, and that the extension $F \subset E$ is $\mathbb{Z}/p\mathbb{Z}$-Galois and of Case-II type. Then it holds that*

$$\mathrm{Sw}_{E/F}(\chi) \in S_F \otimes_{\mathbb{Z}} 1 \subset S_E \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}$$

where χ is an irreducible nontrivial character of $\mathbb{Z}/p\mathbb{Z}$.

1.5 Kato's Hasse–Arf theorem

Our aim for this section is to give the full statement of Kato's Hasse–Arf theorem. We shall state the theorem, then some concepts and definitions which will be useful to us later.

1.5.1 Theorem (Kato [21] Theorem 3.4). *Let E/F be a G-Galois extension of Case-II type and let χ be an irreducible nontrivial character of G. Then it holds that*

$$\mathrm{Sw}_{E/F}(\chi) \in S_F \otimes_{\mathbb{Z}} 1 \subset S_E \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}.$$

1.5.2 Definition. Let $F \subset E$ and G be as in Theorem 1.5.1 and let χ be an irreducible nontrivial character of G. Then there exists a unique integer, denoted by δ_χ, and a unique differential, denoted $\omega_\chi \in \Omega_{\overline{F}}^{\otimes n}$ for some $n \in \mathbb{Z}$, such that

$$\mathrm{Sw}_{E/F}(\chi) = [\pi_F^{\delta_\chi}] - [\omega_\chi] = \delta_\chi.[\pi_K] - [\omega_\chi].$$

The integer δ_χ is called the *depth* of the character χ, and the differential ω_χ will be called the *differential* Swan conductor of the character χ with respect to the choice of the parameter π_F of the discrete valuation field F.

1.5.3 Remark. The integer δ_χ is independent of the choice of the parameter π_F, however, notice that the differential Swan conductor is dependent on the choice of the parameter. For instance, when π_F and π'_F are both parameters for F, then one has the following identity.

$$\begin{aligned}\mathrm{Sw}_{E/F}(\chi) &= [\pi_F^{\delta_\chi}] - [\omega_\chi] \\ &= [(\pi'_F)^{\delta_\chi}] - [(\frac{\pi'_F}{\pi_F})^{\delta_\chi}\omega_\chi].\end{aligned}$$

1.5.4 Convention. In the future, when referring to the field F, we shall always assume that a choice of a parameter π_F has been fixed, and therefore, when referring to the differential Swan conductor we shall always mean the differential Swan conductor with respect to this parameter.

1.6 Vector space property

Let χ_1 and χ_2 be two characters of degree 1 of the p-group G. Kato's Swan conductor $\mathrm{Sw}_{E/F}$ associates to each a differential form in $\Omega_{\overline{F}}$. The following is then known and taken from Kato–Saito [23] Corollary 4.6 and Kato [21] Theorem 3.7 (see also Kato [22] Corollary 5.2 and the remark following Proposition 6.8).

1.6.1 Theorem. *Let χ_1, χ_2 be two characters of degree 1, and let $\delta_{\chi_i} \in \mathbb{Z}$ and $\omega_{\chi_i} \in \Omega_{\overline{F}}$ be the depths and differentials associated to these characters via the Swan conductor. Assume that $\delta_{\chi_1} = \delta_{\chi_2}$. Then we have that*

$$\delta_{\chi_1\chi_2} \leq \delta_{\chi_1} = \delta_{\chi_2}.$$

Furthermore, equality holds if and only if $\omega_{\chi_1} + \omega_{\chi_2} \neq 0$. In this case it also holds that

$$\omega_{\chi_1\chi_2} = \omega_{\chi_1} + \omega_{\chi_2}.$$

1.6.2 Remark. Notice that in the particular cases that $\chi_2 = \chi_1^r$, with r relatively prime to p and χ_i of order p, Theorem 1.6.1 is a special case of Lemma 1.4.2 and Lemma 1.4.3.

1.6.3 Example. Let F be a two-dimensional local field, and let u_0 and u_∞ be two elements of $\mathcal{O}_F^*$ which do not reduce to p^{th}-powers of $\overline{F}$. Let $E_i := F(y_i)$ where y_i satisfies $y^p = 1 + \pi^n u_i$ for $i = 0$ or $i = \infty$, where $n > 0$. Both E_0/F and E_∞/F are $\mathbb{Z}/p\mathbb{Z}$-extensions of F, and the compositum $E := E_0 E_\infty$ is a $(\mathbb{Z}/p\mathbb{Z})^2$-Galois extension. The differentials associated to E_0 and E_∞ are exact and are $\mathrm{d}u_0$ and $\mathrm{d}u_\infty$ respectively.

Let $E_i := F(y_i)$ denote the $\mathbb{Z}/p\mathbb{Z}$-subextension of E/F generated by y_i, where y_i satisfies

$$\begin{aligned} y_i^p &= (1+\pi^n u_0)(1+\pi^n u_\infty)^i \\ &= 1 + \pi^n(u_0 + i u_\infty) + \text{terms of higher } \pi\text{-order.} \end{aligned}$$

We see that $\mathrm{d}u_0 + i\,\mathrm{d}u_\infty \neq 0$ if and only if $u_0 + iu_\infty$ does not reduce to a p^{th}-power inside $\overline{F}$, and in this case the differential of E_i/F is simply the sum $\mathrm{d}(u_0 + iu_\infty) = \mathrm{d}u_0 + i\,\mathrm{d}u_\infty$.

1.7 Examples

1.7.1 Example : $(\mathbb{Z}/p\mathbb{Z})^2$-extensions

Let E/F be a G-Galois extension where $G = (\mathbb{Z}/p\mathbb{Z})^2$. We denote by $H_0, \ldots, H_p$ the $p+1$ subgroups of G of order p, and similarly by $E_i := E^{H_i}$. Notice that each E_i/F is a $\mathbb{Z}/p\mathbb{Z}$-Galois extension of Case-II type.

Let χ be an irreducible nontrivial character of H_0. Then there are exactly p characters $\chi_1, \ldots, \chi_p$ of G such that

$$\chi_i|_{H_0} = \chi.$$

It follows that

$$\mathrm{Ind}_{H_0}^G \chi = \sum \chi_i.$$

Assume now that E/F is of Case-II type. We see from Theorem 1.3.8 that

$$\sum \mathrm{Sw}_{E/F}(\chi_i) = \mathrm{Sw}_{E/F}(\mathrm{Ind}_{H_0}^G \chi) = p \cdot \mathrm{Sw}_{E/E_0}(\chi) + p \cdot \mathcal{D}_{E_0/E}\,.$$

We denote by χ_0 any irreducible character of G with kernel exactly H_0. Then we see from Lemma 1.4.5 that

$$p \cdot \mathcal{D}_{E_0/E} = (p-1) \cdot \mathrm{Sw}_{E_0/E}(\chi_0) + [-1]$$

and hence we obtain

$$\sum \mathrm{Sw}_{E/F}(\chi_i) = p \cdot \mathrm{Sw}_{E/E_0}(\chi) + (p-1) \cdot \mathrm{Sw}_{E_0/E}(\chi_0) + [-1].$$

Let us write $\mathrm{Sw}_{E/F}(\chi_i) = [\pi^{\delta_i}] - [\omega_i]$ where $\omega_i \in \Omega_{\overline{F}}$, and $\mathrm{Sw}_{E/E_0}(\chi) = [\pi^{\delta}] - [\omega]$. We see that we obtain

$$\sum_{1 \le i \le p} \delta_i = p \cdot \delta + (p-1) \cdot \delta_0 \tag{1.1}$$

and

$$\sum_{1 \le i \le p} [\omega_i] = p \cdot [\omega] + (p-1) \cdot [\omega_0] + [-1] \tag{1.2}$$

inside S_{E_0}. From Remark 1.4.4 we obtain the following theorem.

1.7.1 Theorem. *Let E/F be a $(\mathbb{Z}/p\mathbb{Z})^2$-Galois extension such that each E_i/F is of Case-II type. Assume that K'/K is an algebraic Galois extension such that EK'/FK' is also of Case-II type and that this is the smallest extension with this property. If p^2 does not divide $\sum_{1\leq i\leq p} \delta_i - (p-1)\delta_0$ then the ramification index of K'/K is a multiple of p.*

1.7.2 Example. Let $K = \mathbb{Q}_p(\zeta_p)$ with local parameter $\lambda := \zeta_p - 1$. Let F be a discrete valuation field containing K and with parameter λ such that the reduction $\overline{F}$ is the rational function field $k(t)$ in one variable. We define E_0/F to be the $\mathbb{Z}/p\mathbb{Z}$-Galois extension generated by y_0, where y_0 satisfies

$$y_0^p = x. \tag{1.3}$$

For $i = 1, \dots, p$ we define E_i/F to be the $\mathbb{Z}/p\mathbb{Z}$-Galois extension generated by y_i, where y_i satisfies

$$y_i^p = x(1-x)^i. \tag{1.4}$$

We let E/F be the G-Galois extension which is the compositum of the E_i/F, where

$$G = (\mathbb{Z}/p\mathbb{Z})^2.$$

One checks that each E_i/F is of Case-II type. Let $H_i := \mathrm{Gal}(E/E_i)$ and define the δ_i as above. Then a calculation shows that

$$\delta_0 = \ldots = \delta_p = p \cdot v_K(\lambda) = p.$$

Therefore $p^2 \nmid \sum_{1\leq i\leq p} \delta_i - (p-1)\delta_0$ and we see that the smallest extension K'/K such that EK'/FK' is of Case-II type has ramification index divisible by p.

1.7.3 Remark. The example above is inspired by the Fermat curve $x^p + y^p = 1$. It is known that $G := (\mathbb{Z}/p\mathbb{Z})^2$ acts on this curve, and quotients of the curve by the $\mathbb{Z}/p\mathbb{Z}$-subgroups of G are given by the equations (1.3) and (1.4) above.

We continue with the situation before Example 1.7.2. We now assume that the residue field $\overline{F}$ is a complete discrete valuation field, i.e. a local power series field $\overline{F} \simeq k((t))$. We see that $\overline{E_0} = k((v))$ where $v^p = t$. We may therefore write

$$\omega_i = t^{n_i} u_i \,\mathrm{d}t$$

where $n_i \in \mathbb{Z}$ and u_i is a unit inside $k((t))$. Similarly, we may write

$$\omega = v^n u \,\mathrm{d}v$$

where $n \in \mathbb{Z}$ and u is an unit inside $k((v))$. We then obtain from (1.2) that

$$\sum_{1\leq i\leq p} n_i = n + (p-1)\cdot n_0$$

and hence we obtain the following theorem.

1.7.4 Theorem. *Let E/F be a G-Galois cover of Case-II type where G is the group $(\mathbb{Z}/p\mathbb{Z})^2$. Let E_i/F, $i=0,\dots,p$, be the $\mathbb{Z}/p\mathbb{Z}$-subextensions of E/F, and let $\omega_i \in \overline{F}$ be the differential Swan conductor associated with a nontrivial irreducible character of* $\mathrm{Gal}(E_i/F)$. *Let $\omega \in \overline{E_0}$ be the differential Swan conductor associated with a nontrivial irreducible character of* $\mathrm{Gal}(E/E_0)$. *Then we have that*

$$\sum_{1\le i\le p} ord_{\overline{F}}\,\omega_i = ord_{\overline{E_0}}\,\omega + (p-1)\cdot ord_{\overline{F}}\,\omega_0.$$

1.7.2 Example : generalized quaternion extensions

In this section we consider a G-Galois extension E/F of case-II type, where $G = Q_{2^{n+1}}$ is the generalized quaternion group with finite presentation

$$Q_{2^{n+1}} = \left\langle a, b |\ a^{2^n} = 1,\ a^{2^{n-1}} = b^2,\ bab^{-1} = a^{-1} \right\rangle.$$

Let $H = \langle a \rangle$. Notice that H is a normal subgroup of G. We let $L := E^H$ and we notice that L/F is a $\mathbb{Z}/2\mathbb{Z}$-Galois extension of Case-II type.

Let χ be any irreducible character of H of order 2^n. We let ψ be a nontrivial irreducible character of $G/H \simeq \mathbb{Z}/2\mathbb{Z}$. The character $\mathrm{Ind}_H^G\,\chi$ is irreducible of rank 2. From Theorem 1.3.8 it follows that

$$\mathrm{Sw}_{E/F}(\mathrm{Ind}_H^G\,\chi) = 2\cdot \mathrm{Sw}_{E/L}(\chi) + 2\cdot \mathcal{D}_{L/F}\,.$$

Let us write

$$\mathrm{Sw}_{E/L}(\chi) = [\pi^{\delta}] - [\omega],$$

where $\omega \in \Omega_{\overline{E}}$. We also write

$$\mathrm{Sw}_{L/F}(\psi) = [\pi^{\delta_0}] - [\omega_0],$$

where $\omega_0 \in \Omega_{\overline{F}}$. Furthermore, we write

$$\mathrm{Sw}_{E/F}(\mathrm{Ind}_H^G\,\chi) = [\pi^{\delta'}] - [\omega'].$$

Then it follows from Theorem 1.3.8 and Lemma 1.4.5 that

$$[\pi^{\delta'}] - [\omega'] = 2\cdot[\pi^{\delta}] - 2\cdot[\omega] + [\pi^{\delta_0}] - [-\omega_0].$$

It follows that

$$\delta' = 2\cdot\delta + \delta_0 \tag{1.5}$$

and

$$[\omega'] = 2\cdot[\omega] + [-\omega_0].$$

In terms of orders of the differentials, we have that

$$\mathrm{ord}_{\overline{F}}(\omega') = \mathrm{ord}_{\overline{L}}(\omega) + \mathrm{ord}_{\overline{F}}(\omega_0).$$

1.7.3 Example : depths at double points

Let $A := R[\![X, Y]\!]/(XY - \pi^n)$. Notice that there exists exactly two codimension-one prime ideals p_1 and p_2 of A which contain the element π. We denote by F_1 the fraction field of the discrete valuation ring A_{p_1} and similarly F_2 that of A_{p_2}. Assume that B/A is a G-Galois extension such that B is also of the form $B := R[\![X', Y']\!]/(X'Y' - \pi^m)$. One checks that $n := m \cdot |G|$ and furthermore, the Galois extension B/A induces G-Galois extensions E_1/F_1 and E_2/F_2, where both E_1 and E_2 are 2-local fields with parameter π. The aim of this section is to prove the following theorem.

1.7.5 Theorem. *Assume that $G := \mathbb{Z}/p^n\mathbb{Z}$. Let χ be an irreducible character of G and assume that E_1/F_1 is of Case-II type. We write $\mathrm{Sw}_{E_i/F_i}(\chi) = [\pi^{\delta_i}] - [\omega_i]$ where $i = 1, 2$ and where $\omega_i \in \Omega_{\overline{F_i}}$. Then we have that*

$$\delta_2 = \delta_1 - n(ord_{\overline{F_1}}(\omega_1) - 1).$$

PROOF. The case $G := \mathbb{Z}/p\mathbb{Z}$ has already been proved by Green–Matignon [16] and later Henrio [19]. Let us now consider the general case and we proceed by induction on the integer n. Assume that Theorem 1.7.5 has been proved for all $n < s$ where $s \in \mathbb{N}$. We shall now prove it also for $n = s$.

Denote by B_1/A the intermediate $\mathbb{Z}/p\mathbb{Z}$-Galois extension, and notice that we may write

$$B_1 := R[\![X'', Y'']\!]/(X''Y'' - \pi^r)$$

where $rp = n$. The $\mathbb{Z}/p\mathbb{Z}$-Galois extension B_1/A induces a $\mathbb{Z}/p\mathbb{Z}$-Galois subextension L_1/F_1 of the $\mathbb{Z}/p^n\mathbb{Z}$-Galois subextension E_1/F_1, and similarly a $\mathbb{Z}/p\mathbb{Z}$-Galois extension L_2/F_2.

Let χ' be a character of $\mathrm{Gal}(L_1/F_1) \simeq \mathbb{Z}/p\mathbb{Z}$ and let χ'' be the restriction of χ to the group

$$\mathrm{Gal}(E_1/L_1) \simeq \mathbb{Z}/p^{s-1}\mathbb{Z}.$$

We may write $\mathrm{Sw}_{L_i/F_i}(\chi') = [\pi^{\delta'_i}] - [\omega'_i]$ and similarly $\mathrm{Sw}_{E_i/L_i}(\chi'') = [\pi^{\delta''_i}] - [\omega''_i]$ where $\omega'_i \in \Omega_{\overline{F_i}}$ and $\omega''_i \in \Omega_{\overline{L_i}}$.

By the induction hypothesis we see that we have

$$\delta''_2 = \delta''_1 - r(\mathrm{ord}_{\overline{L_1}}(\omega''_1) - 1).$$

Furthermore, since the extension B_1/A is $\mathbb{Z}/p\mathbb{Z}$-Galois we have that

$$\delta'_2 = \delta'_1 - n(\mathrm{ord}_{\overline{F_1}}(\omega'_1) - 1).$$

We finish the induction step by noting that from Proposition 1.3.8 and Lemma 1.4.5 it follows that

$$p\delta_i = p\delta''_i + (p-1)\delta'_i$$

and

$$p\,\mathrm{ord}_{\overline{F_1}}(\omega_1) = (p-1)\,\mathrm{ord}_{\overline{F_1}}(\omega'_1) + \mathrm{ord}_{\overline{L_1}}(\omega''_1)$$

from which the result follows. □

1.7.6 Corollary. *Assume now that B/A is a $G := Q_8$-Galois extension. Let χ be the irreducible character of Q_8 of rank two. We write $\mathrm{Sw}_{E_i/F_i}(\chi) = [\pi^{\delta_i}] - [\omega_i]$. One has that $\omega_1 \in \Omega^{\otimes 2}_{\overline{F_1}}$ and it follows that*

$$\delta_2 = \delta_1 - n(ord_{\overline{F_1}}(\omega_1') - 2).$$

PROOF. The proof is essentially exactly as before. We let $H \subset G$ be a $\mathbb{Z}/4\mathbb{Z}$-subgroup. We let $L_i := E_i^H$ and we let $\chi_H : H \to \mathbb{C}^*$ be an irreducible character of H with trivial kernel. Let χ' be a nontrivial character of $\mathrm{Gal}(L_i/F_i) \simeq \mathbb{Z}/2\mathbb{Z}$. We may write

$$\mathrm{Sw}_{E_i/L_i}(\chi_H) = [\pi^{\delta_i''}] - [\omega_i'']$$

and similarly

$$\mathrm{Sw}_{L_i/F_i}(\chi') = [\pi^{\delta_i'}] - [\omega_i'].$$

This time Proposition 1.3.8 implies that

$$\delta_i = \delta_i' + 2\delta_i''$$

and similarly

$$\mathrm{ord}_{\overline{F_1}}(\omega_1) = \mathrm{ord}_{\overline{F_1}}(\omega_1') + \mathrm{ord}_{\overline{L_1}}(\omega_1''),$$

from which the result now follows exactly as before. □

1.7.7 Remark. In Chapter three (see Proposition 3.4.5) we shall proof a powerful generalization of this result, namely to arbitrary p-groups G.

Chapter 2

Swan conductors II: Ramification groups

In Serre [37] a ramification filtration is introduced for Galois extensions of complete discrete valuation fields of which the residue field extension is separable. In the current chapter we follow Kato–Saito [23] to develop this idea in the case of Case-II type extensions. However, we restrict to the case where the extension is induced by the boundary extension of local power series rings (in a manner to be made more precise in Section 2.1). This has the advantage that it significantly simplifies the proof that the higher ramification groups are normal subgroups of the Galois group and that they are independent of the choices of local parameters used.

In Section 2.2 we start with the definition of Kato–Saito [23]. We also introduce a simplified ramification filtration in Section 2.3 which will prove useful to us later in the third chapter and important for the fifth chapter. In Spriano [39] and Zhukov [42] similar ramification filtrations were also studied, and in Section 2.3.1 we shall reconcile our definition with that of Zhukov. In Section 2.4 we introduce two new characters, the *Artin* and *depth* characters of the Galois extension. We then relate Kato's differential Swan conductor to these as well as to the associated upper ramification filtration. We conclude by giving a structure theorem on the quotients of the simplified upper ramification filtration in terms of the Kato Swan-conductor differentials and vector spaces thereof.

2.1 Notation and setting

As in the previous chapter, we let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic p and ring of integers R. We denote by v_K the valuation of the field K, and by $\pi_K \in R$ a local parameter of K. We shall assume that v_K has been normalized such that $v_K(\pi_K) = 1$. We shall assume $\zeta_p \in K$ and we set $\lambda := \zeta_p - 1$.

We let $A := R[\![t]\!]$ and $F := \operatorname{Frac} \widehat{A_{(\pi A)}}$. Notice that F is a two-dimensional local field with parameter π_K and residue field $\overline{F} = k(\!(t)\!)$. We denote by v_F the discrete valuation of F, and by $v_{\overline{F}}$ the discrete valuation of $\overline{F}$.

2.1.1 Definition. We define the rank-2 valuation $\underline{\mathrm{v}}_F : F^* \to \mathbb{Z}^2$ by the rule:

$$\underline{\mathrm{v}}_F : x \in F \mapsto (v_{\overline{F}}(\tilde{x}), v_F(x)),$$

where $\tilde{x} \in \overline{F}$ is the reduction of $x\pi_K^{-v_F(x)} \in \mathcal{O}_F^*$ to $\overline{F}$.

Let G be a p-group and let B/A be a G-Galois extension such that B is also a power series ring over R, i.e. $B = R[\![z]\!]$. We define $E := B \otimes_A F$. Notice that E/F is a G-Galois extension and that π_K is by assumption also a local parameter of E. Denote by v_E the valuation of E and by $v_{\overline{E}}$ the valuation of the discrete valuation field $\overline{E}$ which is the reduction of F. As in the case of F, we may construct a rank- 2 valuation $\underline{\mathrm{v}}_E$ on E. We assume that $\underline{\mathrm{v}}_E$ has been normalized such that $\underline{\mathrm{v}}_E|_F = \underline{\mathrm{v}}_F$. In order to emphasize this normalization we shall from now on denote $\underline{\mathrm{v}}_E$ simply by $\underline{\mathrm{v}}_F$.

2.1.2 Assumption. We shall from now on always assume that E/F is of Case-II type.

2.1.3 Remark. For an element $x \in E$ we have that

$$\underline{\mathrm{v}}_F(x) = (\frac{m}{e_{\overline{E}/\overline{F}}}, n)$$

where $n := v_F(x)$ and where $m := v_{\overline{E}}(x\pi_K^{-n})$ is the valuation of $x\pi_K^{-n}$ inside the residue field $\overline{E}$. Here $e_{\overline{E}/\overline{F}}$ is defined to be the ramification index of the residue field extension $\overline{F} \subset \overline{E}$. Since we have assumed that E/F is of Case-II type, we see that $e_{\overline{E}/\overline{F}} = [\overline{E} : \overline{F}] = [E : F]$.

2.1.4 Definition. An element $x \in E$ is called a *local geometric parameter* of E if $B = R[\![x]\!]$.

2.2 Ramification groups

Assume the notation and setting as introduced above. We shall very briefly introduce the higher ramification filtration, for more details see Kato–Saito [23]. We remind the reader of the reversed lexicographic ordering $<$ on $\mathbb{Q}^2$.

2.2.1 Definition. For any $(a, b) \in \mathbb{Q}^2$ and $(c, d) \in \mathbb{Q}^2$ we declare that $(a, b) \leq (c, d)$ if and only if either $b = d$ and $a \leq c$ or if $b < d$.

2.2.2 Notation. We shall denote the second projection $\mathbb{Q}^2 \to \mathbb{Q}$ by p_2, i.e. $p_2((a, b)) = b$, where $a, b \in \mathbb{Q}$.

2.2.3 Remark. If $t, t' \in \mathbb{Q}^2$ such that $t \leq t'$, then $p_2(t) \leq p_2(t')$.

We now come to the definition of the ramification filtration of G.

2.2.4 Definition. For $t \in \mathbb{Q}^2$, we define

$$G_t := \{\sigma \in G | \underline{\mathrm{v}}_F(\frac{\sigma(x_E)}{x_E} - 1) \geq t\},$$

where x_E is any geometric local parameter of E.

2.2.5 Proposition. *This definition is independent of the choice of x_E. Furthermore, the subgroup G_t is a normal subgroup of G.*

PROOF. Let x_E and y_E be local geometric parameters of E. Therefore $B = R[\![x_E]\!] = R[\![y_E]\!]$. We see that we can find $(a_i)_{i\in\mathbb{N}\cup\{0\}}$ and $(b_i)_{i\in\mathbb{N}\cup\{0\}}$ such that

$$y_E = a_0 + \sum_{i\geq 1} a_i \cdot x_E^i \text{ and } x_E = b_0 + \sum_{j\geq 1} b_i \cdot y_E^j. \tag{2.1}$$

One checks that $a_0, b_0 \in \pi_K R$ and that a_1 and b_1 are units of R.

Let us write

$$\sigma(x_E) - x_E = \pi_K^m \cdot p(x_E) \cdot u \tag{2.2}$$

where $u \in B$ is a unit and $p(x_E)$ is a distinguished polynomial of degree n. Since E/F is of Case-II type, we see that $m > 0$. Therefore we have that

$$\underline{\mathrm{v}}_F(\sigma(x_E) - x_E) = (\frac{n}{[E:F]}, m).$$

Now we calculate $\sigma(y_E) - y_E$ in terms of x_E. Indeed we have that

$$\sigma(y_E) - y_E = \sum_{1\leq i} a_i \cdot (\sigma(x_E)^i - x_E^i).$$

For each $i \geq 1$ we notice from (2.2) that we may write

$$\sigma(x_E)^i = x_E^i + i \cdot x_E^{i-1} \cdot \pi^m \cdot p(x_E) \cdot u + \alpha_i,$$

where $\alpha_i \in B$ with $v_E(\alpha_i) > m$. Hence we have

$$\sigma(x_E)^i - x_E^i = i \cdot x_E^{i-1} \cdot \pi^m \cdot p(x_E) \cdot u + \alpha_i.$$

It follows that

$$\sigma(y_E) - y_E = \pi^m \cdot p(x) \cdot u \cdot (\sum_{i\geq 1} i \cdot x_E^{i-1}) + \alpha$$

where $\alpha \in B$ with $v_E(\alpha) > m$. It follows that

$$\underline{\mathrm{v}}_F(\sigma(y_E) - y_E) = (\frac{n}{[E:F]}, m) = \underline{\mathrm{v}}_F(\sigma(x_E) - x_E).$$

Therefore we see that the G_t is independent of the choice of local geometric parameter x_E.

Next we check the normality. Let r and s be defined by $t = (\frac{s-1}{[E:F]}, r)$. We define the B-ideal $\mathcal{I} := \langle \pi^r x_E^s, \pi^{r+1} \rangle$. We shall now show that $\mathcal{I} \subset B$ is fixed by the action of G. Let $\tau \in G$. Then since E/F is of Case-II type, we may write

$$\tau(x_E) = x_E + \pi^{r_\tau} \alpha_\tau \tag{2.3}$$

where $r_\tau > 0$ and $\alpha_\tau \in B$. Therefore we have that

$$\pi^r(\tau(x_E))^s = \pi^r x_E^s + \alpha$$

where $\alpha \in \pi^{r+1}B$. We see thus that $\mathcal{I}$ is fixed by G. We may therefore consider the ring $B_0 := B/\mathcal{I}$ and we notice that the G-action on B induces a G-action on B_0.

Assume that $\rho \in G$ acts trivially on B_0. Therefore

$$\rho(x_E) = x_E + \pi^r x_E^s \beta_1 + \pi^{r+1}\beta_2 \tag{2.4}$$

where $\beta_1, \beta_2 \in B$. We see thus that

$$\underline{\mathrm{v}}_F(\rho(x_E) - x_E) \geq (\frac{s}{[E:F]}, r)$$

and hence

$$\underline{\mathrm{v}}_F(\frac{\rho(x_E)}{x_E} - 1) \geq t.$$

Hence we have that $\rho \in G_t$.

Now let $\sigma \in G_t$ and let $\tau \in G$. First we show that σ acts trivially on B_0. Define m, $p(x) \in B$ and $u \in B$ by (2.2). If $m > r$ then we see that σ acts trivially on the ring B_0. Assume now that $m = r$ and let n be the degree of $p(x)$. Then we see that

$$\underline{\mathrm{v}}_F(\frac{\sigma(x_E)}{x_E} - 1) = (\frac{n-1}{[E:F]}, r)$$

and since $\sigma \in G_t$ we see that $n \geq s$. Therefore also in this case we have that σ acts trivially on the ring B_0. Hence $\tau\sigma\tau^{-1}$ also acts trivially on B_0. It follows that $\tau\sigma\tau^{-1} \in G_t$. □

We have the following lemma.

2.2.6 Lemma. *Let $t \in \mathbb{Q}^2$ and $\sigma \in G$. Then the following statements are equivalent.*

- *We have that $\sigma \in G_t$ and $\sigma \notin G_{t'}$ for all $t' \in \mathbb{Q}^2$ with $t < t'$.*
- *We have that $\underline{\mathrm{v}}_F(\frac{\sigma(x_E)}{x_E} - 1) = t$ where x_E is any geometric local parameter.*

PROOF. Assume that $\sigma \in G_t$ and $\sigma \notin G_{t'}$ for all $t' \in \mathbb{Q}^2$ with $t < t'$. By definition of G_t we see that

$$\underline{\mathrm{v}}_F(\frac{\sigma(x_E)}{x_E} - 1) \geq t.$$

Let $\hat{t} := \underline{\mathrm{v}}_F(\frac{\sigma(x_E)}{x_E} - 1) \in \mathbb{Q}^2$. We see that $\sigma \in G_{\hat{t}}$. Therefore, by hypothesis and the fact that $\hat{t} \geq t$, we see that $\hat{t} = t$. The converse direction we leave to the reader. □

2.2.7 Definition. We shall say that $t \in \mathbb{Q}^2$ is a lower ramification jump if $G_t \neq G_{t'}$ for all $t' \in \mathbb{Q}^2$ with $t' > t$.

In order to introduce the higher ramification filtration on G, one defines a generalization $\phi : Q^2 \to Q^2$ of the classical Herbrandt function as follows:

$$\phi(s) := \int\limits_0^s |G_w| dw, \quad s \in \mathbb{Q}^2$$

and

$$\psi := \phi^{-1},$$

for the notion of integration on the ordered $\mathbb{Q}^2$ see Kato–Saito [23] Sections 1 and 2. We briefly recall the definition of the integral $\int\limits_0^s |G_w| dw$. Let $s_1, \ldots, s_j \in \mathbb{Q}^2$ be the set of lower ramification jumps not exceeding t. Then we define

$$\int\limits_0^s |G_w| dw := s_1 \cdot |G_{s_1}| + (s_2 - s_1) \cdot |G_{s_2}| + \ldots + (s_j - s_{j-1}) \cdot |G_{s_j}| + (s - s_j) \cdot |G_s| \in \mathbb{Q}^2.$$

One then defines the higher ramification filtration by setting for $t \in \mathbb{Q}^2$

$$G^t := G_{\psi(t)}$$

for $t \in \mathbb{Q}^2$.

2.2.8 Definition. We shall say that $t \in \mathbb{Q}^2$ is an upper ramification jump if for all $t' \in \mathbb{Q}^2$ with $t' > t$ we have that $G^t \neq G^{t'}$.

2.2.9 Remark. Notice that $s \in \mathbb{Q}^2$ is a lower ramification jump if and only if $\phi(s)$ is a higher ramification jump.

Furthermore, one proves that for a $t \in \mathbb{Q}^2$ we have

$$\psi(t) := \int\limits_0^t (|G^w|)^{-1} dw,$$

see Kato-Saito [23] Lemma 2.3. For convenience, let us make this integral explicit. Let $t_1, \ldots, t_j$ be the set of upper ramification jumps not exceeding t. Then we define

$$\int\limits_0^t (|G^w|)^{-1} dw := \frac{t_1}{|G^{t_1}|} + \frac{t_2 - t_1}{|G^{t_2}|} + \ldots + \frac{t_j - t_{j-1}}{|G^{t_j}|} + \frac{t - t_j}{|G^t|}. \tag{2.5}$$

The following proposition shows that the upper ramification filtration G^t, $t \in \mathbb{Q}^2$, behaves well with respect to quotients.

2.2.10 Proposition (Kato–Saito [23] Corollary 3.3). *Let $H \lhd G$ be a normal subgroup, and consider the G/H-Galois extension of two-dimensional local fields $F \subset E^H$. Then the higher ramification filtration on G/H is compatible with that of G, i.e. for any $t \in \mathbb{Q}^2$ we have that*

$$(G/H)^t = (G^t H)/H.$$

Later in this chapter we shall prove that the set of upper ramification jumps are related to Kato's Swan conductors. More precisely, let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be an irreducible representation of G with associated character χ, and let $t \in \mathbb{Q}^2$ be the maximal upper ramification jump such that $\rho|_{G^t}$ is not trivial. We write $\mathrm{Sw}_{E/F}(\chi) = [\pi^\delta] - [\omega]$. Then we shall see that $\omega \in \Omega^{\mathrm{rk}(\chi)}$, see Theorem 2.4.5, and furthermore, that

$$\mathrm{rk}(\chi) \cdot t = (-\operatorname{ord}_{\overline{F}} \omega - \mathrm{rk}(\chi), \delta),$$

see Corollary 2.4.12. In the case that G is abelian, we obtain the following theorem.

2.2.11 Theorem. *Assume that the group G is abelian. Then the upper ramification jumps are elements of $(\mathbb{Z})^2$.*

2.3 The simplified ramification groups $\hat{G}$

2.3.1 Definitions

We start by defining $\hat{G}^0 = G$. Consider the upper ramification jumps $t_1 < \ldots < t_N$ of the group G. For each j we write $t_j = (m_j, n_j)$, where $m_j, n_j \in \mathbb{Q}$. For $i \in \mathbb{Q}_{\geq 0}$, we define

$$\hat{G}^i := \bigcup_{j : n_j \geq i} G^{t_j}.$$

We shall refer to this filtration $\hat{G}^i$, $i \in \mathbb{Q}$, on $G = \hat{G}^0$ as the *simplified upper ramification* filtration. It behaves well with respect to normal subgroups, as the following consequence of Proposition 2.2.10 states.

2.3.1 Proposition. *Let $H \lhd G$ be a normal subgroup of G. Then the simplified upper filtrations of G and G/H are related via*

$$\widehat{G/H}^i = \hat{G}^i H/H.$$

PROOF. This follows directly from the definition of the simplified upper ramification filtration, as well as the corresponding property Proposition 2.2.10 for the two-dimensional upper ramification filtration. □

2.3.2 Definition. We shall say that $n \in \mathbb{Q}$ is a *simplified upper ramification jump* if $\hat{G}^n \neq \hat{G}^{n'}$ for all $n' \in \mathbb{Q}$ with $n < n'$.

2.3.3 Lemma. *Let n be a simplified upper ramification jump. Then there exists an $a \in \mathbb{Q}$ such that (a, n) is an upper ramification jump.*

PROOF. Indeed, if $\hat{G}^n \neq \hat{G}^{n'}$ for all $n' \in \mathbb{Q}$ with $n' > n$, then there exists a nontrivial $\sigma \in \hat{G}^n$ such that $\sigma \notin \hat{G}^{n'}$. Thus, by the definition of $\hat{G}^n$ for some $a \in \mathbb{Q}$ we have that

$$\sigma \in G^{(a,n)}.$$

Since σ is nontrivial, there exists an upper ramification jump $t \in \mathbb{Q}$ such that $\sigma \in G^t$ and $\sigma \notin G^{t'}$ for all $t' \in \mathbb{Q}^2$ with $t' > t$. Furthermore, we thus have that $(a, n) \leq t$.

However, if $t = (b, n')$ for some $b, n' \in \mathbb{Q}$ with $n' > n$, then $\sigma \in G^{(b',n')}$ for all $b' \in \mathbb{Q}$ with $b' < b$, and hence $\sigma \in \hat{G}^{n'}$. Thus $n' = n$, completing the proof. □

2.3.4 Definition. We define the *simplified* Hasse–Arf function $\hat{\psi}$ by defining for $i \in \mathbb{Q}$

$$\hat{\psi}(i) := \int_0^i |\hat{G}^s|^{-1} ds$$

and setting $\hat{\phi} := \hat{\psi}^{-1}$.

2.3.5 Definition. We define the lower simplified ramification filtration by

$$\hat{G}_i := G^{\hat{\phi}(i)}.$$

Let us now compare the functions ψ and $\hat{\psi}$.

2.3.6 Lemma. *The functions $\psi : \mathbb{Q}^2 \to \mathbb{Q}^2$ and $\hat{\psi} : \mathbb{Q} \to \mathbb{Q}$ are related by*

$$p_2(\psi(T)) = \hat{\psi}(p_2(T)),$$

where $p_2 : \mathbb{Q}^2 \to \mathbb{Q}$ denotes the second projection of $\mathbb{Q}^2$, i.e. $p_2((a, b)) = b$ for $a, b \in \mathbb{Q}^2$.

PROOF. Let $T = (T_1, T_2)$. Let $t_1, \ldots, t_N$ be the set of upper ramification jumps of G not exceeding T. Assume first that $T > t_1$. We write $t_j = (a_j, b_j)$ for $a_j, b_j \in \mathbb{Q}$. We define the indices $j_1 < \ldots < j_s$ by $j_1 = 1$ and

$$b_1 = b_2 = \ldots = b_{j_2 - 1},\ b_{j_2} = b_{j_2+1} = \ldots = b_{j_3-1}, \ldots,\ b_{j_s} = \ldots = b_N$$

and such that $b_{j_1} < \ldots < b_{j_s}$. Notice that the simplified upper ramification jumps not exceeding T_2 are $b_{j_1} < \ldots < b_{j_s}$ and possibly T_2 itself. Furthermore, it follows that

$$G^{t_{j_i}} = \hat{G}^{b_{j_i}}, \quad i = 1, \ldots, s.$$

Define $t_0 = (0, 0) \in \mathbb{Q}^2$, $j_0 = 0$ and $b_0 = 0$. Then

$$\begin{aligned} p_2(\psi(T)) &= p_2\Big(\sum_{1 \le j \le N} \frac{t_j - t_{j-1}}{|G^{t_j}|} + \frac{T - t_N}{|G^T|}\Big) \\ &= \sum_{1 \le j \le N} \frac{b_j - b_{j-1}}{|G^{t_j}|} + \frac{T_2 - b_N}{|G^T|} \\ &= \sum_{1 \le i \le s} \frac{b_{j_i} - b_{j_{i-1}}}{|\hat{G}^{b_{j_i}}|} + \frac{T_2 - b_{j_s}}{|G^T|}. \end{aligned}$$

If $T_2 = b_{j_s}$ then we have that $\frac{T_2 - b_{j_s}}{|G^T|} = 0 = \frac{T_2 - b_{j_s}}{|\hat{G}^{T_2}|}$. Hence we obtain from the definition of $\hat{\psi}$ that $p_2(\psi(T_2)) = \hat{\psi}(T_2)$, since the simplified upper ramification jumps not exceeding T_2 are $b_{j_1}, \ldots, b_{j_s}$. Next assume that $T_2 > b_{j_s}$. Then we see that $G^T = \hat{G}^{T_2}$ and hence the result follows once again. We leave the case $T \le t_1$ to the reader. $\square$

2.3.7 Lemma. *Assume that $i \in \mathbb{Q}$ and that $\sigma \in \hat{G}_i = \hat{G}^{\hat{\phi}(i)}$. Then*

$$v_F\Big(\frac{\sigma(x_E)}{x_E} - 1\Big) \ge i.$$

PROOF. Let $t_l \in \mathbb{Q}^2$ be an upper ramification jump such that $G^{\hat{\phi}(i)} = G^{t_l}$. Thus $p_2(t_l) \geq \hat{\phi}(i)$. Therefore $\sigma \in G^{t_l}$ and hence $\sigma \in G_{\psi(t_l)}$. Thus,

$$\underline{\mathrm{v}}_F(\frac{\sigma(x_E)}{x_E} - 1) \geq \psi(t_l). \tag{2.6}$$

Therefore

$$\begin{aligned} v_F(\frac{\sigma(x_E)}{x_E} - 1) &= p_2(\underline{\mathrm{v}}_F(\frac{\sigma(x_E)}{x_E} - 1)) \quad \text{since } p_2 \circ \underline{\mathrm{v}}_F = v_F \\ &\geq p_2(\psi(t_l)) \quad \text{by (2.6)} \\ &= \hat{\psi}(p_2(t_l)) \quad \text{by Lemma 2.3.6} \\ &\geq \hat{\psi}(\hat{\phi}(i)) \\ &= i. \end{aligned}$$

□

2.3.8 Lemma. *Assume that* $i \in \mathbb{Q}$ *and that* $\sigma \notin \hat{G}_i = \hat{G}^{\hat{\phi}(i)}$. *Then*

$$v_F(\frac{\sigma(x_E)}{x_E} - 1) < i.$$

PROOF. Once again, let $t_1, \ldots, t_N$ be the upper ramification jumps. Let $b_j := p_2(t_j)$ for $j = 1, \ldots, N$. Assume that $\hat{G}^{\hat{\phi}(i)} = G^{t_l}$ for some index l. Therefore, we obtain that by assumption $\sigma \notin G^{t_l}$. Notice that $t_l \neq t_1$, since $G^{t_1} = G$ and would therefore have contained σ. Thus $l \geq 2$.

Since $\hat{G}^{\hat{\phi}(i)} = G^{t_l}$ we see that $b_{l-1} < \hat{\phi}(i) \leq b_l$. Let j be the largest index such that $\sigma \in G_{\psi(t_j)} = G^{t_j}$. We see thus that $t_j < t_l$. By Lemma 2.2.6 we have

$$\underline{\mathrm{v}}_F(\frac{\sigma(x_E)}{x_E} - 1)) = \psi(t_j).$$

Thus we obtain

$$\begin{aligned} v_F(\frac{\sigma(x_E)}{x_E} - 1)) &= p_2(\psi(t_j)) \\ &= \hat{\psi}(p_2(t_j)) \\ &\leq \hat{\psi}(p_2(t_{l-1})) \quad \text{since } t_j \leq t_{l-1} \text{ and hence } p_2(t_j) \leq p_2(t_{l-1}) \\ &= \hat{\psi}(b_{l-1}) \\ &< \hat{\psi}(\hat{\phi}(i)) \quad \text{since } b_{l-1} < \hat{\phi}(i) \\ &= i. \end{aligned}$$

□

As a consequence, we obtain the following proposition

2.3.9 Proposition. *Assume that* $i \in \mathbb{Q}$. *Then*

$$\hat{G}_i = \{\sigma \in G | v_F(\frac{\sigma(x_E)}{x_E} - 1) \geq i\}.$$

2.3.10 Remark. We have shown that our ramification filtration is the same as that of Zhukov [42].

2.3.11 Remark. We see thus that we may also define $\hat{G}_n$ by

$$\hat{G}_n := \bigcup_{b_j \geq n} G_{(a_j,b_j)}, \tag{2.7}$$

where the set $\{(a_1, b_1), \ldots, (a_N, b_N)\} \subset \mathbb{Q}^2$ are the lower ramification jumps.

2.3.2 The quotients $\hat{G}_n/\hat{G}_{n+1}$

We fix a geometric local parameter x_E of the field E. Assume that $n \in \mathbb{Z}$ is a simplified lower ramification jump corresponding to the simplified lower filtration on $G := \mathrm{Gal}(E/F)$. By Remark 2.3.11 there exists a lower ramification jump $t \in \mathbb{Q}^2$ such that $\hat{G}_n = G_t$. Let us write $t = (\frac{n}{[E:F]}, m)$, where $m, n \in \mathbb{Z}$.

Let $\sigma \in \hat{G}_n$. Then by the Weierstrass preparation theorem we may write

$$\sigma(x_E) = x_E + \pi_K^r \cdot p_\sigma \cdot u_\sigma$$

where p_σ is a distinguished polynomial of degree d and u_σ a unit of $B = R[\![x_E]\!]$. Furthermore, since $\sigma \in \hat{G}_n$, we have that $r \geq m$. If $r = m$ then we have that $d \geq n+1$. We may therefore write

$$\sigma(x_E) = x_E + \pi_K^r \cdot x_E^d \cdot u_\sigma + \pi^{r+1} \cdot l_\sigma$$

where $l_\sigma \in \mathcal{O}_F$ (however notice that l_σ need not be in B).

We define $v_\sigma := \pi_K^{r-m} \cdot x_E^{d-n-1} \cdot u_\sigma$ for $\sigma \neq 1$ and $v_1 = 0$.

We have that $v_\sigma \in \mathcal{O}_E^*$ if and only if $\sigma \notin \hat{G}_{m+1}$. After fixing the choice of x_E, each $\sigma \in \hat{G}_m - \hat{G}_{m+1}$ determines an unique $\overline{v_\sigma} \in \overline{E}$, and if $\sigma \in \hat{G}_{m'}$ for some $m' > m$, then we find that $\overline{v_\sigma} = 0$. We leave the proof of the following calculation to the reader.

2.3.12 Lemma. *Let σ and τ be two elements of $\hat{G}_m - \hat{G}_{m+1}$. Then $\tau\sigma \in \hat{G}_m - \hat{G}_{m+1}$ if and only if $\overline{v_\sigma} + \overline{v_\tau} \neq 0$. In this case we also obtain $\overline{v_{\tau\sigma}} = \overline{v_\tau} + \overline{v_\sigma}$. Furthermore we have that $\overline{v_{\sigma^{-1}}} = -\overline{v_\sigma}$.*

A similar calculation yields also the following

2.3.13 Lemma. *Let $\sigma \in \hat{G}_m$ but not in $\hat{G}_{m+1}$ and let $\tau \in \hat{G}_{m+1}$. Then $\sigma\tau \in \hat{G}_m - \hat{G}_{m+1}$, and furthermore,*

$$\overline{v_\sigma} = \overline{v_{\sigma\tau}}.$$

This also holds for the automorphism $\tau\sigma$.

2.3.14 Remark. The assumption that E/F is of Case-II type is essential for the two lemmata above.

We now construct a mapping $\gamma_{E/F}^m : \hat{G}_m \to \overline{E}$ by defining

$$\gamma_{E/F}^m : \sigma \mapsto \overline{v_\sigma} \in \overline{E}, \quad \sigma \in \hat{G}_m.$$

From the lemmata above we see that $\gamma_{E/F}^m$ induces a group homomorphism $\hat{G}_m/\hat{G}_{m+1} \to \overline{E}$. Furthermore $\gamma_{E/F}^n$ is an injective group homomorphism.

2.3.15 Remark. Notice that the definition of $\gamma^m_{E/F}$ depends on the choice of x_E. This is different from the classical construction in Serre [37] Chapter IV for the case of a one-dimensional local field. A further distinction is that our construction only works because we have assumed that E/F is of Case-II type.

We thus obtain the following theorem.

2.3.16 Theorem. *The quotient group $\hat{G}_m/\hat{G}_{m+1}$ can be embedded into the additive group $\overline{E}$. Thus each quotient $\hat{G}_m/\hat{G}_{m+1}$ is an elementary abelian p-group.*

2.3.17 Remark. We stress the point once again that this theorem only holds because we have made the assumption that E/F is of Case-II type.

2.3.18 Corollary. *Assume that G is cyclic of order p^N. Then there exist exactly N distinct lower (respectively upper) simplified ramification jumps.*

PROOF. The group $\mathbb{Z}/p^N\mathbb{Z}$ has exactly N nontrivial subgroups, and hence there are at most N lower ramification jumps, and at most N lower simplified ramification jumps. However, each quotient of consecutive lower ramification groups must be elementary abelian, and hence the result follows. □

2.3.19 Example. Assume that $G = Q_8$, the quaternion group of eight elements. Then there exist at least two simplified ramification jumps.

A consequence of our work is the following lemma.

2.3.20 Lemma. *Let $\sigma \in G$ and $\sigma \neq 1_G$. Then for any $b \in \mathbb{Z}$ relatively prime to p, we have that*

$$\mathrm{sw}_{E/F}(\sigma^b) = \mathrm{sw}_{E/F}(\sigma) - [b].$$

PROOF. Let x_E be a geometric local parameter of E, and assume that $\sigma \in \hat{G}_m$, but not in $\hat{G}_{m+1}$, for $m \in \mathbb{Z}$. Let $t \in \mathbb{Q}^2$ be the least lower ramification jump with second coordinate m, i.e. $p_2(t) = m$, and write

$$t = (\frac{n}{[E:F]}, m),$$

where m and n are integers. We thus have

$$\begin{aligned} \mathrm{sw}_{E/F}(\sigma) &= [\mathrm{d}\overline{x_E}] - [-x_E^{n+1}\pi^m v_\sigma] \\ &= [\mathrm{d}\overline{x_E}] - [-x_E^{n+1}\pi^m] - [\overline{v_\sigma}] \quad \text{since } v_\sigma \in \mathcal{O}_E^* \\ &= [\mathrm{d}\overline{x_E}] - [-x_E^{n+1}\pi^m] - [\gamma^m_{E/F}(\sigma)]. \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{sw}_{E/F}(\sigma^b) &= [\mathrm{d}\overline{x_E}] - [-x_E^{n+1}\pi^m] - [\gamma^m_{E/F}(\sigma^b)] \\ &= [\mathrm{d}\overline{x_E}] - [-x_E^{n+1}\pi^m] - [\gamma^m_{E/F}(\sigma)] - [b] \\ &= \mathrm{sw}_{E/F}(\sigma) - [b]. \end{aligned}$$ □

A consequence of this is

2.3.21 Theorem. *Let χ be a character of degree 1, i.e. χ is a homomorphism $G \to \mathbb{C}^*$. Let $H \subset G$ be a normal subgroup of G and consider the induced tower of Galois extensions $F \subset E^H \subset E$. This induces embeddings $S_F \hookrightarrow S_{E^H} \hookrightarrow S_E$. Let $\mathcal{D}_{E^H/F} \in S_{E^H}$ denote the different of the extension $F \subset E^H$. Then we have the following identity inside S_{E^H}.*

$$\mathrm{Sw}_{E/E^H}(\chi|_H) + \mathcal{D}_{E^H/F} = \mathrm{Sw}_{E/F}(\chi).$$

PROOF. The proof is exactly the proof of Lemma 3.12 of Kato [21]. There Kato uses the equality

$$\mathrm{sw}_{E/F}(\sigma^i) = \mathrm{sw}_{E/F}(\sigma) - [i],$$

which follows directly from Lemma 2.3.20. □

2.3.22 Corollary. *Let χ be a character of degree 1, and let $r \in \mathbb{Z}$ be an integer relatively prime to p. Then*

$$\mathrm{Sw}_{E/F}(\chi^r) = \mathrm{Sw}_{E/F}(\chi) - [r].$$

In particular, restricting to the differential components, we have

$$\omega_{\chi^r} = [\omega_\chi] + [r] = [r\omega_\chi].$$

PROOF. From Theorem 2.3.21 it suffices to prove this in the case that χ is a character of order p. But then the result is an immediate consequence of the explicit calculations in Lemma 1.4.2 and Lemma 1.4.3. □

2.3.23 Remark. In this section we embedded the quotient groups $\hat{G}_m/\hat{G}_{m+1}$ into the ring $\overline{E} = \mathcal{O}_E/\pi\,\mathcal{O}_E$. However, we simply comment that it is in fact possible to refine our calculations and to prove and even stronger statement, namely that it is possible to embed $\hat{G}_m/\hat{G}_{m+1}$ into the ring $E_m := \mathcal{O}_E/\pi^m\,\mathcal{O}_E$.

2.4 Artin and depth characters

Let $f \in E^*$.

2.4.1 Definition. We define the *order* $\#f$ of f as

$$\#f := v_{\overline{E}}(\tilde{f}) = v_{\overline{E}}(\overline{f/\pi^{v_E(f)}}).$$

Let x_E be a local geometric parameter of E.

2.4.2 Definition. We define the *Artin character* of E/F as the class function of G defined by

$$a_{E/F}(\sigma) := -\#(\sigma(x_E) - x_E), \quad \text{for } \sigma \neq 1$$

and

$$a_{E/F}(1) := -\sum_{\sigma \neq 1} a_{E/F}(\sigma).$$

We define the *Swan character* of E/F as the class function of G defined by

$$s_{E/F} := a_{E/F} - u_G,$$

where u_G is the augmentation character of G.

2.4.3 Definition. We define the *depth character* $\delta_{E/F}$ with respect to x_E by

$$\delta_{E/F}(\sigma) := -|G| \cdot v_E(\sigma(x_E) - x_E), \quad \text{for } \sigma \neq 1$$

and

$$\delta_{E/F}(1) := -\sum_{\sigma \neq 1} \delta_{E/F}(\sigma).$$

2.4.4 Remark. By the fact that the G_t for $t \in \mathbb{Q}^2$ are independent of x_E, by Lemma 2.2.6, we see that the $\delta_{E/F}$, $a_{E/F}$ and $s_{E/F}$ are independent of the choice of x_E.

2.4.1 Relation to Kato's Swan conductor

Let χ be a character of G. Let us write $\mathrm{Sw}_{E/F}(\chi) = [\pi^{\delta_\chi}] - [\omega_\chi]$, where $\omega_\chi \in \Omega_{\overline{F}}^{\otimes \mathrm{rk}(\chi)}$. Define $m_\sigma := v_F(x_E - \sigma x_E)$ and $n_\sigma := \#(x_E - \sigma(x_E))$. Notice that for each $\sigma \in G$ with $\sigma \neq 1$, there exists a $u_\sigma \in E$ such that

$$x_E - \sigma(x_E) = \pi^{m_\sigma} \cdot x_E^{n_\sigma} \cdot u_\sigma. \tag{2.8}$$

Furthermore by the definitions of $\delta_{E/F}$ and $a_{E/F}$, we have that both $u_\sigma \in E$ (respectively its reduction $\overline{u_\sigma} \in \overline{E}$) are units, i.e. $v_E(u_\sigma) = 0$ (respectively $v_{\overline{E}}(\overline{u_\sigma}) = 0$).

Let us now calculate δ_χ and ω_χ explicitly from $\delta_{E/F}$ and $a_{E/F}$. Indeed we see that inside $\tilde{S_E}$

$$\begin{aligned}
\mathrm{Sw}_{E/F}(\chi) &= \sum_{\sigma \in G} \mathrm{sw}_{E/F}(\sigma) \otimes \chi(\sigma) + \chi(1) \cdot \epsilon(\zeta_p) \\
&= \sum_{\sigma \neq 1} ([\mathrm{d}\overline{x_E}] - [\pi^{m_\sigma} x_E^{n_\sigma} u_\sigma]) \otimes (\chi(\sigma) - \mathrm{rk}(\chi)) + \chi(1) \cdot \epsilon(\zeta_p) \quad \text{from (2.8).}
\end{aligned}$$

The term $-\mathrm{rk}(\chi)$ in the factors $(\chi(\sigma) - \mathrm{rk}(\chi)$ comes from the fact that we have defined

$$\mathrm{sw}_{E/F}(1_G) = -\sum_{\sigma \neq 1_G} \mathrm{sw}_{E/F}(\sigma)$$

and the fact that $\chi(1) = \mathrm{rk}(\chi)$.

First we concentrate on the terms $\sum_{\sigma \neq 1} [\mathrm{d}\overline{x_E}] \otimes (\chi(\sigma) - \mathrm{rk}(\chi))$. We find

$$\begin{aligned}
\sum_{\sigma \neq 1} [\mathrm{d}\overline{x_E}] \otimes (\chi(\sigma) - \mathrm{rk}(\chi)) &= -|G| \cdot \mathrm{rk}(\chi) \cdot [\mathrm{d}\overline{x_E}] \\
&= -\mathrm{rk}(\chi) \cdot [\mathrm{d}\overline{x_F}]
\end{aligned}$$

where x_F is an element of F such that in the reduction $\overline{x_F} := \overline{x_E}^{|G|}$ (and hence x_F is a local geometric parameter of F). Consider now the terms

$$-\sum_{\sigma \neq 1} [\pi^{m_\sigma} x_E^{n_\sigma} u_\sigma]) \otimes (\chi(\sigma) - \mathrm{rk}(\chi)).$$

We notice that inside $\tilde{S_E}$ we have that

$$[\pi^{m_\sigma} x_E^{n_\sigma} u_\sigma] = [\pi^{m_\sigma}] + [x_E^{n_\sigma}] + [u_\sigma].$$

A calculation reveals that

$$-\sum_{\sigma \neq 1} [\pi^{m_\sigma}] \otimes (\chi(\sigma) - \mathrm{rk}(\chi)) = [\pi^{\delta_{E/F}(\chi)}]$$

and that

$$-\sum_{\sigma \neq 1} [x_E^{n_\sigma}] \otimes (\chi(\sigma) - \mathrm{rk}(\chi)) = [x_E^{-|G| \cdot a_{E/F}(\chi)}] = [x_F^{-a_{E/F}(\chi)}].$$

Therefore we obtain that

$$\begin{aligned} \mathrm{Sw}_{E/F}(\chi) = [\pi^{\delta_{E/F}(\chi)}] - [x_F^{-a_{E/F}(\chi)}] - \mathrm{rk}(\chi) \cdot [\mathrm{d}\overline{x_F}] \\ - \sum_{\sigma \neq 1} [u_\sigma] \otimes (\chi(\sigma) - \mathrm{rk}(\chi)) + \chi(1) \cdot \epsilon(\zeta_p). \end{aligned}$$

Define $u \in \tilde{S_E}$ by

$$u := -\sum_{\sigma \neq 1} [u_\sigma] \otimes (\chi(\sigma) - \mathrm{rk}(\chi)) + \chi(1) \cdot \epsilon(\zeta_p).$$

By Kato's Hasse-Arf theorem (Theorem 1.5.1), we see that $u \in S_F$. Furthermore, we have that

$$\mathrm{ord}_{E,\pi}(u) = \mathrm{ord}_{E,\Omega}(u) = \mathrm{ord}_{E,\overline{E}}(u) = 0$$

and hence we obtain the following theorem.

2.4.5 Theorem. *Let $\mathrm{Sw}_{E/F}(\chi) = [\pi^n] - [\omega]$, where $\omega \in \Omega_{\overline{F}}^{\otimes j}$ and $n \in \mathbb{Z}$. Then we have that*

$$j = \mathrm{rk}(\chi), \quad n = \delta_{E/F}(\chi)$$

and finally the order of ω in $\overline{F}$ is given by

$$\mathit{ord}_{\overline{F}}\, \omega = -a_{E/F}(\chi).$$

We obtain as a corollary of Theorem 2.4.5 the following.

2.4.6 Corollary. *The values of $a_{E/F}$ and $\delta_{E/F}$ at the irreducible characters of G are integers.*

2.4.7 Remark. In the next chapter we shall see that the values of $a_{E/F}$ and $\delta_{E/F}$ at the irreducible characters of G are nonnegative integers.

2.4.2 The upper ramification jumps

In this section we shall relate the Artin and depth characters (and hence by means of Theorem 2.4.5 also Kato's Swan conductor) to the upper ramification jumps of G. Let $t_1 < \ldots < t_m \in \mathbb{Q}^2$ be the sequence of upper ramification jumps of G. To help us reach our goal, we shall also work with the lower ramification jumps, and we denote them by $s_1 < \ldots < s_m \in \mathbb{Q}^2$. We write $s_i = (a_i, b_i)$ where $a_i, b_i \in \mathbb{Q}$.

We define a class function λ on G with values in $\mathbb{Q}^2$ by

$$-|G| \cdot \lambda := t_1 \cdot u_G + \sum_{j<m} (t_{j+1} - t_j) \cdot \mathrm{Ind}_{G^{t_{j+1}}}^{G} u_{G^{t_{j+1}}}. \tag{2.9}$$

Let $\sigma \in G$ be an element of $G_{s_i} - \bigcup_{s'>s_i} G_{s'}$. Then we see that by the definition of $s_{E/F}$ and $\delta_{E/F}$ we have

$$s_{E/F}(\sigma) = -|G| \cdot a_i, \quad \delta_{E/F} = -|G| \cdot b_i. \tag{2.10}$$

Since $\sigma \in G_{s_i} - \bigcup_{s'>s_i} G_{s'}$, we see that $\sigma \in G^{t_i} - \bigcup_{t'>t_i} G^{t'}$. Therefore by the normality of the G^{t_i} in G and by (2.9) we have that

$$\lambda(\sigma) = \frac{t_1}{|G|} + \sum_{j<i} \frac{t_{j+1} - t_j}{|G^{t_{i+1}}|} = s_i \tag{2.11}$$

by the relation (2.5) between the lower and upper ramification jumps. We now write $t_i = (h_i, d_i)$ with $h_i, d_i \in \mathbb{Q}$. Then we obtain the following theorem.

2.4.8 Theorem. *We have that*

$$s_{E/F} = h_1 \cdot u_G + \sum_j (h_{j+1} - h_j) \cdot \mathrm{Ind}_{G^{t_{j+1}}}^{G} u_{G^{t_{j+1}}} \tag{2.12}$$

and

$$\delta_{E/F} = d_1 \cdot u_G + \sum_j (d_{j+1} - d_j) \cdot \mathrm{Ind}_{G^{t_{j+1}}}^{G} u_{G^{t_{j+1}}}. \tag{2.13}$$

PROOF. Let $\sigma \neq 1$. Then there exists a lower jump s_i such that $\sigma \in G_{s_i} - \bigcup_{s'>s_i} G_{s'}$. By (2.11) we see that

$$\lambda(\sigma) = s_i.$$

Therefore by (2.9) and the (2.10) we see that (2.12) and (2.13) holds for all elements of G different from 1. However, since both the left hand sides and right hand sides of (2.12) and (2.13) are orthogonal to the trivial character 1_G, we see that they are also equal at the element 1_G. □

Following Serre [37] we now define $\chi(G^t)$, where t is an upper ramification jump and χ is a class function of G, as follows:

$$\phi(G^t) := \frac{1}{|G^t|} \sum_{\sigma \in G^t} \chi(\sigma).$$

We leave the proof of the following for the reader.

2.4.9 Lemma. *Let t be an upper ramification jump and let χ be a character of G. Then we have that*

$$\left\langle \chi, \operatorname{Ind}_{G^t}^G u_{G^t} \right\rangle = \langle \chi|_{G^t}, u_{G^t} \rangle = \chi(1) - \chi(G^t).$$

Therefore we obtain as a corollary of Theorem 2.4.8 the following.

2.4.10 Corollary. *Let χ be a character of G. Then we have that*

$$s_{E/F}(\chi) = h_1 \cdot (\chi(1) - \chi(G)) + \sum_j (h_{j+1} - h_j) \cdot (\chi(1) - \chi(G^{t_{j+1}}))$$

and

$$\delta_{E/F}(\chi) = d_1 \cdot (\chi(1) - \chi(G)) + \sum_j (d_{j+1} - d_j) \cdot (\chi(1) - \chi(G^{t_{j+1}})).$$

Finally we are able to prove the following theorem.

2.4.11 Theorem. *Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be an irreducible representation of G and let χ be the corresponding character. Assume that $t \in \mathbb{Q}^2$ is the largest upper ramification jump such that $\rho(G^t) \neq 1$. Let $t = (h, d)$ with $h, d \in \mathbb{Q}$. Then we have that*

$$s_{E/F}(\chi) = \mathrm{rk}(\chi) \cdot h$$

and

$$\delta_{E/F}(\chi) = \mathrm{rk}(\chi) \cdot d.$$

In particular, we have the following generalization of Serre [37] Exercise VI.2.2:

$$a_{E/F}(\chi) = \mathrm{rk}(\chi) \cdot (h + 1).$$

PROOF. Let $s \in \mathbb{Q}^2$ be such that $s > t$. Then since $\rho(G^s) = 1$, we have that $\chi(G^s) = \mathrm{rk}(\chi) = \chi(1)$. Therefore we have that

$$s_{E/F}(\chi) = h_1 \cdot (\chi(1) - \chi(G)) + \sum_{j<i} (h_{j+1} - h_j) \cdot (\chi(1) - \chi(G^{t_{j+1}}))$$

where $t_i = t$ (i.e. t is the ith upper jump).

Let us now assume that $s \leq t$ and we shall study $\chi(G^s)$. Since $\rho|_{G^s}$ is not the trivial representation on G^s, there exists a nontrivial irreducible character μ of G^s such that $\langle \chi|_{G^s}, \mu \rangle \neq 0$. We have that the G^s are normal subgroups of G. For a $g \in G$, denote by τ_g the conjugation of $G^s \to G^s$ by g. Then by Clifford's theorem (Dornhoff [11] Theorem 14.1), there exists integers e and f and elements $g_1, \ldots, g_f \in G$ such that

$$\chi|_{G^s} = e \cdot \sum_{1 \leq j \leq f} \mu \circ \tau_{g_i}.$$

Furthermore, each $\mu \circ \tau_{g_i}$ is an irreducible nontrivial character of G^s. Therefore we have that

$$\begin{aligned}\chi(G^s) &= \langle \chi|_{G^s}, 1_{G^s}\rangle \\ &= e \cdot \sum_{1\leq j\leq f} \langle \mu \circ \tau_{g_i}, 1_{G^s}\rangle \\ &= 0.\end{aligned}$$

We see therefore that for each $s \leq t$ we have that

$$\chi(1) - \chi(G^s) = \mathrm{rk}(\chi).$$

The result now follows. □

2.4.12 Corollary. *Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be an irreducible representation of G and let χ be the corresponding character. Assume that $t \in \mathbb{Q}^2$ is the largest upper ramification jump such that $\rho(G^t) \neq 1$. Let $t = (h, d)$ with $h, d \in \mathbb{Q}$. Let $\mathrm{Sw}_{E/F}(\chi) = [\pi^n] - [\omega]$ where $n \in \mathbb{Z}$ and $\omega \in \Omega_{\overline{F}}^{\otimes \mathrm{rk}(\chi)}$. Then we have that*

$$n = \mathrm{rk}(\chi) \cdot d, \quad -ord_{\overline{F}}\, \omega = \mathrm{rk}(\chi) \cdot (1 + h).$$

2.4.3 Example : $G = (\mathbb{Z}/p\mathbb{Z})^2$

For our example we let $G = (\mathbb{Z}/p\mathbb{Z})^2$. What follows is a continuation of Example 1.7.2. Let K be a finite extension of $\mathbb{Q}_p(\zeta_p)$ such that the ramification index $e_{K/\mathbb{Q}_p(\zeta_p)}$ satisfies $e_{K/\mathbb{Q}_p(\zeta_p)} < p$. Let π be a local parameter of K. Let $Y \to X$ be a G-Galois cover of smooth projective K-curves and assume that Y attains semistable reduction over K. Let $\mathcal{Y} \to \mathrm{spec}(\mathcal{O}_K)$ be a semistable model of Y such that the G-action extends to $\mathcal{Y}$. We denote by $\mathcal{X} := \mathcal{Y}/G$ the quotient model of X. Furthermore we denote by $\mathcal{Y}_k$ the special fibre of $\mathcal{Y}$.

2.4.13 Theorem. *There exists no component $\tilde{\Gamma}$ of $\mathcal{Y}_k$ with inertia group G.*

Proof. Assume that such a $\tilde{\Gamma}$ existed, and denote by $\Gamma \subset \mathcal{X}_k$ its image inside the special fibre $\mathcal{X}_k$. Let $x \in \Gamma$ be a smooth point and let y be its preimage. Assume that y is also a smooth point. We set $A := \widehat{\mathcal{O}_{\mathcal{X},x}}$ and $B := \widehat{\mathcal{O}_{\mathcal{Y},y}}$. Notice that B/A is a G-Galois extension. We let $F := \mathrm{Frac}(\hat{A}_{(\pi A)})$ and $E := B \otimes_A F$. By assumption on the inertia group of $\tilde{\Gamma}$, we see that E/F is a G-Galois extension of Case-II type of two-dimensional local fields. We are thus in the situation of this chapter.

We let H_i, $i = 0, 1, \ldots, p$, be the order-p subgroups of G, and for each $i = 0, 1, \ldots, p$, we choose an irreducible G-character χ_i with kernel H_i. We define $\delta_i := \delta_{E/F}(\chi_i)$. Notice that by Theorem 2.4.11 we may assume that $\delta_1 = \ldots = \delta_p$ (since we may assume that none of $H_1, \ldots, H_p$ is a higher ramification group).

Let χ be an irreducible character of H_0. We define $\delta := \delta_{E/E^{H_0}}(\chi)$. Then we have by (1.1) of Section 1.7 that

$$p \cdot \delta_1 = \sum_{1\leq i\leq p} \delta_i = p \cdot \delta + (p-1) \cdot \delta_0.$$

Notice that by Remark 1.4.4 we have that $p|\delta$ and since $\delta_1 = \ldots = \delta_p$, we have that $p^2|\delta_0$. However,

$$0 < \delta_0 \leq v_K(\lambda^p) = p \cdot e_{K/\mathbb{Q}_p(\zeta_p)} < p^2$$

inside K, a contradiction. □

2.5 The local vector-space theorem: trivial filtration case

We now define vector spaces associated to the differentials ω_χ, where the χ range over the irreducible characters of the group G. For an integer i, we define the set $V^i_{E/F} \subset \Omega_{\overline{F}}$ by

$$V^i_{E/F} := \{\omega_\chi | \delta_\chi = i\} \cup \{0\}.$$

From Theorem 1.6.1 it follows that $V^i_{E/F}$ is a $\mathbb{F}_p$-vector space. Our aim is to prove the following theorem

2.5.1 Theorem. *Assume that G is abelian. Then the vector space $V^i_{E/F}$ isomorphic to the group* $\mathrm{Hom}_{\mathbb{F}_p}(\hat{G}^i/\hat{G}^{i+1}, \mathbb{C}^*)$*, and therefore isomorphic to the quotient itself.*

2.5.2 Remark. It is clear that this theorem needs refinement if we are to handle nonabelian G as well.

In the rest of this section we shall deduce this theorem in the case that $F \subset E$ induces a trivial simplified higher ramification filtration, i.e. $\hat{G}^i$ is either G or $\{1_G\}$. We shall leave the general case for Section 2.6.

Let us denote the unique simplified higher ramification jump by n. Notice that by Theorem 2.3.16 our assumption implies that $G \simeq (\mathbb{Z}/p\mathbb{Z})^N$ for some N.

Let $X(G) := \mathrm{Hom}(G, \mathbb{C}^*)$ denote the group of irreducible degree one characters of G.

Let $\chi_1, \chi_2 \in X(G)$ be nontrivial. By assumption on the filtration of G, we see from Corollary 2.4.12 that

$$\delta_{\chi_1} = \delta_{\chi_2} = n.$$

Assume that $\chi_2 \neq \chi_1^{-1}$. In this case we have that $\chi_1\chi_2$ is nontrivial on $G = \hat{G}^n$. Therefore, it too satisfies $\delta_{\chi_1\chi_2} = n$.

Thus by Theorem 1.6.1 we have that

$$\omega_{\chi_1\chi_2} = \omega_{\chi_1} + \omega_{\chi_2} \neq 0.$$

Furthermore, we have that

$$\omega_{\chi_1^{-1}} = -\omega_{\chi_1}.$$

Hence the association $\omega : X(G) \to V^n_{E/F} \subset \Omega_{\overline{F}}$

$$\omega : \chi \mapsto \omega_\chi$$

is a group homomorphism, and is in fact an embedding. However, we see that ω is also onto, since the cardinality of $V^n_{E/F}$ can by definition not exceed the cardinality of $X(G) = X(\hat{G}^n)$. This proves Theorem 2.5.1 in the special case of exactly one simplified upper ramification jump.

2.6 The local vector-space theorem: general case

We now prove Theorem 2.5.1 in the general case. Let G therefore be an abelian group and let the simplified upper ramification jumps be $i_1, \ldots, i_N$.

2.6.1 Claim. *Theorem 2.5.1 holds for the first simplified upper jump i_1, i.e. $X(\hat{G}^{i_1}/\hat{G}^{i_1+1}) \simeq V^{i_1}_{E/F}$.*

PROOF. From Corollary 2.4.12 any character with $\delta_\chi = i_1$ vanishes on $\hat{G}^{i_1+1}$. Therefore, all such characters induce characters of $G/\hat{G}^{i_1+1}$. It therefore suffices to consider the subextension $F \subset E^{G^{i_1+1}}$ with Galois group $G/\hat{G}^{i_1+1}$. But in this case the theorem has already been proved in Section 2.5. $\square$

Assume now that the theorem has been proved for the indices $i = i_1, \ldots, i_{r-1}$, i.e. for each $j \le r-1$ we have that

$$X(\hat{G}^{i_j}/\hat{G}^{i_j+1}) \simeq V^{i_j}_{E/F}.$$

We shall now prove Theorem 2.5.1 for the index $i = i_r$. Consider the subextension $F \subset F' := E^{\hat{G}^{i_r+1}}$ with Galois group $G' := G/\hat{G}^{i_r+1}$. Notice that the simplified upper ramification jumps of G' are given by $i_1 < \ldots < i_r$. Furthermore, by Corollary 2.4.12, every G-character χ with $\delta_\chi \le i_r$ vanishes on $\hat{G}^{i_r+1}$, and therefore comes from a character on $G' = G/\hat{G}^{i_r+1}$. The spaces $V^{i_r}_{F'/F}$ and $V^{i_r}_{E/F}$ are thus isomorphic by Theorem 1.3.9. By restricting our attention to the subextension $F \subset F'$, we may thus assume that $\hat{G}^{i_r+1} = \{0\}$ is trivial.

Consider the exact sequence of groups

$$0 \to \hat{G}^{i_r} \to \hat{G} \to \hat{G}/\hat{G}^{i_r} \to 0.$$

We now use the assumption that G is abelian.

2.6.2 Claim. *By applying $X(-) := \mathrm{Hom}(-, \mathbb{C}^*)$, we obtain an exact sequence*

$$0 \to X(\hat{G}/\hat{G}^{i_r}) \to X(\hat{G}) \to X(\hat{G}^{i_r}) \to 0.$$

PROOF. We have an exact sequence

$$0 \to X(\hat{G}/\hat{G}^{i_r}) \to X(\hat{G}) \to X(\hat{G}^{i_r}).$$

One notes that for an abelian group M, every irreducible representation is of degree 1 and hence we have that $|X(M)| = |M|$. Thus by a cardinality argument, we see that the last homomorphism is also surjective. $\square$

Let $\chi_1, \chi_2 \in X(\hat{G})$ which maps to the same *nontrivial* element of $X(\hat{G}^{i_r})$, i.e.

$$\chi_1|_{\hat{G}^{i_r}} = \chi_2|_{\hat{G}^{i_r}}.$$

We see that $\chi_1\chi_2^{-1}$ vanishes identically on the subgroup $\hat{G}^{i_r}$, and therefore by Corollary 2.4.12 it follows that

$$\delta_{\chi_1\chi_2^{-1}} < \delta_{\chi_1} = \delta_{\chi_2} = i_r.$$

Thus by Theorem 1.6.1 we see that

$$\omega_{\chi_1} = \omega_{\chi_2}.$$

We may therefore define a function $\omega_{i_r} : X(\hat{G}^{i_r}) \to V_{E/F}^{i_r}$ by defining

$$\omega_{i_r} : \chi \mapsto \omega_{\hat{\chi}},$$

where $\hat{\chi}$ is any character of G restricting to χ on $\hat{G}^{i_r}$. By Corollary 2.4.12 we see that this map is injective and surjective. This finishes the proof of Theorem 2.5.1.

Chapter 3

Group actions on the open disc

In Green–Matignon [16] and later Henrio [19], a combinatorial object, the so-called *Hurwitz tree*, was introduced in order to study $\mathbb{Z}/p\mathbb{Z}$-actions on the p-adic open disc. These objects simultaneously reflected the local ramification theory of such an action, as well as the relative positions of the geometric ramification points.

In this chapter we shall partially generalize the concept of Hurwitz trees to general p-groups. Most of the work in this chapter is taken from Brewis–Wewers [6]. Our aim is to introduce a combinatorial object which reflects the ramification theory of a Galois extension of the p-adic open disc, as well as the relative positions of the geometric fixed points. The new ingredient in our approach is the use of the *Artin* and *depth* characters introduced in Chapter two. These were originally introduced by Huber [20] as generalizations of the classical Artin character.

First we define the underlying objects of Hurwitz trees in Section 3.2.2, the *metric* trees. These are essentially ordered trees which are related to the geometry of the fixed points of an action on the open disc. Thereafter in Section 3.2.3 we consider the very technical definition of a Hurwitz tree. This is a metric tree with additional data which measures the ramification around the fixed points in the open disc.

In Section 3.3 we introduce the notion of *density* of a Hurwitz tree. Roughly speaking, this can be interpreted as a measure of the relative distances between the branch points. It will be a fundamental tool in Section 3.6.2 for showing that certain generalized quaternion actions in characteristic 2 do not lift to characteristic 0. We show in particular how the density of the branch points can be determined by means of the representation theory of G and the depth characters (to be introduced later) of the Hurwitz tree.

Thereafter the difficult part of associating a Hurwitz tree to a group action on the open disc begins. This is done in Section 3.5, first by localizing around a single branch point, and then building the tree in an inductive manner by moving away from the branch point.

Finally, in Section 3.6 we introduce a new obstruction to the lifting problem, namely the existence of suitable Hurwitz trees.

Chinburg, Guralnick and Harbater ([8], [9]) call a group G a *local Bertin group* if the Bertin obstruction (see Bertin [1]) of every local G-action vanishes. They call G a *local Oort group* if every local G-action lifts to characteristic zero. They prove that the generalized quaternion groups are local Bertin groups if their orders exceed 8. However, using our results on the density of Hurwitz trees, we shall see in Section 3.6.2 that there exist generalized quaternion actions, the so-called *simple* actions, which cannot be lifted to characteristic 0. This answers Question 1.3 of Chinburg–Guralnick–Harbater [8] negatively. Furthermore, this shows that our necessary condition is strictly stronger than that of Bertin.

Finally, we also generalize an old theorem of Green–Matignon [15] which studies the geometric branch points of a $(\mathbb{Z}/p\mathbb{Z})^2$-Galois cover of the open disc.

3.1 Notations and setting

Let K be a complete discrete valuation field of characteristic 0 with algebraically closed residue field k of characteristic p. We denote by R the ring of integers of K, and by π a local parameter. We denote by v_K the valuation of K, and by $\tilde{K}$ the algebraic closure of K. We shall always normalize v_K such that $v_K(\pi) = 1$. We shall write $|\cdot|_{\tilde{K}}$ for the norm on the normed field $\tilde{K}$.

Let G be a finite group. We denote by $R(G)$ the Grothendieck group of the category of $\mathbb{C}[G]$-modules of finite type. We may identify elements of $R(G)$ with their virtual characters $\chi : G \to \mathbb{C}$. We denote by $R^+(G) \subset R(G)$ the submonoid of true characters.

We write $1_G \in R^+(G)$ for the unit character, $r_G \in R^+(G)$ for the regular character and $u_G = r_G - 1_G \in R^+(G)$ for the augmentation character.

3.2 Hurwitz trees

A Hurwitz tree $\mathcal{T}$ consists of an oriented metric tree T and certain additional data attached to each vertex and edge of T, satisfying certain conditions. These additional data are related to a finite group G. We postpone all motivation and explanation of the following definitions to Section 3.4.

In Section 3.3 we discuss the notion of *density*. Later on in Section 3.6 this will be our main tool for showing that certain Hurwitz trees and, therefore, certain group actions on the disc, are impossible.

3.2.1 The multiplicative character

We start by introducing an important character to be used later in our work. Fix the field K.

3.2.1 Definition. Let $G = \langle\sigma\rangle \cong \mathbb{Z}/p^m\mathbb{Z}$ be a finite cyclic group of order p^m, with $m \geq 0$. We define an element $\delta_G^{\mathrm{mult}} \in R(G)$ via the following class function. For $a \not\equiv 0 \pmod{p^m}$ we set

$$\delta_G^{\mathrm{mult}}(\sigma^a) := -\frac{p^{i+1}}{p-1} \cdot v_K(p),$$

where $i := \mathrm{ord}_p(a) < m$ is the exponent of p in a; furthermore,

$$\delta_G^{\mathrm{mult}}(1) := -\sum_{a=1}^{p^m-1} \delta_G^{\mathrm{mult}}(\sigma^a) = m\, p^m \cdot v_K(p).$$

Let $\chi \in R^+(G)$ be an irreducible character of G of order p^n (with $0 \leq n \leq m$). One checks that

$$\delta_G^{\mathrm{mult}}(\chi) = \begin{cases} \frac{np-n+1}{p-1} \cdot v_K(p) & n > 0, \\ 0 & \chi = 1_G. \end{cases} \tag{3.1}$$

It follows that $\delta_G^{\mathrm{mult}} \in R^+(G)$. The superscript $^{\mathrm{mult}}$ stands for *multiplicative* and was chosen because δ^{mult} describes the ramification of a torsor under the multiplicative group scheme μ_{p^n}. See Lemma 3.5.2.

3.2.2 Metric trees

3.2.2 Definition. Let T be a connected tree, with set of vertices V and set of edges E and with one distinguished vertex $v_0 \in V$, called the *root*. We call T a *rooted tree* if the root v_0 is connected to a unique edge $e_0 \in E$ (which we call the *trunk* of T).

A rooted tree T carries a natural orientation, determined by source and target maps $s, t : E \to V$, as follows. Given an edge $e \in E$, the source $s(e)$ (resp. the target $t(e)$) is the vertex adjacent to e contained in same connected component of $T \backslash \{e\}$ as v_0 (resp. in the connected component not containing v_0). If $v = s(e)$ and $v' = t(e)$ we call v' a *successor* of v; notation: $v \to v'$. There is a natural partial ordering $\leq$ on V, where $v_1 \leq v_2$ if and only if there is an oriented path starting from v_1 and ending at v_2.

It is clear that the root v_0 is the unique minimal vertex with respect to this ordering. A maximal vertex is called a *leaf*. We write $B \subset V$ for the set of all leaves. It follows from Definition 3.2.2 that B is nonempty and does not contain the root v_0. For any vertex v we define

$$B_v := \{\, b \in B \mid v \leq b \,\}$$

as the set of leaves which can be reached from v along an oriented path.

3.2.3 Definition. Let T be a rooted tree. A *metric* on T is given by a map $\epsilon : E \to \mathbb{Z}_{\geq 0}$, $e \mapsto \epsilon_e$ such that $\epsilon_e = 0$ if and only if $t(e)$ is a leaf. We call ϵ_e the *thickness* of the edge e. The pair (T, ϵ) is called a *metric tree*. Sometimes we write T instead of (T, ϵ), if no confusion can arise.

3.2.3 Hurwitz trees

Let G be a finite p-group. Fix the field K.

3.2.4 Definition. A *G-Hurwitz tree* over K is a datum $\mathcal{T} = (T, [G_v], a_e, \delta_v)$, where

- $T = (T, \epsilon)$ is a metric tree (with root v_0, trunk e_0 and set of leaves B),
- $[G_v]$ is the conjugacy class of a subgroup $G_v \subset G$, for every vertex v of T,
- $a_e \in R^+(G)$ is a character of G, for every edge e of T,
- $\delta_v \in R^+(G)$ is a character of G, for all vertices v.

We call G_v the *monodromy group* and δ_v the *depth* of the vertex v. We call a_e the *Artin character* of the edge $e \in E$.

The datum $\mathcal{T}$ is required to satisfy the following conditions:

(H1) Let v be a vertex. Then, up to conjugation in G, we have

$$G_{v'} \subset G_v,$$

for every successor v' of v. Moreover, we have

$$\sum_{v \to v'} [G_v : G_{v'}] > 1,$$

except if $v = v_0$ is the root, in which case there exists exactly one successor v' and we have $G_v = G_{v'} = G$.

(H2) The group G_b is nontrivial and cyclic, for every leaf $b \in B$.

(H3) For all $e \in E$ we have

$$a_e = \begin{cases} \sum_{t(e)=s(e')} a_{e'} & t(e) \notin B, \\ u^*_{G_b} & b = t(e) \in B. \end{cases}$$

(H4) For all $e \in E$ we have

$$\delta_{t(e)} = \delta_{s(e)} + \epsilon_e \cdot s_e,$$

where $s_e := a_e - u^*_{G_{t(e)}} \in R(G)$.

(H5) For $b \in B$ we have that

$$\delta_b = (\delta^{\mathrm{mult}}_{G_b})^*.$$

Here $\delta^{\mathrm{mult}}_{G_b}$ is given by Definition 3.2.1 for the field K.

We set

$$\delta_{\mathcal{T}} := \delta_{v_0}, \quad a_{\mathcal{T}} := a_{e_0},$$

which we call the *depth* and the *Artin character* of the Hurwitz tree $\mathcal{T}$.

3.2.5 Remark. Let $\mathcal{T} = (T, [G_v], a_e, \delta_v)$ be a Hurwitz tree, as in Definition 3.2.4.

1. Condition (H3) is equivalent to the following claim: for all edges e we have

$$a_e = \sum_{b \in B_{t(e)}} u^*_{G_b}.$$

   This follows immediately from induction over the tree T.

2. It follows from (1) that the Artin characters a_e are already determined by the tree T and the conjugacy classes of (cyclic) subgroups $([G_b])_{b \in B}$. Moreover, using (H4) and (H5) we see that the depth δ_v is determined by the metrized tree (T, ϵ) and the conjugacy classes $([G_v])_{v \in V}$.

3.2.6 Definition. Let $\mathcal{T}_K$ and $\mathcal{T}_{K'}$ be two Hurwitz trees over K and K' respectively, where K' is an extension of K. Let $e_{K'/K}$ denote the ramification index of K'/K. We say that $\mathcal{T}_K$ and $\mathcal{T}_{K'}$ are *equivalent* if

1. the underlying trees are isomorphic and the roots and trunks correspond,
2. the metrics are scaled by $e_{K'/K}$, i.e. if e is an edge then the thickness of e in $\mathcal{T}_{K'}$ is $e_{K'/K}$ times its thickness as an edge of $\mathcal{T}_K$,
3. the depth characters of the vertices are scaled by $e_{K'/K}$, i.e. if v is a vertex then its depth character in $\mathcal{T}_{K'}$ is $e_{K'/K}$ times its depth characters as a vertex of $\mathcal{T}_K$,
4. and finally if the monodromy group associated to a vertex of $\mathcal{T}_K$ is the same as that associated to the vertex in $\mathcal{T}_{K'}$.

3.3 Densities

We fix a G-Hurwitz tree $\mathcal{T} = (T, [G_v], a_e, \delta_v)$, with set of leaves B.

3.3.1 Definition. 1. Let $b_1, b_2 \in B$ be two distinct leaves. The *inverse distance* of b_1 and b_2 is the positive rational number $d(b_1, b_2) \in \mathbb{Q}_{>0}$ defined as follows. Let $(v_0, v_1, \ldots, v_r)$ be the longest oriented path in T starting from the root v_0 and ending in a vertex $v_r \notin B$ with $v_r \leq b_1, b_2$. For $i = 1, \ldots, r$ let e_i be the edge with $s(e_i) = v_{i-1}$ and $t(e_i) = v_i$. Then we set

$$d(b_1, b_2) := \sum_{i=1}^{r} \epsilon_{e_i}.$$

2. Let $A \subset B$ be a nonempty set of leaves and $b \in A$. The *density of A at b* is the rational number

$$d(A, b) := \sum_{b' \in A \setminus \{b\}} d(b, b').$$

Note that $d(A, b)$ only depends on A, b and the metrized tree T.

3.3.2 Lemma. *Let A, b be as in Definition 3.3.1.*

1. *Let $(v_0, v_1, \ldots, v_r, b)$ be the unique oriented path from the root to b. For $i = 1, \ldots, r$ let e_i be the edge with $s(e_i) = v_{i-1}$ and $t(e_i) = v_i$. Then*

$$d(A, b) = \sum_{i=1}^{r} \epsilon_{e_i} \cdot n(A, v_i),$$

where

$$n(A, v) := |\{b' \in A \mid b' \neq b,\ v \leq b'\}|.$$

2. *Let* $\chi \in R(G)^+$ *be a character such that*

$$\langle \chi, u^*_{G_a}\rangle_G = \begin{cases} m & a \in A, \\ 0 & a \in B\backslash A, \end{cases}$$

where $m := \langle \chi, u_G\rangle_G$. *Then*

$$m \cdot d(A,b) = \delta_b(\chi) - \delta_{v_0}(\chi).$$

PROOF. The proof of (1) follows from an induction argument which we leave to the reader. For the proof of (2) we may assume that $G_b \subset G_{v_r} \subset G_{v_{r-1}} \subset \ldots \subset G$, by Condition (H1) of Definition 3.2.4. We deduce the following sequence of inequalities

$$m = \langle \chi, u^*_{G_b}\rangle \leq \langle \chi, u^*_{G_{v_r}}\rangle \leq \ldots \leq \langle \chi, u_G\rangle = m,$$

which, a posteriori, turn out to be equalities. Using Remark 3.2.5 and the hypothesis on χ we therefore get

$$s_{e_i}(\chi) = a_{e_i}(\chi) - m = \sum_{a \in B_{v_i}} \langle \chi, u^*_{G_a}\rangle - m = m \cdot n(A, v_i). \tag{3.2}$$

Now we compute:

$$\begin{aligned} \delta_b(\chi) - \delta_{v_0}(\chi) = &\quad \sum_{i=1}^{r} \delta_{v_i}(\chi) - \delta_{v_{i-1}}(\chi) \\ \overset{\text{(H3)}}{=} &\quad \sum_{i=1}^{r} \epsilon_{e_i} \cdot s_{e_i}(\chi) \\ \overset{(3.2)}{=} &\quad m \cdot \sum_i \epsilon_{e_i} \cdot n(A, v_i) \\ \overset{\text{(i)}}{=} &\quad m \cdot d(A,b). \end{aligned}$$

□

3.3.3 Example. Assume that $G = \mathbb{Z}/p^n\mathbb{Z}$ and let $b \in B$ such that $G_b = G$. Let χ be an irreducible character of G with trivial kernel. Then by Lemma 3.3.2, (H5) and (3.1) we have

$$d(B,b) = \frac{np - n + 1}{p - 1} \cdot v_K(p) - \delta_{\mathcal{T}}(\chi).$$

If $\delta_{\mathcal{T}} = 0$ (which is the interesting case for us) we thus get a simple formula for the density $d(B,b)$ which puts a strong restriction on the metric of the tree T.

3.4 Group actions on the disk

3.4.1 Setting

We fix the following notation. We fix an open rigid-analytic disk Y over K and a subgroup $G \subset \mathrm{Aut}_K(\mathsf{Y})$. We assume that there exists at least one fixed point, i.e. a point in Y with a nontrivial stabilizer. The goal of this section is to attach to (Y, G) a G-Hurwitz tree $\mathcal{T} = (T, [G_v], a_e, \delta_v)$.

This construction is based on Huber's theory of Artin and Swan characters for rigid-analytic curves (Huber [20]). But since we only consider a very special case (a disc), we can do everything in an elementary and self-contained way, and we do not have to actually use any of the results of [20].

3.4.2 The depth character

At the beginning we shall work with a slightly more general situation than announced above. The ring B will either denote the ring of formal power series $R[\![z]\!]$ or the ring $R\{z\}$ of convergent powers series in z. It gives rise to a formal scheme $\mathcal{Y} := \operatorname{Spf} B$ and a rigid-analytic space $\mathsf{Y} := \mathcal{Y} \otimes K$. In the first case, Y is an *open disk*, i.e.

$$\mathsf{Y} = \{\, z \in \tilde{K} \;\mid\; |z|_{\tilde{K}} < 1 \,\}.$$

In the second case it is a *closed disk*, and we have a bijection

$$\mathsf{Y} = \{\, z \in \tilde{K} \;\mid\; |z|_{\tilde{K}} \leq 1 \,\}.$$

We let $\mathrm{val}_{\mathsf{Y}} : B\backslash\{0\} \to \mathbb{Z}$ denote the Gauss valuation, i.e.

$$\mathrm{val}_{\mathsf{Y}}\big(\sum a_i z^i\big) = \min_i\, \mathrm{val}(a_i).$$

We set $\bar{B} := B/(\pi)$ and let $\bar{f} \in \bar{B}$ denote the image of $f \in B$. We have $\bar{B} = k[\![z]\!]$ or $\bar{B} = k[z]$.

Suppose we are given a finite subgroup $G \subset \mathrm{Aut}_K(\mathsf{Y})$ of automorphisms of Y. The action of G extends uniquely to the formal model $\mathcal{Y}$ and hence induces an action of G on the ring B.

Our first goal is to define an invariant $\delta_{\mathsf{Y}}^G \in R^+(G)$, called the *depth character*. It measures the ramification of G with respect to $\mathrm{val}_{\mathsf{Y}}$, i.e. the amount to which the induced map $G \to \mathrm{Aut}_k(\bar{B})$ fails to be injective.

Let $I \lhd G$ be the inertia group with respect to $\mathrm{val}_{\mathsf{Y}}$, i.e. the normal subgroup consisting of elements $\sigma \in G$ with $\mathrm{val}_{\mathsf{Y}}(\sigma(z) - z) > 0$.

In Section 2.4 we introduced the *depth* character of a Galois extension. We reintroduce this here in order to emphasize that it is an invariant of Y.

3.4.1 Definition. The *depth character* associated to (Y, G) is the character $\delta_{\mathsf{Y}}^G \in R(G, \mathbb{Q})$ associated to the following class function:

$$\delta_{\mathsf{Y}}^G(\sigma) := -|G| \cdot \mathrm{val}_{\mathsf{Y}}(\sigma(z) - z)$$

for $\sigma \in G\backslash\{1\}$ and

$$\delta_{\mathsf{Y}}^G(1) := -\sum_{\sigma \neq 1} \delta_{\mathsf{Y}}^G(\sigma).$$

By definition we have $\delta_{\mathsf{Y}}^G = 0$ if and only if $I = \{1\}$.

3.4.3 The Artin character

We continue with the notation introduced above. But from now on we assume that $B = R[[z]]$, i.e. that Y is an open disk. Our goal is to define an Artin character $a_{\mathsf{Y}}^G \in R^+(G)$ which describes the action of G on the boundary of Y. We let E be the *boundary* of B, i.e. E is the fraction field of

$\hat{B}_{\pi B} = R[\![z]\!]\left\langle z^{-1}\right\rangle$. We let F be the fixed field of E under the induced action of G. Notice that the situation E/F is that of Chapter two.

We define

$$\#_{\mathsf{Y}} f := \mathrm{ord}_z\left(\overline{f/\pi^{\mathrm{val}_{\mathsf{Y}}(f)}}\right).$$

Here $\mathrm{ord}_z : k[[z]] \to \mathbb{Z} \cup \{\infty\}$ is the usual order function and $\pi^{\mathrm{val}_{\mathsf{Y}}(f)} \in R$ is an arbitrary element with valuation $\mathrm{val}_{\mathsf{Y}}(f)$. The Weierstrass preparation theorem shows that $\#_{\mathsf{Y}} f$ is the number of zeros of f on Y, counted with multiplicity.

3.4.2 Definition. The *Artin character* of (Y, G) is the element of $R^+(G)$ associated to the class function defined by

$$a_{\mathsf{Y}}^G(\sigma) := -\#_{\mathsf{Y}}(\sigma(z) - z), \qquad \text{for } \sigma \neq 1$$

and

$$a_{\mathsf{Y}}^G(1) := -\sum_{\sigma \neq 1} a_{\mathsf{Y}}^G(\sigma).$$

3.4.3 Remark. Notice that $a_{\mathsf{Y}} = a_{E/F}$.

We now relate a_{Y} to the permutation representation arising from the set of fixed points. For $\sigma \in G \backslash \{1\}$ let $\Delta_\sigma \subset \mathsf{Y}(\bar{K})$ denote the set of (geometric) fixed points of σ. Set

$$\Delta := \cup_{\sigma \neq 1} \Delta_\sigma.$$

This is a finite G-set. Let $B := \Delta/G$ denote the orbit space. Choose, for each $b \in B$, an element $y \in \Delta$ belonging to b and let $G_b \subset G$ denote the stabilizer of y.

3.4.4 Proposition. *We have*

$$a_{\mathsf{Y}}^G = \sum_{b \in B} u_{G_b}^*.$$

In particular, a_{Y}^G *is an element of* $R^+(G)$.

PROOF. Fix an element $\sigma \in G \backslash \{1\}$. Then Δ_σ is the set of zeros of the function $f_\sigma := \sigma(z) - z$. An easy local calculation, coupled with the assumption that σ has finite order and that $\mathrm{char}(K) = 0$, shows that all zeros of f_σ are simple (cf. Green–Matignon [16], §II.1). Therefore, by Definition 3.4.2 and the Weierstrass preparation theorem we have

$$a_{\mathsf{Y}}^G(\sigma) = -\#_{\mathsf{Y}} f_\sigma = -|\Delta_\sigma|.$$

The proposition follows immediately. $\square$

3.4.4 Relation between the depth and Artin characters

The next proposition is the key result behind the construction of the Hurwitz tree associated to (Y, G).

3.4.5 Proposition. *Let* $\mathsf{D} \subset \mathsf{Y}$ *be a closed disk which contains the set* Δ *and is fixed by the action of* G*. Let* $\mathsf{E} \subset \mathsf{D}$ *denote the residue class of a* K*-rational point* y *in* D*. Let* $H \subset G$ *denote the stabilizer of* E*. Then*

$$\delta_{\mathsf{D}}^G = \mathrm{Ind}_H^G \, \delta_{\mathsf{E}}^H = \delta_{\mathsf{Y}}^G + |G| \cdot \epsilon \cdot s_{\mathsf{Y}}^G, \tag{3.3}$$

where $s_{\mathsf{Y}}^G := a_{\mathsf{Y}}^G - u_G$ *and where* $\epsilon \in \mathbb{Z}_{>0}$ *is the thickness of the annulus* $\mathsf{Y}\backslash\mathsf{D}$ *(after a possible extension of* K *).*

PROOF. After a change of parameter we may assume that the point y is given by the equation $z = 0$. Then

$$\mathsf{D} = \{\, z \mid \mathrm{val}(z) \geq \epsilon \,\}, \quad \mathsf{E} = \{\, z \mid \mathrm{val}(z) > \epsilon \,\}.$$

After replacing K by some finite extension, we may further assume that there exists an element $a \in R$ with $\mathrm{val}(a) = \epsilon$. We obtain formal models $\mathsf{D} = (\mathrm{Spf}\, R\{w\}) \otimes_R K$ and $\mathsf{E} = (\mathrm{Spf}\, R[[w]]) \otimes K$, where $w := a^{-1}z$. By definition, we have $\mathrm{val}_{\mathsf{D}} = \mathrm{val}_{\mathsf{E}}|_{R\{w\}}$ and therefore

$$\delta_{\mathsf{D}}^G(\sigma) = \begin{cases} [G:H] \cdot \delta_{\mathsf{E}}^H(\sigma) & \sigma \in H\backslash\{1\} \\ \quad 0 & \sigma \in G\backslash H. \end{cases}$$

Now the first equality in (3.3) is obvious.

Fix an element $\sigma \in G\backslash\{1\}$ and set $f_\sigma := \sigma(z) - z \in R[[z]] \subset R\{w\}$. By the assumption on D, the function f_σ has no zero on the annulus $\mathsf{Y}\backslash\mathsf{D}$. It follows that

$$\mathrm{val}_{\mathsf{D}}(f_\sigma) = \mathrm{val}_{\mathsf{Y}}(f_\sigma) + \epsilon \cdot \#_{\mathsf{Y}} f_\sigma, \tag{3.4}$$

see e.g. the proof of [19], Proposition 1.10. We compute:

$$\begin{aligned} \delta_{\mathsf{D}}^G(\sigma) &= -|G| \cdot \mathrm{val}_{\mathsf{D}}(\sigma(w) - w) = -|G| \cdot \big(\mathrm{val}_{\mathsf{D}}(f_\sigma) - \epsilon\big) \\ &\overset{(3.4)}{=} -|G| \cdot \mathrm{val}_{\mathsf{Y}}(f_\sigma) - |G| \cdot \epsilon \cdot (\#_{\mathsf{Y}}(f_\sigma) - 1) \\ &= \delta_{\mathsf{Y}}^G(\sigma) + |G| \cdot \epsilon \cdot s_{\mathsf{Y}}^G. \end{aligned}$$

This proves the second equality in (3.3). □

3.5 Definition of the Hurwitz tree

We can now state and prove our main theorem.

3.5.1 Theorem. *Let* $\mathsf{Y} = (\mathrm{Spf}\, R[[z]]) \otimes K$ *be an open rigid disk over* K *and* $G \subset \mathit{Aut}_K(\mathsf{Y})$ *be a finite* p*-group of automorphisms. Suppose that the set of fixed points* $\Delta \subset \mathsf{Y}$ *is nonempty. Then after possibly extending* K *there exists a* G*-Hurwitz tree* $\mathcal{T}$ *over* K *with*

$$\delta_{\mathcal{T}} = \delta_{\mathsf{Y}}^G, \qquad a_{\mathcal{T}} = a_{\mathsf{Y}}^G. \tag{3.5}$$

PROOF. Our proof is by induction over the number of elements of Δ.
We first assume that $|\Delta| = 1$. In this case the theorem is essentially equivalent to the following lemma.

3.5.2 Lemma. *Let $y \in \Delta$ be the unique fixed point. Then*

1. *the group G is cyclic,*
2. $a^G_{\mathsf{Y}} = u_G$, *and*
3. $\delta^G_{\mathsf{Y}} = \delta^{\mathrm{mult}}_G$.

PROOF. It is clear that every element of G fixes the point y. So (2) follows directly from Proposition 3.4.4.

After a change of parameter we may assume that y is the point $z = 0$. Then for an element $\sigma \in G$ we have

$$\sigma(z) = \chi(\sigma)\, z\, (1 + a_1 z + a_2 z^2 + \ldots), \tag{3.6}$$

where $\chi : G \hookrightarrow K^\times$ is an injective character (Green–Matignon [16], §II.1). This proves (1). Let us fix an element $\sigma \in G$ of order np^m, with $(n, p) = 1$ and $m \geq 0$. By (3.6) we have

$$f_\sigma := \sigma(z) - z = (\chi(\sigma) - 1)\, z + \chi(\sigma)\, a_1 z^2 + \ldots$$

Since $z = 0$ is the only zero of f_σ, we have $\# f_\sigma = 1$ and therefore

$$\mathrm{val}_{\mathsf{Y}}(f_\sigma) = \mathrm{val}(\chi(\sigma) - 1) = \begin{cases} 0 & m = 0, \\ \frac{1}{(p-1)p^{m-1}} \cdot v_K(p) & m > 0. \end{cases}$$

Now (3) follows from Definition 3.2.1 and a direct computation. □

So in the case $|\Delta| = 1$ we define the Hurwitz tree $\mathcal{T} = (T, [G_v], \delta_v, a_e)$ as follows.

- The tree T has two vertices v_0, v_1 and one edge e_0 with $s(e_0) = v_0$ and $t(e_0) = v_1$. The metric ϵ is trivial, i.e. we set $\epsilon_{e_0} := 0$.
- We define
$$\delta_{v_0} = \delta_{v_1} := \delta^G_{\mathsf{Y}}.$$
- We define $G_{v_0} = G_{v_1} := G$ and $a_{e_0} := u_G$.

The validity of the axioms (H2) and (H5) follows from Lemma 3.5.2; all the other axioms and (3.5) hold by definition. This finishes the proof of the theorem in the case $|\Delta| = 1$.

We may now assume that $|\Delta| \geq 2$. Then there exists a smallest closed disk $\mathsf{D} \subset \mathsf{Y}$ which contains Δ. Clearly, D is fixed by the G-action. There also exists a finite family $(\mathsf{E}_j)_{j \in J}$ of residue classes $\mathsf{E}_j \subset \mathsf{D}$ with

$$\Delta_j := \mathsf{E}_j \cap \Delta \neq \emptyset \quad \text{and} \quad \Delta \subset \cup_j \mathsf{E}_j. \tag{3.7}$$

For $j \in J$ we let $G_j \subset G$ denote the stabilizer of E_j. By induction, there exists a Hurwitz tree $\mathcal{T}_j$ for the group G_j with

$$\delta_{\mathcal{T}_j} = \delta_{\mathsf{E}_j}^{G_j}, \qquad a_{\mathcal{T}_j} = a_{\mathsf{E}_j}^{G_j}. \tag{3.8}$$

The Hurwitz tree $\mathcal{T} = (T, [G_v], a_e, \delta_v)$ associated to (Y, G) is defined as follows.

- Let T_j denote the metric tree underlying the Hurwitz tree $\mathcal{T}_j$. Choose a system of representatives $J' \subset J$ of J/G. The metric tree T underlying $\mathcal{T}$ is obtained by patching together the metric trees T_j, $j \in J'$, at their roots, i.e. we identify the set of roots of the trees T_j, $j \in J'$ with one vertex v_1 of T. We complete T by adding another vertex v_0 (the root of T) and an edge e_0 with $s(e_0) = v_0$, $t(e_0) = v_1$. The value of the metric ϵ on the edge e_0 is defined as the thickness of the annulus $\mathsf{Y}\backslash\mathsf{D}$, multiplied with $|G|$. (In fact, ϵ_{e_0} is the thickness of the quotient annulus $(\mathsf{Y}\backslash\mathsf{D})/G$.)
- If v is a vertex of T other than v_0 and v_1, it corresponds to a vertex v' of one of the T_j which is not the root. We define $G_v := G_{v'}$ and $\delta_v := \mathrm{Ind}_{G_{v'}}^G \delta_{v'}$.
- Let e be an edge of T which corresponds to an edge e' of T_j. We define $a_e := \mathrm{Ind}^G a_{e'}$.
- We set $G_{v_0} = G_{v_1} := G$, $\delta_{v_0} := \delta_{\mathsf{Y}}^G$, $\delta_{v_1} := \delta_{\mathsf{D}}^G$ and $a_{e_0} := a_{\mathsf{Y}}^G$.

It remains to show that $\mathcal{T}$ satisfies the axioms (H1)-(H5). Since these axioms hold for the Hurwitz trees $\mathcal{T}_j$, many of them hold for $\mathcal{T}$ by construction. For instance, this is clear for (H1) and (H2).

It follows from (3.7), (3.8) and Proposition 3.4.4 that

$$a_{e_0} = a_{\mathsf{Y}}^G = \sum_{j \in J/G} \mathrm{Ind}_{G_j}^G a_{\mathsf{E}_j}^{G_j} = \sum_{s(e)=v_1} a_e. \tag{3.9}$$

Therefore, (H3) holds for the edge e_0. For the other edges it holds by construction.
To check the axioms (H4) and (H5) we remark that

$$\delta_{v_1} = \delta_{\mathsf{D}}^G = \mathrm{Ind}^G \delta_{\mathcal{T}_j}, \tag{3.10}$$

for all $j \in J$, by the first equality in (3.3). This means that our definition of δ_{v_1} is consistent with the fact that the vertex v_1 corresponds to the roots of the Hurwitz trees $\mathcal{T}_j$, $j \in J'$. It follows that (H5) holds automatically and that we have to check (H4) only for the edge e_0. But for the edge e_0 the statement of (H4) follows directly from Proposition 3.4.5. This concludes the proof of Theorem Theorem 3.5.1. □

3.5.3 Remark. An alternative way to construct the metric tree T is the following (cf. Henrio [19] and Bouw–Wewers [4]). Let $\mathcal{Y}$ be the minimal semistable model of the disk Y which separates the points of Δ. Then the G-action on Y extends to $\mathcal{Y}$, and the quotient $\mathcal{X} := \mathcal{Y}/G$ is a semistable model of the disk $\mathsf{X} = \mathsf{Y}/G$ which separates the points of $B := \Delta/G$. Now there is a standard way to associate to the pair $(\mathcal{X}, B)$ a metric tree T with set of leaves B (see e.g. Bouw–Wewers [4], §3.2). Essentially, T is a modification of the graph of components of the special fiber of $\mathcal{X}$.

The construction of T in the proof of Theorem 3.5.1 avoids the use of semistable models and may therefore be considered as more elementary. However, semistable models become inevitable if one wants to construct G-actions on the disk with given Hurwitz tree.

3.5.4 Remark. Assume that G acts on $R[\![z]\!]$ and that the associated Hurwitz tree $\mathcal{T}_K$ is defined over K. We may also consider an extension K'/K and the induced action of G on $R'[\![z]\!]$, where R' is the discrete valuation ring of K'. Let $\mathcal{T}_{K'}$ be the induced Hurwitz tree. Then one checks that the Hurwitz tree $\mathcal{T}_K$ defined over K and the Hurwitz tree $\mathcal{T}_{K'}$ defined over K' are equivalent (see Definition 3.2.6).

3.6 Applications to the lifting problem

3.6.1 A new obstruction

Let k be an algebraically closed field of characteristic $p > 0$ and G be a finite group. A *local G-action* is a faithful and k-linear action $\phi : G \hookrightarrow \mathrm{Aut}_k(k[[z]])$ on a ring of formal power series in one variable over k.

The *local lifting problem* asks: can ϕ be lifted to an action $\phi_R : G \hookrightarrow \mathrm{Aut}_R(R[[z]])$, where R is some discrete valuation ring of characteristic zero with residue field k. If it does then we say that ϕ *lifts to characteristic zero*.

From our main result we can deduce a new necessary condition for liftability of local G-actions. Before we state it, we recall the definition of the *classical Artin character* (see Serre [37] Chapter VI).

3.6.1 Definition. Let ϕ be a local G-action. The *Artin character* of ϕ is the element $a_\phi \in R^+(G)$ defined by

$$a_\phi(\sigma) := -\operatorname{ord}_z(\sigma(z) - z)$$

for $\sigma \neq 1$ and

$$a_\phi(1) := -\sum_{\sigma \neq 1} a_\phi(\sigma).$$

See [37], VI, §2.

3.6.2 Theorem (Hurwitz-tree obstruction). *Let $\phi : G \hookrightarrow Aut_k(k((t)))$ be a local G-action. If ϕ lifts to characteristic 0 then there exists a G-Hurwitz tree $\mathcal{T}$ over some K such that*

$$a_{\mathcal{T}} = a_\phi \qquad \textit{and} \qquad \delta_{\mathcal{T}} = 0.$$

PROOF. A lift of ϕ gives rise to a G-action on the disk $\mathsf{Y} = (\mathrm{Spf}\, R[[z]]) \otimes K$. Since ϕ is injective by assumption, we have $\delta_{\mathsf{Y}}^G = 0$ (Definition 3.4.1) and $a_{\mathsf{Y}}^G = a_\phi$ (Definition 3.4.2). Therefore, Theorem 3.6.2 is a direct consequence of Theorem 3.5.1. □

By the theorem, the existence of a Hurwitz tree $\mathcal{T}$ with given Artin character $a_{\mathcal{T}} = a_\phi$ and trivial depth $\delta_{\mathcal{T}} = 0$ is a necessary condition for ϕ to lift. In this case, if such that a Hurwitz tree exists, we shall say that the Hurwitz-tree obstruction *vanishes*. If one can show that such a Hurwitz tree does not exist, i.e. that the obstruction does not vanish, then one has found an obstruction against liftability of ϕ.

As a special case of this criterion, we obtain the well-known *Bertin obstruction*, see Bertin [1]. Namely, if $\mathcal{T} = (T, [G_v], a_e, \delta_v)$ is a Hurwitz tree with $a_{\mathcal{T}} = a_\phi$, then Remark 3.2.5 shows that

$$a_\phi = \sum_{b \in B} u^*_{G_b}. \tag{3.11}$$

This equality is easily seen to imply the following statement: there exists a finite G-set Δ, with cyclic stabilizers, such that

$$a_\phi = m \cdot r_G - \chi_\Delta. \tag{3.12}$$

Here $\chi_\Delta \in R^+(G)$ is the character of the permutation representation realized by Δ and $m := |\Delta/G|$. However, there exist local G-actions ϕ whose Artin character can *not* be written in this form (see e.g. Bertin [1] and Chinburg–Guralnick–Harbater [9]). It follows from Theorem 3.6.2 that such a ϕ does not lift to characteristic zero.

The examples presented in the following section show that our new obstruction is strictly stronger than the Bertin obstruction. However, it should be pointed out that the converse of Theorem 3.6.2 does not hold. For $G = \mathbb{Z}/p \times \mathbb{Z}/p$, Pagot has shown in Pagot [31] that certain local G-actions ϕ do not lift to characteristic zero. For such a ϕ it is straightforward to write down a Hurwitz tree $\mathcal{T}$ with $a_{\mathcal{T}} = a_\phi$ and $\delta_{\mathcal{T}} = 0$.

3.6.2 Simple quaternion actions

We fix an integer $n \geq 2$ and let $G = Q_{2^{n+1}}$ denote the generalized quaternion group of order 2^{n+1}, with presentation

$$Q_{2^{n+1}} = \langle\, \sigma, \tau \mid \tau^{2^n} = 1,\, \tau^{2^{n-1}} = \sigma^2,\, \sigma\tau\sigma^{-1} = \tau^{-1} \,\rangle. \tag{3.13}$$

Our base field k is assumed to be of characteristic 2.

Chinburg, Guralnick and Harbater [9] have proved that G is a *local Bertin group* for $n \geq 3$, which means that the Bertin obstruction of every local G-action over k vanishes. The goal of this section is to construct certain G-actions which do not lift to characteristic zero. This result gives a negative answer to Question 1.3 of Chinburg–Guralnick–Harbater [9].

We first introduce some more notation. Set

$$H_0 := \langle \tau \rangle, \quad H_1 := \langle \sigma \rangle, \quad H_2 := \langle \sigma\tau \rangle;$$

these are cyclic subgroups of G of order 2^n, 4 and 4, respectively. For $i = 0, 1, 2$ there exists a unique character $\chi_i : G \to \{\pm 1\}$ of order 2 such that $H_i \subset \mathrm{Ker}(\chi_i)$. Clearly, χ_0, χ_1, χ_2 define pairwise distinct irreducible characters of the quotient group

$$\bar{G} := G/\langle \tau^2 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2.$$

3.6.3 Definition. A local G-action ϕ is called *simple* if

$$a_\phi(\chi_0) = 2, \qquad a_\phi(\chi_1) = a_\phi(\chi_2) \geq 2.$$

3.6.4 Proposition. *There exists a simple G-action over k, for every $n \geq 2$.*

PROOF. Choose an embedding of abelian groups $\bar{G} \hookrightarrow (k,+)$. We obtain a local $\bar{G}$-action $\bar{\phi} : \bar{G} \hookrightarrow \mathrm{Aut}_k(k[[t]])$ by sending $\mu \in \bar{G}$ to the automorphism

$$t \mapsto \frac{t}{1+\mu t} = t - \mu t^2 + \mu^2 t^3 - \ldots .$$

One checks that

$$a_{\bar{\phi}}(\chi_i) = 2, \qquad \text{for } i = 0,1,2$$

By [8], Lemma 2.10, we can extend $\bar{\phi}$ to a local G-action $\phi : G \hookrightarrow \mathrm{Aut}_k(k[[z]])$, such that $k[[t]] = k[[z]]^{\langle \tau^2 \rangle}$. It follows from [37], Proposition IV.3, that

$$a_{\phi}(\chi_i) = a_{\bar{\phi}}(\chi_i), \qquad i = 0,1,2.$$

We conclude that ϕ is simple. □

3.6.5 Remark. It is possible to give an alternative proof of the proposition above using local class field theory. We sketch the idea here and leave the details for the reader. Let $F := k((t))$ and let L/F be a $\mathbb{Z}/2\mathbb{Z}$-extension with local degree of different exactly 2. We know from Pop's theorem that we can find a $\mathbb{Z}/2^n\mathbb{Z}$-Galois extension E/L such that the extension E/F is D_{2^n}-Galois, where D_{2^n} denotes the dihedral group of order 2^{n+1}. This is due to a very special splitting property of the group D_{2^n}.

We also know that we can find a $\mathbb{Z}/4\mathbb{Z}$-Galois extension M/L such that M/F is a Q_8-extension. Let $\chi_E : G_L \to \mathbb{Q}/\mathbb{Z}$ (respectively $\chi_M : G_L \to \mathbb{Q}/\mathbb{Z}$) be the irreducible character of order 2^n (respectively of order 4) associated with the extension E/L (respectively M/L), where G_L denotes the absolute Galois group of L. The composite character $\chi_E \circ \chi_M$ induces a $\mathbb{Z}/2^n\mathbb{Z}$-Galois extension N/L, and using the Verlagerung morphism one proves that N/F is a $Q_{2^{n+1}}$-Galois extension. Since it contains L, it is simple.

3.6.6 Theorem. *Let ϕ be a simple G-action over k. Then ϕ does not lift to characteristic zero.*

PROOF. Suppose that ϕ lifts to characteristic zero. After a possible extension of K, by Theorem 3.6.2, there exists a G-Hurwitz tree $\mathcal{T} = (T, [G_v], a_e, \delta_v)$ over K, with Artin character $a_{\mathcal{T}} = a_\phi$ and vanishing depth $\delta_{\mathcal{T}} = 0$. We will show that such a Hurwitz tree (up to equivalence) cannot exist. Our main tool is the notion of density introduced in Section 3.3.

Let B denote the set of leaves of the tree T. For $i = 0,1,2$ we set

$$B_i := \{ b \in B \mid [G_b] = [H_i] \}, \qquad B' := B_0 \,\dot{\cup}\, B_1 \,\dot{\cup}\, B_2$$

and

$$B^i := B' \backslash B_i.$$

Then for all $b \in B$ we have

$$\langle \mathrm{Ind}_{G_b}^G u_{G_b}, \chi_i \rangle = \begin{cases} 1 & b \in B^i, \\ 0 & b \in B \backslash B^i. \end{cases} \tag{3.14}$$

So Lemma 3.3.2 (ii), Condition (H5) of Definition 3.2.4 and (3.1) show that

$$d(B^i, b) = \delta_b(\chi_i) = 2 \cdot v_K(2), \tag{3.15}$$

for all $b \in B^i$.

From (3.11) and (3.14) we conclude that

$$a_\phi(\chi_i) = |B^i| = \sum_{j \neq i} |B_j|. \tag{3.16}$$

So the assumption that ϕ is simple (Definition 3.6.3) implies that

$$|B_1| = |B_2| = 1, \qquad |B_0| \geq 1.$$

Let b_0 denote the unique element of B_2. Since the B_i are disjoint, we have $B^0 \cap B^1 = B_2 = \{b_0\}$ and $B^0 \cup B^1 = B'$. Using Definition 3.3.1 (ii) and (3.15) we therefore get

$$d(B', b_0) = d(B^0, b_0) + d(B^1, b_0) = 2 \cdot v_K(2) + 2 \cdot v_K(2) = 4 \cdot v_K(2). \tag{3.17}$$

Let $\chi : H_0 \hookrightarrow \mathbb{C}^\times$ be an injective irreducible character. The induced character $\psi := \mathrm{Ind}_{H_0}^G \chi$ has the following property. For any nontrivial cyclic subgroup $C \subset G$, the restriction $\psi|_C$ is the sum of two *nontrivial* irreducible characters of C, $\psi|_C = \psi_1 + \psi_2$. Applying this to $C = G_b$, we obtain

$$\langle \mathrm{Ind}_{G_b}^G u_{G_b}, \psi \rangle_G = \langle \psi_1, u_{G_b} \rangle + \langle \psi_2, u_{G_b} \rangle = 2,$$

for all $b \in B$. We may therefore apply Lemma 3.3.2 (ii) and conclude that

$$d(B, b_0) = \delta_{b_0}(\psi)/2. \tag{3.18}$$

Moreover, the restriction of ψ to $G_{b_0} = H_2 \cong \mathbb{Z}/4$ is the sum of two irreducible characters ψ_1, ψ_2 of order 4. From (H5) and (3.1) we get

$$\delta_{b_0}(\psi) = \delta_{H_2}^{\mathrm{mult}}(\psi_1) + \delta_{H_2}^{\mathrm{mult}}(\psi_2) = 3 \cdot v_K(2) + 3 \cdot v_K(2) = 6 \cdot v_K(2). \tag{3.19}$$

We now obtain a contradiction by comparing (3.17), (3.18) and (3.19):

$$4 = d(B', b_0) \leq d(B, b_0) = 3 \cdot v_K(2).$$

We conclude that there does not exist a G-Hurwitz tree $\mathcal{T}$ (up to equivalence) with $a_{\mathcal{T}} = a_\phi$ and $\delta_{\mathcal{T}} = 0$. Theorem 3.6.6 follows. □

3.6.3 Example : $G = Q_8$

Let $K := \mathbb{Q}_2$ and let R be its ring of integers with parameter π_K. Let $A := R[[t]]$ and assume that B/A is a G-Galois extension such that B is also a local power series ring. We do not however assume that the geometric branch points of B/A are K-rational. Let K'/K now be such that all branch points are K'-rational and consider the induced extension B'/A' of $A' = R'[[t]]$, where R' is the ring of integers of K'. We let $\pi_{K'}$ be a local parameter of K'. We denote by $e_{K'/K}$ the ramification index of R'/R.

We let $F := \hat{A}_{(\pi_K A)}$ and we let $F' := \hat{A}_{(\pi_{K'} A')}$. Furthermore, we let $E := F \otimes_A B$ and $E' := F' \otimes_{A'} B'$. Notice that both E'/F' and E/F are of Case-II type and hence we are in the situation of Chapter two. One checks that

$$\delta_{E'/F'} = e_{K'/K} \cdot \delta_{E/F}. \tag{3.20}$$

Consider the simplified upper ramification filtration of the G-Galois extension E/F. Let i_1 be the first simplified upper ramification jump. We see that since G is not elementary abelian, we have that $\hat{G}^{i_1+1} \neq 1$. Let ϕ be an irreducible one-dimensional character of G with kernel containing $\hat{G}^{i_1+1}$. From Lemma 2.3.3 and Corollary 2.4.12 we see that $\delta_{E/F}(\phi) = i_1$.

Let ϕ have kernel H and let χ be an irreducible character of H of order 4. Let $\psi := \mathrm{Ind}_H^G \chi$. We see from (1.5) in Section 1.7 that

$$\delta_{E/F}(\psi) = 2 \cdot \delta_{E/E^H}(\chi) + \delta_{E/F}(\phi) = 2 \cdot \delta_{E/E^H}(\chi) + i_1. \tag{3.21}$$

From (3.21) and Remark 1.4.4 we see thus that $\delta_{E/F}(\psi) \in \mathbb{Z}$.

Furthermore, let $j \in \mathbb{Q}$ be the largest simplified upper ramification jump such that the underlying representation of ψ is not trivial on $\hat{G}^j$. We conclude that $\hat{G}^{j+1} = \{1\}$. Since $\hat{G}^{i_1+1} \neq 1$ we see that $j > i_1$. Furthermore, from Lemma 2.3.3 there exists some $t := (h, j) \in \mathbb{Q}^2$ such that t is an upper (usual) ramification jump and such that $\hat{G}^j \supset G^t$. Furthermore since $\hat{G}^{j+1} = \{1\}$, we can choose h such that $t = (h, j)$ is the last upper (usual) ramification jump. By Corollary 2.4.12 it follows that $\delta_{E/F}(\psi) = 2 \cdot j$.

Now assume that there exist at least two geometric branch points in the geometric cover $\operatorname{Spf} B' \to \operatorname{Spf} A'$, one of which is a point $b \in \operatorname{Spf} A'$ with inertia group cyclic of order 4. Then using the technique used (3.19) we see that $\delta_{E'/F'}(\psi) < 6.v_{K'}(2)$ (after assuming that K' has been extended such that the Hurwitz tree is defined over K'). However $v_{K'}(2) = e_{K'/K} \cdot v_K(2)$ and hence from (3.20) it follows that

$$j = \frac{\delta_{E/F}(\psi)}{2} < 3 \cdot v_K(2) = 3$$

We see thus that $i_1 < j \leq 2$. However $2|i_1$ from Remark 1.4.4 and hence we obtain a contradiction. We therefore have the following theorem.

3.6.7 Theorem. *There exists no Q_8-Galois extension B/A of formal power series rings over R with at least two geometric branch points (not necessarily R-rational), one of which has inertia $\mathbb{Z}/4\mathbb{Z}$ and which induces a purely inseparable extension of $\overline{A} := k[\![t]\!]$.*

3.7 Theorem of Green–Matignon

Let $G := (\mathbb{Z}/p\mathbb{Z})^2$ and consider a G-action on the ring $B := R[\![z]\!]$. Let $A := B^G$ and notice that we can find a $t \in A$ such that $A = R[\![t]\!]$. We let F be the field of fractions of $R[\![t]\!]\left\langle t^{-1}\right\rangle$ and, similarly, we let E be the field of fractions of $R[\![z]\!]\left\langle z^{-1}\right\rangle$. Notice that E/F is also a G-Galois extension.

Let us enumerate the subgroups of order p of G by $H_0, \ldots, H_p$. We let $B_i := B^{H_i}$ for $i = 0, \ldots, p$. Consider the formal R-scheme $\mathcal{X} := \operatorname{Spf} B$ and the quotient $\mathcal{X}_i := \operatorname{Spf} B_i$. Our aim for this section is study the relations between the geometric branch points of the different $\mathbb{Z}/p\mathbb{Z}$-covers $\mathcal{X}_i \to \mathcal{Y} := \operatorname{Spf} A$.

Let $\mathcal{B}^i$, $i = 0, \ldots, p$, be the set of branch points of $\mathcal{X}_i \to \mathcal{Y}$. We let $n_i := |\mathcal{B}^i|$. In Green–Matignon [15] it is shown that if the residue field extension $k((t)) \subset k((z))$ of $F \subset E$ is separable, then $p | n_0$ where we have assumed without loss of generality that n_0 is the minimum of $\{n_0, n_1, \ldots, n_p\}$. This is a version of the Bertin-obstruction (see Bertin [1]) for the group $G = (\mathbb{Z}/p\mathbb{Z})^2$. Furthermore, it is shown in Green–Matignon [15] that for $i \neq 0$, the covers $\mathcal{X}_i \to \mathcal{Y}$ and $\mathcal{X}_0 \to \mathcal{Y}$ share exactly $\frac{p-1}{p} \cdot n_0$ common geometric branch points.

Our aim for this section is to give an analog of this result for the case that E/F is of Case–II type. After possibly extending R, let $\mathcal{T}$ be the Hurwitz tree defined over R associated with the action of G on $B = R[\![z]\!]$. Let e_0 be its trunk and consider the Artin character a_{e_0} of the trunk e_0. We let $a_{E/F}$ be the Artin character of the extension E/F. By construction of $\mathcal{T}$ we see that $a_{e_0} = a_{E/F}$.

One sees that the leaves b with monodromy group $G_b \neq H_i$ are exactly in correspondence to the branch points of $\mathcal{X}_i \to \mathcal{Y}$. Let us define $m_i := |\{b | G_b = H_i\}|$. Therefore we have that $n_i = \sum_{j \neq i} m_j$.

We consider two cases dependent on the upper ramification filtration of the Galois extension E/F. The first case is when there is exactly one upper jump $t_1 \in \mathbb{Z}^2$, i.e. the filtration is $G = G^{t_1} \supset \{1\}$, and the second case is where there are exactly two upper jumps $t_1 < t_2 \in \mathbb{Z}^2$, i.e. the filtration is $G = G^{t_1} \supset G^{t_2} \supset \{1\}$.

We consider the first case first, i.e. the case of exactly one upper jump $t_1 \in \mathbb{Z}^2$ on the filtration of G. We write $t_1 = (a_1, b_1)$. Let χ_i be a homomorphism $G \to \mathbb{C}^*$ with kernel exactly H_i. Notice that we may view χ_i as an irreducible character of G. By Theorem 2.4.11 we see that

$$a_{e_0}(\chi_i) = a_{E/F}(\chi_i) = a_1 + 1. \tag{3.22}$$

By Definition 3.2.4 (H3) we see that we may also write a_{e_0} as

$$a_{e_0} = \sum_i m_i \cdot \operatorname{Ind}_{H_i}^G u_{H_i}. \tag{3.23}$$

It follows from (3.22) and (3.23) that for all i we have

$$a_1 + 1 = a_{e_0}(\chi_i) = \sum_{i \neq j} m_j. \tag{3.24}$$

We see therefore that $m_0 = m_1 = \ldots = m_p$.

In particular it follows that $p | (a_1 + 1)$ and furthermore that $\mathcal{X}_i \to \mathcal{Y}$ and $\mathcal{X}_j \to \mathcal{Y}$ share $\frac{p-1}{p} \cdot (a_1 + 1)$ common geometric branch points.

Now we consider the case where the upper ramification filtration of G has two upper jumps, namely $t_1 < t_2 \in \mathbb{Z}^2$. We write $t_1 = (a_1, b_1)$ and $t_2 = (a_2, b_2)$. Without loss of generality we may assume that $G^{t_2} = H_0$. It follows that $a_{e_0}(\chi_0) = a_{E/F}(\chi_0) = a_1 + 1$ and $a_{e_0}(\chi_i) = a_{E/F}(\chi_i) = a_2 + 1$ for $i = 1, \ldots, p$. From (3.23) we see that

$$a_1 + 1 = \sum_{j \neq 0} m_j$$

and

$$a_2 + 1 = a_{e_0}(\chi_i) = \sum_{i \neq j} m_j, \quad i = 1, \ldots, p.$$

We see that $m_1 = \ldots = m_p$. Therefore we obtain the following theorem.

3.7.1 Theorem. *We have that $p|(a_1 + 1)$. Furthermore, the covers $\mathcal{X}_0 \to \mathcal{Y}$ and $\mathcal{X}_i \to \mathcal{Y}$, $i = 1, \ldots, p$, share exactly $\frac{p-1}{p} \cdot (a_1 + 1)$ common geometric branch points.*

We can deduce another interesting property. Indeed, it may occur that $a_1 > a_2$ (with $t_1 < t_2$ then due to the second component $b_1 < b_2$). Notice that this cannot occur in the case that the residue field extension of E/F is separable. In this case (where $a_1 > a_2$) we see that $\mathcal{X}_0 \to \mathcal{Y}$ has exactly $a_1 + 1$ geometric branch points, and the covers $\mathcal{X}_i \to \mathcal{Y}$ for $i = 1, \ldots, p$ each have exactly $a_2 + 1$ geometric branch points. However, $\frac{p-1}{p} \cdot (a_1 + 1)$ cannot exceed $a_2 + 1$, therefore we obtain the following inequality on the number of branch points for the different covers $\mathcal{X}_i \to \mathcal{Y}$.

3.7.2 Theorem. *If $n_0 > n_i$ for $i = 1, \ldots, p$, then we have*

$$n_0 > n_i \geq \frac{p-1}{p} \cdot n_0.$$

3.7.3 Remark. The proof of Theorem 3.7.1 using the representation theory approach of the Hurwitz trees was shown to the author by Stefan Wewers in an earlier version of Brewis–Wewers [6]. One can also deduce this result by studying the differentials of the associated differential Hurwitz tree closely together with the results on G-actions of Section 1.7.

Chapter 4

Hurwitz-tree obstruction to cyclic actions

Let $F := k((t))$ be a local power series field, where k is an algebraically closed field of characteristic p. Let n be a positive integer and let E/F be a $\mathbb{Z}/p^n\mathbb{Z}$-Galois extension. Associated with E/F is the Artin character $a_{E/F}$.

The Oort conjecture states that the Galois extension E/F should lift to a $\mathbb{Z}/p^n\mathbb{Z}$-Galois extension of local power series rings over R, where R is some dominant extension of the Witt vectors $W(k)$. Recall from Theorem 3.6.2 that if this is the case, then there exists a Hurwitz tree $\mathcal{T}$ with trunk e such that $a_e = a_{E/F}$ and depth 0. Our main goal for this chapter is to prove that such a Hurwitz tree always exists. This provides some new evidence for the validity of the strong Oort conjecture.

We now give an overview of this chapter. Let χ_j denote an irreducible character of the group $\mathbb{Z}/p^n\mathbb{Z}$ of order p^j, i.e. a homomorphism $G \to \mathbb{C}^*$ with image of order p^j. In Section 4.2 we begin by stating inequalities between the $a_{E/F}(\chi_j)$ as j varies. The main idea is to study the upper ramification jumps of the extension E/F. In particular, we recall a result of Schmid [38] which relates the upper ramification jumps via inequalities. We then interpret these inequalities in terms of the values of the Artin character $a_{E/F}$ at the characters χ_j.

In Section 4.3 we focus on the problem of constructing a Hurwitz tree for the Artin character $a_{E/F}$. The inequalities discussed in Section 4.2 are crucial ingredients in this construction. Essentially, the main problem is to fit the depth characters in such a way that we obtain the multiplicative depth characters at the leaves of the constructed Hurwitz trees (see Definition 3.2.4 (H5)).

Let us briefly outline our strategy for constructing Hurwitz trees for the character $a_{E/F}$. Let the upper jumps of E/F be $r_1, \ldots, r_n$. One knows from Serre [37] Proposition VI.5 that $a_{E/F}(\chi_j) = r_j + 1$. We define integers $m_1, \ldots, m_n$ by writing

$$r_1 = m_1,\ r_2 = m_1 + m_2,\ \ldots,\ r_n = m_1 + \ldots + m_n.$$

Let T_n be a Hurwitz tree with trunk e. If $a_e = a_{E/F}$, then by Definition 3.2.4 (H3), for $j \geq 2$ exactly m_j leaves of T_n have monodromy group $\mathbb{Z}/p^{n+1-j}\mathbb{Z}$ and exactly $m_1 + 1$ leaves have monodromy group $\mathbb{Z}/p^n\mathbb{Z}$. Thus, in order to construct a Hurwitz tree with trunk e and $a_e = a_{E/F}$, we first have to accomplish that the monodromy groups of the leaves agree with this.

It is helpful to think of the monodromy groups associated with the leaves as follows. If the Hurwitz tree T_n was induced by a G-Galois extension of p-adic open discs, then the branch points with inertia group $\mathbb{Z}/p^i\mathbb{Z}$ correspond to the leaves with the same monodromy group. Therefore, there will be exactly m_j branch points with monodromy group $\mathbb{Z}/p^{n+1-j}\mathbb{Z}$ for $j \neq 1$ and exactly $m_1 + 1$ branch points with monodromy group $\mathbb{Z}/p^n\mathbb{Z}$. The distinction for the case $j = 1$ and $j \neq 1$ comes from the expression

$$a_{E/F}(\chi_j) = 1 + m_1 + \ldots + m_j.$$

The construction will be carried out inductively. First we construct a Hurwitz tree T_1 for the group $\mathbb{Z}/p\mathbb{Z}$ with exactly $m_1 + 1$ leaves in Section 4.3.1 and Section 4.3.2. We then assume that a Hurwitz tree T_{n-1} has been constructed for the group $\mathbb{Z}/p^{n-1}\mathbb{Z}$ and the sequence of integers $m_1 < \ldots < m_{n-1}$, where the m_i have to satisfy certain inequalities (see Definition 4.3.1). In this case T_{n-1} has exactly m_i leaves with monodromy group $\mathbb{Z}/p^{n-i}\mathbb{Z}$, where $i = 2, \ldots, n-1$, and exactly $m_1 + 1$ leaves with monodromy group $\mathbb{Z}/p^{n-1}\mathbb{Z}$. We construct a tree T_n for the group $\mathbb{Z}/p^n\mathbb{Z}$ which essentially contains the underlying tree T_{n-1} (with the root and trunk of T_{n-1} deleted), and with m_n new leaves.

The difficult part is now to fit the thicknesses of the edges of T_n in a correct way. This is done in Section 4.3.3. Notice that by requiring that the depth at the root is 0, a choice of the thicknesses of the edges automatically fixes the depth characters at each vertex of T_n. Thus we have to be careful in this step, since the depths at the leaves are required to be the induced multiplicative characters (see Definition 3.2.4 (H5)). We shall check that this is indeed the case in Section 4.3.4.

4.1 Notation

As always, we let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic p. We denote by v_K the valuation of the field K and we denote by π a local parameter for the discrete valuation field K. We shall always normalize v_K such that $v_K(\pi) = 1$. The letter F will denote the local power series field $k(\!(t)\!)$.

4.2 Artin characters of $\mathbb{Z}/p^n\mathbb{Z}$-Galois extensions

Let E/F be a totally ramified $\mathbb{Z}/p^n\mathbb{Z}$-Galois extension of F. Let $r_1, \ldots, r_n$ be the upper ramification jumps associated with E/F. By the classical Hasse–Arf theorem (see Serre [37] IV.3) we have that the r_i are integers. Then the following theorem is known.

4.2.1 Theorem (Schmid [38]). *We have that $r_j \geq p \cdot r_{j-1}$ for $j = 2, \ldots, n$.*

Let us denote by $a_{E/F}$ the Artin character of E/F. We define n positive integers $m_1, \ldots, m_n$ by requiring that $a_{E/F}(\chi_j) = 1 + m_1 + \ldots + m_j$, where χ_j is a character of $\mathbb{Z}/p^n\mathbb{Z}$ of order p^j. It is known (see Serre [37] Proposition VI.5) that

$$a_{E/F}(\chi_j) = r_j + 1.$$

4.2.2 Remark. The condition that $r_j \geq p \cdot r_{j-1}$ for $j = 2, \ldots, n$ is therefore equivalent to $m_j \geq (p-1) \cdot (m_1 + \ldots + m_{j-1})$ for $j = 2, \ldots, n$.

4.3 Vanishing of obstruction

Let $n \in \mathbb{Z}$ be a positive integer and consider a set of positive integers $m_1 < m_2 \ldots < m_n$. In view of Remark 4.2.2 we make the following definition.

4.3.1 Definition. We say that the natural numbers $m_1 < \ldots < m_n$ form an *admissible set* if for every $j \geq 2$ we have that $m_j \geq (p-1) \cdot (m_1 + \ldots + m_{j-1})$.

4.3.2 Definition. Let $\mathcal{T}$ be a Hurwitz tree for the group $\mathbb{Z}/p^n\mathbb{Z}$ with trunk e. The *branching sequence* of $\mathcal{T}$ is the unique sequence of integers $m_1, \ldots, m_n$ such that $a_e(\chi_j) = 1 + m_1 + \ldots + m_j$, where χ_j is an irreducible character of $\mathbb{Z}/p^n\mathbb{Z}$ of order p^j.

4.3.3 Definition. A $\mathbb{Z}/p^n\mathbb{Z}$-Hurwitz tree with root v_0 and trunk e_0 is said to be *admissible* if its branching sequence $m_1, \ldots, m_n$ is an admissible set and such that

[AH 1] $\delta_{v_0} = 0$,

[AH 2] $a_{e_0}(\chi_j) = 1 + m_1 + \ldots + m_j$ where χ_j is a character of G with order p^j (Compare with Remark 4.2.2),

[AH 3] $\epsilon_{e_0} > \frac{v_K(\lambda)}{m_1 + \ldots + m_n}$ (see Remark 4.3.5).

Our main theorem for this chapter is the following.

4.3.4 Theorem. *After a possible extension of K, an admissible Hurwitz tree defined over K exists for every n and every set of admissible positive integers $m_1 < \ldots < m_n$. Therefore, for every $\mathbb{Z}/p^n\mathbb{Z}$-Galois extension E/F with Artin character $a_{E/F}$ there exists a Hurwitz tree for the group $\mathbb{Z}/p^n\mathbb{Z}$ with root v_0 and trunk e_0 such that $\delta_{v_0} = 0$ and $a_{e_0} = a_{E/F}$.*

4.3.5 Remark. The condition (AH3) is a technical condition which will allow us to prove that the thicknesses we associate to the edges are all positive. This will become clear in Section 4.3.3.

We start with the case $n = 1$, i.e. the case $G = \mathbb{Z}/p\mathbb{Z}$. Let $m_1 > 1$ be an integer. By definition, the singleton $\{m_1\}$ is admissible. Let B be the normalization of $A := R[\![t]\!]$ inside the field extension generated by the Kummer equation $Y^p = 1 + \lambda^p t^{-m_1}$. One checks that B is again a formal power series ring over R and that the residue extension $k[\![t]\!] \subset B \otimes k$ is a separable extension of local power series rings. One also checks that the associated Hurwitz tree is admissible with respect to the admissible singleton $\{m_1\}$.

Now we proceed to the case $n > 1$. In order to continue, we make the following assumption.

4.3.6 Assumption. We assume that $p \neq 2$.

At the end of our construction we shall return to the case $p = 2$.

We assume that we are given admissible integers $m_1 < \ldots < m_n$. Let us assume that an admissible Hurwitz tree T_{n-1} has been constructed for the group $G' := \mathbb{Z}/p^{n-1}\mathbb{Z}$ and the admissible integers $m_1 < \ldots < m_{n-1}$. We let v_0' be the root of this tree and e_0' be its trunk. For an edge e of T_{n-1} we write a_e' for its Artin character, s_e' for its Swan character, and for a vertex v of T_{n-1} we write

$G'_v \subset G'$ for the group associated with the vertex. For an edge e of T_{n-1} we write ϵ_e for its thickness. Notice that by assumption (see Definition 4.3.3 (AH3))

$$\epsilon_{e'_0} > \frac{v_K(\lambda)}{m_1 + \ldots + m_{n-1}}.$$

4.3.1 Underlying tree

We define T'_{n-1} to be the tree T_{n-1} with the root v'_0 and trunk e'_0 deleted. Notice that T'_{n-1} is a rooted tree and we denote this root by v'_1. We now distinguish two cases. Since the integers $m_1 < \ldots < m_n$ form an admissible set, we see that $m_n \geq (p-1)\cdot(m_1 + \ldots + m_{n-1})$. The first case we shall consider is the case

$$m_n = (p-1)\cdot(m_1 + \ldots + m_{n-1}) \tag{4.1}$$

and the second case is

$$m_n > (p-1)\cdot(m_1 + \ldots + m_{n-1}). \tag{4.2}$$

The reason we distinguish these two cases is that the construction of the underlying trees for these cases are different. In both cases we shall make use of m_n copies of an auxiliary object called the *new chain* $\tilde{T}$. Each copy $\tilde{T}_i$ is defined as follows. The new chain $\tilde{T}_i$ consists of $n+1$ vertices $\tilde{v}_{i,0}, \tilde{v}_{i,1}, \ldots, \tilde{v}_{i,n-1}, \tilde{v}_{i,n}$ and n edges $\tilde{e}_{i,j},\ j = 1, \ldots, n$, where $\tilde{e}_{i,j}$ connects $\tilde{v}_{i,j-1}$ to $\tilde{v}_{i,j}$.

First case (4.1)

Now we define the underlying tree of the G-Hurwitz tree. We consider the first case first, i.e. the case of (4.1). Let T_n be the tree defined as follows. We let v_0, w_1, w_2 be three vertices. We connect v_0 to w_1 with an edge e_0, and we connect w_1 with w_2 with an edge f_0. Next we connect T'_{n-1} to w_2 by identifying the vertex w_2 with the vertex v'_1. Furthermore, we connect the m_n copies $\tilde{T}_i$ of the new chain to the vertex w_1 by identifying for each copy the vertex $\tilde{v}_{i,0}$ with the vertex w_1. This completes the definition of the tree T_n and we choose v_0 to be its root and e_0 to be its trunk.

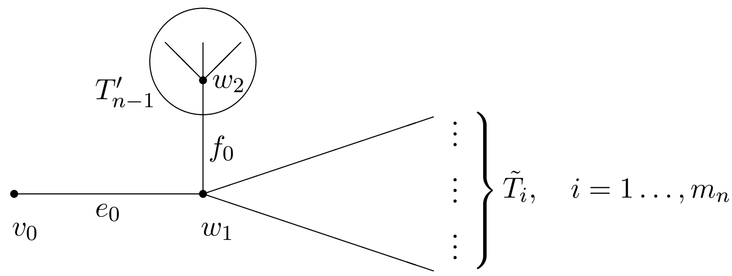

Figure 1: The underlying tree T_n for the first case (4.1).

Second case (4.2)

For the second case, i.e. the case of (4.2), we define the tree T_n as follows. We let v_0, w_1, w_2, w_3 be four vertices. We connect v_0 to w_1 with an edge e_0, and we connected w_1 with w_2 with an edge f_0. Furthermore, we connect w_1 and w_3 via an edge g_0. Next we connect T'_{n-1} to w_2 by identifying the vertex w_2 with the vertex v'_1. Furthermore, we connect the m_n copies $\tilde{T}_i$ to the vertex w_3 by identifying for each copy the vertex $\tilde{v}_{i,0}$ with the vertex w_3. This completes the definition of the tree T_n and we choose v_0 to be its root and e_0 to be its trunk.

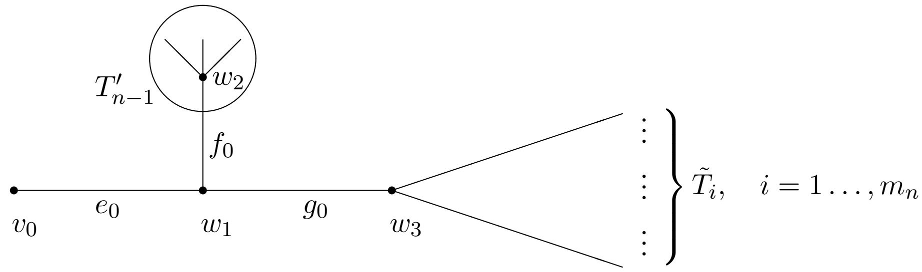

Figure 2: The underlying tree T_n for the second case (4.2).

4.3.7 Remark. In both cases the leaves of T_n are the leaves of T_{n-1} together with m_n new leaves, one for the endpoint of each copy $\tilde{T}_i$, i.e. the vertices $\tilde{v}_{i,n}$.

4.3.2 Monodromy groups

Let us now associate a group with each vertex of the tree T_n. In the first case, with the vertices v_0, w_1 and w_2 we associate the group G, i.e. we set $G_{v_0} = G_{w_1} = G_{w_2} := G$. In the second case, with the vertices v_0, w_1, w_2 and w_3 we associate the group G, i.e. $G_{v_0} = G_{w_1} = G_{w_2} = G_{w_3} := G$.

Let us now consider the vertices of T_n coming from the subtree T'_{n-1}. Since T_{n-1} was a Hurwitz tree for the group $G' = \mathbb{Z}/p^{n-1}\mathbb{Z}$, we see that with each vertex v', subgroup $G'_{v'}$ of $G' = \mathbb{Z}/p^{n-1}\mathbb{Z}$. is associated. Let v' be a vertex of T'_{n-1}. Then with v' considered as a vertex of T_n we associate the preimage of $G'_{v'}$ under the surjection $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z}$. Notice that $\#G_{v'} = p \cdot \#G'_{v'}$.

Since in both cases we identified the vertex w_2 with the vertex v'_1 of T'_{n-1}, we need to check that there is no contradiction at this vertex. Indeed, with w_2 we have associated the group $G = \mathbb{Z}/p^n\mathbb{Z}$. However, with v'_1 the group $G'_{v'_1} = G' = \mathbb{Z}/p^{n-1}\mathbb{Z}$ was associated in the Hurwitz tree T_{n-1}, and hence there is no contradiction in our definition.

Next we consider the vertices $\tilde{v}_{i,1}, \ldots, \tilde{v}_{i,n}$ of the copies $\tilde{T}_i$. With the vertex $\tilde{v}_{i,j}$ we associate the subgroup $G_{\tilde{v}_{i,j}} := \mathbb{Z}/p^{n-j+1}\mathbb{Z} \subset G$. Notice that with the leaf $\tilde{v}_{i,n}$ we have associated the subgroup $G_{\tilde{v}_{i,n}} = \mathbb{Z}/p\mathbb{Z} \subset G$.

The choices of associations of subgroups of G with the vertices now also determine by Definition 3.2.4 (H3) the Artin characters associated to the edges of T_n. Furthermore, setting $s_e := a_e - \mathrm{Ind}_{G_{t(e)}}^G u_{G_{t(e)}}$ for an edge e also defines the Swan character of the edge e. The proof of the following lemma is left to the reader.

4.3.8 Lemma. *Let χ_j be a character of G of order p^j. Let b be a vertex of T_n. Assume that $G_b = \mathbb{Z}/p^m\mathbb{Z}$. Then χ_j restricts to the trivial character on G_b if and only if $n - m + 1 > j$. Therefore we have that*

$$\left\langle \mathrm{Ind}_{G_b}^G u_{G_b}, \chi_j \right\rangle = \begin{cases} 1 & n - m + 1 \leq j, \\ 0 & n - m + 1 > j. \end{cases} \tag{4.3}$$

The following lemma follows from Definition 3.2.4 (H3) and Lemma 4.3.8.

4.3.9 Lemma. *For an edge e of T_n, we have that $a_e(\chi_j) = n_j$ where n_j is the number of leaves $b \geq t(e)$ such that $\mathbb{Z}/p^{n-j+1}\mathbb{Z} \lhd G_b$. In particular for the edge e_0 we have $a_{e_0}(\chi_j) = m_1 + \ldots + m_j + 1$.*

PROOF. There are exactly $1+m_1$ leaves with the group $\mathbb{Z}/p^n\mathbb{Z}$ associated with them, and for $i \geq 2$ there are exactly m_i leaves associated with the group $\mathbb{Z}/p^{n+1-i}\mathbb{Z}$. Therefore, $a_{e_0}(\chi_j)$ is the number of the leaves with associated group containing $\mathbb{Z}/p^{n-j+1}\mathbb{Z}$, and this is therefore $1+m_1+\ldots+m_j$. □

Let us also make Artin and Swan characters explicit for the edge e_0 and f_0, as well as the edge g_0 in the second case (4.2).

4.3.10 Lemma. *We have*

$$\langle a_{e_0}, \chi_j \rangle = 1 + m_1 + \ldots + m_j,\ \langle a_{f_0}, \chi_j \rangle = 1 + m_1 + \ldots + m_{\min\{j,n-1\}}$$

and

$$\langle s_{e_0}, \chi_j \rangle = m_1 + \ldots + m_j,\ \langle s_{f_0}, \chi_j \rangle = m_1 + \ldots + m_{\min\{j,n-1\}}.$$

In the second case, i.e. the case of (4.2), we have for the edge g_0 that

$$\langle a_{g_0}, \chi_n \rangle = m_n,\ \langle s_{g_0}, \chi_n \rangle = m_n - 1.$$

We leave the proof to the reader.

4.3.3 Thicknesses

Next we define the thicknesses of the edges. Since T_{n-1} was originally a Hurwitz tree, we define the thickness of an edge of the subtree T'_{n-1} of T_n to be its thickness in the tree T_{n-1}.

The first case (4.1)

We now consider the thicknesses of the edges e_0 and f_0 in the first case (4.1). We define ϵ_{e_0} such that

$$\epsilon_{e_0} \cdot m_n = v_K(p) \tag{4.4}$$

and we define ϵ_{f_0} such that

$$\epsilon_{e_0} + \epsilon_{f_0} = \epsilon_{e'_0}. \tag{4.5}$$

4.3.11 Remark. To ensure that ϵ_{e_0} and ϵ_{f_0} are integers we might need to extend K.

Our first priority is to check that ϵ_{f_0} is positive. Indeed it suffices to show that $\epsilon_{e_0} < \epsilon_{e'_0}$. For this we note

$$\begin{aligned} \epsilon_{e'_0} &> \frac{v_K(\lambda)}{m_1 + \ldots + m_{n-1}} \quad \text{by Definition 4.3.3 (AH3) for } T_{n-1} \\ &\geq \frac{v_K(\lambda)(p-1)}{m_n} \end{aligned}$$

since $m_n \geq (p-1) \cdot (m_1 + \ldots + m_{n-1})$ and hence we have that

$$\epsilon_{e_0'} > \frac{v_K(p)}{m_n} = \epsilon_{e_0}$$

by (4.4). This shows the positivity of ϵ_{f_0}.

Since $v_K(\lambda) \leq v_K(p)$, we also have that $\epsilon_{e_0} > \frac{v_K(\lambda)}{m_1+\ldots+m_n}$, hence satisfying (AH3) of Definition 4.3.3.

Next we define the thicknesses of the edges $\tilde{e}_{i,j}$ coming from the m_n copies $\tilde{T}_i$ of the new chain. We define

$$\epsilon_{\tilde{e}_{i,j}} := \epsilon_{e_0} \cdot m_j \tag{4.6}$$

for $j = 1, \ldots, n-1$ and $\epsilon_{\tilde{e}_{i,n}} := 0$.

4.3.12 Remark. The reason the thicknesses are defined in this way is to make sure that Definition 3.2.4 (H5) holds for the m_n new leaves of the tree. This will become clear toward the end of Section 4.3.4.

The second case (4.2)

For the second case, we define ϵ_{e_0} respectively ϵ_{f_0} again by (4.4) respectively (4.5). We define ϵ_{g_0} such that

$$\epsilon_{e_0} \cdot (m_1 + \ldots + m_n) + \epsilon_{g_0} \cdot (m_n - 1) = v_K(\lambda^p). \tag{4.7}$$

Our first priority is to show that ϵ_{g_0} is positive. For this we notice that by (4.2) we have

$$m_1 + \ldots + m_n < \frac{p}{p-1} \cdot m_n$$

in the second case. Hence

$$\epsilon_{e_0} \cdot (m_1 + \ldots + m_n) < \frac{p}{p-1} \cdot m_n \cdot \epsilon_{e_0} = v_K(\lambda^p)$$

which shows the positivity of ϵ_{g_0}. As a side remark to be used later, we prove the following lemma now.

4.3.13 Lemma. *We have that* $\epsilon_{e_0} \cdot m_1 > \epsilon_{g_0}$.

PROOF. By (4.7) we have that

$$\epsilon_{g_0} \cdot (m_n - 1) = v_K(\lambda^p) - \epsilon_{e_0} \cdot (m_1 + \ldots + m_n).$$

By (4.4) we have that

$$\begin{aligned}\epsilon_{g_0} \cdot (m_n - 1) &= v_K(\lambda^p) - \epsilon_{e_0} \cdot m_n - \epsilon_{e_0} \cdot (m_1 + \ldots + m_{n-1}) \\ &= v_K(\lambda) - \epsilon_{e_0} \cdot (m_1 + \ldots + m_{n-1}).\end{aligned}$$

It follows that

$$\begin{aligned}\epsilon_{g_0} &= \frac{v_K(\lambda)}{m_n - 1} - \frac{(p-1)\cdot v_K(\lambda)}{m_n(m_n-1)} \cdot (m_1 + \ldots + m_{n-1}) \\ &= \frac{v_K(\lambda)}{m_n} \cdot \frac{m_n - (p-1)\cdot(m_1 + \ldots + m_{n-1})}{m_n - 1} \\ &< \frac{v_K(p)}{m_n},\end{aligned}$$

since $p > 2$ by Assumption 4.3.6. □

4.3.14 Remark. The proof of this lemma also works in the case $p = 2$ and $n \geq 3$. However, it fails in the specific case $p = 2$, $n = 2$ and $m_1 = 1$.

We define $\epsilon_{\tilde{e}_{i,1}} := \epsilon_{e_0} \cdot m_1 - \epsilon_{g_0}$ and $\epsilon_{\tilde{e}_{i,j}} := \epsilon_{e_0} \cdot m_j$ for $j = 2, \ldots, n-1$ and $\epsilon_{\tilde{e}_{i,n}} := 0$. By Lemma 4.3.13 it follows that all $\epsilon_{\tilde{e}_{i,j}}$ are positive.

4.3.4 Depth characters

We have now defined the thickness of each edge of T_n. By defining $\delta_{v_0} := 0$, we see that by Definition 3.2.4 (H4) this also fixes the choices of depth characters at each vertex of the tree T_n. Our aim now is to check Definition 3.2.4 (H5), i.e. to show that for a leaf b we have $\delta_b = \mathrm{Ind}_{G_b}^{G} \delta_{G_b}^{mult}$.

Let us first do this for the leaves coming from the subtree $T'_{n-1} \subset T_n$. Our strategy is as follows. We shall show that for every irreducible character χ of G we have that $\langle \delta_b, \chi \rangle = \left\langle \delta_{G_b}^{mult}, \chi|_{G_b} \right\rangle$. Let us start with the trivial character 1_G. Indeed, we have that $\langle s_e, 1_G \rangle = 0$ for each edge e of the tree T_n, and hence it follows that

$$\langle \delta_b, 1_G \rangle = 0 = \left\langle \delta_{G_b}^{mult}, 1_G|_{G_b} \right\rangle .$$

Let χ_l be an irreducible character of G with order p^l for $l = 1, \ldots, n$ and let χ'_l be an irreducible character of $G' = \mathbb{Z}/p^{n-1}\mathbb{Z}$ of order p^l for $l = 1, \ldots, n-1$. Let e be an edge of T'_{n-1} and let a'_e denote the Artin character of the Hurwitz tree T_{n-1} associated to the edge e. Let us first consider the case that $l \leq n-1$. Our first observation is the following relation between a_e and a'_e which follows from Lemma 4.3.9.

4.3.15 Lemma. *We have that* $\langle a_e, \chi_l \rangle = \langle a'_e, \chi'_l \rangle$.

Our next observation concerns the relation between $\mathrm{Ind}_{G'_{t(e)}}^{G'} u_{G'_{t(e)}}$ and $\mathrm{Ind}_{G_{t(e)}}^{G} u_{G_{t(e)}}$. From Lemma 4.3.8 and the fact that $\#G_{t(e)} = p \cdot \#G_{t(e)'}$ we deduce the following lemma.

4.3.16 Lemma. *We have that*

$$\left\langle \mathrm{Ind}_{G_{t(e)'}}^{G'} u_{G'_{t(e)}}, \chi'_l \right\rangle = \left\langle \mathrm{Ind}_{G_{t(e)}}^{G} u_{G_{t(e)}}, \chi_l \right\rangle .$$

Therefore we obtain the following important lemma.

4.3.17 Lemma. *For* $l \leq n-1$ *we have that*

$$\langle s_e, \chi_l \rangle = \langle s'_e, \chi'_l \rangle .$$

Let us now study explicitly what the defined depth character at a leaf $b \in T'_{n-1}$ is. We treat both cases (4.1) and (4.2) simultaneously. Let $v_0 < w_1 < w_2 < z_1 < \ldots < z_m < b$ be the unique chain of vertices of T_n starting at the root v_0 and ending at the leaf b. The vertices $z_1, \ldots, z_m, b$ are vertices of the subtree T'_{n-1}. Let h_1 be the edge connecting w_2 and z_1, and let h_i be the edge connecting z_{i-1} to z_i, for $i = 2, \ldots, m$. Then from the definition of δ_b (Definition 3.2.4 (H4)), we obtain

$$\begin{aligned}
\langle \delta_b, \chi_l \rangle &= \epsilon_{e_0} \cdot \langle s_{e_0}, \chi_l \rangle + \epsilon_{f_0} \cdot \langle s_{f_0}, \chi_l \rangle + \sum_i \epsilon_{h_i} \cdot \langle s_{h_i}, \chi_l \rangle \quad \text{(Definition 3.2.4 (H4))} \\
&= \epsilon_{e_0} \cdot (m_1 + \ldots + m_l) + \epsilon_{f_0} \cdot (m_1 + \ldots + m_l) + \sum_i \epsilon_{h_i} \cdot \langle s_{h_i}, \chi_l \rangle \quad \text{(Lemma 4.3.10)} \\
&= (\epsilon_{e_0} + \epsilon_{f_0}) \cdot (m_1 + \ldots + m_l) + \sum_i \epsilon_{h_i} \cdot \langle s_{h_i}, \chi_l \rangle \\
&= \epsilon_{e'_0} \cdot \left\langle s'_{e'_0}, \chi'_l \right\rangle + \sum_i \epsilon_{h_i} \cdot \langle s'_{h_i}, \chi'_l \rangle \quad \text{(Lemma 4.3.17)} \\
&= \left\langle \delta^{mult}_{G'_b}, \chi'_l|_{G'_b} \right\rangle \quad \text{(Definition 3.2.4 (H5) for } T_{n-1}).
\end{aligned}$$

Therefore, if we can prove that the order of $\chi'_l|_{G'_b}$ is the same as that of $\chi_l|_{G_b}$, then by (3.1) of Chapter three we have the equality

$$\langle \delta_b, \chi_l \rangle = \left\langle \delta^{mult}_{G_b}, \chi_l|_{G_b} \right\rangle.$$

Let us now check that the order of $\chi'_l|_{G'_b}$ is the same as that of $\chi_l|_{G_b}$. Let $G_b \simeq \mathbb{Z}/p^m\mathbb{Z}$, where $m \in \mathbb{N}$. Furthermore, the kernel of χ_l is the subgroup $\mathbb{Z}/p^{n-l}\mathbb{Z}$ of $G = \mathbb{Z}/p^n\mathbb{Z}$. Therefore, $\chi_l|_{G_b}$ is trivial if and only if $n-l \geq m$. If $n-l < m$, then the kernel of $\chi_l|_{G_b}$ is the subgroup $\mathbb{Z}/p^{n-l}\mathbb{Z}$, and hence $\chi_l|_{G_b}$ has order $m-(n-l)$.

The subgroup G'_b is isomorphic to $\mathbb{Z}/p^{m-1}\mathbb{Z}$. Similarly, $\chi'_l|_{G'_b}$ is trivial if and only if $n-1-l \geq m-1$. If $n-1-l < m-1$, then the kernel of $\chi'_l|_{G'_b}$ is the subgroup $\mathbb{Z}/p^{n-1-l}\mathbb{Z}$, and hence $\chi_l|_{G'_b}$ has order $m-1-(n-1-l)$, which is the same as $m-(n-l)$. We see therefore that the order of $\chi'_l|_{G'_b}$ is the same as that of $\chi_l|_{G_b}$.

We leave the details of the case of $l = n$ to the reader. Next we turn our attention to the m_n new leaves of the tree T_n. Let b be one of the new leaves of T_n, i.e. a leaf of T_n that is not in T'_{n-1}.

The first case (4.1)

We consider the first case first. We consider the chain $v_0 < w_1 < \tilde{v}_{i,1} < \ldots < b$ starting at v_0 and ending at the endpoint b of the copy $\tilde{T}_i$. Notice that $G_b = \mathbb{Z}/p\mathbb{Z}$. First of all we remark once again that for the trivial character 1_G of G, we have that

$$\langle \delta_b, 1_G \rangle = 0 = \left\langle \delta^{mult}_{G_b}, 1_G \right\rangle.$$

It remains to prove that for a nontrivial irreducible character χ of G, we have that

$$\langle \delta_b, \chi \rangle = \left\langle \delta^{mult}_{G_b}, \chi|_{G_b} \right\rangle.$$

Consider the vertex $\tilde{v}_{i,j}$ coming from a new chain $\tilde{T}_i$ in T_n. Since the group $G_{\tilde{v}_{i,j}}$ associated with $\tilde{v}_{i,j}$ is $\mathbb{Z}/p^{n+1-j}\mathbb{Z}$, we see from Lemma 4.3.8 that χ_l restricts to the trivial character on $G_{\tilde{v}_{i,j}}$ exactly when $j > l$. Therefore we obtain that

$$\left\langle \mathrm{Ind}_{G_{\tilde{v}_{i,j}}}^{G} u_{G_{\tilde{v}_{i,j}}}, \chi_l \right\rangle = \begin{cases} 1 & j \leq l, \\ 0 & j > l. \end{cases}$$

Furthermore, the leaf b is the only leaf with $\tilde{v}_{i,1} \leq b$. We see that since $G_b = \mathbb{Z}/p\mathbb{Z}$, we have by the fact that $a_{\tilde{e}_{i,j}} = \mathrm{Ind}_{G_b}^{G} u_{G_b}$ that

$$\left\langle a_{\tilde{e}_{i,j}}, \chi_l \right\rangle = \begin{cases} 0 & l < n, \\ 1 & l = n. \end{cases}$$

Therefore, for $l < n$ we have that

$$\left\langle s_{\tilde{e}_{i,j}}, \chi_l \right\rangle = \begin{cases} -1 & j \leq l, \\ 0 & j > l, \end{cases} \tag{4.8}$$

and for $l = n$ we have that

$$\left\langle s_{\tilde{e}_{i,j}}, \chi_l \right\rangle = 0. \tag{4.9}$$

Let us now compute $\langle \delta_b, \chi_l \rangle$. We note that by Lemma 4.3.10 we have

$$\langle s_{e_0}, \chi_l \rangle = m_1 + \ldots + m_l.$$

Therefore for $l \leq n-1$,

$$\begin{aligned} \langle \delta_b, \chi_l \rangle &= \epsilon_{e_0} \cdot (m_1 + \ldots + m_l) + \sum_{j \leq n} \left\langle s_{\tilde{e}_{i,j}}, \chi_l \right\rangle \cdot \epsilon_{\tilde{e}_{i,j}} \\ &= \epsilon_{e_0} \cdot (m_1 + \ldots + m_l) + \sum_{j \leq l} -1 \cdot \epsilon_{\tilde{e}_{i,j}} + 0 \quad \text{(by (4.8))} \\ &= \epsilon_{e_0} \cdot (m_1 + \ldots + m_l) + \sum_{j \leq l} -1 \cdot \epsilon_{e_0} \cdot m_j \quad \text{(by (4.6))} \\ &= 0. \end{aligned}$$

Hence we see that $\langle \delta_b, \chi_l \rangle = 0$. However $\left\langle \delta_{G_b}^{mult}, \chi_l|_{G_b} \right\rangle = 0$ since χ_l restricts to the trivial character on G_b if $l < n$. Therefore we have that

$$\langle \delta_b, \chi_l \rangle = 0 = \left\langle \delta_{G_b}^{mult}, \chi_l|_{G_b} \right\rangle \quad \forall l \leq n-1.$$

We now consider the remaining case $l = n$, i.e. the case of a character χ_n of order p^n. Indeed, in this case we obtain

$$\begin{aligned} \langle \delta_b, \chi_n \rangle &= \epsilon_{e_0} \cdot (m_1 + \ldots + m_n) + \sum_{j \leq n} \left\langle s_{\tilde{e}_{i,j}}, \chi_n \right\rangle \cdot \epsilon_{\tilde{e}_{i,j}} \\ &= \epsilon_{e_0} \cdot (m_1 + \ldots + m_n) \quad \text{(by (4.9))} \\ &= \epsilon_{e_0} \cdot \frac{p}{p-1} \cdot m_n \quad \text{(by (4.1))} \\ &= v_K(\lambda^p) \\ &= \left\langle \delta_{G_b}^{mult}, \chi_n|_{G_b} \right\rangle. \end{aligned}$$

Finally, we have proved that $\delta_b = \mathrm{Ind}_{G_b}^{G} \delta_{G_b}^{mult}$ for the new leaves b of the tree T_n. We have thus proved that the datum T_n is indeed a Hurwitz tree in the first case.

The second case (4.2)

For the second case, we proceed as follows. Consider the chain $v_0 < w_1 < w_3 < \tilde{v}_{i,1} \ldots < b$ starting from the root v_0 and ending at the new leaf b which is the endpoint of the copy $\tilde{T}_i$. As usual, we shall show that $\delta_b = \mathrm{Ind}_{G_b}^{G} \delta_{G_b}^{mult}$ by showing that

$$\langle \chi, \delta_b \rangle = \left\langle \delta_{G_b}^{mult}, \chi|_{G_b} \right\rangle$$

for every irreducible character χ of G. We consider the case $\chi = \chi_n$, a character of order p^n of G.

We find

$$\langle \delta_b, \chi_n \rangle = \epsilon_{e_0} \cdot \langle s_{e_0}, \chi_n \rangle + \epsilon_{g_0} \cdot \langle s_{g_0}, \chi_n \rangle + \sum_j \epsilon_{\tilde{e}_{i,j}} \cdot \left\langle s_{\tilde{e}_{i,j}}, \chi_n \right\rangle . \tag{4.10}$$

We concentrate on the values $\left\langle s_{\tilde{e}_{i,j}}, \chi_n \right\rangle$. Namely, by (4.9) we see that these are all 0. Therefore, putting this into (4.10) we obtain

$$\begin{aligned} \delta_b(\chi_n) &= \epsilon_{e_0} \cdot (m_1 + \ldots + m_n) + \epsilon_{g_0} \cdot (m_n - 1) + 0 \quad \text{(by Lemma 4.3.10)} \\ &= v_K(\lambda^p) \quad \text{(by definition of } \epsilon_{g_0}) \\ &= \left\langle \delta_{G_b}^{mult}, \chi_n|_{G_b} \right\rangle \end{aligned}$$

since G_b is a cyclic group of order p. We leave the details of the other irreducible characters to the reader.

Finally we say something about the case $p = 2$. One can attempt the exact same construction as above in the case $p = 2$. However one encounters a problem with Lemma 4.3.13 for the case $n = 2$ and $m_1 = 1$. To remedy this problem, we start the induction at $n = 3$ in the case of $p = 2$. Then the base step of the induction process, where $n = 2$, is given by the explicit liftings of Green–Matignon [15] for $\mathbb{Z}/p^2\mathbb{Z}$-extensions. One checks that the Hurwitz trees induced by their liftings are admissible (for instance, by applying the Newton polygon to the polynomial $G(T^{-1})$ of Lemma 5.4 in Green–Matignon [15]). The proof then goes through exactly as we have done it above.

Chapter 5

Towards the lifting problem

In Henrio [19] it was proved that every $\mathbb{Z}/p\mathbb{Z}$-Hurwitz tree is induced by a $\mathbb{Z}/p\mathbb{Z}$-action on the p-adic open disc. Let us explain Henrio's method in some more detail.

A fundamental ingredient that Henrio used was an association of differential forms to the vertices of the Hurwitz tree. Henrio proved that the differentials associated to the vertices are either *exact* or *logarithmic*, and that this depends on the depth associated to the vertex (see for instance Lemma 1.4.2 and Lemma 1.4.3).

A further ingredient was a classification result, which classifies the $\mathbb{Z}/p\mathbb{Z}$-Galois actions on a two-dimensional local field up to conjugation. In particular, Henrio proved that if the depths and differentials associated to two $\mathbb{Z}/p\mathbb{Z}$-actions are equal, then the actions are conjugate.

He then exploited this to construct $\mathbb{Z}/p\mathbb{Z}$-automorphisms of the open disc which induce a given $\mathbb{Z}/p\mathbb{Z}$-Hurwitz tree. The technique is based on patching certain rigid analytic spaces with G-actions along their boundaries. Since the boundaries of spaces that Henrio used are essentially two-dimensional local fields, one needs a classification result, up to conjugation, of the spirit above in order to apply the patching techniques. A similar technique was used in Bouw–Wewers [4] for the case that G is the dihedral group of order $2p$, where p is an odd prime.

In this chapter, we shall consider some problems that are present when one attempts to generalize Henrio's work to the case of general p-group actions. It is possible to extend our definition of a Hurwitz tree, by using Kato's differential Swan conductor, in order to associate differential forms to the vertices of the Hurwitz tree. However, a fundamental problem that we shall illustrate, is that the differential forms need not only be exact or logarithmic. Instead, as we shall see in Section 5.2, it seems that the space in which these differential forms take their values can be very big. Therefore, it is not clear what the correct conditions are that have to be imposed on the differential forms.

The second issue to which we turn our attention is the classification result of Henrio. Unfortunately we shall see in Section 5.3 that this does not generalize to $\mathbb{Z}/p^2\mathbb{Z}$-actions. We shall show that there exist two $\mathbb{Z}/p^2\mathbb{Z}$-actions on a two-dimensional local field, with the same Swan-conductor information, which are not conjugate.

Lastly, we return to the case of a Q_8-action in Section 5.4, where Q_8 denotes the quaternion group of order 8. We shall explicitly write down a Huwritz tree for Q_8 which cannot be induced from an action on the p-adic open disc. It seems that the crucial ingredient that is missing is the simplified ramification filtration, which is trivial for a $\mathbb{Z}/p\mathbb{Z}$-action, together with Theorem 2.4.11.

5.1 Notation and setting

As in the previous chapter, we let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic p and ring of integers R. We denote by v_K the valuation of the field K, and by $\pi \in R$ a local parameter of K. We shall always normalize v_K such that $v_K(\pi) = 1$. In Section 5.2 and Section 5.3 we shall assume $\zeta_{p^2} \in K$ and we set $\lambda := \zeta_p - 1$. In Section 5.4 we shall assume that $p = 2$ and K is such that $v_K(2) = 5$.

5.2 Towards differential Hurwitz trees

Let B be either $R[\![z]\!]$ or the ring of convergent power series $R\{z\}$. We let $E := \mathrm{Frac}\, \hat{B}_{\pi B}$. Let G be a finite p-group and assume that G acts on B fixing R. We let $A := B^G$ and $F := E^G$.

Assume that E/F is a Case-II type G-Galois extension. To every character χ of G, we may now associate a differential $\omega_\chi \in \Omega_{\overline{F}}^{\otimes \mathrm{rk}(\chi)}$ via Kato's Swan conductor. Using this and the technique of Section 3.5, we may now extend the definition of Hurwitz trees to also include differential data.

Before this approach can be pursued further, one needs a better understanding of the values that the differential Swan conductor can take. In this section we would like to illustrate that the space of differentials that can occur as differential Swan conductors can be very big.

5.2.1 Example : nonlogarithmic nonexact differential Swan conductor

Consider the polynomial

$$x^2 + 6x + 6 \tag{5.1}$$

and let $\alpha \in \tilde{\mathbb{Q}}_2$ be a root of (5.1). We let $K := \mathbb{Q}_2(\alpha, i, 2^{\frac{1}{4}}, \sqrt{2\alpha + 2})$ and we denote by R the ring of integers of K. Let $\pi_K \in R$ be a parameter for K. We set $b := 2\alpha + 2$. Let $A := R[\![t]\!]$ and let B be the normalization of A inside the $\mathbb{Z}/4\mathbb{Z}$-extension of A generated by y, where y satisfies

$$y^4 = (1 + 4t^{-1})(1 - bt^{-1})^2. \tag{5.2}$$

5.2.1 Remark. One can prove that the ring B is a local power series ring, i.e. $B = R[\![z]\!]$. Furthermore, one shows that the residue extension $\mathbb{F}_2[\![t]\!] \subset \mathbb{F}_2[\![z]\!]$ is separable. Therefore, the extension B/A has *good reduction*.

5.2.2 Remark. Notice that $v_K(\alpha) = v_K(\sqrt{2})$ and therefore $v_K(b) = v_K(2)$. Lastly we note that

$$v_K(b - 2) > v_K(2).$$

We now consider the ring $A' := R\{v\}$ where v and t are related by $t = 2v$. The $\mathbb{Z}/4\mathbb{Z}$-extension B/A induces a $\mathbb{Z}/4\mathbb{Z}$-extension B'/A' of normal rings.

5.2.3 Remark. One checks again that B' is a ring of convergent power series, i.e. $B' = R\{s\}$ for some $s \in B'$. However this time the residue extension $k[v] \subset k[s]$ is purely inseparable.

Let $F := \mathrm{Frac}(\hat{A}'_{(\pi A')})$ and let $E := F \otimes_{A'} B'$. The fields F and E are discrete valuation fields with parameter π_K. One checks that E/F is a $\mathbb{Z}/4\mathbb{Z}$-Galois extension of Case-II type. Our aim is now to calculate the differential Swan conductors of the characters of $\mathbb{Z}/4\mathbb{Z}$ for the extension E/F. Notice that the residue field $\overline{F}$ is $k(v)$.

Let M/F be the intermediate $\mathbb{Z}/2\mathbb{Z}$-extension of E/F. One checks that M/F is generated by x where x satisfies

$$x^2 = 1 + 2v^{-1}. \tag{5.3}$$

We make the substitution $x = 1 - \sqrt{2}w^{-1}$ and we thus obtain from (5.3) that M/F is generated by

$$(1 - \sqrt{2}w^{-1})^2 = 1 + 2v^{-1}. \tag{5.4}$$

One obtains from (5.4) that

$$w^{-2} - \sqrt{2}w^{-1} = v^{-1}. \tag{5.5}$$

Let ϕ (respectively χ) be characters of order 2 (respectively 4) of the group $\mathbb{Z}/4\mathbb{Z}$. Then one obtains immediately from (5.3) that

$$\mathrm{Sw}_{E/F}(\phi) = [2] - [\mathrm{d}v^{-1}] = [2] - [-v^{-2} \cdot \mathrm{d}v]. \tag{5.6}$$

Let us now also calculate $\mathrm{Sw}_{E/F}(\chi)$. To do this we shall use the identity (see Theorem 2.3.21)

$$p \cdot \mathrm{Sw}_{E/F}(\chi) = p \cdot (\mathrm{Sw}_{E/M}(\chi|_H) + \mathcal{D}_{M/F}) \tag{5.7}$$

inside the group S_M and where $H := \mathrm{Gal}(E/M)$ and $p = 2$. Set $b = 2\beta$ where $\beta \in R^*$. From (5.2) and (5.5) we see that E/M is generated by y where y satisfies

$$\begin{aligned} y^2 &= (1 - \sqrt{2}w^{-1})(1 - \beta w^{-2} + \beta\sqrt{2}w^{-1}) && (5.8)\\ &= 1 - \beta w^{-2} - \sqrt{2}(\beta - 1)w^{-1} + \beta\sqrt{2}w^{-3} - 2\beta w^{-2}. && (5.9) \end{aligned}$$

We set $y' = \frac{y \cdot w}{w + i\sqrt{\beta}}$ to obtain a generating equation

$$(y')^2 = 1 + \sqrt{2}\beta\frac{w^{-1}}{(w + i\sqrt{\beta})^2} + \sqrt{2}N \tag{5.10}$$

for E/M and where $N \in \pi_K \cdot \mathcal{O}_M$. From (5.10) one sees that

$$\mathrm{Sw}_{E/M}(\chi|_H) = [2\sqrt{2}] - [-\beta \cdot w^{-2}(w + i\sqrt{\beta})^{-2} \cdot \mathrm{d}w]. \tag{5.11}$$

Now we substitute (5.6) and (5.11) into (5.7) to find $\mathrm{Sw}_{E/F}(\chi)$. One obtains

$$\begin{aligned} 2 \cdot \mathrm{Sw}_{E/F}(\chi) &= [16] - [\beta w^{-4}(w^2 - \beta)^{-2}v^{-2} \cdot (\mathrm{d}v)^{\otimes 2}]\\ &= [16] - [\beta v^{-4}(v - \beta)^{-2} \cdot (\mathrm{d}v)^{\otimes 2}]. \end{aligned}$$

We find thus that

$$\mathrm{Sw}_{E/F}(\chi) = [4] - [\frac{\sqrt{\beta}}{v^2(v - \beta)}\,\mathrm{d}v].$$

Notice that the differential part is neither exact nor logarithmic.

5.2.2 Example : special extensions

In this section we consider again general K such that $\zeta_{p^2} \in K$. Let F be a two-dimensional local field containing the field K and such that π_K is a parameter for F. Assume that the residue field of F is $\overline{F} = k((t))$. Let $V_E \subset \Omega_{\overline{F}}$ (respectively $V_L \subset \Omega_{\overline{F}}$) denote the space of exact (respectively logarithmic) differentials of F, and let $V := V_E + V_L \subset \Omega_{\overline{F}}$. Our goal for this paragraph is to show the following theorem.

5.2.4 Theorem. *For every $\omega \in V$, there exists a $\mathbb{Z}/p^2\mathbb{Z}$-Galois extension E_ω/F of Case-II type together with a character of* $\mathrm{Gal}(E_\omega/F) = \mathbb{Z}/p^2\mathbb{Z}$ *such that*

$$\mathrm{Sw}_{E_\omega/F}(\chi) = [\pi^n] - [\omega],$$

where $n = v_K(\lambda^p)$.

Our strategy for doing this is as follows. Firstly one knows that for every logarithmic differential $\omega_L \in V_L$, there exists a $\mathbb{Z}/p\mathbb{Z}$-extension E_{ω_L}/F and an irreducible character χ_{ω_L} of $\mathrm{Gal}(E_{\omega_L}/F)$ such that

$$\mathrm{Sw}_{E_{\omega_L}/F}(\chi_{\omega_L}) = [\lambda^p] - [\omega_L].$$

Notice that we may consider χ_{ω_L} as a one-dimensional representation of the absolute Galois group G_F of F. Secondly, we shall show that for every $\omega_E \in V_E$, there exists a $\mathbb{Z}/p^2\mathbb{Z}$-extension E_{ω_E}/F together with a character χ_{ω_E} of $\mathbb{Z}/p^2\mathbb{Z} = \mathrm{Gal}(E_{\omega_E}/F)$ such that

$$\mathrm{Sw}_{E_{\omega_E}/F}(\chi_{\omega_E}) = [\lambda^p] - [\omega_E].$$

Again we may consider χ_{ω_E} as an irreducible character of the absolute Galois group G_F of F. Let $H \subset G_F$ be the kernel of the one-dimensional character $\chi := \chi_{\omega_L} \cdot \chi_{\omega_E}$. The embedding $H \hookrightarrow G_F$ induces a $\mathbb{Z}/p^2\mathbb{Z}$-Galois extension E/F together with a character, also denoted by χ, of the Galois group $\mathrm{Gal}(E/F)$. By Theorem 1.6.1 we then obtain

$$\mathrm{Sw}_{E/F}(\chi) = [\lambda^p] - [\omega_L + \omega_E]$$

giving our result.

Before we proceed to proving Theorem 5.2.4, we make two remarks.

5.2.5 Remark. It seems that the choice $n = v_K(\lambda^p)$ is not special. Indeed, one can find other values for n such that the set of differential Swan conductors still contain logarithmic and exact differentials simultaneously. We leave to the reader to find such examples.

5.2.6 Remark. For the logarithmic differentials $\omega_L \in V_L$ we used a character χ_{ω_L} of order p. However, twisting χ_{ω_L} by an appropriate character of order p^2, we see that we can find a character of order p^2 with differential Swan conductor ω_L.

We now proceed to the proof of Theorem 5.2.4. By the strategy above, it suffices to consider a $\omega_E \in V_E$. Let $\omega_E = \mathrm{d}f$ where $f \in \overline{F}$. Note that f is not a pth-power in $\overline{F}^*$. Let m be an integer such that $v_K(p) = m \cdot p$. After extending K, we may assume that $p|m$. Define E to be the $\mathbb{Z}/p^2\mathbb{Z}$-extension of F generated by y where y satisfies

$$y^{p^2} = 1 + \pi^{mp} f. \tag{5.12}$$

Let M/F be the intermediate $\mathbb{Z}/p\mathbb{Z}$-extension. Let $H := \mathrm{Gal}(E/M)$. One sees from (5.12) that M/F is generated by $w = y^p$, where w satisfies

$$w^p = 1 + \pi^{mp} f \tag{5.13}$$

and hence for an irreducible $\mathrm{Gal}(E/F)$-character χ' of order p, we have

$$\mathrm{Sw}_{E/F}(\chi') = [\lambda^p \pi^{-mp}] - [b \cdot \mathrm{d}f] = [\lambda^p \pi^{-mp}] - [b \cdot \omega_E] \tag{5.14}$$

where $b \in \mathbb{F}_p^*$.

Let χ now be a character of $\mathrm{Gal}(E/F)$ of order p^2. We make the substitution

$$w = 1 + \pi^m v$$

and substituting this into (5.12) we obtain the following generating equation for the $\mathbb{Z}/p\mathbb{Z}$-extension E/M:

$$y^p = 1 + \pi^m v. \tag{5.15}$$

One verifies that since $p|m$ we have by (5.15) that

$$\mathrm{Sw}_{E/M}(\chi|_H) = [\lambda^p \pi^{-m}] - [c \cdot \mathrm{d}v]. \tag{5.16}$$

where $c \in \mathbb{F}_p^*$.

Using (5.7) we see from (5.16) that

$$\begin{aligned} p \cdot \mathrm{Sw}_{E/F}(\chi) &= p \cdot [\lambda^p \pi^{-m}] - p \cdot [c \cdot \mathrm{d}v] + (p-1) \cdot [\lambda^p \pi^{-mp}] - (p-1) \cdot [\mathrm{d}f] \\ &= [\lambda^p \cdot p^p \cdot \pi^{-mp^2}] - [d \cdot (\mathrm{d}f)^{\otimes p}] \end{aligned}$$

where $d \in \mathbb{F}_p^*$. However we have that $m \cdot p = v_K(p)$ and hence the above shows that

$$\mathrm{Sw}_{E/F}(\chi) = [\lambda^p] - [\mathrm{d}f] = [\lambda^p] - [\omega_E].$$

This proves Theorem 5.2.4.

We now raise two questions. Let $\chi \in X(G_F) := \mathrm{Hom}(G_F, \mathbb{C}^*)$ be a one-dimensional representation. Let E_χ be the induced cyclic extension of F and notice that we may consider χ as a character of $\mathrm{Gal}(E_\chi/F)$. We assume that E_χ/F is of Case-II type. We may therefore define the *depth* δ_χ and the associated differential ω_χ via Kato's Swan conductor of the extension E_χ/F.

Define $V_F^i := \{\omega_\chi | \delta_\chi = i\}$.

5.2.7 Question. What is the space $V_F^i \subset \Omega_{\overline{F}}$? Can it happen for instance that the differential $t^{p-1}\,\mathrm{d}t$ occurs as a differential Swan conductor?

The examples above show that $V_F^{v_K(\lambda^p)} \supset E + L$.

Let A be a formal power series ring and identify F with $\mathrm{Frac}(\hat{A}_{\pi A})$ (after an extension of K this is always possible). We say a character $\chi \in X(G_F)$ is *smooth* if there exists a cyclic cover B/A of formal power series rings such that χ induces the extension E/F where $E := B \otimes_A F$. Let $X_0(G_F)$ denote the group of characters generated by all smooth characters.

5.2.8 Question. What is the space $V_{F,0}^i := \{\omega_\chi | \chi \in X_0(G_F) \text{ and } \delta_\chi = i\}$?

One can show for instance that for certain i, $V_{F,0} \supset E_\infty$, where $E_\infty \subset \Omega_{\overline{F}}$ denotes the space of exact differentials on $\overline{F}$ with a pole.

5.3 Classification problem

In this section we let F be again a two-dimensional local field. Let G be a finite p-group and let E_1/F and E_2/F be two G-Galois extensions of Case-II type. Assume that for every character χ of G we have that $\mathrm{Sw}_{E_1/F}(\chi) = \mathrm{Sw}_{E_2/F}(\chi)$. Our goal for this section is to ask whether or not E_1 and E_2 are G-equivariantly conjugate, i.e. if there exists a K-isomorphism $\phi : E_1 \simeq E_2$ which commutes with the action of G. Notice that we do not require ϕ to be a F-isomorphism.

As we have already pointed out in the introduction of this chapter, in Henrio [19] it was proved that this is indeed the case for $G = \mathbb{Z}/p\mathbb{Z}$. There this has been exploited to show that every $\mathbb{Z}/p\mathbb{Z}$-Hurwitz tree is induced by an action on the p-adic open disc.

We shall now show that in the case of $G = \mathbb{Z}/p^2\mathbb{Z}$ this is no longer true. Therefore we see that we are still far away from constructing $\mathbb{Z}/p^2\mathbb{Z}$-automorphisms of the disc which induce a given $\mathbb{Z}/p^2\mathbb{Z}$-Hurwitz tree.

Our strategy for constructing the example is as follows. First in Section 5.3.1 we shall explicitly define the two $\mathbb{Z}/p^2\mathbb{Z}$-Galois covers of F. We shall choose suitable isomorphisms of their Galois groups with $G := \mathbb{Z}/p^2\mathbb{Z}$. Then we make some remarks on the Galois theory of the two G-Galois extensions. This is done in Section 5.3.2. Thereafter in Section 5.3.3 we shall show that there exists no K-isomorphism between the two fields which are also G-equivariant. Finally in Section 5.3.4 we shall show that for every character χ of G we have that the Swan conductors of χ agrees for both extensions.

5.3.1 Remark. We point out that Tossici [40], [41] has introduced stronger invariants than the differential Swan conductor for $\mathbb{Z}/p^2\mathbb{Z}$-actions. One might ask if his invariants are strong enough to classify $\mathbb{Z}/p^2\mathbb{Z}$-actions up to conjugation.

5.3.1 Definitions of the extensions

Let $A := R[\![t]\!]$ and let $F := \mathrm{Frac}\, \hat{A}_{(\pi A)}$. Notice that F is a two-dimensional local field with residue field $\overline{F} = k(\!(t)\!)$. Let $n \in \mathbb{N}$ be a positive integer satisfying the following conditions.

[n1] $v_K(\pi^{np^2}) < v_K(p)$,

[n2] $v_K(\pi^n) < v_K(p\pi^{-np})$,

[n3] $p^2 < n$ and $p^2 | n$.

5.3.2 Remark. Let us give an example of K and n where these conditions are satisfied. Let $n = 2 \cdot p^2$ and choose K to be an extension of $\mathbb{Q}_p(\zeta_{p^2})$ with ramification index $2 \cdot p^4$. Let π be a parameter of K. Since $v_K(\pi) = 1$, we see that

$$v_K(\pi^{np^2}) = n \cdot p^2 < v_K(p).$$

We construct two $\mathbb{Z}/p^2\mathbb{Z}$-extensions of F as follows. Let $f_1 := 1+\pi^{np}t$ and $g_1 := 1+\pi^{p}t$. Let $f_2 := 1+\pi^{np}t$ and $g_2 := 1$. We define E_1/F to be the extension of F generated by y_1, where y_1 satisfies

$$y_1^{p^2} = f_1 \cdot g_1^p. \tag{5.17}$$

We define E_2/F to be the extension of F generated by y_2, where y_2 satisfies

$$y_2^{p^2} = f_2 \cdot g_2^p. \tag{5.18}$$

Throughout what follows let σ be a generator of $G = \mathbb{Z}/p^2\mathbb{Z}$. We identify $\mathrm{Gal}(E_1/F)$ (respectively $\mathrm{Gal}(E_2/F)$) with G by declaring that $\sigma y_1 = \zeta_{p^2} \cdot y_1$ (respectively $\sigma y_2 = \zeta_{p^2} \cdot y_2$).

5.3.2 Galois theory of E_1/F and E_2/F

Assume that there exists a K-isomorphism $\phi : E_1 \simeq E_2$ which is also G-equivariant.

5.3.3 Remark. Since ϕ is G-equivariant, we see that ϕ restricts to an isomorphism $\phi|_F : F \simeq F$. This need not be the identity.

Let H be the unique subgroup of G of order p. Our first result is the following lemma.

5.3.4 Lemma. *We have that* $\frac{\phi(f_1)}{f_2} \in F^*$ *is a* p*-power inside* F^*.

Proof. Let $u_1 = \phi(\frac{y_1^p}{g_1}) \in E_2$. Notice that $u_1 \in E_2^H$ and furthermore, we have that

$$u_1^p = \phi(f_1).$$

Furthermore,

$$\sigma \cdot u_1 = \zeta_p \cdot u_1.$$

Similarly, we have that $u_2 := \frac{y_2^p}{g_2} \in E_2^H$ and

$$\sigma \cdot u_2 = \zeta_p \cdot u_2.$$

We have therefore that $\frac{u_1}{u_2}$ is an element of F. However,

$$(\frac{u_1}{u_2})^p = \frac{\phi(f_1)}{f_2}$$

and the lemma follows. □

Our second result is the following.

5.3.5 Lemma. *We have that* $\frac{\phi|_F(f_1 g_1^p)}{(f_2 g_2^p)} \in F^*$ *is a* p^2*-power of* F^*.

PROOF. Let $w = \phi(y_1) \in E_2$. We have that

$$w^{p^2} = \phi(f_1 g_1^p)$$

and therefore E_2/F is generated by both the Kummer equations

$$w^{p^2} = \phi(f_1 g_1^p)$$

as well as the equation

$$y_2^{p^2} = f_2 g_2^p.$$

Notice that since ϕ is G-equivariant, we have that $\sigma \cdot w = \zeta_{p^2} \cdot w$ and furthermore $\sigma \cdot y_2 = \zeta_{p^2} \cdot y_2$. Therefore the element $\frac{w}{y_2}$ is fixed under σ and therefore $\frac{w}{y_2} \in F$. However,

$$(\frac{w}{y_2})^{p^2} = \frac{\phi(f_1 g_1^p)}{(f_2 g_2^p)}.$$

The lemma follows. □

5.3.3 Conjugation theorem

Our goal for this section is to prove the following theorem.

5.3.6 Theorem. *There exists no G-equivariant K-isomorphism $E_1 \simeq E_2$.*

PROOF. Assume that a G-equivariant R-isomorphism $\phi : E_1 \simeq E_2$ exists. By the Remark 5.3.3, we see that ϕ restricts to an isomorphism of $\phi|_F : F \simeq F$. Since the reduction of t is a parameter of $\overline{F}$, we see that $\phi(t)$ must also reduce to a parameter of $\overline{F}$. We define $f \in F$ to be such that $\phi(t) = t + f$.

Therefore we have

$$\begin{aligned} \phi(f_1 g_1^p) &= \phi((1 + \pi^{np} t)(1 + \pi^p t)^p) \\ &= (1 + \pi^{np}(t + f))(1 + \pi^p(t + f))^p. \end{aligned}$$

By Lemma 5.3.5 we see therefore that

$$\frac{(1 + \pi^{np}(t + f))(1 + \pi^p(t + f))^p}{(1 + \pi^{np} t)}$$

must be a p^2-power. We expand using [n1] and [n3] and we find the following approximation:

$$\frac{1 + \pi^{np}(t + f))(1 + \pi^p(t + f))^p}{(1 + \pi^{np} t)} = 1 + \pi^{p^2}(t^p + f^p) + O(\pi^{p^2+1})$$

where $O(\pi^{p^2+1})$ means a term of valuation exceeding $v_K(\pi^{p^2})$.

Therefore, since $v_K(\pi^{p^2}) < v_K(p)$ by $(n1)$, we see that the reduction of $t^p + f^p$ must be a p^2-power in $\overline{F}$. In particular, we see that the reduction of f cannot be a p-power in $\overline{F}$.

However, from Lemma 5.3.4, we see that

$$\frac{1+\pi^{np}(t+f)}{1+\pi^{np}t}$$

must be a p-power in $\overline{F}$. Expanding, we find

$$\frac{1+\pi^{np}(t+f)}{1+\pi^{np}t} = 1+\pi^{np}f + O(\pi^{np+1}).$$

Therefore, since $v_K(\pi^{np}) < v_K(p)$ by $(n1)$, we see that the reduction of f must be a p-power in $\overline{F}$, a contradiction. $\square$

5.3.4 The Swan conductors of the extensions

In this section we shall show the following theorem.

5.3.7 Theorem. *For every character χ of G we have that* $\mathrm{Sw}_{E_1/F}(\chi) = \mathrm{Sw}_{E_2/F}(\chi)$.

PROOF. In what follows we let H denote the subgroup of G of index p. Let χ be the character of G with $\chi(\sigma) = \zeta_{p^2}$.

The characters of order p

We start with χ^p. Notice that $E_1^H \simeq E_2^H$ as $\mathbb{Z}/p\mathbb{Z}$-extensions of F. Furthermore, this extensions is generated by $w := \frac{y_1^p}{g_1}$, where w satisfies the Kummer-type equation

$$w^p = f_1 = f_2.$$

We see from the definition of w that G/H acts on w by

$$\sigma \cdot w = \zeta_p \cdot w = \chi^p(\sigma) \cdot w.$$

From Lemma 1.4.3 it follows that

$$\mathrm{Sw}_{E_1/F}(\chi^p) = [\lambda^p \pi^{-np}] - [\mathrm{d}t] = \mathrm{Sw}_{E_2/F}(\chi^p).$$

Therefore, for all i with $(i,p) = 1$ we see that $\mathrm{Sw}_{E_1/F}(\chi^{ip}) = \mathrm{Sw}_{E_2/F}(\chi^{ip})$.

The characters of order p^2

In order to calculate $\mathrm{Sw}_{E_1/F}(\chi)$ we use the identity from Lemma 2.3.21

$$\mathrm{Sw}_{E_i/F}(\chi) = \mathrm{Sw}_{E_i/L}(\chi|_H) + \mathcal{D}_{L/F}$$

where $L := E_1^H = E_2^H$.

It therefore suffices to compute $\mathrm{Sw}_{E_i/L}(\chi|_H) \in S_L$.

We may find a $v \in L$ such that

$$(1+\pi^n v)^p = 1 + \pi^{np} t = f_1 = f_2. \tag{5.19}$$

Therefore it follows that

$$(1+\pi^n v)^p = y_2^{p^2}.$$

It follows that $(1+\pi^n v) = \zeta_p^j \cdot y_2^p$ for some $j \in \mathbb{N}$.

Therefore E_2/L is generated by y_2, where y_2 satisfies the Kummer equation

$$y_2^p = \zeta_p^{-j} \cdot (1+\pi^n v) \tag{5.20}$$

and where the identification of $\mathrm{Gal}(E_2/L)$ with H is such that

$$\sigma^p \cdot y_2 = \zeta_p \cdot y_2.$$

Since χ maps σ^p to ζ_p it follows from Lemma 1.4.3 and (5.20) that $\mathrm{Sw}_{E_2/L}(\chi|_H) = [\lambda\pi^{-n}] - [\mathrm{d}v]$.

Next we calculate $\mathrm{Sw}_{E_1/L}(\chi|_H)$. In order to do this, we use again the element v of L. Notice that

$$\frac{y_1^p}{g_1} = \zeta_p^{-l} \cdot (1+\pi^n v)$$

for some $l \in \mathbb{Z}$ and therefore E_1/L is generated by y_1 where y_1 satisfies the Kummer equation

$$y_1^p = \zeta_p^{-l} \cdot (1+\pi^n v) \cdot g_1,$$

and where $\mathrm{Gal}(E_1/L)$ is identified with H via

$$\sigma^p \cdot y_1 = \zeta_p \cdot y_1.$$

The relation (5.19) implies that $v_K(t - v^p) > v_K(p\pi^{-np})$. Set $a := t - v^p$. We see therefore that E_1/L is generated by y_1, where y_1 satisfies the Kummer equation

$$y_1^p = \zeta_p^{-j} \cdot (1+\pi^n v) \cdot (1+\pi^p(v^p + a)).$$

We concentrate on the factor $1 + \pi^p(v^p + a)$ for a moment. We may write

$$\begin{aligned} 1 + \pi^p(v^p + a) &= (1+\pi v)^p - O(p) + \pi^p a \\ &= (1+\pi v)^p[1 + O(p\pi^{-np})] \end{aligned}$$

where $O(p)$ (respectively $O(p\pi^{-np})$) denotes terms of valuation not smaller than $v_K(p)$ (respectively $v_K(p\pi^{-np})$). Therefore, E_1/F is generated by a Kummer equation of the form

$$w^p = \zeta_p^{-j} \cdot (1+\pi^n v) \cdot (1 + O(p\pi^{-np}))$$

where $w = \frac{y_1}{1+\pi v}$ and such that $\sigma^p \cdot w = \zeta_p \cdot w$.

Since $v_K(\pi^n) < v_K(\lambda)$ (from (n1)) and $v_K(\pi^n) < v_K(p\pi^{-np})$ (from (n2)) we see that E_1/L is therefore generated by a Kummer equation

$$w^p = 1 + \pi^n v + O(\pi^{n+1})$$

where the identification of $\mathrm{Gal}(E_1/L)$ with H is via

$$\sigma^p \cdot w = \zeta_p \cdot w = \chi(\sigma^p) \cdot w.$$

Therefore from Lemma 1.4.3

$$\mathrm{Sw}_{E_1/L}(\chi|_H) = [\lambda^p \pi^{-n}] - [\mathrm{d}v]. \qquad \square$$

It follows that $\mathrm{Sw}_{E_1/F}(\chi) = \mathrm{Sw}_{E_2/F}(\chi)$ and hence $\mathrm{Sw}_{E_1/F}(\chi^i) = \mathrm{Sw}_{E_2/F}(\chi^i)$ for all $(i,p) = 1$. We are done.

5.4 The role of the simplified ramification groups

In this section we want to illustrate that in order to have a more complete theory of Hurwitz trees, one still needs to incorporate the simplified ramification filtration into the definition of a Hurwitz tree. To illustrate this, we shall explicitly write down a Hurwitz tree for the group $G := Q_8$, where the depth character at the root will not be 0. However, assuming that such a Hurwitz tree was induced by an action on the open disc (up to equivalence, see Definition 3.2.6), then a consideration of the associated simplified upper ramification filtration will yield a contradiction, hence showing that the Hurwitz tree was not induced by an action on a local power series ring over R.

5.4.1 Remark. The Hurwitz tree that we shall construct will have $\delta(\phi) > 0$, where δ is the depth character at the root of the Hurwitz tree, for all characters ϕ of Q_8. This will imply in particular that if the Hurwitz tree was induced by an action on the p-adic open disc, then the boundary extension (see Chapter two for an explanation of this) will be of Case-II type.

We start with the following situation. Let $B := R[\![z]\!]$ and assume that the group Q_8 acts on B. We let $E := \operatorname{Frac} \hat{B}_{(\pi B)}$ and we let $F := E^G$. Notice that E/F is a Q_8-Galois extension of two-dimensional local fields, and we assume that E/F is of Case-II type.

Let $\delta_{E/F}$ be the depth character of E/F. We denote by χ the unique irreducible character of rank two on Q_8. Our first goal is to prove the following theorem, which will be become useful to us later.

5.4.2 Theorem. *There exists a rank-one character ψ of Q_8 such that $\langle \delta_{E/F}, \chi \rangle > 2 \cdot \langle \delta_{E/F}, \psi \rangle$.*

PROOF. We let the first simplified upper ramification jump of the extension E/F be i_1. Since Q_8 is not abelian, we see from Theorem 2.3.16 that there exists at least a second simplified upper ramification jump i_2 with $i_2 > i_1$.

By Lemma 2.3.3 we can find an upper (usual) ramification jump $t := (h, d) \in \mathbb{Q}^2$ such that $d = i_2$. Notice that $G^t \neq \{1\}$ since t is an upper ramification jump of E/F. Furthermore, we have $\hat{G}^{i_2} \supset G^t$ and hence $G^t \neq G$. Denote by $\rho : G \hookrightarrow \mathrm{GL}_2(\mathbb{C})$ the rank-two representation associated with the character χ. Then we see that ρ is nontrivial on G^t and hence by Theorem 2.4.11 we see that $\langle \delta_{E/F}, \chi \rangle \geq 2 \cdot i_2$.

Let $t' := (h', d') \in \mathbb{Q}^2$ be an upper ramification jump such that $d' = i_1$. Then there exists a rank-one character ψ such that ψ is trivial on $G^{t''}$ for all $t'' \in \mathbb{Q}^2$ with $t'' > t'$. Therefore we see again by Theorem 2.4.11 that $\langle \delta_{E/F}, \psi \rangle \leq i_1$. We see thus that since $i_2 > i_1$ we have that

$$\langle \delta_{E/F}, \chi \rangle > 2 \cdot \langle \delta_{E/F}, \psi \rangle . \qquad \square$$

Let $p = 2$ and let K be such that $v_K(2) = 5$. We define eleven vertices $v_0, \ldots, v_4$ and $w_1, \ldots, w_6$ as well as seven leaves $b_0, \ldots, b_6$.

Let us also associate the monodromy groups with the vertices and leaves. We define $\tau, \sigma \in Q_8$ by the relation (3.13) of Section 3.6.2 in the case that $n = 2$. With all vertices $v_0, \ldots, v_3$ and $w_1, \ldots, w_6$ we associate the group $G = Q_8$. With the leaf b_0 and the vertex v_4 we associate the subgroup $\langle \tau^2 \rangle$. With the leaves b_1 and b_2 we associate the subgroup $\langle \tau \rangle$, with b_3 and b_4 we associate $\langle \sigma \rangle$ and finally with b_5 and b_6 we associate $\langle \tau\sigma \rangle$.

We connect the vertices in a tree $\mathcal{T}$ as follows.

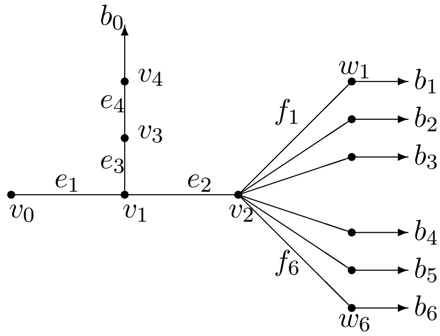

Figure 3: Hurwitz tree $\mathcal{T}$ for the group Q_8

Now we choose the thicknesses of the edges $e_1, \dots, e_4$ and the f_i. We choose $\epsilon_{e_1} = \epsilon_{e_2} = \epsilon_{e_4} := 1$ and $\epsilon_3 := 7$. Furthermore, we choose $\epsilon_{f_i} := 10$.

By Definition 3.2.4 (H3) we can now define the Artin and Swan characters of the tree. Let us make them explicit. Let χ denote the irreducible character of rank two of G. We see that

$$\langle a_{e_1}, \chi \rangle = 14, \ \langle a_{e_2}, \chi \rangle = 12, \ \langle a_{f_i}, \chi \rangle = \langle a_{e_3}, \chi \rangle = \langle a_{e_4}, \chi \rangle = 2.$$

Therefore we obtain

$$\langle s_{e_1}, \chi \rangle = 12, \ \langle s_{e_2}, \chi \rangle = 10, \ \langle s_{f_i}, \chi \rangle = \langle s_{e_3}, \chi \rangle = \langle s_{e_4}, \chi \rangle = 0.$$

Let us also make these characters explicit for the irreducible rank-one characters of G. Let ψ_τ respectively ψ_σ respectively $\psi_{\sigma\tau}$ denote the irreducible rank-one character with kernel $\langle \tau \rangle$ respectively $\langle \sigma \rangle$ respectively $\langle \sigma\tau \rangle$. Then we obtain

$$\langle a_{e_1}, \psi_j \rangle = \langle a_{e_2}, \psi_j \rangle = 4, \ \langle a_{e_3}, \psi_j \rangle = \langle a_{e_4}, \psi_j \rangle = 0$$

where $j \in \{\tau, \sigma, \sigma\tau\}$. Thus we obtain

$$\langle s_{e_1}, \psi_j \rangle = \langle s_{e_2}, \psi_j \rangle = 3, \ \langle s_{e_3}, \psi_j \rangle = -1, \ \langle s_{e_4}, \psi_j \rangle = 0$$

where $j \in \{\tau, \sigma, \sigma\tau\}$. Note the difference between s_{e_3} and s_{e_4}. This is because the monodromy groups of v_3 and v_4 are G and $\langle \tau^2 \rangle$ respectively and hence the induced augmentation characters differ. Finally for the edges f_i we note

$$\langle a_{f_1}, \psi_j \rangle = \langle a_{f_2}, \psi_j \rangle = 1, \ \langle s_{f_1}, \psi_j \rangle = \langle s_{f_2}, \psi_j \rangle = 0$$

for $j \in \{\sigma, \sigma\tau\}$. However for ψ_τ we have that

$$\langle a_{f_1}, \psi_\tau \rangle = \langle a_{f_2}, \psi_\tau \rangle = 0, \ \langle s_{f_1}, \psi_\tau \rangle = \langle s_{f_2}, \psi_\tau \rangle = -1.$$

Similar expressions hold for the other edges.

Let us now also define the depth characters of the tree. We define δ_{v_0} by

$$\delta_{v_0} := 4 \cdot \psi_\tau + 4 \cdot \psi_\sigma + 4 \cdot \psi_{\sigma\tau} + 8 \cdot \chi.$$

Notice that by definition this is a character of G. Using Definition 3.2.4 (H4) we can now extend this definition to define the depth characters of the vertices $v_1, \ldots, v_4$ and $w_1, \ldots, w_6$ as well as the seven leaves $b_0, \ldots, b_6$.

What is left is to verify that our construction is indeed a Hurwitz tree. We need to show that for a leaf b we have that $\mathrm{Ind}_{G_b}^G \delta_{G_b}^{mult} = \delta_b$, where G_b denotes the monodromy group of the leaf b, and δ_b is the depth character as defined above. We do this for the leaf b_1 first. The case of the leaves $b_2, \ldots, b_6$ are similar. Thereafter we do the case of the leaf b_0.

Proceeding to the leaf $b := b_1$, we notice that

$$\left\langle \mathrm{Ind}_{G_b}^G \delta_{G_b}^{mult}, \chi \right\rangle = 6 \cdot v_K(2) = 30.$$

Furthermore, we have that

$$\left\langle \mathrm{Ind}_{G_b}^G \delta_{G_b}^{mult}, \psi_\sigma \right\rangle = \left\langle \mathrm{Ind}_{G_b}^G \delta_{G_b}^{mult}, \psi_{\sigma\tau} \right\rangle = 2 \cdot v_K(2) = 10.$$

and

$$\left\langle \mathrm{Ind}_{G_b}^G \delta_{G_b}^{mult}, \psi_\tau \right\rangle = 0,$$

since ψ_τ restricts to the trivial character on $\langle \tau \rangle$. Let us now also calculate δ_{b_1} as we have defined it above. We obtain

$$\begin{aligned}
\langle \delta_{b_1}, \chi \rangle &= \langle \delta_{v_0}, \chi \rangle + \epsilon_{e_1} \cdot \langle s_{e_1}, \chi \rangle + \epsilon_{e_2} \cdot \langle s_{e_2}, \chi \rangle + \epsilon_{f_1} \cdot \langle s_{f_1}, \chi \rangle \text{ by Definition 3.2.4 (H4)} \\
&= 8 + 1 \cdot 12 + 1 \cdot 10 \\
&= 30 \\
&= \left\langle \mathrm{Ind}_{G_b}^G \delta_{G_b}^{mult}, \chi \right\rangle .
\end{aligned}$$

Furthermore, for ψ_σ we have

$$\begin{aligned}
\langle \delta_{b_1}, \psi_\sigma \rangle &= \langle \delta_{v_0}, \psi_\sigma \rangle + \epsilon_{e_1} \cdot \langle s_{e_1}, \psi_\sigma \rangle + \epsilon_{e_2} \cdot \langle s_{e_2}, \psi_\sigma \rangle + \epsilon_{f_1} \cdot \langle s_{f_1}, \psi_\sigma \rangle \\
&= 4 + 1 \cdot 3 + 1 \cdot 3 + 0 \\
&= 10 \\
&= \left\langle \mathrm{Ind}_{G_b}^G \delta_{G_b}^{mult}, \psi_\sigma \right\rangle .
\end{aligned}$$

It is similar for $\psi_{\sigma\tau}$. Lastly, for ψ_τ we obtain

$$\begin{aligned}
\langle \delta_{b_1}, \psi_\tau \rangle &= \langle \delta_{v_0}, \psi_\tau \rangle + \epsilon_{e_1} \cdot \langle s_{e_1}, \psi_\tau \rangle + \epsilon_{e_2} \cdot \langle s_{e_2}, \psi_\tau \rangle + \epsilon_{f_1} \cdot \langle s_{f_1}, \psi_\tau \rangle \\
&= 4 + 1 \cdot 3 + 1 \cdot 3 + 10 \cdot (-1) \\
&= 0 \\
&= \left\langle \mathrm{Ind}_{G_b}^G \delta_{G_b}^{mult}, \psi_\tau \right\rangle .
\end{aligned}$$

We see therefore that $\mathrm{Ind}_{G_b}^G \delta_{G_b}^{mult} = \delta_{b_1}$ as characters of G. The cases of the leaves $b_2, \ldots, b_6$ are similar.

Let us turn to the leaf b_0. We see that

$$\left\langle \mathrm{Ind}_{G_{b_0}}^{G}\, \delta_{G_{b_0}}^{mult}, \chi \right\rangle = 4 \cdot v_K(2) = 20$$

and furthermore

$$\left\langle \mathrm{Ind}_{G_{b_0}}^{G}\, \delta_{G_{b_0}}^{mult}, \psi_j \right\rangle = 0$$

for $j \in \{\tau, \sigma, \sigma\tau\}$. We calculate δ_{b_0}. For χ we obtain

$$\begin{aligned}
\langle \delta_{b_0}, \chi \rangle &= \langle \delta_{v_0}, \chi \rangle + \epsilon_{e_1} \cdot \langle s_{e_1}, \chi \rangle + \epsilon_{e_3} \cdot \langle s_{e_3}, \chi \rangle + \epsilon_{e_4} \cdot \langle s_{e_4}, \chi \rangle \\
&= 8 + 1 \cdot 12 + 0 + 0 \\
&= 20 \\
&= \left\langle \mathrm{Ind}_{G_{b_0}}^{G}\, \delta_{G_{b_0}}^{mult}, \chi \right\rangle .
\end{aligned}$$

Lastly for $\psi_j,\ j \in \{\tau, \sigma, \sigma\tau\}$, we obtain

$$\begin{aligned}
\langle \delta_{b_0}, \psi_j \rangle &= \langle \delta_{v_0}, \psi_j \rangle + \epsilon_{e_1} \cdot \langle s_{e_1}, \psi_j \rangle + \epsilon_{e_3} \cdot \langle s_{e_3}, \psi_j \rangle + \epsilon_{e_4} \cdot \langle s_{e_4}, \psi_j \rangle \\
&= 4 + 1 \cdot 3 + 7 \cdot (-1) \\
&= 0 \\
&= \left\langle \mathrm{Ind}_{G_{b_0}}^{G}\, \delta_{G_{b_0}}^{mult}, \psi_j \right\rangle .
\end{aligned}$$

We have therefore checked that $\delta_{b_0} = \mathrm{Ind}_{G_{b_0}}^{G}\, \delta_{G_{b_0}}^{mult}$ and hence that our construction is indeed a Hurwitz tree.

Assume now that the Hurwitz tree $\mathcal{T}$ was induced up to equivalence (see Definition 3.2.6) by some Q_8-action on $B := R'[\![z]\!]$, where R'/R is an extension of R. We denote by $e_{R'/R}$ the ramification index of R'/R. Let $E := \mathrm{Frac}\, \hat{B}_{\pi B}$ and let $F := E^G$. We see that $\left\langle \delta_{E/F}, \chi \right\rangle = 8 \cdot e_{R'/R}$ and $\left\langle \delta_{E/F}, \psi_j \right\rangle = 4 \cdot e_{R'/R}$ for all $j \in \{\tau, \sigma, \sigma\tau\}$. A contradiction now follows from Theorem 5.4.2.

Chapter 6

Dihedral actions

Another interesting question with regard to lifting is whether a group G admits *some* local action in characteristic p which can be lifted to characteristic 0. This is the *weak lifting problem* for the group G. Matignon [26] has shown that this is true in the case that G is an elementary abelian p-group, and later Green [14] showed a similar result in the case that G is a cyclic p-group.

In [27], Matignon asks what the situation for nonabelian p-groups is. In this chapter we give the first results in this direction by studying the situation in the case that $p = 2$. This work is taken from Brewis [5]. Let D_4 be the dihedral group of order 8. Our main theorem for this chapter is to prove that there exist examples of D_4-actions on $k[\![z]\!]$, k of characteristic 2, which can be lifted to characteristic 0.

In fact, we exhibit a family of local D_4-actions, the *supersimple* D_4-actions, which can always be lifted. Furthermore, the local degrees of different of these actions are not bounded from above, i.e. the genera of the respective Katz–Gabber compactifications (see Remark 6.2.11) of these actions are not bounded from above. This provides some evidence for a conjecture of Chinburg–Guralnick–Harbater, see for instance Chinburg [7], which states that all actions of dihedral groups D_{p^n} should lift to characteristic 0.

The first part of this chapter is a general overview of the Galois theory that we shall be using. In Section 6.2.1 we study field extensions with Galois group D_4. We also interpret these results in the context of covers of curves. Most notably, we give a method for producing D_4-Galois extensions by composing two $\mathbb{Z}/2\mathbb{Z}$-Galois extensions and taking the Galois closure.

In Section 6.2.2 we focus on the connection between the Galois theory of cyclic extensions and the theory of group cohomology, following Serre [37]. We then focus once again on D_4-Galois extensions. The group D_4 has several subgroups of order (and others of index) 2, and therefore, contained in a D_4-Galois extension are several $\mathbb{Z}/2\mathbb{Z}$-subextensions. We study the Galois-theoretic connections between these subextensions. Finally in Section 6.2.3 we remind the reader of Artin–Schreier theory and its connection with the cohomological interpretation of cyclic Galois theory.

Section 6.2.4 and Section 6.2.5 are concerned exclusively with the supersimple D_4-Galois extensions. Here we deal exclusively with characteristic 2. As we have already pointed out, it is sufficient to study lifting problems in the local context, and therefore, all fields concerned in these two sections will be local power series fields. First we shall define the notion of a local supersimple D_4-extension (Definition 6.2.10), and then we classify them in Theorem 6.2.20.

In the second part of this chapter our focus shifts to questions of good reduction. Although our goal is to eventually lift local actions, this part is of a global nature. In Section 6.3 we consider a D_4-Galois cover $C_3 \to \mathbb{P}^1_K$ of smooth projective curves over a 2-adic field K. Let C_2 be the quotient of C_3 under a nonnormal subgroup of order 2. Given certain good reduction properties of the intermediate cover $C_2 \to \mathbb{P}^1_K$, we ask when we can deduce that the curve C_3 has potentially good reduction. This culminates in Theorem 6.3.7, which is a criterion for potential good reduction.

The assumptions required in Section 6.3 are difficult to check, and the purpose of Section 6.4 is to give a method for producing D_4-Galois covers of curves which satisfy these assumptions. Finally, we conclude by giving an explicit family of examples. Furthermore, by explicitly studying their reductions to characteristic 2, we see that by localizing and completing these families at their ramification points, we obtain liftable examples of local D_4-Galois actions in characteristic 2.

Finally, in Section 6.5 we prove Theorem 6.5.1 which is our main result for this chapter, stating that all supersimple local D_4-actions lift to characteristic 0.

6.1 Notation

Let k be an algebraically closed field of characteristic 2. Let R_0 denote the Witt vectors of k and K_0 its fraction field. We shall reserve the letter K for a finite field extension of the field K_0 and R for the normalization of R_0 inside K_0. The field K will always be assumed to contain $\sqrt{2}$.

If C is a smooth projective curve over a field F, then we write $g(C)$ to mean its genus, and $F(C)$ its function field. Lastly, if A is ring, then we write $\mathbb{P}^1_{A,z}$ for the projective A-line with distinguished parameter z. For the notion of a local degree of different see Serre [37] Chapter III.

Let D_4 denote the dihedral group of order 8. We fix once and for all two generators $a, b \in D_4$ with the relations

$$a^4 = b^2 = 1,\ bab = a^3.$$

Whenever L is a field, G_L will denote the absolute Galois group of L. If G denotes a finite group, then by saying that two G-Galois extensions of L are isomorphic, we are implicitly assuming that a field isomorphism can be found which respects the identification of the respective Galois groups with G.

6.2 Some Galois theory

6.2.1 General Galois theory of D_4-extensions

The aim of this section is to state and prove two facts on constructing D_4-Galois extensions. We start by studying the situation for field extensions. Later we shall also interpret the results in the context of covers of smooth algebraic curves.

Let L_0 be a field. We assume that we are given two $\mathbb{Z}/2\mathbb{Z}$-Galois extensions $L_0 \subset L_1$ and $L_1 \subset L_2$. The extension $L_0 \subset L_2$ is of degree 4, but not necessarily Galois. We denote the Galois closure of this extension by $L_0 \subset \tilde{L}_2$ and the Galois group by G. The following lemma will be crucial to our studies later on in this work.

6.2.1 Lemma. *Assume that $L_0 \subset L_2$ is not a Galois extension, i.e. $L_2 \neq \tilde{L}_2$. Then the Galois extension $L_0 \subset \tilde{L}_2$ is a D_4-extension, i.e. $G \simeq D_4$.*

PROOF. Since $L_0 \subset L_2$ is an extension of degree 4, we notice that $G \subset S_4$. We also see that G must be nonabelian since $L_0 \subset L_2$ is not Galois. Furthermore, the fact that $L_0 \subset L_1$ and $L_1 \subset L_2$ are both Galois immediately places restrictions on the subgroups of G. One checks that all subgroups of S_4 satisfying all these conditions are isomorphic to D_4. □ □

We leave the proof of the following lemma to the reader.

6.2.2 Lemma. *Assume the notation of above. Then there exists an isomorphism*

$$\text{Gal}(\tilde{L}_2/L_0) \simeq D_4$$

such that L_2 is the fixed field under the subgroup

$$\langle b \rangle \subset D_4 \simeq \text{Gal}(\tilde{L}_2/L_0).$$

6.2.3 Remark. Notice that the results of this section can also be applied to separable covers of curves. Let us make this precise. Let C_0 be a smooth projective curve over an algebraically closed field k. Assume that we are given two $\mathbb{Z}/2\mathbb{Z}$-Galois covers $C_1 \to C_0$ and $C_2 \to C_1$, where C_1 and C_2 are smooth projective k-curves, such that the composite cover $C_2 \to C_0$ is of degree 4, but not Galois.

This tower of Galois covers induces a field extension $k(C_0) \subset k(C_1) \subset k(C_2)$ of degree 4 which is separable. Let us denote the Galois closure of this extension by $k(C_0) \subset \tilde{L}$. By Lemma 6.2.1, we see that $k(C_0) \subset \tilde{L}$ is a D_4-Galois extension. Since $\tilde{L}$ is an algebraic function field of transcendence degree one over k, there exists a smooth projective k-curve $\tilde{C}$, such that the function field of $\tilde{C}$ is exactly $\tilde{L}$. Therefore, the separable Galois extension of fields $k(C_0) \subset \tilde{L}$ induces a Galois cover $\tilde{C} \to C_0$ of curves, and this is a D_4-Galois cover.

6.2.2 Cohomological Galois theory of fields

In this section we shall gather some more facts on the Galois theory of fields, and in particular its cohomological interpretation. Our reference is essentially the book of Serre [37].

Let L_1/L_0 be a $\mathbb{Z}/2\mathbb{Z}$-Galois extension of the field L_0. It is known that we have the inflation-restriction exact sequence (see Serre [37] p.118)

$$\begin{aligned} 0 \to \mathrm{H}^1(\text{Gal}(L_1/L_0), \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z})^{\text{Gal}(L_1/L_0)} \\ \to \mathrm{H}^2(\text{Gal}(L_1/L_0), \mathbb{Q}/\mathbb{Z}) \to \dots \end{aligned}$$

Furthermore, one sees that

$$\mathrm{H}^1(\mathrm{Gal}(L_1/L_0), \mathbb{Q}/\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$$

and

$$\mathrm{H}^2(\mathrm{Gal}(L_1/L_0), \mathbb{Q}/\mathbb{Z}) \simeq 0,$$

see for instance Serre [37] p.134. We thus obtain the exact sequence

$$0 \to \mathrm{H}^1(\mathrm{Gal}(L_1/L_0), \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z})^{\mathrm{Gal}(L_1/L_0)} \to 0. \tag{6.1}$$

Let us very briefly remind ourselves what these cohomology groups mean and how they relate to Galois theory.

Consider the absolute Galois group G_{L_0} of the field L_0. The set of $\mathbb{Z}/n\mathbb{Z}$-Galois extensions of L_0 corresponds bijectively to the elements of the group $\mathrm{Hom}_{\mathbb{Z}}(G_{L_0}, \mathbb{Q}/\mathbb{Z})$, and therefore, if we consider the group $\mathbb{Q}/\mathbb{Z}$ as a trivial G_{L_0}-module, the set of *cyclic* Galois extensions of L_0 corresponds bijectively to the set of elements of the group

$$\lim_{n\to\infty} \mathrm{Hom}_{\mathbb{Z}}(G_{L_0}, \mathbb{Z}/n\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(G_{L_0}, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z}), \tag{6.2}$$

i.e. to the group of G_{L_0}-*characters*.

We can now interpret the exact sequence (6.1) in terms of Galois theory. Let L'/L_0 be a cyclic extension of the field L_0. This extension corresponds to an element

$$\chi \in \mathrm{Hom}_{\mathbb{Z}}(G_{L_0}, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z}).$$

Notice that the compositum $L'L_1$ of L' and L_1 over L_0 is a cyclic extension of L_1 of degree dividing n, and therefore, this corresponds to a character

$$\chi' \in \mathrm{Hom}_{\mathbb{Z}}(G_{L_1}, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z}).$$

One checks that the image of $\chi \in \mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z})$ under the restriction map

$$\mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z}) \tag{6.3}$$

is exactly χ'.

Let us now list two properties which will be used later on. We give only a short proof of the last of these and leave the other for the reader.

6.2.4 Lemma. *Let $\chi \in \mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z})$ be an element which maps to an element of order 2 inside $\mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z})$ under the restriction map (6.3). Then the order of χ is a divisor of 4.*

PROOF. Use (6.1). □ □

6.2.5 Lemma. *Let $\chi_i \in \mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z})$ for $i = 1, 2$ be two elements of order 4 which map to elements of order 2 inside $\mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z})$. Then the difference $\chi_1 - \chi_2$ is an element of order at most 2 inside $\mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z})$.*

PROOF. By assumption and exactness of (6.1), we see that both $2\chi_1$ and $2\chi_2$ are of order 2 and in fact contained in the group

$$\mathbb{Z}/2\mathbb{Z} \simeq \mathrm{Gal}(L_1/L_0) \subset \mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z}). \tag{6.4}$$

Therefore $2\chi_1 = 2\chi_2$ and the result follows. □ □

Let us now apply this formalism of characters to study D_4-Galois extensions of a field L_0. Let $L_0 \subset L$ be D_4-Galois and fix an isomorphism $\mathrm{Gal}(L/L_0) \simeq D_4$. Let L_1 be the field $L^{\langle a^2,b\rangle}$ fixed by the subgroup $\langle a^2, b\rangle \subset D_4$. Notice that $L_1 \subset L$ is a $(\mathbb{Z}/2\mathbb{Z})^2$-Galois extension.

There are exactly three proper subfields of L containing L_1 other than L_1 itself. These are $L^{\langle a^2\rangle}$, $L^{\langle b\rangle}$ and $L^{\langle a^2b\rangle}$. Each is a $\mathbb{Z}/2\mathbb{Z}$-Galois extension of L_1 and therefore, these fields correspond to order 2 characters χ_{a^2}, χ_b and χ_{a^2b}, respectively, of the group G_{L_1}. Hence we may regard the χ_* as elements of the group

$$\mathrm{Hom}_{\mathbb{Z}}(G_{L_1}, \mathbb{Q}/\mathbb{Z}) \simeq H^1(G_{L_1}, \mathbb{Q}/\mathbb{Z}). \tag{6.5}$$

We leave the proof of the following lemma to the reader.

6.2.6 Lemma. *We have the following relations.*

1. *The character χ_{a^2} is fixed under the Galois action $\mathrm{Gal}(L_1/L_0)$ on the group $\mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z})$.*
2. *The characters χ_b and χ_{a^2b} are conjugate under this action.*
3. *The sum of χ_b and χ_{a^2b} is χ_{a^2}.*

The following lemma will be useful for lifting D_4-actions later on.

6.2.7 Lemma. *Let $L_0 \subset L$ and $L_0 \subset L'$ be two D_4-Galois extensions. Assume that there exists a L_0-isomorphism between $L^{\langle b\rangle}$ and $L'^{\langle b\rangle}$. Then there is also a L_0-isomorphism between L and L'.*

PROOF. Use the uniqueness of the Galois closure. □ □

6.2.8 Remark. Although a simple lemma, the above tells us that the essential information of the D_4-Galois extension $L_0 \subset L$ is stored inside the subextension $L_0 \subset L^{\langle b\rangle}$.

6.2.9 Notation. From now on, whenever we are given extensions as above and a character

$$\chi \in \mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z}),$$

we shall denote by $\mathrm{N}\,\chi$ the norm (some reference refer to this as the *trace*) of χ under the action of $\mathrm{Gal}(L_1/L_0)$ on $\mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z})$, i.e. the sum of χ and its conjugate $\sigma^*\chi$, where σ is the generator of $\mathrm{Gal}(L_1/L_0)$. Furthermore, we reserve the notation χ_{a^2}, χ_b and χ_{a^2b} for the characters corresponding to the $\mathbb{Z}/2\mathbb{Z}$-extensions $L^{\langle a^2\rangle}$, $L^{\langle b\rangle}$ and $L^{\langle a^2b\rangle}$ of the field $L_1 = L^{\langle a^2,b\rangle}$.

6.2.3 Artin–Schreier theory of power series fields in characteristic 2

Let L_z be the local field $k((z))$ with parameter z. The following identification will be used often:

$$k((z))/\wp k((z)) \simeq \mathrm{H}^1_{et}(\mathrm{spec}(k((z))), \mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{H}^1(G_{L_z}, \mathbb{Q}/\mathbb{Z})[2] \tag{6.6}$$

where

$$\wp : k((z)) \to k((z)), \quad y \mapsto y^2 - y$$

is the Artin–Schreier operator in characteristic 2. The identification (6.6) associates to the element

$$f \in k((z))$$

the $\mathbb{Z}/2\mathbb{Z}$-extension of $k((z))$ generated by w, where w satisfies

$$w^2 - w = f.$$

Often we shall denote the associated class of f simply by

$$[f] \in \mathrm{H}^1(G_{L_z}, \mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{H}^1(G_{L_z}, \mathbb{Q}/\mathbb{Z})[2].$$

Notice that for f_1 and f_2 both elements of $k((z))$, we have that

$$[f_1] = [f_2]$$

if and only if there exists a $q \in k((z))$, such that

$$f_1 = q^2 - q + f_2$$

inside the field $k((z))$.

Let $f \in k[z] \subset k((z))$. In this case, one can always find a $f_0 \in k((z))$ such that

$$f_0^2 - f_0 = f,$$

and therefore,

$$[f] = 0$$

in this case. Therefore, if $f := \sum_{-N \leq i} c_i z^i \in k((z))$ is a general element of the field $k((z))$ for $c_i \in k$, then

$$[f] = [\sum_{-N \leq i} c_i z^i] = [\sum_{N \leq i < 0} c_i z^i]. \tag{6.7}$$

Furthermore, we also have

$$[c_{2m} z^{-2m}] = [\sqrt{c_{2m}} z^{-m}]$$

since k is assumed to be algebraically closed. Therefore, we can also get rid of the terms of f in the expansion (6.7) of degree $-2m$, where m ranges over the natural numbers.

6.2.4 Supersimple D_4-extensions

We now define and study the type of D_4-extensions that we are interested in lifting. Assume throughout this section that L_0 is a local power series field with characteristic 2.

6.2.10 Definition. A local D_4-Galois extension $L_0 \subset L$ is said to be *supersimple* if the following conditions hold.

1. The local degree of different of $L^{\langle a^2,b\rangle} \subset L^{\langle a^2\rangle}$ is 2,
2. the local degree of different of $L_0 \subset L^{\langle a^2,b\rangle}$ is 2.

6.2.11 Remark. Let G be a finite p-group and consider a G-Galois extension of local power series fields

$$k((z)) \subset L_G.$$

We use L_G with the subscript G to emphasize that we are not restricted to supersimple extensions in this remark.

It is known that there exists a G-Galois cover of smooth curves

$$C \to \mathbb{P}^1_{k,z}$$

which is étale over $\mathbb{A}^1_k \subset \mathbb{P}^1_k$, completely branched over the complement $(z = 0) \in \mathbb{P}^1_k$, and which induces $k((z)) \subset L_G$ after localization and completion at $z = 0$. This cover is known as the Katz–Gabber cover associated to the extension $k((z)) \subset L_G$. For details on this and for the more general Katz–Gabber compactification, see for instance the account in Gille [12].

Applying this to the case $G = D_4$ with $L_0 = k((z))$ and $L_G = L$, one sees that $L_0 \subset L$ is supersimple if and only if

$$C/\left\langle a^2\right\rangle \simeq \mathbb{P}^1_k,$$

where $C \to \mathbb{P}^1_k$ is the Katz–Gabber cover associated to $L_0 \subset L$. Notice that this compactification is therefore a hyperelliptic curve.

Let us now construct some examples of supersimple extensions. First we set some notation.

6.2.12 Notation. From now on, we shall reserve the notation L_0 for the local power series field $k((t))$, and the notation L_1 for the local power series field $k((v))$, where the variables t and v are related by

$$v^{-2} - v^{-1} = t^{-1} \tag{6.8}$$

Also, we shall let σ denote the generator of $\mathrm{Gal}(L_1/L_0) \simeq \mathbb{Z}/2\mathbb{Z}$.

6.2.13 Example. In view of Lemma 6.2.1, we now construct some $\mathbb{Z}/2\mathbb{Z}$-extensions of L_1 which, when considered as degree 4-extensions of L_0, are not Galois.

Let $\eta \in k$ and consider the element $f_\eta \in L_1$ given by

$$f_\eta = \eta^2 v^{-3} - \eta^2 v^{-2} = \eta^2 t^{-1} v^{-1}.$$

Notice that the sum of f_η and its conjugate $\sigma^* f_\eta$ is simply $\eta^2 t^{-1}$. One checks quickly that for $\eta \notin \mathbb{F}_2$, the Artin–Schreier class of $\eta^2 t^{-1}$ is non-trivial in the group $\mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z})$. Therefore, the extension L_2 of L_1, defined by

$$w^2 - w = \eta^2 t^{-1} v^{-1},$$

induces an extension of L_0 which is of degree 4 and not Galois. As we have already pointed out in Lemma 6.2.1, this then produces a D_4-Galois extension $L_0 \subset L$ by taking the Galois closure. One notes that, by Lemma 6.2.2 the Galois group can be identified with D_4 such that L_1 is the fixed field of $\langle a^2, b \rangle$ and also L_2 that of $\langle b \rangle$. The extension of L_1 defined by

$$s^2 - s = \eta^2 t^{-1} \tag{6.9}$$

is, by Lemma 6.2.6 (3), exactly the field extension defined by $L_1 = L^{\langle a^2, b \rangle} \subset L^{\langle a^2 \rangle}$, which one checks has local degree of different exactly 2. Therefore, the D_4-Galois extension $L_0 \subset L$ is supersimple.

The idea of this example is that it is somewhat representative of supersimple D_4-actions. In fact, it will be useful for classifying them (Theorem 6.2.20).

6.2.14 Definition. For a $\eta \in k$, we denote by ψ_η the character of $G_{k((v))}$ corresponding to the $\mathbb{Z}/2\mathbb{Z}$-extension generated by $w^2 - w = \eta^2 t^{-1} v^{-1}$, i.e. ψ_η denotes the image of the polynomial $\eta^2 t^{-1} v^{-1}$ in $\mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z})$ under (6.6).

6.2.15 Remark. It is important to note that Definition 6.2.14 depends on the choices of the parameters t and v.

6.2.16 Remark. For the value $\eta = 1$, the character ψ_1 induces a $\mathbb{Z}/4\mathbb{Z}$-Galois extension of L_0. Furthermore, one can show that there exists a character ψ_1' of the group G_{L_0}

$$\psi_1' : G_{L_0} \twoheadrightarrow \mathbb{Z}/4\mathbb{Z},$$

which maps to ψ_1 under the restriction mapping

$$\mathrm{Hom}_{\mathbb{Z}}(G_{L_0}, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(G_{L_1}, \mathbb{Q}/\mathbb{Z})$$

of (6.1). Furthermore, the character ψ_1' generates the torsion subgroup of

$$\mathrm{H}^1(G_{L_1}, \mathbb{Z}/4\mathbb{Z}) \subset \mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(G_{L_1}, \mathbb{Q}/\mathbb{Z}) \tag{6.10}$$

of order-4 characters.

6.2.17 Remark. Notice that we have the following identity in $\mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z})$ for all $\eta \in k$.

$$\psi_{\eta+1} = \psi_\eta + \psi_1. \tag{6.11}$$

6.2.5 Classifying supersimple D_4-extensions

The aim of this section is to classify the local supersimple D_4-Galois extensions. Assume throughout that $k((t)) = L_0 \subset L$ is a supersimple D_4-Galois extension.

6.2.18 Lemma. *By possibly changing the parameter t of L_0, we may assume that the intermediate field extension $L_0 \subset L^{\langle a^2,b\rangle}$ is generated by v, where v and t are related by*

$$v^{-2} - v^{-1} = t^{-1}. \tag{6.12}$$

PROOF. This result follows from the fact that the local degree of different of $L_0 \subset L^{\langle a^2,b\rangle}$ is 2.
□ □

From now on we set $L_1 = L^{\langle a^2,b\rangle}$. We consider the elements t and v fixed, and use the notation of Definition 6.2.14.

6.2.19 Lemma. *Consider the $\mathbb{Z}/2\mathbb{Z}$-Galois extension $L^{\langle b\rangle}/L_1$ and the associated G_{L_1}-character*

$$\chi_b \in \mathrm{H}^1(G_{L_1}, \mathbb{Q}, \mathbb{Z})$$

of order 2. Then there exists an $\eta \in k$ such that the G_{L_1}-character $\chi_b - \psi_\eta$ is the image of a 2-torsion element of $\mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z})$ under the restriction map

$$\mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z}).$$

PROOF. By definition the local degree of different of $L_1 \subset L^{\langle a^2\rangle}$ is 2 and therefore, this extension is generated by an Artin–Schreier equation of the form

$$s^2 - s = \alpha v^{-1}, \tag{6.13}$$

for some $\alpha \in k$. By Lemma 6.2.6 the norm of χ_b is the character χ_{a^2}. The latter corresponds to the field extension $L_1 \subset L^{a^2}$ and therefore corresponds to the Artin–Schreier class $[\alpha v^{-1}]$.

Choose $\eta \in k$ such that $\eta^2 + \eta = \alpha$. One checks that the Artin–Schreier classes of $[\alpha v^{-1}]$ and $[\eta^2 t^{-1}]$ are the same inside $\mathrm{H}^1(G_1, \mathbb{Q}/\mathbb{Z})$.

Consider the norm $\mathrm{N}\,\psi_\eta$ of the character $\psi_\eta \in \mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z})$. We see that this corresponds to the $\mathbb{Z}/2\mathbb{Z}$-Galois extension of L_1 generated by $\tilde{s}$, where $\tilde{s}$ satisfies

$$\tilde{s}^2 - \tilde{s} = \eta^2 t^{-1}.$$

However, by definition of α and η, this is exactly the extension $L_1 \subset L^{a^2}$, see (6.13).

Hence the norm $\mathrm{N}\,\psi_\eta$ and the character $\chi_{a^2} = \mathrm{N}\,\chi_b$ are equal inside the group $\mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z})$, and hence the difference $\chi_b - \psi_\eta$ is fixed under the action of $\mathrm{Gal}(L_1/L_0)$.

Therefore, $\chi_b - \psi_\eta$ is an element of $\mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z})^{\mathrm{Gal}(L_1/L_0)}$, and thus, by the right exactness of (6.1), the image of some $\chi' \in \mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z})$ under the restriction map

$$\mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z}). \tag{6.14}$$

By Lemma 6.2.4, we may conclude that χ' has order a divisor of 4.

By Remark 6.2.16, we notice that ψ_1 is also the image of an order 4 element ψ_1' of $\mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z})$. Hence, by Lemma 6.2.5, either χ' or $\chi' - \psi_1'$ is an order 2 element of $\mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z})$.

If χ' is of order 2, then we have found a suitable η satisfying the hypothesis of the lemma.

Assume this is not the case, i.e. $\chi' - \psi_1'$ is of order 2. Then the image of $\chi' - \psi_1'$ inside $\mathrm{H}^1(G_1, \mathbb{Q}/\mathbb{Z})$ under the restriction map (6.14) is exactly

$$\chi_b - \psi_\eta - \psi_1 = \chi_b - \psi_{\eta+1},$$

and therefore, the value $\eta + 1$ satisfies the hypothesis of the lemma. □

6.2.20 Theorem. *There exists a polynomial*

$$Q(t^{-1}) \in k[t^{-1}] \subset k(\!(t)\!)$$

and an $\eta \in k$, such that the field extension

$$L_1 = L^{\langle a^2, b\rangle} \subset L^{\langle b\rangle}$$

is generated by an Artin–Schreier equation of the form

$$w^2 - w = \eta^2 t^{-1} v^{-1} + Q(t^{-1}). \tag{6.15}$$

Furthermore, the polynomial Q can be chosen to have only odd degree terms in the variable t^{-1}.

PROOF. We let η be as in Lemma 6.2.19. Let χ' be an element of $\mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z})$ which maps to

$$\chi_b - \psi_\eta \in \mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z})$$

under the restriction map

$$\mathrm{H}^1(G_{L_0}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(G_{L_1}, \mathbb{Q}/\mathbb{Z})$$

and which has order at most 2.

The character χ' corresponds to a cyclic Galois extension of the field L_0 of degree at most 2. Therefore, we can find an element Q of the field $L_0 = k(\!(t)\!)$ with associated Artin–Schreier class inducing this extension.

As remarked in Section 6.2.3, we see that we can even choose Q to be inside the subring

$$k[t^{-1}] \subset k(\!(t)\!) = L_0$$

of polynomials in the variable t^{-1}. The comments of Section 6.2.3 also allow us to find a Q with only odd degree terms. We have proved the lemma. □ □

The following lemma will be useful later on and we shall leave the proof to the reader.

6.2.21 Lemma. *We use the notations of Theorem 6.2.20. If the degree of* Q *is denoted by* d *for some odd integer* d*, then the degree of local different of*

$$L_1 = L^{\langle a^2, b\rangle} \subset L^{\langle b\rangle}$$

is exactly the maximum $\max(4, 2d)$.

PROOF. One uses the relation (6.12) together with the Artin–Schreier equation (6.15) for the field extension $L_1 \subset L^{\langle b\rangle}$. □ □

6.2.22 Remark. Recall (Theorem 6.2.20) that the extension $L_1 \subset L^{\langle b\rangle}$ is given by (6.15). The proof of Lemma 6.2.21 shows that if $\deg(Q) \leq 1$, then the term $\eta^2 t^{-1} v^{-1}$ of (6.15) dominates the degree of different of $L_1 \subset L^{\langle b\rangle}$, i.e. it is then 4. If $\deg(Q) \geq 3$, then the term Q dominates this. In Section 6.5, we shall prove that all supersimple actions lift to characteristic 0. There we shall distinguish a supersimple action according to the distinction remarked here, i.e. according to the degree of different of $L_1 \subset L^{\langle b\rangle}$, and we shall need to adapt our lifting technique according to the case we are considering.

6.3 Good reduction of Galois closures

Before we give a brief introduction and overview on this section, we first set some notation. Let

$$C_1 \to \mathbb{P}^1_K =: C_0, \quad C_2 \to C_1$$

be two $\mathbb{Z}/2\mathbb{Z}$-Galois covers of smooth projective K-curves. We shall assume that the composite extension $C_2 \to C_0 \simeq \mathbb{P}^1_K$ of degree 4 is not a Galois cover. We let $C_3 \to C_0 \simeq \mathbb{P}^1_K$ be the Galois closure.

In Section 6.2.1 it was shown that we can identify the Galois group $\mathrm{Gal}(C_3/\mathbb{P}^1_K)$ with D_4 in such a manner that C_2 is the quotient of C_3 under the subgroup

$$\langle b\rangle \subset D_4 \simeq \mathrm{Gal}(C_3/C_0).$$

From now on we shall assume this to be the case.

In this section we shall be concerned with the following question: what reduction conditions on the intermediate cover $C_2 \to C_0 \simeq \mathbb{P}^1_K$ are necessary to conclude that the curve C_3 has good reduction? In Section 6.4, we shall make specific choices for the curves C_0, C_1 and C_2 which will satisfy these conditions. These choices will be such that after studying their reductions, we shall show that by localizing and completing these covers at their branch points, we obtain lifts for all supersimple D_4-actions. For our purposes it is convenient to assume that $g(C_2) \geq 1$. In this section we shall place no restrictions on the genus of C_1, however, in Section 6.4 we shall work only with the case that $g(C_1) = 0$, i.e. $C_1 \simeq \mathbb{P}^1_K$.

One sees that if C_3 has potentially good reduction, then so must the curve C_2. Therefore, we shall always assume that C_2 admits a smooth model $\mathcal{C}_2$. We introduce the following assumption on the cover $C_2 \to C_0$.

6.3.1 Assumption (‘Good reduction’ Assumption). There exists smooth models $\mathcal{C}_i,\ i = 0, 1,$ of the curves $C_i,\ i = 0, 1$, together with *finite* maps

$$\mathcal{C}_2 \to \mathcal{C}_1 \to \mathcal{C}_0 \tag{6.16}$$

which have generic fibre $C_2 \to C_1 \to C_0$. The induced map of smooth k-curves

$$\mathcal{C}_{2,k} \to \mathcal{C}_{1,k} \to \mathcal{C}_{0,k} \simeq \mathbb{P}^1_k \tag{6.17}$$

is a *separable* cover of degree 4. Furthermore, we assume that (6.17) is *totally* branched at some point $x \in \mathcal{C}_{0,k}$.

6.3.2 Remark. It follows from Liu–Lorenzini [25] Proposition 1.6 that since $\mathcal{C}_2$ is smooth, the quotients $\mathcal{C}_1$ and $\mathcal{C}_0$ are also smooth R-curves. Furthermore, it follows from Liu [24] Proposition 10.3.38 that if furthermore $g(C_1) \geq 2$, then (6.17) is separable.

Furthermore, one sees that the following assumption, which does not necessarily hold, is necessary to deduce potentially good reduction for the curve C_3.

6.3.3 Assumption (‘NonGalois reduction’ Assumption). The special fibre cover (6.17)

$$\mathcal{C}_{2,k} \to \mathcal{C}_{1,k} \to \mathcal{C}_{0,k} \simeq \mathbb{P}^1_k$$

is not Galois.

Let us now study the stable model $\hat{\mathcal{C}}_3$ of the curve C_3. The group D_4 acts on this model, and we denote by $\hat{\mathcal{C}}_i,\ i \in \{0, 1, 2\}$, the quotients of this model corresponding to the K-curves C_0, C_1 and C_2 respectively. It is known that all these are themselves semistable R-curves, see for instance Raynaud [33] Appendice.

Since $\mathcal{C}_2$ is a smooth R-curve with positive genus, we see that there exists a birational blowup morphism

$$\hat{\mathcal{C}}_2 \to \mathcal{C}_2. \tag{6.18}$$

Therefore, we may conclude by the universal property of quotient schemes, that similar blowup morphisms

$$\hat{\mathcal{C}}_1 \to \mathcal{C}_1, \quad \hat{\mathcal{C}}_0 \to \mathcal{C}_0 \tag{6.19}$$

exist for $\mathcal{C}_1$ and $\mathcal{C}_0$, even if their genera are 0.

We denote the strict transform of the smooth k-curve $\mathcal{C}_{2,k}$ under the map of (6.18) by Γ_2, and using (6.19) we define the components Γ_1 and Γ_0 similarly. Each Γ_i, for $i = 0, 1, 2$, is therefore a smooth k-curve, and furthermore, we have a separable degree-4 covering

$$\Gamma_2 \to \Gamma_1 \to \Gamma_0 \tag{6.20}$$

which is nothing else than the covering

$$\mathcal{C}_{2,k} \to \mathcal{C}_{1,k} \to \mathcal{C}_{0,k} \simeq \mathbb{P}^1_k. \tag{6.21}$$

Let Γ_3 be any component of $\hat{\mathcal{C}}_{3,k}$ which maps surjectively onto Γ_2 under the finite map $\hat{\mathcal{C}}_{3,k} \to \hat{\mathcal{C}}_{2,k}$. Since $\hat{\mathcal{C}}_{3,k}$ was assumed to be the stable model of C_3, we see that each component of $\hat{\mathcal{C}}_{3,k}$ is reduced, and, in particular, the closed subscheme Γ_3 is an integral scheme. We may therefore consider the extension of function fields

$$k(\Gamma_0) \subset k(\Gamma_1) \subset k(\Gamma_2) \subset k(\Gamma_3). \tag{6.22}$$

6.3.4 Proposition. *The component Γ_3 is the only component of $\hat{\mathcal{C}}_{3,k}$ mapping surjectively onto Γ_2. Furthermore, the field extension (6.22) is a D_4-Galois extension.*

PROOF. Let $D(\Gamma_3)$ (respectively $I(\Gamma_3)$) denote the decomposition (respectively inertia) group of Γ_3. Let L be the separable closure of $k(\Gamma_0)$ inside the normal field extension (6.22). There exists an exact sequence of groups (see Serre [37] Proposition I.20)

$$0 \to I(\Gamma_3) \to D(\Gamma_3) \to \mathrm{Gal}(L/k(\Gamma_0)) \to 0.$$

Notice that by Assumption 6.3.1 the Galois extension $k(\Gamma_0) \subset L$ contains the subextension

$$k(\Gamma_0) \subset k(\Gamma_2).$$

Furthermore, by Assumption 6.3.3, we see that $[L : k(\Gamma_0)] > 4$ and therefore, the order of $\mathrm{Gal}(L/k(\Gamma_0))$ must exceed 4. However, $D(\Gamma_3)$ is a subgroup of D_4, and therefore, the result follows. □

Our next step is to study the normalization of the component Γ_3. Let $\tilde{\Gamma}_3$ denote the normalization of Γ_3. In order to deduce smoothness of the stable R-curve $\hat{\mathcal{C}}_3$, we shall now ask for a condition under which the geometric genus $g(\tilde{\Gamma}_3)$ of $\tilde{\Gamma}_3$ is equal to the geometric genus $g(C_3)$ of the generic fibre C_3. It is known that the latter is never strictly less than the former. Furthermore, since $\hat{\mathcal{C}}_3$ is assumed to be the stable model of the K-curve C_3, equality of $g(\tilde{\Gamma}_3)$ and $g(C_3)$ would imply smoothness of $\hat{\mathcal{C}}_3$. We thus proceed to bounding $g(\tilde{\Gamma}_3)$ from below.

6.3.5 Assumption ('Different' Assumption). We assume the degree of geometric different of the cover $C_3 \to C_2$ is 2.

6.3.6 Lemma. *The genus of C_3 is $2g(C_2)$. In particular, we have the following inequalities.*

$$g(\tilde{\Gamma}_3) \leq 2g(C_2). \tag{6.23}$$

PROOF. Apply the Hurwitz Formula to the cover of smooth K-curves $C_3 \to C_2$. □

6.3.7 Theorem. *The curve $\hat{\mathcal{C}}_3$ is a smooth R-curve.*

PROOF. By Assumption 6.3.5, the cover of K-curves $C_3 \to C_2$ has exactly two geometric branch points x_1, x_2, and after possibly extending K, we may assume that these two points are distinct points of $C_2(K)$. From Theorem 1 of Saïdi [35], we see that since $\Gamma_3 \to \Gamma_2$ is a separable covering, both x_1 and x_2 specialize to the same point x of $\mathcal{C}_{2,k}$. Note that $\tilde{\Gamma}_3 \to \Gamma_2$ is branched at this point. This implies that

$$g(\tilde{\Gamma}_3) \geq 2g(\Gamma_2) = 2g(C_2), \tag{6.24}$$

and hence is equal to exactly this. Therefore, $\hat{\mathcal{C}}_3$ is a smooth R-curve. □ □

6.4 Lifting supersimple D_4-actions.

In this section we shall give a method for producing covers of curves which satisfy the assumptions needed to apply the results in the previous section. Let us first set some notation and then we explain our goals and strategy.

In Section 6.3, we dealt with towers of $\mathbb{Z}/2\mathbb{Z}$-covers $C_2 \to C_1 \to C_0$ of composite degree 4. Our first step is to construct suitable choices for the curves C_1 and C_0. We let $\mathcal{C}_0$ denote the projective R-line $\mathbb{P}^1_{R,t}$ with parameter t.

To define the R-curve $\mathcal{C}_1$, we define an algebraic extension of $K(t)$ by adjoining the element v, where v satisfies the relation

$$t^{-1} = v^{-2} - v^{-1}.$$

We now define $\mathcal{C}_1$ to be the normalization of the projective line $\mathcal{C}_0$ inside the field $K(t)(v)$. We leave for the reader to verify that $\mathcal{C}_1$ is again a projective R-line with parameter v, and that the induced special fibre cover $\mathcal{C}_{1,k} \to \mathcal{C}_{0,k}$ is a separable cover of smooth projective lines. By localizing and completing at the point $t = 0$, we see that the $\mathbb{Z}/2\mathbb{Z}$-Galois cover

$$\mathbb{P}^1_{R,v} = \mathcal{C}_1 \to \mathcal{C}_0 = \mathbb{P}^1_{R,t}$$

already provides a lift for the $\mathbb{Z}/2\mathbb{Z}$-Galois extension of local fields $L_0 \subset L_1$ of Notation 6.2.12.

Now we want to construct some $\mathbb{Z}/2\mathbb{Z}$-Galois extensions of the curve $\mathcal{C}_1$. Let F and G be two elements of $R[v^{-1}] \subset K(v^{-1})$. We denote the reductions of F and G to the ring $k[v^{-1}]$ by $\overline{F}$ and $\overline{G}$, respectively. We define a field extension $K(v) \subset K(v, w)$ where w satisfies

$$w^2 - wG = F. \tag{6.25}$$

We let $\mathcal{C}_2^{F,G}$ be the normalization of $\mathcal{C}_1 = \mathbb{P}^1_{R,v}$ inside $K(v, w)$. We have included the superscripts F and G to emphasize that our definition depends on the choices of F and G.

Our strategy now is to find suitable F and G such that the generic fibre of the finite tower of $\mathbb{Z}/2\mathbb{Z}$-Galois extensions

$$\mathcal{C}_2^{F,G} \to \mathcal{C}_1 = \mathbb{P}^1_{R,v} \to \mathcal{C}_0 = \mathbb{P}^1_{R,t} \tag{6.26}$$

satisfies Assumptions 6.3.1, 6.3.3 and 6.3.5.

To check the 'good reduction' (Assumption 6.3.1), the form of equation (6.25) will be useful (Lemma 6.4.1). However, to check Assumption 6.3.5, we shall need to rewrite this equation in a Kummer form. Here we shall restrict the choices of F and G (Lemma 6.4.2). A further restriction (Lemmas 6.4.3 and 6.4.4) on the choices of F and G will also aid us in checking that the 'reduction is not Galois' (Assumption 6.3.3).

Assume that the degrees of F and $\overline{F}$ are both $2g+1$, where g is some positive integer. Furthermore, assume that the degree of G does not exceed $2g$, and that the reduction $\overline{G}$ is a unit of k (and hence of degree 0, but that $\overline{G} \neq 0$ inside $k[v^{-1}]$).

6.4.1 Lemma. (*a*) *The scheme $\mathcal{C}_2^{F,G}$ is a smooth projective R-curve of genus g. Furthermore, the action of the Galois group $\mathrm{Gal}(K(v,w)/K(v))$ extends to the scheme $\mathcal{C}_2^{F,G}$, and the quotient of $\mathcal{C}_2^{F,G}$ by this action is $\mathcal{C}_1 = \mathbb{P}^1_{R,v}$. Lastly, the induced map of special fibres*

$$\mathcal{C}_{2,k}^{F,G} \to \mathcal{C}_{1,k} \simeq \mathbb{P}^1_{k,v}$$

is generically separable, and is in fact branched uniquely at the point $v = 0$.
(*b*) *By localizing and completing at this point, the cover $\mathcal{C}_2^{F,G} \to \mathcal{C}_1$ induces a $\mathbb{Z}/2\mathbb{Z}$-Galois extension of $k((v))$ generated by w, where w satisfies*

$$w^2 - w\overline{G} = \overline{F}. \tag{6.27}$$

The local degree of different is $2g + 2$.

PROOF. This is essentially Exercise 10.1.9 of Liu [24]. □ □

So far, we have constructed a tower of smooth projective curves

$$\mathcal{C}_2^{F,G} \to \mathcal{C}_1 \to \mathcal{C}_0 \simeq \mathbb{P}^1_R. \tag{6.28}$$

For convenience we set $C_2^{F,G} := \mathcal{C}_{2,K}^{F,G}$ and similarly for C_1 and C_0. As in Section 6.3, we define $C_3^{F,G}$ to be the Galois closure of $C_2^{F,G} \to C_0$. Notice that in order to apply the results of Section 6.3, we also need to know that $C_2^{F,G} \neq C_3^{F,G}$. This will be true for the choices of F and G that we shall later choose.

6.4.2 Lemma. *Assume that we can find an element*

$$H \in R[t^{-1}] \subset K(t) \subset K(v),$$

as well as an $\eta \in R$ such that the following identity holds.

$$4F + G^2 = (1 - 2\eta v^{-1})H.$$

If $\eta \neq 1$, then $C_2^{F,G} \neq C_3^{F,G}$, i.e. the cover $C_2^{F,G} \to C_0$ is not Galois, and furthermore, the degree of geometric different of $C_3^{F,G} \to C_2^{F,G}$ is 2.

PROOF. By construction, the cover

$$\mathbb{P}^1_{K,v} \simeq C_1 \to C_0 \simeq \mathbb{P}^1_{K,t}$$

is ramified at exactly $v = 0$ and $v = 2$. Notice that the function field $K(v, w)$ of $C_2^{F,G}$ is also generated over $K(v)$ by w', where $w' = -2w + G$, i.e. w' satisfies the following Kummer equation

$$(w')^2 = (-2w + G)^2 = 4F + G^2 = (1 - 2\eta v^{-1})H.$$

Therefore, the cover $C_2^{F,G} \to C_1$ is branched at exactly $v = 0$, $v = 2\eta$ and the zeros of

$$H \in K(t) \subset K(v).$$

If $\eta \neq 1$, then the conjugate of the point $v = 2\eta$ (under the action of $\mathrm{Gal}(C_1/C_0)$) is not branched in the cover $C_2^{F,G} \to C_1$. This already implies that $C_2^{F,G} \to C_0$ is not Galois, i.e. $C_2^{F,G} \neq C_3^{F,G}$. Furthermore, one checks that the points of $C_2^{F,G}$ lying above the conjugate of the point $v = 2\eta$ are exactly the branch points of the cover $C_3^{F,G} \to C_2^{F,G}$. There are exactly two points, and hence the degree of geometric different of $C_3^{F,G} \to C_2^{F,G}$ is 2. □ □

Before we state the main theorems of this section, we give two computational results. Let $L_0 \subset L$ be a local supersimple D_4-Galois extension in characteristic 2. Recall from Lemma 6.2.21 and Remark 6.2.22 that we can distinguish between two cases, namely the case where the local different degree of $L^{\langle a^2,b\rangle} \subset L^b$ is 4, and the case where it is $2d$, for some odd integer d. In proving that all supersimple actions lift to characteristic 0, we shall deal with these two cases separately. In both cases, we shall need a similar computation, and it is these that we state in the following two lemmas. Both of these results are essentially computations, and we used the computer package Magma to verify our calculations.

6.4.3 Lemma. *Let $\eta \in R^*$. We assume R has been extended to include a solution, β, of the following equation.*

$$\beta^2 + \sqrt{2}\beta + \eta = 0. \tag{6.29}$$

Let $Q' \in R[t^{-1}]$ of degree less than or equal to 1. Then we have the following identity.

$$(1 - 2\eta v^{-1})(1 + 2\beta^2 t^{-1} + 4Q') = G^2 + 4F,$$

where

$$G := 1 + \sqrt{2}\beta v^{-1}$$

and

$$F := Q' - \eta\beta^2 v^{-1} t^{-1} - 2\eta v^{-1} Q'.$$

6.4.4 Lemma. *Let η and β be as in Lemma 6.4.3. Let m be a positive integer, and let Q' be any element of $R[t^{-1}]$ of degree* strictly *less than $2m$. Furthermore, let $\gamma \in R^*$ be any unit of R. Then we have the following identity.*

$$(1 - 2\eta v^{-1})(1 + 2\beta^2 t^{-1} + 2\gamma^2 t^{-2m} + 2\sqrt{2}\gamma t^{-m} + 4Q') = G^2 + 4F,$$

where

$$G := 1 + \sqrt{2}\beta v^{-1} + \sqrt{2}\gamma t^{-m}$$

and

$$F := Q' - \eta\beta^2 v^{-1} t^{-1} - 2\eta v^{-1} Q' - \eta\gamma^2 t^{-2m} v^{-1} - \sqrt{2}\eta\gamma v^{-1} t^{-m} - \gamma\beta v^{-1} t^{-m}.$$

6.4.5 Remark. The equation (6.29) implies that we have the following equality in k after reduction

$$\overline{\beta}^2 = \overline{\eta}. \tag{6.30}$$

6.4.6 Remark. The polynomial F in Lemma 6.4.4 reduces to

$$\overline{F} := \overline{Q'} + \eta\beta^2 v^{-1} t^{-1}.$$

If F is selected as in Lemma 6.4.4, then it reduces to

$$\overline{F} := \overline{Q'} + \eta\beta^2 v^{-1} t^{-1} + \eta\gamma^2 t^{-2m} v^{-1} + \beta\gamma t^{-m} v^{-1}.$$

In both cases G reduces to the constant polynomial $1 \in k$.

The following theorem is our first main result. It constructs a family of D_4-Galois covers which, by localizing and completing at branch points, induce local supersimple extensions after reduction. We define the normal R-curve $\mathcal{C}_3^{F,G}$ to be the normalization of $\mathcal{C}_2^{F,G}$ inside the extension $C_3^{F,G} \to C_2^{F,G}$.

6.4.7 Theorem. *Let either η, Q' be as in Lemma 6.4.3, or let η, γ, m and Q' be as in Lemma 6.4.4, and let F and G be selected as in these lemmas. Consider the D_4-Galois extension of normal projective R-schemes*

$$\mathcal{C}_3^{F,G} \to \mathcal{C}_2^{F,G} \to \mathcal{C}_1 \to \mathcal{C}_0 = \mathbb{P}^1_{R,t}. \tag{6.31}$$

Then each $\mathcal{C}_i^{F,G}$ is a smooth R-scheme. Furthermore, by localizing and completing at the point $t = 0$ of the scheme $\mathcal{C}_0 = \mathbb{P}^1_{R,t}$, we obtain a lifting of the local D_4-Galois extension obtained by taking the Galois closure of

$$k((t)) \subset k((v)) \subset k((v))(w),$$

where w satisfies

$$w^2 - w = \overline{F}.$$

Here $\overline{F}$ denotes the reduction of the polynomial F, refer to Remark 6.4.6 for an explicit expression of $\overline{F}$.

PROOF. We shall proof the theorem in the case that F and G have been selected as in Lemma 6.4.4 and leave the (easier) case of Lemma 6.4.3 to the reader.

First we see from Lemma 6.4.1 that Assumption 6.3.1 is satisfied for the extension (6.31). In fact, the model $\mathcal{C}_2^{F,G}$ is a smooth model for its generic fibre $C_2^{F,G}$, and by construction the special fibre subcover

$$\mathcal{C}_{2,k}^{F,G} \to \mathcal{C}_{1,k} = \mathbb{P}^1_{k,v} \to \mathcal{C}_0 = \mathbb{P}^1_{k,t} \tag{6.32}$$

is separable. Lemma 6.4.2 tells us that Assumption 6.3.5 is also satisfied for this extension.

Let us check that the induced cover

$$\mathcal{C}_{2,k}^{F,G} \to \mathcal{C}_{1,k} \to \mathcal{C}_{0,k} \tag{6.33}$$

is not a Galois cover, thereby verifying Assumption 6.3.3. By localizing and completing at the point $v = 0$ of $\mathcal{C}_{1,k} \simeq \mathbb{P}^1_{k,v}$, we obtain a cover of $k((v))$ generated by w, where w satisfies

$$w^2 - w = \overline{F} = \eta^2 t^{-1} v^{-1} + Q' + \eta\gamma^2 t^{-2m} v^{-1} + \beta\gamma t^{-m} v^{-1}.$$

One checks that the composite field extension $k((t)) \subset k((v)) \subset k((v))(w)$ is not Galois if $\overline{\eta} \neq 1$. Therefore, the composite cover (6.33) cannot be Galois. By Theorem 6.3.7, we see that the curve $C_3^{F,G}$ has potentially good reduction. Since the smooth model $\mathcal{C}_2^{F,G}$ of $C_2^{F,G}$ is unique (recall that $g(C_2) \geq 1$), we see that $\mathcal{C}_3^{F,G}$ is smooth. We are done. □ □

6.5 Proof of main result

The aim of this section is to prove our main result.

6.5.1 Theorem. *All supersimple* D_4*-actions lift to characteristic* 0.

Assume throughout this section that we have been given a supersimple D_4-Galois extension of local power series fields

$$L_0 := k((t)) \subset L.$$

We use the notation of Section Section 6.2.4 and Section 6.2.5. In particular, we set $L_1 := L^{\langle a^2, b\rangle}$ with parameter v, where v and t are related by

$$v^{-2} - v^{-1} = t^{-1}. \tag{6.34}$$

We have already pointed out (Remark 6.2.7) that the field extension $L_0 \subset L$ is completely determined by the subextension

$$L_0 \subset L_1 \subset L_2 := L^{\langle b\rangle}.$$

By Lemma 6.2.21, we see that there are two cases to consider, namely the case that the degree of different of $L_1 \subset L_2$ is 4 or the case that it is $2d$, where $d > 1$ is an odd integer.

In both cases, we apply Theorem 6.4.7 for suitable choices of η, γ, m and Q'. In the first case, we shall choose the F and G as in Lemma 6.4.3, and in the second as in Lemma 6.4.4. We shall give the details only for the second case, and leave the detailed proof of the first (easier) case to the reader.

Proof of Theorem 6.5.1 We assume that the local different degree of $L_1 \subset L_2$ is of the form $2d$, where $d > 1$ is an odd integer. Define m by the relation $d = 2m + 1$.

From Theorem 6.2.20 and Lemma 6.2.21, we see that we can find a polynomial $Q \in k[t^{-1}]$ of degree exactly $2m+1$, as well as an $\eta \in k - \mathbb{F}_2$, such that the extension $L_1 \subset L_2$ is generated by w, where w satisfies

$$w^2 - w = \eta^2 t^{-1} v^{-1} + Q. \tag{6.35}$$

Recall (Theorem 6.2.20) that we can choose Q to have only odd degree terms in t^{-1}. Furthermore, since k was assumed algebraically closed, we can find a $\gamma \in k^*$ such that

$$Q = \gamma^2 \eta t^{-(2m+1)} + Q',$$

where $Q' \in k[t^{-1}]$ has degree strictly smaller than $2m$.

Let us lift the elements η and γ to units of R. We abuse notation and denote these lifts again by η and γ, respectively. We then choose a polynomial $Q' \in R[t^{-1}]$, of degree less than $2m$, which lifts the polynomial $Q' \in k[t^{-1}]$.

We now apply Theorem 6.4.7 with these choices of η, γ, m and Q' and we choose F and G as in Lemma 6.4.4. In view Theorem 6.4.7, we only need to check that the Artin–Schreier class of $\overline{F}$ is the same as $\eta^2 t^{-1} v^{-1} + Q$.

From Remark 6.4.6, we have the following equality of Artin–Schreier classes inside $\mathrm{H}^1(G_{L_1}, \mathbb{Z}/2\mathbb{Z})$

$$[\overline{F}] = [\eta\beta^2 v^{-1} t^{-1} + Q' + \eta\gamma^2 t^{-2m} v^{-1} + \beta\gamma t^{-m} v^{-1}].$$

Since $\beta^2 = \eta$ inside k (Remark 6.4.5), we see that

$$[\overline{F}] = [\eta^2 v^{-1} t^{-1} + Q' + \eta\gamma^2 t^{-2m-1}] = [\eta^2 v^{-1} t^{-1} + Q].$$

This is exactly the class of the extension $L_1 \subset L_2$, see (6.35). We conclude by applying Theorem 6.4.7.

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Zusammenfassung

Sei k ein Körper von Charakteristik p und sei $C \to \mathrm{spec}(k)$ eine glatte projektive Kurve mit einer G-Aktion $\overline{\phi} : G \hookrightarrow \mathrm{Aut}_k(C)$ auf C, wobei G eine endliche Gruppe ist. Das *globale Hochhebungsproblem* fragt, ob es eine Erweiterung R der Witt-Vektoren $W(k)$, eine Kurve $\mathcal{C} \to \mathrm{spec}(R)$ und eine Aktion $\phi : G \hookrightarrow \mathrm{Aut}_R(\mathcal{C})$ gibt, die nach $\overline{\phi}$ reduziert. Man sagt, dass die Aktion $\overline{\phi}$ nach Charakteristik 0 *hochhebt*, falls solch ein ϕ existiert.

Analog dazu gibt es auch das *lokale Hochhebungsproblem*: Sei $\overline{\phi} : G \hookrightarrow \mathrm{Aut}_k(k[\![t]\!])$ eine lokale G-Aktion. Das lokale Hochhebungsproblem fragt, ob es eine Aktion $\phi : G \hookrightarrow \mathrm{Aut}_R(R[\![t]\!])$ gibt, die nach $\overline{\phi}$ reduziert. Durch eine Betrachtung von Verzweigungsgruppen sieht man, dass jedes globale Hochhebungsproblem ein lokales Problem induziert. Ferner zeigt der *Lokal-Global-Prinzip* von Green und Matignon [15], dass diese zwei Probleme äquivalent sind.

Es ist bekannt, dass das lokale Lifting-Problem sehr schwer ist, falls p die Ordnung von G teilt. Einige Ergebnisse sind bekannt. Oort, Sekiguchi und Suwa [36] haben gezeigt, dass alle lokalen $\mathbb{Z}/p\mathbb{Z}$-Aktionen nach Charakteristik 0 hochheben. Green und Matignon [15] haben dieses Ergebnis auf den Fall $G = \mathbb{Z}/p^2\mathbb{Z}$ veralgemeinert.

Für nicht-zyklische Gruppen haben Bouw und Wewers [4] gezeigt, dass alle G-Aktionen hochheben, falls G die Dieder Gruppe der Ordnung $2p$ ist (wobei p eine ungerade Primzahl ist). Pagot [32] hat das gleiche Ergebnis für $G = (\mathbb{Z}/2\mathbb{Z})^2$ gezeigt.

In dieser Dissertation studieren wir das lokale Hochhebungsproblem durch die Verzweigungstheorie der p-adischen offenen Kreisscheibe. Unsere Theorie basiert auf den Theorien von Kato [21], [22], Kato und Saito [23] und Huber [20]. Im ersten Kapitel zeigen wir insbesondere, eine Idee von Irene Bouw folgend, dass Katos differentieller Swan-Führer die Differenziale von Henrio (im Fall von $\mathbb{Z}/p\mathbb{Z}$-Erweiterungen) verallgemeinert.

Im zweiten Kapitel betrachten wir die Verzweigunsgruppen von Kato und Saito [23]. Wir führen eine *vereinfachte* Verzweigungsfiltrierung ein, und wir zeigen, dass die Quotienten der Filtrierung elementare abelsche Gruppen sind. Wir führen auch zwei Charaktere, die *Artin*- und *Tiefen*-Charaktere, ein, und wir beweisen einen Zusammenhang zwischen denselben, Katos Swan-Führern und der Verzweigungsfiltrierung. Diese Ergebnisse verwenden wir in einem Beispiel im dritten Kapitel, und wir zeigen im fünften Kapitel, dass sie auch relevant für das lokale Hochhebungsproblem sind.

Henrio hat in [16], eine Idee von Green und Matignon [16] folgend, kombinatorische Objekte, die sogenannten Hurwitz-Bäume, für $\mathbb{Z}/p\mathbb{Z}$-Aktionen auf der p-adischen Kreisscheibe eingeführt. Ein Hurwitz-Baum spiegelt die Geometrie der Verzweigungspunkte einer $\mathbb{Z}/p\mathbb{Z}$-Aktion wider. Im dritten Kapitel dieser Dissertation verallgemeinern wir die Konstruktion von Green und Matignon auf den algemeinen Fall mit beliebigen p-Gruppen. Die in diesem Kapitel erzielten Ergebnisse sind in Zusammenarbeit mit Stefan Wewers. Die Neuerung bei unserer Konstruktion ist der Artin-Charakter der p-Gruppe. Weiterhin führen wir ein neues Hindernis für Hochhebbarkeit einer lokalen Aktion in Charakteristik p ein. Wir verwenden unsere neue Bedingung, um ein Problem von Chinburg, Guralnick und Harbater (siehe [8], Frage 1.3) zu studieren. Wir zeigen insbesondere, dass lokale verallgemeinerte Quaternion-Aktionen in Charakteristik 2 existieren, die nicht nach Charakteristik 0 hochheben. Unser Ergebnis beantwortet Frage 1.3 von [8] damit negativ.

Im vierten Kapitel zeigen wir, dass das neue Hindernis für zyklische p-Gruppen verschwindet. Dies ist ein neues Indiz für die Gültigkeit der Oort-Vermutung, welche sagt, dass alle zyklischen Aktionen nach Charakteristik 0 hochheben.

Eine weitere Frage ist, ob für eine gegebene Gruppe G eine lokale Aktion in Charakteristik p existiert, die sich nach Charakteristik Null hochheben lässt. In [27] fragt Matignon, was für nicht Abelshe p-Gruppen gilt. Im sechsten Kapitel geben wir das erste Ergebnis in dieser Richtung. Sei D_4 die Dieder Gruppe der Ordnung 8. Wir zeigen, dass es Beispiele von D_4-Aktionen in Charakteristik 2 gibt, die nach Charakteristik 0 hochheben. Insbesondere geben wir eine Familie von solche Aktionen, sogenannte *sehr einfachen* D_4-Aktionen, die alle nach Charakteristik 0 hochheben. Der Führer der von uns konstruierten Aktionen ist nicht nach oben beschränkt.

Selbständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbständig und nur mit den angegebenen Hilfsmitteln angefertigt habe. Alle Stellen, die anderen Werken entnommen sind, wurden durch Angabe der Quellen kenntlich gemacht.

Die Arbeit wurde bisher keiner anderen Prüfungsbehörde vorgelegt und noch nicht veröffentlicht.

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