Point-sources and Multipoles in Inverse Scattering Theory

Roland Potthast

Habilitation Thesis
Göttingen 1999

Acknowledgements

I would like to give thanks to all who made this thesis possible and accompanied me over the last years.

A large part of the scientific ideas have been developed in interaction with Prof. Dr. Rainer Kreß, Göttingen, and Prof. Dr. David Colton, Delaware. I also enjoyed visits and discussions with Dr. Simon Chandler-Wilde, London, and Dr. Bo Zhang, Coventry, and the work on the implementation of the methods with Dr. Klaus Giebermann, Bonn. Thanks are given to Dr. Imre Paulovits, ORGALOGIC GmbH / Systems Research Laboratories, Köln, for the support to finish the habilitation thesis.

Also, I am grateful to *Deutsche Forschungsgemeinschaft* (DFG), who financed my research over four years and paid for several trips to the US, including one year at the University of Delaware. I appreciate the hospitality of this institution, of Prof. Dr. Peter Monk and his family and of Heidi and Lothar Jeromin, Delaware.

Finally, I would like to express special thanks to my wife Regina, who strongly supported and encouraged me over the years.

Roland Potthast, Göttingen June 1999

Contents

As long as a branch of knowledge offers an abundance of problems, it is full of vitality. *David Hilbert*

Before you generalize, formalize, and axiomatize, there must be mathematical substance. *Hermann Weyl*

Our science, in contrast to others, is not founded on a single period of human history, but has accompanied the development of culture through all its stages. Mathematics is as much interwoven with Greek culture as with the most modern problems in engineering. It not only lends a hand to the progressive natural sciences but participates at the same time in the abstract investigations of logicians and philosophers. *Felix Klein*

1 Introduction.

Scattering of acoustic or electromagnetic waves plays an important role in many fields of applied sciences. Acoustic and electromagnetic waves are used and investigated in such different areas as medical imaging, ultrasound tomography, material science, nondestructive testing, radar, remote sensing, aeronautics and seismic exploration.

In the last twenty years the development of computational power has had a strong impact also on the classical fields of direct and inverse scattering. The computational simulation of scattering processes has become accessible using microcomputers and the field of *inverse scattering problems*, which is concerned with the reconstruction of scattering objects or their properties, grew from its early beginnings in the middle of the century to a large and fastly developing area of applied mathematics.

In the first part of this Section we give a brief introduction into inverse scattering theory and outline our main results. In the second part we collect definitions and tools from functional analysis and integral equations, which are the basis for the further sections.

1.1 A review of the main results.

Scattering by obstacles and media. The classical area of acoustic and electromagnetic scattering is concerned mainly with two different problems, which are studied and applied in many different settings and applications.

The first problem is the scattering of time-harmonic acoustic or electromagnetic waves by an *impenetrable* scatterer, i.e. the waves do not significantly penetrate into the interior of the scattering obstacle D. In this case the scattering process is determined by the shape of D and boundary conditions.

The second problem consists in the scattering of time-harmonic acoustic or electromagnetic waves by a *penetrable* scatterer, where the waves penetrate the obstacle and the interior structure of the obstacle strongly influences the scattering process. If the scatterer is homogeneous, the second problem leads to *transmission problems*, however if the scatterer is inhomogeneous, we speak of *scattering by an inhomogeneous medium.*

We will use the letter $\mathcal{D}$ to denote the full scatterer with its physical properties and D to denote the interior of the support of the scatterer in $\mathbb{R}^m$, $m = 2, 3$. We will always assume that the scatterer $\mathcal{D}$ is bounded in $\mathbb{R}^m$.

Mathematically the behavior of a time-harmonic acoustic wave $u(x)e^{-i\omega t}$ in a homogeneous background medium is governed by the *Helmholtz equation*

$$\triangle u + \kappa^2 u = 0, \tag{1.1.1}$$

where $\kappa = \omega / c_0 > 0$ is the *wave number* of the acoustic wave, w its frequency and c_0 the speed of sound. For scattering of an incident wave u^i by an impenetrable scatterer $\mathcal{D}$ a mathematical model also needs to take into account the behavior of the *total field*

$$u = u^i + u^s \tag{1.1.2}$$

at the boundary ∂D of the scatterer. Here u^s denotes the scattered acoustic field. Different boundary conditions are used to model the underlying physical behavior. For a *sound-soft* scatterer the total field vanishes at the boundary, which leads to the *Dirichlet boundary condition*

$$u(x) \ = \ 0, \ \ x \in \partial D, \tag{1.1.3}$$

for a *sound-hard* scatterer the *Neumann boundary condition*

$$\frac{\partial u}{\partial \nu}(x) \ = \ 0, \ \ x \in \partial D, \tag{1.1.4}$$

is used, where ν denotes the exterior unit normal vector to the boundary ∂D. The model is completed by the *Sommerfeld radiation condition*

$$\lim_{r \to \infty} r^{\frac{m-1}{2}} \left(\frac{\partial u^s(x)}{\partial r} - i\kappa u^s(x) \right) \ = \ 0, \ \ r = |x|, \tag{1.1.5}$$

uniformly for all directions $x/|x|$ for the scattered field u^s. It physically implies that energy is transported to infinity and it is an important ingredient to obtain the physical solution of the scattering problem.

To model scattering by inhomogeneous media the equations (1.1.1), (1.1.3) or (1.1.1), (1.1.4) are replaced by

$$\triangle u + \kappa^2 n(x) u = 0 \quad \text{in} \tag{1.1.6}$$

in the whole space or $I\!R^m \setminus \partial D$. Here

$$n(x) := \frac{c_0^2}{c(x)^2} + i\sigma(x)$$

is the *refractive index*, emerging from the sound speed c_0 in the homogeneous host medium and $c(x)$ in the inhomogeneous medium and a term $\sigma(x)$ to model absorption.

For scattering of *electromagnetic* waves in $\mathbb{R}^3$ the corresponding governing equations are the time-harmonic *Maxwell equations*

$$\operatorname{curl} E - i\kappa H = 0, \qquad \operatorname{curl} H + i\kappa E = 0 \tag{1.1.7}$$

for the electric field E and the magnetic field H in a homogeneous medium, where $\kappa := \omega\sqrt{\epsilon_0\mu_0}$ is the *wave number* and ω the frequency of the time-harmonic wave, ϵ_0 the electric permittivity and μ_0 the magnetic permeability of the host medium. For scattering of an incident electromagnetic field E^i, H^i by a *perfect conductor* $\mathcal{D}$ the boundary condition

$$\nu(x) \times E(x) \;=\; 0, \;\; x \in \partial D, \tag{1.1.8}$$

for the *total field*

$$E := E^i + E^s \tag{1.1.9}$$

models the behavior of the electric field at the boundary ∂D of D. The tangential components of the electric field E vanish at the boundary ∂D of the perfect conductor $\mathcal{D}$. The appropriate radiation condition is the *Silver-Müller radiation condition*

$$\lim_{r\to\infty}(H^s \times x - rE^s) \;=\; 0 \tag{1.1.10}$$

for the scattered electromagnetic field E^s, H^s.

To describe scattering of electromagnetic waves by an inhomogeneous medium the equations (1.1.7) and (1.1.8) are replaced by

$$\operatorname{curl} E - i\kappa H = 0, \qquad \operatorname{curl} H + i\kappa n(x) E = 0, \tag{1.1.11}$$

where the *refractive index*

$$n(x) := \frac{1}{\epsilon_0}\left(\epsilon(x) + i\frac{\sigma(x)}{w}\right)$$

is defined using the permittivity $\epsilon(x)$ of the the inhomogeneous medium, the permittivity ϵ_0 of the homogeneous background medium, the conductivity $\sigma(x)$ and the frequency w of the wave. The magnetic permeability μ is assumed to be constant.

The radiation conditions (1.1.5) or (1.1.10) together with the governing equations (1.1.1) or (1.1.7) imply the behavior

$$u^s(x) \;=\; \frac{e^{i\kappa r}}{r^{\frac{m-1}{2}}}\left\{u^\infty(\hat{x}) \;+\; O\left(\frac{1}{r}\right)\right\}, \;\; r = |x| \to \infty, \tag{1.1.12}$$

where $\hat{x} := x/|x|$ and

$$E^s(x) = \frac{e^{i\kappa r}}{r} \left\{ E^\infty(\hat{x}) + O\left(\frac{1}{r}\right) \right\}, \quad r = |x| \to \infty, \tag{1.1.13}$$

of the scattered acoustic or electric field $u^s(x)$ or $E^s(x)$, respectively. Here u^∞ and E^∞ are known as the acoustic and electromagnetic *far field pattern* or *scattering amplitude*. In the acoustic case the far field pattern is a scalar function defined on the unit sphere Ω, in the electromagnetic case the far field pattern is a tangential vector field on Ω.

Direct and inverse scattering problems. For a *direct* scattering problem the scatterer and the incident field is assumed to be given. The problem is to compute the scattered field or the far field pattern, respectively. Direct scattering problems have been studied for a long time and a number of different approaches for their solution have been developed (see [1], [5], [27], [43], [44], [48], [53], [64], [67], [75]). However, it is an important problem of current research to develop efficient algorithms for the numerical computation of the scattered field, especially in three dimensions.

There is a whole range of different *inverse* problems, which are of interest in this framework. Given the far field pattern for scattering of plane waves we may try to reconstruct special properties of $\mathcal{D}$ or the full scatterer with all its properties. Different settings for the measurements lead to a variety of practically relevant mathematical and algorithmical problems.

In this work we will focus on the reconstruction of the *shape* D of the scatterer $\mathcal{D}$, i.e. the scattering domain for obstacle scattering and the support of the inhomogeneity for scattering by an inhomogeneous medium. For a wide range of applications it is not necessary to reconstruct the full behavior and actual values of the refractive index $n(x)$, but it is sufficient to approximately determine the support $\overline{D} := supp(n)$. For example in nondestructive testing often this information is all that is needed.

As scattering data we use the far field pattern $u^\infty(\cdot, d)$ or $E^\infty(\cdot, d, p)$ for scattering of incident plane waves

$$u^i(x, d) := e^{i\kappa x \cdot d}, \quad x \in I\!R^m, \tag{1.1.14}$$

in the acoustic case or

$$\begin{array}{rcl} E^i(x) & = & i\kappa(d \times p) \times d\, e^{i\kappa \cdot d} \\ H^i(x) & = & i\kappa d \times p\, e^{i\kappa x \cdot d} \end{array} \quad , x \in I\!R^3, \tag{1.1.15}$$

in the electromagnetic case, respectively, where $d \in \Omega$ denotes the direction of incidence and p the polarization of the electromagnetic wave. We will investigate the cases where the far field pattern is given for *one*, *a finite number* or *all* incident plane waves either on the whole *unit sphere* Ω or for a given *finite number* n of observation points $\hat{x}_j \in \Omega$, $j = 1, ..., n$.

Basic mathematical problems. The investigation of an inverse problem consists of several basic mathematical questions, which are strongly related to the inverse nature of the task.

First, it has to be asked which data sets uniquely determine the object, i.e. we have to answer the problem of *uniqueness*.

Second, one will ask whether there exists a solution of the inverse problem for a given data set. This is the problem of *existence*.

Third, usually there is an error in the measurements or in the numerical storage of data and thus we have to ask whether we have *stability* for the reconstruction of D from the given far field data. Mathematically this is the question of *continuity* of the nonlinear inverse operator under appropriate assumptions.

Fourth, there is the need to develop efficient and stable reconstruction algorithms. This leads us to the *numerical* and *algorithmical analysis*.

We will now place our results in a historical context and give a brief introduction to our contributions to the above questions. This includes a sketch of related methods and a description of differences and similarities.

The ill-posedness of inverse scattering problems. First we introduce one of the main features of inverse scattering problems. Let B denote a ball with fixed radius R_e around the origin. We assume the *a-priori information* that B contains the scatterer $\mathcal{D}$ in its interior. By

$$\mathcal{S} : u^s|_{\partial B} \mapsto u^\infty \tag{1.1.16}$$

we denote the operator, which maps the scattered field $u^s(x)$, $x \in \partial B$, onto its far field pattern $u^\infty(\hat{x})$, $\hat{x} \in \Omega$. Computing $\mathcal{S}$ explicitly, it can be seen to be *compact* in any reasonable function space, for example from $C(\partial B)$ into $L^2(\Omega)$ (see Section 2 for further details). Thus by functional analytic arguments the range of the operator $\mathcal{S}$ cannot be the whole space and the inverse $\mathcal{S}^{-1}$ of the operator $\mathcal{S}$ cannot be bounded. This indicates that the inverse scattering problem is an *ill-posed* problem in the sense of Hadamard [15], i.e. that the demands of *existence and stability* are violated. The uniqueness problem, i.e. does u^∞ determine D, may also be violated.

The problem of existence for given measured data. First, consider the problem of *existence*. We already pointed out that in general we do not have existence. The most we can expect is the denseness or completeness of the far field patterns for a given set of incident waves or of scatterers $\mathcal{D}$. *Completeness* of the set of far field patterns for the above acoustic and electromagnetic scattering problems for a set of incident plane waves $u^i(\cdot, d_n)$, $n \in \mathbb{N}$, where $\{d_n, n \in \mathbb{N}\}$ is dense in the unit sphere Ω, has been investigated in detail by Colton, Kirsch, Kress, Blöhbaum and Päivärinta between 1984 and 1990 (see [6] for further references). Necessary conditions for a function in $L^2(\Omega)$ to be a far field pattern can be given in terms of the decay of the Fourier coefficients with respect to spherical harmonics (see [6], Theorem 2.16). This can be obtained by an expansion of the scattered field outside of the ball B. Further necessary conditions have been given by Müller [49] (see also Colton and Kress [5]) using entire functions of exponential type. More recently Kirsch [32], [33] obtained a characterization of the set of far field patterns for a given scatterer in terms of its series representation with respect to the eigenfunctions of the corresponding far field operator

$$(F\varphi)(\hat{x}) := \int_\Omega u^\infty(\hat{x}, d)\ \varphi(d) ds(d), \quad \hat{x} \in \Omega. \tag{1.1.17}$$

A corresponding method for the reconstruction of the support D of the scatterer $\mathcal{D}$ is described below. To the author's knowledge no general characterization of the set of far field patterns for arbitrary scatterers $\mathcal{D}$ is known.

In this work we will not pursue further the problem of existence of a solution to the inverse problem or the characterization of the set of far field patterns. We will assume that the given data are either the exact far field data u^∞ for scattering by a scatterer $\mathcal{D}$ or some measured data

$$u_\delta^\infty \in L^2(\Omega \times \Omega)$$

with

$$||u^\infty(\cdot,\cdot) - u_\delta^\infty(\cdot,\cdot)||_{L^2(\Omega\times\Omega)} \leq \delta. \tag{1.1.18}$$

We will also study the *finite data case*, where a finite number of measured data $u^\infty_{(n_o,n_i),\delta} \in L^2(\Omega_{n_o} \times \Omega_{n_i})$ are given with

$$\left(\frac{c_m}{n_o}\frac{c_m}{n_i} \sum_{\hat{x}\in\Omega_{n_o}} \sum_{d\in\Omega_{n_i}} \left| u^\infty(\hat{x}, d) - u^\infty_{(n_o,n_i),\delta}(\hat{x}, d) \right|^2 \right)^{\frac{1}{2}} \ \leq \ \delta \tag{1.1.19}$$

for $\delta \geq 0$ with some constant c_m. Here we assume $(\Omega_n)_{n\in\mathbb{N}}$ to be a sequence of finite subsets Ω_n of Ω, such that Ω_n consists of n elements and for given ϵ we can find n such that the distance

$$d(\hat{x}, \Omega_n) := \inf_{d\in\Omega_n} |\hat{x} - d|$$

is smaller than ϵ for all $\hat{x} \in \Omega$. The left-hand side of (1.1.19) defines a norm $||\cdot||_{L^2(\Omega_{n_o}\times\Omega_{n_i})}$. The positive real number δ is refered to as the *data error*.

Uniqueness results for reconstructions. The origin of *uniqueness results* for inverse obstacle scattering problems can be found in the works of Rellich in the 40's. He proved that the far field pattern uniquely determines the (analytic) scattered field in the exterior of the scatterer $\mathcal{D}$ (which we refer to as *Rellich's Lemma*). Then Schiffer (see [43]) showed for the inverse acoustic obstacle scattering problem with Dirichlet boundary condition that the far field pattern $u^\infty(\hat{x}, d)$, $\hat{x}, d \in \Omega$, for all incident plane waves and for one fixed wave number κ uniquely determines the domain of the scatterer. The corresponding result for the reconstruction of the acoustic refractive index in three dimensions was obtained by Nachman [52], Novikov [54] and Ramm [65] and considerably simplified by Hähner [16] (see also [17]) in 1996. Analogous results for the electromagnetic problems were first obtained by Colton and Päivärinta [8] in 1990, Colton and Kress [6] in 1993 and by Ola, Päivärinta and Somersalo [55] in 1993.

In 1983 Colton and Sleeman [11] investigated the case where the sound-soft scatterer is known to be a subset of a ball with given radius R_e. They showed that the support is determined by a finite number N of incident plane waves depending on R_e. If R_e is small enough, *one* wave is sufficient to determine the scatterer.

So far it has not been possible to extend Schiffers approach or the ideas of Colton and Sleeman to the sound-hard scatterer or to the case of an inhomogeneous medium. In 1992 Isakov [23] obtained uniqueness results for penetrable obstacles using different techniques, which were simplified and applied to impenetrable sound-soft and sound-hard scatterers by Kirsch and Kress [29] in 1993. The results could also be successfully transfered to the case of electromagnetic obstacle scattering [6]. Since these ideas will be the starting point of a large part of this work (with contributions to uniqueness, stability, the finite data problem and reconstruction algorithms), we briefly want to describe the main ingredients.

Consider the scattered acoustic field for a point-source $\Phi(\cdot, z)$ with source point $z \in I\!R^m \setminus \overline{D}$, where Φ is the fundamental solution of the Helmholtz equation in two or three dimensions, respectively. From the sound-soft boundary condition and the singularity of the incident point-source we derive that for a point $x \in \partial D$ we have

$$\Phi^s(x, z) \to \infty, \quad z \to x. \tag{1.1.20}$$

Kirsch and Kress used (1.1.20) to show that, if the far field patterns of two scatterers $\mathcal{D}_1$ and $\mathcal{D}_2$ for scattering of plane waves coincide for all directions of incidence $d \in \Omega$, then the domains D_1 and D_2 are the same.

In Section 3 we will further develop the techniques of Kirsch and Kress to derive uniqueness of the support D of an inhomogeneous medium n, if n has a jump in one of its derivatives at the boundary of the scatterer $\mathcal{D}$. For the three-dimensional case this also can be obtained from the results of Nachman [52], Novikov [54] and Ramm [65] for the acoustic and from Colton and Päivairinta [9] in the electromagnetic case. In two dimensions Sun and Uhlmann [71] proved uniqueness of the support of n, if n has a jump at the boundary. They use Fourier techniques, which are different from our approach.

ϵ-Uniqueness for finite data. Uniqueness results for inverse scattering problems usually assume the *full far field pattern* on the unit sphere to be given. Often it is assumed that the far field pattern is known for *all* or a *full open set* of directions $d \in \Omega$ of the incident plane waves. Since most proofs use Rellich's Lemma, the knowledge of the far field pattern at least in an open subset of Ω seems to be necessary to uniquely determines the scattered field u^s.

From a practical perspective it is reasonable to ask the question what can be said if the far field pattern is given only at a *finite* number of measurement points and for a finite number of waves. In Section 3, we develop a technique to answer this question and thus avoid the use of Rellich's uniqueness results. It leads to a relaxed concept of uniqueness, a preliminary version of which was first proposed in 1998 (see [59]). We will prove *ϵ-uniqueness* for the reconstruction of the shape of a scatterer, i.e. given $\epsilon > 0$ there are $n_o, n_i \in I\!N$ such that, if for two scatterers $\mathcal{D}_1$ and $\mathcal{D}_2$ the far field patterns for all n_i directions of incidence $d \in \Omega_{n_i}$ coincide at the n_o observations points $\hat{x} \in \Omega_{n_o}$, the Hausdorff distance $d(D_1, D_2)$ of the scatterers D_1 and D_2 satisfies the estimate

$$d(D_1, D_2) < \epsilon.$$

Since the concept of ϵ-uniqueness is close to stability, let us postpone further discussion and first discuss the stability question.

Stability estimates. We have already pointed out that inverse scattering problems are *ill-posed* problems, i.e. that for the inverse of the nonlinear scattering operator in general we do not have stability. There exist two main approaches to restore stability results for this ill-posed problem.

The first approach consists in a modification of the norm used for the far field pattern. In inverse scattering it was used for example in 1990 by Stefanov [70] to study stability for inverse scattering by a medium and has been extended by Hähner [17] in 1998 to electromagnetic and elastic scattering. In principle they consider a space $\mathcal{F}$ of functions on Ω with a very strong norm $||\cdot||_{\mathcal{F}}$ involving all derivatives of functions $\varphi \in C^\infty(\Omega)$, such that the inverse $\mathcal{S}^{-1} : \mathcal{F} \to C(\partial B)$ of

(1.1.16) becomes a bounded operator. But since for real data derivatives of the far field pattern are not available, from a practical point of view this approach only shifts the ill-posedness to the mapping of the data space into $\mathcal{F}$. We will not further pursue the idea.

Another approach is the use of a-priori bounds on the set of fields or objects to be determined (see for example [26]). This approach has been applied to scattering theory by Isakov [23], [24] (see also [25]). We will use the well known fact that the inverse of a continuous mapping is continuous if it is defined on a *compact* subset of a Banach space. Thus with appropriate restrictions on the set of scatterers stability can be restored and stability estimates can be obtained.

Isakov's restrictions mainly consist in a uniform bound on the $C^{2,\alpha}$-norm of all boundaries in a special parametrization. For the reconstruction of the shape of a sound-soft scatterer from the knowledge of the far field pattern for *one* incident wave Isakov derives a double-logarithmic estimate

$$d(D_1, D_2) \ < \ C\Big(\ln\Big|\ln ||u_1^\infty(\cdot) - u_2^\infty(\cdot)||_{C(\Omega)}\Big|\Big)^{-\gamma} \tag{1.1.21}$$

for the Hausdorff distance $d(D_1, D_2)$ of the domains D_1, D_2 with positive constants C, γ depending on a bound for the $C^{2,\alpha}$-norm of the boundary. So far Isakov's techniques could not be used to treat other boundary conditions or electromagnetic scattering problems.

In Section 3 we will pursue the idea of imposing appropriate restrictions on the set of scatterers under consideration. With techniques different from Isakov we will be able to derive stability estimates for the reconstruction of the shape of either a penetrable or impenetrable scatterer from the knowledge of the far field patterns for all incident plane waves. More explicitly we prove an estimate of the form

$$d(D_1, D_2) \ \leq \ F\Big(||u_1^\infty(\cdot,\cdot) - u_2^\infty(\cdot,\cdot)||_{L^2(\Omega)\times L^2(\Omega)}\Big), \tag{1.1.22}$$

where F is a function with the property

$$F(\delta) \to 0, \quad \delta \to 0, \tag{1.1.23}$$

which can be computed according to some a-priori knowledge on the class of scatterers. For the convex hulls $\mathcal{H}(D)$ of the shape D of scatterers $\mathcal{D}$ we derive a logarithmic estimate

$$d\Big(\mathcal{H}(D_1), \mathcal{H}(D_2)\Big) \ \leq \ C\ \Big|\ln ||u_1^\infty(\cdot,\cdot) - u_2^\infty(\cdot,\cdot)||_{L^2(\Omega)\times L^2(\Omega)}\Big|^{-\gamma} \tag{1.1.24}$$

with constants C and γ.

Our technique is inspired by the uniqueness proof of Isakov, Kirsch and Kress. The first step is the explicit estimation of the behavior of $\Phi^s(z,z)$ for $z \to \partial D$. As a second step we develop a method for the approximate reconstruction of $\Phi^s(z,z)$ in the exterior of the domain

$$D_\rho := \{y \in I\!R^m, d(y, D) < \rho\} \tag{1.1.25}$$

with some small parameter $\rho > 0$ from the scattering data $u^\infty(\hat{x}, d)$ for all $\hat{x}, d$ in Ω . Estimating the bound of the approximate reconstruction operator

$$Q : L^2(\Omega) \times L^2(\Omega) \to C(B \setminus D_\rho)$$

and using the fact that $\Phi^s(z,z)$ is large only in a neighborhood of the boundary, stability estimates will be obtained in Section 3. For the acoustic sound-soft and sound-hard scatterer the results can be found in [60]. Similar stability results for the reconstruction of the support of media and for electromagnetic scattering problems will be derived in Section 3.

At this point we would like to relate our results to some demands on the degree of ill-posedness of an inverse problem formulated by Fritz John [26] in 1960. For purposes of computation John demands *Hölder continuous dependence* of a problem on the data. Here for the reconstruction of the domains we obtained *logarithmic continuity*, which is a typical type of estimate for continuing solutions of the wave equation in space-like directions.

As shown in [60], for the reconstruction of the scattered field u^s on fixed compact subsets U of the open exterior of the convex hull $\mathcal{H}(D)$ of D

$$\left|\left|u_1^\infty - u_2^\infty\right|\right|_{L^2(\Omega)} \leq \delta$$

yields the estimate

$$\left|u_1^s(x) - u_2^s(x)\right| \leq \alpha \delta^{\frac{\beta}{|\ln(-\gamma \ln(\delta))|}}, \quad x \in U,$$

with constants $\alpha, \beta, \gamma > 0$. This can be proven with the same techniques which we will use in Section 3. These estimates come close to Hölder continuity demanded by John and are reflected by the numerical results of Sections 5 and 6.

Stability for the case of finite data: ϵ-stability. With the help of the stability estimates it is not difficult to derive related statements for the case of finite data.

We will work with the same assumptions as for stability or ϵ-uniqueness, i.e. a uniform bound on the C^2-norm of the boundaries (and corresponding assumptions on the uniform smoothness of the refractive index n), to derive a uniform bound for the far field patterns in $C^1(\Omega \times \Omega)$. This bound can be used to relate the distance of two far field patterns at a finite number of points to the distance of the full far field patterns. Then for the case of finitely many measurements we derive ϵ-uniqueness and a modified stability statement, which we will refer to as *ϵ-stability.*

Consider a simple example for the derivation of ϵ-uniqueness. Given $\epsilon > 0$ we can use (1.1.24) to obtain a $\delta > 0$ such that

$$||u_1^\infty(\cdot,\cdot) - u_2^\infty(\cdot,\cdot)||_{L^2(\Omega)\times L^2(\Omega)} \leq \delta \tag{1.1.26}$$

implies

$$d\Big(\mathcal{H}(D_1), \mathcal{H}(D_2)\Big) \ \leq \epsilon. \tag{1.1.27}$$

Now given δ, we choose $n \in I\!N$ sufficiently large such that with the help of the bound on $||u_j^\infty(\cdot,\cdot)||_{C^1(\Omega\times\Omega)}$ for $j = 1, 2$, the equation

$$u_1^\infty(\hat{x}, d) = u^\infty(\hat{x}, d) \text{ for all } \hat{x}, d \in \Omega_n \tag{1.1.28}$$

yields (1.1.26). Thus given ϵ we may choose $n \in I\!N$ such that (1.1.28) implies (1.1.27), i.e. we have proven the statement of ϵ-uniqueness for the convex hulls of the scattering domains as a simple consequence of stability.

Now, we describe the concept of *ϵ-stability.* Since in general we do not have uniqueness, for a finite data set we will not be able obtain stability. More explicitly, we can not obtain a function $F(\delta)$ which satisfies (1.1.23) and an estimate of the type (1.1.22) when δ is the data error at finitely many points. But for given $\epsilon > 0$ it is possible to find $n_o, n_i \in I\!N$ and a function $F_{(n_o,n_i)}(\delta)$, such that F has the behavior

$$\limsup_{\delta\to 0} F_{(n_o,n_i)}(\delta) \ \leq \ \epsilon \tag{1.1.29}$$

and the domains D_1 and D_2 satisfy the estimate

$$d(D_1, D_2) \ \leq \ F_{(n_o,n_i)}\Big(||u^\infty_{1,(n_o,n_i)} - u^\infty_{2,(n_o,n_i)}||_{L^2(\Omega_{n_o}\times\Omega_{n_i})}\Big) \tag{1.1.30}$$

with $||\cdot||_{L^2(\Omega_{n_o}\times\Omega_{n_i})}$ given by (1.1.19) and

$$u^\infty_{j,(n_0,n_i)} := u_j^\infty\Big|_{\Omega_{n_o}\times\Omega_{n_i}} \ \in L^2(\Omega_{n_o} \times \Omega_{n_i})$$

for $j = 1, 2$. We call a statement of this form *ϵ-stability.* We will prove ϵ-stability for the reconstruction of the shape of the scatterer for the scattering problems described above and explicitly study the behavior of $F_{(n_o,n_i)}(\delta)$ for the reconstruction of the convex hull of a scatterer.

Three main categories of reconstruction methods. Consider now the problem of the actual reconstruction of the shape of the unknown scatterer. Mainly three different types of reconstruction methods have been developed. We will summarize their main features and use them as a background to explain our results.

The first category. There is *first* the approach to consider the inverse problem as a *nonlinear ill-posed operator equation* and adapt *iterative methods of gradient- or Newton-type* to solve this equation. For inverse obstacle scattering this approach mainly relies on the *Fréchet differentiability* or *domain derivative* of the scattered field with respect to variations of the boundary of the scatterer and a *characterization of the derivative* as a corresponding boundary value proble. Results for interior boundary value problems were obtained in 1980 by Simon [69] with the help of the implicit function theorem, for the scattering problems in 1993 by Kirsch [30] using variational methods and in 1994 by the author [57] by means of integral equations.

For a discussion of the large number of papers on the numerical implementation of these type of methods (for example Murch, Tan and Wall [51], Roger [66], Tobocman [73], Wang and Chen [74], Kirsch [30], [31], Kress and Rundell [40], [41], [42], Kress [38], [36], [39] Mönch [47], Hohage [21], Hanke, Hettlich und Scherzer [18] and Hettlich [19]) we refer to the second edition of the book of Colton and Kress [6]. The present work will not be concerned with this category of methods.

The second category and the point-source method. The second category in principle splits the inverse scattering problem into the *linear ill-posed part to reconstruct the scattered field* in the exterior of the scatterer and a *nonlinear well-posed part to find the boundary of the scatterer* or the refractive index using the boundary condition or the partial differential equation, respectively.

Typical examples in this category are the methods proposed by Colton and Monk 1985 and by Kirsch and Kress 1986, both described in [6]. In Section 5 with the *point-source method* we describe a method of this second category, developed by the author since 1995. Different steps in this development can be found in [61], [62] and [63].

The main aim of the point-source method is the explicit construction of a kernel $g(z,d)$, such that in the domain $B \setminus D_\rho$, where D_ρ is given by (1.1.25) and B by (1.1.16), the scattered field u^s is approximated in the form Au^∞ with a *linear integral operator*

$$(A\varphi)(z) := \int_{\Omega} g(z,\hat{x})\varphi(\hat{x})\, ds(\hat{x}), \quad z \in B. \tag{1.1.31}$$

To this end the far field pattern $\Phi^{\infty}(\cdot, z)$ for incident point-sources $\Phi(\cdot, z)$ with source-point z is considered. Given some a-priori knowledge on the size of the scatterer, the kernel g will be constructed in the following three steps.

1. An approximation for a *point-source* by a *superposition of plane waves*

   $$\Phi(x,z) \;\approx\; \int_{\Omega} e^{i\kappa x\cdot d}\tilde{g}(z,d)\, ds(d), \quad x \in \overline{D}, \; z \in B \setminus D_{\rho}. \tag{1.1.32}$$

   is computed.

2. Passing to the far field patterns, an approximation for the *far field pattern due to point-sources* by a superposition of the far field patterns of plane waves

   $$\Phi^{\infty}(\hat{x},z) \;\approx\; \int_{\Omega} u^{\infty}(\hat{x},d)\tilde{g}(z,d)\, ds(d), \quad \hat{x} \in \Omega, \; z \in B \setminus D_{\rho} \tag{1.1.33}$$

   is obtained.

3. Using the *mixed reciprocity relation* $\Phi^{\infty}(\hat{x},z) = \gamma u^{s}(z,-\hat{x})$, the *far field reciprocity relation* $u^{\infty}(\hat{x},d) = u^{\infty}(-d,-\hat{x})$ and the substitution $d \to -d$, an approximation

   $$u^{s}(z,-\hat{x}) \;\approx\; \int_{\Omega} u^{\infty}(d,-\hat{x}) \left\{ \frac{1}{\gamma}\tilde{g}(z,-d) \right\} ds(d), \quad \hat{x} \in \Omega, \; z \in B \setminus D_{\rho}. \tag{1.1.34}$$

   for the scattered field u^{s} from its far field pattern u^{∞} is derived.

With the help of rotations and translations of the approximating functions, the computation of g can be performed efficiently. Given the reconstruction of the scattered field u^{s}, parts of the unknown boundary ∂D of D can be found using the total field $u = u^{i} + u^{s}$ and the boundary condition. For electromagnetic waves a corresponding operator will be constructed (see also [63]).

To get a better insight into the method, we briefly compare some features of the point-source method with the method of Kirsch and Kress as described in [6], Section 5.4. This method approximates the scattered field u^{s} by a single-layer potential $S\varphi$ on a curve or surface Γ, which has to be located in the interior of the unknown scatterer D. Then the boundary ∂D of the unknown scatterer is found by first minimizing the functional $||S^\infty\varphi_\delta - u_\delta^\infty||^2_{L^2(\Omega)}$, i.e. fitting the far field pattern $S^\infty\varphi$ of the single-layer potential to the given measured data u_δ^∞, and then minimizing $||S\varphi_\delta + u^i||^2_{L^2(\partial D)}$ with respect to ∂D, i.e. fitting the scattered field on the boundary of the scatterer to the incident field To obtain the *convergence*

$$S\varphi_\delta \to u^s, \text{ for } u_\delta^\infty \to u^\infty \text{ and } \delta \to 0,$$

on subsets of $I\!R^m \setminus D$ and

$$\partial D_\delta \to \partial D \text{ for } u_\delta^\infty \to u^\infty \text{ and } \delta \to 0,$$

one has to combine the minimization of these functionals with respect to φ_δ in $L^2(\Gamma)$ and the boundary ∂D of D.

The last point is a central difference of the two methods. For the point-source approach we obtain convergence for the reconstruction of the scattered field u^s without the simultaneous reconstruction of the unknown scatterer.

As a second difference, the point-source method for the reconstruction of ∂D does not need a parametrization of the whole boundary. We are able to reconstruct parts of the boundary independently.

Third, it is possible to reconstruct scatterers which consist of an unknown number of components. However, some restrictions on the location of these components are required due to the exterior cone condition.

Fourth, the reconstruction operator A given by (1.1.31) is computed according to some a-priori knowledge. We do not have to invert a linear system involving the far field data or to minimize a functional with a possibly large number of unknown coefficients.

Note that there are some similarities between the construction procedure of the point-source method and the *Backus-Gilbert method* or *mollifier methods* [35], [12], [45].

The third category and the method of singular sources. Since 1996 methods for the reconstruction of the shape of a scatterer have been developed, which are based on *characterizations of the boundary* of the scatterer independent of its physical properties. For an algorithm of category III the boundary condition or physical properties of the scatterer do not need to be known. The independence of a reconstruction method on the physical properties of the scatterers is of great practical importance, since in many cases a knowledge about the properties of the searched objects does not exist.

A *linear sampling method* has been proposed 1996 by Colton and Kirsch [4], see also Colton and Monk [7] and Colton, Piana and Potthast [10]. The idea is to characterize the boundary ∂D of a scatterer D by the behavior of the solution $g = g(z,\cdot) \in L^2(\Omega)$ of a linear integral equation of the first kind

$$(Fg)(\hat{x}) \;=\; e^{-i\kappa z\cdot\hat{x}}, \;\; \hat{x} \in \Omega, \tag{1.1.35}$$

for $z \in D$, where F is the far field operator

$$(Fg)(\hat{x}) := \int_{\Omega} u^{\infty}(\hat{x}, d)g(d)\; ds(d), \;\; \hat{x} \in \Omega. \tag{1.1.36}$$

Examining either an interior boundary value problem or an *interior transmission problem*, in [10] (see also [6]) it is shown, that there exists an approximate solution of (1.1.35) with

$$||g(z,\cdot)||_{L^2(\Omega)} \to \infty \quad \text{and } ||Hg(z,\cdot)||_{L^2(D)} \to \infty \quad \text{for} \quad z \to \partial D, \tag{1.1.37}$$

where Hg denotes the *Herglotz wave function*

$$(Hg)(x) := \int_{\Omega} e^{i\kappa x\cdot d} g(d)\; ds(d), \;\; x \in I\!R^m.$$

Colton and Kirsch propose to compute a regularized approximate solution of (1.1.35) on a grid containing D. The domain D then can be found as the set of points where $||g(z,\cdot)||_{L^2(\Omega)}$ is large. Numerical experiments can be found in [4], [7], [10] and [2].

Colton, Piana and Potthast [10] also applied *Morozov's discrepancy principle* for the solution of (1.1.35) and used either $||g(z,\cdot)||$ or the values of the regularization parameter $\alpha(z)$ to determine the shape D of the scatterer $\mathcal{D}$. Numerical results of this method can also be found in [2].

As mentioned above, more recently Kirsch [32], [33] was able to derive a characterization of the shape D of the scatterer $\mathcal{D}$ for acoustic scattering in the case of obstacles or non-absorbing media, i.e. a real-valued refractive index n. Kirsch showed that the domain D is the set of points z, where the equation

$$(F^*F)^{1/4}g(\hat{x}) = e^{-i\kappa z\cdot\hat{x}}, \;\; \hat{x} \in \Omega, \tag{1.1.38}$$

is solvable. This characterization holds both for scattering by obstacles [32] or for a non-absorbing medium [33]. The support of the scatterer can then be found as above, computing approximate regularized solutions of (1.1.38) and using either the norm $||g(z,\cdot)||$ of the solution or the size of the regularization parameter $\alpha(z)$, which is chosen according to Morozov's discrepancy principle. A comparison of numerical results for the method of Colton and Kirsch (1.1.35) and the version proposed by Kirsch (1.1.38) can be found in [2]. We will not further investigate the method of Colton and Kirsch, but develop a different method in this third category of reconstruction methods.

In Section 6, we propose a *method of singular sources* for the reconstruction of the support of obstacles or scattering media from the knowledge of the far field pattern $u^\infty(\hat{x}, d)$, $\hat{x}, d \in \Omega$. It is based on the ideas used in the uniqueness and stability proofs. For the reconstruction of obstacles in the case of acoustic waves theoretical and numerical results can be found in [60].

We want to sketch the main ideas of this method. As described above, the boundary ∂D of an obstacle D is the set of points where the scattered field $\Phi^s(z, z)$ for incident point-sources becomes singular. We construct a kernel $g_\tau(x, d) g_\eta(z, \tilde{d})$, such that for $x, z \in B \setminus D_\rho$ an approximation for the field $\Phi^s(x, z)$ is obtained in the form $(Qu^\infty)(x, z)$ with the *bounded linear integral operator* Q defined by

$$(Q\varphi)(x, z) := \int_\Omega \int_\Omega g_\tau(x, d) g_\eta(z, \tilde{d}) \varphi(d, \tilde{d}) \; ds(d) ds(\tilde{d}), \quad x, z \in B. \tag{1.1.39}$$

The boundary ∂D is found as the set of points, where $(Qu^\infty)(z, z)$ is large. We investigate the method both for the reconstruction of obstacles and the support of media and give numerical examples.

The methods of the second and the third category have important differences on a fundamental level. Methods of the second category use the scattered field and the boundary condition to determine the scatterer. The boundary condition does not need to be known for the reconstructions with methods of the third category. The missing knowledge leads to a different behavior of the algorithm and (as we show and discuss in Section 6) influences the ill-posedness of the inverse scattering problem.

Contents. We split our presentation into six Sections. The Section 1.1 of this Section has already been used to introduce the main ideas and results. Section 1.2 serves to present basic definitions and tools for further use.

In *Section 2* we study the solutions to the direct scattering problems and derive properties, on which our investigation of the inverse problems will be based. The main results of Section 2 will be uniform bounds for integral operators and scattering maps and estimates for the behavior of the scattered field $\Phi^s_{\mu,q}(z, z)$ for incident multipoles of order μ and polarization $q \in \Omega$.

The themes of *Section 3* are uniqueness and stability for the reconstruction of the shape of a scatterer. We first derive uniqueness results from the knowledge of the full far field patterns $u^\infty(\hat{x}, d)$, $\hat{x}, d \in \Omega$. In a second part we develop a technique to derive stability estimates for the reconstruction of the shape of both impenetrable and penetrable scatterers from the knowledge of the far field patterns for all incident plane waves.

The *finite data case* is investigated in *Section 4*. We investigate the question of uniqueness if the far field patterns are known only for a finite number of observation points and a finite number of incident plane waves. For this situation we propose a concept which we call ϵ-*uniqueness*: given ϵ there are numbers n_o, n_i such that the far field patterns for n_i directions of incidence measured at n_o observations directions determines the unknown shape up to an error ϵ in the Hausdorff distance of the domains.

In a second part stability for a finite set of measured data is studied. For this case we propose a concept of ϵ-*stability*: given ϵ there are numbers n_o, n_i and a function $F_{(n_i,n_o)}$, such that (1.1.29) and (1.1.30) are satisfied.

In *Section 5* a *point-source method* is introduced for the reconstruction of a scattered field u^s from its far field pattern u^∞ and the construction of the shape of an unknown impenetrable scatterer $\mathcal{D}$. We explicitly construct a family of bounded linear integral operators (1.1.31) for the reconstruction of u^s and prove error estimates and convergence to the true scattered field. Numerical examples for reconstructions in three dimensions are given.

A method for the reconstruction of the shape of both impenetrable and penetrable scatterers is proposed in *Section 6*. We call it the *method of singular sources*, since it uses the singular behavior of the scattered fields $\Phi^s_{\mu,q}(z,z)$ of multipoles, if the source point z of the incident multipole $\Phi_{\mu,q}(\cdot,z)$ tends to the boundary of the scatterer. Some numerical examples in two dimensions demonstrate the applicability of the ideas.

1.2 Basic definitions and tools.

We now introduce some basic definitions, notations and theorems from analysis, functional analysis, scattering and potential theory for later use. An introduction into these areas can be found in [37], [20] and [6].

By D we denote a bounded open set in $I\!R^m$, $m \in I\!N$, with boundary ∂D and closure $\overline{D}$, such that the exterior of D in $I\!R^m$ is connected. $B_r(x)$ is the open ball with radius r and center x in $I\!R^m$. The lower half plane is given by

$$H := \{x = (x_1, ..., x_m), x_m \leq 0\}$$

and we define $H_r := B_r(0) \cap H$. Let $Z_{a,r}$ be the open finite cylinder

$$Z_{a,r} := \Big\{x = (x_1, ..., x_m) \in I\!R^m, \sqrt{x_1^2 + ... + x_{m-1}^2} < r,\ |x_m| < a\Big\}. \tag{1.2.1}$$

We use the notation $Z_{a,r}(x,p)$ for the cylinder defined by (1.2.1) in the coordinate system with origin x and the x_m-axis given by $p \in I\!R^m$.

The spaces of continuous or l-times continuously differentiable functions on D or ∂D are denoted by $C(D)$, $C^l(D)$ or $C(\partial D)$, $C^l(\partial D)$, respectively. The space of *l-times Hölder continuously differentiable functions* with *Hölder coefficient α* is $C^{l,\alpha}(D)$ or $C^{l,\alpha}(\partial D)$, respectively.

For a *multi-index*

$$\gamma := (\gamma_1, ..., \gamma_m)$$

we define

$$|\gamma| := \sum_{j=1}^{m} \gamma_j. \tag{1.2.2}$$

The l-th derivatives of a function $f \in C^l(D)$ are given by

$$f^{(\gamma)} := \frac{\partial^{|\gamma|} f}{\partial^{\gamma_1} x_1 ... \partial^{\gamma_m} x_m}$$

for all $\gamma \in I\!N_0^m$ with $|\gamma| = l$. We will need the space $L^2(D)$ of *square-integrable functions* on D and the *Sobolev spaces* $H^l(D)$, which are defined as the closure of $C^l(D)$ with respect to the norm

$$||f||_{H^l(D)} := \sum_{|\gamma| \leq l} ||f^{(\gamma)}||_{L^2(D)}. \tag{1.2.3}$$

Given $m \in I\!N_0$ and $\alpha \in [0,1]$ the boundary ∂D is said to be of *class* $C^{l,\alpha}$, if for each point $x \in \partial D$ there is an open set $V \in I\!R^m$ with $x \in V$ and a bijective mapping $\psi \in C^{l,\alpha}(B_1(0))$ such that $\psi(B_1(0)) = V$ and $\psi(H_1) = V \cap D$. Since ∂D is compact, we can always find a finite number of such domains V which cover ∂D. A parametrization of $V \cap \partial D$ is given by $\psi|_{U_1}$ with

$$U_1 := \{x \in B_1(0), x_m = 0\}. \tag{1.2.4}$$

Then $\{U_1, \psi, V \cap \partial D\}$ is called a set of *local coordinates* for ∂D

For domains of class $C^{l,\alpha}$ with $l \geq 1$ and $\alpha \in [0,1]$ let $\nu(x)$ be the *exterior unit normal vector* to the boundary ∂D of D in the point $x \in \partial D$. We use special local coordinates for ∂D. For a point $x \in \partial D$ we can find a coordinate system K_x with origin x and the x_m-axis given by $\{x + h\nu(x), h \in I\!R\}$. In this special coordinate system in a neighborhood $Z_{r,a}$ of 0 with $r, a > 0$, a parametrization of $\partial D \cap Z_{r,a}$ is given by

$$\partial D \cap Z_{r,a} \;=\; \Big\{\Big(t_1, ..., t_{m-1}, f(t_1, ..., t_{m-1})\Big),\ (t_1, ..., t_{m-1}) \in B_r(0)\Big\} \tag{1.2.5}$$

with a mapping $f \in C^{l,\alpha}(B_r(0))$ defined on the open set $B_r(0) \subset I\!R^{m-1}$. The function $f \in C^{l,\alpha}(B_r(0))$ is uniquely determined and we have

$$D \cap Z_{r,a} = \{(t_1, ..., t_m) \in Z_{r,a},\ t_m \leq f(t_1, ..., t_{m-1})\}. \tag{1.2.6}$$

For later use we need to specify classes of domains, for which the properties of special functions and integral operators are valid uniformly.

DEFINITION 1.2.1 *Given constants $R_e, r_0, a_0,\ l \in I\!N,\ \alpha \in [0,1],\ C_0 > 0$ and $\beta_e > 0$ we define the class*

$$\mathcal{A} = \mathcal{A}(R_e, r_0, a_0, l, \alpha, C_0) \tag{1.2.7}$$

of domains in $I\!R^m$, which satisfy the following conditions:

1. **(Boundedness.)** *For all $D \in \mathcal{A}$ we have*

$$D \subset B_{R_e}(0). \tag{1.2.8}$$

2. **(Smoothness.)** *For each point $x \in \partial D$ there is a special coordinate system K_x defined above, such that for $r = r_0$, $a = a_0$ and with the function $f \in C^{l,\alpha}(B_{r_0}(0))$ the equations (1.2.5) and (1.2.6) and the estimate*

$$||f||_{C^{l,\alpha}(B_r(0))} \;\leq\; C_0 \tag{1.2.9}$$

are satisfied.

3. **(Exterior cone condition.)** *For each $x \in I\!R^m \setminus D$ there is a cone*

$$\mathrm{co}(x, p, \beta_e) := \left\{ y \in I\!R^m, \frac{y-x}{|y-x|} \cdot p \geq \cos(\beta_e) \right\} \tag{1.2.10}$$

with direction $p \in \Omega$, and opening angle β_e in the exterior $I\!R^m \setminus D$ of D.

In contrast to the theory of partial differential equations, where cone conditions are used to describe the regularity of the boundary of a domain, here the infinite cone condition is a *geometrical condition*. It can be seen as a condition to limit non-convexity and, simultaneously, allow scatterers consisting of several separate components. It guaranties that each point x on the boundary ∂D can be reached by an exterior infinite cone $\mathrm{co}(x, p, \beta_e)$ with a direction $p \in \Omega$ depending on x and a fixed given opening angle β_e. We use the notation $B := B_{R_e}(0)$.

To work with the class $\mathcal{A}$ of domains we need to note some of its properties.

THEOREM 1.2.2 *For parameters $R_e, r_0, a_0 > 0$, $l \geq 2$, $\alpha \in [0, 1]$ and $C_0 > 0$ we obtain for the class of domains $\mathcal{A} = \mathcal{A}(R_e, r_0, a_0, l, \alpha, C_0)$ the following properties.*

1. *There is a radius $r_i = r_i(r_0, a_0, C_0)$, such that for each domain $D \in \mathcal{A}$ and each point $x \in \partial D$ we have*

$$B_{r_i}(x + \nu(x) r_i) \subset I\!R^m \setminus \overline{D} \tag{1.2.11}$$

and

$$B_{r_i}(x - \nu(x) r_i) \subset D. \tag{1.2.12}$$

2. *Each domain $D \in \mathcal{A}$ is a union of balls with radius r_i.*

3. *For each domain $D \in \mathcal{A}$ and each point $x \in \partial D$ we have*

$$x + \nu(x) s \notin D, \quad s \in [0, r_i] \tag{1.2.13}$$

and

$$x - \nu(x) s \in D, \quad s \in (0, r_i]. \tag{1.2.14}$$

4. *There is a number $L_1 = L_1(R_e, r_0, a_0, C_0) \in I\!N$, such that for each domain D in $\mathcal{A}$ the boundary ∂D of D is piecewise parametrized by L_1 twice continuously differentiable injective mappings*

$$\psi_j : B_{r_0}(0) \to I\!R^m, \quad j = 1, ..., L_1, \tag{1.2.15}$$

of the form (1.2.5) with a norm (1.2.9) bounded uniformly for all domains $D \in \mathcal{A}$.

5. *There is a number $L_2 = L_2(R_e, r_0, a_0, C_0)$, such that for every scatterer D in $\mathcal{A}$ the domain D is parametrized by L_2 injective mappings*

$$\psi_j : Z_{r_0,a} \to I\!R^m, \;\; j = 1, ..., L_2, \tag{1.2.16}$$

where ψ_j is an element of $C^{l,\alpha}(Z_{r_0,a})$, $j = 1, ..., L_2$ with a norm bounded uniformly for all domains $D \in \mathcal{A}$.

Proof. Given $x \in \partial D$ the property 1 is obtained with the help of the special coordinate system K_x and the bound C_0 on the norm

$$||f||_{C^{2,0}(B_{r_0}(0))}$$

by elementary calculations as follows. We only need to consider the line $x_2 = 0$ in a neighborhood of 0. Here the boundary of $B_{r_i}(0 + \nu(0)r_i)$ is given by

$$g(x_1) := r_i - \sqrt{r_i^2 - x_1^2}, \;\; 0 \le x_1 < r_i.$$

For $r_i < \frac{1}{C_0}$ we estimate the derivative of g by

$$\frac{x_1}{\sqrt{r_i^2 - x_1^2}} \;\; \ge \;\; \frac{x_1}{r_i} \;\; > \;\; C_0 x_1, \;\; 0 \le x_1 < r_i. \tag{1.2.17}$$

From

$$\frac{\partial f(t,0)}{\partial t}(x_1, 0) \;\; = \;\; \int_0^{x_1} \frac{\partial^2 f(t,0)}{\partial t^2} dt$$

for the derivative of f we derive

$$|\frac{\partial f(t,0)}{\partial t}(x_1, 0)| \;\; \le \;\; C_0 x_1, \;\; 0 \le x_1 < r_i. \tag{1.2.18}$$

Thus for the derivatives of g and f from (1.2.17) and (1.2.18) we obtain the estimate

$$\frac{\partial g}{\partial x_1}(x_1) > |\frac{\partial f(x_1,0)}{\partial x_1}(x_1)|, \;\; 0 \le x_1 < r_i,$$

which yields

$$g(x_1) \;\; > \;\; f(x_1, 0), \;\; 0 \le x_1 < r_i,$$

and thus property 1.

The properties 2 and 3 are an immediate consequences of property 1. Property 4 can be obtained by compactness of

$$\overline{B_{R_e}(0)} \subset I\!R^m, \tag{1.2.19}$$

since there is a finite number L_1 of balls $B_{r_0}(x)$, $x \in B_{R_e}(0)$, which cover (1.2.19) and for each ball $B_{r_0}(x)$ the set $\partial D \cap B_{r_0}(x)$ is parametrized by special local coordinates of the form (1.2.5) with norm bounded by (1.2.9).

To prove 5 we first proceed as in 4 and obtain L_1 local coordinate systems K_{x_j} and corresponding local coordinates of the form (1.2.5) covering ∂D. For the local coordinates K_{x_j} we obtain a mapping $\psi_{1j} : Z_{r_0,a} \to Z_{r_0,a} \cap D$ by

$$(t_1, ..., t_m) \mapsto \Big(t_1, ..., t_{m-1}, \frac{t_m + a}{2a}\Big[f(t_1, ..., t_{m-1}) + a\Big] - a\Big).$$

For this mapping from (1.2.9) we derive $||\psi_{1j}||_{C^{l,\alpha}(Z_{r_o,a})} \leq C$ with some constant C uniformly for $D \in \mathcal{A}$. The set

$$G := D \setminus \Big(\bigcup_{j=1}^{L_1} \psi_{1j}(Z_{r_0,a})\Big).$$

is a compact subset of D with

$$d(G, \partial D) \geq \tau > 0. \tag{1.2.20}$$

uniformly for all domains $D \in \mathcal{A}$.

Second, there is a finite number $\tilde{L}$ of cylinders $Z_{\tau/3,\tau/3}(y_j, p_j)$, $y_j, p_j \in I\!\!R^m$, which cover $B_{R_e}(0)$. We choose those cylinders which are contained in D. Because of (1.2.20) they cover G. Using translation, rotation and multiplication with a diagonal matrix with diagonal terms $\tau/(3r_0)$ and $\tau/(3a)$ we obtain continuously differentiable parametrizations $\psi_{2j} : Z_{r_0,a} \to Z_{\tau/3,\tau/3}(y_j, p_j)$, $j = 1, ..., \tilde{L}$, such that

$$G \subset \Big(\bigcup_{j=1}^{L_2} \psi_{2j}(Z_{r_0,a}).$$

The proof is now completed with $L_2 := L_1 + \tilde{L}$ by combining step one and two.

□

For the work with the exterior cone condition the following technical lemma will be useful.

Lemma 1.2.3 *Given the class $\mathcal{A}$ of domains, there is $0 < \beta_0 \leq \beta_e$ and $\rho_0 > 0$ such that all domains*

$$D_\rho := \{\ y \in I\!\!R^m,\ d(y, D) < \rho\ \} \tag{1.2.21}$$

with $D \in \mathcal{A}$ and $0 \leq \rho \leq \rho_0$ satisfy the exterior cone condition with angle β_0.

Proof. We will show that there is $0 < \beta_0 < \beta_e$, such that for each point x in $I\!R^m \setminus D_\rho \subset I\!R^m \setminus D$ and each cone $\operatorname{co}(x,p,\beta_e) \in I\!R^m \setminus D$ the cone $\operatorname{co}(x,p,\beta_0)$ satisfies

$$\operatorname{co}(x,p,\beta_0) \subset I\!R^m \setminus D_\rho.$$

According to Theorem 1.2.2 the domain D is the union of balls with radius r_i. Thus D is a subset of

$$G := \bigcup_{B_{r_i}(y) \subset I\!R^m \setminus (B_\rho(x) \cup \operatorname{co}(x,p,\beta_e}) B_{r_i}(y).$$

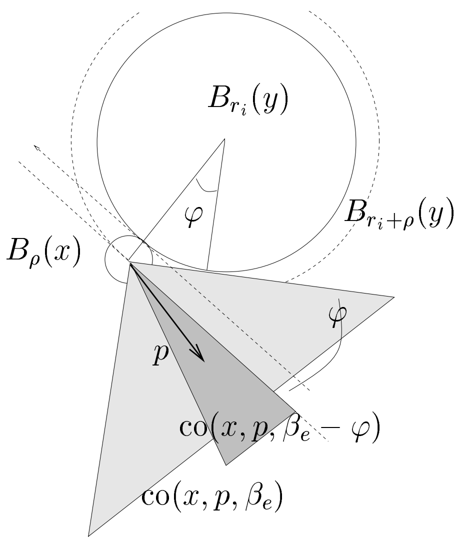

Figure 1

Considering balls $B_{r_i}(y)$, which touch both $B_\rho(x)$ and $\operatorname{co}(x,p,\beta_e)$, we obtain by elementary geometric arguments from Figure 1, that for angles

$$\varphi := \arccos\left(\frac{r_i}{r_i+\rho}\right) \qquad (1.2.22)$$

with

$$\varphi < \beta_e \qquad (1.2.23)$$

we have

$$B_{r_i+\rho}(y) \cap \operatorname{co}(x,p,\beta_e-\varphi) \;=\; \emptyset.$$

Thus we have proven

$$\operatorname{co}(x,p,\beta_e-\varphi) \cap D_\rho \;\subset\; \operatorname{co}(x,p,\beta_e-\varphi) \cap G_\rho \;=\; \emptyset,$$

i.e. we have

$$\operatorname{co}(x,p,\beta_e-\varphi) \subset I\!R^m \setminus D_\rho.$$

The proof is complete by observing that $\arccos(r_i/(r_i+\rho)) \to 0, \rho \to 0.$ and choosing ρ_0, β_0 such that for all $\rho < \rho_0$ and φ defined by (1.2.22) the estimate (1.2.23) is satisfied.

□

An indispensable tool for the investigation of both direct and inverse scattering problems are special solutions to the Helmholtz equation. For later use we now introduce Legendre polynomials, associated Legendre polynomials, spherical harmonics, Bessel and Neumann functions and note some of their properties. If not pointed out otherwise, for a proof of these properties we refer to [6].

Let Y_n^l for $l = -n, ..., n$ and $n = 0, 1, 2, ...$ be a complete orthonormal system of spherical harmonics, as for example given by

$$Y_n^l(\theta, \varphi) := \sqrt{\frac{2n+1}{4\pi} \frac{(n-|l|)!}{(n+|l|)!}} \, P_n^{|l|}(\cos(\theta)) e^{il\varphi} \tag{1.2.24}$$

for $l = -n, ..., n$, $n = 0, 1, 2, ...$ Here, P_n^l are the *associated Legendre functions*, which can be derived from the Legendre polynomials P_n by

$$P_n^l(t) := (1-t^2)^{l/2} \frac{d^l P_n(t)}{dt^l}, \quad l = 0, 1, ..., n. \tag{1.2.25}$$

The Legendre polynomials

$$P_n(x) := \frac{1}{2^n \, n!} \frac{d^n}{dx^n} \Big\{ (x^2-1)^n \Big\}, \ n = 0, 1, ... \tag{1.2.26}$$

form an orthogonal system in $L^2[-1, 1]$, more explicitly we have

$$\int_{-1}^{1} P_n(t) P_l(t) dt \ = \ \frac{2}{2n+1} \, \delta_{nl}, \quad n, l = 0, 1, 2, ... \tag{1.2.27}$$

They satisfy the inequality

$$|P_n(t)| \ \leq \ 1, \quad -1 \leq t \leq 1, \quad n = 0, 1, 2, ... \ . \tag{1.2.28}$$

For $2n+1$ orthonormal spherical harmonics of order n the *addition theorem*

$$\sum_{l=-n}^{n} Y_n^l(\hat{x}) \overline{Y_n^l(\hat{y})} \ = \ \frac{2n+1}{4\pi} \, P_n(\cos(\theta)) \tag{1.2.29}$$

holds for $\hat{x}, \hat{y} \in \Omega$, where θ is the angle between $\hat{x}$ and $\hat{y}$. For the surface gradient of spherical harmonics we note the estimate

$$\Big| \mathrm{Grad} \, Y_n^l(\hat{x}) \Big| \ \leq \ C \, n^{3/2} ||Y_n^l||_{L^2(\Omega)}, \quad \hat{x} \in \Omega, \tag{1.2.30}$$

(see Section X, Lemma 6.1 of [46]).

The *spherical Bessel functions* and *spherical Neumann functions* of order n are given by the series

$$j_n(t) := \sum_{p=0}^{\infty} \frac{(-1)^p t^{n+2p}}{2^p p! 1 \cdot 3 \cdots (2n+2p+1)} \tag{1.2.31}$$

and

$$y_n(t) := -\frac{(2n)!}{2^n n!} \sum_{p=0}^{\infty} \frac{(-1)^p t^{2p-n-1}}{2^p p!(-2n+1)(-2n+3)\cdots(-2n+2p-1)}, \tag{1.2.32}$$

where the first coefficient in the series (1.2.32) has to be set equal to one. The linear combinations

$$h_n^{(1,2)} := j_n \pm i y_n \tag{1.2.33}$$

are known as *spherical Hankel functions* of the first and second kind of order n. By straightforward calculation from the series (1.2.31) and (1.2.32) it is possible to derive the *differentiation formula*

$$t^{n+1} f_{n-1}(t) = \frac{d}{dt}\left\{t^{n+1} f_n(t)\right\} \tag{1.2.34}$$

for both $f_n = j_n$ and $f_n = y_n$, and together with *Stirlings formula*

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + o(1)), \quad n \to \infty \tag{1.2.35}$$

we obtain the behavior

$$j_n(t) \;=\; \frac{1}{2n+1}\left(\frac{et}{2n}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right), \quad n \to \infty, \tag{1.2.36}$$

and

$$h_n^{(1)}(t) \;=\; \frac{-\sqrt{2}}{t}\left(\frac{2n}{et}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right), \quad n \to \infty, \tag{1.2.37}$$

uniformly on compact subsets of $(0, \infty)$. One can use the spherical harmonics and the spherical Bessel or Hankel functions to construct special solutions to the Helmholtz equation. Given a spherical harmonic Y_n of order n, the function

$$u_n(x) := j_n(\kappa|x|) Y_n(\hat{x}) \tag{1.2.38}$$

is an entire solution to the Helmholtz equation. The *multipole of order n*

$$v_n(x) := h_n^{(1)}(\kappa|x|) Y_n(\hat{x}) \tag{1.2.39}$$

is a radiating solution to the Helmholtz equation in $I\!R^m \setminus \{0\}$. Modulo a constant the threedimensional multipole of order zero is the *point-source*

$$\Phi(x,z) := \frac{1}{4\pi} \frac{e^{i\kappa|x-z|}}{|x-z|}, \tag{1.2.40}$$

the multipole of order one is a *dipole*, the multipole of order two a *quadrupole* etc.

Multipole expansions, i.e. expansions of solutions of the Helmholtz equation with respect to the functions v_n, are used both in direct and inverse scattering. For the proof of explicit stability estimates in Section 3, we will need the *multipole-expansion of the fundamental solution*

$$\Phi(x,y) = i\kappa \sum_{n=0}^{\infty} \sum_{l=-n}^{n} h_n^{(1)}(\kappa|x|) Y_n^l(\hat{x}) j_n(\kappa|y|) \overline{Y_n^l(\hat{y})}, \tag{1.2.41}$$

where $\hat{x} = x/|x|$ and $\hat{y} = y/|y|$. Here, the series and its term by term derivatives with respect to x and y are absolutely and uniformly convergent on compact subsets of $|x| > |y|$. Further tools are given by the *Funk-Hecke formula*

$$\int_{\Omega} e^{-i\kappa x\cdot\hat{z}} Y_n(\hat{z})\, ds(\hat{z}) = \frac{4\pi}{i^n} j_n(\kappa|x|) Y_n(\hat{x}), \quad x \in I\!R^m \tag{1.2.42}$$

for *spherical harmonics* Y_n of order n and the *Jacobi-Anger expansion*

$$e^{i\kappa x\cdot d} = \sum_{n=0}^{\infty} i^n (2n+1) j_n(\kappa|x|) P_n(\cos(\theta)), \quad x \in I\!R^m, \tag{1.2.43}$$

where d is a unit vector, θ denotes the angle between x and d and the convergence is uniform on compact subsets of $I\!R^m$.

For scattering in $I\!R^2$ the multipoles and some constants have to be modified. The functions j_n and y_n are replaced by the *Bessel function* of order n

$$J_n(t) := \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(n+p)!} \left(\frac{t}{2}\right)^{n+2p} \tag{1.2.44}$$

and the *Neumann function* of order n

$$\begin{aligned} Y_n(t) &:= \frac{2}{\pi}\left\{\ln\frac{t}{2} + C\right\} J_n(t) - \frac{1}{\pi}\sum_{p=0}^{n-1} \frac{(n-1-p)!}{p!}\left(\frac{2}{t}\right)^{n-2p} \\ &\quad -\frac{1}{\pi}\sum_{p=0}^{\infty} \frac{(-1)^p}{p!(n+p)!}\left(\frac{t}{2}\right)^{n+2p} \{\psi(n+p) + \psi(p)\} \end{aligned} \tag{1.2.45}$$

for $n = 0, 1, 2, ...$, where we define $\psi(0) := 0$,

$$\psi(p) := \sum_{l=1}^{p} \frac{1}{l}, \quad p = 1, 2, ..., \tag{1.2.46}$$

and

$$C := \lim_{p \to \infty} \left\{ \sum_{l=1}^{p} \frac{1}{l} - \ln p \right\} \tag{1.2.47}$$

denotes *Euler's constant*, and if $n = 0$ the finite sum in (1.2.45) is set equal to zero. The linear combinations

$$H_n^{(1,2)} := J_n \pm iY_n \tag{1.2.48}$$

are called *Hankel functions* of the first and second kind of order n, respectively. The *multipoles* in $I\!R^2$ are given by the functions

$$V_n(x) := H_n^{(1)}(\kappa r) e^{\pm in\varphi} \tag{1.2.49}$$

with the polar coordinates (r, φ). For the twodimensional fundamental solution

$$\Phi(x, z) := \frac{i}{4} H_0^{(1)}(\kappa |x - z|) \tag{1.2.50}$$

we obtain the *multipole-expansion*

$$\Phi(x, y) = \frac{i}{4} H_0^{(1)}(\kappa |x|) J_0(\kappa |y|) + \frac{i}{2} \sum_{n=1}^{\infty} H_n^{(1)}(\kappa |x|) J_n(\kappa |y|) \cos(n\theta), \tag{1.2.51}$$

where θ denotes the angle between x and y. It is valid uniformly on compact subsets of $|x| > |y|$. The Jacobi-Anger expansion (1.2.43) in $I\!R^2$ assumes the form

$$e^{i\kappa x \cdot d} = J_0(\kappa |x|) + 2 \sum_{n=1}^{\infty} i^n J_n(\kappa |x|) \cos(n\theta), \quad x \in I\!R^2. \tag{1.2.52}$$

We will use the *boundary-layer approach* to investigate the properties of the solutions to scattering problems for impenetrable scatterers. With the help of boundary-layer potentials the scattering problems are reduced to integral equations on the boundary of the scatterer.

For a domain $D \subset I\!R^m$ with boundary of class C^2 consider the *single-layer potential*

$$u(x) := \int_{\partial D} \varphi(y) \Phi(x, y) ds(y), \quad x \in I\!R^m, \tag{1.2.53}$$

and the *double-layer potential*

$$v(x) \ := \ \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi(y) ds(y), \ \ x \in I\!R^m \setminus \partial D. \tag{1.2.54}$$

Later we will also use $P_1\varphi$ and $P_2\varphi$ for the single-layer or double-layer potential, respectively. The behavior of the single- and double-layer potentials at the boundary ∂D is described by the following *jump relations.*

Theorem 1.2.4 **(Jump relations.)** *The single-layer potential u with continuous density φ is continuous throughout $I\!R^m$. On the boundary we have*

$$u(x) = \int_{\partial D} \varphi(y) \Phi(x,y) \, ds(y), \ \ x \in \partial D, \tag{1.2.55}$$

and

$$\frac{\partial u_{\pm}}{\partial \nu}(x) \ = \ \int_{\partial D} \varphi(y) \frac{\partial \Phi(x,y)}{\partial \nu(x)} \, ds(y) \mp \frac{1}{2}\varphi(x), \ \ x \in \partial D, \tag{1.2.56}$$

where

$$\frac{\partial u_{\pm}}{\partial \nu}(x) \ := \ \lim_{h \to +0} \ \nu(x) \cdot \text{grad} \ u(x \pm h\nu(x)) \tag{1.2.57}$$

is to be understood in the sense of uniform convergence on ∂D. The double-layer potential v with density φ can be continuously extended from D to $\overline{D}$ and from $I\!R^m \setminus \overline{D}$ to $I\!R^m \setminus D$ with limiting values

$$\frac{\partial v_{\pm}}{\partial \nu}(x) \ = \ \int_{\partial D} \varphi(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} \, ds(y) \pm \frac{1}{2}\varphi(x), \ \ x \in \partial D, \tag{1.2.58}$$

where

$$v_{\pm}(x) \ := \ \lim_{h \to +0} \ v(x \pm h\nu(x))$$

and where the integral exists as an improper integral. For a density $\varphi \in L^2(\partial D)$ the jump relations (1.2.55) to (1.2.58) have to be replaced by

$$\lim_{h \to +0} \int_{\partial D} \Big| u(x \pm h\nu(x)) - \int_{\partial D} \varphi(y) \Phi(x,y) \, ds(y) \Big|^2 \, ds(x) \ = \ 0 \tag{1.2.59}$$

$$\lim_{h \to +0} \int_{\partial D} \Big| \frac{\partial u}{\partial \nu}(x \pm h\nu(x)) \ - \ \int_{\partial D} \varphi(y) \frac{\partial \Phi(x,y)}{\partial \nu(x)} \, ds(y) \ \frac{1}{2} \pm \ \varphi(x) \Big|^2 \, ds(x) \ = \ 0 \tag{1.2.60}$$

and

$$\lim_{h \to +0} \int_{\partial D} \Big| \frac{\partial u}{\partial \nu}(x \pm h\nu(x)) \ - \ \int_{\partial D} \varphi(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} \, ds(y) \ \mp \ \frac{1}{2}\varphi(x) \Big|^2 \, ds(x) \ = \ 0. \tag{1.2.61}$$

Proof. The proof for continuous densities can be found in [5], Theorems 2.12, 2.13 and 2.19, the proof for $\varphi \in L^2(\partial D)$ is due to Kersten [28]. □

With the help of boundary layer potentials and jump relations the acoustic and electromagnetic scattering problems can be reduced to *boundary integral equations of the second kind*, i.e. operator equations of the form

$$(I - A)\varphi = f \tag{1.2.62}$$

with a compact linear operator $A : X \to X$ defined on a normed space X. Integral equations of this kind can be solved using the following theorem of Riesz.

THEOREM 1.2.5 **(Riesz Theorem.)** *Let X be a normed space and $A : X \to X$ a compact linear operator. If the homogeneous equation*

$$(I - A)\varphi = 0$$

only has the trivial solution $\varphi = 0$, then for all $f \in X$ the inhomogeneous equation

$$(I - A)\varphi = f$$

has a unique solution $\varphi \in X$ and this solution depends continuously on f.

Proof. See Corollary 1.17 of [5]. □

According to the Riesz theorem the injectivity of an operator $I - A$ yields its continuous invertibility. Usually the injectivity of an integral operators corresponding to a scattering problem is obtained from the uniqueness of the solution to this scattering problem.

For the investigation of special scattered fields for scattering from inhomogeneous medium scatterers we will need to study the integral equations of the scattering problems both in the spaces of continuous and square-integrable functions. The injectivity of the integral operators in $L^2(D)$ will be obtained from the results in $C(D)$ with the help of *dual systems*, defined on subspaces of $L^2(D)$ by the sesquilinear form

$$\langle \varphi, \psi \rangle := \int_D \varphi(y) \overline{\psi(y)} dy \tag{1.2.63}$$

for $\varphi, \psi \in L^2(D)$.

Theorem 1.2.6 **(Fredholm Alternative Theorem)** *Let X and Y be normed spaces, $\langle X, Y \rangle$ a dual system and $A : X \to X$, $B : Y \to Y$ compact adjoint operators. We have either*

$$N(I - A) = \{0\} \quad \text{and} \quad N(I - B) = \{0\}$$

and

$$(I - A)(X) = X \quad \text{and} \quad (I - B)(Y) = Y$$

or

$$\dim N(I - A) = \dim N(I - B) \in \mathbb{N}$$

and

$$(I - A)(X) = N(I - B)^{\perp} \quad \text{and} \quad (I - B)(Y) = N(I - A)^{\perp}.$$

Proof. For a proof we refer the reader to [37]. □

In general, inverse problems are *ill-posed* in the sense of Hadamard [15], i.e. the demands of *uniqueness, existence and stability* are violated. Ill-posed equations of the type

$$A(\varphi) = f \tag{1.2.64}$$

with a compact (linear or nonlinear) operator $A : X \to Y$ are usually solved approximately by a family of bounded (linear or nonlinear) *regularization operators*

$$R_\alpha : Y \to X, \quad \alpha > 0, \tag{1.2.65}$$

with the property

$$\lim_{\alpha \to 0} R_\alpha(A(x)) = x \quad \text{for all} \quad x \in X, \tag{1.2.66}$$

i.e., the operators $R_\alpha A$ converge pointwise to the identity, if the *regularization parameter* α tends to zero. In this case the family of operators R_α is called a *regularization strategy* (see [35]).

As a main tool for the investigation of inverse scattering problems we will use the approximation of multipoles by a superposition of plane waves. This leads to the approximate solution of ill-posed *linear* operator equations of the form

$$H\varphi = f \tag{1.2.67}$$

in a Hilbert space X, where the ill-posedness of the equation is due to the unboundedness of the operator H^{-1}. A standard regularization strategy for the approximate solution of equation (1.2.67) is given by the *Tikhonov regularization scheme*, which computes an approximate solution φ_α by

$$\varphi_\alpha := (\alpha I + H^* H)^{-1} H^* f. \tag{1.2.68}$$

For more details we refer the reader to [35] or [6].

Another possibility to approximately solve (1.2.67) are minimum norm solutions. For a bounded linear operator $H : X \to Y$ between two normed spaces X and Y, $\tau > 0$ and $f \in Y$ an element $\varphi_0 \in X$ is called *minimum norm solution* of $H\varphi = f$ with *discrepancy* τ, if $||H\varphi_0 - f|| \le \tau$ and

$$||\varphi_0|| = \inf \{||\varphi||,\ ||H\varphi - f|| \le \tau\}.$$

THEOREM 1.2.7 **(Minimum norm solutions)** *Let X, Y be Hilbert spaces. If H has dense range in Y, then for each $f \in Y$ there is a unique minimum norm solution φ_0 of $H\varphi = f$ with discrepancy τ. The minimum norm solution φ_0 can be calculated by*

$$\varphi_0 = (\alpha I + H^* H)^{-1} H^* f, \tag{1.2.69}$$

where α is a zero of the function

$$G(\alpha) := \left|\left|(\alpha I + H^* H)^{-1} H^* f - f\right|\right|^2 - \tau^2. \tag{1.2.70}$$

Proof. A proof is given in [37]. □

The preceding theorem can be interpreted as an *a-posteriori strategy* for the choice of the parameter α in the Tikhonov regularization scheme for the approximate solution of $H\varphi = f$.

An impenetrable acoustic or electromagnetic scatterer is given by a *domain* D and a *boundary condition*. We will use the letter $\mathcal{D}$ for the full scatterer with all its properties. The *type* of a scatterer is either *sound-soft* or *sound-hard* for the acoustic problems or *perfect-conductor* for electromagnetic scattering. Thus, an impenetrable scatterer $\mathcal{D}$ can be viewed as a pair

$$\mathcal{D} = (D, type) \tag{1.2.71}$$

of its domain D and its boundary condition.

For penetrable scatterers the situation slightly more complicated. Again, we use $\mathcal{D}$ for the full scatterer. The scatterer $\mathcal{D}$ is given by a *domain* D, defined as the interior of the support of the inhomogeneity, and a refractive index n with $n|_{\mathbb{R}^m \setminus \overline{D}} = 1$ and $n|_{\overline{D}} \in C^{0,\alpha}(\overline{D})$. The full scatterer $\mathcal{D}$ is the pair

$$\mathcal{D} = (D, n). \tag{1.2.72}$$

We will study uniqueness, stability and algorithms for the reconstruction of the domain D of impenetrable and penetrable scatterers $\mathcal{D}$ for both acoustic and electromagnetic scattering problems.

2 Direct scattering problems.

For the investigation and solution of *inverse* scattering problems a good knowledge about the *direct* scattering problems is indispensable. Thus in this Section on *direct* scattering problems we collect and derive definitions and results for further use in the following sections.

2.1 Acoustic obstacle scattering.

We consider acoustic scattering from a bounded *sound-soft* or *sound-hard* impenetrable scatterer $\mathcal{D}$. The scatterer $\mathcal{D}$ consists of a domain $D \subset \mathbb{R}^m$, $m = 2, 3$ and a boundary condition for the total field on ∂D. We always assume the boundary of ∂D of D to be of class C^2 and the open exterior $\overline{\mathbb{R}^m \setminus D}$ of D to be connected. An incident field u^i is a solution to the *Helmholtz equation*

$$\triangle u + \kappa^2 u = 0 \tag{2.1.1}$$

with *wave number* $\kappa > 0$ on a domain containing D in its interior.

Definition 2.1.1 *Given an incident field u^i and a scatterer $\mathcal{D}$, the* direct acoustic obstacle scattering problem *is to find a scattered field*

$$u^s \in C^2(\mathbb{R}^m \setminus \overline{D}) \cap C(\mathbb{R}^m \setminus D),$$

which solves the Helmholtz equation (2.1.1) in $\mathbb{R}^m \setminus \overline{D}$ and satisfies the Sommerfeld radiation condition

$$r^{\frac{m-1}{2}} \Big(\frac{\partial u^s}{\partial r} - i\kappa u^s \Big) \to 0, \quad r = |x| \to \infty, \tag{2.1.2}$$

uniformly in all directions $\hat{x} = x/|x|$, such that the total field

$$u = u^i + u^s \tag{2.1.3}$$

satisfies the sound-soft *boundary condition*

$$u^i + u^s = 0 \ \text{on} \ \partial D \tag{2.1.4}$$

or the sound-hard *boundary condition*

$$\frac{\partial}{\partial \nu}\Big(u^i + u^s\Big) = 0 \ \text{on} \ \partial D. \tag{2.1.5}$$

Here ν denotes the unit outward normal vector to ∂D and the normal derivative in (2.1.5) is understood in the sense of (1.2.57). A solution u of the Helmholtz equation in the exterior of some ball B satisfying (2.1.2) is called radiating.

The main tools for the investigation and solution of the direct scattering problem are Green's integral theorems. In particular, for $u, v \in C^2(\overline{G})$ we have *Green's second theorem*

$$\int_G \{u \triangle v - v \triangle u\}\, dx = \int_{\partial G} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) ds, \tag{2.1.6}$$

where G denotes a domain of class C^1 and ν the unit normal vector to the boundary ∂G directed into the exterior of G. Green's integral theorems can be used to derive for a radiating solution u^s of the direct acoustic scattering problem *Green's formula*

$$u^s(x) = \int_{\partial D} \left\{ u^s(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u^s}{\partial \nu}(y) \Phi(x,y) \right\} ds(y), \quad x \in I\!R^m \setminus \overline{D} \tag{2.1.7}$$

and the asymptotic behavior

$$u^s(x) = \frac{e^{i\kappa|x|}}{|x|^{\frac{m-1}{2}}} \left\{ u^\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\}, |x| \to \infty, \tag{2.1.8}$$

where $\hat{x} := x/|x| \in \Omega$ and $\Omega := \{x \in I\!R^m, |x| = 1\}$, see [6], Theorem 2.5. The function u^∞ is called the *far field pattern* of the scattered acoustic wave.

As incident fields u^i plane waves and point-sources are of special interest. We denote the scattered field for an *incident plane wave*

$$u^i(x,d) := e^{i\kappa x \cdot d}, \quad x \in I\!R^m,$$

with direction $d \in \Omega$ by $u^s(x,d)$, $x \in I\!R^m \setminus D$, and the corresponding far field pattern by $u^\infty(\hat{x},d)$, $\hat{x} \in \Omega$. An incident *point-source* $\Phi(\cdot,z)$ with source point $z \in I\!R^m$ is given by the *fundamental solution* to the Helmholtz equation (1.2.40). The scattered field for an incident point-source $\Phi(\cdot,z)$ with source point z is denoted by $\Phi^s(\cdot,z)$ and the corresponding far field pattern by $\Phi^\infty(\hat{x},z)$, $\hat{x} \in \Omega$.

Several approaches have been developed to solve the direct scattering problem. We will use *integral equations* to obtain a representation of the solution in terms of boundary-layer potential and to study properties of the scattered fields. To this end, let us introduce the classical boundary integral operators. We use the *single-layer operator*

$$(S\varphi)(x) := 2 \int_{\partial D} \Phi(x,y) \varphi(y) ds(y), \quad x \in \partial D, \tag{2.1.9}$$

the *double-layer operator*

$$(K\varphi)(x) := 2 \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi(y) ds(y), \quad x \in \partial D, \tag{2.1.10}$$

the L^2-*adjoint of the double-layer operator*

$$(K^*\varphi)(x) := 2\int_{\partial D} \frac{\partial\Phi(x,y)}{\partial\nu(x)}\varphi(y)ds(y), \quad x \in \partial D, \tag{2.1.11}$$

and the *normal derivative of the double-layer operator*

$$(T\varphi)(x) := 2\frac{\partial}{\partial\nu(x)}\int_{\partial D} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi(y)ds(y), \quad x \in \partial D. \tag{2.1.12}$$

For a derivation of the following uniqueness and existence results for the direct acoustic scattering problems we refer to [6], Section 3. Here we only summarize the results.

THEOREM 2.1.2 *The direct acoustic scattering problem with sound-soft or sound-hard boundary condition has a* unique solution *and the solution depends* continuously on the boundary data $u^i|_{\partial D}$ *or* $\frac{\partial u^i}{\partial\nu}|_{\partial D}$, *respectively, of the incident field in* $C(\partial D)$ *with respect to uniform convergence of the solution and all its derivatives on closed subsets of* $\mathbb{R}^m \setminus \overline{D}$.

In particular, for the case of the sound-soft *boundary condition, the solution can be represented as a* combined acoustic double- and single-layer potential

$$u^s(x) = \int_{\partial D}\left\{\frac{\partial\Phi(x,y)}{\partial\nu(y)} - i\Phi(x,y)\right\}\varphi(y)ds(y), \quad x \in \mathbb{R}^m \setminus \overline{D}, \tag{2.1.13}$$

where the density $\varphi \in C(\partial D)$ *is a solution of the* boundary integral equation

$$(I + K - iS)\varphi = -2u^i|_{\partial D}. \tag{2.1.14}$$

Here I *stands for the identity operator. For the case of the* sound-hard boundary *condition, a representation of the solution is given by the* modified acoustic single- and double-layer potential

$$u^s(x) := \int_{\partial D}\left\{\Phi(x,y)\varphi(y) + i\frac{\partial\Phi(x,y)}{\partial\nu(y)}(S_0^2\varphi)(y)\right\}ds(y), \quad x \in \mathbb{R}^m \setminus \partial D, \tag{2.1.15}$$

where S_0 *denotes the operator* S *in the case* $\kappa = 0$ *and the density* $\varphi \in C(\partial D)$ *solves the* boundary integral equation

$$(I - K^* - iTS_0^2)\varphi = 2\frac{\partial u^i}{\partial\nu}. \tag{2.1.16}$$

Proof. We refer to Theorem 3.7, Theorem 3.9 and Theorem 3.10 of [6]. □

Before we investigate more details of the behavior of integral operators for the solution to the direct scattering problem, we introduce some symmetry properties of the scattered fields or far field patterns, respectively, which are called *reciprocity relations*. Due to reciprocity relations, the role of *source* and *receiver* in the scattering process can be exchanged.

Reciprocity relations play an important role for the investigation of both direct and inverse scattering problems. We will use reciprocity relations in nearly all further sections, for example for a proof of the Isakov-Kirsch-Kress uniqueness theorem, to prove uniqueness for the support of scattering media, to obtain stability estimates, to introduce the point-source method and also to derive the method of singular sources.

For further argumentation we distinguish different reciprocity relations according to the location of the source and receiver in the near field, the far field or a mixed location with either the source in the near field and the observations in the far field or vice versa.

THEOREM 2.1.3 **(Far field reciprocity relation.)** *The far field patterns for scattering of plane waves by a sound-soft or sound-hard scatterer satisfy*

$$u^\infty(\hat{x}, d) = u^\infty(-d, -\hat{x}), \quad \hat{x}, d \in \Omega. \tag{2.1.17}$$

Proof. We refer to [6], Theorem 3.13 for the sound-soft scatterer. The sound-hard boundary condition can be treated analogously, see for example the proof of the mixed reciprocity relation below.

□

For the mixed reciprocity relations we need the constant

$$\gamma_m = \begin{cases} \dfrac{e^{i\pi/4}}{\sqrt{8\pi\kappa}}, & m = 2 \\[2ex] \dfrac{1}{4\pi}, & m = 3, \end{cases} \tag{2.1.18}$$

depending on the dimension $m = 2, 3$.

THEOREM 2.1.4 **(Mixed reciprocity relation.)** *For acoustic scattering of plane waves $u^i(\cdot, d)$, $d \in \Omega$ and point-sources $\Phi(\cdot, z)$, $z \in I\!R^m \setminus \overline{D}$ from a sound-soft and a sound hard scatterer $\mathcal{D}$ we have*

$$\Phi^\infty(\hat{x}, z) = \gamma_m \, u^s(z, -\hat{x}), \quad z \in I\!R^m \setminus \overline{D}, \ \hat{x} \in \Omega. \tag{2.1.19}$$

Proof. The proof for the sound-soft scatterer is due to Kress [36]. By Green's second theorem (2.1.6) we have that

$$\int_{\partial D} \Big(\Phi^s(y,z) \frac{\partial u^s(y,d)}{\partial \nu(y)} - \frac{\partial \Phi^s(y,z)}{\partial \nu(y)} u^s(y,d) \Big) ds(y) = 0, \tag{2.1.20}$$

for $z \in I\!R^m \setminus \overline{D},\ d \in \Omega$. Passing to the limit $|x| \to \infty$ in Green's formula (2.1.7) we obtain the representation

$$\Phi^\infty(\hat{x}, z) = \gamma_m \int_{\partial D} \Big(\Phi^s(y,z) \frac{\partial e^{-i\kappa \hat{x}\cdot y}}{\partial \nu(y)} - \frac{\partial \Phi^s}{\partial \nu}(y,z) e^{-i\kappa \hat{x}\cdot y} \Big) ds(y) \tag{2.1.21}$$

for $z \in I\!R^m \setminus \overline{D},\ \hat{x} \in \Omega$. Let $u(.,d)$ denote the total field for the sound-soft or the sound-hard scattering problem with incident plane wave of direction d. Adding γ_m times (2.1.20) with d replaced by $-\hat{x}$ to equation (2.1.21) with the help of the boundary conditions we obtain

$$\Phi^\infty(\hat{x}, z) = \gamma_m \int_{\partial D} \Phi^s(y,z) \frac{\partial u(y,-\hat{x})}{\partial \nu(y)} ds(y), \quad z \in I\!R^m \setminus \overline{D}, \quad \hat{x} \in \Omega, \tag{2.1.22}$$

for the sound-soft scatterer and

$$\Phi^\infty(\hat{x}, z) = -\gamma_m \int_{\partial D} \frac{\partial \Phi^s(y,z)}{\partial \nu(y)} u(y,-\hat{x}) ds(y), \quad z \in I\!R^m \setminus \overline{D}, \quad \hat{x} \in \Omega, \tag{2.1.23}$$

for the sound-hard scatterer. Again from Green's theorem have the representation formula

$$u^s(x,d) = -\int_{\partial D} \Phi(x,y) \frac{\partial u(y,d)}{\partial \nu} ds(y), \quad x \in I\!R^m \setminus \overline{D}, \quad d \in \Omega \tag{2.1.24}$$

for the sound-soft boundary condition and

$$u^s(x,d) = \int_{\partial D} \frac{\partial \Phi}{\partial \nu}(x,y) u(y,d) ds(y), \quad x \in I\!R^m \setminus \overline{D}, \quad d \in \Omega \tag{2.1.25}$$

for the sound-hard boundary condition. Now from (2.1.22), (2.1.24) and (2.1.23), (2.1.25) using the boundary condition for Φ^s we obtain (2.1.19) both for the sound-soft and sound-hard boundary condition. □

Before we can start to investigate the direct scattering problems, we have to think about the scatterers under consideration for reconstruction. Appropriate assumptions on the scatterers, as for example a bound on the size of the scatterer, occur in uniqueness theorems (see for example [6], Theorem 5.2). As indicated in the introduction, bounds on the curvature of the scatterer can be used to obtain results on stability. To prove convergence of reconstruction algorithms, it is usually required to know some geometric properties of the scatterers under consideration. For example, it is required to know a part of the interior of the unknown domain for the method of Kirsch and Kress (Section 5.4 of [6]).

We now define classes of scatterers for further investigation. They do not describe the weakest possible restrictions for the different statements and algorithms, but they play the role of some simple limitations, which are adequate for the behavior of the inverse scattering problems under consideration.

DEFINITION 2.1.5 *Given positive constants* R_e, r_0, a_0, C_0, β_e, *we define the class*

$$\mathcal{C}_{ss} = \mathcal{C}_{ss}(R_e, r_0, a_0, C_0, \beta_e)$$

as the set of sound-soft *scatterers with domain*

$$D \in \mathcal{A}(R_e, r_0, a_0, l, \alpha, C_0, \beta_e) \tag{2.1.26}$$

for $l = 2$, $\alpha = 0$ *and* $\mathcal{A}$ *given by (1.2.7). The class*

$$\mathcal{C}_{sh} = \mathcal{C}_{sh}(R_e, r_0, a_0, C_0, \beta_e)$$

is the set of sound-hard *scatterers* $\mathcal{D}$ *with domain* D *satisfying (2.1.26). The classes* $\mathcal{C}_{ss}$ *and* $\mathcal{C}_{sh}$ *together form the class*

$$\mathcal{C}_{obst} := \mathcal{C}_{ss} \cup \mathcal{C}_{sh}.$$

To define the convergence of a sequence of domains or boundaries, respectively, we use the parametrizations $(\psi_j)_{j=1,...,L_1}$, which we constructed in Theorem 1.2.2.

DEFINITION 2.1.6 *The convergence*

$$\partial\tilde{D} \to \partial D$$

is understood as a convergence of the parametrizations $\tilde{\psi}_j \to \psi_j$, $j = 1, ..., L_1$ *in the norm of* $C^{l,\alpha}(B_{r_0}(0))$.

We first derive a compactness property of the class $\mathcal{C}_{obst}$.

LEMMA 2.1.7 *Given a sequence* $(\mathcal{D}_j)_{j\in\mathbb{N}}$ *of scatterers* $\mathcal{D}_j \in \mathcal{C}_{obst}$, *there is a subsequence* $(\mathcal{D}_{j_k})_{k\in\mathbb{N}}$ *of* $(\mathcal{D}_j)_{j\in\mathbb{N}}$, *for which the sequence of domains* $(D_{j_k})_{k\in\mathbb{N}}$ *converges in the* $C^{1,\alpha}$*-norm to a domain* $D \subset B$.

Proof. Consider a sequence $(\mathcal{D}_j)_{j\in I\!N}$ of scatterers. Due to Theorem 1.2.2 we obtain mappings $\psi_{jl} : B_{r_i}(0) \to I\!R^m$ with $||\psi_{jl}||_{C^2(B_{r_i}(0))} \leq C_0$, such that the sets

$$V_{jl} := \psi_{jl}(B_{r_i}(0)), \ \ l = 1, ..., L_1,$$

cover ∂D_j. Since a bounded subset of $C^2(B_{r_i}(0))$ is a relatively compact subset of $C^{1,\alpha}(B_{r_i}(0))$, we can find a convergent subsequence $(\psi_{j_k1})_{k\in I\!N}$ of $(\psi_{j1})_{j\in I\!N}$ in $C^{1,\alpha}(B_{r_i}(0))$. Then, we consider the corresponding subsequence $(\psi_{j_k2})_{k\in I\!N}$ of $(\psi_{j2})_{j\in I\!N}$ and again choose a convergent subsequence $(\psi_{j_{(k_n)}2})_{n\in I\!N}$ of $(\psi_{j_k2})_{k\in I\!N}$. We proceed in the same way with $l = 3, ..., L_1$ and after L_1 steps we obtain a subsequence $(\mathcal{D}_{j_k})_{k\in I\!N}$ of $(\mathcal{D}_j)_{j\in I\!N}$ with a convergent sequence $(D_{j_k})_{k\in I\!N}$ of domains. This completes the proof. □

We now collect some results on the integral operators S, K, K^* and T. We have to be careful with bounds, since we need most bounds and constants to hold uniform for scatterers $\mathcal{D} \in \mathcal{C}_{obst}$. We first summarize the classical mapping properties of the potential operators.

THEOREM 2.1.8 *For $\alpha \in (0, 1)$ the operators S, K, K^* and $T - T_0$ are bounded operators from $C(\partial D)$ into $C^{0,\alpha}(\partial D)$. The operators S and K are also bounded from $C^{0,\alpha}(\partial D)$ into $C^{1,\alpha}(\partial D)$. The operator T is bounded from $C^{1,\alpha}(\partial D)$ into $C^{0,\alpha}(\partial D)$. The double-layer potential defines a bounded operator from $C^{0,\alpha}(\partial D)$ into $C(B \setminus \overline{D})$. All bounds hold uniformly for scatterers $\mathcal{D} \in \mathcal{C}_{obst}$.*

Proof. The proofs for the mapping properties of the operators can be found in the Theorems 2.12, 2.15, 2.30 and 2.31 of [5]. Using the properties of the class $\mathcal{A}$ it has been worked out in [58] that the estimates are satisfied uniformly for scatterers $\mathcal{D} \in \mathcal{C}_{obst}$. □

For the use of continuity and compactness arguments, we need to investigate the dependence of the operators S, K, K^* and $T - T_0$ on the domain ∂D. We investigate the operators as bounded linear operators on $C(\partial D)$.

To treat functions and operators on ∂D in dependence on ∂D we need to define appropriate reference spaces. This will be the spaces

$$X \ := \ \left[C^{1,\alpha}(B_{r_0}(0))\right]^{L_1},$$

and for $l \in \{0, 1\}$

$$Y \ := \ [C^{l,\alpha}(B_{r_0}(0))]^{L_1}.$$

For each scatterer $\mathcal{D} \in \mathcal{C}_{obst}$ there is a parametrization $(\psi_l)_{l=1,\dots,L_1} \in X$. This defines a mapping

$$\Pi : \mathcal{C}_{obst} \to X, \ \ \partial D \mapsto (\psi_l)_{l=1,\dots,L_1}.$$

Later we will need the set

$$V := \Pi(\mathcal{C}_{obst}).$$

Let $\Psi_{\partial D}$ be given by

$$\Psi_{\partial D} : C^{l,\alpha}(\partial D) \to Y, \ \ \varphi \mapsto (\varphi \circ \psi_l)_{l=1,\dots,L_1}$$

For fixed ∂D the linear operator $\Psi_{\partial D}$ is injective. We define

$$W := \Psi_{\partial D}(C^{l,\alpha}(\partial D)). \tag{2.1.27}$$

Since for $\varphi \in C^{l,\alpha}(\partial D)$ by

$$\tilde{\varphi}(\tilde{x}) := \varphi(x) \ \ for \ \ \tilde{x} = \tilde{\psi}_l\Big(\psi_l^{-1}(x)\Big), \ \ x \in \partial D,$$

we can define $\tilde{\varphi} \in C^{l,\alpha}(\partial\tilde{D})$ with

$$\Psi_{\partial D}(\varphi) = \Psi_{\partial\tilde{D}}(\tilde{\varphi}),$$

the set W is independent of $D \in \mathcal{C}_{obst}$ and W is well defined. We have

$$\varphi_1(\psi_l(x)) + \varphi_2(\psi_l(x)) \ \ = \ \ (\varphi_1 + \varphi_2)(\psi_l(x))$$

and

$$\lambda\varphi(\psi_l(x)) = (\lambda\varphi)(\psi_l(x)),$$

thus the set W is a linear space. Equipped with the norm of Y the space $W \subset Y$ becomes a normed space. Easily W can be seen to be complete, i.e. W is a Banach space, on which for each $\mathcal{D} \in \mathcal{C}_{obst}$ the mapping

$$\Psi_{\partial D}^{-1} : W \to C^{l,\alpha}(\partial D)$$

is well defined and bounded.

Definition 2.1.9 *For functions $\varphi \in C^{l,\alpha}(\partial D)$ and $\varphi_n \in C^{l,\alpha}(\partial D_n), n \in I\!N$, we say that*

$$\varphi_n \to \varphi, \ \ n \to \infty,$$

if $\partial D_n \to \partial D, \ \ n \to \infty$, and

$$\Psi_{\partial D_n}(\varphi_n) \to \Psi_{\partial D}(\varphi), \ \ n \to \infty,$$

in Y. *A bounded linear operator* J *on* $C^{l,\alpha}(\partial D)$, $l \in \{0,1\}, \alpha \in [0,1]$, *is said to depend continuously on* ∂D, *if the mapping*

$$\hat{J}(\partial D) := \Psi_{\partial D} J(\partial D) \Psi_{\partial D}^{-1} \quad \in BL(W,W)$$

depends continuously on ∂D.

For the investigation of continuity properties of boundary integral operators with weakly singular kernel, we need some further technical tools.

LEMMA 2.1.10 *For scatterers* $\mathcal{D} \in \mathcal{C}_{obst}$ *we uniformly have the estimate*

$$|\nu(z_0) \cdot (y - z_0)| \ \leq \ L\,|y - z_0|^2, \ \ y, z_0 \in \partial D, \tag{2.1.28}$$

with a constant L *and there are constants* $\tau, C > 0$ *such that*

$$\left| \frac{1}{|y - z_h|} - \frac{1}{|y - z_{-h}|} \right| \leq C, \ \ y, z_0 \in \partial D, 0 < h \leq \tau, \tag{2.1.29}$$

where $z_h := z_0 + \nu(z_0)h$.

Proof. First note, that for each point $z_0 \in \partial D$ the statements have to be shown only for y in a neighborhood $B_{r_i}(z_0)$ of z_0, since for $y \notin B_{r_i}(z_0)$ the terms are bounded by a constant uniformly for all domains $D \in \mathcal{C}_{obst}$. For $y \in B_{r_i}(z_0)$ we will give a proof by choosing the special coordinate system K_{z_0} introduced in Section 1.2. The origin of this system is z_0 and the direction of the third axis e_3 is given by $\nu(z_0)$. By rotation around the third axis we can obtain 0 for the second coordinate of y. According to Theorem 1.2.2 in a neighborhood of z_0 the intersection of the boundary ∂D with the $e_1 - e_3$-plane of the new coordinate system is given in the form $(t, 0, f(t))$ with a function f with $|f(t)| \leq C_0 t^2$ for all t with $|t| \leq r_0$.

To prove the first part of the theorem we estimate

$$\begin{aligned} |\nu(z_0) \cdot (y - z_0)| &= \Big|(0,0,1) \cdot \Big((t, 0, f(t)) - (0,0,0)\Big)\Big| \\ &= |f(t)| \\ &\leq C_0 t^2 \\ &\leq C_0\,|y - z_0|^2. \end{aligned} \tag{2.1.30}$$

For the second statement we calculate and estimate with the help of the mean value theorem

$$\begin{aligned}
&\left| \frac{1}{|y - z_h|} - \frac{1}{|y - z_{-h}|} \right| \\
&\quad = \left| \Big|(t, 0, f(t)) - (0, 0, h)\Big|^{-1} - \Big|(t, 0, f(t)) - (0, 0, -h)\Big|^{-1} \right| \\
&\quad = \left| \Big(t^2 + [f(t)]^2 + h^2 - 2f(t)h\Big)^{-\frac{1}{2}} - \Big(t^2 + [f(t)]^2 + h^2 + 2f(t)h\Big)^{-\frac{1}{2}} \right| \\
&\quad \leq \frac{2f(t)h}{(t^2 + [f(t)]^2 + h^2 - 2|f(t)|h)^{\frac{3}{2}}} \\
&\quad \leq \frac{2C_0 t^2 h}{((1 - 2C_0 h)t^2 + h^2)^{\frac{3}{2}}}. \qquad (2.1.31)
\end{aligned}$$

We now use spherical coordinates (r, φ) for (t, h), i.e. we insert $t = r\sin(\varphi)$ and $h = r\cos(\varphi)$ into (2.1.31). With the help of the estimate

$$\Big(\frac{1}{2} r^2 \sin^2(\varphi) + r^2 \cos^2 \varphi\Big)^{\frac{3}{2}} \;\geq\; \sqrt{\frac{1}{8}} r^3$$

we derive

$$\frac{2C_0 t^2 h}{((1 - 2C_0 h)t^2 + h^2)^{\frac{3}{2}}} \leq 2\sqrt{8}\, C_0 \sin^2(\varphi)\cos(\varphi) \;\leq\; 2\sqrt{8}\, C_0 \qquad (2.1.32)$$

for all $0 < h \leq \frac{1}{4C_0}$ and all $0 \leq t \leq r_0$. This ends the proof. □

Lemma 2.1.11 *Consider normed spaces X, Y, a subset $V \subset X$ and a function $f : V \to Y$. We assume that for each $\epsilon > 0$ we have the decomposition*

$$f = f_{1,\epsilon} + f_{2,\epsilon},$$

with functions $f_{1,\epsilon} : V \to Y$ and $f_{2,\epsilon} : V \to Y$. If for each fixed $\epsilon > 0$ the function $f_{2,\epsilon}$ depends continuously on x at a point $x_0 \in V$ and if the family $(f_{1,\epsilon})_{\epsilon>0}$ satisfies the estimate

$$||f_{1,\epsilon}(x)|| \leq c\epsilon \qquad (2.1.33)$$

with a constant c in a neighborhood V' of x_0, then the function f depends continuously on x at the point x_0.

Proof. Given $\tau > 0$ we have to find $\delta > 0$, such that $||f(x) - f(x_0)|| \leq \tau$ for all $x \in B_\delta(x_0) \subset V \subset X$. We first choose $\epsilon := \tau/(4c)$ and obtain

$$||f_{1,\epsilon}(x)|| \leq \tau/4$$

for all $x \in V'$. From the continuity of $f_{2,\epsilon}$ we obtain a $\delta > 0$ such that

$$||f_{2,\epsilon}(x) - f_{2,\epsilon}(x_0)|| \leq \tau/2$$

for all $||x - x_0|| \leq \delta$. Adding $f_{1,\epsilon}$ and $f_{2,\epsilon}$ from

$$||f(x) - f(x_0)|| \leq ||f_{1,\epsilon}(x)|| + ||f_{1,\epsilon}(x_0)|| + ||f_{2,\epsilon}(x) - f_{2,\epsilon}(x_0)||$$

we derive the statement of the lemma.

□

THEOREM 2.1.12 *The operators S, K, K^* and $T - T_0$ in $BL(C(\partial D), C(\partial D))$ depend continuously on the boundary ∂D of the scatterer $\mathcal{D} \in \mathcal{C}_{obst}$ with respect to the $C^{1,\alpha}$-norm for ∂D.*

Proof. The continuity statement for C^2-boundaries is a consequence of more general results on the Fréchet differentiability of the operators with respect to the domain, see [57] and [58]. Here, we will give a proof for boundaries of class $C^{1,\alpha}$, which does not use Fréchet derivatives.

We consider the operator K^* in the three dimensional case $m = 3$. The operator K^* has the kernel

$$\nu(x) \cdot \nabla_x \Phi(x,y), \quad x, y \in \partial D.$$

Let Φ_0 denote the fundamental solution in the case $\kappa = 0$ and K_0^* the corresponding operator K^*. First we note that by (2.1.28) the difference

$$\begin{aligned} \nu(x) \cdot \nabla_x \left(\Phi(x,y) - \Phi_0(x,y)\right) & \\ &= \nu(x) \cdot \nabla_x \left(\frac{e^{i\kappa|x-y|}}{4\pi|x-y|} - \frac{1}{4\pi|x-y|} \right) \\ &= \nu(x) \cdot \nabla_x \left(\sum_{n=1}^{\infty} \frac{(i\kappa)^n \ |x-y|^{n-1}}{4\pi \ n!} \right) \\ &= \frac{\nu(x) \cdot (x-y)}{4\pi \ |x-y|} \left(\sum_{n=2}^{\infty} \frac{(i\kappa)^n \ (n-1) \ |x-y|^{n-2}}{n!} \right) \end{aligned} \qquad (2.1.34)$$

is a continuous function in $x, y \in \partial D$. For such kernels the continuity statements are straightforward to prove, i.e. the difference $K^* - K_0^*$ depends continuously on the boundary ∂D. We investigate the weakly singular part K_0^* with the kernel

$$\nu(x) \cdot \nabla_y \Phi_0(x,y) = \frac{\nu(y) \cdot (x-y)}{|x-y|^3}, \quad x, y \in \partial D. \tag{2.1.35}$$

By definition of the class $\mathcal{A}$ we can choose a local coordinate system with $x = 0$ and $\nu(x) = (0,0,1)$. Then the boundary ∂D in a neighborhood of x is represented in the form $(s,t,f(s,t))$, $s,t \in I\!R$, with $|f(s,t)| \leq C_0|(s,t)|^2$. We obtain the estimate

$$\begin{aligned} \left|\frac{\nu(x)\cdot(x-y)}{|x-y|^3}\right| &= \left|\frac{f(s,t)}{(s^2+t^2+[f(s,t)]^2)^{\frac{3}{2}}}\right| \\ &\leq C_0(s^2+t^2)^{-\frac{1}{2}} \\ &\leq C_0|x-y|^{-1} \end{aligned} \tag{2.1.36}$$

in a neighborhood $|x-y| \leq \frac{1}{\sqrt{2}\,C_0}$ of x. Thus for all scatterers $\mathcal{D} \in \mathcal{C}_{obst}$ the kernel (2.1.35) of the integral operator K_0^* has a weak singularity, which is bounded by the weakly singular function (2.1.36). Decomposing the domain of integration into $B_\epsilon(x) \cap \partial D$ and $\partial D \setminus B_\epsilon(x)$ for all sufficiently small $\epsilon > 0$ we obtain

$$\begin{aligned} (K_0^*\varphi)(x) &= \int_{B_\epsilon(x)\cap\partial D} k_0(x,y)\varphi(y)ds(y) \\ &\quad + \int_{\partial D\setminus B_\epsilon(x)} k_0(x,y)\varphi(y)ds(y) \end{aligned} \tag{2.1.37}$$

with the kernel k_0 given by (2.1.35). We can use the bound (2.1.36) for the singularity of the kernel to estimate the first term of (2.1.37) uniformly for all scatterers $\mathcal{D} \in \mathcal{C}_{obst}$ and all $x \in \partial D$ by

$$\left|\int_{B_\epsilon(x)\cap\partial D} k_0(x,y)\varphi(y)ds(y)\right| \leq c\epsilon||\varphi||_{C(\partial D)} \tag{2.1.38}$$

with some constant c. For each fixed $\epsilon > 0$ the second integral depends continuously on the boundary ∂D and it is bounded uniformly for $\epsilon > 0$ and $\mathcal{D} \in \mathcal{C}_{obst}$ by

$$\left|\int_{B_\epsilon(x)\cap\partial D} k_0(x,y)\varphi(y)ds(y)\right| \leq c||\varphi||_{C(\partial D)} \tag{2.1.39}$$

with some constant c. Let the subset U of X be given by $U := \Pi(\mathcal{C}_{obst})$. We define the mapping $K_j : U \to BL(W,W)$ for $j = 1,2$ by

$$(K_{1,\epsilon}^*(\partial D)\varphi)(x) := \int_{B_\epsilon(x)\cap\partial D} k_0(x,y)\varphi(y)ds(y), \quad x \in \partial D,$$

and

$$(K^*_{2,\epsilon}(\partial D)\varphi)(x) := \int_{\partial D \setminus B_\epsilon(x)} k_0(x,y)\varphi(y)ds(y), \;\; x \in \partial D.$$

From (2.1.37) we derive $K^*_0 = K^*_{1,\epsilon} + K^*_{2,\epsilon}$. Now we apply Lemma 2.1.11 to the mapping

$$V \to BL(W,W), \;\; \partial D \mapsto K^*(\partial D)$$

to obtain with the help of (2.1.38) and (2.1.39) the continuous dependence of K^*_0 on ∂D. Thus we have proven that $K^* = (K^* - K^*_0) + K^*_0$ depends continuously on the boundary ∂D with respect to $C^{1,\alpha}$-norm for ∂D.

The statement for S, K and $T - T_0$ and for the two-dimensional case $m = 2$ can be proven analogously. □

To obtain uniform bounds for the mapping of the incident field onto the scattered field or its far field pattern, respectively, we have to study the inverse of the integral operators $I + K - iS$ for the sound-soft and $I - K' - iTS_0^2$ for the sound-hard boundary condition. Their existence and boundedness is obtained from the Riesz theory. But the Riesz theory does not give a possibility to control the bounds. To derive uniform boundedness in $\mathcal{C}_{obst}$ we use compactness and continuity arguments in the following theorem.

As a preparation we consider the operator strongly singular integral operator T used for the solution of the sound-hard scattering problem. Strongly singular operators are more difficult to handle than operators with weakly singular kernels. Due to the following relations, for the products ST and TS it is possible to obtain a representation in terms of weakly singular integral operators K and K^*. The integral operators S and T satisfy the relations

$$ST = K^2 - I \tag{2.1.40}$$

and

$$TS = (K^*)^2 - I. \tag{2.1.41}$$

A proof can be found in [6], equations (3.12) and (3.13).

Theorem 2.1.13 *In the space $BL(C(\partial D), C(\partial D))$ the integral operators*

$$(I + K - iS)^{-1} \tag{2.1.42}$$

and

$$(I - K^* - iTS_0^2)^{-1} \tag{2.1.43}$$

are bounded by a constant c uniformly for scatterers $\mathcal{D} \in \mathcal{C}_{obst}$.

Proof. First we consider $(I+K-iS)^{-1}$. The boundary integral operators S and K depend continuously on the boundary with respect to the $C^{1,\alpha}$-norm. Since the inverse A^{-1} of an operator A depends continuously on A, the operators (2.1.42) and (2.1.43) depend continuously on the boundary ∂D with respect to the $C^{1,\alpha}$-norm. Using the compactness of the imbedding of $C^2(B_{r_i}(0))$ into $C^{1,\alpha}(B_{r_i}(0))$, the statement is a consequence of the fact that continuous functions on compact sets are bounded.

To treat the operator $(I-K^*-iTS_0^2)^{-1}$ we use (2.1.41) to obtain

$$\begin{aligned}(I-K^*-iTS_0^2)^{-1} &= (I-K^*-i(T-T_0)S_0^2+T_0S_0^2)^{-1}\\ &= (I-K^*-i(T-T_0)S_0^2+[(K_0^*)^2-I]S_0)^{-1}\end{aligned}$$

and proceed in the same way as for $(I+K-iS)^{-1}$. □

THEOREM 2.1.14 *The mappings of the incident field $u^i \in C(\partial D)$ onto the far field pattern $u^\infty \in C^1(\Omega)$ of the scattered field u^s for scattering by a* sound-soft *scatterer and of the normal derivative $\frac{\partial u^i}{\partial \nu} \in C(\partial D)$ onto the far field pattern $u^\infty \in C^1(\Omega)$ of u^s for scattering by a* sound-hard *scatterer are bounded uniformly for $\mathcal{D} \in \mathcal{C}_{obst}$ by a constant c_∞.*

Proof. We use the representations (2.1.13) and (2.1.15) for the solution of the scattering problems. Then the far field patterns of the scattered fields u^s are given by

$$u^\infty(\hat{x}) = \gamma_m \int_{\partial D} \left\{ \frac{\partial e^{i\kappa\hat{x}\cdot y}}{\partial \nu(y)} - ie^{i\kappa\hat{x}\cdot y} \right\} \varphi(y)\, ds(y), \quad \hat{x} \in \Omega \tag{2.1.44}$$

with density $\varphi = -2(I+K-iS)^{-1}u^i$ in the *sound-soft* case and by

$$u^\infty(\hat{x}) := \gamma_m \int_{\partial D} \left\{ e^{i\kappa\hat{x}\cdot y}\varphi(y) + i\frac{\partial e^{i\kappa\hat{x}\cdot y}}{\partial \nu(y)} (S_0^2\varphi)(y) \right\} ds(y), \quad \hat{x} \in \Omega \tag{2.1.45}$$

with density $\varphi = 2(I-K^*-iTS_0^2)^{-1}\frac{\partial u^i}{\partial \nu}$ for the *sound-hard* scatterer. The mapping $u^i \mapsto u^\infty$ can be split into the inversion of the corresponding boundary integral equation and the mapping (2.1.44) or (2.1.45), respectively. We will show that each of these mappings is bounded uniformly for $\mathcal{D} \in \mathcal{C}_{obst}$.

The boundary integral operators $(I+K-iS)^{-1}$ and $(I-K^*-iTS_0^2)^{-1}$ are uniformly bounded according to Theorem 2.1.13. The functions (2.1.44) and (2.1.45) are considered as linear operators from $C(\partial D)$ into $C^1(\Omega)$. The first operator (2.1.44) has continuous kernel and thus depends continuously on $\partial D \in \mathcal{C}_{obst}$ with respect to the $C^{1,\alpha}$-norm. The second operator (2.1.45) is a sum and composition of integral operators with continuous kernels and the operator S_0, which due to Theorem 2.1.12 depends continuously on ∂D. Thus (2.1.45) depends continuously on ∂D.

As in the proof of Theorem 2.1.13, we may use the compactness of the imbedding $C^2(B_{r_i}(0))$ into $C^{1,\alpha}(B_{r_i}(0))$ and the boundedness of continuous functions on compact sets to obtain a uniform bound for scatterers $\mathcal{D} \in \mathcal{C}_{obst}$. □

Our main idea and the ongoing theme will be the use of point-sources, dipoles or multipoles for the investigation and solution of inverse scattering problems. As indicated by the mixed reciprocity relations above, the *far field pattern* of point-sources can be useful for reconstructions, since, up to a constant factor, it is equal to the scattered field of a plane wave. This observation will lead us to the *point-source method* in Section 5. For contributions to the questions of *uniqueness*, *stability* and for the *method of singular sources* we will investigate the *scattered field* of incident point-sources, dipoles or multipoles.

We now investigate the behavior of the scattered field for incident point-sources. In the sound-soft case, for points x on the boundary ∂D from the boundary condition we have

$$\Phi^s(x,z) = -\Phi(x,z), \quad x \in \partial D,$$

and thus

$$\Phi^s(x,z) \to \infty, \quad z \to x.$$

This leads to the idea to use $\Phi^s(x,z)$ for reconstructions of the unknown boundary. But from the viewpoint of the inverse problem, where the boundary ∂D is not known, we would like to replace the point $x \in \partial D$ by something which is known. This new function should not assume knowledge about the boundary, but lead to the same singular behavior. In the following theorem a corresponding behavior is found for

$$\Phi^s(z,z)$$

and we estimate the nature of the singularity.

By $d(z,D)$ or $d(D_1,D_2)$ we denote the *Hausdorff distance*

$$d(z,D) := \inf\{|z-y|, \quad y \in D\} \tag{2.1.46}$$

or

$$d(D_1,D_2) := \inf\{|z_1-z_2|, \quad z_1 \in D_1, \ z_2 \in D_2\} \tag{2.1.47}$$

between z and the domain D or between the two domains D_1 and D_2, respectively.

THEOREM 2.1.15 *Consider the scattering of a point-source* $\Phi(\cdot, z)$ *by a sound-soft or sound-hard scatterer* $\mathcal{D} \in \mathcal{C}_{obst}$. *In* $I\!R^2$ *there exist constants* $\tau, c > 0$, *such that the scattered field* Φ^s *satisfies the* lower estimate

$$\left|\Phi^s(z, z)\right| \;\geq\; c \left|\ln d(z, D)\right| \tag{2.1.48}$$

in the strip $0 < d(z, D) < \tau$. *With constants* $C, E > 0$ *we have the* upper estimate

$$\left|\Phi^s(z, z)\right| \;\leq\; C \left|\ln d(z, D)\right| \;+\; E \tag{2.1.49}$$

for all $z \in B \setminus \overline{D}$. *In* $I\!R^3$ *the corresponding estimates are*

$$\left|\Phi^s(z, z)\right| \;\geq\; \frac{c}{|d(z, D)|} \tag{2.1.50}$$

and

$$\left|\Phi^s(z, z)\right| \;\leq\; \frac{C}{|d(z, D)|}. \tag{2.1.51}$$

All estimates hold uniformly for domains $D \in \mathcal{C}_{obst}$.

Proof. Consider the sound-hard boundary condition. We will investigate the behavior of Φ^s using the solution of the direct problem by means of boundary integral equations.

We abbreviate the modified acoustic single- and double-layer potential (2.1.15) by

$$(P\varphi)(x) := \int_{\partial D} \left\{ \Phi(x, y)\varphi(y) + i \frac{\partial \Phi(x, y)}{\partial \nu(y)} (S_0^2 \varphi)(y) \right\} ds(y), \quad x \in I\!R^m \setminus \partial D. \tag{2.1.52}$$

Then from Theorem 2.1.2 we obtain a representation

$$\Phi^s(\cdot, z) = 2P(I - K' - iTS_0^2)^{-1} \frac{\partial \Phi(\cdot, z)}{\partial \nu}, \quad z \in I\!R^m \setminus \partial D, \tag{2.1.53}$$

for the scattered field of point-sources.

For a point $z \in I\!R^m \setminus D$ with $d(z, D)$ sufficiently small we have the unique representation $z = z_h$ with

$$z_h := z_0 + \nu(z_0)h, \quad z_0 \in \partial D$$

and $h \geq 0$.

We use z_h to decompose

$$
\begin{aligned}
&2P(I - K^* - iTS_0^2)^{-1} \frac{\partial \Phi(\cdot, z_h)}{\partial \nu(\cdot)} \\
&= -\, 2P(I - K^* - iTS_0^2)^{-1} \frac{\partial \Phi(\cdot, z_{-h})}{\partial \nu(\cdot)} \\
&\qquad + 2P(I - K^* - iTS_0^2)^{-1} \left\{ \frac{\partial \Phi(\cdot, z_h)}{\partial \nu(\cdot)} + \frac{\partial \Phi(\cdot, z_{-h})}{\partial \nu(\cdot)} \right\} \\
&= -\Phi^s(\cdot, z_{-h}) \\
&\qquad + 2P(I - K^* - iTS_0^2)^{-1} \left\{ \frac{\partial \Phi(\cdot, z_h)}{\partial \nu(\cdot)} + \frac{\partial \Phi(\cdot, z_{-h})}{\partial \nu(\cdot)} \right\}.
\end{aligned} \tag{2.1.54}
$$

First, consider the term $-\Phi^s(\cdot, z_{-h})$ of (2.1.54). Since z_{-h} is in the interior of the scatterer, we have

$$
-\Phi^s(x, z_{-h}) \;=\; \Phi(x, z_{-h}), \;\; x \in I\!R^m \setminus D. \tag{2.1.55}
$$

In two dimensions Φ has a logarithmic singularity, in three dimensions its singularity is of first order.

To obtain the estimates of the theorem we will show that the singularity of the second term of (2.1.54) is weaker than the singularity of the first term. To this end, in the three-dimensional case $m = 3$ we establish that

$$
\left| 2P(I - K^* - iTS_0^2)^{-1} \left\{ \frac{\partial \Phi(\cdot, z_h)}{\partial \nu(\cdot)} + \frac{\partial \Phi(\cdot, z_{-h})}{\partial \nu(\cdot)} \right\} (z_h) \right| \;\leq\; C \left| \ln h \right| \tag{2.1.56}
$$

for all sufficiently small $h > 0$ with some constant C independent of ∂D. We decompose

$$
P(I - K^* - iTS_0^2)^{-1} \;=\; P \;+\; P(I - K^* - iTS_0^2)^{-1}(K^* + iTS_0^2), \tag{2.1.57}
$$

where P, given by (2.1.52), is the sum

$$
P \;=\; P_1 \;+\; iP_2 S_0^2 \tag{2.1.58}
$$

of a single-layer potential P_1 as defined in (1.2.53) and a term $P_2 S_0^2$ with a double-layer potential P_2 given by (1.2.54). Abbreviating

$$
\Psi_h(y) := \left\{ \frac{\partial \Phi(y, z_h)}{\partial \nu(y)} + \frac{\partial \Phi(y, z_{-h})}{\partial \nu(y)} \right\}, \;\; y \in \partial D, \;\; h \in [0, 1], \tag{2.1.59}
$$

we obtain

$$
\begin{aligned}
P(I-K^*-iTS_0^2)^{-1}\Psi_h &= \Big(P_1\Psi_h\Big) + P_2 S_0\Big(S_0\Phi_h\Big) \\
&\quad + P(I-K^*-iTS_0^2)^{-1}\Big(K^*\Psi_h\Big) \\
&\quad + iP(I-K^*-iTS_0^2)^{-1}\Big(TS_0\Big)\Big(S_0\Psi_h\Big)
\end{aligned}
\tag{2.1.60}
$$

Below we will derive the logarithmic bounds

$$
\begin{aligned}
\Big|(P_1\Psi_h)(z_h)\Big| &\le c\,|\ln(h)|, \\
\Big|(S_0\Psi_h)(x)\Big| &\le c\,|\ln(h)|, x \in \partial D, \\
\Big|(K^*\Psi_h)(x)\Big| &\le c\,|\ln(h)|, x \in \partial D,
\end{aligned}
\tag{2.1.61}
$$

with some constant c uniformly for $\mathcal{D} \in \mathcal{C}_{obst}$. To keep the main line of reasoning, here we first use (2.1.61) and finish the proof. We note that by

$$
\begin{aligned}
TS_0 &= (T-T_0)S_0 + T_0 S_0 \\
&= (T-T_0)S_0 \ + \ [(K_0^*)^2 - I],
\end{aligned}
$$

the operator TS_0 can be extended from $C^{0,\alpha}(\partial D)$ to $C(\partial D)$ and is bounded in $C(\partial D)$ uniformly for all $\mathcal{D} \in \mathcal{C}_{obst}$. According to Theorems 2.1.8 and 2.1.13 the operators

$$
\begin{gathered}
(I-K^*-iTS_0^2)^{-1} : C(\partial D) \to C(\partial D), \\
TS_0 : C(\partial D) \to C(\partial D), \\
S_0 : C(\partial D) \to C^{0,\alpha}(\partial D)
\end{gathered}
$$

and

$$
P_2 : C^{0,\alpha}(\partial D) \to C(B \setminus \overline{D})
$$

are bounded uniformly for $\mathcal{D} \in \mathcal{C}_{obst}$. The single-layer

$$
P_1 : C(\partial D) \to C(B \setminus \overline{D})
$$

is bounded uniformly for $\mathcal{D} \in \mathcal{C}_{obst}$. Using the properties of P_2 and S_0 the same is obtained for the operator

$$
P : C(\partial D) \to C(B \setminus \overline{D}).
$$

Then from the decomposition (2.1.60) using the bounds (2.1.61) we derive the estimate (2.1.56). Finally, we combine (2.1.56), (2.1.55) and (2.1.54) to obtain the statements (2.1.50) and (2.1.51) of the Theorem.

We now reduce (2.1.61) to the potential theoretic case $\kappa = 0$ and estimate the corresponding integrals explicitly. The reduction is possible, since the difference

$\Phi(x, y) - \Phi_0(x, y)$ and its normal derivative are continuous functions. Thus, to estimate the singularity in (2.1.61), we can replace Φ by Φ_0. We proceed in three steps.

1. For Ψ_h in the case $\kappa = 0$ we calculate

$$\begin{aligned} \Psi_h(y) &= \left\{ \frac{\partial \Phi_0(y, z_h)}{\partial \nu(y)} + \frac{\partial \Phi_0(y, z_{-h})}{\partial \nu(y)} \right\} \\ &= \frac{\nu(y) \cdot (y - z_0)}{|y - z_h|^3} + \frac{\nu(y) \cdot (y - z_0)}{|y - z_{-h}|^3} \\ &\quad - \nu(y)\, \nu(z)\, h \left(\frac{1}{|y - z_h|^3} - \frac{1}{|y - z_{-h}|^3} \right). \end{aligned} \tag{2.1.62}$$

With the help of $|\nu(y) \cdot (y - z_0)| \leq L|y - z_0|^2$ given in Lemma 2.1.10 the first two termes of (2.1.62) can be estimated by

$$c\, |y - z_{\pm h}|^{-1} \tag{2.1.63}$$

uniformly for scatterers $\mathcal{D} \in \mathcal{D}_{obst}$. The last term of (2.1.62) can be decomposed into

$$\begin{aligned} &\nu(y)\, \nu(z)\, h \left(\frac{1}{|y - z_h|^3} - \frac{1}{|y - z_{-h}|^3} \right) \\ &= \frac{\nu(y)\, \nu(z)\, h}{|y - z_h|^2} \left(\frac{1}{|y - z_h|} - \frac{1}{|y - z_{-h}|} \right) \\ &\quad + \frac{\nu(y)\, \nu(z)\, h}{|y - z_h| \cdot |y - z_{-h}|} \left(\frac{1}{|y - z_h|} - \frac{1}{|y - z_{-h}|} \right) \\ &\quad + \frac{\nu(y)\, \nu(z)\, h}{|y - z_{-h}|^2} \left(\frac{1}{|y - z_h|} - \frac{1}{|y - z_{-h}|} \right). \end{aligned} \tag{2.1.64}$$

Then with the help of (2.1.29) and

$$h < |y - z_{\pm h}|$$

we derive the bound (2.1.63) also for the third term of (2.1.62).

2. The kernel $\Phi(x_\eta, y)$ with $x_\eta = x + \eta \nu(x)$, $\eta \in [0, h]$, of the single-layer potential P_1 or the operator S_0 can be estimated by

$$\left| \Phi(x_\eta, y) \right| \leq \frac{c}{|x_\eta - y|},$$

for $x, y \in \partial D$ and $h \geq 0$ sufficiently small. For the kernel $\frac{\partial \Phi(x,y)}{\partial \nu(x)}$ of K^* we have

$$\left| \frac{\partial \Phi(x,y)}{\partial \nu(x)} \right| \leq \frac{c}{|x-y|}$$

for $x, y \in \partial D$. Thus to prove (2.1.61) we need to estimate

$$\left| \int_{\partial D} \frac{1}{|x_\eta - y|} \frac{1}{|y - z_{\pm h}|} \, ds(y) \right| \tag{2.1.65}$$

for $\eta \in [0, h]$ and $0 < h \leq \tau$ with some sufficiently small parameter τ.

3. We decompose the domain of integration

$$\int_{\partial D} \dots ds(y) = \int_{\partial D \cap B_R(x)} \dots ds(y) + \int_{\partial D \setminus B_R(x)} \dots ds(y) \tag{2.1.66}$$

with R chosen sufficiently small. Let us first consider the second integral. It is bounded by

$$\frac{1}{R} \left| \int_{\partial D \setminus B_R(x_0)} \frac{1}{|y - z_{\pm h}|} ds(y) \right|.$$

The weak singularity of $|y - z_{\pm h}|^{-1}$ is integrable with bounded integral. Thus the second integral of (2.1.66) is bounded uniformly for $\mathcal{D} \in \mathcal{C}_{obst}$ and $x \in \partial D$.

For the first integral of (2.1.66) we use the special coordinate system with origin x_0, third axis given by $\nu(x_0) = (0, 0, 1)$ and z_0 on the e_1-axis. The tangent plane T_{x_0} in x_0 coincides with the $e_1 - e_2$-plane. The tangent plane in z_0 is denoted by T_{z_0}. We consider only the case $|x_0 - z_0| \leq 2R$, since otherwise we may proceed as above.

Let $\hat{y}$ denote the projection of y onto T_{z_0}. Using the bound on the C^2-norm of the surface by some lines of computation we estimate

$$|y - z_{\pm h}| \geq c \sqrt{|\hat{y} - z_0|^2 + h^2}, \quad |y - z_0| \leq 4R, \tag{2.1.67}$$

with some positive constant c uniformly for $\mathcal{D} \in \mathcal{C}_{obst}$. We now project $\hat{y}$ and z_0 onto the plane T_{x_0}. Let $\tilde{y}$ and $\tilde{z}_0$ be the projection of $\hat{y}$ and z_0, respectively. With the help of polar coordinates (r, ϕ) in T_{x_0} we obtain the estimate

$$\begin{aligned} |\hat{y} - z_0| &\geq |\tilde{y} - \tilde{z}_0| \\ &= \sqrt{r^2 - 2|x_0 - \tilde{z}_0| r \cos(\phi) + |x_0 - \tilde{z}_0|^2} \end{aligned} \tag{2.1.68}$$

and

$$|y - x_\eta| \geq |\tilde{y} - x_0| = r \tag{2.1.69}$$

Thus with $b := |x_0 - \tilde{z}_0| \in [0, 2R]$ an estimate for (2.1.65) is given by a constant C times the integral

$$\int_0^R \frac{1}{\sqrt{r^2 - 2br + b^2 + h^2}}\, dr = -\ln(\sqrt{b^2 + h^2} - b) + \ln\Big(\sqrt{h^2 + (R-b)^2} + R - b\Big).$$

We derive

$$\left| \int_{\partial D} \frac{1}{|x_\eta - y|} \frac{1}{|y - z_h|}\, ds(y) \right| \leq C|\ln h| \tag{2.1.70}$$

for all $\eta \in [0, h]$ and sufficiently small $h > 0$ with a constant C uniformly for scatterers $\mathcal{D} \in \mathcal{C}_{obst}$. This completes the proof for the three-dimensional case.

For scattering in two dimensions $m = 2$ and in the sound-soft case the statements can be proven analogously. To avoid repetitions we leave this part to the reader. □

In the following Theorem we investigate the scattered fields $\Phi^s_{\mu,q}(z, z)$ for scattering of multipoles by a sound-soft or sound-hard impenetrable scatterer. We will use the results for the reconstruction of the shape of a scatterer, if the physical properties of the scatterer are unknown.

THEOREM 2.1.16 *We consider the scattering of a multipole $\Phi_{\mu,q}(\cdot, z)$ by a sound-soft or sound-hard scatterer $\mathcal{D} \in \mathcal{C}_{obst}$. In a sufficiently small neighborhood $0 < d(z, D) < \tau$ of the boundary ∂D let z_0 be defined by the unique representation $z = z_0 + h\nu(z_0)$.*

There are constants $\tau, c > 0$, such that in the strip $0 < d(z, D) < \tau$ the scattered field $\Phi^s_{\mu,-\nu(z_0)}$ satisfies the lower estimate

$$\left|\Phi^s_{\mu,-\nu(z_0)}(z, z)\right| \geq c \left|d(z, D)\right|^{-\mu-m+2} \tag{2.1.71}$$

uniformly for $\mathcal{D} \in \mathcal{C}_{obst}$. For all $z \in B \setminus \overline{D}$ we have the upper estimate

$$\left|\Phi^s_{\mu,-\nu(z_0)}(z, z)\right| \leq C \left|d(z, D)\right|^{-\mu-m+2} \tag{2.1.72}$$

with a constant C uniformly for $\mathcal{D} \in \mathcal{C}_{obst}$.

Proof. For a proof we can go along the lines of the proof of Theorem 2.1.15. To avoid repetitions we will only sketch the differences for the case of the sound-hard scatterer. As in (2.1.54) we decompose into

$$\Phi^s_{\mu,-\nu(z_0)}(\cdot, z) = -\Phi^s_{\mu,\nu(z_0)}(\cdot, z_{-h}) + 2P(I - K^* - iTS_0^2)^{-1}\left\{\frac{\partial \Phi_{\mu,-\nu(z_0)}(\cdot, z_h)}{\partial \nu(\cdot)} + \frac{\partial \Phi_{\mu,\nu(z_0)}(\cdot, z_{-h})}{\partial \nu(\cdot)}\right\}. \tag{2.1.73}$$

With the same arguments as in the proof of Theorem 2.1.15 the second term of (2.1.73) can be estimated by

$$\Big|2P(I-K^*-iTS_0^2)^{-1}\left\{\frac{\partial\Phi_{\mu,-\nu(z_0)}(\cdot,z_h)}{\partial\nu(\cdot)}\right. \qquad (2.1.74)$$

$$\left.+\frac{\partial\Phi_{\mu,\nu(z_0)}(\cdot,z_{-h})}{\partial\nu(\cdot)}\right\}\Big| \;\leq\; C\,\frac{|\ln h|}{h^{\mu+m-3}}.$$

Then both estimates (2.1.71) and (2.1.72) are a consequence of (2.1.73) and (2.1.74).

□

2.2 The inhomogeneous acoustic medium.

We now consider the scattering of time harmonic acoustic waves $u^i(x)e^{-i\omega t}$ by a penetrable inhomogeneous scatterer. The inhomogeneity of the scatterer is described by a *refractive index*

$$n(x) := \frac{c_0^2}{c^2(x)} + i\sigma(x),$$

where $c(x)$ is the *sound speed* at the point $x \in I\!R^m$, c_0 denotes the sound speed of a homogeneous background medium and $\sigma \geq 0$ is a function which models the influence of absorption. We assume the scatterer to be bounded, i.e. we have $n(x) = 1$ for x in the open exterior of a bounded domain D. The refractive index may have jumps on the boundary ∂D of D. Let the boundary ∂D of the inhomogeneity be of class C^2 and $n \in C^{0,\alpha}(\overline{D})$. As described in (1.2.72) we use the letter $\mathcal{D}$ for the full inhomogeneous scatterer and $\chi := 1 - n$.

DEFINITION 2.2.1 *Given an incident field u^i with wave number $\kappa = \omega/c_0$ and an inhomogeneous penetrable scatterer $\mathcal{D}$, the* direct acoustic inhomogeneous medium scattering problem *is to find a radiating scattered field*

$$u^s \in C^2(I\!R^m \setminus \partial D) \cap C^1(I\!R^m),$$

such that the total field $u = u^i + u^s$ *satisfies*

$$\triangle u + \kappa^2 n(x) u = 0 \tag{2.2.1}$$

in $I\!R^m \setminus \partial D$.

Since $n(x) = 1$ for $x \in I\!R^m \setminus \overline{D}$, in the open exterior of D the scattered field is a radiating solution of the Helmholtz equation (1.1.1). Thus Green's formula (2.1.7) and the asymptotic behavior (2.1.8) remain valid for scattering from an inhomogeneous medium, i.e. the scattered field u^s has a far field pattern u^∞.

To solve the scattering problem by means of integral equations we introduce the *volume potential*

$$(V\varphi)(x) := \int_D \Phi(x,y)\varphi(y)\, dy, \quad x \in I\!R^m, \tag{2.2.2}$$

defined on a bounded domain $D \subset I\!R^m$. For a proof of the following uniqueness and existence result for the inhomogeneous medium scattering problem we refer to Theorems 8.3 and 8.7 of [6].

Theorem 2.2.2 *The inhomogeneous medium scattering problem has a unique solution and the solution u depends continuously on the incident field u^i with respect to the norm in $C(D)$. In particular, the scattered field u^s can be represented as a* volume potential

$$u^s(x) = -\kappa^2 \int_D \Phi(x,y)\chi(y)u(y)\,dy, \quad x \in \mathbb{R}^m, \tag{2.2.3}$$

where the total field u is a solution of the Lippmann-Schwinger equation

$$(I + \kappa^2 V\chi)u = u^i. \tag{2.2.4}$$

The integral operator $I + \kappa^2 V\chi$ is continuously invertible in $C(D)$. □

Recall that for incident plane waves $u^i(x,d) = e^{i\kappa x\cdot d}$ we denote the scattered field by $u^s(\cdot,d)$ and its far field pattern by $u^\infty(\cdot,d)$. If the incident field is given by a point-source $\Phi(\cdot,z)$ with source-point $z \in \mathbb{R}^m \setminus \overline{D}$, for the scattered field we write $\Phi^s(\cdot,z)$ and for its far field pattern $\Phi^\infty(\cdot,z)$. As for scattering by obstacles the symmetry between source and receiver is expressed by far field and mixed reciprocity relations.

Theorem 2.2.3 **(Far field reciprocity relation.)** *The far field patterns for scattering of plane waves by an inhomogeneous medium $\mathcal{D}$ satisfy*

$$u^\infty(\hat{x},d) = u^\infty(-d,-\hat{x}), \quad \hat{x}, d \in \Omega. \tag{2.2.5}$$

Proof. We refer to [6], Theorem 8.8. □

Theorem 2.2.4 **(Mixed reciprocity relation.)** *For acoustic scattering of plane waves $u^i(\cdot,d)$, $d \in \Omega$ and point-sources $\Phi(\cdot,z)$, $z \in \mathbb{R}^m \setminus \overline{D}$ from an inhomogeneous medium $\mathcal{D}$ we have*

$$\Phi^\infty(\hat{x},z) = \gamma_m\, u^s(z,-\hat{x}), \quad z \in \mathbb{R}^m \setminus \overline{D},\ \hat{x} \in \Omega, \tag{2.2.6}$$

where the constant γ_m is defined in (2.1.18).

Proof. By Green's theorem (2.1.6) we have that

$$\int_{\partial D} \Big(\Phi^s(y,z)\frac{\partial u^s(y,d)}{\partial \nu(y)} - \frac{\partial \Phi^s(y,z)}{\partial \nu(y)} u^s(y,d)\Big) ds(y) = 0, \quad z \in \mathbb{R}^m \setminus \overline{D},\ d \in \Omega. \tag{2.2.7}$$

Green's formula (2.1.7) yields the representation

$$\Phi^\infty(\hat{x},z) = \gamma_m \int_{\partial D} \Big(\Phi^s(y,z) \frac{\partial e^{-i\kappa \hat{x}\cdot y}}{\partial \nu(y)} - \frac{\partial \Phi^s}{\partial \nu}(y,z) e^{-i\kappa \hat{x}\cdot y} \Big) ds(y), \quad \hat{x} \in \Omega \qquad (2.2.8)$$

with γ_m given by (2.1.18). Adding γ_m times (2.2.7) with d replaced by $-\hat{x}$ to equation (2.2.8) we obtain

$$\Phi^\infty(\hat{x},z) = \gamma_m \int_{\partial D} \Big(\Phi^s(y,z) \frac{\partial u(y,-\hat{x})}{\partial \nu(y)} - \frac{\partial \Phi^s}{\partial \nu}(y,z) u(y,-\hat{x}) \Big) ds(y) \qquad (2.2.9)$$

for $z \in \mathbb{R}^m \setminus \overline{D}$ and $\hat{x} \in \Omega$. We can now use Green's second theorem (2.1.6), the differential equation (2.2.1) and the representation (2.2.3) of the scattered field to derive (2.2.6). □

We now define an appropriate class of media for further investigation. For the interior D of the support of the inhomogeneity we will have to demand the same restrictions as for obstacle scattering. To obtain stability estimates for the support of the function $\chi = 1 - n$ we will also need to uniformly specify the behavior of n at the boundary ∂D and to assume uniform smoothness of n or χ, respectively, on D.

Definition 2.2.5 *Given positive constants R_e, r_0, a_0, C_0, β_e, C_n, c_{min}, c_{max}, $\mu_0 \in \mathbb{N}_0$, $l = 2$ and $\alpha \in [0,1]$ we define the class $\mathcal{C}_m$ of inhomogeneous medium scatterers $\mathcal{D} = (D, n)$ by the following assumptions.*

1. *The domain D is of class $\mathcal{A}(R_e, r_0, a_0, l, \alpha, C_0, \beta_e)$.*
2. *The refractive index n is in $C^{\mu_0,\alpha}(\overline{D})$ with*

$$||n||_{C^{\mu_0,\alpha}(D)} \leq C_n. \qquad (2.2.10)$$

If $\mu_0 \geq 1$, the condition

$$n \in C^{\mu_0 - 1}(B_{R_e}(0))$$

is satisfied.

3. *At the boundary ∂D the function $\chi = 1 - n$ has a jump in its μ_0-th derivatives uniformly for $\mathcal{D} \in \mathcal{C}_m$ in the sense that*

$$0 < c_{min} \leq \Big|\frac{\partial^{\mu_0} \chi}{\partial \nu^{\mu_0}}(x)\Big| \leq c_{max}, \quad x \in \overline{D}. \qquad (2.2.11)$$

We will now use the assumptions to derive uniform bounds for the solution of the scattering problem.

THEOREM 2.2.6 *For scatterers $\mathcal{D} \in \mathcal{C}_m$ the norms*

$$\left|\left|(I + \kappa^2 V\chi)^{-1}\right|\right|_{C(B)}$$

of the integral operators $(I+\kappa^2 V\chi)^{-1}$ are uniformly bounded by some constant c.

Proof. We first note that due to [13], Section II, §13, the operator

$$V\chi : C(D) \to C^1(B)$$

is well defined. Thus $V\chi : C(B) \to C(B)$ is compact and by Riesz' theory the injectivity of $I + \kappa^2 V\chi$ yields its invertibility and the boundedness of the inverse operator. The injectivity of $I + \kappa^2 V\chi$ in $C(B)$ is proven in [6], Theorem 8.7. The operator $V\chi$ depends continuously on the function $\chi \in L^\infty(B)$, since the singularity of Φ is integrable and we can estimate

$$\begin{aligned} ||V\chi\varphi||_{C(B)} &= \left|\left| \int_B \Phi(x,y)\chi(y)\varphi(y)\, dy \right|\right|_{C(B)} \\ &\leq c\, ||\chi||_{L^\infty(B)} ||\varphi||_{C(B)} \end{aligned} \tag{2.2.12}$$

with some constant c. Then also the operator $(I+\kappa^2 V\chi)^{-1}$ depends continuously on χ for all $\chi \in L^\infty(B)$ for which $I + \kappa^2 V\chi$ is invertible. We will show that the set

$$\mathcal{M} := \{\chi,\ \ n = 1 - \chi \text{ is refractive index of } \mathcal{D} \in \mathcal{C}_m\}$$

is relatively compact in $L^\infty(B)$. Let $(\mathcal{D}_j)_{j\in I\!N} \in \mathcal{C}_m$ be a sequence of scatterers and n_j the refractive index of $\mathcal{D}_j$. Then $\chi_j = 1 - n_j \in \mathcal{M}$, $j \in I\!N$. With the parametrizations ψ_l constructed in part 5 of Lemma 1.2.2 we have

$$\left|\left|\chi_j \circ \psi_l\right|\right|_{C^{0,\alpha}(Z_{r_0,a_0})} \leq c \tag{2.2.13}$$

for all $j \in I\!N$ and $l = 1, ..., L_2$ with some constant c depending only on $\mathcal{C}_m$. We use the compactness of the imbedding of $C^{0,\alpha}(Z_{r_0,a_0})$ into $C(Z_{r_0,a_0})$ to successively construct convergent subsequences of $(\chi_j \circ \psi_l)_{j\in I\!N}$ for $l = 1, ..., L_2$. We obtain a subsequence $(\mathcal{D}_{j_k})_{k\in I\!N}$ of $(\mathcal{D}_j)_{j\in I\!N}$, such that $(\chi_{j_k})_{k\in I\!N}$ is convergent towards a function $\chi \in L^\infty(B)$, i.e. we have shown that the set $\mathcal{M} \subset L^\infty(B)$ is relatively compact.

From the construction of the convergent subsequence above we derive that each element χ of $\overline{\mathcal{M}} \subset L^\infty(B)$ is in $C(D)$ with a domain $D \subset B$ of class C^1

and $\chi = 0$ in the exterior of $\overline{D}$. We have shown above that for such functions the operator $I + \kappa^2 V\chi$ is injective and thus $(I + \kappa^2 V\chi)^{-1}$ is well defined. Since on the compact set $\overline{\mathcal{M}} \subset L^\infty(B)$ the operator

$$(I + \kappa^2 V\chi)^{-1} \in BL(C(B), C(B))$$

depends continuously on χ, it is bounded uniformly for $\mathcal{D} \in \mathcal{C}_m$ and the proof is complete. □

THEOREM 2.2.7 *The mapping of the incident field $u^i \in C(D)$ onto the far field pattern $u^\infty \in C^1(\Omega)$ of the scattered field u^s for scattering from an inhomogeneous medium is bounded by a constant c_∞ uniformly for $\mathcal{D} \in \mathcal{C}_m$.*

Proof. We use the representation (2.2.3) for the scattered field. Passing to the limit $|x| \to \infty$ a representation of the far field pattern of u^s is given by

$$u^\infty(\hat{x}) = -\gamma_m \int_B e^{-i\kappa\hat{x}\cdot y}\chi(y)u(y)\, ds(y), \quad \hat{x} \in \Omega \tag{2.2.14}$$

with $u = (I + \kappa^2 V\chi)^{-1}u^i$. By

$$\begin{aligned} &\Big|\Big| \int_B e^{-i\kappa\hat{x}\cdot y}\chi(y)\varphi(y)\, ds(y)\Big|\Big|_{C^1(\Omega)} \\ &\quad \leq \Big|\Big| \int_B (-i\kappa y)e^{-i\kappa\hat{x}\cdot y}\, ds(y)\Big|\Big|_{C^1(\Omega)} ||\varphi||_{C(B)}\, ||\chi||_{L^\infty(B)} \end{aligned} \tag{2.2.15}$$

the integral operator (2.2.14) from $C(B)$ into $C^1(\Omega)$ depends continuously on $\chi \in L^\infty(B)$. We can use the compactness arguments of Theorem 2.2.6 to derive its boundedness uniformly for scatterers $\mathcal{D} \in \mathcal{C}_m$. From the bound for (2.2.14) and the uniform bound for the operators $(I + \kappa^2 V\chi)^{-1}$ given in Theorem 2.2.6 we obtain the statement of the theorem. □

We will also need the mapping properties of $(I + \kappa^2 V\chi)^{-1}$ in $L^2(B)$ and of the potential $V\chi$ from $L^2(B)$ into $C(B)$.

THEOREM 2.2.8 *For scatterers $\mathcal{D} \in \mathcal{C}_m$ the operator $I + \kappa^2 V\chi$ is invertible in $L^2(B)$. The norms of*

$$(I + \kappa^2 V\chi)^{-1} :\ L^2(B) \to L^2(B) \tag{2.2.16}$$

and

$$V\chi :\ L^2(B) \to C(B) \tag{2.2.17}$$

are bounded uniformly for $\mathcal{D} \in \mathcal{C}_m$ by some constant c.

Proof. As shown in [6], Theorem 8.2, the operator V defines a bounded operator from $L^2(B)$ into $H^2(B)$. Thus also $V\chi$ is bounded from $L^2(B)$ into $H^2(B)$ and it is compact in $L^2(B)$. By the Riesz Theory for compact operators, injectivity of $I + \kappa^2 V\chi$ in $L^2(B)$ yields its invertibility. To show injectivity let us assume $u + \kappa^2 V\chi u = 0$ for $u \in L^2(B)$. Then we have $u = -\kappa^2 V\chi u \in H^2(B)$. Since $H^2(B)$ is a subset of $C(B)$ we obtain $u = 0$ from the injectivity of $I + \kappa^2 V\chi$ in $C(B)$.

To get uniform bounds for the norms with the help of the Cauchy Schwarz inequality we first derive the estimate

$$\begin{aligned} ||V\chi\varphi||^2_{C(B)} &= \Big|\Big| \int_B \Phi(x,y)\chi(y)\varphi(y)\,dy \Big|\Big|^2_{C(B)} \\ &\le \left(\sup_{x\in B} \int_B |\Phi(x,y)|^2\,dy \right) \int_B |\chi(y)\varphi(y)|^2\,dy \\ &\le c\,||\chi||^2_{L^\infty(B)}\,||\varphi||^2_{L^2(B)} \end{aligned} \tag{2.2.18}$$

with some constant c. Thus $V\chi : L^2(B) \to C(B)$ depends continuously on χ and we can proceed as in the proof of Theorem 2.2.6 to obtain the bounds uniformly for $\mathcal{D} \in \mathcal{C}_m$. □

In the case of obstacle scattering two different ideas lead to the investigation of point-sources. First, with the help of the mixed reciprocity relation 2.1.4 we want to reconstruct the scattered field u^s in a constructive way from its far field pattern u^∞. Then we can use u^s to detect unknown scatterers using the boundary condition for the total field (see the point-source method in Section 5). Second, in Theorem 2.1.15 we obtained a characterization of the unknown scatterer by the behavior of the scattered field of point-sources $\Phi^s(z,z)$, which shall lead us to the method of singular sources for reconstructions in Section 6.

In the case of scattering by an inhomogeneous medium the situation is different. First, we will have to observe that the reconstruction of the scattered field u^s in the exterior of the unknown scatterer does *not* provide a straightforward possibility to detect the unknown scattering domain D or the size of the refractive index n. Second, we will show that the scattered field $\Phi^s(z,z)$ is bounded for $z \in I\!R^m \setminus D$. Thus the behavior of Φ^s does *not* characterize the unknown boundary.

LEMMA 2.2.9 *For scattering of a point-source $\Phi(\cdot,z)$ by an inhomogeneous medium we have*

$$\Big|\Phi^s(x,z)\Big| \le c, \quad x \in I\!R^m, z \in I\!R^m \setminus \overline{D}, \tag{2.2.19}$$

with a constant c uniformly for scatterers $\mathcal{D} \in \mathcal{C}_m$.

Proof. We use the representation (2.2.3) of the solution to the scattering problem to derive the decomposition

$$\begin{aligned} \Phi^s(\cdot,z) &= -\kappa^2 V\chi(I+\kappa^2 V\chi)^{-1}\Phi(\cdot,z) \\ &= -\kappa^2 V\chi\Phi(\cdot,z) \ + \ \kappa^4 V\chi(I+\kappa^2 V\chi)^{-1} V\chi\Phi(\cdot,z). \end{aligned} \tag{2.2.20}$$

Estimating the singularity of the kernel of the volume potential we derive that $||V\chi\Phi(\cdot,z)||_{C(B)}$ is bounded uniformly for $z \in \mathbb{R}^m \setminus D$ and $\mathcal{D} \in \mathcal{C}_m$. Now the statement of the theorem follows with the help of Theorem 2.2.6. □

For scattering by inhomogeneous media to obtain a singular behavior for the scattered field of an incident singular source when the source point tends to the boundary of the scatterer we have to work with multipoles of higher order. With γ_m given by (2.1.18) for the *multipole of order* $\mu \in \mathbb{N}_0$ we use the notation

$$\Phi_{\mu,q}(x,y) := \begin{cases} \gamma_2 H^{(1)}_\mu(\kappa|x-y|)e^{i\mu\theta}, & m=2, \\ \\ \gamma_3 h^{(1)}_\mu(\kappa|x-y|)P_\mu(\cos(\theta)), & m=3, \end{cases} \tag{2.2.21}$$

where θ is the angle between $x-y$ and q. Point-sources Φ are the multipoles $\Phi_{0,q}$ of order zero. If the incident field is $\Phi_{\mu,q}(\cdot,z)$, we denote the scattered field by $\Phi^s_{\mu,q}(\cdot,z)$ and its far field pattern by $\Phi^\infty_{\mu,q}(\cdot,z)$.

We first calculate some integrals, which are needed for the proof of the following Lemma 2.2.11.

Lemma 2.2.10 *For $\mu \in \mathbb{N}$ we have*

$$\int_{-\pi/2}^{\pi/2} \cos^\mu(\phi)\cos((\mu+2)\phi)\ d\phi \ = \ 0, \tag{2.2.22}$$

$$\int_{-\pi/2}^{\pi/2} \ln(\cos(\phi))\cos^\mu(\phi)\cos((\mu+2)\phi)\ d\phi \ = \ \frac{\pi}{(\mu+1)\ 2^{\mu+1}}, \tag{2.2.23}$$

$$\int_{0}^{\pi/2} \cos^\mu(\theta)P_{\mu+1}(\cos(\theta))\sin(\theta)\ d\theta \ = \ \frac{1}{(\mu+1)\ 2^{\mu+1}}. \tag{2.2.24}$$

Proof.: We first treat the integral (2.2.22). By straightforward differentiation we verify

$$\int_{-\pi/2}^{\pi/2} \cos^\mu(\theta)\cos((\mu+2)\theta)\ d\theta \ = \ \left[\frac{\cos^{\mu+1}(\theta)\sin((\mu+1)\theta)}{\mu+1}\right]_{-\pi/2}^{\pi/2}. \tag{2.2.25}$$

Since $\cos(\pm\pi/2) = 0$, we obtain (2.2.22). By partial integration we calculate

$$\begin{aligned}
&\int_{-\pi/2}^{\pi/2} \ln(\cos(\phi))\ \cos^\mu(\phi)\ \cos((\mu+2)\phi)\ d\phi \\
&= \Big[\ln(\cos(\phi))\ \frac{\cos^{\mu+1}(\phi)\sin((\mu+1)\phi)}{\mu+1}\Big]_{-\pi/2}^{\pi/2} \\
&\qquad + \int_{-\pi/2}^{\pi/2} \frac{\sin(\phi)}{\cos(\phi)}\ \frac{\cos^{\mu+1}(\phi)\ \sin((\mu+1)\phi)}{\mu+1}\ d\phi \\
&= \frac{1}{\mu+1} \int_{-\pi/2}^{\pi/2} \sin(\phi)\ \sin((\mu+1)\phi)\ \cos^\mu(\phi)\ d\phi. \\
&= \frac{1}{2(\mu+1)} \int_0^{2\pi} \sin(\phi)\ \sin((\mu+1)\phi)\ \cos^\mu(\phi)\ d\phi.
\end{aligned} \tag{2.2.26}$$

With the help of the identity

$$\sin(\phi)\ \sin((\mu+1)\phi) = \frac{1}{2}\Big(\cos(\mu\phi) - \cos((\mu+2)\phi)\Big), \tag{2.2.27}$$

the expansion

$$\begin{aligned}
\cos^\mu\phi &= \frac{1}{2^\mu}\left(e^{i\phi} + e^{-i\phi}\right)^\mu \\
&= \frac{1}{2^{\mu-1}} \sum_{k=0}^{\mu} \binom{\mu}{k} \cos(k\phi) \\
&= \frac{1}{2^{\mu-1}} \cos(\mu\phi) + \text{ lower terms,}
\end{aligned} \tag{2.2.28}$$

the integral

$$\int_0^{2\pi} \cos^2(\mu\phi) d\phi \ = \ \pi, \quad \mu \in I\!N, \tag{2.2.29}$$

and the orthogonality of $\cos(k\phi)$ and $\cos(j\phi)$ for $k \neq j$ from (2.2.26) we derive (2.2.23). With the substitution $\cos(\theta) = x$ we get

$$\int_0^{\pi/2} \cos^\mu(\theta) P_{\mu+1}(\cos(\theta)) \sin(\theta)\ d\theta = \int_0^1 x^\mu P_{\mu+1}(x)\ dx, \tag{2.2.30}$$

where the Legendre polynomial $P_{\mu+1}(x)$ is given by formula (1.2.26). By induction we calculate

$$\begin{aligned}
&\int_0^1 x^\mu \frac{d^{\mu+1}}{dx^{\mu+1}} \Big\{(x^2-1)^{\mu+1}\Big\} dx \\
&\quad = (-1)^j\ \frac{\mu!}{(\mu-j)!} \int_0^1 x^{\mu-j} \frac{d^{\mu+1-j}}{dx^{\mu+1-j}} \Big\{(x^2-1)^{\mu+1}\Big\}\ dx,
\end{aligned} \tag{2.2.31}$$

$1 \leq j \leq \mu$. For $j = \mu$ this yields

$$\int_0^1 x^\mu P_{\mu+1}(x)\,dx = \frac{(-1)^\mu}{(\mu+1)2^{\mu+1}} \int_0^1 \frac{d}{dx}\left\{(x^2-1)^{\mu+1}\right\}\,dx = \frac{1}{(\mu+1)\,2^{\mu+1}} \tag{2.2.32}$$

and the proof is complete. □

Lemma 2.2.11 *With μ_0 given by the definition of $\mathcal{C}_m$ and*

$$\mu := \begin{cases} \mu_0 + 2 & m = 2 \\ \mu_0 + 1 & m = 3, \end{cases} \tag{2.2.33}$$

we obtain the estimate

$$\left|V\chi\Phi_{\mu,q}(\cdot,z)(x)\right| \leq C \begin{cases} |\ln|x-z|| + E & m = 2 \\ \frac{1}{|x-z|} & m = 3 \end{cases} \tag{2.2.34}$$

for all $x \in B, z \in B \setminus \overline{D}$, $x \neq z$, and $q \in \Omega$ with constants C, E uniformly for scatterers $\mathcal{D} \in \mathcal{C}_m$.

There are constants $\tau, c > 0$, such that in a strip $0 < d(z,D) \leq \tau$ for the special choice $q(z) := -\nu(z_0)$ with z_0 defined by the unique representation

$$z = z_0 + \nu(z_0)h$$

the lower estimate

$$\left|V\chi\Phi_{\mu,-\nu(z_0)}(\cdot,z)(z)\right| \geq c\left|\ln d(z,D)\right| \tag{2.2.35}$$

holds uniformly for $\mathcal{D} \in \mathcal{C}_m$. For $z \in B \setminus \overline{D}$ we have the upper estimate

$$\left|V\chi\Phi_{\mu,q}(\cdot,z)(z)\right| \leq C\left|\ln d(z,D)\right| + E \tag{2.2.36}$$

with constants C, E uniformly for scatterers $\mathcal{D} \in \mathcal{C}_m$.

Proof. We first investigate the behavior of χ near a point $z_0 \in \partial D$. From the continuity of the derivatives $\chi^{(\gamma)}$ for $|\gamma| < \mu_0$ on B we obtain $\chi^{(\gamma)}(z_0) = 0$ for $|\gamma| < \mu_0$. Thus the tangential derivatives of χ up to the order μ_0 vanish in z_0. Then in a neighborhood $B_R(z_0) \cap \overline{D}$ of z_0 in $\overline{D}$, $0 < R < r_o < 1$, Taylor's expansion of $\chi \in C^{\mu_0,\alpha}(\overline{D})$ assumes the form

$$\chi(y) = \frac{(-1)^{\mu_0}}{\mu_0!}\frac{\partial^{\mu_0}\chi}{\partial\nu^{\mu_0}}(z_0)(r\cos(\theta))^{\mu_0} + O\left(r^{\mu_0+\alpha}\right) \tag{2.2.37}$$

uniformly for $\mathcal{D} \in \mathcal{C}_m$ with

$$r := |y - z_0|, \quad \cos(\theta) := -\nu(z_0) \cdot (y - z_0)/r \tag{2.2.38}$$

and

$$c_{min} \leq \left| \frac{\partial^{\mu_0} \chi}{\partial \nu^{\mu_0}}(z_0) \right| \leq c_{max}.$$

To prove (2.2.35) we split the domain of integration into $D \setminus B_R(z)$ and $D \cap B_R(z)$. The integral

$$\int_{D \setminus B_R(z)} \Phi(z, y)\chi(y)\Phi_{\mu,q}(y, z)\, dy. \tag{2.2.39}$$

has a bounded integrand and is bounded uniformly for $\mathcal{D} \in \mathcal{C}_m$. To treat the integral on the ball $B_R(z_0)$ we define

$$\chi_0(y) := \frac{(-1)^{\mu_0}}{\mu_0!} \frac{\partial^{\mu_0} \chi}{\partial \nu^{\mu_0}}(z_0)(r \cos(\theta))^{\mu_0}, \tag{2.2.40}$$

where r and θ are given by (2.2.38), and decompose

$$\begin{aligned} & \int_{D \cap B_R(z)} \Phi(z, y)\chi(y)\Phi_{\mu,q}(y, z)dy \\ = & \int_{D \cap B_R(z)} \Phi(z, y)\chi_0(y)\Phi_{\mu,q}(y, z)dy \\ & \qquad + \int_{D \cap B_R(z)} \Phi(z, y)\Big(\chi(y) - \chi_0(y)\Big)\Phi_{\mu,q}(y, z)dy. \end{aligned} \tag{2.2.41}$$

We may use (2.2.37), the definition (2.2.33) of μ and the singularity of the multipoles $\Phi_{\mu,q}$ as given by (1.2.32) and (1.2.45) to estimate the second integral on the right-hand side of (2.2.41) by

$$\begin{aligned} & \left| \int_{D \cap B_R(z)} \Phi(z, y)\Big(\chi(y) - \chi_0(y)\Big)\Phi_{\mu,q}(y, z)dy \right| \\ & \leq c \begin{cases} \displaystyle\int_h^R |\ln(r)| r^{\alpha-1}\, dr \leq R^\alpha \frac{|\ln(R)|}{\alpha} + \frac{R^\alpha}{\alpha^2}, & m = 2, \\ \displaystyle\int_h^R r^{\alpha-1}\, dr \leq \frac{1}{\alpha} R^\alpha, & m = 3, \end{cases} \end{aligned} \tag{2.2.42}$$

with some constant c, i.e. for $z \in B \setminus \overline{D}$ and $q \in \Omega$ the integral (2.2.42) is bounded by a constant uniformly for $\mathcal{D} \in \mathcal{C}_m$.

We now investigate the first integral on the right-hand side of (2.2.41). In a neighborhood of a point $z_0 \in \partial D$ a domain D with C^2-boundary is close to a half space

$$H(z_0) := \{y \in \mathbb{R}^m, (y - z_0) \cdot \nu(z_0) < 0\}.$$

To estimate the difference of the integral

$$\int_{D \cap B_R(z)} \Phi(z, y)\chi_0(y)\Phi_{\mu,q}(y, z)\, dy \tag{2.2.43}$$

to the corresponding integral where D is replaced by the half-space $H(z_0)$, we define the set

$$\triangle := (D \cap B_R(z)) \setminus H(z_0) \ \cup\ (H(z_0) \cap B_R(z)) \setminus D.$$

Using the bound on the second derivatives of the parametrizations of ∂D by straightforward calculations we obtain the bound

$$\left| \int_{\triangle} \Phi(z, y)\chi_0(y)\Phi_{\mu,q}(y, z)\, dy \right| \ \le\ c \tag{2.2.44}$$

with some constant c uniformly for $z \in B \setminus \overline{D}$ and $\mathcal{D} \in \mathcal{C}_m$. Thus a lower bound for (2.2.43) is provided by

$$\begin{aligned} &\left| \int_{D \cap B_R(z)} \Phi(z, y)\chi_0(y)\Phi_{\mu,q}(y, z)\, dy \right| \\ &\quad \ge \left| \int_{H(z_0) \cap B_R(z)} \Phi(z, y)\chi_0(y)\Phi_{\mu,q}(y, z) \right| \ -\ c \end{aligned} \tag{2.2.45}$$

with c given by (2.2.44). For $q = -\nu(z_0)$ we now explicitly calculate the leading term of (2.2.45). In two dimensions we derive

$$\begin{aligned} &\left| \int_{H(z_0) \cap B_R(z)} \Phi(z, y)\chi_0(y)\Phi_{\mu,q}(y, z) \right| \\ &= c \left| \int_{-\psi_0}^{\psi_0} \int_{\frac{d(z,D)}{\cos(\psi)}}^{R} \ln(r)\ (r\cos(\psi))^{\mu_0}\ r^{-\mu_0 - 2} \cos((\mu_0 + 2)\psi)\ r dr\ d\psi \right| \ +\ O(1) \\ &= \frac{c}{2} \left| \int_{-\psi_0}^{\psi_0} \Big[\ln^2(R) - \ln^2 d(z, D) - \ln^2 \cos(\psi) \right. \\ &\qquad \left. + 2 \ln d(z, D) \ln \cos(\psi) \Big]\ \cos^{\mu_0}(\psi)\ \cos((\mu_0 + 2)\psi)\ d\psi \right| \ +\ O(1) \end{aligned} \tag{2.2.46}$$

with ψ_0 defined by $h = R\cos(\psi_0)$ and some constant c. We can now use Lemma 2.2.10 to calculate the integrals in (2.2.46) and derive

$$\left| \int_{H(z_0)\cap B_R(z)} \Phi(z,y)\chi_0(y)\Phi_{\mu,q}(y,z) \right| \geq c\, |\ln d(z,D)| \tag{2.2.47}$$

for $d(z,D)$ sufficiently small with some constant c uniformly for $\mathcal{D} \in \mathcal{C}_m$. From (2.2.39), (2.2.42), (2.2.45) and (2.2.47) we obtain (2.2.35).

The estimate (2.2.36) can be proven analogously to (2.2.35), where now an upper estimate for (2.2.43) has to be calculated. Since basically all arguments are the same as above, we leave this part to the reader.

In three dimensions for (2.2.45) we calculate

$$\begin{aligned} &\left| \int_{H(z_0)\cap B_R(z)} \Phi(z,y)\chi_0(y)\Phi_{\mu,q}(y,z) \right| \qquad (2.2.48) \\ &= c \left| \int_0^{\theta_0} \int_{\frac{d(z,D)}{\cos(\theta)}}^{R} \int_0^{2\pi} \frac{1}{r}\, (r\cos(\theta))^{\mu_0}\, r^{-\mu_0-2} P_{\mu_0+1}(\cos(\theta))\, \sin(\theta)\, r^2 d\varphi\, dr\, d\theta \right| \\ &\qquad + O(1) \\ &= 2\pi c \left| \int_0^{\theta_0} \Big[\ln(R) - \ln d(z,D) + \ln\cos(\theta) \Big] \right. \\ &\qquad \left. \cos^{\mu_0}(\theta)\, P_{\mu_0+1}(\cos(\theta))\, \sin(\theta)\, d\theta \right| \; + \; O(1) \end{aligned}$$

with θ_0 defined by $h = R\cos(\theta_0)$ and some constant c. We use equation (2.2.24) of Lemma 2.2.10 and proceed as in the two-dimensional case.

We now prove the upper estimates (2.2.34). For the function

$$\Psi(y,z) \;:=\; \chi(y)\Phi_{\mu,q}(y,z)$$

we have

$$|\Psi(y,z)| \;\leq\; c\,|y-z|^{-2}, \;\; y \in B, z \in B\setminus\overline{D},\; x \neq y \tag{2.2.49}$$

with some constant c uniformly for $\mathcal{D} \in \mathcal{C}_m$. We split the domain of integration into three parts. For $x \in I\!R^m \setminus \{z\}$ we define $R := |x-z|/2$ and use the sets $D_1 := D \cap B_R(x)$, $D_2 := D \cap B_R(z)$ and $D_3 := D \setminus (D_1 \cup D_2)$. We first treat the three-dimensional case $m = 3$. For the following calculations we use C as a generic constant, i.e. C may vary from line to line. The integral over D_1 can be estimated by

$$\begin{aligned} \left| \int_{D_1} \Phi(x,y)\Psi(y,z)\, dy \right| \;&\leq\; C\, R^{-2} \int_0^R r\, dr \\ &\leq\; C \end{aligned} \tag{2.2.50}$$

uniformly for $\mathcal{D} \in \mathcal{C}_m$. To estimate the integral over D_2 we use the decomposition

$$\Big|\int_{D_2} \Phi(x,y)\Psi(y,z)\,dy\Big| \;\le\; \Big|\int_{D_2} \Big(\Phi(x,y)-\Phi(x,z)\Big)\Psi(y,z)\,dy\Big| + \Big|\int_{D_2} \Phi(x,z)\Psi(y,z)\,dy\Big|. \qquad (2.2.51)$$

From the mean value theorem we derive

$$|\Phi(x,y)-\Phi(x,z)| \le \frac{C}{|x-z|^2}|y-z|, \quad y \in D_2$$

and estimate (2.2.51) by

$$\Big|\int_{D_2} \Phi(x,y)\Psi(y,z)\,dy\Big| \;\le\; C\Big[\frac{1}{R^2}\int_0^R r\,dr \;+\; \frac{1}{R}\int_0^R dr\Big] \;\le\; C. \qquad (2.2.52)$$

The integral over D_3 can be estimated by

$$\Big|\int_{D_3} \Phi(x,y)\Psi(y,z)\,dy\Big| \;\le\; \frac{C}{R}\int_{D_3}\frac{1}{|y-z|^2}ds(y) \;\le\; \frac{C}{R} \qquad (2.2.53)$$

with some constant C. Note that the last estimate is not sharp, but sufficient for our purposes. Now, from (2.2.50), (2.2.52) and (2.2.53) we obtain the estimate (2.2.34) for the case $m = 3$.

In principle in two dimensions we can proceed analogously. A modification is necessary where $\Psi(\cdot, z)$ is considered, since for $m = 2$ the function $|y|^{-2}$ is not integrable. To obtain the estimate (2.2.51) we need to proceed as in (2.2.46) and estimate

$$\Big|\int_{D_2} \chi(y)\Phi_{\mu,q}(y,z)\,dy\Big| \le C. \qquad (2.2.54)$$

Since all other parts of the proof are analogous to $m = 3$, to avoid repetitions we leave these parts to the reader.

□

We are now prepared to prove estimates for the behavior of the scattered fields of multipoles.

Theorem 2.2.12 *With* μ *given by* (2.2.33) *consider the scattering of a multipole* $\Phi_{\mu,q}$ *from an inhomogeneous medium scatterer* $\mathcal{D} \in \mathcal{C}_m$*. There are constants* $\tau, c > 0$*, such that in the strip* $0 < d(z,D) < \tau$ *the scattered field* $\Phi^s_{\mu,q}$ *satisfies the* lower estimate

$$\left|\Phi^s_{\mu,-\nu(z_0)}(z,z)\right| \;\geq\; c\,\left|\ln d(z,D)\right| \tag{2.2.55}$$

uniformly for scatterers $\mathcal{D} \in \mathcal{C}_m$*, where* $z_0 \in \partial D$ *is defined by the unique representation* $z = z_0 + h\nu(z_0)$*. With constants* C, E *for all* $z \in B \setminus \overline{D}$ *and* $q \in \Omega$ *we have the* upper estimate

$$\left|\Phi^s_{\mu,q}(z,z)\right| \;\leq\; C\,\left|\ln d(z,D)\right| \;+\; E \tag{2.2.56}$$

uniformly for $\mathcal{D} \in \mathcal{C}_m$*.*

Proof. As in (2.2.20) we decompose

$$\begin{aligned} \Phi^s_{\mu,q}(\cdot,z) \;&=\; -\kappa^2 V\chi\Phi_{\mu,q}(\cdot,z) \\ &\quad + \kappa^4 V\chi(I+\kappa^2 V\chi)^{-1} V\chi\Phi_{\mu,q}(\cdot,z). \end{aligned} \tag{2.2.57}$$

We have shown in Lemma 2.2.11 that the function $V\chi\Phi_{\mu,q}(\cdot,z)$ is bounded in $L^2(B)$ uniformly for $\mathcal{D} \in \mathcal{C}_m$. From Theorem 2.2.8 we know that $(I+\kappa^2 V\chi)^{-1}$ is bounded in $L^2(B)$ and that $V\chi$ is bounded from $L^2(B)$ into $C(B)$ uniformly for $\mathcal{D} \in \mathcal{C}_m$. Thus

$$\kappa^4 V\chi(I+\kappa^2 V\chi)^{-1} V\chi\Phi_{\mu,q}(\cdot,z)$$

is bounded for $z \in B \setminus \overline{D}$ by some constant c uniformly for $\mathcal{D} \in \mathcal{C}_m$.

Since the second term of (2.2.57) is bounded, the statements of Theorem 2.2.12 are a consequence of the estimates (2.2.35) and (2.2.36) of Lemma 2.2.11 for the function $\big(V\chi\Phi_{n,q}(\cdot,z)\big)(z)$. □

2.3 Electromagnetic scattering by a perfect conductor.

We now switch from the Helmholtz equation to Maxwell's equations to investigate the scattering of electromagnetic waves. As in the case of acoustic scattering we separately study the cases of impenetrable and penetrable scatterers. Here we start with the problem of scattering by a perfectly conducting obstacle D in $\mathbb{R}^3$. We assume the boundary ∂D of the support of the scatterer $\mathcal{D}$ to be of class $C^{2,\alpha}$. A time-harmonic incident field E^i, H^i is a solution to the *reduced Maxwell equations*

$$\text{curl } E - i\kappa H = 0, \qquad \text{curl } H + i\kappa E = 0 \tag{2.3.1}$$

where the *wave number* κ is a constant given by

$$\kappa^2 = \left(\epsilon + \frac{i\sigma}{\omega}\right)\mu\omega^2 \tag{2.3.2}$$

with the electric permittivity ϵ, the magnetic permeability μ, the electric conductivity σ, the frequency ω of the time-harmonic wave and the sign of κ chosen such that $\text{Im}\kappa \geq 0$.

DEFINITION 2.3.1 *Given an incident electromagnetic field E^i, H^i and a domain D, the* direct electromagnetic scattering problem *with* perfect conductor boundary condition *is to find a scattered electromagnetic field E^s, H^s which solves the reduced Maxwell equations (2.3.1) in $\mathbb{R}^3 \setminus \overline{D}$ and satisfies the Silver-Müller radiation condition*

$$\lim_{r\to\infty}(E \times x + rH) \;=\; 0, \quad r = |x|, \tag{2.3.3}$$

where the limit is assumed to hold uniformly in all directions $x/|x|$, such that the total field

$$E = E^i + E^s, \qquad H = H^i + H^s \tag{2.3.4}$$

satisfies the perfect conductor *boundary condition*

$$\nu \times E = 0 \text{ on } \partial D. \tag{2.3.5}$$

A solution of the reduced Maxwell equations in the exterior of some ball B which satisfies (2.3.3) is called radiating.

An important tool for the treatment of the direct and inverse scattering problems will be again Green's first and second theorem and a version of Green's formula for electromagnetic waves. For a radiating solution

$$E, H \in C^1(\mathbb{R}^3 \setminus \overline{D}) \cap C(\mathbb{R}^3 \setminus D)$$

to Maxwell's equations we have the *Stratton-Chu* formulas

$$\begin{aligned} E(x) &= \operatorname{curl} \int_{\partial D} \nu(y) \times E(y)\Phi(x,y)\, ds(y) \\ &\quad - \frac{1}{i\kappa} \operatorname{curl} \operatorname{curl} \int_{\partial D} \nu(y) \times H(y)\Phi(x,y)\, ds(y), \quad x \in I\!R^3 \setminus \overline{D}, \end{aligned} \tag{2.3.6}$$

and

$$\begin{aligned} H(x) &= \operatorname{curl} \int_{\partial D} \nu(y) \times H(y)\Phi(x,y)\, ds(y) \\ &\quad + \frac{1}{i\kappa} \operatorname{curl} \operatorname{curl} \int_{\partial D} \nu(y) \times E(y)\Phi(x,y)\, ds(y), \quad x \in I\!R^3 \setminus \overline{D}. \end{aligned} \tag{2.3.7}$$

Every radiating solution E, H to the Maxwell equations has the asymptotic form

$$\begin{aligned} E(x) &= \frac{e^{i\kappa|x|}}{|x|} \left\{ E^\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \to \infty \\ H(x) &= \frac{e^{i\kappa|x|}}{|x|} \left\{ H^\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \to \infty \end{aligned} \tag{2.3.8}$$

uniformly for all directions $\hat{x} = x/|x|$. The vector fields E^∞ and H^∞ are defined on the unit sphere Ω and known as the *electric* and *magnetic far field pattern*, respectively. They satisfy

$$H^\infty = \nu \times E^\infty \quad \text{and } \nu \cdot E^\infty = \nu \cdot H^\infty = 0. \tag{2.3.9}$$

Solutions E, H to Maxwells equations are divergence free and satisfy the vector Helmholtz equation

$$\triangle E + \kappa^2 E = 0 \quad \text{and} \quad \triangle H + \kappa^2 H = 0. \tag{2.3.10}$$

Passing to the far fields in (2.3.6) and (2.3.7) we obtain

$$\begin{aligned} E^\infty(x) &= i\kappa \operatorname{curl} \int_{\partial D} \nu(y) \times E(y) e^{-i\kappa \hat{x}\cdot y}\, ds(y) \\ &\quad - \frac{\kappa}{i} \operatorname{curl} \operatorname{curl} \int_{\partial D} \nu(y) \times H(y) e^{-i\kappa \hat{x}\cdot y}\, ds(y), \quad x \in I\!R^3 \setminus \overline{D}, \end{aligned} \tag{2.3.11}$$

and

$$\begin{aligned} H^\infty(x) &= i\kappa \operatorname{curl} \int_{\partial D} \nu(y) \times H(y) e^{-i\kappa \hat{x}\cdot y}\, ds(y) \\ &\quad + \frac{\kappa}{i} \operatorname{curl} \operatorname{curl} \int_{\partial D} \nu(y) \times E(y) e^{-i\kappa \hat{x}\cdot y}\, ds(y), \quad x \in I\!R^3 \setminus \overline{D}. \end{aligned} \tag{2.3.12}$$

As for acoustic scattering we will use *integral equations* to solve the direct scattering problem and study the properties of the scattered fields. With the fundamental solution $\Phi(x, y)$ of the Helmholtz equation we use the *magnetic dipole operator*

$$(Ma)(x) := 2 \int_{\partial D} \nu(x) \times \operatorname{curl}_x \{a(y)\Phi(x,y)\}\ ds(y), \quad x \in \partial D \tag{2.3.13}$$

and the *electric dipole operator*

$$(Nb)(x) := 2\nu(x) \times \operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times b(y)\ \Phi(x,y)\ ds(y), \tag{2.3.14}$$

$x \in \partial D$. Further, we define the *projection operator* P by

$$(Pb)(x) := (\nu(x) \times b(x)) \times \nu(x), \quad x \in \partial D. \tag{2.3.15}$$

THEOREM 2.3.2 *The* direct electromagnetic scattering problem *with* perfect conductor boundary condition *has a unique solution and the solution depends continuously on the incident field in the sense that the mapping of the boundary data* $\nu \times E^i$ *onto the scattered fields is continuous from*

$$T_d^{0,\alpha}(\partial D) := \{a \in C^{0,\alpha}(\partial D), \operatorname{Div} a \in C^{0,\alpha}(\partial D), \nu \cdot a = 0\} \tag{2.3.16}$$

into $C^{0,\alpha}(\mathbb{R}^3 \setminus D) \times C^{0,\alpha}(\mathbb{R}^3 \setminus D)$.

In particular, the combined magnetic and electric dipole potential

$$\begin{aligned} (P_E a)(x) &= \operatorname{curl} \int_{\partial D} a(y)\Phi(x,y)ds(y) \\ &\quad + i \operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times (S_0^2 a)(y)\Phi(x,y)ds(y), \\ (P_H a)(x) &= \frac{1}{i\kappa}\operatorname{curl} E^s(x), \quad x \in \partial D, \end{aligned} \tag{2.3.17}$$

with density $a \in T_d^{0,\alpha}(\partial D)$ *and* S_0 *defined as in (2.1.15) solves the electromagnetic scattering problem provided the density* a *solves the integral equation*

$$a + Ma + i\ NPS_0^2 a = -2\nu \times E^i. \tag{2.3.18}$$

The inverse operator $(I + M + iNPS_0^2)^{-1}$ *exists and is bounded in* $T_d^{0,\alpha}(\partial D)$.

Proof. We refer to Theorem 6.18 and 6.19 of [6]. □

For the electromagnetic scattering problems we will proceed along the lines of the acoustic problems. We first prove reciprocity relations; i.e. we show that for electromagnetic scattering the role of *source* and *receiver* can be exchanged.

In the following D is a perfect conductor. Let $q \in \Omega$ be a constant vector. The electric field of incident plane waves with polarization q and direction of incidence d is given by

$$E^i_{pl}(x,d,q) = i\kappa(d\times q)\times d\; e^{i\kappa x\dot{d}},\quad H^i_{pl}(x,d,q) = i\kappa d\times q\; e^{i\kappa x\times d}. \tag{2.3.19}$$

The corresponding scattered fields and far field patterns are E^s_{pl}, H^s_{pl} and $E^\infty_{pl}, H^\infty_{pl}$, respectively.

First we formulate reciprocity for the far field patterns.

THEOREM 2.3.3 **(Far field reciprocity relation.)** *For scattering of electromagnetic plane waves by a perfect conductor we have the reciprocity relation*

$$q\cdot E^\infty_{pl}(\hat{x},d,p) = p\cdot E^\infty_{pl}(-d,-\hat{x},q) \tag{2.3.20}$$

for all $\hat{x}, d, p, q \in \Omega$.

Proof. See [6], Theorem 6.28. □

To obtain a reciprocity relation if either the source or the receiver is in the near field, we consider the electromagnetic field of an *electric dipole* with polarization p, which is given by

$$\begin{aligned} E^i_{edp}(x,z,p) &:= \frac{-1}{i\kappa}\,\mathrm{curl}\,_y\mathrm{curl}\,_y\Big(p\Phi(x,z)\Big),\\ H^i_{edp}(x,z,p) &:= \mathrm{curl}\,_y\Big(p\Phi(x,z)\Big) \end{aligned} \tag{2.3.21}$$

for $x \neq z$. We denote the corresponding scattered field by

$$E^s_{edp}(.,z,p),\ H^s_{edp}(.,z,p),$$

its far field pattern by

$$E^\infty_{edp}(.,z,p),\ H^\infty_{edp}(.,z,p).$$

For the total field, i.e. the sum of incident and scattered field, we use

$$E_{edp}(.,z,p),\ H_{edp}(.,z,p).$$

The total field for plane waves is denoted by

$$E_{pl}(.,d,q),\ H_{pl}(.,d,q).$$

THEOREM 2.3.4 **(Mixed electromagnetic reciprocity.)** *For scattering by a perfect conductor we have*

$$q \cdot E^{\infty}_{edp}(\hat{x}, z, p) = \gamma \, p \cdot E^{s}_{pl}(z, -\hat{x}, q) \tag{2.3.22}$$

for $\hat{x} \in \Omega$, $z \in I\!R^M \setminus \overline{D}$ *and* $p, q \in \Omega$, *where* $\gamma = \frac{1}{4\pi}$.

Proof. We proceed in two steps. First, from Green's Vector Theorem for electromagnetic plane waves we derive the equation

$$\begin{aligned} 0 &= \text{curl} \int_{\partial D} \nu(y) \times E^{i}_{pl}(y, -\hat{x}, q) \Phi(z, y) ds(y) \\ &\quad - \frac{1}{i\kappa} \text{curl curl} \int_{\partial D} \nu(y) \times H^{i}_{pl}(y, -\hat{x}, q) \Phi(z, y) ds(y), \quad z \in I\!R^3 \setminus \overline{D}. \end{aligned} \tag{2.3.23}$$

We add (2.3.23) to the representation formula (Stratton-Chu formula)

$$\begin{aligned} E^{s}_{pl}(z, -\hat{x}, q) &= \text{curl} \int_{\partial D} \nu(y) \times E^{s}_{pl}(y, -\hat{x}, q) \Phi(z, y) ds(y) \\ &\quad - \frac{1}{i\kappa} \text{curl curl} \int_{\partial D} \nu(y) \times H^{s}_{pl}(y, -\hat{x}, q) \Phi(z, y) ds(y), \end{aligned} \tag{2.3.24}$$

$z \in I\!R^3 \setminus \overline{D}$, and calculate

$$\begin{aligned} E^{s}_{pl}(z, -\hat{x}, q) &= \text{curl} \int_{\partial D} \nu(y) \times E_{pl}(y, -\hat{x}, q) \Phi(z, y) ds(y) \\ &\quad - \frac{1}{i\kappa} \text{curl curl} \int_{\partial D} \nu(y) \times H_{pl}(y, -\hat{x}, q) \Phi(z, y) ds(y) \\ &= - \frac{1}{i\kappa} \text{curl curl} \int_{\partial D} \nu(y) \times H_{pl}(y, -\hat{x}, q) \Phi(z, y) ds(y), \end{aligned} \tag{2.3.25}$$

$z \in I\!R^3 \setminus \overline{D}$, where for the last equality we used the boundary condition for a perfect conductor. By elementary calculations we may verify the equality

$$p \cdot \text{curl}_z \text{curl}_z (a(y) \Phi(y, z)) = a(y) \cdot \text{curl}_z \text{curl}_z (p \Phi(y, z)). \tag{2.3.26}$$

Using (2.3.26) we derive from (2.3.25)

$$\begin{aligned} p \cdot E^{s}_{pl}(z, -\hat{x}, q) &= \int_{\partial D} \nu(y) \times H_{pl}(y, -\hat{x}, q) \cdot E^{i}_{edp}(y, z, p) \, ds(y), \\ &= - \int_{\partial D} \nu(y) \times E^{i}_{edp}(y, z, p) \cdot H_{pl}(y, -\hat{x}, q) \, ds(y) \end{aligned} \tag{2.3.27}$$

for $z \in I\!R^3 \setminus \overline{D}$ and $p, q \in \Omega$.

Second, from the Stratton-Chu formula (2.3.11) and (2.3.12) for far field patterns and the definition of the electromagnetic plane waves we get

$$\begin{aligned} q \cdot E^{\infty}_{edp}(\hat{x}, z, p) &= \gamma \int_{\partial D} \Big\{ \nu(y) \times E^s_{edp}(y, z, p) \cdot H^i_{pl}(y, -\hat{x}, q) \\ &\quad + \nu(y) \times H^s_{edp}(y, z, p) \cdot E^i_{pl}(y, -\hat{x}, q) \Big\}\, ds(y) \end{aligned} \tag{2.3.28}$$

for $\hat{x} \in \Omega$ and $z \in I\!R^3 \setminus \overline{D}$. Analogously to (2.3.23) we derive the formula

$$\begin{aligned} 0 &= i\kappa \int_{\partial D} \Big\{ \nu(y) \times E^s_{edp}(y, z, p) \cdot H^s_{pl}(y, -\hat{x}, q) \\ &\quad + \nu(y) \times H^s_{edp}(y, z, p) \cdot E^s_{pl}(y, -\hat{x}, q) \Big\}\, ds(y) \end{aligned} \tag{2.3.29}$$

from Green's vector formula and the Maxwell equations applied to the scattered electromagnetic fields. We multiply (2.3.29) by $\gamma / i\kappa$ and add it to (2.3.28) to obtain

$$\begin{aligned} q \cdot E^{\infty}_{edp}(\hat{x}, z, p) &= \gamma \int_{\partial D} \Big\{ \nu(y) \times E^s_{edp}(y, z, p) \cdot H_{pl}(y, -\hat{x}, q) \\ &\quad + \nu(y) \times H^s_{edp}(y, z, p) \cdot E_{pl}(y, -\hat{x}, q) \Big\}\, ds(y) \\ &= \gamma \int_{\partial D} \nu(y) \times E^s_{edp}(y, z, p) \cdot H_{pl}(y, -\hat{x}, q) ds(y) \end{aligned} \tag{2.3.30}$$

for $z \in I\!R^3 \setminus \overline{D}$ and $p, q \in \Omega$, where we used the boundary condition for the total field $E_{pl}(., -\hat{x}, q)$.

Now, (2.3.27) and (2.3.30) and the boundary condition for $E_{edp}(., z, p)$ yield the statement of the theorem.

□

We now discuss appropriate assumptions on the boundary of a perfectly conducting scatterer D. For the electromagnetic scattering problems we will need slightly more regularity than for the acoustic problems, since we need to work with the mapping properties of M and N and S_0 in appropriate Hölder spaces on the boundary ∂D of the domain D.

Definition 2.3.5 *Given positive constants R_e, r_0, a_0, C_0, β_e, $l = 2$ and $\alpha \in (0, 1]$ we define the class*

$$\mathcal{C}_{pc} = \mathcal{C}_{pc}(R_e, r_0, a_0, l, \alpha, C_0, \beta_e)$$

of impenetrable perfectly conducting *scatterers $\mathcal{D}$ as the set of perfect conductors with scattering domain $D \in \mathcal{A}(R_e, r_0, a_0, l, \alpha, C_0, \beta_e)$.*

Clearly, for electromagnetic scatterers of class $\mathcal{C}_{pc}$ the statements of Lemmas 1.2.3 and 1.2.2 remain true. Instead of Lemma 2.1.7 we need the following property.

LEMMA 2.3.6 *Given a sequence $(\mathcal{D}_j)_{j\in\mathbb{N}}$ of scatterers $\mathcal{D}_j \in \mathcal{C}_{pc}$ and $0 < \alpha' < \alpha$, there is a subsequence $(\mathcal{D}_{j_k})_{k\in\mathbb{N}}$ of $(\mathcal{D}_j)_{j\in\mathbb{N}}$, for which the sequence of scattering domains $(D_{j_k})_{k\in\mathbb{N}}$ converges in the $C^{2,\alpha'}$-norm to a domain $D \subset B$.*

Proof. The proof is analogous to the proof of Lemma 2.1.7, where we now use the compactness of the imbedding of $C^{2,\alpha}(B_{r_0}(0))$ into $C^{2,\alpha'}(B_{r_0}(0))$.

□

We need to collect some of the well-known mapping properties of the operators M and N to use results of [6]. Let $CT(\partial D)$ be the space of continuous tangential vector fields on the boundary ∂D of the scatterer $\mathcal{D}$. $CT^{n,\alpha}(\partial D)$ are the tangential vector fields of $C^{n,\alpha}(\partial D)$ for $n \in \mathbb{N}_0$. The space $T_d^{0,\alpha}(\partial D)$ has been defined in (2.3.16). We also need the space

$$T_r^{0,\alpha}(\partial D) := \Big\{ b \in CT^{0,\alpha}(\partial D), \nu \times b \in T_d^{0,\alpha}(\partial D) \Big\}. \tag{2.3.31}$$

For a more detailed study we refer to [6]

THEOREM 2.3.7 *The operator M is bounded from $CT(\partial D)$ into $CT^{0,\alpha}(\partial D)$, $CT^{0,\alpha}(\partial D)$ into $CT^{1,\alpha}(\partial D)$ and $T_d(\partial D)$ into $T_d^{0,\alpha}(\partial D)$. The operator N is bounded from $T_r^{0,\alpha}(\partial D)$ into $T_d^{0,\alpha}(\partial D)$.*

Proof. The first statement for M is given by Theorem 3.32 of [5], the second by Theorem 3.3 of [34], the third by Theorem 6.16 of [6]. The mapping properties of N are proven in Theorem 6.17 of [6].

□

Since the kernel of N has a strongly singular part, for $M + iNPS_0^2$ it is more difficult to obtain continuity statements with respect to the boundary of the domain than for the weakly singular acoustic potential operators S, K, K^* or $T - T_0$. In the following we use the results of [58] on the Fréchet differentiability of the operators with respect to the boundary of the domain, which imply the continuous dependence.

Theorem 2.3.8 *For each* $0 < \alpha < 1$ *the operators*

$$\begin{aligned} S_0 &: \ CT(\partial D) \to C^{0,\alpha}(\partial D), \\ S_0 &: \ C^{0,\alpha}(\partial D) \to C^{1,\alpha}(\partial D) \\ M &: \ CT(\partial D) \to CT^{0,\alpha}(\partial D) \end{aligned}$$

depend continuously on the boundary ∂D *of the scatterer* $\mathcal{D}$ *with respect to the* C^2-*norm of* ∂D. *The operator*

$$N: \ C^{1,\alpha}(\partial D) \to CT^{0,\alpha}(\partial D)$$

depends continuously on the boundary ∂D *of the scatterer* $\mathcal{D}$ *with respect to the* $C^{2,\alpha}-$*norm of* ∂D.

Proof. The proof for the first statement for S_0 and M is a consequence of Theorem 3.14 of [58]. The second statement for S_0 is given by Theorem 3.18 of [58]. The statement for N can be found in Corollary 3.15 of [58]. □

Theorem 2.3.9 *For* $0 < \alpha' < \alpha$ *the operators*

$$\begin{aligned} S_0 &: \ CT(\partial D) \to C^{0,\alpha'}(\partial D), \\ S_0 &: \ C^{0,\alpha'}(\partial D) \to C^{1,\alpha'}(\partial D) \\ M &: \ CT(\partial D) \to CT^{0,\alpha'}(\partial D) \\ N &: \ C^{1,\alpha'}(\partial D) \to CT^{0,\alpha'}(\partial D) \end{aligned}$$

are bounded uniformly for $\mathcal{D} \in \mathcal{C}_{pc}$.

Proof. By the compactness of the imbedding $C^{2,\alpha}(B_{r_i}(0))$ into $C^{2,\alpha'}(B_{r_i}(0))$ for $\alpha' < \alpha$ the uniform bounds are a consequence of the continuous dependence given by Theorem 2.3.8.

□

Theorem 2.3.10 *For* $0 < \alpha' < \alpha$ *the boundary integral operators*

$$(I + M + \ iNPS_0^2)^{-1}: \ CT(\partial D) \to CT(\partial D)$$

and

$$(I + M + \ iNPS_0^2)^{-1}: \ CT^{0,\alpha'}(\partial D) \to CT^{0,\alpha'}(\partial D)$$

are bounded by a constant c *uniformly for scatterers* $\mathcal{D} \in \mathcal{C}_{pc}$.

Proof. We first prove the invertibility of the integral operator $I+M+\, iNPS_0^2$ in $CT(\partial D)$ and $CT^{0,\alpha'}(\partial D)$. Since M and NPS_0^2 are compact in $CT(\partial D)$ and in $CT^{0,\alpha'}(\partial D)$ we have to prove injectivity of the integral operator to obtain invertibility by the Riesz-Theory for compact operators. Injectivity in $T_d^{0,\alpha'}(\partial D)$ is stated in Theorem 2.3.2. We assume that a is a density in $CT(\partial D)$ such that

$$(I + M + \, iNPS_0^2)a = 0.$$

Then we have $a = -Ma - iNPS_0^2 a$. By the mapping properties of M, N and S_0^2 as given by Theorem 2.3.7 we first obtain $a \in CT^{0,\alpha'}(\partial D)$, in a second step $a \in T_d^{0,\alpha'}(\partial D)$. We obtain $a = 0$ from Theorem 2.3.2, i.e. the integral operator is injective in $CT(\partial D)$ and $CT^{0,\alpha'}(\partial D)$ and thus continuously invertible.

In $CT(\partial D)$ and $CT^{0,\alpha'}(\partial D)$ for $0 < \alpha' < \alpha$ the integral operator depends continuously on the boundary ∂D with respect to the $C^{2,\alpha'}$-norm for the domain. We use the compactness of the imbedding $C^{2,\alpha}(U_0) \to C^{2,\alpha'}(U_0)$ and the fact that continuous functions on compact sets are bounded to derive the statement of the theorem. □

The preceding theorems enable us to derive bounds for the mapping of the incident field onto the far field pattern of the scattered electromagnetic wave uniformly for scatterers $\mathcal{D} \in \mathcal{C}_{pc}$.

THEOREM 2.3.11 *For scattering by a perfect conductor the mapping of the incident electromagnetic field E^i, H^i in $T_d^{0,\alpha}(\partial D)$ onto the far field pattern E^∞ in $C^1(\Omega, I\!R^3)$ of the scattered electric field E^s is bounded by a constant c_∞ uniformly for scatterers $\mathcal{D} \in \mathcal{C}_{pc}$.*

Proof. We use the combined magnetic and electric dipole potential (2.3.17) for the solution of the scattering problem. Then the far field pattern of the electric field is given by

$$\begin{aligned}(P_E^\infty a)(\hat{x}) \;:=\; & \frac{i\kappa}{4\pi}\;\hat{x} \times \int_{\partial D} a(y) e^{-i\kappa \hat{x}\cdot y}\, ds(y) \\ & + i\,\frac{\kappa^2}{4\pi}\;\hat{x} \times \int_{\partial D} \Big(\nu(y) \times (S_0^2 a)(y)\Big) \times \hat{x}\; e^{-i\kappa\hat{x}\cdot y}\, ds(y), \hat{x} \in \Omega,\end{aligned} \tag{2.3.32}$$

with a density a given by (2.3.18). By the now well-known compactness arguments, the mapping (2.3.32) is bounded from $T_d^{0,\alpha}(\partial D)$ into $C^1(\Omega, I\!R^3)$ uniformly for $\mathcal{D} \in \mathcal{C}_{pc}$. Together with the uniform bound for the integral operator $(I + M + \, iNPS_0^2)^{-1}$ we obtain the statement of the theorem. □

We now study the behavior of the scattered field for incident electric dipoles. The following theorem is the basis and main ingredient for the proof of stability in Section 4 and the convergence properties of the method of singular sources for the reconstruction of electromagnetic scatterers in Section 6.

THEOREM 2.3.12 *Consider scattering of an incident electric dipole* E^i_{edp}, H^i_{edp} *by a perfect conductor* $\mathcal{D} \in \mathcal{C}_{pc}$. *There are constants* $\tau, c > 0$, *such that in the strip* $0 < d(z, D) < \tau$ *the scattered electric field* $E^s_{edp}(z, z, \nu(z_0))$, *with* $z_0 \in \partial D$ *defined by the unique representation* $z = z_0 + h\nu(z_0)$, *satisfies the* lower estimate

$$\left| E^s_{edp}(z, z, \nu(z_0)) \right| \geq \frac{c}{|d(z, D)|^3}. \tag{2.3.33}$$

With a constant C *we have for all* $p \in \Omega$ *and* $z \in B \setminus \overline{D}$ *the* upper estimate

$$\left| E^s_{edp}(z, z, p) \right| \leq \frac{C}{|d(z, D)|^3}. \tag{2.3.34}$$

The estimates are satisfied uniformly for scatterers $\mathcal{D} \in \mathcal{C}_{pc}$.

Proof. With $z_h := z_0 + h\nu(z_0)$ and a representation of the scattered electric field by means of the combined potential P_E we decompose

$$\begin{aligned}
& E^{s_{edp}}(\cdot, z_h, p) \\
&= -2\, P_E\, (I + M + iNPS_0^2)^{-1}\Big(\nu \times E^i_{edp}(\cdot, z_h, p)\Big) \\
&= 2\, P_E\, (I + M + iNPS_0^2)^{-1}\Big(\nu \times E^i_{edp}(\cdot, z_{-h}, p)\Big) \\
&\quad - 2\, P_E\, (I + M + iNPS_0^2)^{-1}\Big(\nu \times \Big\{E^i_{edp}(\cdot, z_h, p) + E^i_{edp}(\cdot, z_{-h}, p)\Big\}\Big) \\
&= -E^s_{edp}(\cdot, z_{-h}, p) \\
&\quad - 2\, P_E\, \Big(\nu \times \Big\{E^i_{edp}(\cdot, z_h, p) + E^i_{edp}(\cdot, z_{-h}, p)\Big\}\Big) \\
&\quad +2\, P_E\, (I + M + iNPS_0^2)^{-1}(M + iNPS_0^2) \\
&\qquad\qquad \Big(\nu \times \Big\{E^i_{edp}(\cdot, z_h, p) + E^i_{edp}(\cdot, z_{-h}, p)\Big\}\Big).
\end{aligned} \tag{2.3.35}$$

For the first term of (2.3.35) we have $-E^s_{edp}(\cdot, z_{-h}, p) = E^i_{edp}(\cdot, z_{-h}, p)$. By straightforward differentiation with the help of

$$\text{curl curl } = -\triangle + \text{grad div} \tag{2.3.36}$$

we calculate

$$E^i_{edp}(x, z, p) \;=\; -\frac{c\, p}{|x - z|^3} + \frac{3c\, p \cdot (x - z)\, (x - z)}{|x - z|^5} + O\Big(\frac{1}{|x - z|^2}\Big). \tag{2.3.37}$$

with some constant c. We estimate

$$E^i_{edp}(z_h, z_{-h}, \nu(z_0)) = O(\frac{1}{h^2}) \; + \; \frac{c}{4\, h^3}\nu(z_0), \tag{2.3.38}$$

i.e. the first term of the right-hand side of (2.3.35) has a singularity of order three in h. To obtain the statement of the theorem we will show that all other terms can be estimated by a constant times h^{-2} uniformly for scatterers $\mathcal{D} \in \mathcal{C}_{pc}$. We will proceed in four steps.

1. As a first step from (2.3.37) by some lines of computation as in (2.1.64) we derive the estimate

$$\Big|\Big|\nu \times \Big\{E^i_{edp}(\cdot, z_h, \nu(z_0) + E^i_{edp}(\cdot, z_{-h}, \nu(z_0))\Big\} \;\Big|\Big|_{C(\partial D)} \;=\; O(\frac{1}{h^2}) \tag{2.3.39}$$

uniformly for scatterers $\mathcal{D} \in \mathcal{C}_{pc}$. Using the bounds of Theorem 2.3.9 we obtain

$$\begin{aligned} &\Big|\Big| \; (M + iNPS_0^2) \\ &\quad \Big(\nu \times \Big\{E^i_{edp}(\cdot, z_h, \nu(z_0) + \;\; E^i_{edp}(\cdot, z_{-h}, \nu(z_0))\Big\}\Big) \;\Big|\Big|_{CT^{0,\alpha'}(\partial D)} \;=\; O(\frac{1}{h^2}) \end{aligned} \tag{2.3.40}$$

and from Theorem 2.3.10 we calculate the estimate

$$\begin{aligned} &\Big|\Big| \; (I + M + \; iNPS_0^2)^{-1}(M + iNPS_0^2) \\ &\quad \Big(\nu \times \Big\{E^i_{edp}(\cdot, z_h, \nu(z_0) + E^i_{edp}(\cdot, z_{-h}, \nu(z_0))\Big\}\Big)\Big|\Big|_{CT^{0,\alpha'}(\partial D)} \;=\; O(\frac{1}{h^2}). \end{aligned} \tag{2.3.41}$$

2. We now investigate the potential P_E defined by (2.3.17). We first note that using

$$\text{grad}\,_x\Phi(x, y) = -\text{grad}\,_y\Phi(x, y), \tag{2.3.42}$$

the relation

$$\text{grad}\; v \;=\; \text{Grad}\; v + \frac{\partial v}{\partial \nu}\nu \tag{2.3.43}$$

in a neighborhood of ∂D and the partial integration

$$\int_{\partial D} v\text{Div}\; a\; ds \;=\; -\int_{\partial D} \text{Grad}\; v \cdot a\; ds \tag{2.3.44}$$

for continuously differentiable functions v and tangential fields a and for points $x \in I\!R^3 \setminus \partial D$ we obtain

$$\text{div} \int_{\partial D} a(y)\Phi(x, y)\; ds(y) \;=\; \int_{\partial D} \text{Div}\; a(y)(\Phi(x, y)\; ds(y). \tag{2.3.45}$$

By an application of (2.3.36) with the help of (2.3.45) we derive

$$\begin{aligned}(P_E a)(x) &= \text{curl} \int_{\partial D} a(y)\Phi(x,y)\, ds(y) \\ &+ \kappa^2 \int_{\partial D} a(y)\Phi(x,y)\, ds(y) \\ &+ \text{grad} \int_{\partial D} \text{Div}\,(\nu(y) \times S_0^2 a(y))\Phi(x,y)\, ds(y),\end{aligned} \tag{2.3.46}$$

$x \in I\!R^3 \setminus \partial D$.

3. We now investigate the behavior of the potential P_E for a Hölder continuous density a with $||a||_{CT^{0,\alpha}(\partial D)} = O(h^{-2})$. Since the gradient of the single-layer potential with Hölder continuous density is bounded (see Theorem 2.17 of [5] for the boundedness and [58] for the uniformity of the bounds), we obtain

$$\Big|(P_E a)(\cdot)\Big| = O(\frac{1}{h^2})$$

uniformly for scatterers $\mathcal{D} \in \mathcal{C}_{pc}$, i.e. we obtain the desired estimate for the last term of (2.3.35).

4. Finally, we investigate the term

$$P_E\left(\nu \times \left\{E^i_{edp}(\cdot, z_h, p) + E^i_{edp}(\cdot, z_{-h}, p)\right\}\right), \tag{2.3.47}$$

where no smoothing operators are involved. For the second and third summand of (2.3.46) appropriate bounds can be found by the same arguments as above. We will have to explicitly calculate the leading term of the first summand of P_E, i.e.

$$\int_{\partial D} \left(\nu(y) \times \left\{E^i_{edp}(y, z_h, p) + E^i_{edp}(y, z_{-h}, p)\right\}\right) \times \nabla_x \Phi(x,y)\Big|_{x=z_h} ds(y), \tag{2.3.48}$$

since for the gradient of the single-layer potential with mere continuous density we have no general estimates available. Proceeding as in (2.1.64) and (2.3.39) we derive a bound

$$\Big|\nu(y) \times \left\{E^i_{edp}(y, z_h, p) + E^i_{edp}(y, z_{-h}, p)\right\}\Big| \le \frac{c}{|y - z_{\pm h}||y - z_{\pm h}|}. \tag{2.3.49}$$

The leading term of (2.3.48) is thus bounded by

$$\int_{\partial D} \frac{C}{|y - z_h|^2 |y - z_{\pm h}||y - z_{\pm h}|}\, ds(y) \tag{2.3.50}$$

with some constant C. We proceed analogously to the part 3 of the proof of Theorem 2.1.15. With the help of the integral

$$\int_0^R \int_0^{2\pi} \frac{r d\varphi dr}{(r^2 + h^2)^2} = \frac{1}{2h^2} \frac{1}{2(R^2 + h^2)} \tag{2.3.51}$$

we obtain $O(h^{-2})$ for (2.3.50) and the proof is complete. □

2.4 Electromagnetic waves in an inhomogeneous medium.

The inhomogeneity of an inhomogeneous medium for scattering of an electromagnetic wave is described by the refractive index

$$n(x) := \frac{1}{\epsilon_0}\left(\epsilon(x) + i\frac{\sigma(x)}{\omega}\right), \tag{2.4.1}$$

where $\epsilon = \epsilon(x) > 0$ denotes the electric permittivity, $\sigma = \sigma(x)$ the electric conductivity of the medium, and where ω is the frequency of the wave. The magnetic permeability is considered to be a constant $\mu = \mu_0 > 0$. We assume the medium to be bounded, i.e. $\epsilon(x) = \epsilon_0$ and $\sigma(x) = 0$ for $x \notin \overline{D}$ for some domain D contained in a fixed ball $B = B_{R_e}(0)$. Let the domain D have the boundary ∂D of class $C^{2,\alpha}$ and $n \in C^{1,\alpha}(I\!R^3)$ for some $0 < \alpha < 1$. The inhomogeneous electromagnetic scatterer is denoted by $\mathcal{D} = (D, n)$ and we use $\chi := 1 - n$.

Definition 2.4.1 *For an incident time-harmonic electromagnetic field E^i, H^i,*

$$\operatorname{curl} E^i - i\kappa H^i = 0, \qquad \operatorname{curl} H^i + i\kappa E^i = 0 \tag{2.4.2}$$

with wave number $\kappa = \epsilon_0\mu_0\omega^2$ and an inhomogeneous penetrable scatterer $\mathcal{D}$, the electromagnetic inhomogeneous medium scattering problem *is to find a radiating scattered field $E^s, H^s \in C^1(I\!R^3)$, such that the* total field

$$E = E^i + E^s, \qquad H = H^i + H^s \tag{2.4.3}$$

satisfies the time-harmonic Maxwell equations

$$\operatorname{curl} E - i\kappa H = 0, \qquad \operatorname{curl} H + i\kappa n(x) E = 0 \tag{2.4.4}$$

in $I\!R^3 \setminus \partial D$.

Outside of the support of the scatterer $\mathcal{D}$ the scattered field E^s, H^s solves the Maxwell equations (2.3.1). Thus the *Stratton-Chu formulas* (2.3.6) and (2.3.7) are valid and the scattered field has the asymptotic behavior (2.3.8) uniformly for all directions, i.e. the scattered fields have a *far field pattern* E^∞, H^∞.

To study the properties of the scattered field E^s, H^s we use a solution of the direct scattering problem by means of volume integral equations. For a continuous vector field a we define the potential

$$\begin{aligned}(T_e a)(x) \; &:= -\kappa^2 \int_D \Phi(x,y)\chi(y)a(y)\,dy \\ &+ \operatorname{grad} \int_D \frac{1}{n(y)} \operatorname{grad} n(y) \cdot a(y)\Phi(x,y)\,dy, \quad x \in I\!R^3.\end{aligned} \tag{2.4.5}$$

We summarize uniqueness and existence results in the following theorem.

THEOREM 2.4.2 *The electromagnetic inhomogeneous medium scattering problem has a unique solution and the solution depends continuously on the incident field with respect to the maximum norm on D. In particular, the scattered electric field E^s can be represented as a potential $T_e E$, where the total electric field E satisfies the integral equation*

$$(I - T_e)E = E^i \tag{2.4.6}$$

on D. The integral operator $I - T_e$ is continuously invertible in $C(D)$.

Proof. We refer to Theorems 9.1, 9.2, 9.4 and 9.5 of [6]. □

As for obstacle scattering we denote an incident plane wave with polarization $q \in \Omega$ and direction of incidence $d \in \Omega$ by $E^i_{pl}(\cdot, d, q)$, $H^i_{pl}(\cdot, d, q)$. The corresponding scattered fields and far field patterns are E^s_{pl}, H^s_{pl} and $E^\infty_{pl}, H^\infty_{pl}$, respectively. An incident electric dipole E^i_{edp}, H^i_{edp} produces the scattered field E^s_{edp}, H^s_{edp} with far field pattern $E^\infty_{edp}, H^\infty_{edp}$. We obtain electromagnetic reciprocity relations as follows.

THEOREM 2.4.3 **(Far field reciprocity relation.)** *The far field patterns for scattering of plane waves by an inhomogeneous medium $\mathcal{D}$ satisfy*

$$q \cdot E^\infty_{pl}(\hat{x}, d, p) \ = \ p \cdot E^\infty_{pl}(-d, -\hat{x}, q) \tag{2.4.7}$$

for $\hat{x}, d, p, q \in \Omega$.

Proof. See [6], Theorem 9.6. □

THEOREM 2.4.4 **(Mixed reciprocity relation.)** *The far field patterns for scattering of plane waves by an inhomogeneous medium $\mathcal{D}$ satisfy*

$$q \cdot E^\infty_{edp}(\hat{x}, z, p) \ = \ \gamma \, p \cdot E^s_{pl}(z, -\hat{x}, q) \tag{2.4.8}$$

for $\hat{x}, p, q \in \Omega$ and $z \in I\!R^3 \setminus \overline{D}$, where $\gamma = \frac{1}{4\pi}$.

Proof. The proof is literally the same than for Theorem 2.3.4. □

To detect the boundary of an inhomogeneous medium using incident singular sources, in acoustic scattering we needed a jump in one of the derivatives of n. For the treatment of electromagnetic inhomogeneous medium scattering by means of integral equations, to avoid the use of boundary integral terms we will have to restrict our presentation to a refractive index $n \in C^{1,\alpha}(I\!R^3)$. Thus the order of the derivatives of n, where jumps can occur, must be larger or equal to two. Here we will restrict ourselves to a jump in the second derivative of the refractive index at the boundary of the inhomogeneous medium. In addition, we need some smoothness conditions on n to obtain stability estimates.

DEFINITION 2.4.5 *Given positive constants R_e, r_0, a_0, C_0, β_e, C_n, c_{min}, c_{max}, $l = 2$ and $\alpha \in (0, 1]$ we define the class $\mathcal{C}_{elm}$ of electromagnetic inhomogeneous medium scatterers $\mathcal{D}$ by the following assumptions.*

1. *The scattering domain D is of class $\mathcal{A}(R_e, r_0, a_0, l, \alpha, C_0, \beta_e)$.*

2. *The refractive index n is in $C^{1,\alpha}(I\!R^3)$ and in $C^{2,\alpha}(\overline{D})$ with*

$$||n||_{C^{2,\alpha}(D)} \leq C_n. \tag{2.4.9}$$

3. *At the boundary ∂D the function $\chi = 1 - n$ has a jump in its second derivatives uniformly for $\mathcal{D} \in \mathcal{C}_m$ in the sense that*

$$0 < c_{min} \leq |\frac{\partial^2 \chi}{\partial \nu^2}(x)| \leq c_{max}, \quad x \in \overline{D}. \tag{2.4.10}$$

We need to study the mapping properties of the operator T_e and $(I - T_e)^{-1}$ in the spaces of continuous and of L^2-integrable functions.

THEOREM 2.4.6 *For the integral operator $(I - T_e)^{-1}$ the norms*

$$||(I - T_e)^{-1}||_{C(B)} \tag{2.4.11}$$

are bounded by some constant c uniformly for scatterers $\mathcal{D} \in \mathcal{C}_{elm}$.

Proof. The operator T_e as a mapping from $C(B)$ into $C(B)$ depends continuously on the refractive index $n \in C^1(B)$ and the same is true for the inverse of the operator $I - T_e$. Thus the uniform bound is obtained by a compactness argument using the imbedding from $C^{1,\alpha}(B)$ into $C^1(B)$ as in Theorem 2.2.6. $\square$

With the same compactness arguments we derive the following uniform bound for the scattering map. We leave the straightforward proof to the reader.

THEOREM 2.4.7 *For scattering of electromagnetic waves from an inhomogeneous medium the mapping of the incident electric $E^i \in C(D)$ onto the electric far field patterns $E^\infty \in C^1(\Omega)$ is bounded uniformly for $\mathcal{D} \in \mathcal{C}_{elm}$ by a constant c_∞.*

Please note that the singularity of the operator T_e is one order stronger than the singularity of the acoustic volume potential V. Thus the proofs of upper and lower estimates for E^s_{edp} will be more complicated than in the acoustic case.

The following theorem investigates the behavior of $E^s_{edp}(z, z, p)$, if z tends to the boundary of an electromagnetic inhomogeneous medium scatterer $\mathcal{D} \in \mathcal{C}_{elm}$. As a preparation we prove a lemma.

LEMMA 2.4.8 *For the kernel $\Psi(y, z)$ we assume*

$$|\Psi(y,z)| \leq c|y-z|^{-2}, \;\; y \in D, \; z \in B, \; y \neq z \tag{2.4.12}$$

with some constant c. Then we have

$$\Big| \int_D \operatorname{grad}_x \Phi(x,y)\Psi(y,z)\, dy \Big| \leq \frac{C}{|x-z|}, \;\; x, z \in B, \; x \neq z, \tag{2.4.13}$$

with a constant C uniformly for $\mathcal{D} \in \mathcal{C}_{elm}$.

Proof. We need to work out the proof only for the potential theoretic case $\kappa = 0$. We split the domain of integration into three parts. For $x \in I\!R^3 \setminus \{z\}$ we define $R := |x-z|/2$ and use $D_1 := D \cap B_R(x)$, $D_2 := D \cap B_R(z)$ and $D_3 := D \setminus (D_1 \cup D_2)$.

1. The integral over D_1 can be estimated by

$$\Big| \int_{D_1} \nabla_x \Phi(x,y)\Psi(y,z)\, dy \Big| \leq C\, R^{-2} \int_0^R dr \leq \frac{C}{R} \tag{2.4.14}$$

with some constant C.

2. To estimate the integral over D_2 we use the decomposition

$$\Big| \int_{D_2} \nabla_x \Phi(x,y)\Psi(y,z)\, dy \Big| \leq \Big| \int_{D_2} \Big(\nabla_x \Phi(x,y) - \nabla_x \Phi(x,z)\Big)\Psi(y,z)\, dy \Big| + \Big| \int_{D_2} \nabla_x \Phi(x,z)\Psi(y,z)\, dy \Big|. \tag{2.4.15}$$

With

$$|\nabla_x \Phi(x,y) - \nabla_x \Phi(x,z)| \le \frac{c}{|x-z|^3} |y-z|, \quad |y-z| \le R$$

for some constant c we estimate (2.4.15) by

$$\begin{aligned} \left| \int_{D_2} \nabla_x \Phi(x,y) \Psi(y,z) \, dy \right| &\le C \left[\frac{1}{R^3} \int_0^R r \, dr + \frac{1}{R^2} \int_0^R dr \right] \\ &\le \frac{C}{R}, \end{aligned} \tag{2.4.16}$$

where C is a generic constant; i.e. C may change from line to line.

3. The domain D_3 can be decomposed again into the subdomains $D_4 := D \setminus B_{3R}(z)$ and $D_5 := D \cap (B_{3R}(z) \setminus (B_R(x) \cup B_R(z))$. We use polar coordinates with origin z and third axis given by $x - z$ to calculate

$$|y-x|^2 = 4R^2 + r^2 - 4rR\cos(\theta).$$

Now the integral over D_4 is estimated by a constant times

$$\int_{3R}^{R_e} \frac{1}{4R^2 + r^2 - 4rR} \, dr = \left[-\frac{1}{r-2R} \right]_{3R}^{R_e}. \tag{2.4.17}$$

The integral over D_5 can be estimated by a constant times

$$\frac{1}{R^2} \int_R^{3R} dr = \frac{2}{R}. \tag{2.4.18}$$

The estimates (2.4.17) and (2.4.18) yield

$$\left| \int_{D_3} \nabla_x \Phi(x,y) \Psi(y,z) \, dy \right| \le \frac{C}{R} \tag{2.4.19}$$

with some constant C. Now from (2.4.14), (2.4.16) and (2.4.19) we obtain the estimate (2.4.13). □

Theorem 2.4.9 *Consider scattering of an electric dipole by an inhomogeneous medium scatterer $\mathcal{D} \in \mathcal{C}_{elm}$. There are constants $\tau, c > 0$ such that in the strip $0 < d(z,D) < \tau$ the scattered field $E^s_{edp}(z,z,\nu(z_0))$ satisfies the* lower estimate

$$\left| E^s_{edp}(z,z,\nu(z_0)) \right| \ge \frac{c}{|d(z,D)|}, \tag{2.4.20}$$

where $z_0 \in \partial D$ is defined by the unique representation $z = z_0 + h\nu(z_0)$. With a constant C we hve for all $z \in B \setminus \overline{D}$ the upper estimate

$$\left| E^s_{edp}(z,z,\nu(z_0)) \right| \le \frac{C}{|d(z,D)|}. \tag{2.4.21}$$

Proof. By Theorem 2.4.2 we have a representation of the scattered field of an incident electric dipole by

$$\begin{aligned} E^s_{edp}(\cdot,z,p) &= T_e(I-T_e)^{-1}E^i_{edp}(\cdot,z,p) \\ &= T_eE^i_{edp}(\cdot,z,p) \;+\; T_e(I-T_e)^{-1}T_eE^i_{edp}(\cdot,z,p) \\ &= T_eE^i_{edp}(\cdot,z,p) \;+\; T_eT_eE^i_{edp}(\cdot,z,p) \qquad (2.4.22) \\ &\quad + T_e(I-T_e)^{-1}T_eT_eE^i_{edp}(\cdot,z,p), \end{aligned}$$

where we twice inserted the identity operator $I=(I-T_e)+T_e$. We first give upper and lower estimates for the singularity of $T_eE^i_{edp}(\cdot,z,p)$, in a second step prove the boundedness of $T_eT_eE^i_{edp}(\cdot,z,p)$ and in a third step derive bounds for $T_e(I-T_e)^{-1}T_eT_eE^i_{edp}(\cdot,z,p)$ in $\mathbb{R}^3$.

1. We need to investigate the refractive index near the boundary. From Definition 2.4.5 as in (2.2.37) we obtain

$$\chi(y) \;=\; \frac{1}{2}\frac{\partial\chi}{\partial\nu}(z_0)\Big(r\cos(\theta)\Big)^2 \;+\; O(r^{2+\alpha}) \qquad (2.4.23)$$

with polar coordinates $r=|y-z_0|$, $\cos(\theta)=-\nu(z_0)\cdot(y-z_0)/r$ and

$$\begin{aligned} \operatorname{grad} n(y) \;=\; \frac{\partial\chi}{\partial\nu}(z_0)\; r \Bigg\{ &\cos(\theta)^2\cdot\begin{pmatrix}\sin(\theta)\cos(\varphi)\\ \sin(\theta)\sin(\varphi)\\ \cos(\theta)\end{pmatrix} \qquad (2.4.24) \\ &-\;\cos(\theta)\sin(\theta)\cdot\begin{pmatrix}\cos(\theta)\cos(\varphi)\\ \cos(\theta)\sin(\varphi)\\ -\sin(\theta)\end{pmatrix}\Bigg\} \;+\; O(r^{1+\alpha}) \end{aligned}$$

uniformly for scatterers $\mathcal{D}\in\mathcal{C}_{elm}$.

Consider the operator T_e as defined by (2.4.5). It consists of two terms, the first one of which is a single-layer potential and the second the gradient of a single-layer potential. From (2.4.23) we derive

$$\Big|\chi(y)E^i_{edp}(y,p,z)\Big| \;\le\; \frac{c}{|y-z|},\quad y\in D, \qquad (2.4.25)$$

with some constant c. Thus by standard arguments the potential

$$\int_D \Phi(x,y)\chi(y)E^i_{edp}(z,p,z)\;dy$$

is bounded uniformly for $z\in B\setminus\overline{D}$ and for all scatterers $\mathcal{D}\in\mathcal{C}_{elm}$. We now investigate the second term of T_e. We use Lemma 2.4.8 to estimate the integral

$$\int_D \operatorname{grad}_x\Phi(x,y)\;\underbrace{\frac{1}{n(y)}\operatorname{grad} n(y)\cdot E^i_{edp}(y,p,z)}_{=:\Psi(y,z)}\;dy \qquad (2.4.26)$$

and derive the *upper estimate*

$$\left|\Big(T_e E^i_{edp}(\cdot, z, p)\Big)(x)\right| \;\leq\; \frac{C}{|x-z|}, \quad p \in \Omega, \quad x, z \in B, \; x \neq z, \tag{2.4.27}$$

with some constant C uniformly for scatterers $\mathcal{D} \in \mathcal{C}_{elm}$.

We now calculate a lower bound for $T_e E^i_{edp}(z, z, \nu(z_0))$. The leading term of the potential is given by

$$\int_D \frac{z-y}{|z-y|^3} \, \frac{1}{n(y)} \, \mathrm{grad}\, n(y) \cdot \tag{2.4.28}$$
$$\left\{ \frac{3c\,\nu(z_0)\cdot(y-z)\,(y-z)}{|y-z|^5} - \frac{c\,\nu(z_0)}{|y-z|^3} \right\} dy, \quad z \in I\!R^3 \setminus \overline{D}.$$

We insert (2.4.24) into (2.4.28) and use an argumentation analogous to the derivation of (2.2.46) and (2.2.48) to derive

$$\begin{aligned} &T_e E^i_{edp}(z, z, \nu(z_0)) \\ &= C \int_0^{2\pi} \int_0^{\theta_0} \int_{d(z,D)/\cos(\theta)}^{R_0} \left\{ \cos(\theta)^2 \begin{pmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{pmatrix} - \cos(\theta)\sin(\theta) \begin{pmatrix} \cos(\theta)\cos(\varphi) \\ \cos(\theta)\sin(\varphi) \\ -\sin(\theta) \end{pmatrix} \right\} \\ &\cdot \left\{ 3\cos(\theta) \begin{pmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \begin{pmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{pmatrix} \frac{1}{r^2} \, dr \, d\theta \, d\varphi \;+\; O(1) \end{aligned}$$

with a sufficiently small fixed constant R_0 and θ_0 defined by $R_0 \cos(\theta_0) = d(z, D)$. We expand the products and first integrate over φ to obtain zero for the e_1 and e_2 components of the vector. Evaluating the integral over r we derive

$$\begin{aligned} &T_e E^i_{edp}(z, z, \nu(z_0)) \\ &\quad = \frac{C}{d(z,D)} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \int_0^{\theta_0} \cos^3(\theta) \left\{ 3\cos^2(\theta) - 1 \right\} d\theta \;+\; O(1) \\ &\quad = \frac{C}{d(z,D)} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \;+\; O(1), \quad z \in I\!R^3 \setminus \overline{D}, \end{aligned}$$

with a generic constant C. This yields

$$\left| T_e E^i_{edp}(z, z, \nu(z_0)) \right| \;\geq\; \frac{C}{|d(z,D)|}, \quad z \in I\!R^3 \setminus \overline{D}, \tag{2.4.29}$$

with some constant C uniformly for scatterers $\mathcal{D} \in \mathcal{C}_{elm}$.

2. We need to investigate $T_e T_e E^i_{edp}(\cdot, z, p)$. From (2.4.27) we derive

$$\left| \Big(T_e T_e E^i_{edp}(\cdot, z, p) \Big)(x) \right| \quad \leq \quad C \int_D \frac{1}{|x-y|^2} \frac{\text{grad}\, n(y)}{|y-z|} \, dy \; + \; O(1). \qquad (2.4.30)$$

Since the integral in (2.4.30) can be estimated by a constant, the right-hand side is bounded. These bounds hold uniformly for $\mathcal{D} \in \mathcal{C}_{elm}$.

3. To complete the proof we collect the estimates (2.4.27), (2.4.29) and (2.4.30), use the decomposition (2.4.22) and the uniform bounds for $(I - T_e)^{-1}$ in $C(B)$ as given by Theorem 2.4.6 to derive the upper and lower bounds (2.4.20) and (2.4.21).

□

3 Uniqueness and stability in inverse scattering.

Uniqueness theorems usually investigate the amount of data necessary to determine scattering objects uniquely. Clearly, if we do not have uniqueness, we cannot expect to obtain stable numerical algorithm for the computation of the scattering objects. Thus uniqueness is a question of practical importance.

In this chapter we first investigate the question of uniqueness of the support of obstacle scatterers, given the far field patterns for incident plane waves at a fixed wave number κ. We will show that the shape D of the scatterer $\mathcal{D}$ is uniquely determined by the far field patterns $u^{\infty}(\cdot, d)$ of the scattered fields for *all incident plane waves* $u^i(\cdot, d)$ with directions of incidence $d \in \Omega$. Uniqueness for penetrable scatterers will be obtained as a consequence of stability, which is the second problem of this section.

The question of *stability* leads right into the center of the difficulties of inverse problems. How do errors in the measurements affect the reconstructions? We already indicated that inverse scattering problems are ill-posed, because a radiating solution to the Helmholtz equation in the exterior of a ball does not depend continuously on its far field pattern. We also indicated that with appropriate assumptions on the scatterers under consideration stability can be restored and stability estimates can be derived.

In this section we will derive stability estimates for the reconstruction of the domain D of a scatterer $\mathcal{D}$ from the far field patterns $u^{\infty}(\hat{x}, d)$ for $\hat{x}, d$ in Ω, where for measurements of the far field patterns we will use the practically relevant L^2-norm. A stability estimate consists of a function $F : \mathbb{R}^+ \to \mathbb{R}^+$ with the property

$$F(\delta) \to 0, \quad \delta \to 0, \tag{3.0.1}$$

such that the Hausdorff distance $d(D_1, D_2)$ of two scatterers can be estimated by

$$d(D_1, D_2) \ \leq \ F\Big(||u_1^{\infty}(\cdot,\cdot) - u_2^{\infty}(\cdot,\cdot)||_{L^2(\Omega\times\Omega)} \Big). \tag{3.0.2}$$

For the convex hulls $\mathcal{H}(D_1)$ and $\mathcal{H}(D_2)$ of impenetrable acoustic or arbitrary electromagnetic scatterers $\mathcal{D}_1$ and $\mathcal{D}_2$ we will derive a logarithmic estimate, i.e.

$$F(\delta) = \frac{C}{|\ln(\delta)|^c}$$

with constants $C > 0$ and $0 < c < 1$ uniformly for the given classes of scatterers.

Clearly stability inplies uniqueness and thus as a consequence of stability we obtain *uniqueness* statements. In particular, this will yield new uniqueness results for inhomogeneous medium scatterers.

3.1 Acoustic scattering.

The following uniqueness theorem for the reconstruction of domains in obstacle scattering was first proven in 1993 by Kirsch and Kress [29], simplifying techniques of Isakav [22]. It will be the starting point for our further considerations, and we will give an even simpler proof using mixed reciprocity relations.

THEOREM 3.1.1 *Let $\mathcal{D}_1, \mathcal{D}_2$ be sound-soft or sound-hard scatterers. If the far field patterns $u_1^\infty(\hat{x}, d)$ and $u_2^\infty(\hat{x}, d)$ for scattering of plane waves coincide for all $\hat{x}, d \in \Omega$, then $D_1 = D_2$.*

Proof. Let D_e be the unbounded component of $I\!R^m \setminus (\overline{D_1 \cup D_2})$. From

$$u_1^\infty(\hat{x}, d) = u_1^\infty(\hat{x}, d), \ \ \hat{x}, d \in \Omega,$$

and the Rellich Lemma we obtain

$$u_1^s(z, d) = u_1^s(z, d), \ \ z \in D_e, \ d \in \Omega.$$

We use the mixed reciprocity relation (2.1.4) on both sides to derive for the far field patterns of incident point-sources the equality

$$\Phi_1^\infty(d, z) = \Phi_2^\infty(d, z), \ \ z \in D_e, \ d \in \Omega.$$

Again we use the Rellich lemma to get

$$\Phi_1^s(x, z) = \Phi_2^s(x, z), \ \ x, z \in D_e. \tag{3.1.1}$$

From (3.1.1) and Theorem 2.1.15 we derive $D_1 = D_2$ in the following way. We assume that $\partial D_1 \setminus \overline{D_2} \neq \emptyset$ and $z_0 \in \partial D_1 \setminus \overline{D_2}$. Then

$$\infty > \Phi_2(z_0, z_0) = \lim_{z \to z_0, z \in D_e} \Phi_2(z, z) = \lim_{z \to z_0, z \in D_e} \Phi_1(z, z) = \infty$$

and we obtain a contradiction. In the same way we treat the case $z_0 \in \partial D_2 \setminus \overline{D_1}$. Thus we obtain $D_1 = D_2$ and the proof is complete. □

Since most further results will hold both for scattering from an impenetrable and penetrable scatterers, we define the class

$$\mathcal{C} := \mathcal{C}_{obst} \cup \mathcal{C}_m. \tag{3.1.2}$$

We will now focus on the stability question and develop techniques to prove stability for the reconstruction of the domain D of both impenetrable and penetrable scatterers. Since stability implies uniqueness, we also will obtain *uniqueness* statements for D. All further results will use the following two steps.

As the *first step*, we consider approximations of a multipole by a continuous superposition of plane waves, i.e. by a *Herglotz wave function*

$$(Hg)(x) := \int_{\Omega} e^{i\kappa x\cdot d} g(d)\, ds(d), \quad x \in I\!R^m. \tag{3.1.3}$$

To obtain these approximations in a uniform way we use the exterior cone condition as follows. With the help of the cone (1.2.10) we define the domain

$$G_{z,p,\rho} := B_{2R_e}(z - \frac{p\rho}{\cos(\beta_0)}) \setminus \mathrm{co}(z - \frac{p\rho}{\cos(\beta_0)}, p, \beta_0) \tag{3.1.4}$$

for $z \in I\!R^m, p \in \Omega$ and $\rho > 0$ as shown in Figure 2. Here, since the parameter β_0 is kept constant throughout this work, we do not explicitly note the dependence of G on β_0. From the exterior cone condition for $D \subset B_{R_e}(0)$ for each $z \in B \setminus D_\rho$ we obtain a vector $p \in \Omega$, such that $D \subset G_{z,p,\rho}$.

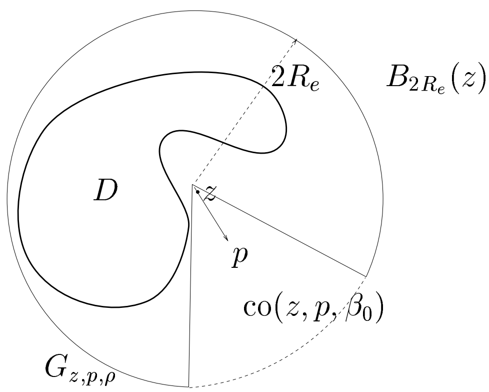

Figure 2

We first consider the operator H from $L^2(\Omega)$ into $L^2(\partial G)$, where G denotes a appropriately chosen domain G with $G \supset \overline{G_{0,p,\rho}}$ and $0 \notin \overline{G}$.

LEMMA 3.1.2 *It is possible to choose a domain $G \supset \overline{G_{0,p,\rho}}$ with $0 \notin \overline{G}$ and boundary ∂G of class C^2, such that the homogeneous interior Dirichlet problem*

$$-\triangle u = \kappa^2 u \text{ in } G, \quad u = 0 \text{ on } \partial G \tag{3.1.5}$$

has only the trivial solution $u = 0$. In this case the operator H has dense range in $L^2(\partial G)$.

Proof. Due to Theorem 4.7 of [44] for the l-th eigenvalue λ_l of the problem (3.1.5), where the eigenvalues are ordered according to their magnitude and multiplicity

$$\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_l \leq . .., $$

we have the *monotonicity property*

$$G_1 \overset{\neq}{\subset} G_2 \;\; \Rightarrow \;\; \lambda_{1,l} > \lambda_{2,l}, \;\; l \in I\!N. \tag{3.1.6}$$

Thus, if for a domain

$$G_1 \supset G_{0,p,\rho}, \; 0 \notin \overline{G_1}$$

and $l \in I\!N$ we have $\lambda_{1,l} = \kappa^2$, we can choose a domain

$$G_2 \supset G_1, \; 0 \notin \overline{G_2},$$

such that $\lambda_{2,l'} \neq \kappa^2$ for all $l' \in I\!N$. Then κ^2 is not an eigenvalue for the domain G_2 and every solution u of (3.1.5) with $G := G_2$ must vanish identically.

To show denseness for the range of H in $L^2(\partial G)$ we prove injectivity of the adjoint H^* of the operator H. The adjoint H^* of H is given by

$$(H^*\varphi)(\hat{x}) = \int_{\partial G} e^{-i\kappa\hat{x}\cdot y}\varphi(y)\, ds(y), \;\; \hat{x} \in \Omega.$$

The function $H^*\varphi$ is the far field pattern of the single-layer potential

$$(S\varphi)(x) := \int_{\partial G} \Phi(x,y)\varphi(y)\, ds(y), \;\; x \in I\!R^m.$$

We assume $H^*\varphi = 0$ on Ω for $\varphi \in L^2(\partial G)$. Then Rellich's Lemma yields $(S\varphi)(x) = 0$ for $x \in I\!R^m \setminus \overline{G}$. Now from the jump-relations for L^2-densities Theorem 1.2.4 we get $(I - K^*)\varphi = 0$. We can apply the Fredholm alternative 1.2.6 first in the dual system

$$\Big\langle C(\partial D), L^2(\partial D)\Big\rangle, \;\; \langle \varphi, \psi \rangle := \int_D \varphi\overline{\psi}\, dy,$$

and then in the dual system

$$\Big\langle L^2(\partial D), L^2(\partial D)\Big\rangle, \;\; \langle \varphi, \psi \rangle := \int_D \varphi\overline{\psi}\, dy,$$

to conclude that the null spaces of $I - K^*$ in $L^2(\partial G)$ and $C(\partial G)$ have the same finite dimension. Since $C(D)$ is a subset of $L^2(D)$, the null spaces coincide and φ is in $C(\partial G)$. Now continuity of the single-layer potential with continuous density implies that $S\varphi$ solves the homogeneous Dirichlet problem in G. From the first part of the lemma we obtain $S\varphi = 0$ on G. Now the jump relations of Theorem 1.2.4 yield $\varphi = 0$ and thus the injectivity of H^*. $\square$

We now complete the first step and show, that in the space $C^s(G_{0,p,\rho})$ the function $\Phi_{\mu,q}(\cdot,0)$ can be approximated by a Herglotz wave function.

LEMMA 3.1.3 *Given $\mu, s \in I\!N_0$, $\rho > 0$ and $\tau > 0$ there is a finite set*

$$\mathcal{G} = \mathcal{G}(\tau, \rho, \mu, s, \beta_0)$$

of densities g in $L^2(\Omega)$, such that for each $p, q \in \Omega$ there is a density $g \in \mathcal{G}$ with

$$\left|\left|\Phi_{\mu,q}(\cdot,0) - Hg\right|\right|_{C^s(G_{0,p,\rho})} \leq \tau. \tag{3.1.7}$$

Proof. According to Lemma 3.1.2, given $\epsilon > 0$, we can find $g \in L^2(\Omega)$ such that

$$\left|\left|\Phi_{\mu,q}(\cdot,0) - Hg\right|\right|_{L^2(\partial G)} \leq \epsilon. \tag{3.1.8}$$

Let g be the minimum norm solution of (3.1.8), which is unique according to Theorem 1.2.7. We note that both $\Phi_{\mu,q}$ and Hg solve an interior Dirichlet problem for the Helmholtz equation in the domain G and can be represented as the combined acoustic double- and single-layer potential (2.1.13). Differentiating under the integral sign and using the Cauchy-Schwarz inequality we observe that on compact subsets of G the solution of the interior Dirichlet problem depends continuously on the boundary values in $L^2(\partial G)$; i.e. from (3.1.8) with ϵ replaced by ϵ/c with some constant c we obtain (3.1.7).

So far, the density g in (3.1.8) depends on the parameters $p, q \in \Omega$. We observe that for fixed g the norm

$$\left|\left|\Phi_{\mu,q}(\cdot,0) - Hg\right|\right|_{C^s(G_{0,p,\rho})}$$

depends continuously on p and q. Thus given $p, q \in \Omega$ and $g \in L^2(\Omega)$ such that

$$\left|\left|\Phi_{\mu,q}(\cdot,0) - Hg\right|\right|_{C^s(G_{0,p,\rho})} \leq \frac{\tau}{2} \tag{3.1.9}$$

is satisfied, there is a neighborhood U of p and a neighborhood V of q, such that

$$\left|\left|\Phi_{\mu,q'}(\cdot,0) - Hg\right|\right|_{C^s(G_{0,p',\rho})} \leq \tau \tag{3.1.10}$$

is satisfied for all $q' \in U$ and all $p' \in V$. The compact set Ω is covered by a finite number of such domains U and V; i.e. there is a finite set $\mathcal{G}$ of densities g, such that for all $p, q \in \Omega$ the estimate (3.1.7) is satisfied with a density $g \in \mathcal{G}$. □

In the *second step* estimates for the difference of the fields $\Phi^s_{1,\mu,q}$ and $\Phi^s_{2,\mu,q}$ for scattering of a multipole from two scatterers $\mathcal{D}_1$ and $\mathcal{D}_2$ will be used to estimate the Hausdorff difference $d(D_1, D_2)$ of the domains D_1 and D_2.

We need to consider six different situations according to the physical properties of the scatterers $\mathcal{D}_1$ and $\mathcal{D}_2$ in two or three dimensions. For each situation we need a corresponding choice of the parameter μ.

Situation	Properties of $\mathcal{D}_1$ and $\mathcal{D}_2$	μ
S1	$\mathcal{D}_1, \mathcal{D}_2$ impenetr. scatterers in $\mathbb{R}^2$	0
S2	$\mathcal{D}_1, \mathcal{D}_2$ impenetr. scatterers in $\mathbb{R}^3$	0
S3	$\mathcal{D}_2$ inhom. medium scatterer and $\mathcal{D}_1$ impenetr. scatterer or vice versa in $\mathbb{R}^2$	$\mu_0 + 2$
S4	$\mathcal{D}_2$ inhom. medium scatterer and $\mathcal{D}_1$ impenetr. scatterer or vice versa in $\mathbb{R}^3$	$\mu_0 + 1$
S5	$\mathcal{D}_1, \mathcal{D}_2$ inhom. medium scatterers in $\mathbb{R}^2$	$\mu_0 + 2$
S6	$\mathcal{D}_1, \mathcal{D}_2$ inhom. medium scatterers in $\mathbb{R}^3$.	$\mu_0 + 1$

(3.1.11)

For technical reasons we introduce the set

$$U(D_1, D_2, \rho, \beta_0) \;:=\; \{z \in B \setminus (D_{1,\rho} \cup D_{2,\rho}), \quad \exists p \in \Omega \text{ such that } \mathrm{co}(z, p, \beta_0) \subset \mathbb{R}^m \setminus (D_{1,\rho} \cup D_{2,\rho})\}\,. \tag{3.1.12}$$

Lemma 3.1.4 *We consider scattering of acoustic waves by two scatterers $\mathcal{D}_1, \mathcal{D}_2$ in $\mathcal{C}$ and choose $\mu \in \mathbb{N}_0$ according to the table above . Assume that with parameters $\rho, \sigma > 0$ the scattered fields $\Phi^s_{1,\mu,q}$ and $\Phi^s_{2,\mu,q}$ satisfy*

$$\left|\Phi^s_{1,\mu,q}(z, z) - \Phi^s_{2,\mu,q}(z, z)\right| \;\leq\; \sigma \tag{3.1.13}$$

for all $q \in \Omega$ and $z \in U(D_1, D_2, \rho, \beta_0)$. Then we conclude

$$d(D_1, D_2) \;\leq\; F_1(\rho, \sigma) \tag{3.1.14}$$

where the function F_1 is defined according to the situations $S1$ to $S6$ by

$$F_1(\rho, \sigma) \;:=\; \begin{cases} \rho + Bb^\sigma \rho^s & \textit{S1, S5 and S6}\,, \\ \rho + \frac{C\rho}{c - \sigma\rho} & \textit{S2}\,, \\ \rho + \left(\frac{c}{Cb}|\ln \rho| - \frac{\sigma}{Cb}\right)^{-1/(\mu_0 + m + 2)} & \textit{S3 and S4} \end{cases} \tag{3.1.15}$$

with constants B, b, s, c and C uniformly for $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{C}$.

Proof. We split the proof into two parts.
1. We consider the scattered fields for a point $z \in U(D_1, D_2, \rho, \beta_0)$. From (2.1.48), (2.1.49) and (3.1.13) we obtain for situation S1 of two impenetrable scatterers in the two dimensions the estimate

$$\begin{aligned} C \left| \ln d(z, D_1) \right| + E &\geq \left| \Phi^s_{1,\mu,\nu(z_0)}(z, z) \right| \\ &\geq \left| \Phi^s_{2,\mu,\nu(z_0)}(z, z) \right| - \sigma \\ &\geq c \left| \ln d(z, D_2) \right| - \sigma, \end{aligned} \tag{3.1.16}$$

which can be transformed into

$$d(z, D_1) \leq B\, b^\sigma\, d(z, D_2)^s \tag{3.1.17}$$

with constants $B = e^{E/C}, b = e^{1/C}$ and $1 > s = c/C > 0$. In the same way for scattering by impenetrable scatterers in three dimensions (situation S2) we obtain from (2.1.50), (2.1.51) and (3.1.13)

$$\frac{C}{d(z, D_1)} \geq \frac{c}{d(z, D_2)} - \sigma \tag{3.1.18}$$

and transform it into

$$d(z, D_1) \leq \frac{C d(z, D_2)}{c - \sigma d(z, D_2)}. \tag{3.1.19}$$

Consider an inhomogeneous medium scatterer $\mathcal{D}_2$ and an impenetrable scatterer $\mathcal{D}_1$; i.e. the situations S3 and S4. From the definition of the multipoles and the choice of μ we derive

$$\left|\left| \Phi_{\mu,q}(\cdot, z) \right|\right|_{C^1(\overline{D_1})} \leq C d(z, D_1)^{-\mu_0 - 3} \tag{3.1.20}$$

with some constant C. The scattering map $u^i \mapsto u^s$ from $C^1(\overline{D_1})$ into $C(B \backslash D_{1,\rho})$is bounded by b/ρ^{m-1} with some constant b uniformly for $\mathcal{D} \in \mathcal{C}_{obst}$. Thus we obtain

$$\left| \Phi^s_{\mu,q}(z, z) \right| \leq C\, b\, |d(z, D_1)|^{-\mu_0 - m - 2}. \tag{3.1.21}$$

Now from the lower estimate (2.2.55) we derive

$$C\, b\, |d(z, D_1)|^{-\mu_0 - m - 2} \geq c\, | \ln d(z, D_2)| - \sigma \tag{3.1.22}$$

with constants $c, C, b > 0$, which yields

$$d(z, D_1) \leq \Big(\frac{c}{Cb}\, | \ln d(z, D_2)| - \frac{\sigma}{Cb} \Big)^{-\frac{1}{\mu_0 + m + 2}} \tag{3.1.23}$$

If $\mathcal{D}_2$ is an impenetrable scatterer and $\mathcal{D}_1$ an inhomogeneous medium, the estimates can be obtained as in (3.1.17) and (3.1.19). Since the functions are dominated by (3.1.23), a bound is again given by (3.1.23). For two inhomogeneous medium scatterers $\mathcal{D}_1$ and $\mathcal{D}_2$ we derive from (2.2.55), (2.2.56) and (3.1.13) the bound (3.1.17)

2. We now come to the second step of the proof. According to the cases S1 to S6 we choose $\epsilon := F_1(\rho, \sigma)$ with F_1 defined in (3.1.15).

First, we consider a point $z_0 \in \partial D_2$, such that for $z' := z_0 + \rho\nu(z_0)$ we have $z' \in U(D_1, D_2, \rho, \beta_0)$. From the triangle inequality, $\rho = d(z', D_2)$ and the first part of the proof we obtain

$$\begin{aligned} d(z_0, D_1) \ &\leq \ d(z_0, z') \ + \ d(z', D_1) \\ &\leq \ F_1(\rho, \sigma) \ = \ \epsilon. \end{aligned} \tag{3.1.24}$$

Second, we investigate an arbitrary cone

$$\mathrm{co}(z, p, \beta_0) \subset I\!R^m \setminus \overline{D_{1,\epsilon}}. \tag{3.1.25}$$

To show

$$\mathrm{co}(z, p, \beta_0) \cap D_2 = \emptyset \tag{3.1.26}$$

by contradiction, we assume that

$$\mathrm{co}(z, p, \beta_0) \cap D_2 \neq \emptyset. \tag{3.1.27}$$

Then we have

$$r_1 := \inf \Big\{ \ r > 0, \ \Big(D_2 \cap \mathrm{co}(z, p, \beta_0)\Big) \ \subset \ \Big(B_r(x) \cap \mathrm{co}(z, p, \beta_e)\Big) \ \Big\} \ > 0,$$

and we can find a point $z_1 \in \partial D_2 \cap \partial B_{r_1}(x) \cap \mathrm{co}(z, p, \beta)$. On the line

$$L = \{z_1 + tp, t \in (0, \infty)\}$$

there is a point z' with $d(z', D_2) = \rho$. Then we have $z' = z_0 + \rho\nu(z_0)$ for some point $z_0 \in \partial D_2$ and $z' \in U(D_1, D_2, \rho, \beta_0)$. Now, the estimate (3.1.24) yields $d(z_0, D_1) \leq \epsilon$. But from (3.1.25) we obtain $\epsilon < d(z_0, D_1)$ and thus a contradiction i.e. the assumption (3.1.27) is wrong and (3.1.26) is shown.

Finally, we note that for $D_1 \in \mathcal{C}$ and ϵ sufficiently small the open exterior of $D_{1,\epsilon}$ can be covered by cones $\mathrm{co}(z, p, \beta_0)$ with $z \in I\!R^m \setminus D_{1,\epsilon}$ and $p \in \Omega$; i.e. from (3.1.26) we obtain $D_2 \subset D_{1,\epsilon}$. Since we can go through all arguments with D_1 and D_2 exchanged, we obtain (3.1.14). $\square$

We will define and study an operator Q, which is built to approximate the *scattered fields of multipoles* by a superposition of the far field patterns of the scattered fields of *plane waves*. To this end we need some further preparations.

a) Consider scattering of an incident field given by a superposition Hg of plane waves (3.1.3) with density $g \in L^2(\Omega)$. Using the scattered field $u^s(\cdot, d)$ and the far field pattern $u^\infty(\cdot, d)$ for the scattering of plane waves, by linearity and boundedness of the scattering maps $u^i \mapsto u^s$ and $u^i \mapsto u^\infty$ the scattered field and far field pattern can be expressed by

$$(H^s g)(x) = \int_\Omega u^s(x, d) g(d)\, ds(d), \quad x \in I\!R^m \setminus D \tag{3.1.28}$$

and

$$(H^\infty g)(x) = \int_\Omega u^\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \Omega, \tag{3.1.29}$$

respectively.

b) Lemma 3.1.3 states the possibility of approximations of multipoles with source-point $x = 0$ on a domain $G_{0,p,\rho}$ by a superposition of plane waves. To approximate a multipole with source-point $x \neq 0$ on $G_{x,p,\rho}$ we consider translations of both the multipole and the Herglotz wave function.

Translations of a Herglotz wave function Hg_0 with a translation vector x can be performed by multiplication of the density $g_0(d)$ with the complex factor $e^{-i\kappa x \cdot d}$. Clearly, the density

$$g(x, d) := e^{-i\kappa x \cdot d} g_0(d) \tag{3.1.30}$$

of the translated Herglotz wave function has the same norm as the original density g_0. From (3.1.7) by translation we derive

$$\Big|\Big|\Phi_{\mu,q}(\cdot, x) - Hg(x, \cdot)\Big|\Big|_{C^s(G_{x,p,\rho})} \leq \tau. \tag{3.1.31}$$

c) The density g defined by (3.1.30) and Lemma 3.1.3 is a function

$$g = g(x, d, p, q, \tau, \rho, \mu, s, \beta_0).$$

We will need the density g with two different sets of values for x, d, μ and τ and for vectors p and q depending on x. We use the abbreviation

$$g_\tau(x, d) := g\Big(x, d, p(x), q(x), \tau, \rho, \mu, s, \beta_0\Big) \tag{3.1.32}$$

d) To estimate the error of the approximations we need the bound

$$b_{\tau,\rho,\mu,s,\beta_0} := \max\Big\{||g||_{L^2(\Omega)}, \quad g \in \mathcal{G}(\tau, \rho, \mu, s, \beta_0)\Big\} \tag{3.1.33}$$

with the set $\mathcal{G}(\tau, \rho, \mu, s, \beta_0)$ of densities given by Lemma 3.1.3. We use the abbreviation

$$b_\tau = b_{\tau,\rho,\mu,s,\beta_0}$$

or

$$b_{\tau,\rho} = b_{\tau,\rho,\mu,s,\beta_0},$$

if the dependence on ρ, μ, s, β_0 or on μ, s, β_0 is not needed. For the set $\mathcal{G}$ we will write

$$\mathcal{G}_\tau := \mathcal{G}(\tau, \rho, \mu, s, \beta_0)$$

e) For a scatterer $\mathcal{D} \in \mathcal{C}$ and a point $x \in B \setminus D_\rho$, according to the boundedness of D and the exterior cone condition we can find $p \in \Omega$ with

$$D \subset G_{x,p,\rho}. \tag{3.1.34}$$

Thus it is possible to define a function $p : B \to \Omega$, such that with $p = p(x)$ the condition (3.1.34) is satisfied for all $x \in B \setminus D_\rho$. The condition (3.1.34) will be needed to estimate the approximation of $\Phi^s_{\mu,q}(x,z)$ by Qu^∞. Later we will work with two different domains D_1 and D_2. Then for a fixed functions p only at points $x \in B$, for which (3.1.34) is satisfied for both D_1 and D_2, the approximation properties of the corresponding operator Q will be valid for both scattered fields $\Phi^s_{1,\mu,q}$ and $\Phi^s_{2,\mu,q}$.

We are now ready to formulate the definition of the operator Q and to investigate its approximation properties.

Definition 3.1.5 *Given a set of parameters $\mu \in I\!N_0$, $s = 1$, $\rho, \tau > 0$ and functions $p, q : B \to \Omega$ with the help of*

$$g_\tau(x,d) \ = \ g(x, d, p(x), q(x), \tau, \rho, 0, s, \beta_0)$$

and

$$g_\eta(z,\tilde{d}) = g(z, \tilde{d}, p(z), q(z), \eta, \rho, \mu, s, \beta_0)$$

we define the operator

$$Q : L^2(\Omega \times \Omega) \to L^\infty(B)$$

by

$$(Qw)(x,z) \ := \ \frac{1}{\gamma_m} \int_\Omega \int_\Omega \Big\{ g_\tau(x,\tilde{d}) g_\eta(z,d) \Big\} \ w(-d,\tilde{d}) \ ds(d) ds(\tilde{d}), \ \ x, z \in B. \tag{3.1.35}$$

Theorem 3.1.6 *Consider scattering by a sound-soft, sound-hard or inhomogeneous medium scatterer $\mathcal{D} \in \mathcal{C}$. The error for the approximation of $\Phi^s_{\mu,q}(x,z)$ by Qu^∞ is estimated by*

$$\left|\Phi^s_{\mu,q}(x,z) - (Qu^\infty)(x,z)\right| \;\leq\; c\,\frac{\eta}{\rho^{(m-1)}} + C\, b_{\eta,\rho,\mu,s,\beta_0}\,\tau \tag{3.1.36}$$

for all $x, z \in B \setminus D_\rho$, for which (3.1.34) is satisfied for $p = p(x)$ or $p = p(z)$, respectively. The constants c and C depend on μ, but not on $\mathcal{D} \in \mathcal{C}$.

Remark. For an appropriate choice of τ and η the error in (3.1.36) can be made arbitrary small. Given $\epsilon > 0$ we first choose $\eta = \epsilon\rho^{(m-1)}/(2c)$ and then $\tau = \epsilon/(2Cb_\eta)$ to obtain

$$\left|\Phi^s_{\mu,q}(x,z) - (Qu^\infty)(x,z)\right| \;\leq\; \epsilon, \;\; x, z \in B \setminus D_\rho. \tag{3.1.37}$$

Proof. First, for $g_\eta(z,\cdot)$ by definition we have

$$\left\|\Phi_{\mu,q}(\cdot,z) - Hg_\eta(z,\cdot)\right\|_{C^1(D)} \;\leq\; \eta. \tag{3.1.38}$$

The scattering map $u^i \mapsto u^s$ is bounded from $C^1(\overline{D})$ into $C(B \setminus D_\rho)$. We use the solutions of the scattering problems as given in Section 2 and estimate the scattered field u^s with the help of the Cauchy-Schwarz inequality to obtain

$$\left\|\Phi^s_{\mu,q}(\cdot,z) - H^s g_\eta(z,\cdot)\right\|_{C(B\setminus D_\rho)} \;\leq\; c\,\frac{\eta}{\rho^{m-1}} \tag{3.1.39}$$

for all $z \in B \setminus D_\rho$, for which (3.1.34) is satisfied, with some constant c not depending on ρ or $\mathcal{D} \in \mathcal{C}$. In the same way for $g_\tau(x,\cdot)$ we derive

$$\left\|\Phi^\infty(\cdot,x) - H^\infty g_\tau(x,\cdot)\right\|_{C(\Omega)} \;\leq\; c\,\tau \tag{3.1.40}$$

for all $x \in B \setminus D_\rho$, for which (3.1.34) is satisfied, with some constant c uniformly for $\mathcal{D} \in \mathcal{C}$. We use the mixed reciprocity relations (2.1.4) and (2.2.4) to transform (3.1.40) into

$$\left|u^s(x,d) - \frac{1}{\gamma_m}\Big(H^\infty g_\tau(x,\cdot)\Big)(-d)\right| \;\leq\; \frac{c}{\gamma_m}\,\tau \tag{3.1.41}$$

for all $d \in \Omega$. Using the Cauchy-Schwarz inequality from (3.1.41) we obtain the estimate

$$\begin{aligned}
&\left|\Big(H^s g_\eta(z,\cdot)\Big)(x) - \Big(Qu^\infty\Big)(x,z)\right| \\
&\quad = \left|\int_\Omega \left(u^s(x,d) - \frac{1}{\gamma_m}\int_\Omega u^\infty(-d,\tilde{d})\; g_\tau(x,\tilde{d})\; ds(\tilde{d})\right) g_\eta(z,d)\; ds(d)\right| \\
&\quad \leq\; C\left\|g_\eta(z,\cdot)\right\|_{L^2(\Omega)}\tau
\end{aligned} \tag{3.1.42}$$

with some constant C. Now from (3.1.42) and (3.1.39) we derive

$$\begin{aligned} \Big|\Phi^s_{\mu,q}(x,z) - \Big(Qu^\infty\Big)(x,z)\Big| &\leq \Big|\Phi^s(x,z) - \Big(H^s g_\eta(z,\cdot)\Big)(x)\Big| \\ &\quad + \Big|\Big(H^s g_\eta(z,\cdot)\Big)(x) \ - \ \Big(Qu^\infty\Big)(x,z)\Big| \\ &\leq c\,\frac{\eta}{\rho^{m-1}} + C\,\Big|\Big|g_\eta(z,\cdot)\Big|\Big|_{L^2(\Omega)}\,\tau \qquad (3.1.43) \end{aligned}$$

with constants $c, C > 0$, which yields (3.1.36). □

The operator Q is a bounded operator from $L^2(\Omega \times \Omega)$ into $L^\infty(B)$. We will exploit the behavior of the bounds to derive stability estimates for the reconstruction of the scattered field $\Phi^s_{\mu,q}$ and use these estimates to obtain stability estimates for the reconstruction of the shape D of a scatterer $\mathcal{D}$.

Lemma 3.1.7 *Let $u_1^\infty(\hat{x},d)$ and $u_2^\infty(\hat{x},d)$, $\hat{x},d \in \Omega$, be the far field patterns for scattering of acoustic plane waves two from scatterers $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{C}$. If for some parameter $\delta > 0$ the far field patterns satisfy*

$$\Big|\Big|u_1^\infty - u_2^\infty\Big|\Big|_{L^2(\Omega\times\Omega)} \leq \delta, \qquad (3.1.44)$$

then the fields $\Phi^s_{1,\mu,q}$ and $\Phi^s_{2,\mu,q}$, $\mu \in I\!N_0$, for scattering of multipoles by $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively, satisfy the estimate

$$\Big|\Phi^s_{1,\mu,q}(x,z) - \Phi^s_{2,\mu,q}(x,z)\Big| \leq 2c\frac{\eta}{\rho^{m-1}} + 2C\,b_\eta\,\tau + \frac{1}{\gamma_m}b_\eta b_\tau\,\delta \qquad (3.1.45)$$

for all $q \in \Omega$, $\rho, \tau, \eta > 0$ and all points $x, z \in U$ defined by (3.1.12), where the constants c, C are given by Theorem 3.1.6. .

Proof. We define the function $p : B \to \Omega$ such that the condition (3.1.34) for both scatterers $\mathcal{D}_1$ and $\mathcal{D}_2$ is satisfied in $x \in U$. Then for the operator Q given by (3.1.35) we use (3.1.36) for each scatterer $\mathcal{D}_1$ and $\mathcal{D}_2$ and the Cauchy-Schwarz inequality applied to $Q(u^\infty_{1,\mu,q} - u^\infty_{2,\mu,q})$ to derive from

$$\begin{aligned} \Big|\Phi^s_{1,\mu,q}(x,z) - \Phi^s_{2,\mu,q}(x,z)\Big| &\leq \Big|\Phi^s_{1,\mu,q}(x,z) - \Big(Qu_1^\infty\Big)(x,z)\Big| \qquad (3.1.46) \\ &\quad \Big|Q\Big(u_1^\infty - u_2^\infty\Big)(x,z)\Big| + \Big|\Big(Qu_1^\infty\Big)(x,z) - \Phi^s_{2,\mu,q}(x,z)\Big| \end{aligned}$$

the estimate (3.1.45). □

It is now possible to use Lemma 3.1.4 to obtain estimates for the Hausdorff distance $d(D_1, D_2)$ between the two scatterers $\mathcal{D}_1$ and $\mathcal{D}_2$ and derive stability estimates for the reconstruction of the shape of scatterers. For positive parameter ρ, τ, η and δ let the function F_2 be given by

$$F_2(\rho, \tau, \eta, \delta) := 2c\frac{\eta}{\rho^{m-1}} + 2C\, b_{\eta,\rho}\, \tau + \frac{1}{\gamma_m} b_{\eta,\rho} b_{\tau,\rho}\, \delta, \tag{3.1.47}$$

where the bounds $b_{\eta,\rho}$ and $b_{\tau,\rho}$ are defined by (3.1.33) and the constants c and C are chosen according to Theorem 3.1.6. We define

$$F(\delta) := \inf\Big\{F_1(\rho, F_2(\rho, \tau, \eta, \delta)),\ \ \tau, \eta, \rho > 0\Big\} \tag{3.1.48}$$

with F_1 given by (3.1.15).

The following lemma and corollary will be needed to study the behavior of the function F. For a function $f : (0, \epsilon_0) \times (0, \epsilon_0) \to I\!R$ we define

$$f^*(s, t) := \sup_{\xi \in [t, \epsilon_0)} f(s, \xi), \tag{3.1.49}$$

i.e. we build the supremum in the second coordinate.

Lemma 3.1.8 *For every function $f : (0, \epsilon_0) \to I\!R$ with $f(t) \to \infty$ for $t \to 0$ there exists a function $g : (0, \delta_0) \to (0, \epsilon_0)$ with*

$$f^*(g(\delta))\, \delta \to 0 \quad \textit{and} \quad g(\delta) \to 0 \quad \textit{for } \delta \to 0. \tag{3.1.50}$$

Proof. Let f' be a strictly monotone function with $f'(t) \geq f(t)$. On (a, ∞) with $a := f'(\epsilon_0)$ there is an *inverse* function

$$(f')^{-1} : (a, \infty) \to (0, \epsilon_0)$$

with $(f')^{-1}(f'(s)) = s$ and $f'((f')^{-1}(t)) = t$ for all $s \in (0, \epsilon_0)$ and $t \in (a, \infty)$. Then for the function

$$g : (0, \delta_0) \to (0, \epsilon_0), \quad g(\delta) := (f')^{-1}\Big(\frac{1}{\sqrt{\delta}}\Big),$$

with δ_0 defined by $(f')^{-1}(1/\sqrt{\delta_0}) = \epsilon_0$ we obtain the behavior (3.1.50). □

Corollary 3.1.9 *For every function $f : (0, \epsilon_0) \times (0, \epsilon_0) \to I\!R$ with $f(t, s) \to \infty$ for $(t, s) \to 0$ there exists a function $g : (0, \delta_0) \to (0, \epsilon_0) \times (0, \epsilon_0)$, $g = (g_1, g_2)$ which satisfies*

$$f^*(g(\delta))\,\delta \;\to\; 0, \;\; g(\delta) \to 0 \quad \textit{and} \quad \frac{g_1(\delta)}{g_2(\delta)^{m-1}} \to 0 \quad \textit{for } \delta \to 0. \tag{3.1.51}$$

Proof. Use the preceding Lemma for $\tilde{f}(t) := f^*(t^2, t^{1/(m-1)})$ to obtain a function $\tilde{g}$ and define g by $g_1(\delta) := \tilde{g}(\delta)^2$ and $g_2(\delta) := \tilde{g}(\delta)^{1/(m-1)}$. □

After these preparations we investigate the behavior of the function F.

Lemma 3.1.10 *The function F defined by (3.1.48) satisfies*

$$F(\delta) \to 0, \;\; \delta \to 0. \tag{3.1.52}$$

Proof. The function $F(\delta)$ is dominated by

$$F_1\Big(h_1(\delta), F_2(h_1(\delta), h_2(\delta), h_3(\delta))\Big)$$

with arbitrary positive functions h_1, h_2 and h_3 on $(0, \delta_0)$. We will show, that we can choose h_1, h_2 and h_3 such that

$$F_1\Big(h_1(\delta), F_2(h_1(\delta), h_2(\delta), h_3(\delta))\Big) \to 0, \;\; \delta \to 0. \tag{3.1.53}$$

The function F_2 can be decomposed into the sum and product

$$F_2(\rho, \tau, \eta, \delta) \;=\; 2c\frac{\eta}{\rho^{m-1}} \;+ b_{\eta,\rho}\,\Big(2C\,\tau \;+\; \frac{1}{\gamma_m} b_{\tau,\rho}\,\delta\Big).$$

By an application of Corollary 3.1.9 to $f(s,t) := b_{s,t}$ we obtain functions h_4 and h_5 with

$$h_4(\delta) \to 0, \;\; \delta \to 0,$$

and

$$h_5(\delta) \to 0, \;\; \delta \to 0, \tag{3.1.54}$$

such that the functions

$$h_6(\delta) \;:=\; 2C\,h_4(\delta) \;+\; \frac{1}{\gamma_m} b^*_{h_4, h_5(\delta)}\,\delta$$

and

$$h_7(\delta) \;:=\; 2c\frac{h_4(\delta)}{h_5(\delta)^{m-1}} \;+ b^*_{h_4(\delta),h_5(\delta)}\;\delta$$

with b^* defined in (3.1.49) satisfy

$$h_6(\delta) \;\to\; 0, \quad \delta \to 0,$$

and

$$h_7(\delta) \to\; 0, \quad \delta \to 0.$$

For

$$\begin{aligned} h_1(\delta) &:= \max\{h_5(\delta), h_5(h_6(\delta))\}, \\ h_2(\delta) &:= h_4(\delta), \\ h_3(\delta) &:= h_4(h_6(\delta)) \end{aligned}$$

we obtain

$$\begin{aligned} &F_2(h_1(\delta), h_2(\delta), h_3(\delta), \delta)\delta \\ &\quad= 2c\frac{h_3(\delta)}{h_1(\delta)^{m-1}} \;+\; b_{h_3(\delta),h_1(\delta)}\Big(2Ch_2(\delta) + \frac{1}{\gamma_m} b_{h_2(\delta),h_1(\delta)}\delta\Big) \\ &\quad\leq 2c\frac{h_4(h_6(\delta))}{h_5(h_6(\delta))^{m-1}} \;+\; b^*_{h_4(h_6(\delta)),h_5(h_6(\delta))}\Big(2Ch_4(\delta) + \frac{1}{\gamma_m} b^*_{h_4(\delta),h_5(\delta)}\delta\Big) \\ &\quad\to 0, \quad \delta \to 0. \end{aligned} \tag{3.1.55}$$

Thus we have

$$F_1\Big(h_1(\delta), F_2(h_1(\delta), h_2(\delta), h_3(\delta), \delta)\Big) \to 0, \quad \delta \to 0 \tag{3.1.56}$$

and the proof is complete. □

We now put all previous steps together to obtain the main stability result.

Theorem 3.1.11 *Let $u_1^\infty(\hat{x}, d)$ and $u_2^\infty(\hat{x}, d)$, $\hat{x}, d \in \Omega$, be the far field patterns for scattering of acoustic plane waves two from scatterers $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{C}$. We assume that with a nonnegative parameter δ the difference between the far field patterns satisfies*

$$\Big|\Big|u_1^\infty - u_2^\infty\Big|\Big|_{L^2(\Omega\times\Omega)} \;\leq\; \delta. \tag{3.1.57}$$

Then with the function F defined by (3.1.48) the Hausdorff distance $d(D_1, D_2)$ of the two scatterers $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{C}$ can be estimated by

$$d(D_1, D_2) \;\leq\; F(\delta) \tag{3.1.58}$$

Proof. With F_2 defined by (3.1.47) we use Lemma 3.1.7 to derive

$$\left|\Phi^s_{1,\mu,q}(x,z) - \Phi^s_{2,\mu,q}(x,z)\right| \leq F_2(\rho,\tau,\eta,\delta) \tag{3.1.59}$$

for all $q \in \Omega$, $\rho, \tau, \eta > 0$ and all points $x, z \in U$ with U defined by (3.1.12). We can now apply Lemma 3.1.4 to derive

$$d(D_1, D_2) \leq F_1(\rho, F_2(\rho,\tau,\eta,\delta)) \tag{3.1.60}$$

for every choice of parameters $\rho, \tau, \eta > 0$. With the definition (3.1.48) for F from (3.1.60) we obtain (3.1.58). □

Lemma 3.1.10 shows that the estimate (3.1.58) in fact is a stability estimate for the reconstruction of D from the far field patterns for scattering of plane waves. The function F can be calculated or estimated according to the a-priori knowledge about the unknown scatterers as given by the class $\mathcal{C}$ defined in Definitions 2.1.5 and 2.2.5.

The natural problem is the investigation of the dependence of $F(\delta)$ on δ. Unfortunately, for general angles β_0 we were not yet able to estimate explicitly the behavior of F, but for $\beta_0 = \frac{\pi}{2}$ and for impenetrable domains we will now use the asymptotics of Bessel and Hankel functions to derive a logarithmic bound. More general, we will explicitly estimate the dependence of the function $F(\delta)$ on δ for the estimate

$$d\Big(\mathcal{H}(D_1), \mathcal{H}(D_2)\Big) \leq F(\delta) \tag{3.1.61}$$

for the convex hulls $\mathcal{H}(D_1)$ and $\mathcal{H}(D_2)$ of two scatterers $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{C}_{obst}$.

THEOREM 3.1.12 *Let $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{C}_{obst}$ be impenetrable scatterers with scattering data $u_1^\infty(\hat{x}, d)$ and $u_2^\infty(\hat{x}, d)$, $\hat{x}, d \in \Omega$. If*

$$\left|\left|u_1^\infty - u_2^\infty\right|\right|_{L^2(\Omega\times\Omega)} \leq \delta, \tag{3.1.62}$$

then we have

$$d\Big(\mathcal{H}(D_1), \mathcal{H}(D_2)\Big) \leq \frac{C}{|\ln\delta|^c} \tag{3.1.63}$$

with constants $C > 0$ and $0 < c < 1$ uniformly for all $\mathcal{D} \in \mathcal{C}_{obst}$.

Proof. We consider the three-dimensional problem. Let $z_0 \in \partial D_2 \cap \partial\mathcal{H}(D_2)$ be a point, such that for $z = z_0 + \nu(z_0)\rho$ we can choose a domain of approximation $G_{(z,p,\rho)}$, $p \in \Omega$, with the angle $\beta_e = \pi/2$ and

$$\overline{D_j} \subset G_{z,p,\rho}, \quad j = 1, 2.$$

Then we can use Lemma 3.1.7 and proceed along the lines of the proof of Lemma 3.1.4 to obtain

$$d(z_0, D_1) \;\leq\; F_1\Big(\rho, F_2(\rho, \tau, \eta, \delta\Big) \tag{3.1.64}$$

for every choice of parameters $\rho, \tau, \eta > 0$ with the functions F_1 defined by (3.1.15) and F_2 defined by (3.1.47). For the special choice

$$\rho = |\ln \delta|^{-\frac{c_1}{2}}, \quad \eta = e^{-|\ln \delta|^{c_2}} \quad \text{and} \quad \tau = e^{-|\ln \delta|^{c_3}} \tag{3.1.65}$$

with $0 < c_1 < c_2 < 1$, $c_1 + c_2 < 1$ and $0 < c_3 < c_2 - c_1$ we will derive

$$|F_2(\rho, \eta, \epsilon, \delta)| \;\leq\; C \tag{3.1.66}$$

for all sufficiently small $\delta > 0$. Then from (3.1.64), (3.1.65) and (3.1.66) we obtain the explicit stability estimate (3.1.63).

To prove (3.1.66) we proceed in six steps.

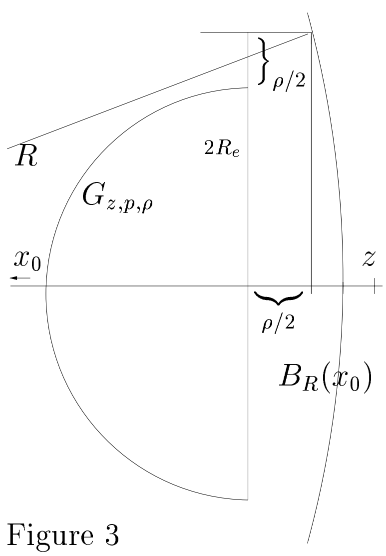

Figure 3

1. We explicitly construct the domain G of Lemma 3.1.2. Consider the domain $G_{z,p,\rho}$ and define $G := B_R(x_0)$ by

$$\begin{aligned} R^2 &= (R - \rho/4)^2 + (2R_e + \rho/2)^2, \\ x_0 &:= z - (\tfrac{\rho}{4} + R)p, \end{aligned} \tag{3.1.67}$$

where the first equation yields

$$R := \frac{5\rho}{8} + \frac{2(2R_e)^2}{\rho} + 2R_e \tag{3.1.68}$$

Then, by straightforward geometric arguments with the help of Figure 2 we derive

$$G_{z,p,\rho} \subset B_R(x_0), \quad d(z, B_R(x_0)) = \frac{\rho}{4}$$

and

$$d(G_{z,p,\rho}, \Omega_R(x_0)) \geq \frac{\rho}{2}$$

for sufficiently small ρ.

2. We need to estimate the bounds b_τ defined in (3.1.33). To this end we investigate special solutions of the interior Dirichlet problem for the Helmholtz

equation in the domain $B_R(x_0)$. We choose a coordinate system centered at $x_0 = 0$ and define the special functions

$$u_n(x) := \sum_{k=0}^{n} \sum_{l=-k}^{k} a_k^l \; j_k(\kappa|x|) \; Y_k^{-l}(\hat{x}), \quad x \in I\!R^3 \tag{3.1.69}$$

with coefficients $a_k^l := i\kappa h_k^{(1)}(\kappa|z|) Y_k^l(\hat{z})$ for $k = 0, ..., n$, $l = -k, ..., k$. From the expansion

$$\Phi(x,z) = i\kappa \sum_{k=0}^{\infty} \sum_{l=-k}^{k} h_k^{(1)}(\kappa|z|) Y_k^l(\hat{z}) j_k(\kappa|x|) Y_k^{-l}(\hat{x}), \tag{3.1.70}$$

$$= \sum_{k=0}^{\infty} \sum_{l=-k}^{k} a_k^l \; j_k(\kappa|x|) Y_k^{-l}(\hat{x}) \tag{3.1.71}$$

of $\Phi(x,z)$ for $|x| < |z|$ with respect to the spherical harmonics, the addition theorem

$$\sum_{m=-k}^{k} Y_k^l(\hat{z}) \; Y_k^{-l}(\hat{x}) \;=\; \frac{2k+1}{4\pi} P_k(\cos(\theta)) \tag{3.1.72}$$

with the Legendre Polynomial P_k, where θ denotes the angle between $\hat{z}$ and $\hat{x}$, the asymptotic behavior (1.2.36) and (1.2.37) of the Hankel and Bessel functions and $|P_k(t)| \leq 1$ for $t \in [-1,1]$ we obtain

$$\begin{aligned}
\Big|\Big|\Phi(\cdot,z) - u_n\Big|\Big|_{L^2(\Omega_R(x_0))} &= \Big|\Big|\kappa \sum_{k=n+1}^{\infty} \sum_{m=-k}^{k} h_k^{(1)}(\kappa|z|) \; Y_k^l(\hat{z}) \; j_k(\kappa R) \; Y_k^{-l}(\hat{x})\Big|\Big|_{L^2(\Omega)} \\
&= \Big|\Big|\kappa \sum_{k=n+1}^{\infty} h_k^{(1)}(\kappa|z|) \; j_k(\kappa R) \; \frac{2k+1}{4\pi} P_k(\cos(\theta))\Big|\Big|_{L^2(\Omega)} \\
&\leq \frac{c_0}{|z|} \sum_{k=n+1}^{\infty} q^k \;=\; \frac{c_0}{|z|} \frac{q^{n+1}}{1-q}
\end{aligned} \tag{3.1.73}$$

with $q := R/|z| < 1$ and a constant c_0 not depending on R, $|z|$ or n. Since for compact subsets $\overline{G_{z,p,\rho}}$ of $B_R(x_0)$ the solution of the interior Dirichlet problem in $B_R(x_0)$ with L^2-boundary data on $\Omega_R(x_0)$ defines a bounded mapping from $L^2(\Omega_R(x_0))$ into $C(\overline{G_{z,p,\rho}})$, we get a factor λ such that

$$\Big|\Big|\Phi(\cdot,z) - u_n\Big|\Big|_{C(\overline{G_{z,p,\rho}})} \leq \lambda \Big|\Big|\Phi(\cdot,z) - u_n\Big|\Big|_{L^2(\Omega_R(x_0))} \tag{3.1.74}$$

$$\leq \frac{\lambda c_0}{|z|} \frac{q^{n+1}}{1-q}. \tag{3.1.75}$$

The factor λ is a function of z, R and ρ.

3. To evaluate the dependence of λ on ρ let u be a solution to the interior Dirichlet problem for the Helmholtz equation in $B_R(x_0)$. We will show

$$\Big|u(x)\Big| \;\leq\; \frac{C}{\rho^2}\,||u||_{L^2(\Omega_R(x_0))} \tag{3.1.76}$$

for $x \in B_{R-\rho/2}(x_0)$, i.e. $\lambda \leq C\rho^{-2}$ with some constant C. We present a proof by means of spherical harmonics and Bessel functions. Again using a coordinate system centered at $x_0 = 0$, from Green's formula and the expansion (3.1.70) we observe that

$$u(x) = \sum_{k=0}^{\infty}\sum_{l=-k}^{k} a_k^l j_k(\kappa|x|)Y_k^l(\hat{x}), \quad x \in \overline{B_R(x_0)}, \tag{3.1.77}$$

where the sum (3.1.77) converges uniformly on compact subsets of $B_R(x_0)$. We use the Cauchy-Schwarz inequality to calculate for $|u(x)|$ with $x = r\hat{x} \in B_{R-\rho/2}(x_0)$ the estimate

$$\begin{aligned} |u(x)|^2 \;&=\; \Big|\sum_{k=0}^{\infty}\sum_{l=-k}^{k} a_k^m\; j_k(\kappa|x|)\; Y_k^l(\hat{x})\;\Big|^2 \\ &\leq\; \left(\sum_{k=0}^{\infty}\sum_{l=-k}^{k}\Big|a_k^l\; j_k(\kappa R)\Big|^2\right)\left(\sum_{k=0}^{\infty}\sum_{l=-k}^{k}\Big|Y_k^l(\hat{x})\frac{j_k(\kappa r)}{j_k(\kappa R)}\Big|^2\right). \end{aligned} \tag{3.1.78}$$

For the first term of (3.1.78) we observe that

$$\sum_{k=0}^{\infty}\sum_{l=-k}^{k}\Big|a_k^l\; j_k(\kappa R)\Big|^2 \;=\; ||\;u\;||^2_{L^2(\Omega_R)} \;<\; \infty. \tag{3.1.79}$$

For the second term of (3.1.78) we use the estimate

$$\Big|Y_k^l(\hat{x})\Big| \;\leq\; C\;k^{1/2}||Y_k^l||_{L^2(\Omega)}, \quad \hat{x} \in \Omega, \tag{3.1.80}$$

for spherical harmonics of order $k \in I\!N$ (see Chapter X, Lemma 6.1 of [46]) and the asymptotic behavior of the spherical Bessel functions to derive

$$\begin{aligned} &\sum_{k=1}^{\infty}\sum_{l=-k}^{k}\Big|Y_k^l(\hat{x})\frac{j_k(\kappa r)}{j_k(\kappa R)}\Big|^2 \\ &\leq \sum_{k=0}^{\infty}\sum_{l=-k}^{k}\;|k|\;\Big|\frac{j_k(\kappa r)}{j_k(\kappa R)}\Big|^2 \;\leq\; c\sum_{k=0}^{\infty}\;k\;q^{2k} \;\leq\; c\sum_{k=0}^{\infty}\;(2k+1)\;q^{2k} \\ &=\; c\frac{d}{dq}\sum_{k=0}^{\infty}\;q^{2k+1} \;=\; c\frac{d}{dq}\left(\frac{q}{1-q^2}\right) \;=\; c\left(\frac{1}{1-q^2}+\frac{2q^2}{(1-q^2)^2}\right) \end{aligned} \tag{3.1.81}$$

with $q := \frac{r}{R}$ and a constant c. From (3.1.68) and $0 \le r \le R - \rho$ we obtain

$$c\rho^2 \;\le\; 1 - q \;\le\; 1, \quad |\ln q| \ge B\rho^2 \tag{3.1.82}$$

for sufficiently small $\rho > 0$ with constants c, C and B. We now use $1 - q^2 = (1-q)(1+q)$ and $1 + q \ge 1$ to estimate (3.1.81) by $c\rho^{-4}$ with a constant c. We take the square root of (3.1.78) to obtain (3.1.76), i.e. we have proven $\lambda \le C\rho^{-2}$ with a constant C not depending on ρ.

4. For the function

$$g_n(\hat{x}) = \sum_{k=0}^{n} \sum_{l=-k}^{k} \frac{1}{4\pi i^k}\, a_k^l \, Y_k^{-l}(\hat{x}), \quad \hat{x} \in \Omega. \tag{3.1.83}$$

we calculate $u_n = Hg_n$. With n chosen as

$$n \;:=\; \left[\frac{\ln(\tau \, \frac{(1-q)|z|}{\lambda c_0})}{\ln(q)} \right] - 1, \tag{3.1.84}$$

where $[a]$ denotes the smallest integer larger than $a \in I\!R$, by straightforward calculation from the estimate (3.1.75) we derive

$$\Big|\Big| \Phi(\cdot, z) - Hg_n \Big|\Big|_{C^1(\overline{G_{z,p,\rho}})} \;\le\; \tau. \tag{3.1.85}$$

With the help of (3.1.82) and $\lambda \le c\rho^{-2}$ we derive for n the estimate

$$n \le C \frac{|\ln(c\tau\rho^3)|}{\rho^2} \tag{3.1.86}$$

with constants C, c.

5. We now estimate the norm of the function g_n with n given by (3.1.84). We calculate

$$\begin{aligned} \Big|\Big| g_n \Big|\Big|^2_{L^2(\Omega)} &= \Big(\frac{1}{4\pi}\Big)^2 \sum_{k=0}^{n} \sum_{l=-k}^{k} |a_k^l|^2 \\ &\le c_1 \sum_{k=0}^{n} (2k+1) \Big(\frac{2k}{e\kappa |z|} \Big)^{2k} \\ &\le c_1 \, (2n+1)(n+1) \Big(\frac{2n}{e\kappa|z|} \Big)^{2n} \end{aligned} \tag{3.1.87}$$

with a constant c_1. We insert (3.1.84) into (3.1.87) to obtain

$$\left|\left|g_n\right|\right|_{L^2(\Omega)} \;\leq\; C \left(\frac{|\ln(c\tau\rho^3)|}{\rho^2}\right)^2 \left(\frac{C|\ln(c\tau\rho^3)|}{\rho}\right)^{C\frac{|\ln(c\tau\rho^3)|}{\rho^2}} \tag{3.1.88}$$

with constants c and C not depending on ρ.

6. We need to investigate the set of densities $\mathcal{G}$ which we constructed in Lemma 3.1.3. Since we chose $g \in \mathcal{G}$ to be minimum norm solutions of (3.1.8), the norm $||g_n||_{L^2(\Omega)}$ is an upper bound for b_τ.

Finally, the bound (3.1.66) for $|F_2(\rho, \eta, \tau, \delta)|$ can be obtained by straightforward (but lengthy) calculation from definition (3.1.47) of F_2, the choice of parameters (3.1.65) and the estimate (3.1.88) for the bounds b_τ and b_η.

The proof in two dimensions can be worked out in the same way with obvious modifications. □

From stability as a corollary we obtain *uniqueness of the support* of penetrable or impenetrable scatterers.

Corollary 3.1.13 *For acoustic scatterers $\mathcal{D} \in \mathcal{C}$ the domain D is uniquely determined by the far field pattern for scattering of all plane waves.*

3.2 Electromagnetic scattering.

In this section we will prove stability estimates for inverse electromagnetic scattering. Consider the *electromagnetic Herglotz pair*

$$(Vg)(x,p) := \int_\Omega \frac{i}{\kappa}(p \cdot \nabla_x)\nabla_x e^{i\kappa x\cdot d} g(d)\, ds(d), \quad \frac{1}{i\kappa}\text{curl}\, Vg \tag{3.2.1}$$

with a density $g \in L^2(\Omega)$. With the aid of

$$\nabla \times \nabla \times (pw) = (p \cdot \nabla)\nabla\, w, \quad p \in \Omega, \tag{3.2.2}$$

for solutions w of the Helmholtz equation and

$$E^i_{pl}(x,d,p) \;=\; i\kappa(d \times p) \times d e^{i\kappa x\cdot d} \;=\; \frac{i}{\kappa}\nabla_x \times \nabla_x \times (pe^{i\kappa x\cdot d}) \tag{3.2.3}$$

we derive

$$(Vg)(x,p) \;=\; \int_\Omega E^i_{pl}(x,d,p)g(d)ds(d), \quad x \in \mathbb{R}^3. \tag{3.2.4}$$

By linearity and continuity of the direct scattering problem the incident electromagnetic wave

$$E^i := Vg, \quad H^i := \frac{1}{i\kappa}\text{curl}\, Vg$$

has the scattered field

$$(V^s g)(x,p) := \int_\Omega E^s_{pl}(x,d,p)g(d)ds(d), \quad \frac{1}{i\kappa}\text{curl}\,(V^s g)(\cdot,p) \tag{3.2.5}$$

and far field pattern

$$(V^\infty g)(x,p) := \int_\Omega E^\infty_{pl}(x,d,p)g(d)ds(d), \quad \nu \times (V^\infty g)(\cdot,p). \tag{3.2.6}$$

For fixed $\rho > 0$ we will construct an approximation

$$E^i_{edp}(x,z,p) \;\approx\; (Vg(z,\cdot))(x,p), \quad x \in \overline{D}, \tag{3.2.7}$$

for an electric dipole uniformly for $z \in B \setminus D_\rho$. Since both plane waves and multipoles in acoustic and electromagnetic are strongly related to each other, we will be able to use the results of the acoustic parts. From Lemma 3.1.3 we first derive the following lemma.

Lemma 3.2.1 *Given $\rho, \tau > 0$ there exists a density $g \in L^2(\Omega)$, such that for each $p, q \in \Omega$ there is an orthogonal matrix M with*

$$\left|\left|E^i_{edp}(\cdot, 0, q) - \Big(Vg(M^{-1}\cdot)\Big)(\cdot, q)\right|\right|_{C^2(G_{0,p,\rho})} \leq \tau. \tag{3.2.8}$$

Proof. We use Lemma 3.1.3 for $s = 4$ and $\mu = 0$. For the special case $\mu = 0$ the multipole $\Phi_{0,q} = \Phi$ has rotational symmetry. Thus for $p \in \Omega$ we obtain the approximation of $\Phi(\cdot, 0)$ by Hg on $G_{0,p,\rho}$ from the approximation on $G_{0,e_1,\rho}$, $e_1 = (1, 0, 0)$, by a simple rotation of the Herglotz wave function. The domain of approximation $G_{0,p,\rho}$ is obtained from the domain of approximation $G_{0,e_1,\rho}$ by rotation with an orthogonal matrix M satisfying $p = Me_1$. Clearly, the rotated Herglotz wave function $v(x) := (Hg)(M^{-1}x)$ approximates the point-source $\Phi(\cdot, 0)$ on $G_{0,p,\rho}$. We calculate

$$\begin{aligned} v(x) &= \int_\Omega e^{i\kappa M^{-1}x\cdot d} g(d)\, ds(d) \\ &= \int_\Omega e^{i\kappa x\cdot Md} g(d)\, ds(d) \\ &= \int_\Omega e^{i\kappa x\cdot \tilde{d}} g(M^{-1}\tilde{d})\, ds(d). \end{aligned} \tag{3.2.9}$$

To obtain the statement of the lemma we now apply the differential operator $\frac{i}{\kappa}(q \cdot \nabla)\nabla$ to both $\Phi(\cdot, 0)$ and $Hg(M^{-1}\cdot)$. □

Given the density g of the preceding lemma and an orthogonal matrix function M on B we define the translated and rotated density by

$$g(x, d, M, \tau, \rho, \beta_0) \ := \ e^{-i\kappa x\cdot d} g(M^{-1}(x)d), \ \ d \in \Omega. \tag{3.2.10}$$

We use the abbreviations

$$g_\tau(x, d) = g(x, d, M, \tau, \rho, \beta_0),$$

$$b_{\tau,\rho} := ||g_\tau(x, \cdot)||_{L^2(\Omega)},$$

and

$$b_\tau = b_{\tau,\rho},$$

where $b_{\tau,\rho}$ is well defined, since the norm of $g_\tau(x, \cdot)$ is independent of $x \in I\!R^3$.

DEFINITION 3.2.2 *Given a set of parameters $\mu \in I\!N_0$, $s = 4$, $\rho, \tau, \eta > 0$, $\gamma = \frac{1}{4\pi}$ and an orthogonal matrix function M on B with the help of*

$$g_\tau(x,d) \ = \ g(x,d,M,\tau,\rho,0,s,\beta_0)$$

and

$$g_\eta(z,\tilde{d}) = g(z,\tilde{d},M,\eta,\rho,\mu,s,\beta_0)$$

we define the operator

$$Q : L^2(\Omega \times \Omega, I\!R^3) \to L^\infty(B, I\!R^3) \tag{3.2.11}$$

by

$$(Qw)(x,z) \ := \ \frac{1}{\gamma} \int_\Omega \int_\Omega \Big\{ g_\tau(x,d) g_\eta(z,\tilde{d}) \Big\} \, w(-d,\tilde{d}) \; ds(d) ds(\tilde{d}), \quad x,z \in B \tag{3.2.12}$$

The Operator (3.2.12) is basically the operator given by (3.1.35), but now applied to vector-valued functions. It can be used to construct E^s_{edp} from the knowledge of E^∞_{pl}. Again, we observe the strong relationship between acoustic and electromagnetic scattering. We use the class $\mathcal{C}$ of electromagnetic scatterers defined by

$$\mathcal{C} := \mathcal{C}_{pc} \cup \mathcal{C}_{elm}. \tag{3.2.13}$$

THEOREM 3.2.3 *Consider electromagnetic scattering by a perfect conductor or an inhomogeneous medium scatterer $\mathcal{D} \in \mathcal{C}$. The error for the approximation of E^s_{edp} by QE^∞_{pl} satisfies*

$$\Big| \tilde{q} \cdot E^s_{edp}(x,z,q) - (q \cdot QE^\infty_{pl}(\cdot,\cdot,\tilde{q}))(x,z) \Big| \ \le \ c\frac{\eta}{\rho^3} \ + \ Cb_{\eta,\rho}\tau \tag{3.2.14}$$

for all $q, \tilde{q} \in \Omega$ and all points $x, z \in B \setminus D_\rho$, for which (3.1.34) is satisfied with $p = M(x)e_1$ or $p = M(z)e_1$, respectively. The constants c and C hold uniformly for $\mathcal{D} \in \mathcal{C}$.

Proof. For $\eta > 0$ and $q \in \Omega$ by definition of the kernel g_η we have

$$\Big|\Big| E^i_{edp}(\cdot,z,q) - \Big(V g_\eta(z,\cdot) \Big)(\cdot,q) \Big|\Big|_{C^2(\overline{D})} \ \le \ \eta \tag{3.2.15}$$

for all $z \in B \setminus D_\rho$, for which (3.1.34) is satisfied with $p = M(z)e_1$.

For the perfect conductor we use the boundedness of the integral operator $(I + M + iNPS_0^2)^{-1}$, the singularity of the kernels of the potential P_E and the Cauchy-Schwartz inequality to estimate the corresponding scattered fields by

$$\Big|\Big|E^s_{edp}(\cdot, z, q) - \Big(V^s g_\eta(z, \cdot)\Big)(\cdot, q)\Big|\Big|_{C(B \setminus \overline{D_\rho})} \leq c\,\frac{\eta}{\rho^3} \tag{3.2.16}$$

with some constant c not depending on ρ or $\mathcal{D} \in \mathcal{C}$. If the scatterer is an inhomogeneous medium we obtain an analogous estimate by consideration of the volume integral equation and the potential T_e.

In the same way for a second independent parameter τ, $\tilde{q} \in \Omega$ and all points $x \in B \setminus D_\rho$, for which (3.1.34) is satisfied with $p = M(x)e_1$, we derive

$$\Big|\Big|E^\infty_{edp}(\cdot, x, \tilde{q}) - \Big(V^\infty g_\tau(x, \cdot)\Big)(\cdot, \tilde{q})\Big|\Big|_{C(\Omega)} \leq c\,\tau \tag{3.2.17}$$

with some constant c. With the help of the mixed reciprocity relations Theorems 2.3.4 and 2.4.4 we transform (3.2.17) into

$$\Big|\tilde{q} \cdot E^s_{pl}(x, -d, q) \; - \; \Big(q \cdot \frac{1}{\gamma} V^\infty g_\tau(x, \cdot)\Big)(d, \tilde{q})\Big| \leq \frac{c}{\gamma}\,\tau \tag{3.2.18}$$

with some constant c. We use (3.2.18) to estimate

$$\begin{aligned}
&\Big|\Big(\tilde{q} \cdot V^s g_\eta(z, \cdot)\Big)(x, q) \; - \; \Big(q \cdot QE^\infty(\cdot, \cdot, \tilde{q})\Big)(x, z)\Big| \\
&= \Big|\int_\Omega \Big(\tilde{q} \cdot E^s_{pl}(x, d, q) - \frac{1}{\gamma}\int_\Omega q \cdot E^\infty(-d, \tilde{d}, \tilde{q}) g_\tau(x, \tilde{d})\, ds(\tilde{d})\Big)\, g_\eta(z, d)\, ds(d)\Big| \\
&\leq C\Big|\Big|g_\eta(z, \cdot)\Big|\Big|_{L^2(\Omega)}\tau
\end{aligned} \tag{3.2.19}$$

with some constant C. Now from (3.2.19) and (3.2.16) we derive

$$\Big|\tilde{q} \cdot E^s_{edp}(x, z, q) - \Big(q \cdot QE^\infty(\cdot, \cdot, \tilde{q})\Big)(x, z)\Big| \leq c\frac{\eta}{\rho^3} + C\Big|\Big|g_\eta(z, \cdot)\Big|\Big|_{L^2(\Omega)}\tau$$

with constants $c, C > 0$, and the proof is complete.

□

We will use the preceding theorem to estimate the difference between the scattered fields of electric dipoles for two scatterers $\mathcal{D}_1$ and $\mathcal{D}_2$ in $\mathcal{C}$.

Lemma 3.2.4 *Let $E^{\infty}_{pl,1}(\hat{x}, d, q)$ and $E^{\infty}_{pl,2}(\hat{x}, d, q)$ for $\hat{x}, d, q \in \Omega$ be the far field patterns for scattering of electromagnetic plane waves from two scatterers $\mathcal{D}_1, \mathcal{D}_2$ in $\mathcal{C}$. If for some parameter $\delta > 0$ the far field patterns satisfy*

$$\left|\left|E^{\infty}_{pl,1}(\cdot,\cdot,p) - E^{\infty}_{pl,2}(\cdot,\cdot,p)\right|\right|_{L^2(\Omega\times\Omega)} \leq \delta, \quad p \in \Omega, \tag{3.2.20}$$

then with the constants c, C given by Theorem 3.2.3 we have

$$\begin{aligned} &\left|E^{s}_{edp,1}(x,z,q) - E^{s}_{edp,2}(x,z,q)\right| \\ &\quad \leq 2c\frac{\eta}{\rho^3} + 2Cb_{\eta,\rho}\,\tau + \frac{1}{\gamma}b_{\eta,\rho}\,b_{\tau,\rho}\,\delta \end{aligned} \tag{3.2.21}$$

for all $q \in \Omega$, $\rho, \tau, \eta > 0$ and all points $x, z \in U$, where U is defined by (3.1.12).

Proof. Let M be an orthogonal matrix function on B, such that for all $x \in U$ with $p = M(x)e_1$ the condition (3.1.34) is satisfied for both scatterers $\mathcal{D}_1$ and $\mathcal{D}_2$. We use (3.2.14) for each scatterer $\mathcal{D}_1$ and $\mathcal{D}_2$. Then as in (3.1.46) the estimate (3.2.21) is obtained with the help of the Cauchy-Schwarz inequality. □

We now prove the electromagnetic counterpart of Lemma 3.1.4. From estimates for the difference of $E^s_{edp,1}$ and $E^s_{edp,2}$ for scattering by two scatterers $\mathcal{D}_1$ and $\mathcal{D}_2$ we derive estimates for the Hausdorff distance $d(D_1, D_2)$ of the scatterers. Again we have to consider different situations according to the physical properties of the scatterers $\mathcal{D}_1$ and $\mathcal{D}_2$.

Situation	Properties of $\mathcal{D}_1$ and $\mathcal{D}_2$
S1	$\mathcal{D}_1, \mathcal{D}_2$ are impenetrable scatterers
S2	$\mathcal{D}_2$ is an inhomogeneous medium scatterer and $\mathcal{D}_1$ is an impenetrable scatterer or vice versa
S3	$\mathcal{D}_1, \mathcal{D}_2$ are inhomogeneous medium scatterers.

(3.2.22)

Lemma 3.2.5 *We consider scattering of electromagnetic waves by two scatterers $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{C}$. Assume that with parameters $\sigma, \rho > 0$ the scattered fields $E^s_{edp,1}$ and $E^s_{edp,2}$ for scattering of electric dipoles by $\mathcal{D}_1$ or $\mathcal{D}_2$, respectively, satisfy*

$$\left|E^{s}_{edp,1}(z,z,q) - E^{s}_{edp,2}(z,z,q)\right| \leq \sigma \tag{3.2.23}$$

for all $q \in \Omega$ *and for points* $z \in B \setminus (D_{1,\rho} \cup D_{2,\rho})$*, for which a cone* $\text{co}(z, p_z, \beta_0)$*,* $p_z \in \Omega$*, in the exterior of* $D_{1,\rho} \cup D_{2,\rho}$ *exists. Then we conclude*

$$d(D_1, D_2) \;\leq\; F_1(\rho, \sigma), \tag{3.2.24}$$

where the function F_1 *is defined according to the situations S1 to S3 by*

$$F_1(\rho, \sigma) \;:=\; \begin{cases} \rho + \frac{C\rho}{(c-\sigma\rho^3)^{1/3}}, & S1, \\ \rho + \left(\frac{C\rho}{c-\sigma\rho}\right)^{1/3}, & S2, \\ \rho + \frac{C\rho}{c-\sigma\rho}, & S3 \end{cases} \tag{3.2.25}$$

with constants c *and* C *not depending on the scatterers* $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{C}$*.*

Proof. We split the proof in two parts. First, consider the scattered fields at a special point z in $B \setminus (\overline{D_1 \cup D_2})$, such that $z = z_0 + \rho\nu(z_0)$ with $z_0 \in \partial D_2$ and $\rho = d(z, D_2)$ sufficiently small and there exists a cone

$$\text{co}(z, p, \beta_0) \subset I\!R^m \setminus (\overline{D_{1,\rho} \cup D_{2,\rho}}), \quad p \in \Omega.$$

Then from (2.3.33), (2.3.34) and (3.2.23) for situation S1 of two impenetrable scatterers we obtain the estimate

$$\begin{aligned} \frac{C}{|d(z, D_1)|^3} &\geq \left| E^s_{edp,1}(z, z, p) \right| \\ &\geq \left| E^s_{edp,1}(z, z, p) \right| - \sigma \qquad (3.2.26) \\ &\geq \frac{c}{|d(z, D_2)|^3} - \sigma, \qquad (3.2.27) \end{aligned}$$

which can be transformed into

$$d(z, D_1) \;\leq\; \frac{C d(z, D_2)}{\Big(c - \sigma d(z, D_2)^3\Big)^{1/3}} \tag{3.2.28}$$

with constants C and c. In the same way for situation S2 we we use (2.3.33), (2.3.34), (2.4.20) and (2.4.21) to derive

$$d(z, D_1) \;\leq\; \left(\frac{C d(z, D_2)}{c - \sigma d(z, D_2)} \right)^{\frac{1}{3}}. \tag{3.2.29}$$

For situation S3 we estimate

$$d(z, D_1) \;\leq\; \frac{C d(z, D_2)}{c - \sigma d(z, D_2)}. \tag{3.2.30}$$

The second part of the proof is literally the same as in Lemma 3.1.4 and thus from the estimates (3.2.28), (3.2.29) and (3.2.30) we obtain Lemma 3.2.5. □

Let the function F_2 be given by

$$F_2(\rho, \tau, \eta, \delta) \ := \ 2c\frac{\eta}{\rho^3} \ + \ 2Cb_{\eta,\rho}\tau \ + \ \frac{1}{\gamma}b_{\eta,\rho}\, b_{\tau,\rho}\, \delta \tag{3.2.31}$$

for $\rho, \tau, \eta, \delta > 0$, where the constants c, C are chosen according to Theorem 3.2.3. Then we define the function

$$F(\delta) \ := \ \inf\Big\{F_1(\rho, F_2(\rho, \tau, \eta, \delta)), \ \ \tau, \eta, \rho > 0\Big\} \tag{3.2.32}$$

with F_1 given by (3.2.25). The functions F for acoustic and electromagnetic scattering differ by constants and by the definitions of the function $F_1(\rho, \sigma)$. The proof and statement of Lemma 3.1.10 is literally the same for the electromagnetic case (3.2.32), i.e. we have

$$F(\delta) \to 0, \ \ \delta \to 0. \tag{3.2.33}$$

We now collect all previous results to obtain the main stability result for electromagnetic scattering by a perfect conductor or an inhomogeneous electromagnetic medium.

Theorem 3.2.6 **(Stability estimate.)** *Let $E^{\infty}_{pl,1}(\hat{x}, d, p)$ and $E^{\infty}_{pl,2}(\hat{x}, d, p)$ for $\hat{x}, d, p \in \Omega$ be the far field patterns for scattering of electromagnetic plane waves from two scatterers $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{C}$. We assume that with a nonnegative parameter δ the difference between the far field patterns satisfies*

$$\Big|\Big|E^{\infty}_{pl,1}(\cdot, \cdot, p) - E^{\infty}_{pl,2}(\cdot, \cdot, p)\Big|\Big|_{L^2(\Omega\times\Omega)} \ \leq \ \delta \tag{3.2.34}$$

for all $p \in \Omega$. Then with the function F defined by (3.2.32) the Hausdorff distance $d(D_1, D_2)$ of the domains D_1 and D_2 can be estimated by

$$d(D_1, D_2) \ \leq \ F(\delta) \tag{3.2.35}$$

Proof. With F_2 defined by (3.2.31) we use Lemma 3.2.4 to derive

$$\Big|E^s_{edp,1}(x, z, p) - E^s_{edp,2}(x, z, p)\Big| \ \leq \ F_2(\rho, \tau, \eta, \delta) \tag{3.2.36}$$

for all $p \in \Omega$, $\rho, \tau, \eta > 0$ and all points $x, z \in U$ with U defined by (3.1.12). We now apply Lemma 3.2.5 to derive (3.2.35). □

We would like to explicitly estimate the behavior of the function $F(\delta)$ for $\delta \to 0$. For the convex hulls we will now derive an logarithmic estimate both for a perfect conductor and an inhomogeneous electromagnetic medium.

Theorem 3.2.7 **(Explicit stability estimate.)** *Let $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{C}$ be two electromagnetic scatterers with scattering data $E^\infty_{pl,1}(\hat{x}, d, p)$ and $E^\infty_{pl,2}(\hat{x}, d, p)$, $\hat{x}, d, p \in \Omega$. Then*

$$\Big|\Big| E^\infty_{pl,1}(\cdot,\cdot,p) - E^\infty_{pl,2}(\cdot,\cdot,p) \Big|\Big|_{L^2(\Omega\times\Omega)} \;\leq\; \delta \tag{3.2.37}$$

for all $p \in \Omega$ yields

$$d\Big(\mathcal{H}(D_1), \mathcal{H}(D_2)\Big) \;\leq\; \frac{C}{|\ln \delta|^c} \tag{3.2.38}$$

with constants $C > 0$ and $0 < c < 1$.

Proof. The theorem can be proven analogously to Theorem 3.1.12. Here we paraphrase the proof and point out the places where changes have to be made.

The explicit estimate (3.1.63) is obtained by explicitly estimating the behavior of the functions $F_2(\rho, \tau, \eta, \delta)$, and $F_1(\rho, \sigma)$. To this end the domain of approximation $G_{z,p,\rho}$ is placed in a ball $B_R(x_0)$. In the ball $B_R(x_0)$ an expansion of both the point-source $\Phi(\cdot, z)$ and the incident Herglotz wave function with respect to spherical harmonics and (spherical) Bessel functions is used. With the help of the asymptotic behavior of the spherical Bessel functions, the norm of the minimum norm solution g with discrepancy τ of

$$(Hg)(x) = \Phi(x, z), \;\; x \in \Omega_R(x_0),$$

with $H : L^2(\Omega) \to L^2(\Omega_R(x_0)$ is estimated in (3.1.88).

For the electromagnetic cases we first note that the incident electric dipole is obtained by

$$E^i_{edp}(x, z, p) = \frac{i}{\kappa}(p \cdot \nabla_x)\nabla_x \Phi(x, z) \tag{3.2.39}$$

as a second derivative of the acoustic point-source and that the electric field

$$(Vg)(x, p) \;=\; \frac{i}{\kappa}(p \cdot \nabla_x)\nabla_x \int_\Omega e^{i\kappa x\cdot d} g(d)\, ds(d) \tag{3.2.40}$$

of the electromagnetic Herglotz pair is obtained by an application of the same differential operator to the acoustic Herglotz wave function Hg. Thus to obtain an approximation of E^i_{edp} by Vg instead of equation (3.1.74) we have to derive the estimate

$$\Big|\Big|\Phi(\cdot, z) - u_n\Big|\Big|_{C^4(\overline{G_{z,p,\rho}})} \;\leq\; \lambda \, \Big|\Big|\Phi(\cdot, z) - u_n\Big|\Big|_{L^2(\Omega_R(x_0))} \tag{3.2.41}$$

and investigate the behavior of the corresponding constant λ with respect to variations of ρ. Instead of a singularity of second order here we obtain $\lambda \leq C\rho^{-6}$. Then we use the special choice (3.1.65) to derive (3.2.38) in the same way as in Theorem 3.1.12. □

From stability as a corollary we obtain *uniqueness of the support* of penetrable or impenetrable scatterers.

COROLLARY 3.2.8 *For electromagnetic scatterers $\mathcal{D} \in \mathcal{C}$ the domain D is uniquely determined by the far field pattern for scattering of all plane waves.*

4 The case of finite data.

In this chapter we will investigate uniqueness and stability of inverse problems in the case where only a *finite number of measurements* of the far field patterns $u^\infty(\cdot, d)$ are given for a *finite number of incident plane waves.*

Since we search for domains in a space of infinite dimension, in this case in general we will not obtain a full uniqueness or stability result. We introduce a concept to treat the situation appropriately, which we call ϵ-uniqueness or ϵ-stability, respectively.

First, we show that for given $\epsilon > 0$ there are integers n_i, n_o in $I\!N$, such that, if for n_i incident plane waves the far field patterns for two scatterers $\mathcal{D}_1$ and D_2 coincide for n_o observation directions, the Hausdorff distance $d(D_1, D_2)$ of the domains D_1 and D_2 satisfies

$$d(D_1, D_2) \ \leq \ \epsilon.$$

The distance $d(D_1, D_2)$ tends to zero, if n_i and n_o tend to infinity. This is a kind of *continuity statement*: with more measurements we obtain better reconstructions, and in the limit $n_i, n_o \to \infty$ we obtain precise reconstructions. We call the concept ϵ-uniqueness in analogy to the ϵ-δ-formulation of continuity.

We would like to point out the difference of ϵ-uniqueness and stability. Stability investigates the continuity of the mapping from the data space into the space of domains. Stability implies full uniqueness. In contrast to stability, ϵ-uniqueness investigates a sequence of finite data-spaces for which uniqueness is not necessarily satisfied.

Second, we will investigate stability for the case of finitely many measurements and for a finite number of incident plane waves. In this case, where we do not have uniqueness, it cannot be possible to obtain full stability results. We develop a concept of ϵ-stability as follows. Given $\epsilon > 0$ it is possible to find $n_o, n_i \in I\!N$ and a function $F_{(n_0,n_i)} : I\!R^+ \to I\!R^+$ with the behavior

$$\limsup_{\delta \to 0} F_{(n_0,n_i)}(\delta) \ \leq \ \epsilon \tag{4.0.1}$$

such that the Hausdorff distance $d(D_1, D_2)$ of the domains D_1 and D_2 of the scatterers $\mathcal{D}_1$ and D_2 can be estimated by

$$d(D_1, D_2) \ \leq \ F_{(n_0,n_i)}\Big(\, ||u_1^\infty(\cdot,\cdot) - u_2^\infty(\cdot,\cdot)||_{L^2(\Omega_{n_i} \times \Omega_{n_o})} \, \Big). \tag{4.0.2}$$

Clearly, each function $F_{(n_0,n_i)}$ with (4.0.1) and (4.0.2) provides an ϵ-stability estimate for the reconstruction of the shapes of scatterers.

4.1 Acoustic scattering.

The proof of the uniqueness theorem 3.1.1 is based on the application of Rellich's lemma and it cannot be applied to the case where the far field pattern is known only for a *finite number* of observation points and incident waves. Since Rellich's lemma includes an analyticity argument, it is not possible to use approximations to treat a finite data set. We now develop modified techniques to derive the results of ϵ-uniqueness, which in the limit-case $\epsilon \to 0$ also yield the above uniqueness results.

We will replace the role of Rellich's lemma by an operator $Q_{(n_o,n_i)}$ for the approximate reconstruction of point-sources $\Phi^s(x,z)$ and, more general, the scattered field $\Phi^s_{\mu,q}(x,z)$ of multipoles. We will obtain

$$\Phi^s_{q,\mu}(x,z) \;\approx\; \Big(Q_{(n_o,n_i)} u^\infty_{(n_o,n_i)}\Big)(x,z), \quad x,z \in B \setminus D_\rho,$$

on a set $B \setminus D_\rho$ with $\rho > 0$, where for $d_j, d_k \in \Omega$

$$u^\infty_{(n_o,n_i)} := \Big(u^\infty(d_j,d_k)\Big)_{j=1,\dots,n_o,k=1,\dots,n_i} \quad \in \mathbb{C}^{n_o \cdot n_i} \tag{4.1.1}$$

denotes a *finite set of measured far field patterns.* We consider subsets of the unit sphere

$$\Omega_n \;:=\; \{d_j, j = 1, \dots n\} \quad \subset \Omega \tag{4.1.2}$$

with $d_i \neq d_j$ for $i \neq j$, $d_i, d_j \in \Omega_n$. For simplicity, for the sequence $(\Omega_n)_{n\in\mathbb{N}}$ we demand the *denseness property*

$$d(\hat{x}, \Omega_n) \to 0, \quad n \to \infty \tag{4.1.3}$$

for all $\hat{x} \in \Omega$, the *symmetry property*

$$d \in \Omega_n \quad \Rightarrow \quad -d \in \Omega_n \tag{4.1.4}$$

if n is even and the *monotonicity property*

$$\Omega_{n'} \subset \Omega_n \text{ for } n > n'. \tag{4.1.5}$$

The function space $L^2(\Omega_n)$ is defined as the space of functions

$$w : \Omega_n \to \mathbb{C}$$

equipped with the norm

$$||w||_{L^2(\Omega_n)} := \left(\frac{c_m}{n} \sum_{k=1}^{n} |w(d_k)|^2\right)^{\frac{1}{2}},$$

where c_m is given by

$$c_m = \begin{cases} 2\pi & m = 2 \\ \\ 4\pi & m = 3 \end{cases} . \tag{4.1.6}$$

The mapping

$$\iota : w \mapsto (w(d_k))_{k=1,\ldots,n} \in \mathbb{C}^n$$

is a *norm isomorphism* from the space $L^2(\Omega_n)$ onto the space $\mathbb{C}^n$ equipped with the norm

$$||a||_{L^2(\mathbb{C}^n)} := \left(\frac{c_m}{n} \sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}} . \tag{4.1.7}$$

In the same way $L^2(\Omega_{n_o} \times \Omega_{n_i})$ with the norm

$$||w||_{L^2(\Omega_{n_o} \times \Omega_{n_i})} := \left(\frac{c_m}{n_o} \frac{c_m}{n_i} \sum_{k=1}^{n_o} \sum_{j=1}^{n_i} |w(d_k, d_j)|^2 \right)^{\frac{1}{2}} .$$

is defined and shown to be isomorphic to $\mathbb{C}^{n_o \cdot n_i}$ equipped with the norm (4.1.7), where c_m has to be replaced by c_m^2. Usually, we will identify the two spaces and treat $u^\infty_{(n_o,n_i)}$ (defined in (4.1.1)) as an element of $L^2(\Omega_{n_o} \times \Omega_{n_i})$. By $\tilde{A} := \iota^{-1} \circ A \circ \iota$ Operators $A : \mathbb{C}^n \to Y$ can be considered as operators

$$\tilde{A} : L^2(\Omega_{n_o} \times \Omega_{n_i}) \to Y.$$

We will usually identify A and $\tilde{A}$.

To construct the approximation operator $Q_{(n_o,n_i)}$ in principle we have two possibilities. First, we may discretize the continuous operator Q. Second, we may approximate point-sources by a finite superposition of plane waves and proceed analogously to the derivation of the operator Q. Here, we will choose the second approach.

A finite superposition of plane waves is given by the *finite Herglotz wave function*

$$(H_n a)(x) := \frac{c_m}{n} \sum_{j=1}^{n} e^{i\kappa x \cdot d_j} a_j, \quad x \in \mathbb{R}^m \tag{4.1.8}$$

with *density vector* $a \in \mathbb{C}^n$ and c_m given by (4.1.6). For scattering of the functions $H_n a$ from a scatterer $\mathcal{D}$ we can exploit the linearity of the scattering problem. If $H_n a$ is the incident field, the corresponding scattered field and its far field pattern are given by

$$(H_n^s a)(x) = \frac{c_m}{n} \sum_{j=1}^{n} u^s(x, d_j) a_j, \quad x \in \mathbb{R}^m \setminus D, \tag{4.1.9}$$

and

$$(H_n^\infty a)(\hat{x}) = \frac{c_m}{n} \sum_{j=1}^{n} u^\infty(x, d_j) a_j, \quad \hat{x} \in \Omega. \tag{4.1.10}$$

In the next two lemmas we investigate the approximation of a multipole $\Phi_{\mu,q}$ of order μ by a *finite* superposition of plane waves $H_n g$ on the domains $G_{z,p,\rho}$ introduced in (3.1.4).

LEMMA 4.1.1 *For fixed $\mu, s \in I\!N_0$ and $\rho > 0$ the error*

$$E(n) := \sup_{p,q\in\Omega} \inf_{a\in\mathbb{C}^n} \Big|\Big|\Phi_{\mu,q}(\cdot, z) - H_n a\Big|\Big|_{C^s(G_{z,p,\rho})} \tag{4.1.11}$$

for an proximation of the multipole of order μ by a finite superposition of plane waves is independent of $z \in I\!R^m$ and satisfies

$$\lim_{n\to\infty} E(n) = 0. \tag{4.1.12}$$

Define

$$b(\epsilon, n) := \sup_{p,q\in\Omega} \inf_{a\in\mathbb{C}^n} \Big\{ ||a||_{L^2(\Omega_n)} \Big| \ ||\Phi_{\mu,q}(\cdot, z) - H_n a||_{C^s(G_{z,p,\rho})} \le \epsilon \Big\} \tag{4.1.13}$$

if

$$\Big\{ a \in \mathbb{C}^n \Big| \ ||\Phi_{\mu,q}(\cdot, z) - H_n a||_{C^s(G_{z,p,\rho})} \le \epsilon \Big\} \neq \emptyset \ \ \forall p, q \in \Omega$$

and $b(\epsilon, n) := 0$ otherwise. For fixed $\epsilon > 0$ the function $b(\epsilon, \cdot)$ is bounded.

Proof. As for the continuous case we use the fact that $G_{z,p,\rho}$ is obtained from $G_{0,p,\rho}$ by translation. If we translate a finite Herglotz wave function (4.1.8) by $z \in I\!R^m$, the result is again a finite Herglotz wave function with density vector

$$\tilde{a}_j := e^{-i\kappa z \cdot d_j} a_j, \quad j = 1, .., n. \tag{4.1.14}$$

Hence, an approximation of $\Phi(\cdot, z)$ on $\overline{D} \subset G_{z,q,\rho}$ can be derived from an approximation of $\Phi(\cdot, 0)$ on $G_{0,q,\rho}$.

Because of (4.1.14) the error $E(n)$ is independent of z and we can restrict our investigation to the case $z = 0$. We will use an approximation argument to derive the statements from the continuous case.

For $\epsilon > 0$ consider the finite set $\mathcal{G}(\epsilon/3, \rho, \mu, s, \beta_0)$ given by Lemma 3.1.3. By the definition of L^2 a function $g \in \mathcal{G} \subset L^2(\Omega)$ can be approximated by a function $\tilde{g} \in C^1(\Omega)$ such that

$$\Big|\Big|Hg - H\tilde{g}\Big|\Big|_{C^s(G_{0,p,\rho})} \le \frac{\epsilon}{3}$$

For the density $\tilde{g} \in C^1(\Omega)$ we can apply the standard convergence theorems for quadrature rules to approximate the integral $H\tilde{g}$ by a finite sum $H_n a$ with error

$$\Big|\Big|H\tilde{g} - H_n a\Big|\Big|_{C^s(G_{0,p,\rho})} \leq \frac{\epsilon}{3},$$

where $a \in \mathbb{C}^n$ is given by $a_j := \alpha_j^{(n)}\tilde{g}(d_j),\ j = 1, ..., n$, with appropriate weights $\alpha_j^{(n)}$. For the weights we assume $|\alpha_j^{(n)}| \leq c$, $j = 1, ..., n$, $n \in \mathbb{N}$ with some constant c, which is valid for example for the approximate computation of integrals by Riemann sums. Hence, given $\epsilon > 0$ we can find $n \in \mathbb{N}$, such that for every pair $p, q \in \Omega$ there is $a \in \mathbb{C}^n$ with

$$\begin{aligned} \Big|\Big|\Phi_{\mu,q}(\cdot, 0) - H_n a\Big|\Big|_{C^s(G_{0,p,\rho})} &\leq \Big|\Big|\Phi_{\mu,q}(\cdot, 0) - Hg\Big|\Big|_{C^s(G_{0,p,\rho})} \\ &\quad + \Big|\Big|Hg - H\tilde{g}\Big|\Big|_{C^s(G_{0,p,\rho})} + \Big|\Big|H\tilde{g} - H_n a\Big|\Big|_{C^s(G_{0,p,\rho})} \\ &\leq \epsilon, \end{aligned} \tag{4.1.15}$$

where $g \in \mathcal{G}(\epsilon/3, \rho, \mu, s, \beta_0)$ has to be chosen appropriately. Clearly, the function on the left-hand side of (4.1.15) is an upper bound for the function $E(n)$ defined in (4.1.11). By (4.1.5) the function E is monotonous. Thus we obtain $E(n) \to 0$ for $n \to \infty$. For the norm of a we estimate for sufficiently large n

$$\begin{aligned} ||a||^2_{L^2(\Omega_n)} &= \frac{c_m}{n} \sum_{j=1}^{n} |\alpha_j^{(n)} \tilde{g}(d_j)|^2 \\ &\leq c\, \frac{c_m}{n} \sum_{j=1}^{n} |\alpha_j^{(n)}|\ |\tilde{g}(d_j)|^2 \ \to\ c\ ||g||^2_{L^2(\Omega)}, \quad n \to \infty; \end{aligned} \tag{4.1.16}$$

i.e. the norm of a is bounded uniformly for $n \in \mathbb{N}$. Thus we obtain the boundedness of $b(\epsilon, \cdot)$ and the proof of Lemma 4.1.1 is complete. □

For $\mu, s \in \mathbb{N}_0$ and $\rho, \tau > 0$ according to the behavior (4.1.12) of $E(n)$ there is an even integer $n \in \mathbb{N}$ such that

$$\sup_{p,q\in\Omega} \inf_{a\in\mathbb{C}^n} \Big\{ \Big|\Big|\Phi_{\mu,q}(\cdot, 0) - H_n a\Big|\Big|_{C^s(G_{0,p,\rho})} \Big\} \leq \frac{\tau}{2} \tag{4.1.17}$$

is satisfied.

Lemma 4.1.2 *Given $\mu, s \in \mathbb{N}_0$, $\rho, \tau > 0$ and an even integers $n \in \mathbb{N}$ such that (4.1.17) is satisfied, there is a finite set*

$$\mathcal{G} = \mathcal{G}(n, \tau, \rho, \mu, s, \beta_0)\ \subset \mathbb{C}^n$$

such that for each $p, q \in \Omega$ there is a vector $a \in \mathcal{G}$ with

$$\left|\left|\Phi_{\mu,q}(\cdot, 0) - H_n a\right|\right|_{C^s(G_{0,p,\rho})} \leq \tau. \tag{4.1.18}$$

If for μ, s, ρ, τ and n the condition (4.1.17) is not *satisfied, we define*

$$\mathcal{G}(n, \tau, \rho, \mu, s, \beta_0) := \{0\}.$$

Proof. Consider $p, q \in \Omega$ and $a \in \mathbb{C}^n$ with

$$\left|\left|\Phi_{\mu,q}(\cdot, 0) - H_n a\right|\right|_{C^s(G_{0,p,\rho})} \leq \frac{\tau}{2}. \tag{4.1.19}$$

Since for fixed $a \in \mathbb{C}^n$ the function on the left-hand side of (4.1.19) depends continuously on p and q, by compactness of Ω as in the proof of Lemma 3.1.3 we obtain a finite set $\mathcal{G}$ of vectors $a \in \mathbb{C}^n$, such that for $p, q \in \Omega$ there is a vector $a \in \mathcal{G}$ which satisfies the estimate (4.1.21). □

As shown in (4.1.14), an approximation of $\Phi_{\mu,q}(\cdot, 0)$ on $G_{0,p,\rho}$ yields an approximation of $\Phi_{\mu,q}(\cdot, x)$ on $G_{x,p,\rho}$. For a vector a in $\mathbb{C}^n$ we define the vector function $a(\cdot) : B \to \mathbb{C}^n$ by

$$a_j(x) = e^{-i\kappa x \cdot dj} a_j, \quad j = 1, ..., n, \quad x \in B, \tag{4.1.20}$$

where d_j is given by (4.1.2). From (4.1.18) by translation we derive

$$\left|\left|\Phi_{\mu,q}(\cdot, x) - H_n a(x)\right|\right|_{C^s(x,p,\rho)} \leq \tau. \tag{4.1.21}$$

Following Lemma 4.1.2 and (4.1.20) the vector a is a function

$$a = a(x, p, q, \tau, \rho, \mu, s, \beta_0). \tag{4.1.22}$$

We will need the vector with two different sets of values for x, τ and μ and with vectors p, q depending on x. To indicate the dependence we will use the notation

$$a_\tau(x) = a(x, p(x), q(x), \tau, \rho, \mu, s, \beta_0). \tag{4.1.23}$$

The j-th component of $a_\tau(x) \in \mathbb{C}^n$ is denoted by

$$a_{\tau,j}(x).$$

For the set $\mathcal{G}$ we write as in the continuous case

$$\mathcal{G}_\tau = \mathcal{G}(n, \tau, \rho, \mu, s, \beta_0). \tag{4.1.24}$$

We are now prepared to define the operator $Q_{(n_o,n_i)}$.

Definition 4.1.3 *Given a set of parameters $\mu \in \mathbb{N}_0$, $s = 1$, $\rho, \tau, \eta > 0$, even integers $n_o, n_i \in \mathbb{N}$, for which (4.1.17) is satisfied for $(0, \tau, n_o)$, (μ, η, n_i), and functions*

$$p, q : B \to \Omega$$

with the help of

$$a_\tau(x) \; = \; a(x, p(x), q(x), \tau, \rho, 0, s, \beta_0).$$

and

$$a_\eta(z) \; = \; a(z, p(z), q(z), \tau, \rho, \mu, s, \beta_0).$$

we define the operator

$$Q_{(n_o,n_i)} : \; L^2(\Omega_{n_o} \times \Omega_{n_i}) \to L^\infty(B)$$

by

$$(Q_{(n_o,n_i)} w)(x, z) \; := \; \frac{1}{\gamma_m} \frac{c_m}{n_o} \frac{c_m}{n_i} \sum_{j=1}^{n_o} \sum_{k=1}^{n_i} a_{\tau,j}(x) a_{\eta,k}(z) w(-d_k, d_j) \tag{4.1.25}$$

for $x, z \in B$.

Theorem 4.1.4 *Consider scattering by a sound-soft, sound-hard or inhomogeneous medium scatterer $\mathcal{D} \in \mathcal{C}$. For all $x, z \in B \setminus D_\rho$, for which (3.1.34) is satisfied for $p = p(x)$ or $p = p(z)$, respectively, the error for the approximation of $\Phi^s_{\mu,q}(x, z)$ by $Q_{(n_o,n_i)} u^\infty_{(n_o,n_i)}$ is estimated by*

$$\left| \Phi^s_{\mu,q}(x, z) - \left(Q_{(n_o,n_i)} u^\infty_{(n_o,n_i)} \right)(x, z) \right| \; \leq \; c \, \frac{\eta}{\rho^{m-1}} \; + C \left|\left| a_\eta(z) \right|\right|_{L^2(\Omega_{n_i})} \tau \tag{4.1.26}$$

uniformly for $\mathcal{D} \in \mathcal{C}$ with constants c and C depending on μ.

Proof. We start with the estimate

$$\left|\left| \Phi_{\mu,q}(\cdot, z) - H_{n_i} a_\eta(z) \right|\right|_{C^1(D)} \; \leq \; \eta \tag{4.1.27}$$

derived from (4.1.21). The scattering map $u^i \mapsto u^s$ is bounded from $C^1(\overline{D})$ into $C(B \setminus D_\rho)$. Estimating the combined single- and double-layer potential with the help of the Cauchy-Schwarz inequality we derive a constant c, such that (4.1.27) yields

$$\left|\left| \Phi^s_{\mu,q}(\cdot, z) - H^s_{n_i} a_\eta(z) \right|\right|_{C(B \setminus D_\rho)} \; \leq \; \frac{c}{\rho^{m-1}} \, \eta \tag{4.1.28}$$

uniformly for all scatterers $\mathcal{D} \in \mathcal{C}$. We exploit the estimate (4.1.21) a second time, now for τ, n_o and $\mu = 0$, to obtain a constant c with

$$\left|\left|\Phi^{\infty}(\cdot, x) - H^{\infty}_{n_o} a_{\tau}(x)\right|\right|_{C(\Omega)} \leq c\,\tau \tag{4.1.29}$$

uniformly for all scatterers $\mathcal{D} \in \mathcal{C}$. We use the mixed reciprocity relation (2.1.4) to transform (4.1.29) into

$$\left|u^s(x, d) - \frac{1}{\gamma_m}\Big(H^{\infty}_{n_o} a_{\tau}(x)\Big)(-d)\right| \leq \frac{c}{\gamma_m}\,\tau \tag{4.1.30}$$

for all $d \in \Omega$. We now insert the approximation (4.1.30) for $u^s(x, -d)$ into

$$\Big(H^s_{n_i} a_{\eta}(z)\Big)(x) = \frac{c_m}{n_i} \sum_{k=1}^{n_i} u^s(x, d_k) a_{\eta,k}(z).$$

From the Cauchy-Schwarz inequality we obtain the estimate

$$\begin{aligned} &\left|\Big(H^s_{n_i} a_{\eta}(z)\Big)(x) - \Big(Q_{(n_o,n_i)} u^{\infty}_{(n_o,n_i)}\Big)(x, z)\right| \\ &\qquad = \left|\frac{c_m}{n_i} \sum_{j=k}^{n_i} \left(u^s(x, d_k) \;-\; \frac{c_m}{\gamma_m n_o} \sum_{j=1}^{n_0} u^{\infty}(-d_k, d_j) a_{\tau,j}(x)\right) a_{\eta,k}(z)\right| \\ &\qquad \leq C\, \left|\left|a_{\eta}(z)\right|\right|_{L^2(\Omega_{n_i})}\,\tau \end{aligned} \tag{4.1.31}$$

with some constant C uniformly for scatterers $\mathcal{D} \in \mathcal{C}$. We can now use (4.1.31) and (4.1.28) to estimate the distance between $\Phi^s_{n,q}$ and $Q_{(n_o,n_i)} u^{\infty}_{(n_o,n_i)}$. We calculate

$$\left|\Phi^s_{\mu,q}(x, z) - \Big(Q_{(n_o,n_i)} u^{\infty}_{(n_o,n_i)}\Big)(x, z)\right| \tag{4.1.32}$$

$$\begin{aligned} &\leq \left|\Phi^s_{\mu,q}(x, z) - \Big(H^s_n a_{\eta}(z)\Big)(x)\right| \\ &\quad + \left|\Big(H^s_n a_{\eta}(z)\Big)(x) - \Big(Q_{(n_o,n_i)} u^{\infty}_{(n_o,n_i)}\Big)(x, z)\right| \\ &\leq \frac{c}{\rho^{m-1}}\,\eta \;+\; C\left|\left|a_{\eta}(z)\right|\right|_{L^2(\Omega_{n_i})}\,\tau \end{aligned} \tag{4.1.33}$$

with constants c and C uniformly for scatterers $\mathcal{D} \in \mathcal{C}$. □

In the finite data case it is of interest to explicitly formulate the approximation properties of $Q_{(n_o,n_i)}$ in a corollary.

Corollary 4.1.5 *For scattering by a sound-soft, sound-hard or inhomogeneous medium scatterer $\mathcal{D} \in \mathcal{C}$ we consider the approximation of $\Phi^s_{\mu,q}$ by $Q_{(n_o,n_i)} u^\infty_{(n_o,n_i)}$. Given $\mu \in I\!N_0$, $\rho, \sigma > 0$ and functions $p, q : B \to \Omega$ there are parameter $\tau, \eta > 0$ and even integers $n_i, n_o \in I\!N$, such that*

$$\left|\Phi^s_{\mu,q}(x,z) - \left(Q_{(n_o,n_i)} u^\infty_{(n_o,n_i)}\right)(x,z)\right| \leq \sigma, \tag{4.1.34}$$

for all $x, z \in B \setminus D_\rho$, for which (3.1.34) is satisfied for $p = p(x)$ or $p = p(z)$, respectively.

Proof. Given $\rho, \sigma > 0$ for $\eta = \sigma/(2c)$ we use Lemma 4.1.2 and first obtain $n_i \in I\!N$ such that (4.1.17) is satisfied with (τ, n) replaced by (η, n_i). We then construct the function $a_\eta(\cdot)$ as in (4.1.23). In the same way for $\tau = \sigma/(2C||a_\eta(x)||)$ and $\mu = 0$ we get $n_o \in I\!N$ to obtain the estimate (4.1.17) with n replaced by n_o. Now an application of Theorem 4.1.4 yields (4.1.34). □

We can use the Operator $Q_{(n_o,n_i)}$ to obtain estimates for the difference of the scattered fields $\Phi^s_{1,\mu,q}$ or $\Phi^s_{2,\mu,q}$ by scatterers $\mathcal{D}_1$ or $\mathcal{D}_2$, respectively, from the knowledge of a finite data set of far field patterns for scattering of plane waves.

Lemma 4.1.6 *Let $u_1^\infty(\hat{x}, d)$ and $u_2^\infty(\hat{x}, d)$, $\hat{x}, d \in \Omega$, be the far field patterns for scattering of acoustic plane waves from scatterers $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{C}$. Given $\mu \in I\!N_0$, $\rho > 0$ and $\sigma > 0$ there are even integers $n_i, n_o \in I\!N$, such that*

$$u_1^\infty(\hat{x}, d) = u_2^\infty(\hat{x}, d), \;\; \hat{x} \in \Omega_{n_o}, \; d \in \Omega_{n_i}, \tag{4.1.35}$$

yields

$$\left|\Phi^s_{1,\mu,q}(x,z) - \Phi^s_{2,\mu,q}(x,z)\right| \leq \sigma \tag{4.1.36}$$

for all $q \in \Omega$ and all points $x, z \in U$, where U is defined by (3.1.12).

Proof. First we remark, that we can choose a function $p : B \to \Omega$ such that (3.1.34) is satisfied for both scatterers $\mathcal{D}_1$ and $\mathcal{D}_2$ and for all $x \in U$. We choose an arbitrary constant function $q(\cdot) : B \to \Omega$. With $Q_{(n_o,n_i)}$ given by (4.1.25) we estimate

$$\begin{aligned}
&\left|\Phi^s_{1,\mu,q}(x,z) - \Phi^s_{2,\mu,q}(x,z)\right| \\
&\leq \left|\Phi^s_{1,\mu,q}(x,z) - \left(Q_{(n_o,n_i)} u^\infty_{1,(n_o,n_i)}\right)(x,z)\right| \\
&\quad + \left|\left(Q_{(n_o,n_i)} u^\infty_{1,(n_o,n_i)}\right)(x,z) - \left(Q_{(n_o,n_i)} u^\infty_{2,(n_o,n_i)}\right)(x,z)\right| \\
&\quad + \left|\left(Q_{(n_o,n_i)} u^\infty_{2,(n_o,n_i)}\right)(x,z) - \Phi^s_{2,\mu,q}(x,z)\right|.
\end{aligned} \tag{4.1.37}$$

According to Corollary 4.1.5 given $\sigma/2$ we obtain $\tau, \eta > 0$, $n_o, n_i \in I\!N$, such that the first and the last term of (4.1.37) are bounded by $\sigma/2$ for all points $x, z \in U$, where according to Lemma 4.1.2 we may choose n_i, n_o and τ, η uniformly for $q \in \Omega$. From (4.1.35) we derive that the central term of (4.1.37) is zero, thus we obtain the estimate (4.1.36). □

We are now prepared to formulate the main theorem on ϵ-uniqueness for the reconstruction of the shape of a sound-soft, a sound-hard or an inhomogeneous medium scatterer $\mathcal{D} \in \mathcal{C}$ from a finite data set $u^{\infty}_{(n_o,n_i)}$ of scattered plane waves.

THEOREM 4.1.7 **(ϵ-uniqueness)** *Let $u_1^{\infty}(\hat{x}, d)$ and $u_2^{\infty}(\hat{x}, d)$, $\hat{x}, d \in \Omega$, be the far field patterns for scattering of plane waves from two acoustic scatterers $\mathcal{D}_1, \mathcal{D}_2$ in $\mathcal{C}$. Given $\epsilon > 0$ there is $n_o, n_i \in I\!N$, such that*

$$u_1^{\infty}(\hat{x}, d) = u_2^{\infty}(\hat{x}, d), \;\; \hat{x} \in \Omega_{n_o}, \; d \in \Omega_{n_i}, \tag{4.1.38}$$

yields the estimate

$$d(D_1, D_2) \;\leq\; \epsilon \tag{4.1.39}$$

for the Hausdorff distance $d(D_1, D_2)$ of D_1 and D_2.

Proof. Consider one of the situations S1 to S6. Given $\epsilon > 0$ we choose ρ and σ such that $\epsilon = F_1(\rho, \sigma)$ with F_1 given by (3.1.15). We use Lemma 4.1.6 to obtain integers n_i and n_o such that

$$\left| \Phi^s_{1,\mu,q}(z, z) - \Phi^s_{2,\mu,q}(z, z) \right| \;\leq\; \sigma \tag{4.1.40}$$

is satisfied for all $q \in \Omega$ and for all points $z \in U$ with U defined by (3.1.12). Then the estimate (4.1.39) for the chosen situation is given by (3.1.14) of Lemma 3.1.4. To obtain the statement (4.1.39) for arbitrary situations we take the maximum of all n_i, n_o for S1 to S6. □

We have shown that a sufficiently large *finite* number of measurements of the far field patterns determine the boundary of an acoustic scatterer up to a given error ϵ in the Hausdorff distance. This is independent of the physical properties of the scatterer, i.e. the scatterer may be an impenetrable sound-soft or sound-hard obstacle or an inhomogeneous medium scatterer.

If more information about the physical properties of the scatterer is known, we may obtain better estimates for the dependence of n_o and n_i on ϵ; i.e. less measurements are necessary to determine the boundary up to an error ϵ. This is due to the fact that we do not have to build a maximum of all bounds for the different possible situations S1 to S6. The observation indicates, that the ill-posedness of an inverse problem is influenced by the amount of information given for reconstructions.

As a simple consequence of ϵ-uniqueness we obtain the *uniqueness statements* of Corollary 3.1.13 for the support of either impenetrable or inhomogeneous medium scatterers $\mathcal{D} \in \mathcal{C}$.

The second theme of this section is stability in the case of finite data. In this case we do not have uniqueness and thus it cannot be possible to obtain full stability statements. We will show that it is still possible to derive statements close to stability which we refer to as *ϵ-stability*.

The concept of ϵ-stability is closely related both to full stability and to ϵ-uniqueness as introduced in Section 3. To emphasize and enlighten the connections we will give two proofs for the main theorem on ϵ-stability. The *first approach* does not build on stability estimates, but derives the statements of ϵ-stability in a way close to ϵ-uniqueness. The *second approach* shows how the statements of ϵ-stability can be derived from full stability.

The main tool of the first approach is the operator $Q_{(n_0,n_i)}$ defined in (4.1.25). We need some further preparations. To estimate the norm of $Q_{(n_0,n_i)}$ we use the bound

$$b_{n,\tau,\rho} := \max\Big\{ ||a||_{L^2(\mathbb{C}^n)}, a \in \mathcal{G}(n,\tau,\rho,\mu,s,\beta_0) \Big\} \tag{4.1.41}$$

where the norm $||\cdot||_{L^2(\mathbb{C}^n)}$ is given by (4.1.7) and the set $\mathcal{G}(n,\tau,\rho,\mu,s,\beta_0)$ by (4.1.24). In Lemma 4.1.1 it has been shown, that for fixed $\rho,\tau > 0$ the constant $b_{n,\tau,\rho}$ is bounded uniformly for $n_o, n_i \in I\!N$. We define

$$\begin{aligned} F_2(n_o,n_i,\rho,\tau,\eta,\delta) \;\; := \;\; & 2c\frac{\eta}{\rho^{m-1}} \; + \; 2C\; b_{n_i,\eta,\rho}\,\tau \\ & + \frac{1}{\gamma_m} b_{n_i,\eta,\rho} b_{n_o,\tau,\rho}\,\delta \end{aligned} \tag{4.1.42}$$

and

$$\begin{aligned} F_{(n_o,n_i)}(\delta) \; := \; \inf\Big\{ & F_1\Big(\rho, F_2(n_o,n_i,\rho,\tau,\eta,\delta)\Big)\Big|\; \rho,\tau,\eta > 0 \\ & \text{for which (4.1.17) is satisfied for } (\mu,\eta,n_i) \text{ and } (0,\tau,n_o) \Big\} \end{aligned} \tag{4.1.43}$$

with the function F_1 given by (3.1.15) according to the situations S1 to S6. We first study the behavior of $F_{(n_o,n_i)}(\delta)$ for $\delta \to 0$.

Lemma 4.1.8 *Given $\epsilon > 0$ there is $n_o, n_i \in I\!N$, such that the function $F_{(n_o,n_i)}$ defined by (4.1.43) satisfies*

$$\limsup_{\delta\to 0} F_{(n_o,n_i)}(\delta) \le \epsilon. \tag{4.1.44}$$

Proof. For $n_o, n_i \in I\!N$ the function $F_{(n_o,n_i)}(\delta)$ is dominated by

$$F_1\Big(\rho_0, F_2(n_o, n_i, \rho_0, \tau_0, \eta_0, \delta)\Big)$$

for all positive parameters ρ_0, τ_0 and η_0, for which the condition (4.1.17) is satisfied. We show that for $\epsilon > 0$ we can find $n_o, n_i \in I\!N$ and parameters ρ_0, τ_0 and η_0, such that

$$\limsup_{\delta\to 0} F_1\Big(\rho_0, F_2(n_o, n_i, \rho_0, \tau_0, \eta_0, \delta)\Big) \le \epsilon \tag{4.1.45}$$

and (4.1.17) is satisfied for (μ, η_0, n_i), $(0, \tau_0, n_o)$ and ρ_0. We proceed in two steps.

1. The function F_2 can be decomposed into the sum and product

$$F_2(n_o, n_i, \rho, \tau, \eta, \delta) = \; 2c\frac{\eta}{\rho^{m-1}} \; + b_{n_i,\eta,\rho} \, \left(2C\,\tau \; + \; \frac{1}{\gamma_m} b_{n_o,\tau,\rho}\,\delta\right)$$

Since for fixed $\rho, \tau > 0$ the constant $b_{n_o,\tau,\rho}$ is bounded independently of $n_o \in I\!N$, by an application of Corollary 3.1.9 to $f(t,s) := sup_{n\in I\!N} b_{n,t,s}$ we obtain functions h_4 and h_5 with

$$h_4(\delta) \to 0, \;\; \delta \to 0,$$

and

$$h_5(\delta) \to 0, \;\; \delta \to 0, \tag{4.1.46}$$

such that the functions

$$h_6(\delta) \;:=\; 2C\; h_4(\delta) \;+\; \frac{1}{\gamma_m} b^*_{n_o,h_4,h_5(\delta)}\;\delta$$

and

$$h_7(\delta) \;:=\; 2c\frac{h_4(\delta)}{h_5(\delta)^{m-1}} \;+ b^*_{n_i,h_4(\delta),h_5(\delta)}\;\delta$$

with b^* defined in (3.1.49) satisfy

$$h_6(\delta) \;\to\; 0, \;\; \delta \to 0,$$

and

$$h_7(\delta) \to\; 0, \;\; \delta \to 0,$$

uniformly for $n_i, n_o \in I\!N$. For

$$\begin{aligned} h_1(\delta) &:= \max\{h_5(\delta), h_5(h_6(\delta))\}, \\ h_2(\delta) &:= h_4(\delta) \\ h_3(\delta) &:= h_4(h_6(\delta)) \end{aligned}$$

we obtain

$$\begin{aligned} &F_2(n_o, n_i, h_1(\delta), h_2(\delta), h_3(\delta), \delta)\delta \qquad (4.1.47) \\ &= 2c\frac{h_3(\delta)}{h_1(\delta)^{m-1}} + b_{n_i,h_3(\delta),h_1(\delta)}\Big(2Ch_2(\delta) + \frac{1}{\gamma_m} b_{n_o,h_2(\delta),h_1(\delta)}\delta\Big) \\ &\leq 2c\frac{h_4(h_6(\delta))}{h_5(h_6(\delta))^{m-1}} + b^*_{n_i,h_4(h_6(\delta)),h_5(h_6(\delta))}\Big(2Ch_4(\delta) + \frac{1}{\gamma_m} b^*_{n_o,h_4(\delta),h_5(\delta)}\delta\Big) \\ &\to 0, \quad \delta \to 0. \end{aligned}$$

Thus we have

$$F_1\Big(h_1(\delta), F_2(n_o, n_i, h_1(\delta), h_2(\delta), h_3(\delta), \delta)\Big) \to 0, \quad \delta \to 0, \qquad (4.1.48)$$

uniformly for all $n_o, n_i \in I\!N$.

2. For $\epsilon > 0$ from (4.1.48) with the explicit form of F_2 and F_1 we get parameters $\rho_0, \tau_0, \eta_0 > 0$ and some constant C, such that we have

$$\sup_{n_o, n_i \in I\!N} F_2(n_o, n_i, \rho_0, \tau_0, \eta_0, \delta) \leq C$$

and

$$F_1(\rho_0, \sigma) \leq \epsilon$$

for all δ sufficiently small and $0 < \sigma \leq C$. This yields

$$\limsup_{\delta \to 0} \sup_{n_o, n_i \in I\!N} F_1\Big(\rho_0, F_2(n_o, n_i, \rho_0, \tau_0, \eta_0, \delta)\Big) \leq \epsilon.$$

We now explicitly take into account the condition (4.1.17). For ρ_0, τ_0 and η_0 following Lemma 4.1.1 we choose $n_o, n_i \in I\!N$ such that (4.1.17) is satisfied for (μ, η_0, n_i) and $(0, \tau_0, n_0)$. Then we obtain (4.1.45) and thus (4.1.44). □

To prove the following Theorem on ϵ-stability we will use the approximation of the fields Φ^s by the operator $Q_{(n_0,n_i)}$ applied to

$$u^\infty_{(n_o,n_i)} = \Big(u^\infty(d_j, d_k)\Big)_{j=1,..,n_0, k=1,...,n_i} \in L^2(\Omega_{n_o} \times \Omega_{n_i})$$

to derive an estimate for the Hausdorff distance $d(D_1, D_2)$ between the domains of two scatterers $\mathcal{D}_1$ and $\mathcal{D}_2$ in $\mathcal{C}$.

THEOREM 4.1.9 (**ϵ-stability**) *Let $u_1^\infty(\hat{x}, d)$ and $u_2^\infty(\hat{x}, d)$, $\hat{x}, d \in \Omega$, be the far field patterns for scattering of acoustic plane waves from two scatterers $\mathcal{D}_1, \mathcal{D}_2$ in $\mathcal{C}$. Given $\epsilon > 0$ there are even integers $n_o, n_i \in I\!N$ such that, if for a nonnegative parameter δ the far field patterns satisfy*

$$\left|\left|u_{1(n_o,n_i)}^\infty - u_{2(n_o,n_i)}^\infty\right|\right|_{L^2(\Omega_{n_o} \times \Omega_{n_i})} \;\le\; \delta, \tag{4.1.49}$$

the Hausdorff distance $d(D_1, D_2)$ of the two scatterers can be estimated by

$$d(D_1, D_2) \;\le\; F_{(n_o,n_i)}(\delta). \tag{4.1.50}$$

Proof. Let $\mu \in I\!N_0$ be given according to the situations S1 to S6 defined in (3.1.11). For $\epsilon > 0$ following Lemma 4.1.8 there are even integers $n_o, n_i \in I\!N$, such that the function $F_{(n_o,n_i)}$ satisfies (4.1.44). We now keep n_i, n_o fixed and apply Theorem 4.1.2 to each choice of $\rho, \tau, \eta > 0$, for which the conditions (4.1.17) and (4.1.17) are satisfied. Then for a function $p : B \to \Omega$ and $q \in \Omega$ an application of Theorem 4.1.4 to the scatterer $\mathcal{D}_j$, $j = 1, 2$, yields

$$\left|\Phi_{j,\mu,q}^s(x, z) - \left(Q_{(n_o,n_i)} u_{j,(n_o,n_i)}^\infty\right)(x, z)\right| \;\le\; c\,\frac{\eta}{\rho^{m-1}} \;+ C\, b_{n_i,\eta,\rho}\,\tau \tag{4.1.51}$$

for all $x, z \in B$, for which (3.1.34) is satisfied. We use (4.1.51) for $\mathcal{D}_1$ and $\mathcal{D}_2$ and the Cauchy-Schwarz inequality applied to

$$Q(u_{1,\mu,q}^\infty - u_{2,\mu,q}^\infty)$$

to derive from

$$\begin{aligned} \left|\Phi_{1,\mu,q}^s(x, z) - \Phi_{2,\mu,q}^s(x, z)\right| \;\le\; & \left|\Phi_{1,\mu,q}^s(x, z) - \left(Q u_1^\infty\right)(x, z)\right| \\ & + \left|Q\left(u_1^\infty - u_2^\infty\right)(x, z)\right| \;+\; \left|\left(Q u_1^\infty\right)(x, z) - \Phi_{2,\mu,q}^s(x, z)\right| \end{aligned} \tag{4.1.52}$$

the estimate

$$\left|\Phi_{1,\mu,q}^s(x, z) - \Phi_{2,\mu,q}^s(x, z)\right| \;\;\le\;\; F_2(n_o, n_i, \rho, \tau, \eta, \delta) \tag{4.1.53}$$

for all $q \in \Omega$ and all points $x, z \in B \setminus \left(D_{1,\rho} \cup D_{2,\rho}\right)$, for which cones $\mathrm{co}(x, p_x, \beta_0)$ and $\mathrm{co}(z, p_z, \beta_0)$, $p_x, p_z \in \Omega$, in the exterior of $D_{1,\rho} \cup D_{2,\rho}$ exist. Now we apply Lemma 3.1.4 to obtain

$$d(D_1, D_2) \;\le\; F_1(\rho, F_2(n_o, n_i, \rho, \tau, \eta, \delta)). \tag{4.1.54}$$

Since this is valid for each choice of the parameters ρ, τ, η satisfying (4.1.17) and (4.1.17), we obtain the estimate (4.1.50).

□

We now come to the second approach to prove the estimates of ϵ-stability. For Ω_n given by (4.1.2) we know that

$$\tau(n) := \max_{x\in\Omega} \min_{j=1,..,n} d(x, d_j) \ \to\ 0, \quad n\to\infty. \tag{4.1.55}$$

We will prove Theorem 4.1.9 with the function $F_{(n_o,n_i)}$ replaced by

$$F_{(n_o,n_i)}(\delta) \ := \ F\Big(16\pi^2(4\pi C_\infty(\tau(n_o)+\tau(n_i))+\delta)\Big), \quad \delta\geq 0, \tag{4.1.56}$$

where F is given by (3.1.48).

From Theorem 2.1.14 for impenetrable scatterers and Theorem 2.2.7 for the inhomogeneous medium scatterer we obtain that the set of far field patterns $u^\infty(\cdot, d)\in C^1(\Omega)$ is bounded uniformly for $d\in\Omega$ and scatterers $\mathcal{D}\in\mathcal{C}$. By the same arguments this can be seen to be true for the surface gradient $\mathrm{Grad}_d\, u^\infty(\cdot,d)$. Thus the functions $u^\infty(\cdot,\cdot)$ are bounded in $C^1(\Omega\times\Omega)$ by a constant C_∞ uniformly for all scatterers $\mathcal{D}\in\mathcal{C}$. Using the mean value theorem we have for $n_o, n_i\in I\!N$ and $j=1,2$

$$\Big|u_j^\infty(\hat{x},\hat{y}) - u_j^\infty(d_l,d_k)\Big| \ \leq\ 2\pi||u_j^\infty||_{C^1(\Omega\times\Omega)}\Big(d(\hat{x},d_l)+d(\hat{y},d_k)\Big). \tag{4.1.57}$$

From (4.1.49) and (4.1.57) we obtain

$$\begin{aligned}\Big|u_1^\infty(\hat{x},\hat{y}) - u_2^\infty(\hat{x},\hat{y})\Big| \ &\leq\ \Big|u_1^\infty(\hat{x},\hat{y}) - u_1^\infty(d_l,d_k)\Big| \\ &\quad + \Big|u_1^\infty(d_l,d_k) - u_2^\infty(d_l,d_k)\Big| \\ &\quad + \Big|u_2^\infty(d_l,d_k) - u_2^\infty(\hat{x},\hat{y})\Big| \\ &\leq\ 4\pi C_\infty\Big(d(\hat{x},d_l)+d(\hat{y},d_k)\Big)+\delta. \end{aligned} \tag{4.1.58}$$

By integration of $|u_1^\infty - u_2^\infty|^2$ with the help of (4.1.55) we now derive

$$\Big|\Big|u_1^\infty - u_2^\infty\Big|\Big|_{L^2(\Omega\times\Omega)} \ \leq\ 16\pi^2\Big(4\pi C_\infty(\tau(n_o)+\tau(n_i))+\delta\Big). \tag{4.1.59}$$

Given $\epsilon>0$ because of (4.1.55) and the behavior (3.1.52) of F it is possible to choose $n_o, n_i\in I\!N$ such that

$$F\Big(64\pi^3 C_\infty(\tau(n_o)+\tau(n_i))\Big) \ \leq\ \epsilon.$$

Then for $F_{(n_0,n_i)}$ defined by (4.1.56) we calculate (4.1.44). Finally, from (4.1.59) and Theorem 3.1.11 we derive (4.1.50) with $F_{(n_o,n_i)}$ defined by (4.1.56).

It is an important question to explicitly determine the behavior of $F_{(n_o,n_i)}(\delta)$ for $\delta \to 0$ and for $n_i, n_0 \to \infty$. For the reconstruction of the convec hull $\mathcal{H}(\mathcal{D})$ of a scatterer $\mathcal{D} \in \mathcal{D}$ we will now answer this question.

THEOREM 4.1.10 *Let $u_1^\infty(\hat{x}, d)$ and $u_2^\infty(\hat{x}, d)$, $\hat{x}, d \in \Omega$, be the far field patterns for scattering of acoustic plane waves two from scatterers $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{C}$. Given $\epsilon > 0$ there is $n_o, n_i \in I\!N$ such that, if for a nonnegative parameter δ the far field patterns satisfy*

$$\left|\left| u_{1(n_o,n_i)}^\infty - u_{2(n_o,n_i)}^\infty \right|\right|_{L^2(\Omega_{n_o} \times \Omega_{n_i})} \leq \delta, \tag{4.1.60}$$

the Hausdorff distance $d(\mathcal{H}(D_1), \mathcal{H}(D_2))$ of the convex hulls of two scatterers $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{C}$ can be estimated by

$$d(\mathcal{H}(D_1), \mathcal{H}(D_2)) \leq \frac{C}{\left| \ln \left[16\pi^2 \left(4\pi C_\infty (\tau(n_o) + \tau(n_i)) + \delta \right) \right] \right|^c} \tag{4.1.61}$$

with constants $C > 0$, $0 < c < 1$ and C_∞.

Proof. We proceed analogously to the second proof of Theorem 4.1.9, where the role of Theorem 3.1.11 has to be replaced by Theorem 3.1.12. □

4.2 Electromagnetic scattering.

For electromagnetic scattering to obtain uniqueness and ϵ-uniqueness in principle we can proceed in the same way as for acoustic scattering. For diversity here we use a second approach to ϵ-uniqueness via the results of stability. The following theorem includes scattering from a perfect conductor or from an inhomogeneous electromagnetic medium.

THEOREM 4.2.1 (**ϵ-uniqueness**) *Let $E^{\infty}_{pl,1}(\hat{x}, d, q)$ and $E^{\infty}_{pl,2}(\hat{x}, d, q)$ be the electric far field patterns for scattering of plane waves from two electromagnetic scatterers $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{C}$. Given $\epsilon > 0$ there is $n_o, n_i, n_{pol} \in \mathbb{N}$, such that*

$$E^{\infty}_{pl,1}(\hat{x}, d, q) = E^{\infty}_{pl,2}(\hat{x}, d, q), \quad \hat{x} \in \Omega_{n_o}, \ d \in \Omega_{n_i}, \ q \in \Omega_{n_{pol}} \tag{4.2.1}$$

yields the estimate

$$d(D_1, D_2) \leq \ \epsilon \tag{4.2.2}$$

for the Hausdorff distance $d(D_1, D_2)$ of D_1 and D_2.

Proof. We base our proof on the stability estimates. A stability estimate consists of a function $F : \mathbb{R}^+ \to \mathbb{R}^+$

$$F(\delta) \to 0, \ \ \delta \to 0, \tag{4.2.3}$$

such that for two scatterers $\mathcal{D}_1$ and $\mathcal{D}_2$ in $\mathcal{C}$ with electric far field patterns $E^{\infty}_{pl,1}(\hat{x}, d, q)$ and $E^{\infty}_{pl,2}(\hat{x}, d, q)$, respectively, the estimate

$$\Big|\Big| E^{\infty}_{pl,1}(\cdot, \cdot, q) - E^{\infty}_{pl,2}(\cdot, \cdot, q) \Big|\Big|_{L^2(\Omega \times \Omega)} \ \leq \ \delta, \ \ q \in \Omega, \tag{4.2.4}$$

yields

$$d(D_1, D_2) \ \leq \ F(\delta). \tag{4.2.5}$$

From Theorem 2.3.11 for the perfect conductor and Theorem 2.4.7 for the inhomogeneous electromagnetic medium we obtain bounds for the far field patterns $E^{\infty}_{pl,j}(\cdot, \cdot, \cdot)$, $j = 1, 2$, in $C^1(\Omega \times \Omega \times \Omega)$ uniformly for scatterers $\mathcal{D}_j \in \mathcal{C}$. Thus given $\epsilon > 0$ and δ with $F(\delta) = \epsilon$, we can find n_o, n_i and n_{pol} in $\mathbb{N}$ such that

$$E^{\infty}_{pl,1}(\hat{x}, d, q) = E^{\infty}_{pl,2}(\hat{x}, d, q), \quad \hat{x} \in \Omega_{n_o}, \ d \in \Omega_{n_i}, \ q \in \Omega_{n_{pol}} \tag{4.2.6}$$

yields

$$\Big|\Big| E^{\infty}_{pl,1}(\cdot, \cdot, q) - E^{\infty}_{pl,2}(\cdot, \cdot, q) \Big|\Big|_{L^2(\Omega \times \Omega)} \ \leq \ \delta \tag{4.2.7}$$

for all $q \in \Omega$. Now from (4.2.7) and (4.2.5) we obtain (4.2.2) and the proof is complete. □

The brevity of the preceding proof compared to the lengthy estimates for the acoustic case may be surprising at a first glance. Of course, the main work to obtain estimates has to be done at some place. Here it is hidden in the stability estimate (4.2.5), proven in the preceding chapter.

In a similar way it is possible to derive the results of ϵ-stability for the above inverse electromagnetic scattering problems. To avoid repetitions we omit the presentation.

5 A point-source method in inverse scattering.

In the preceding sections we used point-sources to obtain uniqueness and stability results for inverse scattering problems. We will now investigate the reconstruction of the domain D of scatterers $\mathcal{D}$ from the algorithmical and numerical viewpoint.

As a starting point we develop a *point-source method* to reconstruct the scattered field $u^s(z,d)$ in the exterior of a scatterer $\mathcal{D} \in \mathcal{C}$. More explicitly, given a scatterer $\mathcal{D} \in \mathcal{C}$ and $\rho, \tau > 0$ we will construct a kernel

$$g_\tau(z,\hat{x}), \quad z \in B, \quad \hat{x} \in \Omega,$$

such that the reconstruction of the scattered field $u^s(z,d)$ by

$$(Au^\infty(\cdot,d))(z) := \int_\Omega g_\tau(z,\hat{x}) u^\infty(\hat{x},d) ds(\hat{x}), \quad z \in B \setminus D_\rho.$$

satisfies the error estimate

$$\Big|\Big| u^s(\cdot,d) - Au^\infty(\cdot,d) \Big|\Big|_{L^\infty(B\setminus D_\rho)} \leq \tau. \tag{5.0.1}$$

Analogous estimates will be derived for electromagnetic scattering from a *perfect conductor* or an *inhomogeneous electromagnetic medium*. We will prove the *convergence* of the point-source method to reconstruct u^s, i.e. given a family of measured far field patterns u_δ^∞, $\delta \in (0,1)$, with

$$||u^\infty(\cdot,d) - u_\delta^\infty||_{L^2(\Omega)} \leq \delta$$

we will show that we can choose $\tau = \tau(\delta)$ such that

$$\Big|\Big| u^s(\cdot,d) - Au_\delta^\infty \Big|\Big|_{L^\infty(B\setminus D_\rho)} \to 0, \quad \delta \to 0.$$

The operator A is a kind of backprojection operator as used for the Backus-Gilbert or mollifier methods, see [35], [12] and [45].

In a second step we will use the scattered field u^s, the known incident field u^i and the boundary conditions of an impenetrable scatterer $\mathcal{D}$ for reconstructions of the domain D of $\mathcal{D}$. Numerical examples are provided for acoustic scattering in three dimensions.

A different approach to the point-source method, which does not use the reciprocity relations, has been developed in [61], [62] and [63].

5.1 Acoustic obstacle scattering.

For the construction of the operator A we need to investigate the approximation of point-sources by a superposition of plane waves as in Section 3.

DEFINITION 5.1.1 *For $\mu = 0, s = 1,\ \rho, \tau > 0$ and a function $p : B \to \Omega$ let the density $g_\tau(z, \cdot)$ be given by (3.1.32). We define the operator*

$$A : L^2(\Omega) \to L^\infty(B)$$

by

$$(Aw)(z) \ := \ \frac{1}{\gamma_m} \int_\Omega g_\tau(z, d) w(-d)\, ds(d), \quad z \in B. \tag{5.1.1}$$

The error for the reconstruction of a scattered field $u^s(\cdot, d)$ from its far field pattern $u^\infty(\cdot, d)$ is estimated in the following theorem.

THEOREM 5.1.2 *Consider scattering by a sound-soft, sound-hard or inhomogeneous medium scatterer $\mathcal{D} \in \mathcal{C}$. For $\mu = 0, s = 1,\ \rho, \tau > 0$ and a vector field $p : B \to \Omega$ with*

$$D \subset G_{x,p(x),\rho}, \quad x \in B \setminus D_\rho, \tag{5.1.2}$$

let the operator A be given by (5.1.1). Then for fixed $d \in \Omega$ and a measured far field pattern $u_\delta^\infty(\cdot, d)$ with

$$\Big|\Big| u^\infty(\cdot, d) - u_\delta^\infty(\cdot, d) \Big|\Big|_{L^2(\Omega)} \ \le \ \delta \tag{5.1.3}$$

the error for the approximation of $u^s(\cdot, d)$ by $Au_\delta^\infty(\cdot, d)$ satisfies

$$\Big| u^s(z, d) - (Au_\delta^\infty(\cdot, d))(z) \Big| \ \le \ c\tau + Cb_{\tau,\rho}\, \delta, \quad z \in B \setminus D_\rho, \tag{5.1.4}$$

with the constant $b_{\tau,\rho}$ defined by (3.1.33) and constants $c, C > 0$ uniformly for scatterers $\mathcal{D} \in \mathcal{C}$.

Remark. The elements of the following proof have already been used to investigate the reconstruction properties of the operator Q.

Proof. First, for $\tau > 0$ and $z \in B \setminus D_\rho$ by definition of g we have

$$\Big|\Big|\Phi_{\mu,q}(\cdot,z) - Hg_\tau(z,\cdot)\Big|\Big|_{C^1(D)} \leq \tau. \tag{5.1.5}$$

The scattering map maps the incident field in $C^1(\overline{D})$ boundedly onto the far field pattern of the scattered field in $L^2(\Omega)$. Thus we obtain a constant c such that

$$\Big|\Big|\Phi^\infty(\cdot,z) - H^\infty g_\tau(z,\cdot)\Big|\Big|_{C(\Omega)} \leq c\,\tau \tag{5.1.6}$$

We use the mixed reciprocity relations (2.1.4) and (2.2.4) to transform (5.1.6) into

$$\Big|u^s(z,d) - \frac{1}{\gamma_m}\Big(H^\infty g_\tau(z,\cdot)\Big)(-d)\Big| \leq \frac{c}{\gamma_m}\tau \tag{5.1.7}$$

for all $d \in \Omega$. From the decomposition

$$\begin{aligned} \Big|u^s(z,d) - \Big(Au_\delta^\infty(\cdot,d)\Big)(z)\Big| \;\leq\; & \Big|u^s(z,d) - \frac{1}{\gamma_m}\Big(H^\infty g_\tau(z,\cdot)\Big)(-d)\Big| \\ & + \Big|\frac{1}{\gamma_m}\int_\Omega g_\tau(z,\tilde{d})\Big\{u^\infty(-\tilde{d},d) - u_\delta^\infty(-\tilde{d},d)\Big\}\,ds(\tilde{d})\Big| \end{aligned} \tag{5.1.8}$$

with the help of the Cauchy-Schwarz inequality we derive (5.1.4). □

We use the operator A to reconstruct the field $u^s(\cdot,d)$ from a measured far field pattern u_δ^∞. Given the error δ in the measurements, the error of the reconstructions is given by (5.1.4). For a family of measurements u_δ^∞, $\delta \in (0,1)$, with

$$||u_\delta^\infty - u^\infty(\cdot,d)||_{L^2(\Omega)} \to 0,\ \delta \to 0,$$

by an application of Lemma 3.1.8 there is a function h with

$$h(\delta) \to 0, \quad \delta \to 0,$$

such that A defined with $\tau = h(\delta)$ satisfies

$$\Big|\Big|u^s(\cdot,d) \;-\; Au_\delta^\infty\Big|\Big|_{C(B\setminus D_\rho)} \to 0, \quad \delta \to 0.$$

We have proven the *convergence* of the point-source method for the reconstruction of the scattered field $u^s(\cdot,d)$.

The kernel $g_\tau(z,\cdot)$ of the operator A can be calculated according to the a-priori knowledge about the unknown scatterers $\mathcal{D}$ given by the class $\mathcal{C}$ of scatterers (see Definitions 2.1.5 and 2.2.5). In a first step the set of densities $\mathcal{G}$ has to be computed according to Lemma 3.1.3 . Then $g_\tau(z,\cdot)$ is given by (3.1.30). Clearly the computation of the function $g_0 \in L^2(\Omega)$ with

$$\Big|\Big|\Phi(\cdot,0) - Hg_0\Big|\Big|_{C^s(G_{0,p,\rho})} \leq \tau \tag{5.1.9}$$

for each $p \in \Omega$ and $s = 1$ is not effective from a computational point of view. We now describe a possibility to obtain $g_\tau(z,\cdot)$ by the computation of only *one* solution $g_0 \in L^2(\Omega)$ of (5.1.9).

LEMMA 5.1.3 *For a vector $p_0 \in \Omega$ let g_0 be a solution of (5.1.9) and let $p_1 \in \Omega$ be given. Then the minimum norm solution g_1 for (5.1.9) with p_0 replaced by p_1 can be obtained from g_0 by rotation*

$$g_1(d) := g_0(M^{-1}d) \tag{5.1.10}$$

with an orthogonal matrix M which satisfies $p_1 = Mp_0$.

Proof. The proof is obtained by a simple rotation of the domain of approximation and the Herglotz wave function from the fact that the rotated Herglotz wave function

$$\begin{aligned} \int_\Omega e^{i\kappa M^{-1}x\cdot d} g(d)ds(d) &= \int_\Omega e^{i\kappa x\cdot Md} g(d)ds(d) \\ &= \int_\Omega e^{i\kappa x\cdot \tilde{d}} g(M^{-1}\tilde{d})ds(\tilde{d}) \end{aligned} \tag{5.1.11}$$

is again a Herglotz wave function with the the rotated density $g(M^{-1}d)$. □

We now investigate the reconstruction of an unknown impenetrable scatterer $\mathcal{D}$ from the knowledge of the far field pattern for scattering of a plane wave. As described in Theorem 5.1.2 it is possible to use the operator A to reconstruct $u^s(\cdot,d)$ in the exterior of $\mathcal{D}$.

We first consider the reconstruction of a *sound-soft* scatterer. In this case we can use the boundary condition (2.1.4) to find the unknown scatterer as a minimum curve of the total field

$$u^s(\cdot,d) + u^i(\cdot,d). \tag{5.1.12}$$

But so far the construction of the operator A to compute an approximation for u^s is built on the knowledge of D. The construction of the kernel $g_\tau(z,\cdot)$ of A to obtain the approximation (5.1.4) assumes the knowledge of adequate directions $p = p(z)$ such that

$$\overline{D} \subset G_{z,p(z),\rho}, \quad z \in B \setminus D_\rho, \tag{5.1.13}$$

is satisfied, i.e. it assumes the knowledge of the unknown scatterer. Clearly we do not know the scatterer and a knowledge for the choice of $p(z)$ is not available, when we start the algorithm. This gives rise to different strategies to choose $p(z)$ in a multistep procedure.

We now describe one simple strategy to reconstruct u^s and p in several steps. We start with a number of fixed directions $p(z) = p_\xi$ for $\xi = 1, ..., N$, compute the corresponding operators A_ξ and the fields

$$\Big(A_\xi u_\delta^\infty\Big)(z) + u^i(z,d). \tag{5.1.14}$$

For each $\xi = 1, ..., N$ we search for parts of the unknown boundary ∂D of the domain D as the minimum curve of (5.1.14) and obtain a first approximation ∂D_1 to the boundary ∂D. In each further step we adapt the choice of $p(z)$ according to the reconstruction ∂D_n of the n-th step to achieve the condition (5.1.13) for further points $z \in B$ and obtain an approximation ∂D_{n+1}. A stopping criterion is provided by the condition (5.1.13), which has to be satisfied for the current choice of p and the current reconstruction D_n. Efficient adaptive algorithms for the choice of p will have to be part of further research (see also [14]).

For the *sound-hard* scatterer we have to modify the approach according to the different boundary condition

$$\frac{\partial}{\partial \nu}\Big(u^s(\cdot,d) + u^i(\cdot,d)\Big) = 0. \tag{5.1.15}$$

So far, we described the reconstruction of u^s. But to use the normal derivative in (5.1.15) the reconstruction of ∇u^s is needed.

As a consequence of the linearity and boundedness of the scattering operator $\mathcal{S} : u^i \mapsto u^\infty$ for scattering of point-sources $\Phi(\cdot,z)$ the operator $\mathcal{S}$ and the differentiation ∇_z with respect to the source point can be exchanged. This is a simple consequence of the linearity and boundedness of $\mathcal{S}$. Thus we have

$$\begin{aligned} \nabla_z \Phi^\infty(-d,z) &= \nabla_z \mathcal{S}\Big(\Phi(\cdot,z)\Big)(-d) \\ &= \mathcal{S}\Big(\nabla_z \Phi(\cdot,z)\Big)(-d). \\ &= -\mathcal{S}\Big(\nabla \Phi(\cdot,z)\Big)(-d). \end{aligned} \tag{5.1.16}$$

Due to Lemma 3.1.3 there is an approximation of $\nabla\Phi(\cdot, z)$ by a Herglotz wave function

$$\nabla_y \int_\Omega e^{i\kappa y\cdot d} g_\tau(z,d)\, ds(d) \;=\; \int_\Omega i\kappa d\, e^{i\kappa y\cdot d} g_\tau(z,d)\, ds(d).\;\; y \in B \setminus D_\rho, \qquad (5.1.17)$$

We define

$$(A'w)(z) \;:=\; \frac{1}{\gamma_m} \int_\Omega (-i\kappa d)\; g_\tau(z,d)\; w(-d)\; ds(d),\;\; z \in B \setminus D_\rho \qquad (5.1.18)$$

and obtain an approximation of $\nabla u^s(\cdot, d)$ by $A' u^\infty(\cdot, d)$ with the error estimate

$$\Big|\nabla_z u^s(z,d) - (A' u_\delta^\infty(\cdot, d))(z)\Big| \;\le\; c\tau + C b_{\tau,\rho}\, \delta, \qquad (5.1.19)$$

analogous to (5.1.4). For the reconstruction of a scatterer $\mathcal{D}$ with sound-hard boundary condition we can now proceed analogously to the sound-soft case, where the sound-soft boundary condition (5.1.12) has to be replaced by the sound-hard condition (5.1.15).

Before we present numerical results, let us compare the design of the above *point-source method* to the reconstruction of the unknown scatterer with a related method of Kirsch and Kress [6].

First, Kirsch and Kress search for the unknown scattered field $u^s(\cdot, d)$ as a single-layer potential $S\varphi$ on a curve Γ with has to be in the interior of the unknown scatterer $\mathcal{D}$. Thus the method needs to know some apriori-knowledge about the location of the scatterer. In contrast to this, the point-source method does need to know only a rough bound on the size of the scatterer but no information about its location. Also the point-source method allows the reconstruction of scatterers, which consist of an unknown number of separated components.

Second, for the reconstruction of $u^s(\cdot, d)$ by a single-layer potential without the simultaneous reconstruction of the full domain we do not obtain convergence of the reconstructions, if the data error tends to zero. Even if the exact far field pattern u^∞ of a scattered field u^s is given, in general we will not observe convergence of the approximations to u^s. This is due to the ill-posedness of this inverse problem and the fact that in general the far field pattern u^∞ is not in the range of the single-layer potential operator. Using the single-layer potential we cannot control the regularization error, which due to (5.1.4) and (5.1.19) is possible for the point-source method.

To obtain convergence, Kirsch and Kress had to combine the solution of the far field equation

$$S\varphi = u^{\infty}$$

and the minimization problem

$$\min_{\partial D} \; ||u^i + S\varphi||_{L^2(\partial D)}$$

to find the unknown boundary ∂D from the approximation u_{approx} for the total field u into *one* nonlinear optimization problem. Thus convergence needs the reconstruction of the full scatterer.

With the point-source method it is possible to reconstruct the scattered field $u^s(\cdot, d)$ on arbitrary subsets of the exterior of D. It is possible to obtain reconstructions of parts of ∂D without consideration of other parts. The method still involves optimization problems for the reconstructions, that is the search of parts of the unknown boundary as a minimum curve from the reconstruction u_{approx} of the total field u. But the reconstruction of u is given by an application of the integral operator A and the search for parts of ∂D can be performed locally, thus we may split the problem into a series of optimization problems and the dimension of each of these optimization problems can be chosen arbitrarily small. Especially in three dimensions this reduces the reconstruction time considerably.

In the following we show numerical results by Giebermann and Potthast [14], who used the point-source method for the reconstruction of scatterers in three dimensions. The measured data consist of the far field patterns for six different incident plane waves at 256 observation points.

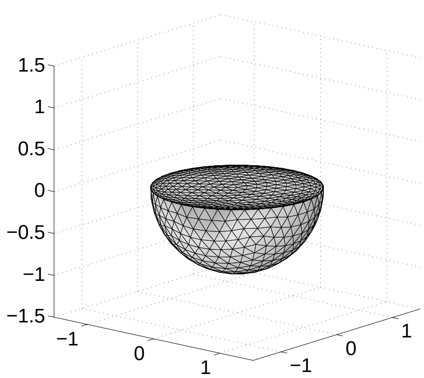

Figure 5.1

Figure 5.2

Figure 5.2 shows a cut of the reconstruction in the x-z-plane in the second second step of the reconstruction algorithm. The first step is given by Figure 5.3 to Figure 5.8, i.e. the minimum curves for different waves and orientations of the domain of approximation. With the direction of incidence d and $p = -d$ we show the minimum curves of $|u_{approx}|$ for $d = (0, 0, -1)$ in Figure 5.3, $d = (0, 0, 1)$ in Figure 5.4, $d = (0, 1, 0)$ in Figure 5.5, $d = (-1, 0, 0)$ in Figure 5.6, $d = (0, -1, 0)$ in Figure 5.7 and $d = (0, 0, 1)$ in Figure 5.8.

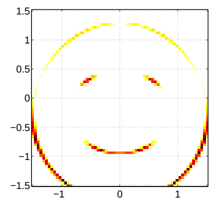

Figure 5.3

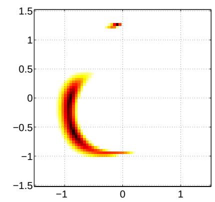

Figure 5.4

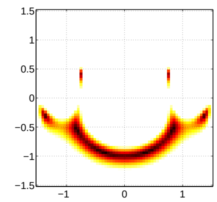

Figure 5.5

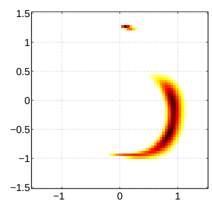

Figure 5.6

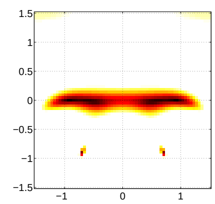

Figure 5.7

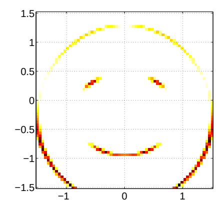

Figure 5.8

Instead of a further minimization here we built the union of the minimum points and then removed those points from the union, which according to the orientation of the domain of approximation in comparison with the union of minimum points for the six steps could not be part of the boundary of the scatterer. The union of the minimum points and the result of the removal-step can be seen in Figure 5.9, 5.10 and 5.2.

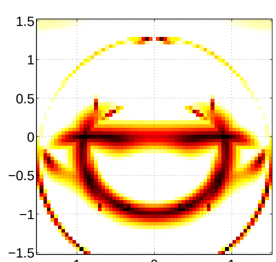

Figure 5.9

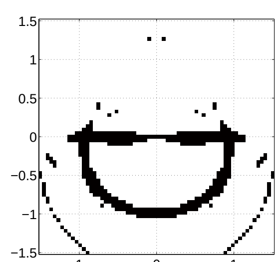

Figure 5.10

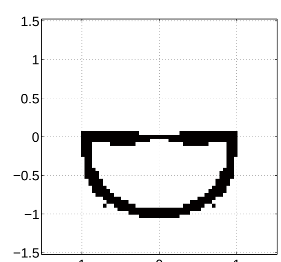

Figure 5.2 (see above)

In a second example we want to show, that we can reconstruct obstacles which consist out of several separate components. Figure 5.11 shows two balls which are reconstructed with the point-source method. Here we again use six different waves and show the second step, i.e. the union of the minimum points, which are this time shown in a full three-dimensional plot in Figure 5.12 without an application of a removal-step.

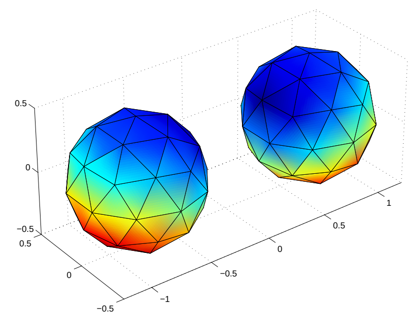

Figure 5.11

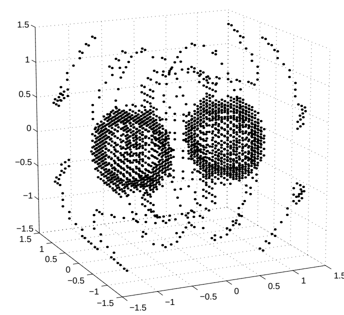

Figure 5.12

5.2 Electromagnetic scattering by a perfect conductor.

The operator A defined by (5.1.1) for the reconstruction of the scattered acoustic fields can be used also for the reconstruction of the scattered electromagnetic field $E^s_{pl}(z,d,p)$, $z \in B \setminus D_\rho$, from its far field pattern $E^\infty_{pl}(\cdot,d,p)$. For the electromagnetic case we have to choose the kernel $g_\tau(z,\cdot)$ such that the estimates (3.1.31) or (5.1.9), respectively, are satisfied with $s=4$. The following theorem investigates not only the perfect conductor, but also an inhomogeneous electromagnetic medium.

THEOREM 5.2.1 *Consider the scattering of electromagnetic waves by a perfect conductor or an inhomogeneous medium scatterer $\mathcal{D} \in \mathcal{C}$. For $\mu = 0, s = 4$, $\rho, \tau > 0$ and a vector field $p : B \to \Omega$ satisfying (5.1.2) the operator A is defined by (5.1.1). Let the measured far field pattern E^∞_δ satisfy the data error estimate*

$$\Big|\Big| E^\infty_{pl}(\cdot,d,p) \; - \; E^\infty_\delta \Big|\Big|_{L^2(\Omega,I\!R^3)} \; \le \; \delta. \tag{5.2.1}$$

Then the error for the approximation of $E^s_{pl}(\cdot,d,p)$ by AE^∞_δ is estimated by

$$\Big| E^s(z,d,p) \; - \; \Big(AE^\infty_\delta\Big)(z)\Big| \; \le \; c\tau \; + \; C||g_0||_{L^2(\Omega)}\delta, \;\; z \in B \setminus D_\rho, \tag{5.2.2}$$

with constants $c, C > 0$ and g_0 defined by Lemma 5.1.3.

Proof. Given $\rho, \tau > 0$ and $z \in B \setminus D_\rho$ by the definition of g_τ we have

$$\begin{aligned}
&\Big|\Big| E^i_{edp}(\cdot,z,q) - \int_\Omega E^i_{pl}(\cdot,\tilde{d},p) g_\tau(z,\tilde{d}) \; ds(\tilde{d}) \Big|\Big|_{C^2(\overline{D})} \\
&= \Big|\Big| (q\cdot\nabla)\nabla \left(\Phi(\cdot,z) - \int_\Omega e^{i\kappa\cdot\cdot\tilde{d}} g_\tau(z,\tilde{d}) \; ds(\tilde{d}) \right) \Big|\Big|_{C^2(\overline{D})} \\
&\le \Big|\Big| \Phi(\cdot,z) - \int_\Omega e^{i\kappa\cdot\cdot\tilde{d}} g_\tau(z,\tilde{d}) \; ds(\tilde{d}) \Big|\Big|_{C^4(\overline{D})} \\
&\le \; \tau
\end{aligned} \tag{5.2.3}$$

for all $q \in \Omega$. The mapping of the incident electric field in $C^2(\overline{D})$ onto the far field pattern of the scattered electric field in $C^1(\Omega)$ is bounded uniformly for scatterers $\mathcal{D} \in \mathcal{C}$. Thus there is a constant c such that

$$\Big|\Big| E^\infty_{edp}(\cdot,z,q) - \int_\Omega E^\infty_{pl}(\cdot,\tilde{d},p) g_\tau(z,\tilde{d}) \; ds(\tilde{d}) \Big|\Big|_{C^1(\Omega)} \; \le \; c\,\tau. \tag{5.2.4}$$

With the help of the reciprocity relations (2.3.20), (2.3.22), (2.4.7) and (2.4.8) from (5.2.4) we obtain the estimate

$$\Big| E^s_{pl}(z,d,q) - \frac{1}{\gamma} \int_\Omega E^\infty_{pl}(-\tilde{d},d,p) g_\tau(z,\tilde{d}) \; ds(\tilde{d}) \Big|_{C^1(\Omega)} \; \le \; \frac{c}{\gamma}\,\tau \tag{5.2.5}$$

for $z \in B \setminus D_\rho$. Using the Cauchy-Schwarz inequality we calculate the norm of the operator A and estimate

$$\begin{aligned} \Big|E^s(z,d,p) \ - \ \Big(AE_\delta^\infty\Big)(z)\Big| \ &\leq \ \Big|E^s(z,d,p) \ - \ \Big(AE_{pl}^\infty(\cdot,d,p)\Big)(z)\Big| \\ &\quad + \Big|A\Big(E_{pl}^\infty(\cdot,d,p) \ - \ E_\delta^\infty\Big)(z)\Big| \\ &\leq \ \frac{c}{\gamma}\tau \ + \ \frac{4\pi}{\gamma}\,||g_0||_{L^2(\Omega)}\delta. \end{aligned} \tag{5.2.6}$$

This completes the proof. □

We can use the preceding theorem to formulate a *point-source method* for the reconstruction of a perfect conductor $\mathcal{D}$ from the knowledge of a measured far field pattern E_δ^∞ for an incident plane wave $E_{pl}^i(\cdot,d,p)$. According to (5.2.2) for an appropriate choice of the parameters ρ,τ and the function $p : B \to \Omega$ the function AE_δ^∞ approximates the scattered field $E_{pl}^s(\cdot,d,p)$ on $B \setminus D_\rho$. We can use the boundary condition (2.3.5) to search for parts Λ of the unknown boundary ∂D of the scatterer $\mathcal{D}$ as a surface where

$$\Big|\nu_\Lambda(\cdot) \times \Big(E_{pl}^i(\cdot,d,p) + AE_\delta^\infty\Big)\Big|$$

is small. Here ν_Λ is a function $B \to \Omega$, which coincides with the normal vector ν to Λ on Λ.

We face similar problems as for the reconstruction of sound-hard acoustic scatterers. First, the choice of $g_\tau(z,\cdot)$ involves the orientation $p \in \Omega$ of the domain of approximation $G_{z,p,\rho}$, which has to be chosen such that the unknown scatterer $\mathcal{D}$ is in the interior of $G_{z,p(z),\rho}$. Second, the boundary condition involves the normal vector to the unknown surface.

For the reconstruction of a perfect conductor in principle the strategy, which we suggested for the acoustic inverse scattering problems, can be used to overcome the problems and reconstruct the scatterer in a multistep procedure.

6 Singular sources and shape reconstruction.

The methods discussed in the preceding Section 5 need to know the boundary condition to reconstruct the unknown scatterer. But in many practical situations the physical properties of the scatterer are not known and these methods are not applicable. It is therefore of practical interest to develop reconstruction methods which do not need to know the boundary condition or the physical properties of a scatterer.

We will develop a *method of singular sources* for the reconstruction of a scattering object when the physical properties of the scatterer are not known. In a first step we will consider a sound-soft or sound-hard impenetrable scatterer and show in a second step how the results can be extended to the reconstruction of the shape of an unknown inhomogeneous medium scatterer.

6.1 Acoustic scattering.

The main idea of the method of singular sources is the use of the field $\Phi^s(z,z)$ to reconstruct the shape of the scattering object. In a first part of this section we use the operator Q given by (3.1.35) for the reconstruction of $\Phi^s(z,z)$ for scattering by impenetrable scatterers. We estimate the error for the reconstruction of $\Phi^s(z,z)$ by $(Qu^\infty)(z,z)$ in the following theorem.

THEOREM 6.1.1 *Consider the far field patterns $u^\infty(\cdot,d)$ for scattering of plane waves $u^i(\cdot,d)$, $d \in \Omega$, by a sound-soft or sound-hard scatterer $\mathcal{D} \in \mathcal{C}$. Let the function $p: B \to \Omega$ be chosen such that (5.1.2) is satisfied. Given $\mu = 0, s = 1$, $\tau, \eta, \rho > 0$ and a measured far field pattern u_δ^∞ with data error*

$$\left|\left|u_\delta^\infty - u^\infty\right|\right|_{L^2(\Omega\times\Omega)} \leq \delta, \tag{6.1.1}$$

the error for the approximation of $\Phi^s(x,z)$ by Qu_δ^∞ is estimated by

$$\left|\Phi^s(x,z) - (Qu_\delta^\infty)(x,z)\right| \leq c\,\frac{\eta}{\rho^{m-1}} + C\,b_{\eta,\rho}\,\tau + \frac{1}{\gamma_m} b_{\eta,\rho} b_{\tau,\rho}\,\delta, \tag{6.1.2}$$

for $x, z \in B \setminus D_\rho$ with the constants $b_{\eta,\rho}$ defined by (3.1.33) and constants c, C uniformly for domains $\mathcal{D} \in \mathcal{C}$.

Proof. We use Theorem 3.1.6 and the Cauchy-Schwarz inequality applied to the first and second term of the right-hand side of the decomposition

$$\begin{aligned} \left|\Phi^s(x,z) - (Qu_\delta^\infty)(x,z)\right| \;\le\; & \left|\Phi^s(x,z) - (Qu^\infty)(x,z)\right| \\ & + \left|\Big(Q\left(u^\infty - u_\delta^\infty\right)\Big)(x,z)\right| \end{aligned}$$

to obtain (6.1.2). □

According to Theorem 6.1.1 we can use the operator Q to reconstruct Φ^s from the knowledge of the far field patterns $u^\infty(\cdot, d)$ of the scattered fields $u^s(\cdot, d)$ of incident plane waves for all directions of incidence $d \in \Omega$. Given the error δ in the measurements, the error for the reconstruction of Φ^s from u_δ^∞ is estimated by (6.1.2). The density functions $g_\eta(x, \cdot)$ and $g_\tau(z, \cdot)$ used to define Q can be computed according to some apriori-knowledge on the unknown scatterer $\mathcal{D}$ as given by the class $\mathcal{C}$ of scatterers defined in Definition 2.1.5 and Definition 2.2.5.

The operator Q is strongly related to the operator A defined by (5.1.1) to set up the point-source method. We can use Lemma 5.1.3 to efficiently compute the densities g_τ and g_η using rotations and translations. The computation of g_τ for different $p \in \Omega$ is reduced to the computation of one density function g_τ of (5.1.9) with $p = p_0 \in \Omega$ and discrepancy τ and another density function g_η of (5.1.9) with $p = p_0 \in \Omega$ and discrepancy η.

Given an approximating function $a(z)$ for $\Phi^s(z, z), z \in B \setminus D_\rho$, we may follow Theorem 2.1.15 and search for the boundary of the unknown scatterer $\mathcal{D}$ as the set of points z where $a(z)$ is larger than a constant $C > 0$. We call this the *method of singular sources*, since the main reason for the behavior of Φ^s is the singularity of the source $\Phi(\cdot, z)$. The constant C plays the role of a *regularization parameter* and has to be chosen according to the other regularization parameters of the reconstruction, i.e. depending on ρ, τ and η.

The method of singular sources has some features, which are similar to the point-source method. The choice of the orientations $p(z)$ of the domain of approximation needs a knowledge about the unknown scatterer. The following multistep procedure describes a method to successively construct both the scatterer and an appropriate function $p(z)$.

- As a *first step* compute for a number of fixed orientations p_ξ, $\xi = 1, .., N$ approximations $a_\xi^{(1)}(z)$ for the field $\Phi^s(z, z)$ using the operator Q, where Q is depending on p_ξ via the densities $g_\tau(x, \cdot)$ and $g_\eta(z, \cdot)$. Search an approximation D_1 for the shape D of the unknown scatterer $\mathcal{D}$ as the set

$$D_1 := \Big\{ z \in B, |a_\xi^{(1)}(z)| > C \text{ for } \xi = 1, ..., N \Big\}$$

of the points where all approximations $a_\xi^{(1)}(z)$ are larger than C.

- Adapt in each further step $n > 1$ the orientation $p_n(z)$ of the domain of approximation to the knowledge about the unknown scatterer of the previous steps $1, ..., n-1$ and repeat the procedure to compute an approximation $a^{(n)}(z)$ to $\Phi^s(z, z)$ and to search an approximation D_n for D according to

$$D_n := \Big\{ z \in B, |a^{(n)}(z)| > C \Big\}.$$

A background of this procedure is the observation, that the values of $a_\xi(z)$ in general are very large, if the unknown scatterer $\mathcal{D}$ is not contained in the domain of approximation G_{z,p_ξ,β_0}. Thus we obtain large values of $a_\xi(z)$, if z is in the interior of the unknown scatterer and we can replace the search for ∂D by the search for D.

An obvious advantage of the method of singular sources is the fact, that we do not need to take into account the boundary condition of the impenetrable scatterer. In contrast to the point-source method, for the method of singular sources the reconstruction algorithm is the same for the sound-soft and sound-hard boundary condition, i.e. the boundary condition does not need to be known for the algorism. This feature is one of the main reasons which leads to the distinction of category II and III of reconstruction methods. The method of singular sources thus belongs to the *third category*.

One of the prices paid for the advantage of the method of singular sources is the amount of data necessary for reconstructions. The method of singular sources needs the far field pattern for a large number of incident plane waves, whereas with the point-source method we obtain reconstructions even for one or a small number of measured far field patterns.

Another price, which has to be paid to be independent of the boundary condition, is the ill-posedness of the reconstruction operator Q. The norm of Q is given by

$$||g_\tau(z, \cdot)||_{L^2(\Omega)} \cdot ||g_\eta(z, \cdot)||_{L^2(\Omega)}$$

with the densities g_τ and g_η, which form the kernel of Q. For the point-source method the norm of the operator A is equal to $||g_\tau(z, \cdot)||_{L^2(\Omega)}$. This is only the square root of the norm of Q. We will observe below, that the ill-posedness of the problem is becoming even worse, if we consider at the same time impenetrable and penetrable scatterers.

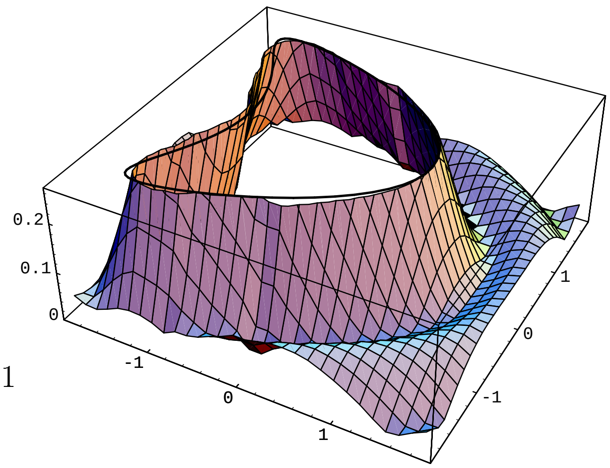

Figure 6.1

Figure 6.1 shows a surface plot of the approximation $a(z)$ and the boundary ∂D of the unknown sound-soft scatterer for $\kappa = 2$ in the second step of the reconstruction algorithm. Here the orientation $p(z)$ of the domain of approximation is chosen as $p(z) = z/|z|$.

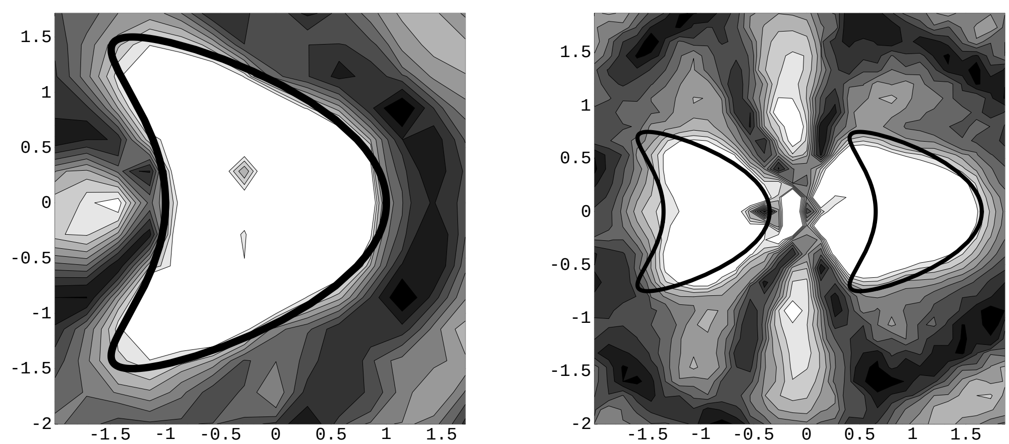

Figure 6.2

Figure 6.2 shows a contour plots of $a(z)$ in step 2 of the algorithm for one or two sound-soft domains, $\kappa = 3$ and $p(z) = z/|z|$. Here we used the same set of regularization parameters for the different data sets for one or two obstacles.

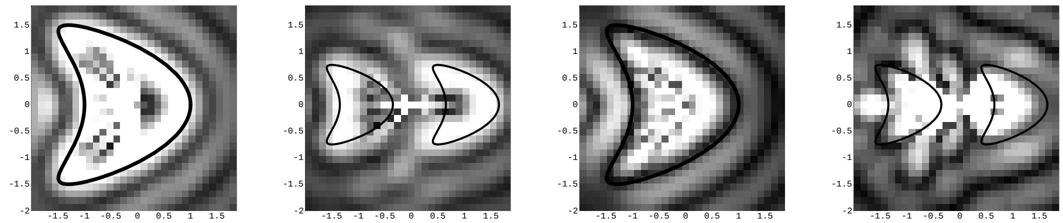

Figure 6.3

Figure 6.3 shows density plots of $a(z)$ in the second step of the reconstruction algorithm for one or two sound-soft domains and one or two sound-hard domains, $\kappa = 4$ and $p(z) = z/|z|$. We used the same set of regularization parameters for all four images.

We now investigate the reconstruction of the shape of an object, which may be either an *impenetrable* or an *inhomogeneous medium scatterer* $\mathcal{D} \in \mathcal{C}$. From the sections on uniqueness and stability we know that the shape of the scatterer is uniquely determined by the far field patterns for all incident plane waves and that the shape depends stable on the far field patterns with respect to the data error in $L^2(\Omega \times \Omega)$. Results on ϵ-uniqueness and ϵ-stability for a large finite number of measurements investigate the practical situation, where only a finite number of measurements of the far field pattern for a finite number of incident plane waves are possible.

For the reconstruction of the shape D of an inhomogeneous medium scatterer $\mathcal{D}$ we will use Theorem 2.2.12 and the operator Q defined by (3.1.35). We have to adapt the order μ of the multipole to the behavior of the refractive index n at the boundary of the unknown scatterer according to the situation S5 or S6 in (3.1.11) and Definition 2.2.5. In the same way as Theorem 6.1.1 we obtain

THEOREM 6.1.2 *Consider the far field patterns $u^\infty(\cdot, d)$ for scattering of plane waves $u^i(\cdot, d)$, $d \in \Omega$, by a sound-soft, sound-hard or inhomogeneous medium scatterer $\mathcal{D} \in \mathcal{C}$. For a measured far field pattern u^∞_δ with data error*

$$\left\| u^\infty_\delta - u^\infty \right\|_{L^2(\Omega\times\Omega)} \leq \delta, \tag{6.1.3}$$

the error for the approximation of $\Phi^s_{\mu,q}(x,z)$ by Qu^∞_δ is estimated by

$$\left| \Phi^s_{\mu,q}(x,z) - \left(Qu^\infty_\delta\right)(x,z) \right| \leq c\, \frac{\eta}{\rho^{m-1}} + C\, b_{\eta,\rho}\, \tau + \frac{1}{\gamma_m} b_{\eta,\rho} b_{\tau,\rho}\, \delta, \tag{6.1.4}$$

$x, z \in B \setminus D_\rho$, with $b_{\eta,\rho}$ defined by (3.1.33) depending on μ, $\rho > 0$ and $\eta > 0$ and constants c and C uniformly for $\mathcal{D} \in \mathcal{C}$.

Following Theorem 6.1.2 we can use the operator Q to reconstruct

$$\Phi^s_{\mu,-\nu(z_0)}(z, z) \tag{6.1.5}$$

from the knowledge of the far field patterns $u^\infty(\cdot, d)$ of the scattered fields $u^s(\cdot, d)$ for all directions of incidence $d \in \Omega$. Given the error δ in the measurements, the error for the reconstruction of (6.1.5) from u^∞_δ is estimated by (6.1.4).

With these preparations we can formulate a *method of singular sources* for the reconstruction of the shape of an object, which may be either an impenetrable or an inhomogeneous medium scatterer. Due to Theorems 2.2.12 and 2.1.16 the singular behavior of (6.1.5) can be used to find the support of the unknown scatterer as the set of points where the approximation $a(z)$ for (6.1.5) is sufficiently large. A corresponding constant C has to be chosen according to the other parameters of the reconstruction, i.e. depending on ρ, τ and η. We have to use a strategy to adapt the orientation $p = p(z)$ of the domain of approximation and the polarization $q = q(z)$ of the multipole to the knowledge about the scatterer in each step of a multistep procedure.

At this point we have to realize a difference between the cases $\mu = 0$ and $\mu > 0$. For $\mu = 0$ the multipole has a rotational symmetry. In this case we can work with rotations and do not need to compute different densities for different vectors q or $\nu(z_0) \in \Omega$, respectively. For the cases $\mu > 0$ we need to compute different densities for the reconstruction of (6.1.5), if the angle between the orientation p of the domain of approximation and the polarization q of the approximated multipole changes. From a practical point of view this problem is not as serious as it first seems, since usually a small number of different angles is sufficient to obtain a reasonable reconstruction of the shape of the unknown scatterer.

We close with a remark on the ill-posedness of the method of singular sources. If it is not known whether the unknown scatterer is penetrable or impenetrable, we have to work with multipoles of order μ for reconstructions. Since the ill-posedness of the reconstruction of D is mainly influenced by the norm of the densities g_τ and g_η, and since these norms increase with μ, the ill-posedness for the general problem is considerably larger than the ill-posedness for the reconstruction of arbitrary impenetrable scatterers or the ill-posedness for the case of a given boundary condition of an impenetrable scatterer.

6.2 Electromagnetic scattering.

In Theorems 2.3.12 and 2.4.9 of Section 2 the behavior of the scattered field $E^s_{edp}(z,z,p)$ for points near a scatterer $\mathcal{D}$ is estimated. The boundary ∂D of the scatterer $\mathcal{D}$ is the set of points, for which

$$\left|E^s_{edp}(z,z,\nu(z_0))\right|$$

becomes infinite, where in a neighborhood of the boundary $z_0 \in \partial D$ is given by the unique representation

$$z = z_0 + h\nu(z_0).$$

In Section 4, Theorem 3.2.3, we constructed the operator Q for the reconstruction of E^s_{edp} from the far field patterns $E^\infty_{pl}(\cdot,\cdot,q)$, $q \in \Omega$. For a measured far field patterns E^∞_δ the error for the reconstruction of E^s_{edp} by QE^∞_δ is estimated in the following theorem.

THEOREM 6.2.1 *We consider the scattering of electromagnetic plane waves $E^i_{pl}(\cdot,d,p)$ for $d,p \in \Omega$ by a scatterer $\mathcal{D} \in \mathcal{C}$. Given a measured far field pattern E^∞_δ with data error*

$$\left|E^\infty_{pl}(\cdot,\cdot,p) \; - \; E^\infty_\delta(\cdot,\cdot,p)\right|_{L^2(\Omega\times\Omega)} \;\le\; \delta, \;\; p \in \Omega, \tag{6.2.1}$$

the error for the approximation of $E^s_{edp}(x,z,p)$ by QE^∞_δ is estimated by

$$\begin{aligned} &\left|q \cdot E^s_{edp}(x,z,p) - (p \cdot QE^\infty_\delta(\cdot,\cdot,q))(x,z)\right| \\ &\qquad\le\; c\frac{\eta}{\rho^3} \;+\; C||g_\eta||_{L^2(\Omega)}\tau \;+\; \frac{1}{\gamma}||g_\tau||_{L^2(\Omega)}||g_\eta||_{L^2(\Omega)}\,\delta, \end{aligned} \tag{6.2.2}$$

$x,z \in B \setminus D_\rho$, with constants c and C uniformly for $\mathcal{D} \in \mathcal{C}$ and functions g_τ and g_η depending on ρ, τ and η.

Proof. The estimate can be obtained from (3.2.14) with the help of

$$\begin{aligned} &\left|q \cdot E^s_{edp}(x,z,p) - (p \cdot QE^\infty_\delta(\cdot,\cdot,q))(x,z)\right| \\ &\le\; \left|q \cdot E^s_{edp}(x,z,p) - (p \cdot QE^\infty_{pl}(\cdot,\cdot,q))(x,z)\right| \\ &\qquad + \left|\left\{p \cdot Q\Big(E^\infty_{pl}(\cdot,\cdot,q) - E^\infty_\delta(\cdot,\cdot,q)\Big)\right\}(x,z)\right| \end{aligned} \tag{6.2.3}$$

and the Cauchy-Schwarz inequality. □

A reconstruction of the boundary of a scatterer $\mathcal{D} \in \mathcal{C}$ with the help of the singular scattered field $E^s_{edp}(z, z, p)$ can be formulated in analogy to the acoustic case. As a difference to the acoustic case both for the reconstruction of a perfect conductor and the reconstruction of the shape of an inhomogeneous medium we can use the scattered field of electric dipoles. For inverse scattering from an *acoustic* inhomogeneous medium we had to use multipoles of higher order even for the simplest case of a medium with a jump at the boundary ∂D.

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