On Acoustic, Electromagnetic, and Elastic Scattering Problems in Inhomogeneous Media

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Introduction

I first realized the importance of solution operators for the differential equation $\Delta u + 2i\zeta \cdot \nabla u = f$ which depend in a suitable way on the complex parameter vector ζ when I studied the paper [32] by Nachman. He based the reconstruction of coefficients in a partial differential equation from boundary measurements on these solution operators. Unfortunately, for the construction of the solution operators he gives a reference to a paper that has never been published and a reference to a similar construction in the paper of Sylvester and Uhlmann [46] where only the special case $\zeta \cdot \zeta = 0$ is examined. Other authors dealing with parameter identification problems cite these two papers when they need the solution operators (see [8, 38, 39]).

The proof of Sylvester and Uhlmann uses Fourier transform techniques in weighted Sobolev spaces and is quite involved as are the proofs of other authors who prove related results.

Therefore, it was a great simplification when I discovered how to obtain solution operators with the help of Fourier series in a straightforward and elementary way. The solution operators became even more attractive when it turned out that they are not only useful in parameter identification problems but also in proving unique continuation results which are needed to show uniqueness for direct scattering problems in an inhomogeneous medium.

This thesis explains the construction of the solution operators via Fourier series and then applies them to some direct and inverse scattering problems in inhomogeneous media. We shall examine acoustic, electromagnetic and elastic scattering problems. In order to give an idea what kind of problems will occur in the sequel let us now sketch the acoustic scattering problem. The acoustic scattering problem is a good example for the other problems, it is probably the one which is known best by the reader, and it needs less technical and distracting details than the electromagnetic or elastic scattering problem.

The direct acoustic scattering problem consists in finding the scattered wave u^s, given the wave number $\kappa > 0$, the refractive index n, and the incident wave u^i. The total wave $u = u^i + u^s$ must obey the differential equation $\Delta u + \kappa^2 nu = 0$ in $\mathbb{R}^3$ and u^s must satisfy a radiation condition at infinity.

Of course, the first questions to ask are whether there exists a solution and whether it is unique. Since we assume $n(x) = 1$ in the exterior of a large ball, we can use Rellich's lemma to obtain $u(x) = 0$ in the exterior of that ball, if u is a radiating solution to the above differential equation. The second step of the uniqueness proof is a unique continuation principle, i.e., a solution u to $\Delta u + \kappa^2 nu = 0$ in $\mathbb{R}^3$, which has compact support, must vanish everywhere. At this point our solution operator is very useful because it allows a short and elementary derivation of the desired unique continuation principle (see Theorem 1.2).

We establish the existence of a solution via an integral equation which is known as the Lippmann-Schwinger equation. It is derived by applying Green's representation theorem to u.

The solution u^s to the direct scattering problem has the asymptotic behavior

$$u^s(x) = \frac{e^{i\kappa|x|}}{|x|}\Big(u^s_\infty(\hat{x}) + O\Big(\frac{1}{|x|}\Big)\Big) \ , \ |x| \to \infty \ ,$$

with $\hat{x} = |x|^{-1}x$ denoting the direction of x. u^s_∞ is known as the far field or scattering amplitude of u^s.

For the inverse scattering problem we assume that we have measured the scattering amplitude u^s_∞ for sufficiently many incident waves u^i and that n is unknown. The task is to reconstruct n from these data. We start with a more modest result, namely a uniqueness theorem which was first established by Novikov in [37]: two refractive indices n and $\tilde{n}$ producing the same far field data must coincide. Actually, we prove that all Fourier coefficients of n and $\tilde{n}$ must coincide. To this end we construct for $\zeta \in \mathbb{C}^3$ with $\zeta \cdot \zeta = \kappa^2$ solutions of the form

$$u(x) = e^{i\zeta\cdot x}(1 + O(|\zeta|^{-1}) \ , \ |\zeta| \to \infty \ ,$$

to the equation $\Delta u + \kappa^2 nu = 0$. Solutions depending in this way on a parameter ζ had already been considered by Faddeev in connection with quantum mechanical scattering problems. We refer the reader to [32, p. 536] for a brief review of their history. Sylvester and Uhlmann used them in [46] to prove that the conductivity is uniquely determined by boundary measurements of voltage and current. We shall construct the special solutions by essentially replacing in the Lippmann-Schwinger equation the usual fundamental solution to the Helmholtz equation,

$$\Phi_\kappa(x,y) = \frac{e^{i\kappa|x-y|}}{4\pi|x-y|}\ ,\ x \neq y\ ,$$

by a different one,

$$\Psi_\zeta(x-y) = e^{i\zeta\cdot(x-y)} g_\zeta(x-y)\ ,\ x \neq y\ .$$

It is immediately seen that, if Ψ_ζ is a fundamental solution to the Helmholtz equation, then g_ζ must be a fundamental solution to the operator $(\Delta + 2i\zeta\cdot\nabla)$, whence our solution operator comes in.

Next, we turn to the stability of the inverse scattering problem. We consider the far field patterns $u_{\infty,n}$ originating from a small set of C^2-smooth refractive indices n. We are then able to prove that the inverse mapping $u_{\infty,n} \mapsto n$ is continuous when we use a very strong norm on the set of far field patterns and the maximum norm for the refractive indices. Here, we shall use the Faddeev-type solutions again in order to estimate the Fourier coefficients of the difference $n - \tilde{n}$ of two refractive indices. The Fourier coefficients in turn yield estimates of the difference $n - \tilde{n}$ itself. Alessandrini [2] proved this result for the problem considered by Sylvester and Uhlmann, and Stefanov [42] investigated the acoustic case.

Finally, we give a procedure how n can be reconstructed from a knowledge of the far field pattern $u_{\infty,n}$ associated with it. This reconstruction goes back to Nachman [32]. The main idea is to compute the boundary data of the Faddeev-type solutions with the help of integral equations whose kernels originate from the fundamental solution Ψ_ζ and with the help of the knowledge of the scattering amplitude. Then, the boundary data are used to compute the Fourier coefficients of $(1-n)$. Since all these questions concerning the inverse acoustic problem in an inhomogeneous medium are examined with the help of the special solutions to the perturbed Helmholtz equation, our solution operators play an essential role during the analysis.

We carry out the analogous program for the direct and the inverse electromagnetic scattering problem in an inhomogeneous medium. For the inverse elastic scattering problem we omit the reconstruction procedure because a rigorous examination would have extended the length of this thesis even more.

Before we proceed with the direct acoustic scattering problem in the first chapter let us point out that the results concerning the direct scattering problems have been known for a long time [50, 31, 51, 23, 7]. We have included those chapters because a good understanding of the direct problems facilitates the understanding of the inverse problems. Moreover, a self-contained presentation of the material may help the reader not to be distracted too often by searching for references. Finally, since we want to apply our solution operators during the uniqueness proofs for the direct problems, we have to deal with the direct problems anyway.

Concerning the inverse scattering problems the results for the elasticity equation and the stability result for the electromagnetic case seem to be new. Furthermore, our approach, using Fourier series and carrying out the analysis in classical function spaces, i.e., spaces of continuous, Hölder continuous or differentiable functions and L^2-spaces, is new. Contrary to the papers mentioned above we have avoided Sobolev spaces throughout. After having established the existence and the properties of the fundamental solution Ψ_ζ, which differs from Φ_κ by a smooth function, it is possible to use all the results from classical potential theory. We hope that this way to present the material simplifies the technical details and contributes to the clarity of the main ideas.

Of course, there are limits for being self-contained. We assume that the reader has a good knowledge in analysis and functional analysis (as provided by many textbooks) and in boundary integral equations (as provided by the first four chapters in [6] together with [7]). These assumptions reflect the author's mathematical education and background.

Nevertheless, we have included proofs for results which can be found in the monographs [6, 7], if the proofs differ from the ones given there or if they are important for the understanding of the subject. The above choice of what is assumed to be known also required a discussion of volume potentials and of Weyl's lemma in this thesis although the reader might argue that these are standard results and can be found in classical monographs. For similar reasons we have included an appendix dealing with the elastic single-layer potential though [23] is a standard reference.

The reader can infer from the table of contents the organisation of the material. We have devoted one chapter to each problem, the direct and the inverse scattering problem in the acoustic, electromagnetic, and elastic case.

Finally, I want to thank all my relatives, friends, and colleagues who have helped in some way during my work. Especially, I gratefully acknowledge the help of Professor Dr. David Colton and of my teacher Professor Dr. Rainer Kreß. Their research and their books inspired my own research, and the enthusiasm of the former and the steady encouragement of the latter provided much support to write this thesis.

Chapter 1

The Direct Acoustic Scattering Problem

Let us start with a brief physical motivation of the main mathematical problems we shall examine in this section. We consider an inhomogeneous medium in $\mathbb{R}^3$ and assume that the inhomogeneity is compactly supported. The propagation of time harmonic acoustic waves in the medium is governed by the equation

$$\Delta u(x) + \kappa^2 n(x) u(x) = 0 \; , \;\; x \in \mathbb{R}^3 \; . \tag{1.1}$$

u describes the pressure field, $\kappa > 0$ is the wave number and n is the refractive index of the medium. κ and $n(x)$ are related to the frequency ω of the wave and to the speed of sound of the medium via $\kappa = \omega / c_0$ and $n(x) = c_0^2 / c^2(x)$. Here, $c(x)$ is the speed of sound at the point $x \in \mathbb{R}^3$ and c_0 is the speed of sound in the homogeneous part of the medium (see [7, chapt. 8] or [50]). In order to model absorbing media, too, we allow $\Im(n(x)) \geq 0$, $x \in \mathbb{R}^3$.

In the direct acoustic scattering problem we know the wave number κ and the refractive index n and we are given an incident wave u^i which is scattered by the inhomogeneity. The task is to find the scattered field u^s such that the total field $u := u^i + u^s$ satisfies equation (1.1) and such that u^s satisfies a radiation condition.

In the following sections we shall provide the tools to prove that the direct scattering problem has a unique solution. Our first aim is the uniqueness proof. It turns out that solution operators for the differential equation $\Delta v + 2i\zeta \cdot \nabla v = f$ whose L^2-norms depend in a suitable way on the parameter $\zeta \in \mathbb{C}^3$ are useful during the uniqueness proof. Since these operators also play a central role when we examine the inverse problem, we introduce them at the beginning. However, in the next section we shall only prove a result which is absolutely necessary for the uniqueness proof. When we need stronger results in later sections we improve our assertions then. Since we employ Fourier series techniques for the norm estimates of the solution operators, we start with a brief review about Fourier series in the next section. We proceed with the norm estimates and prove a unique continuation principle as a first application of the solution operators.

In the second section we review Green's formula and then present a version of Rellich's lemma. Both uniqueness theorems for the direct as for the inverse scattering problems are based on this lemma.

After giving a precise formulation of the direct scattering problem in the third section we establish its uniqueness. Next, we turn to the existence proof for the direct scattering problem. We use volume potentials and integral equation techniques. Thus, we investigate the mapping properties of volume potentials and finally obtain the unique solvability for the direct scattering problem.

Although all the results can be found in the literature we have included this chapter because we want to give a self-contained exposition of the direct scattering problem. Furthermore, some proofs of well-known results are new (e.g. the unique continuation principle), and finally the existence proof for the direct problem will suggest proofs when we examine the inverse problem in the second chapter.

1.1 Fourier Series and a Unique Continuation Principle

The purpose of this section is to give a brief account on Fourier series which are then used to derive formally a solution operator G'_ξ for the equation $\Delta u + 2i\xi \cdot \nabla u = f$ and to estimate its norm. As a first application of the norm estimate we derive a unique continuation principle.

If $D \subset \mathbb{R}^3$ is an open set we denote by $L^2(D)$ the linear space of complex-valued functions on D which are measurable and square integrable on D with respect to the Lebesgue measure. We shall tacitly identify functions which coincide in D except on a set having Lebesgue measure zero. $L^2(D)$ endowed with the scalar product

$$(f,g) := \int_D \overline{f(x)} g(x) dx \ , \ f, g \in L^2(D) \ ,$$

is a Hilbert space. $\|f\|_{L^2(D)} = \|f\|_{L^2}$ denotes its norm.

For a fixed $R' > 0$ we define the cube $C := (-R', R')^3 \subset \mathbb{R}^3$. The usual orthogonal basis in the space $L^2(C)$ are the trigonometric polynomials $e^{i\alpha \cdot x}$, $x \in C$, $(R'/\pi)\alpha \in \mathbb{Z}^3$. However, for reasons which become obvious in Theorem 1.1 it is more suitable for us to shift the grid $(\pi/R')\mathbb{Z}^3$ and to work with a slightly modified basis.

We denote by Γ the grid

$$\Gamma := \left\{ \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \colon \frac{R'}{\pi}\alpha_2 - \frac{1}{2} \in \mathbb{Z} \ , \ \frac{R'}{\pi}\alpha_1, \frac{R'}{\pi}\alpha_3 \in \mathbb{Z} \right\} \ ,$$

i.e., we have shifted $(\pi/R')\mathbb{Z}^3$ by $\pi/(2R')$ in the direction of the second coordinate. Furthermore, we define $e_\alpha(x) := (2R')^{-3/2} \exp(i\alpha \cdot x)$, $x \in C$, $\alpha \in \Gamma$.

Straightforward calculations show that we have for α, $\beta \in \Gamma$:
$(e_\alpha, e_\beta) = 1$ if $\alpha = \beta$, and $(e_\alpha, e_\beta) = 0$ if $\alpha \neq \beta$, i.e., the functions e_α, $\alpha \in \Gamma$, are an orthonormal system in $L^2(C)$.

They are also a complete system. For $f \in L^2(C)$ the function $g(x) := e^{-i\pi x_2/(2R')} f(x)$, $x \in C$, satisfies $g \in L^2(C)$. Hence, by the Weierstrass approximation theorem for trigonometric polynomials there is a sequence p_j, $j \in \mathbb{N}$, of trigonometric polynomials converging to g, $\|p_j - g\|_{L^2} \to 0$, $j \to \infty$. The functions q_j, defined by $q_j(x) := e^{i\pi x_2/(2R')} p_j(x)$, $x \in C$, $j \in \mathbb{N}$, belong to span$\{e_\alpha\colon \alpha\in\Gamma\}$ and

$$\begin{aligned}\|q_j - f\|_{L^2}^2 &= \int_C |e^{i\pi x_2/(2R')}p_j(x) - e^{i\pi x_2/(2R')}g(x)|^2 dx \\ &= \|p_j - g\|_{L^2}^2 \to 0\ ,\ j\to\infty\ ,\end{aligned}$$

due to $|e^{i\pi x_2/(2R')}| = 1$, $x\in C$.

Then e_α, $\alpha\in\Gamma$, are an orthonormal basis in $L^2(C)$ and for $f\in L^2(C)$ we have the Fourier expansion $f = \sum_{\alpha\in\Gamma}\hat f(\alpha)e_\alpha$ with Fourier coefficients $\hat f(\alpha) := (e_\alpha, f)$. Moreover, the Fourier coefficients $\hat f(\alpha)$, $\hat g(\alpha)$ satisfy Parseval's identities

$$\sum_{\alpha\in\Gamma}|\hat f(\alpha)|^2 = \|f\|_{L^2}^2\ , \tag{1.2}$$

$$\sum_{\alpha\in\Gamma}\overline{\hat f(\alpha)}\hat g(\alpha) = (f,g) \tag{1.3}$$

for f, $g\in L^2(C)$. By the Riesz-Fischer theorem any sequence c_α, $\alpha\in\Gamma$, with $\sum_{\alpha\in\Gamma}|c_\alpha|^2<\infty$ corresponds to a uniquely defined function $f=\sum_{\alpha\in\Gamma}c_\alpha e_\alpha\in L^2(C)$ having Fourier coefficients $\hat f(\alpha)=c_\alpha$.

Let us introduce some more notation. For any set $G\subset\mathbb{R}^3$ we denote by $C(G)$ the space of continuous functions on G. For a function u defined on an open set $D\subset\mathbb{R}^3$ we denote by $\partial_j u = \partial u/\partial x_j$ its partial derivative with respect to the coordinate x_j, $j=1,2,3$. $\nabla u := (\partial_1 u, \partial_2 u, \partial_3 u)$ is the gradient of u and $\Delta u := \partial_1^2 u + \partial_2^2 u + \partial_3^2 u$ is the Laplacian of u. $C^k(D)$ denotes the set of functions defined on D having continuous derivatives up to order $k\in\mathbb{N}$ in D. $C^k(\overline{D})$ consists of those functions from $C^k(D)$ whose derivatives can be continuously extended to $\overline{D}$ and $C_0^k(D)$ is the subspace of functions from $C^k(D)$ having compact support in D.

Let us now turn to the differential equation

$$\Delta u + 2i\xi\cdot\nabla u = f \tag{1.4}$$

in the cube $C := (-R', R')^3\subset\mathbb{R}^3$ where the vector $\xi\in\mathbb{C}^3$ is defined as $\xi := (s, it, 0)$ with the real parameters $s\in\mathbb{R}$, $t>0$. The differential operator in (1.4) occurs immediately, if one tries to find a solution v of the Poisson equation $\Delta v = g$ which has the form $v(x) = e^{i\xi\cdot x}u(x)$, $x\in C$. We shall need solutions of this form when we study the inverse problem in later sections. At the moment we are content in deriving operators G'_ξ which map (formally) the function f to a solution u of (1.4) such that the operator norms $\|G'_\xi\|_{L^2}$ converge to zero as $|\Im(\xi)| = t \to \infty$. This behavior allows to give a simple proof for a unique continuation principle which in turn is a basic ingredient in the uniqueness proof of the direct scattering problem.

In order to obtain G'_ξ let us insert the Fourier expansions of f and u into equation (1.4), formally reverse the order of differentiation and summation and compare the Fourier coefficients. We arrive at the equations

$$-(\alpha \cdot \alpha + 2\xi \cdot \alpha)\hat{u}(\alpha) = \hat{f}(\alpha),\ \alpha \in \Gamma,$$

hence

$$u = -\sum_{\alpha \in \Gamma} \frac{\hat{f}(\alpha)}{(\alpha \cdot \alpha + 2\xi \cdot \alpha)} e_\alpha \ .$$

We do not worry whether and in what sense u is a solution. Instead, we derive the two properties of the suggested operator G'_ξ which we need for the unique continuation principle.

Theorem 1.1 *Let $s \in \mathbb{R}$, $t > 0$ be real numbers and $\xi := (s, it, 0) \in \mathbb{C}^3$. Then, the operator*

$$G'_\xi \colon L^2(C) \to L^2(C) \qquad G'_\xi f := -\sum_{\alpha \in \Gamma} \frac{\hat{f}(\alpha)}{(\alpha \cdot \alpha + 2\xi \cdot \alpha)} e_\alpha$$

is well defined and has the following properties:

(a) $\|G'_\xi f\|_{L^2} \leq \dfrac{R'}{\pi t}\|f\|_{L^2}$ *for all* $f \in L^2(C)$,

(b) $G'_\xi(\Delta f + 2i\xi \cdot \nabla f) = f$ *for all* $f \in C_0^2(C)$.

Proof: From $(R'/\pi)\alpha_2 - (1/2) \in \mathbb{Z}$ we conclude $|\alpha_2| \geq \pi/(2R')$ for all $\alpha \in \Gamma$ and then

$$|\alpha \cdot \alpha + 2\xi \cdot \alpha| \geq |\Im(\alpha \cdot \alpha + 2\xi \cdot \alpha)| = 2t|\alpha_2| \geq (\pi t)/R' \qquad (1.5)$$

for all $\alpha \in \Gamma$. Note that this is the reason for the shift of the usual grid $((\pi/R')\mathbb{Z})^3$ when defining Γ and e_α, $\alpha \in \Gamma$. Then $(\alpha \cdot \alpha + 2\xi \cdot \alpha)^{-1}$ exists for all $\alpha \in \Gamma$ and we can estimate

$$\sum_{\alpha \in \Gamma} \left| \frac{\hat{f}(\alpha)}{(\alpha \cdot \alpha + 2\xi \cdot \alpha)} \right|^2 \leq \frac{{R'}^2}{\pi^2 t^2} \sum_{\alpha \in \Gamma} |\hat{f}(\alpha)|^2 < \infty \ ,$$

for all $f \in L^2(C)$. Hence, by Parseval's identity (1.2) $G'_\xi : L^2(C) \to L^2(C)$ is a well defined linear operator which satisfies the norm inequality (a).

For the proof of assertion (b) we use integration by parts and obtain for $f \in C_0^2(C)$

$$\begin{aligned}(\Delta f + 2i\xi \cdot \nabla f)\hat{\ }(\alpha) &= (2R')^{-3/2} \int_C (\Delta f + 2i\xi \cdot \nabla f)(x) e^{-i\alpha\cdot x} dx \\ &= (2R')^{-3/2} \int_C f(x)(-\alpha\cdot\alpha - 2\xi\cdot\alpha) e^{-i\alpha\cdot x} dx \\ &= -(\alpha\cdot\alpha + 2\xi\cdot\alpha)\hat{f}(\alpha)\ .\end{aligned}$$

Now, part (b) follows from the definition of G'_ξ.

□

Since the differential equation (1.4) has constant coefficients, one might try to use the Fourier transform for the construction of a solution operator. Employing formally the Fourier transform F to (1.4) we obtain with $p_\xi(y) := -y \cdot y - 2\xi \cdot y$, $y \in \mathbb{R}^3$, the equation $p_\xi(y)(Fu)(y) = (Ff)(y)$, $y \in \mathbb{R}^3$. Using the inverse Fourier transform we arrive at the solution operator $G''_\xi f := F^{-1}((1/p_\xi)Ff)$. However, contrary to our derivation with the help of Fourier series we now have $p_\xi(y) = 0$ for certain $y \in \mathbb{R}^3$. This fact causes difficulties to verify the norm estimate (a) from Theorem 1.1. Several authors have found ways to deal with this difficulty. We refer the reader to [46, 40] and [1, 49] where differential operators of the form (1.4) are studied. In his paper [17] Isakov has pointed out that there is another very general method to construct fundamental solutions for partial differential operators with constant coefficients via Fourier transform techniques which is given in Theorem 7.3.10 in Hörmander's book [14]. This method also yields the right behavior for large $|\Im(\xi)|$ according to the proof of Theorem 10.3.7 in [15]. All those proofs need a more advanced machinery than our elementary considerations in Theorem 1.1.

The second theorem of this section is a unique continuation principle: a function $u \in C_0^2(\mathbb{R}^3)$ satisfying the inequality $|\Delta u| \le M|u|$ in $\mathbb{R}^3$ must vanish identically. This is a very weak form of the unique continuation principle and much better results can be found in the literature ([7, Lemma 8.5], [27, p. 65], [16, Theorem 17.2.6 and further references therein]). However, it turns out that our strong assumption $u \in C_0^2(\mathbb{R}^3)$ is satisfied in the problems we shall consider in the sequel. Moreover, our proof, which is built on the operator G'_ξ from Theorem 1.1, is quite simple.

Theorem 1.2 *If $u \in C_0^2(\mathbb{R}^3)$ satisfies $|\Delta u(x)| \leq M|u(x)|$ for all $x \in \mathbb{R}^3$ with a constant M, then u vanishes in all of $\mathbb{R}^3$. This is also true, if $u = (u_1, \ldots, u_l)$ is a vector valued function, $\Delta u := (\Delta u_1, \ldots, \Delta u_l)$ and $|\cdot|$ denotes the euclidean norm of a vector in $\mathbb{C}^l$.*

Proof: We choose $R' > 0$ large enough to ensure $\operatorname{supp}(u) \subset C = (-R', R')^3$. Furthermore, we define $t := ((MR')/\pi) + 1$, and $\xi := (t, it, 0) \in \mathbb{C}^3$. Defining $v(x) := \exp(-i\xi \cdot x)u(x)$, $x \in \mathbb{R}^3$, a simple computation shows

$$\Delta u(x) = \exp(i\xi \cdot x)(\Delta v + 2i\xi \cdot \nabla v)(x) \ ,$$

whence

$$|(\Delta + 2i\xi \cdot \nabla)v(x)| \leq M|v(x)| \quad \text{for all } x \in \mathbb{R}^3 \ . \tag{1.6}$$

Moreover, using Theorem 1.1 (b) we obtain for $v \in C_0^2(C)$

$$v = G'_\xi((\Delta + 2i\xi \cdot \nabla)v) \ .$$

Combining the last equality with (1.6) and Theorem 1.1 (a) we arrive at

$$\|v\|_{L^2} \leq \frac{R'}{\pi t}\|(\Delta + 2i\xi \cdot \nabla)v\|_{L^2} \leq \frac{MR'}{\pi t}\|v\|_{L^2} \ .$$

Since $(MR')/(\pi t) < 1$, the function v must vanish, and then u must vanish.

If u is a vector valued function, we can use the same reasoning where we understand that a differential operator or the operator G'_ξ is applied to each cartesian component of a vector valued function.

□

1.2 Green's Formula and Rellich's Lemma

Green's formula allows to represent smooth functions in a smooth, bounded domain by a superposition of certain potentials. In order to give a precise statement of Green's formula (Green's representation theorem, Helmholtz representation) we define that a nonempty, bounded, open set $D \subset \mathbb{R}^3$ with boundary ∂D is C^2-smooth (or that ∂D is C^2-smooth), if for all $x \in \partial D$ there exist an open ball $B_r(x) \subset \mathbb{R}^3$ and a bijective mapping $\psi\colon B_r(x) \to V \subset \mathbb{R}^3$ such that ψ and its inverse ψ^{-1} are twice continuously differentiable in the closure of their respective domains of definition and such that

$$\psi(B_r(x) \cap D) \subset \{y \in \mathbb{R}^3\colon y_3 > 0\}\ ,\ \ \psi(B_r(x) \cap \partial D) \subset \{y \in \mathbb{R}^3\colon y_3 = 0\}\ .$$

For a C^2-smooth surface ∂D we can define the outward unit normal vector $\nu(x) = (\nu_1(x), \nu_2(x), \nu_3(x)) \in \mathbb{R}^3$ to ∂D at the point $x \in \partial D$. By ds we indicate the two-dimensional area element in ∂D.

If $D \subset \mathbb{R}^3$ is a C^2-smooth nonempty, bounded, open set and if $w \in C^1(\overline{D})$, then Gauss' theorem (integration by parts) yields

$$\int_{\partial D} \nu_j(x) w(x) ds(x) = \int_D (\partial_j w)(x) dx\ ,\ \ j = 1, 2, 3\ . \tag{1.7}$$

Applying equation (1.7) to the functions $w = u(\partial_l v)$, $l = 1, 2, 3$, with $j = l$ and adding the results yields Green's first theorem

$$\int_{\partial D} u \frac{\partial v}{\partial \nu} ds = \int_D (\nabla u \cdot \nabla v + u \Delta v) dx \tag{1.8}$$

for u, $v \in C^2(\overline{D})$.

If we interchange u and v in the above formula and subtract, we obtain Green's second theorem

$$\int_{\partial D} (u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu}) ds = \int_D (u \Delta v - v \Delta u) dx \tag{1.9}$$

for u, $v \in C^2(\overline{D})$.

Note, that the smoothness assumptions on u and v can be relaxed somewhat. In equation (1.8) it suffices to suppose u, $v \in C^1(\overline{D})$, $u \in C^2(D)$ and $\Delta u \in C(\overline{D})$. For equation (1.9) the assumptions u, $v \in C^2(D) \cap C^1(\overline{D})$, and Δu, $\Delta v \in C(\overline{D})$ are sufficient.

For $\kappa \in \mathbb{C}$, $x,\, y \in \mathbb{R}^3$, $x \neq y$, we denote by

$$\Phi_\kappa(x,y) := \frac{e^{i\kappa|x-y|}}{4\pi|x-y|}$$

the fundamental solution to the Helmholtz equation $\Delta u + \kappa^2 u = 0$. If $y \in \mathbb{R}^3$ is fixed, $\Phi_\kappa(\cdot, y)$ is a solution to the Helmholtz equation with respect to the variable x in $\mathbb{R}^3 \setminus \{y\}$ and similarly, if x is fixed.

Now, assume D is C^2-smooth and bounded, $u \in C^2(\overline{D})$ and $x \in D$ is fixed. Applying Green's second theorem with $v := \Phi_\kappa(x,\cdot)$ in the smooth open set $D \setminus \overline{B_\epsilon(x)}$ and taking $\epsilon \to 0$ yields the following representation of u which is known as Green's formula.

Theorem 1.3 *Let $D \subset \mathbb{R}^3$ be a nonempty, bounded, open set with C^2-smooth boundary. Then, for $\kappa \in \mathbb{C}$ and for a function $u \in C^2(\overline{D})$ we have Green's formula*

$$\begin{aligned} u(x) &= \int_{\partial D} \{\frac{\partial u}{\partial \nu}(y)\Phi_\kappa(x,y) - u(y)\frac{\partial \Phi_\kappa(x,y)}{\partial \nu(y)}\} ds(y) \\ &\quad - \int_D (\Delta u(y) + \kappa^2 u(y))\Phi_\kappa(x,y)dy \ , \ x \in D \ . \end{aligned} \tag{1.10}$$

Green's formula is also true if u only satisfies $u \in C^2(D) \cap C^1(\overline{D})$ and $\Delta u + \kappa^2 u \in C(\overline{D})$.

A detailed proof can be found in [7, Theorem 2.1].

Note that Φ_κ is analytic for $x \neq y$, i.e., if $x_0,\, y_0 \in \mathbb{R}^3$, $x_0 \neq y_0$, then there exists an $\epsilon > 0$ such that for all $x,\, y \in \mathbb{R}^3$ with $|x - x_0| + |y - y_0| < \epsilon$ the series expansion

$$\Phi_\kappa(x,y) = \sum_{\alpha,\beta \in \mathbb{N}_0^3} a_{\alpha\beta}(x_0,y_0)(x-x_0)^\alpha (y-y_0)^\beta$$

holds true with certain coefficients $a_{\alpha\beta}(x_0,y_0) \in \mathbb{C}$. The series converges absolutely and uniformly. Here, we use $z^\beta := z_1^{\beta_1} z_2^{\beta_2} z_3^{\beta_3}$ for a multi-index $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}_0^3$ and $z \in \mathbb{C}^3$. Hence we can infer from the above theorem that solutions to the Helmholtz equation $\Delta u + \kappa^2 u = 0$ in D are analytic functions in D.

$\Phi_\kappa(\cdot, y)$ and $\Phi_{-\kappa}(\cdot, y)$ are both solutions to the Helmholtz equation in $\mathbb{R}^3 \setminus \{y\}$ which have a different behavior at infinity. We can therefore guess that, in order to have a unique solution to our scattering problem, we have to specify its behavior at infinity. For $\kappa \in \mathbb{C} \setminus \mathbb{R}$ it is reasonable to expect the scattered waves being bounded for large x. This excludes $\Phi_{-\kappa}$ if $\Im(\kappa) > 0$. But for $\kappa > 0$ both $\Phi_\kappa(\cdot, y)$ and $\Phi_{-\kappa}(\cdot, y)$ are bounded for large x. The necessary distinction is made by the Sommerfeld radiation condition: let $u \in C^2(\mathbb{R}^3 \setminus B_r)$ be a solution of $\Delta u + \kappa^2 u = 0$ in $\mathbb{R}^3 \setminus \overline{B_r}$. Then u satisfies the Sommerfeld radiation condition (u is a radiating solution to the Helmholtz equation) if

$$\left|\hat{x} \cdot \nabla u(x) - i\kappa u(x)\right| = o(\frac{1}{|x|}) \ , |x| \to \infty \ , \tag{1.11}$$

uniformly for all directions $\hat{x} := |x|^{-1} x$.

Our next aim is a representation formula as in (1.10) for a radiating solution to the Helmholtz equation in the exterior of a ball. To this end we first prove the following useful lemma.

Lemma 1.4 *Let u, $v \in C^2(\mathbb{R}^3 \setminus B_R)$ be radiating solutions to $\Delta u + \kappa^2 u = 0$ in $\mathbb{R}^3 \setminus \overline{B_R}$ where $\kappa > 0$. Then,*

$$\int\limits_{|x|=r} (u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}) ds = 0$$

for all $r \geq R$.

Proof: We first show that there is a constant bounding the integrals

$$\int\limits_{|y|=r} |u(y)|^2 ds(y)$$

for all $r \geq R$ from above. From the radiation condition we know

$$\begin{aligned} 0 &= \lim_{r\to\infty} \int\limits_{|x|=r} |\hat{x} \cdot \nabla u(x) - i\kappa u(x)|^2 ds \\ &= \lim_{r\to\infty} \left\{ \int\limits_{|x|=r} \left\{ |\hat{x} \cdot \nabla u(x)|^2 + \kappa^2 |u(x)|^2 \right\} ds + 2\Re\{ i\kappa \int\limits_{|x|=r} \frac{\partial u}{\partial \nu} \overline{u} ds \} \right\} . \end{aligned} \tag{1.12}$$

Applying Green's first theorem in the set $\{R < |x| < r\}$ we have

$$
\begin{aligned}
&2\Re\{i\kappa \int_{|x|=r} \frac{\partial u}{\partial \nu}\overline{u}ds\} \\
&= \quad -2\kappa\Im\{\int_{|x|=r} \frac{\partial u}{\partial \nu}\overline{u}ds - \int_{|x|=R} \frac{\partial u}{\partial \nu}\overline{u}ds\} - 2\kappa\Im\{\int_{|x|=R} \frac{\partial u}{\partial \nu}\overline{u}ds\} \\
&= \quad -2\kappa\Im\{\int_{|x|=R} \frac{\partial u}{\partial \nu}\overline{u}ds\} \ .
\end{aligned}
$$

Inserting this into equation (1.12) we can conclude that

$$
\limsup_{r\to\infty} \int_{|y|=r} |u(y)|^2 ds(y) < \infty \ .
$$

Of course the same holds true for v.

Employing Green's second theorem in the spherical shell $\{r \leq |x| \leq r'\}$ we arrive at

$$
0 = \int_{|x|=r} (u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu})ds - \int_{|x|=r'} (u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu})ds \ .
$$

The assertion then follows as $r' \to \infty$ because the radiation condition together with the Cauchy-Schwarz inequality implies

$$
\begin{aligned}
&\int_{|x|=r'} (u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu})ds \\
&= \int_{|x|=r'} \{u(\frac{\partial v}{\partial \nu} - i\kappa v) - v(\frac{\partial u}{\partial \nu} - i\kappa u)\}ds \to 0 \ , \ r' \to \infty \ .
\end{aligned}
$$

□

Now we can prove that for a radiating solution to the Helmholtz equation in the exterior of a ball a similar representation as in (1.10) holds true.

Theorem 1.5 *Let $u \in C^2(\mathbb{R}^3 \setminus B_R)$ be a radiating solution to $\Delta u + \kappa^2 u = 0$ in $\mathbb{R}^3 \setminus \overline{B_R}$ where $\kappa > 0$.*

(a) *For* $x \in \mathbb{R}^3 \setminus \overline{B_R}$ *the representation*

$$u(x) = \int_{\partial B_R} \{u(y)\frac{\partial \Phi_\kappa(x,y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y)\Phi_\kappa(x,y)\} ds(y) \tag{1.13}$$

is valid.

(b) *For* $x \in B_R$ *we have the relation*

$$0 = \int_{\partial B_R} \{u(y)\frac{\partial \Phi_\kappa(x,y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y)\Phi_\kappa(x,y)\} ds(y)\ . \tag{1.14}$$

Proof: We fix $x \in \mathbb{R}^3 \setminus \overline{B_R}$ and use Green's formula in the domain $\{R < |x| < r\}$ to arrive at

$$\begin{aligned} u(x) \quad &= \int_{|y|=r} \{\frac{\partial u}{\partial \nu}(y)\Phi_\kappa(x,y) - u(y)\frac{\partial \Phi_\kappa(x,y)}{\partial \nu(y)}\} ds(y) \\ &\quad - \int_{|y|=R} \{\frac{\partial u}{\partial \nu}(y)\Phi_\kappa(x,y) - u(y)\frac{\partial \Phi_\kappa(x,y)}{\partial \nu(y)}\} ds(y) \end{aligned}$$

Since $\Phi_\kappa(x,\cdot)$ is a radiating solution to the Helmholtz equation in the exterior of $\overline{B_r}$, the integral over ∂B_r vanishes due to the preceding lemma and we have proved assertion (a).

For the proof of part (b) we apply the preceding lemma with $v = \Phi_\kappa(x,\cdot)$. □

The last theorem allows to study more precisely the behavior of a radiating solution to the Helmholtz equation. We denote by

$$S^2 := \partial B_1 = \{x \in \mathbb{R}^3 : |x| = 1\}$$

the unit sphere in $\mathbb{R}^3$.

Lemma 1.6 *Let* $u \in C^2(\mathbb{R}^3 \setminus B_R)$ *be a radiating solution to* $\Delta u + \kappa^2 u = 0$ *in* $\mathbb{R}^3 \setminus \overline{B_R}$ *where* $\kappa > 0$. *Then, there exists a function* $u_\infty : S^2 \to \mathbb{C}$ *such that*

$$u(x) = \frac{e^{i\kappa|x|}}{|x|}\Big\{u_\infty(\hat{x}) + O\Big(\frac{1}{|x|}\Big)\Big\}\ ,\ \ |x| \to \infty\ ,$$

uniformly for all directions $\hat{x} = |x|^{-1}x \in S^2$.

Proof: With the help of

$$|x-y| = \sqrt{|x|^2 - 2x\cdot y + |y|^2} = |x| - \hat{x}\cdot y + O\Big(\frac{1}{|x|}\Big)\ ,\ |x|\to\infty\ ,$$

which holds uniformly for all $y \in \overline{B_R}$, $|x| \geq 2R+1$, we obtain the asymptotic behavior

$$\begin{aligned}\Phi_\kappa(x,y) &= \frac{e^{i\kappa|x|}}{4\pi|x|}\Big\{e^{-i\kappa\hat{x}\cdot y} + O\Big(\frac{1}{|x|}\Big)\Big\}\ ,\ |x|\to\infty\ , \qquad (1.15)\\ \frac{\partial\Phi_\kappa(x,y)}{\partial\nu(y)} &= \frac{e^{i\kappa|x|}}{4\pi|x|}\Big\{-i\kappa\hat{x}\cdot\nu(y)e^{-i\kappa\hat{x}\cdot y} + O\Big(\frac{1}{|x|}\Big)\Big\}\ ,\ |x|\to\infty\ .\end{aligned}$$

The proof now follows by inserting the above expressions into the representation (1.13).

□

From $u_\infty(\hat{x}) = \lim_{r\to\infty} re^{-i\kappa r}u(r\hat{x})$ we conclude that u_∞ is uniquely determined by u and we define that u_∞ is the far field pattern (far field, scattering amplitude) of u.

Now, the natural question arises whether two different radiating solutions to the Helmholtz equation can have the same far field pattern. The next lemma which is a variant of Rellich's lemma (see [41, 22, 7]) states that this is not the case, i.e., the far field uniquely determines the radiating solution to the Helmholtz equation. We use stronger assumptions than those employed in the above references and give a proof whose main idea is due to Miranker ([28]). This proof avoids spherical harmonics and solutions to the spherical Bessel differential equation and is based on Green's formula and the behavior of the functions Φ_κ.

Lemma 1.7 *Assume $r > 0$ and $u \in C^2(\mathbb{R}^3 \setminus B_r)$ is a solution to $\Delta u(x) + \kappa^2 u(x) = 0$, $|x| > r$, that satisfies*

$$\int_{|x|=r'} |u(x)|^2 ds(x) \to 0\ ,\ r'\to\infty\ , \qquad (1.16)$$

and the Sommerfeld radiation condition (1.11). Then $u = 0$ in $\mathbb{R}^3 \setminus B_r$. Especially, any radiating solution to the Helmholtz equation with vanishing far field pattern must vanish identically.

Proof: The proof consists of three steps. First, we show that u can be represented as

$$\begin{aligned} u(x) &= \int_{|y|=r} \{\frac{\partial \Phi_\kappa(x,y)}{\partial \nu(y)} u(y) - \Phi_\kappa(x,y) \frac{\partial u}{\partial \nu}(y)\} ds(y) \\ &= \int_{|y|=r} \{\frac{\partial \Phi_{-\kappa}(x,y)}{\partial \nu(y)} u(y) - \Phi_{-\kappa}(x,y) \frac{\partial u}{\partial \nu}(y)\} ds(y) \ , \ |x| > r \ . \end{aligned} \tag{1.17}$$

Next, we define the functions

$$\begin{aligned} f_+(\hat{x},t) &:= e^{-i\kappa/t} u(\frac{1}{t}\hat{x}) \ , \ \hat{x} \in S^2 \ , \ 0 < t < t_0 \ , \\ f_-(\hat{x},t) &:= e^{i\kappa/t} u(\frac{1}{t}\hat{x}) \ , \ \hat{x} \in S^2 \ , \ 0 < t < t_0 \ , \end{aligned}$$

where $t_0 > 0$ is sufficiently small, and then deduce from the above representations that f_+ and f_- are real analytic in t. In regard of the expansion theorem for radiating solutions to the Helmholtz equation due to Atkinson [3] and Wilcox [52, 53],

$$u(x) = \frac{e^{i\kappa|x|}}{|x|} \sum_{j=0}^{\infty} \frac{u_j(\hat{x})}{|x|^j}$$

that is valid for all sufficiently large $|x|$, the assertion for f_+ immediately follows by replacing $|x|$ by $1/t$. We actually repeat their proof. In the last step we use the analyticity of f_+ and f_- to lead the assumption $f_+ \neq 0$ to a contradiction. This yields the assertion.

In Theorem 1.5 (a) we have proved the first equality of (1.17). From the representation (1.10), which also holds for $-\kappa$, applied to u in the set $\{r < |x| < r'\}$ we have for $r < |x| < r'$

$$\begin{aligned} u(x) = &\int_{|y|=r'} \{\frac{\partial u}{\partial \nu}(y) \Phi_{-\kappa}(x,y) - u(y) \frac{\partial \Phi_{-\kappa}(x,y)}{\partial \nu(y)}\} ds(y) \\ &- \int_{|y|=r} \{\frac{\partial u}{\partial \nu}(y) \Phi_{-\kappa}(x,y) - u(y) \frac{\partial \Phi_{-\kappa}(x,y)}{\partial \nu(y)}\} ds(y) \ . \end{aligned}$$

The first integral can be written as

$$\int\limits_{|y|=r'} \{(\frac{\partial u}{\partial \nu}(y) - i\kappa u(y))\Phi_{-\kappa}(x,y) \\ -u(y)(\frac{\partial \Phi_{-\kappa}(x,y)}{\partial \nu(y)} - i\kappa\Phi_{-\kappa}(x,y))\}ds(y) \ .$$

Now we use the radiation condition, the condition (1.16) and the estimates $|\Phi_{-\kappa}(x,y)|+|\partial\Phi_{-\kappa}(x,y)/\partial\nu(y)| = O(|y|^{-1})$, $|y| \to \infty$, and deduce for $r' \to \infty$ with the help of the Cauchy-Schwarz inequality the second representation in (1.17) for u.

From (1.17) we obtain

$$\begin{aligned} f_+(\hat{x},t) &:= e^{-i\kappa/t}u(\frac{1}{t}\hat{x}) \\ &= \int\limits_{|y|=r} \{\frac{\partial\Phi_\kappa(\frac{1}{t}\hat{x},y)}{\partial\nu(y)}e^{-i\kappa/t}u(y) - \Phi_\kappa(\frac{1}{t}\hat{x},y)e^{-i\kappa/t}\frac{\partial u}{\partial\nu}(y)\}ds(y) \ . \end{aligned} \tag{1.18}$$

Since

$$\begin{aligned} |\frac{1}{t}\hat{x} - y| &= \frac{1}{t}\sqrt{1 - 2t\hat{x}\cdot y + t^2|y|^2} \\ &= \frac{1}{t}(1 + \sum_{j=1}^{\infty} c_j(\hat{x},y)t^j) \ , \ 0 < t < t_0 \ , \end{aligned}$$

we can conclude

$$\begin{aligned} \Phi_\kappa(\frac{1}{t}\hat{x},y)e^{-i\kappa/t} &= \frac{e^{i\kappa(|\frac{1}{t}\hat{x}-y|-\frac{1}{t})}}{4\pi|\frac{1}{t}\hat{x}-y|} \\ &= \sum_{j=1}^{\infty} d_j(\hat{x},y)t^j \ , \ 0 < t < t_0 \ , \end{aligned}$$

where c_j, d_j denote continuous functions in $y \in \partial B_r$, $\hat{x} \in S^2$. The convergence is uniform in y, $\hat{x}$ and t if t_0 is sufficiently small. Similarly, we can obtain a series expansion for $e^{-i\kappa/t}\partial\Phi_\kappa(\frac{1}{t}\hat{x},y)/\partial\nu(y)$. Inserting these expansions in (1.18) yields the expansion

$$f_+(\hat{x},t) = \sum_{j=1}^{\infty} a_j(\hat{x})t^j \ , \ 0 < t < t_0 \ , \ \hat{x} \in S^2 \ .$$

Analogously, we can derive an expansion for f_-.

Now, fix $\hat{x} \in S^2$ and assume $f_+(\hat{x}, \cdot)$ does not vanish identically in $(0, t_0)$. Then, the quotient $f_-(\hat{x}, \cdot)/f_+(\hat{x}, \cdot)$ has at most a pole at $t = 0$. However, this is a contradiction because $f_-(\hat{x}, t)/f_+(\hat{x}, t) = e^{2i\kappa/t}$ has an essential singularity at $t = 0$. Therefore, we conclude $f_+(\hat{x}, t) = 0$ for all $0 < t < t_0$, $\hat{x} \in S^2$, hence $u(x) = 0$ for $|x| > 1/t_0$, and then $u = 0$ in $\mathbb{R}^3 \setminus B_r$ because solutions to the Helmholtz equation are analytic, and we have proved the first assertion of the lemma.

If u is a radiating solution to the Helmholtz equation with vanishing far field pattern, we know that $|u(x)| = O(|x|^{-2})$, $|x| \to \infty$, hence

$$\int\limits_{|x|=r} |u(x)|^2 ds(x) \to 0 \ , \ r \to \infty \ .$$

Then u vanishes and we have proved the lemma.

□

1.3 Unique Solvability of the Direct Acoustic Scattering Problem

In this section we give a precise formulation of the direct acoustic scattering problem, prove its uniqueness and then turn to its existence proof. Since the existence proof requires more regularity than continuity for the refractive index n, we state the problem with the higher regularity assumption on n although the uniqueness proof works for continuous n, too. The existence proof is based on integral equations containing a volume potential. We therefore study volume potentials more closely before presenting the existence result.

We need a regularity that is somewhat between continuous and continuously differentiable. To this end let us introduce some function spaces.

If $G \subset \mathbb{R}^3$ is bounded, $C(\overline{G})$ is a Banach space with the norm

$$\|\varphi\|_\infty = \|\varphi\|_{\infty,G} := \sup_{x \in G} |\varphi(x)| \ .$$

A complex-valued function φ defined on a set $G \subset \mathbb{R}^3$ is called uniformly Hölder continuous with Hölder exponent γ, $0 < \gamma < 1$, if there is a positive constant M such that $|\varphi(x) - \varphi(y)| \leq M|x-y|^\gamma$ for all x, $y \in G$. For $0 < \gamma < 1$ we denote by $C^{0,\gamma}(G)$ the linear space of bounded and uniformly Hölder continuous functions on G with Hölder exponent γ. Equipped with the norm

$$\|\varphi\|_{0,\gamma} = \|\varphi\|_{0,\gamma,G} := \sup_{x \in G} |\varphi(x)| + \sup_{\substack{x,y \in G \\ x \neq y}} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\gamma}$$

$C^{0,\gamma}(G)$ is a Banach space.

Similarly, we can introduce functions having uniformly Hölder continuous derivatives. If $G \subset \mathbb{R}^3$ is an open set, we define $C^{1,\gamma}(\overline{G})$ to be the Banach space of all bounded and continuously differentiable functions φ on G for which the gradient $\nabla\varphi$ is a bounded and uniformly Hölder continuous vector field on G with exponent γ. The norm in $C^{1,\gamma}(\overline{G})$ is

$$\|\varphi\|_{1,\gamma} = \|\varphi\|_{1,\gamma,G} := \sup_{x \in G} |\varphi(x)| + \sup_{x \in G} |\nabla\varphi(x)| + \sup_{\substack{x,y \in G \\ x \neq y}} \frac{|\nabla\varphi(x) - \nabla\varphi(y)|}{|x-y|^\gamma} \ .$$

We define $C^{1,\gamma}(\partial D)$ and its norm analogously by replacing the gradient by the surface gradient.

Let us now formulate our model for the scattering of an incident acoustic wave u^i in an inhomogeneous medium in $\mathbb{R}^3$ with compact inhomogeneity. Assume $\kappa > 0$ and $n \in C^{0,\gamma}(\mathbb{R}^3)$, $0 < \gamma < 1$, are given with $\mathrm{supp}(1-n) \subset B_R$ and $\Im(n) \geq 0$. Moreover, $u^i \in C^2(\mathbb{R}^3)$ with $\Delta u^i + \kappa^2 u^i = 0$ in $\mathbb{R}^3$ is known. Then the direct acoustic scattering problem (DAP) consists in finding $u^s \in C^2(\mathbb{R}^3)$ such that $u := u^i + u^s$ satisfies

$$\Delta u(x) + \kappa^2 n(x) u(x) = 0 \ , \ x \in \mathbb{R}^3 \ , \tag{1.19}$$

and such that u^s satisfies the Sommerfeld radiation condition (1.11)

$$\Big| \hat{x} \cdot \nabla u^s(x) - i\kappa u^s(x) \Big| = o(\frac{1}{|x|}) \ , \ |x| \to \infty \ ,$$

uniformly for all directions $\hat{x} := |x|^{-1} x$.

We regard u^i as an incident wave, u^s as the scattered wave and u as the total wave.

The first theorem states uniqueness for the problem (DAP).

Theorem 1.8 *If $u^s \in C^2(\mathbb{R}^3)$ satisfies $\Delta u^s + \kappa^2 n u^s = 0$ in $\mathbb{R}^3$ and the Sommerfeld radiation condition (1.11), then $u^s = 0$ in $\mathbb{R}^3$, especially, the direct acoustic scattering problem has at most one solution.*

Proof: As in (1.12) we know from the radiation condition

$$0 = \lim_{r\to\infty} \Big\{ \int\limits_{|x|=r} \{ |\frac{\partial u^s}{\partial \nu}|^2 + \kappa^2 |u^s|^2 \} ds + 2\Re\{ i\kappa \int\limits_{|x|=r} \frac{\partial u^s}{\partial \nu} \overline{u^s} ds \} \Big\} \ , \tag{1.20}$$

and Green's first theorem (1.8) yields

$$\begin{aligned} 2\Re\{ i\kappa \int\limits_{|x|=r} \frac{\partial u^s}{\partial \nu} \overline{u^s} ds \} &= -2\Im\{ \kappa \int\limits_{B_r} \{ |\nabla u^s|^2 + \overline{u^s} \Delta u^s \} dx \} \\ &= -2\Im\{ \kappa \int\limits_{B_r} \{ |\nabla u^s|^2 - \kappa^2 n |u^s|^2 \} dx \} \\ &= 2\kappa^3 \{ \int\limits_{B_r} \Im(n) |u^s|^2 dx \} \geq 0 \ . \end{aligned}$$

Hence, we can infer from equation (1.20) that

$$\int\limits_{|x|=r} |u^s|^2 ds \to 0 \ , \ r \to \infty \ ,$$

and Lemma 1.7 yields $u^s(x) = 0$ for $|x| > R$, i.e., $u \in C_0^2(\mathbb{R}^3)$.

From $\Delta u^s + \kappa^2 n u^s = 0$ we obtain the inequality $|\Delta u^s| \le M|u^s|$ in $\mathbb{R}^3$ where $M := \max\{\kappa^2 |n(x)| \colon x \in \mathbb{R}^3\}$. Now, applying Theorem 1.2 we arrive at $u^s = 0$ in $\mathbb{R}^3$ and have proved the theorem.

□

Now, we are going to show that the direct scattering problem (DAP) has a solution. To this end assume $u = u^i + u^s$ is the solution to the direct scattering problem (DAP). For $x \in \mathbb{R}^3$ we choose $r > |x| + R$. Applying Green's formula (1.10) we arrive at

$$\begin{aligned} u(x) &= \int\limits_{\partial B_r} \{\frac{\partial u}{\partial \nu}(y)\Phi_\kappa(x,y) - u(y)\frac{\partial \Phi_\kappa(x,y)}{\partial \nu(y)}\} ds(y) \\ &\quad -\kappa^2 \int\limits_{B_r} (1 - n(y)) u(y) \Phi_\kappa(x,y) dy \ . \end{aligned}$$

Inserting $u^i + u^s$ for u into the integral over ∂B_r and observing that the contribution from u^s is zero due to (1.14) whereas the remaining integrals over ∂B_r represent u^i due to (1.10) we obtain the integral equation

$$u(x) = u^i(x) - \kappa^2 \int\limits_{B_R} (1 - n(y)) u(y) \Phi_\kappa(x,y) dy \ , \ x \in \mathbb{R}^3 \ , \tag{1.21}$$

which is known as the Lippmann-Schwinger equation. This is an integral equation of the second kind in B_R for the unknown total field u. Our aim is to show that a solution u of the Lippmann-Schwinger equation yields a solution u^s to the scattering problem via $u^s = u - u^i$. We can then obtain the solvability of the integral equation by the Riesz theory and the previous uniqueness theorem. Hence we have proved the existence of a solution to the scattering problem (DAP). To this end we have to study the properties of the volume potential

$$(V_\kappa \varphi)(x) := \int\limits_{B_R} \Phi_\kappa(x,y) \varphi(y) dy \ , \ x \in \mathbb{R}^3 \ ,$$

which appears in the above equation. We replace the kernel Φ_κ by a more general kernel because in later sections we shall encounter volume potentials with different kernels again. In [13, IV 4.1] the reader can find the proofs for volume potentials with a kernel $k(x,y)$ which are not of a convolution type.

Let us start with an examination of the first derivatives of a volume potential and with the proof for the compactness of volume potential operators.

Theorem 1.9 *Assume* $0 < R_1 < R_2$. *Let* $k \in C^2(B_{2R_2} \setminus \{0\})$ *satisfy*

$$|k(x)| \leq M|x|^{-1} \ , \ |\partial_j k(x)| \leq M|x|^{-2} \ , \ |\partial_j \partial_l k(x)| \leq M|x|^{-3} \ ,$$

for $j,\ l = 1,2,3,\ 0 < |x| \leq 2R_1$, *with a suitable constant* M. *Define for a density* $\varphi \in C(\overline{B_{R_1}})$ *the volume potential* $V\varphi$ *by*

$$(V\varphi)(x) := \int\limits_{B_{R_1}} k(x-y)\varphi(y)dy \ , \ x \in B_{R_1} \ .$$

Then $V\varphi$ *has the following properties:*

(a) *For all* $\gamma \in (0,1)$ $V\varphi \in C^{1,\gamma}(\overline{B_{R_1}})$ *and there is a suitable constant* C_γ *such that* $\|V\varphi\|_{1,\gamma} \leq C_\gamma \|\varphi\|_\infty$ *for all* $\varphi \in C(\overline{B_{R_1}})$.

(b) *The derivatives have the form*

$$(\partial_j(V\varphi))(x) = \int\limits_{B_{R_1}} (\partial_j k)(x-y)\varphi(y)dy \ , \ x \in B_{R_1} \ , \ j = 1,2,3 \ .$$

For $\varphi \in C_0^1(B_{R_1})$ *the relation* $\partial_j(V\varphi) = V(\partial_j \varphi)$ *holds true.*

(c) *The operators*

$$\begin{aligned} V\colon (C(\overline{B_{R_1}}), \|\cdot\|_\infty) &\rightarrow (C(\overline{B_{R_1}}), \|\cdot\|_\infty) \ , \\ V\colon (C(\overline{B_{R_1}}), \|\cdot\|_{L^2(B_{R_1})}) &\rightarrow (C(\overline{B_{R_1}}), \|\cdot\|_{L^2(B_{R_1})}) \quad \textit{and} \\ V\colon (C(\overline{B_{R_1}}), \|\cdot\|_{L^2(B_{R_1})}) &\rightarrow (C(\overline{B_{R_1}}), \|\cdot\|_\infty) \end{aligned}$$

are compact.

Proof: Due to the weak singularities of k and $\partial_j k$ at $x = 0$ the integrals in the assertion exist as improper integrals. We choose a function $\chi \in C^1(\mathbb{R}^3)$ such that $0 \leq \chi \leq 1$, $\chi(x) = 0$ for $|x| \leq 1$, and $\chi(x) = 1$ for $|x| \geq 2$. Then we define

$$(V_l\varphi)(x) := \int\limits_{B_{R_1}} \chi(l(x-y))k(x-y)\varphi(y)dy \ , \ x \in B_{R_1} \ , \ l \in \mathbb{N} \ .$$

Since $\chi(l\cdot)k \in C^1(B_{2R_2})$, we know $V_l\varphi \in C^1(\overline{B_{R_1}})$ and

$$\begin{aligned}(\partial_j(V_l\varphi))(x) &= \int_{B_{R_1}} \chi(l(x-y))(\partial_j k)(x-y)\varphi(y)dy \\ &\quad + \int_{B_{R_1}} l(\partial_j\chi)(l(x-y))k(x-y)\varphi(y)dy \ , \ x \in B_{R_1} \ . \end{aligned} \tag{1.22}$$

From

$$|V_l\varphi(x) - V\varphi(x)| \le \int_{\{|y-x|\le 2/l\}} \frac{M}{|x-y|}\|\varphi\|_\infty dy \le cl^{-2} \ , \ l \to \infty \ ,$$

we know $\|V_l\varphi - V\varphi\|_\infty \to 0$, $l \to \infty$. c denotes various positive constants during the proof which may vary from inequality to inequality. A similar estimate shows that the second integral in (1.22) converges uniformly to zero, whereas the first integral in (1.22) converges uniformly to

$$\int_{B_{R_1}} (\partial_j k)(x-y)\varphi(y)dy \ .$$

This implies $V\varphi \in C^1(\overline{B_{R_1}})$ and the first formula for the derivative in part (b). For $\varphi \in C_0^1(B_{R_1})$ integration by parts yields $\partial_j(V_l\varphi) = V_l(\partial_j\varphi)$. Passing to the limit $l \to \infty$ we can derive the second assertion of part (b) by the previous considerations.

Next, we show the Hölder continuity of $\partial_j(V\varphi)$ and the norm estimate. The inequality

$$\begin{aligned}|(V\varphi)(x)| &\le \int_{B_{R_1}} |k(x-y)|\|\varphi\|_\infty dy \\ &\le M \int_{|x-y|\le 2R_1} |x-y|^{-1} dy\|\varphi\|_\infty \le c\|\varphi\|_\infty\end{aligned}$$

and a similar estimate for $\partial_j(V\varphi)$ provide bounds for the supremum norms of $V\varphi$ and $\partial_j(V\varphi)$. For the Hölder continuity we first observe that if x, z, $y \in B_{R_1}$ with $2|x-z| \le |x-y|$ are given and if $x^* = x + t(z-x)$, $t \in [0,1]$, lies on the line between x and z, then $|x^*-y| \ge (1/2)|x-y|$ because

$$\begin{aligned}|x^*-y| = |x+t(z-x)-y| &\ge |x-y| - t|z-x| \\ &\ge |x-y| - (t/2)|x-y| \ge (1/2)|x-y| \ .\end{aligned}$$

Then we obtain for x, y, $z \in B_{R_1}$ with $2|x-z| \leq |x-y|$ the estimate

$$\begin{aligned} |(\partial_j k)(x-y) - (\partial_j k)(z-y)| &= |\int_0^1 (x-z)\cdot(\nabla \partial_j k)(x+t(z-x))dt| \\ &\leq c\frac{|x-z|}{|x-y|^3} \ . \end{aligned} \tag{1.23}$$

Finally, we compute for x, $z \in B_{R_1}$ with $\delta := |x-z| > 0$:

$$\begin{aligned} &|\int_{B_{R_1}} ((\partial_j k)(x-y) - (\partial_j k)(z-y))\varphi(y)dy| \\ &\leq \int_{B_{R_1}} |(\partial_j k)(x-y) - (\partial_j k)(z-y)|dy \|\varphi\|_\infty \\ &\leq \|\varphi\|_\infty \{ \int_{B_{R_1}\cap\{|x-y|\leq 2\delta\}} (|(\partial_j k)(x-y)| + |(\partial_j k)(z-y)|)dy \\ &\qquad + \int_{B_{R_1}\cap\{|x-y|\geq 2\delta\}} |(\partial_j k)(x-y) - (\partial_j k)(z-y)|dy\} \end{aligned}$$

and bound the first integral by

$$\int_{|x-y|\leq 2\delta} \frac{M}{|x-y|^2}dy + \int_{|z-y|\leq 3\delta} \frac{M}{|z-y|^2}dy \leq c\delta$$

and the second with the help of (1.23) by

$$\int_{2\delta\leq|x-y|\leq 2R_1} \frac{c|x-z|}{|x-y|^3}dy \leq c|x-z|\log\delta \leq c|x-z|^\gamma$$

This completes the proof of assertion (a).

Since by the Arzelà-Ascoli theorem the imbedding $(C^{1,\gamma}(\overline{B_{R_1}}), \|\cdot\|_{1,\gamma})$ into $(C(\overline{B_{R_1}}), \|\cdot\|_\infty)$ is compact, we conclude from part (a) the first assertion of part (c). The Arzelà-Ascoli theorem also implies that the operators V_l are compact operators from $(C(\overline{B_{R_1}}), \|\cdot\|_{L^2(B_{R_1})})$ to $(C(\overline{B_{R_1}}), \|\cdot\|_\infty)$. Using the Cauchy-Schwarz inequality we can estimate

$$|V_l\varphi(x) - V\varphi(x)|^2 \leq \Big[\int_{\{|y-x|\leq 2/l\}} \frac{M}{|x-y|}|\varphi(y)|dy\Big]^2$$

$$\leq \int\limits_{\{|y-x|\leq 2/l\}} \frac{M^2}{|x-y|^2} dy \int\limits_{\{|y-x|\leq 2/l\}} |\varphi(y)|^2 dy$$

$$\frac{c}{l} \|\varphi\|^2_{L^2(B_{R_1})}$$

for all $x \in B_{R_1}$, whence

$$\sup_{\|\varphi\|_{L^2}=1} \|V_l\varphi - V\varphi\|_\infty \to 0 \ , \ l \to \infty \ .$$

Then, $V\colon (C(\overline{B_{R_1}}), \|\cdot\|_{L^2(B_{R_1})}) \to (C(\overline{B_{R_1}}), \|\cdot\|_\infty)$ is compact and so is $V\colon (C(\overline{B_{R_1}}), \|\cdot\|_{L^2(B_{R_1})}) \to (C(\overline{B_{R_1}}), \|\cdot\|_{L^2(B_{R_1})})$ due to the continuous imbedding of $(C(\overline{B_{R_1}}), \|\cdot\|_\infty)$ into $(C(\overline{B_{R_1}}), \|\cdot\|_{L^2(B_{R_1})})$. This ends the proof of the theorem.

□

The next theorem deals with the second derivatives of a volume potential.

Theorem 1.10 *Assume* $0 < R_1 < R_2$.

(a) If $k \in C^2(B_{2R_2} \setminus \{0\})$ *satisfies the assumptions from Theorem 1.9 and if the density* $\varphi \in C^{0,\gamma}(\overline{B_{R_1}})$, $\gamma \in (0,1)$, *is uniformly Hölder continuous, then the volume potential*

$$(V\varphi)(x) := \int\limits_{B_{R_1}} k(x-y)\varphi(y)dy \ , \ x \in B_{R_1} \ ,$$

is twice continuously differentiable in B_{R_1} *and we have*

$$\begin{aligned}
(\partial_l \partial_j (V\varphi))(x) &= \int\limits_{B_{R_1}} (\partial_l \partial_j k)(x-y)[\varphi(y) - \varphi(x)]dy \\
&\quad -\varphi(x) \int\limits_{\partial B_{R_1}} \nu_l(y)(\partial_j k)(x-y)ds(y) \\
&= \int\limits_{B_{R_1}} (\partial_l \partial_j k)(x-y)[\varphi(y) - \varphi(x)]dy \\
&\quad -\varphi(x) \int\limits_{\partial B_{R_1}} \nu_j(y)(\partial_l k)(x-y)ds(y) \ .
\end{aligned}$$

(b) If k has the form $k(x) = (1/(4\pi|x|)) + \tilde{k}(x)$, $0 < |x| < 2R_2$, where $\tilde{k} \in C^3(B_{2R_2} \setminus \{0\})$ satisfies

$$|\tilde{k}(x)| \le M \ , \ |\partial_j \tilde{k}(x)| \le M|x|^{-1} \ ,$$

$$|\partial_j \partial_l \tilde{k}(x)| \le M|x|^{-2} \ , \ |\partial_j \partial_l \partial_m \tilde{k}(x)| \le M|x|^{-3} \ ,$$

for j, l, $m = 1, 2, 3$, $0 < |x| \le 2R_1$, with a suitable constant M, then $V\varphi \in C^2(B_{R_1})$ and

$$(\Delta(V\varphi))(x) = -\varphi(x) + \int\limits_{B_{R_1}} (\Delta\tilde{k})(x-y)\varphi(y)dy \ , \ x \in B_{R_1} \ .$$

Proof: We know from Theorem 1.9 that $V\varphi \in C^1(\overline{B_{R_1}})$ and that we have for a fixed j

$$v(x) := (\partial_j(V\varphi))(x) = \int\limits_{B_{R_1}} (\partial_j k)(x-y)\varphi(y)dy \ , \ x \in \overline{B_{R_1}} \ .$$

Using the function χ from Theorem 1.9 we define

$$v_m(x) := \int\limits_{B_{R_1}} (\partial_j k)(x-y)\chi(m(x-y))\varphi(y)dy \ , \ x \in \overline{B_{R_1}} \ , \ m \in \mathbb{N} \ .$$

Proceeding as in the previous theorem we obtain $\|v_m - v\|_\infty \to 0$, $m \to \infty$, $v_m \in C^1(B_{R_1})$ and

$$\begin{aligned}
\partial_l v_m(x) &= \int\limits_{B_{R_1}} \frac{\partial}{\partial x_l}\Big((\partial_j k)(x-y)\chi(m(x-y))\Big)\varphi(y)dy \\
&= \int\limits_{B_{R_1}} \frac{\partial}{\partial x_l}\Big((\partial_j k)(x-y)\chi(m(x-y))\Big)(\varphi(y)-\varphi(x))dy \\
&\quad - \varphi(x) \int\limits_{B_{R_1}} \frac{\partial}{\partial y_l}\Big((\partial_j k)(x-y)\chi(m(x-y))\Big)dy \\
&= \int\limits_{B_{R_1}} \frac{\partial}{\partial x_l}\Big((\partial_j k)(x-y)\chi(m(x-y))\Big)(\varphi(y)-\varphi(x))dy \\
&\quad - \varphi(x) \int\limits_{\partial B_{R_1}} \nu_l(y)(\partial_j k)(x-y)\chi(m(x-y))ds(y) \ , \ x \in B_{R_1} \ .
\end{aligned}$$

Here, we have used integration by parts in the last step. As $m \to \infty$, due to the uniform Hölder continuity of φ, the first integral converges uniformly to

$$\int\limits_{B_{R_1}} (\partial_l \partial_j k)(x-y)(\varphi(y)-\varphi(x))dy \ ,$$

whereas the second term converges locally uniformly to

$$-\varphi(x) \int\limits_{\partial B_{R_1}} \nu_l(y)(\partial_j k)(x-y)ds(y) \ .$$

Hence, $v \in C^1(B_{R_1})$, i.e., $V\varphi \in C^2(B_{R_1})$, and

$$\begin{aligned}
(\partial_l \partial_j (V\varphi))(x) &= \partial_l v(x) \\
&= \int\limits_{B_{R_1}} (\partial_l \partial_j k)(x-y)(\varphi(y)-\varphi(x))dy \\
&\quad -\varphi(x) \int\limits_{\partial B_{R_1}} \nu_l(y)(\partial_j k)(x-y)ds(y) \ , \ x \in B_{R_1} \ .
\end{aligned}$$

Moreover, from $\partial_l \partial_j (V\varphi) = \partial_j \partial_l (V\varphi)$ we can infer the second formula in assertion (a) for the derivatives and we have proved part (a).

For assertion (b) we note that $\tilde{k}$ and $\partial_j \tilde{k}$ satisfy the assumptions of Theorem 1.9. Therefore,

$$(\tilde{V}\varphi)(x) := \int\limits_{B_{R_1}} \tilde{k}(x-y)\varphi(y)dy \ , \ x \in \overline{B_{R_1}} \ ,$$

is twice continuously differentiable in B_{R_1} and integration and differentiation may be interchanged. Hence, it remains to investigate the Newton potential

$$(V_0\varphi)(x) := \int\limits_{B_{R_1}} \frac{1}{4\pi|x-y|}\varphi(y)dy \ , \ x \in B_{R_1} \ .$$

We can conclude from part (a) that $V_0\varphi$ is twice continuously differentiable in B_{R_1} and that

$$\begin{aligned}
\Delta(V\varphi) &= \int\limits_{B_{R_1}} \Delta_x\Big(\frac{1}{4\pi|x-y|}\Big)(\varphi(y)-\varphi(x))dy \\
&\quad + \varphi(x) \int\limits_{\partial B_{R_1}} \frac{\nu(y)\cdot(x-y)}{4\pi|x-y|^3} ds(y) = -\varphi(x) \ , \ x \in B_{R_1} \ .
\end{aligned}$$

In the last step we have used $\Delta(1/|x|) = 0$, $|x| > 0$, and the representation formula (1.10) in B_{R_1} for $\kappa = 0$ and the function $u(x) = 1$, $x \in B_{R_1}$.

□

Straightforward calculations show that for $\kappa \in \mathbb{C}$ the kernel

$$k(x) := \frac{e^{i\kappa|x|}}{4\pi|x|} \ , \ |x| > 0 \ ,$$

satisfies the assumptions in Theorem 1.9. Furthermore,

$$\tilde{k}(x) := \frac{e^{i\kappa|x|} - 1}{4\pi|x|} = \frac{1}{4\pi} \sum_{j=1}^{\infty} \frac{(i\kappa)^j}{j!} |x|^{j-1} \ , \ |x| > 0 \ , \ \kappa \in \mathbb{C} \ ,$$

satisfies the assumptions of Theorem 1.10. And finally,

$$\Delta\tilde{k}(x) = \Delta \frac{e^{i\kappa|x|}}{4\pi|x|} = -\kappa^2 \frac{e^{i\kappa|x|}}{4\pi|x|} \ , \ |x| > 0 \ .$$

Consequently, the following theorem holds true for the volume potential

$$(V_\kappa\varphi)(x) := \int_{B_R} \Phi_\kappa(x,y)\varphi(y)dy \ , \ x \in \mathbb{R}^3 \ .$$

Theorem 1.11

(a) *If $\varphi \in C(\overline{B_R})$, then $V_\kappa\varphi \in C^{1,\gamma}(\overline{B_R})$ for all $0 < \gamma < 1$, the order of differentiation and integration can be interchanged, and $\|V_\kappa\varphi\|_{1,\gamma} \leq C_\gamma\|\varphi\|_\infty$.*

(b) *If $\varphi \in C^{0,\gamma}(\overline{B_R})$, then $V_\kappa\varphi \in C^2(B_R)$ and $\Delta(V_\kappa\varphi) + \kappa^2(V_\kappa\varphi) = -\varphi$. Especially, if $\varphi \in C^{0,\gamma}(\overline{B_R})$ has compact support in B_R, then $V_\kappa\varphi \in C^2(\mathbb{R}^3)$ and $\Delta(V_\kappa\varphi) + \kappa^2(V_\kappa\varphi) = -\varphi$ in $\mathbb{R}^3$.*

(c) *If $\varphi \in C_0^1(B_R)$, then $\partial_j(V_\kappa\varphi) = V_\kappa(\partial_j\varphi)$.*

We are now in a position to prove that a solution to the Lippmann-Schwinger equation (1.21)

$$u(x) = u^i(x) - \kappa^2 \int_{B_R} (1 - n(y))u(y)\Phi_\kappa(x,y)dy \ , \ x \in B_R \ .$$

yields a solution to the direct acoustic scattering problem.

Lemma 1.12 *Let $u \in C(\overline{B_R})$ be a solution of the Lippmann-Schwinger equation (1.21) in B_R. Then*

$$u^s(x) := -\kappa^2 \int_{B_R} (1 - n(y))u(y)\Phi_\kappa(x, y)dy \ , \ x \in \mathbb{R}^3 \ , \qquad (1.24)$$

is the solution of the direct acoustic scattering problem (DAP).

Proof: First, we conclude from the Lippmann-Schwinger equation and the regularity of the volume potential that $u \in C^1(B_R)$ and then $u^s \in C^2(\mathbb{R}^3)$. Moreover, u^s is a radiating solution to the Helmholtz equation in the exterior of B_R because Φ_κ is. Finally, due to the Lippmann-Schwinger equation, we can extend u by $u := u^i + u^s$ in $\mathbb{R}^3$ and we compute

$$\Delta u + \kappa^2 u = \Delta u^s + \kappa^2 u^s = \kappa^2(1 - n)u \ ,$$

where we have used Theorem 1.11 again.

□

The existence of a solution to the direct scattering problem (DAP) is an easy consequence of the previous lemma and the uniqueness proof.

Theorem 1.13 *The direct acoustic scattering problem (DAP) has a unique solution u^s. The total field $u := u^i + u^s$ is the unique solution to the Lippmann-Schwinger equation (1.21) in $\mathbb{R}^3$.*

Proof: We have established uniqueness for (DAP) in Theorem 1.8. Due to the compact imbedding of $C^{1,\gamma}(\overline{B_R})$ into $C(\overline{B_R})$ we can conclude from the mapping properties of the volume potential that the equation

$$u(x) = u^i(x) - \kappa^2 \int_{B_R} (1 - n(y))u(y)\Phi_\kappa(x, y)dy \ , \ x \in \overline{B_R} \ ,$$

is a Fredholm integral equation of the second kind with a compact integral operator in $C(\overline{B_R})$. Consequently, by the Riesz theory it has a unique solution if it has a trivial nullspace. If $u \in C(\overline{B_R})$ is a solution of the integral equation with $u^i = 0$, we define u^s as in (1.24) and conclude from Lemma 1.12 that u^s is a solution of the homogeneous problem (DAP). This implies $u = u^s = 0$ by the uniqueness for (DAP). Since by Lemma 1.12 a solution of the Lippmann-Schwinger equation yields a solution to the scattering problem (DAP), we have proved the theorem.

□

From the representation

$$u^s(x) = -\kappa^2 \int_{B_R} (1 - n(y))u(y)\Phi_\kappa(x,y)dy \ , \ x \in \mathbb{R}^3 \ ,$$

and the asymptotic behavior of the fundamental solution (1.15) we obtain

$$u^s_\infty(\hat{x}) = -\frac{\kappa^2}{4\pi} \int_{B_R} (1 - n(y))u(y)e^{-i\kappa\hat{x}\cdot y}dy \ , \ \hat{x} \in S^2 \ . \qquad (1.25)$$

For the incident wave $u^i(x,d) := e^{i\kappa\hat{x}\cdot d}$, $x \in \mathbb{R}^3$, which represents a plane wave travelling in direction $d \in S^2$ we denote by $u^s(\cdot,d)$ and $u_\infty(\cdot,d)$ the corresponding scattered wave and far field pattern, respectively.

The next chapter is devoted to the question how much information about the refractive index n can be recovered from $u_\infty \colon S^2 \times S^2 \to \mathbb{C}$.

Chapter 2

The Inverse Acoustic Scattering Problem

This chapter is devoted to an inverse acoustic scattering problem. We assume that the refractive index n is unknown. In order to obtain some information about n we probe the medium with plane incident waves and measure the corresponding far field patterns of the scattered waves. Assuming that the far field patterns at a fixed wave number for all incident directions are available, i.e., $u_\infty: S^2 \times S^2 \to \mathbb{C}$ is known, the task is to reconstruct n from these data.

We follow the historic development and start with a more modest result. Namely, these data suffice to determine n uniquely. To this end we have to construct special solutions to the equation $\Delta u + \kappa^2 n u = 0$. The construction for a very similar case, due to Sylvester and Uhlmann, is worked out in [46]. We modify their method by using Fourier series techniques instead of the Fourier transform. The special solutions allow to prove that the Fourier coefficients of two refractive indices producing the same far field patterns must coincide (see [20, 37, 40] and [47, 46, for related problems]). Meanwhile, in [44, 35] there are even uniqueness results for more general operators available, like the Schrödinger operator in the presence of a magnetic field. However, the proofs require a more elaborate analysis due to the first order perturbations of the Laplacian.

Then, we proceed to the question what norm should be used on the data set in order to have continuous dependence of n on the data. In a first step we construct certain boundary integral operators S_n from the far field belonging to n. It turns out that this problem is severely ill-posed and we have to employ a very strong norm on the data set. In a second step we give a logarithmic stability estimate of $\|n - \tilde{n}\|_\infty$ in terms of $\|S_n - S_{\tilde{n}}\|_\infty$ and in terms of the distance of the far field patterns. Inspired by the paper [2] of Alessandrini about the continuous dependence of the conductivity on boundary measurements Stefanov has investigated the question of stability for the inverse scattering problem in [42]. Our approach, though built on his proof, avoids the Dirichlet-to-Neumann map in order to circumvent difficulties which arise from interior Dirichlet eigenvalues. Moreover, we are able to obtain the stability result without employing a lemma concerning estimates of holomorphic functions in $\mathbb{C}^3$.

Finally, we shall give a constructive procedure to recover n. Again, we first construct certain boundary integral operators from the far field pattern. Then, we use these operators to compute the Fourier coefficients of $n - 1$. The main ideas of the proof for the second part can be found in Nachman's paper [32]. Of course, we base our analysis on Fourier series whereas he uses Fourier transformation techniques. Furthermore, Nachman works with the Dirichlet-to-Neumann map which is not possible in the presence of interior Dirichlet eigenvalues. Since we start with the far field pattern, we can use a different map and therefore avoid this problem.

Inspite of the fact that the reconstruction of n implies its uniqueness we deal with the uniqueness question separately because the reconstruction procedure grows out of the ideas from the uniqueness proof. Hence, beginning with uniqueness and then proceeding to the reconstruction seems to be the less difficult (but longer) way to understand the reconstruction.

2.1 Uniqueness for the Inverse Acoustic Scattering Problem

We assume n, $\tilde{n} \in C^{0,\gamma}(\mathbb{R}^3)$, $0 < \gamma < 1$, satisfy $\Im(n) \geq 0$, $\Im(\tilde{n}) \geq 0$, $\operatorname{supp}(1-n) \subset B_R$, and $\operatorname{supp}(1-\tilde{n}) \subset B_R$. For a fixed wave number $\kappa > 0$ we denote by $u^i(x,d) := e^{i\kappa d \cdot x}$, $x \in \mathbb{R}^3$, a plane incident wave propagating into the direction $d \in S^2$. For the incident wave $u^i(\cdot,d)$ we define $u^s(\cdot,d)$, $\tilde{u}^s(\cdot,d)$ to be the solutions to the direct scattering problem (DAP) from page 26 corresponding to the incident wave $u^i(\cdot,d)$ and to the refractive index n, $\tilde{n}$, respectively. Similarly, $u(\cdot,d)$, $\tilde{u}(\cdot,d)$, $u_\infty(\cdot,d)$ and $\tilde{u}_\infty(\cdot,d)$ are the total waves and the far field patterns of the scattered waves belonging to the incident wave $u^i(\cdot,d)$ and the refractive index n, $\tilde{n}$, respectively. Let us now assume, that for a fixed wave number $\kappa > 0$ and for all directions $d \in S^2$ the far field patterns $u_\infty(\cdot,d)$ and $\tilde{u}_\infty(\cdot,d)$ coincide on S^2. It is the aim of this section to prove that then n and $\tilde{n}$ must coincide.

Before we start with the proof we want to give a brief outline. The first step is the relation

$$\int_{B_R} (n(x) - \tilde{n}(x))u(x)\tilde{u}(x)dx = 0 \tag{2.1}$$

for all solutions u, $\tilde{u}$ to $\Delta u + \kappa^2 n u = 0$ and $\Delta \tilde{u} + \kappa^2 \tilde{n}\tilde{u} = 0$ in B_{R_1}, respectively, where we choose $R_1 > R$. If u has the special form $u(\cdot,d)$, we obtain relation (2.1) with the help of Green's theorem from the coincidence of the far field patterns in the next lemma. In order to prove the relation for general solutions u we show in Lemma 2.3 that an arbitrary solution u can be approximated by elements from $\operatorname{span}\{u(\cdot,d) \colon d \in S^2\}$ with respect to the L^2-norm.

We can infer the uniqueness of the refractive index from relation (2.1) if we know that a function $q \in C(\overline{B_R})$ satisfying $\int_{B_R} q u \tilde{u} dx = 0$ for all u and $\tilde{u}$ as above must vanish identically. To this end we have to construct special solutions for the equation $\Delta u + \kappa^2 n u = 0$ which depend in a suitable way on a parameter $\zeta \in \mathbb{C}^3$. This construction is the second step of the proof and it will take the largest amount of work in this section. Finally, in Theorem 2.10 by inserting these special solutions for u and $\tilde{u}$ we can show that the Fourier coefficients of q vanish, whence q is zero.

Let us begin with the relation (2.1) for the special case $u = u(\cdot,d)$.

Lemma 2.1 *Let* $0 < R < R_1$ *and let* $\tilde{u} \in C^2(B_{R_1})$ *be a solution to* $\Delta\tilde{u} + \kappa^2\tilde{n}\tilde{u} = 0$ *in* B_{R_1}*. If for a fixed* $d \in S^2$ *the far field patterns* $u_\infty(\cdot, d)$ *and* $\tilde{u}_\infty(\cdot, d)$ *for the refractive indices* n*,* $\tilde{n}$ *coincide on* S^2*, then we have*

$$\int\limits_{B_R} (n(x) - \tilde{n}(x))u(x,d)\tilde{u}(x)dx = 0 \ . \tag{2.2}$$

Proof: We consider the function $u(\cdot, d) - \tilde{u}(\cdot, d) = u^s(\cdot, d) - \tilde{u}^s(\cdot, d) \in C^2(\mathbb{R}^3)$ which is a radiating solution to the Helmholtz equation in the exterior of B_R with vanishing far field pattern due to $\tilde{u}_\infty(\cdot, d) = u_\infty(\cdot, d)$. Hence, for $R < R_2 < R_1$, we know from Rellich's lemma (Lemma 1.7) $u(\cdot, d) - \tilde{u}(\cdot, d) \in C_0^2(B_{R_2})$ and therefore $u(\cdot, d) = \tilde{u}(\cdot, d)$ and $(\partial u/\partial\nu)(\cdot, d) = (\partial\tilde{u}/\partial\nu)(\cdot, d)$ on ∂B_{R_2}. Using these identities we obtain

$$\begin{aligned} 0 &= \int\limits_{\partial B_{R_2}} \Big(\frac{\partial\tilde{u}}{\partial\nu}\tilde{u}(\cdot, d) - \frac{\partial\tilde{u}}{\partial\nu}(\cdot, d)\tilde{u}\Big)ds \\ &= \int\limits_{\partial B_{R_2}} \Big(\frac{\partial\tilde{u}}{\partial\nu}u(\cdot, d) - \frac{\partial u}{\partial\nu}(\cdot, d)\tilde{u}\Big)ds \\ &= \kappa^2 \int\limits_{B_{R_2}} (n - \tilde{n})u(\cdot, d)\tilde{u}\, dx \ , \end{aligned}$$

where in the first and in the last equation we have also employed Green's second theorem together with the partial differential equations for $\tilde{u}$, $\tilde{u}(\cdot, d)$ and $u(\cdot, d)$. Since $n - \tilde{n}$ vanishes in the exterior of B_R, we have proved the lemma.

□

Remark: We shall show in the appendix to this section that (2.2) is actually equivalent to the assumption $\tilde{u}_\infty(\cdot, d) = u_\infty(\cdot, d)$. In the sequel we work with equation (2.2).

Our next goal is to replace the function $u(\cdot, d)$ in (2.2) by an arbitrary solution $u \in C^2(B_{R_1})$ to $\Delta u + \kappa^2 nu = 0$ in B_{R_1}. This can be achieved by proving that u can be approximated by elements from $\text{span}\{u(\cdot, d) \colon d \in S^2\}$ with respect to the $L^2(B_R)$-norm. In the next lemma we prove this approximation result for the special case $n = 1$. From this we derive the general approximation result with the help of the Lippmann-Schwinger equation. We essentially follow the proof of [20, Lemma 5.20]. We give a different proof for

$n = 1$ with the help of series expansions in the appendix. A third proof can be found in [21, Lemma 3.2].

Before we prove the completeness of the plane waves in the space of solutions to the Helmholtz equation we want to give an informal outline of the main idea. Assume that $v_0 \in L^2(B_R)$ satisfies $\Delta v_0 + \kappa^2 v_0 = 0$ in B_R and that v_0 is orthogonal to all plane waves, i.e.,

$$\int\limits_{B_R} \overline{v_0(x)} e^{i\kappa d\cdot x} dx = 0 \ , \ \ d \in S^2 \ .$$

Defining

$$w(x) = \int\limits_{B_R} \overline{v_0(y)} \Phi_\kappa(x,y) dy \ , \ \ x \in \mathbb{R}^3 \ ,$$

we compute $\Delta w + \kappa^2 w = -\overline{v_0}$ and $4\pi w_\infty(-d) = \int\limits_{B_R} \overline{v_0(x)} e^{i\kappa d\cdot x} dx = 0$, $d \in S^2$, i.e., $w(x) = 0$ for $|x| > R$. Green's second theorem then implies

$$-\int\limits_{B_R} |v_0|^2 dx = \int\limits_{B_R} v_0(\Delta w + \kappa^2 w) dx = \int\limits_{B_R} w(\Delta v_0 + \kappa^2 v_0) dx = 0 \ .$$

Let us now give a rigorous proof of this idea.

Lemma 2.2 *Let $0 < R < R_2$ and let $u^i \in C^2(B_{R_2})$ satisfy $\Delta u^i + \kappa^2 u^i = 0$ in B_{R_2}. Then, there exists a sequence $u^i_j \in span\{u^i(\cdot, d) \colon d \in S^2\}$, $j \in \mathbb{N}$, such that $\|u^i - u^i_j\|_{L^2(B_R)} \to 0$, $j \to \infty$.*

Proof: We define the linear subspace

$$X := \{v|_{B_R} \colon v \in C^2(B_{R_2}) \text{ and } \Delta v + \kappa^2 v = 0 \text{ in } B_{R_2}\} \subset L^2(B_R)$$

and $\overline{X}$ to be the completion of X in $L^2(B_R)$. It suffices to prove that $\text{span}\{u^i(\cdot, d)|_{B_R} \colon d \in S^2\} \subset X$ is dense in $\overline{X}$. Since $\overline{X}$ is a Hilbert space, this is equivalent to the assertion that any $v_0 \in \overline{X}$ which is orthogonal to $\text{span}\{u^i(\cdot, d)|_{B_R} \colon d \in S^2\}$ must be zero.

Now let $v_0 \in \overline{X}$ be orthogonal to the plane waves and define

$$w(x) := \int\limits_{B_R} \overline{v_0(y)} \Phi_\kappa(x,y) dy \ , \ \ x \in \mathbb{R}^3 \setminus \overline{B_R} \ .$$

Then, $w \in C^2(\mathbb{R}^3 \setminus \overline{B_R})$ is a radiating solution to the Helmholtz equation in the exterior of $\overline{B_R}$ and we compute

$$w_\infty(-d) = \frac{1}{4\pi} \int\limits_{B_R} \overline{v_0(y)} e^{i\kappa d\cdot y} dy = 0 \ , \ d \in S^2 \ .$$

Rellich's lemma implies $w = 0$ and especially for $R < R_3 < R_2$: $w|_{\partial B_{R_3}} = (\partial w/\partial\nu)|_{\partial B_{R_3}} = 0$.

Next, we choose a sequence $v_k \in C^2(B_{R_2})$ with $\Delta v_k + \kappa^2 v_k = 0$ in B_{R_2} and $\|v_k - v_0\|_{L^2(B_R)} \to 0$, $k \to \infty$. We know from Green's formula (1.10)

$$v_k(x) = \int\limits_{\partial B_{R_3}} \Big(\frac{\partial v_k}{\partial\nu}(y)\Phi_\kappa(x,y) - v_k(y)\frac{\partial \Phi_\kappa}{\partial\nu(y)}(x,y)\Big) ds(y) \ , \ x \in B_R \ .$$

Inserting this expression for v_k and interchanging the orders of integration we derive

$$\begin{aligned}
\int\limits_{B_R} v_k(x)\overline{v_0(x)}dx &= \int\limits_{\partial B_{R_3}} \Big\{\frac{\partial v_k}{\partial\nu}(y) \int\limits_{B_R} \overline{v_0(x)}\Phi_\kappa(x,y)dx \\
&\qquad - v_k(y)\frac{\partial}{\partial\nu(y)} \int\limits_{B_R} \overline{v_0(x)}\Phi_\kappa(x,y)dx\Big\} ds(y) \\
&= \int\limits_{\partial B_{R_3}} \Big\{ w(y)\frac{\partial v_k}{\partial\nu}(y) - v_k(y)\frac{\partial w}{\partial\nu(y)}\Big\} ds(y) = 0 \ .
\end{aligned}$$

Now, taking $k \to \infty$ implies the desired result $v_0 = 0$.

□

The next lemma uses the Lippmann-Schwinger equation to extend the result of the preceding lemma to the general case with an arbitrary refractive index n.

Lemma 2.3 *Let* $0 < R < R_1$ *and let* $u \in C^2(B_{R_1})$ *satisfy* $\Delta u + \kappa^2 n u = 0$ *in* B_{R_1}. *Then, there exists a sequence* $u_j \in span\{u(\cdot,d) : d \in S^2\}$, $j \in \mathbb{N}$, *such that* $\|u - u_j\|_{L^2(B_R)} \to 0$, $j \to \infty$.

Proof: We choose $R < R_2 < R_1$ and define

$$u^i(x) := \int\limits_{\partial B_{R_2}} \Big(\frac{\partial u}{\partial\nu}(y)\Phi_\kappa(x,y) - u(y)\frac{\partial \Phi_\kappa}{\partial\nu(y)}(x,y)\Big) ds(y) \ , \ x \in B_{R_2} \ .$$

Green's formula (1.10) applied to u in the domain B_{R_2} yields together with the differential equation $\Delta u + \kappa^2 u = \kappa^2(1-n)u$ the integral equation

$$u(x) = u^i(x) - \kappa^2 \int\limits_{B_{R_2}} (1-n(y))u(y)\Phi_\kappa(x,y)dy \ , \ x \in B_R \ , \tag{2.3}$$

i.e., $u = (I-T)^{-1}u^i$ on B_R, where $T: C(\overline{B_R}) \to C(\overline{B_R})$ denotes the integral operator

$$(T\varphi)(x) := -\kappa^2 \int\limits_{B_R} (1-n(y))\varphi(y)\Phi_\kappa(x,y)dy \ , \ x \in B_R \ .$$

Theorem 1.9 (c) implies that T is a compact linear operator from $C(\overline{B_R})$ equipped with the $\|\cdot\|_{L^2}$-norm into itself, whence $(I-T)^{-1}$ is bounded in that space. Due to Lemma 2.2 there is a sequence u_j^i from $\mathrm{span}\{u^i(\cdot,d): d \in S^2\}$ with $\|u_j^i - u^i\|_{L^2(B_R)} \to 0$, $j \to \infty$. We define u_j to be the solution of the Lippmann-Schwinger equation (2.3) with u^i replaced by u_j^i. Then, we have $u_j \in \mathrm{span}\{u(\cdot,d): d \in S^2\}$, $j \in \mathbb{N}$, and

$$\|u-u_j\|_{L^2(B_R)} = \|(I-T)^{-1}(u^i - u_j^i)\|_{L^2(B_R)} \leq c\|u^i - u_j^i\|^2_{L^2(B_R)} \ ,$$

i.e., $\|u-u_j\|_{L^2(B_R)} \to 0$, $j \to \infty$.

□

If we approximate an arbitrary solution u of $\Delta u + \kappa^2 nu = 0$ in B_{R_1} by elements from $\mathrm{span}\{u(\cdot,d): d \in S^2\}$ in $L^2(B_R)$ and use Lemma 2.1, we obtain the desired relation (2.1) which we state in the next lemma.

Lemma 2.4 *Let $0 < R < R_1$. If the far field patterns for the refractive indices n, $\tilde{n}$ coincide on $S^2 \times S^2$, i.e., $u_\infty = \tilde{u}_\infty$, then for all solutions $\tilde{u} \in C^2(B_{R_1})$ to $\Delta\tilde{u} + \kappa^2\tilde{n}\tilde{u} = 0$ in B_{R_1} and for all solutions $u \in C^2(B_{R_1})$ to $\Delta u + \kappa^2 nu = 0$ in B_{R_1} the relation*

$$\int\limits_{B_R} (n(x) - \tilde{n}(x))u(x)\tilde{u}(x)dx = 0 \tag{2.4}$$

holds true.

Our next aim is to show that a function $q \in C(\overline{B_R})$, satisfying $\int\limits_{B_R} q u \tilde{u} dx = 0$ for all u, $\tilde{u}$ as in the above lemma, must vanish. This result together with relation (2.4) implies our desired uniqueness theorem for the inverse acoustic scattering problem.

In order to motivate the following analysis let us first give a proof for the case that u and $\tilde{u}$ are solutions to the Helmholtz equation, i.e., $n = \tilde{n} = 1$, which goes back to Calderón [5].

Lemma 2.5 *Let $q \in C(\overline{B_R})$ satisfy*

$$\int\limits_{B_R} q(x) u(x) \tilde{u}(x) dx = 0$$

for all solutions u, $\tilde{u} \in C^2(\mathbb{R}^3)$ to the Helmholtz equation $\Delta v + \kappa^2 v = 0$ in $\mathbb{R}^3$. Then $q = 0$.

Proof: We fix $R' > 0$ sufficiently large to ensure $\overline{B_R} \subset (-R', R')^3$. Then, for a fixed vector $\alpha \in \Gamma \subset \mathbb{R}^3$ we choose d_1, $d_2 \in \mathbb{R}^3$ such that α, d_1 and d_2 are orthogonal and such that $|d_1| = |d_2| = 1$. Finally, we define

$$\zeta := -\frac{1}{2}\alpha + \frac{i|\alpha|}{2} d_1 + \kappa d_2 \ , \ \tilde{\zeta} := -\frac{1}{2}\alpha - \frac{i|\alpha|}{2} d_1 - \kappa d_2 \ \in \mathbb{C}^3$$

and compute $\zeta + \tilde{\zeta} = -\alpha$, $\zeta \cdot \zeta = \tilde{\zeta} \cdot \tilde{\zeta} = \kappa^2$. Hence, $u(x) := e^{i\zeta \cdot x}$ and $\tilde{u}(x) := e^{i\tilde{\zeta} \cdot x}$, $x \in \mathbb{R}^3$, are solutions to the Helmholtz equation. Using the assumption of the lemma this implies

$$0 = \int\limits_{B_R} q(x) u(x) \tilde{u}(x) dx = \int\limits_{B_R} q(x) e^{-i\alpha \cdot x} dx \ .$$

Consequently, the Fourier coefficients of q must vanish, i.e., $q = 0$ by Parseval's relation (1.2).

□

We have some freedom in the choice of the vectors ζ and $\tilde{\zeta}$ in the above lemma, e.g. for any $t > \kappa$ the vectors

$$\begin{aligned} \zeta &:= -\frac{1}{2}\alpha + i\sqrt{t^2 - \kappa^2 + \frac{|\alpha|^2}{4}} d_1 + t d_2 \in \mathbb{C}^3 \ , \\ \tilde{\zeta} &:= -\frac{1}{2}\alpha - i\sqrt{t^2 - \kappa^2 + \frac{|\alpha|^2}{4}} d_1 - t d_2 \ \in \mathbb{C}^3 \end{aligned}$$

also satisfy $\zeta + \tilde{\zeta} = -\alpha$, $\zeta \cdot \zeta = \tilde{\zeta} \cdot \tilde{\zeta} = \kappa^2$ and can be used to define u and $\tilde{u}$.

For arbitrary n and $\tilde{n}$ we try to imitate the reasoning in Lemma 2.5. We look for solutions to $\Delta u + \kappa^2 n u = 0$ having the form $u(x) = e^{i\zeta \cdot x}(1 + v(x, \zeta))$, $x \in B_R$, with a suitable function $v(\cdot, \zeta)$ depending on the parameter $\zeta \in \mathbb{C}^3$, $\zeta \cdot \zeta = \kappa^2$. We shall use the freedom in the choice of ζ and show that we can find functions $v(\cdot, \zeta)$ such that $\|v(\cdot, \zeta)\|_{L^2(B_R)} \to 0$ as $|\Im(\zeta)| \to \infty$. Inserting these solutions and analogous solutions $\tilde{u}$ into (2.4) and using the limit $t \to \infty$ implies that the Fourier coefficients of $n - \tilde{n}$ must vanish at the point $\alpha \in \Gamma$. Since α is arbitrary, we can conclude $n = \tilde{n}$.

Note that in two dimensions the freedom in the choice of the parameters ζ and $\tilde{\zeta}$ no longer exists. Therefore this proof fails in the two-dimensional case. To the authors knowledge the question whether u_∞ uniquely determines n in two dimensions is still open. It is possible to show uniqueness if u_∞ is known for many different frequencies (see [18]) or if some more assumptions on n are made (see [34]).

Inserting $u(x) = e^{i\zeta \cdot x}(1 + v(x, \zeta))$, $x \in B_R$, into the equation $\Delta u + \kappa^2 n u = 0$ and using $\zeta \cdot \zeta = \kappa^2$ we obtain the differential equation

$$\Delta v(\cdot, \zeta) + 2i\zeta \cdot \nabla v(\cdot, \zeta) = \kappa^2(1 - n)v(\cdot, \zeta) + \kappa^2(1 - n)$$

for the function $v(\cdot, \zeta)$. For special vectors $\xi = (s, it, 0)$ we had formally derived solution operators G'_ξ for the equation $\Delta w + 2i\xi \cdot \nabla w = f$ in Theorem 1.1, namely

$$G'_\xi : L^2(C) \to L^2(C) \qquad G'_\xi f := -\sum_{\alpha \in \Gamma} \frac{\hat{f}(\alpha)}{(\alpha \cdot \alpha + 2\xi \cdot \alpha)} e_\alpha \ .$$

The L^2-norms of those operators converge to zero as $t \to \infty$ which seems to match the desired behavior of $v(\cdot, \zeta)$. In Theorem 1.1 we did not worry whether these operators really yield solutions to the differential equation. Since we want to employ them for the construction of $v(\cdot, \zeta)$, we have to deal with this problem now. The second difficulty is that we have to allow more general vectors ζ than those of the special form $(s, it, 0)$. However, it is easy to reduce the general case to the special one with the help of unitary transformations.

Inserting $\hat{f}(\alpha) = \int_C f(y)\overline{e_\alpha(y)}dy$ into the definition of G'_ξ we formally obtain

$$G'_\xi f(x) = -\int_C g_\xi(x - y) f(y) dy \ ,$$

where the function g_ξ is defined by

$$g_\xi := \sum_{\alpha\in\Gamma} \frac{1}{(\alpha\cdot\alpha + 2\xi\cdot\alpha)} e_\alpha \ . \tag{2.5}$$

Our next aim is to show that g_ξ satisfies the assumptions from Theorems 1.9 and 1.10. Then, the results for volume potentials apply to G'_ξ and we can work with G'_ξ the way we did during the proof of Lemma 1.12 with the operator V_κ. The basic tool in proving regularity of g_ξ is the following lemma which is known as Weyl's lemma (see [13, IV 4.2]).

Lemma 2.6 *Let $D \subset \mathbb{R}^3$ be an open set and $q \in C^{0,\gamma}(\overline{D})$, $0 < \gamma < 1$, be a uniformly Hölder continuous function in $\overline{D}$. Furthermore, assume $u \in L^2(D)$ satisfies*

$$\int_D (\Delta\varphi + q\varphi) u dx = 0$$

for all functions $\varphi \in C_0^\infty(D)$. Then the following assertions hold true:

(a) $u \in C^2(D)$ and $\Delta u + qu = 0$ in D.

(b) For any open subset $D' \subset D$ such that $\overline{D'} \subset D$ is compact there exists a constant M depending on D, D', and $\|q\|_{\infty,\overline{D}}$, but not on u, such that $\|u\|_{1,\gamma,\overline{D'}} \leq M\|u\|_{L^2(D)}$. Similarly, there is a constant M', depending on D, D', and $\|q\|_{0,\gamma,\overline{D}}$ such that $\max\limits_{j,l} \|\partial_j\partial_l u\|_{\infty,\overline{D'}} \leq M'\|u\|_{L^2(D)}$.

Proof: The first step of the proof consists in the construction of appropriate test functions φ. Let $B_\epsilon(x^*) \subset D$, $\epsilon > 0$, be a ball and let $\chi \in C_0^\infty(\mathbb{R})$ be a cut-off function satisfying $\chi(t) = 0$, if $|t| \geq \epsilon/2$, and $\chi(t) = 1$, if $|t| \leq \epsilon/4$. For a function $\psi \in C_0^\infty(B_{\epsilon/4}(x^*))$ we define

$$\varphi(x) := \int_{\mathbb{R}^3} \chi(|x-y|)\Phi_0(x,y)\psi(y)dy \ , \ x \in \mathbb{R}^3 \ .$$

Using the transformation $z = x - y$ we arrive at

$$\varphi(x) = \int_{\mathbb{R}^3} \chi(|z|)\frac{1}{4\pi|z|}\psi(x-z)dz = \int_{B_{\epsilon/2}(0)} \chi(|z|)\frac{1}{4\pi|z|}\psi(x-z)dz \ , \ x \in \mathbb{R}^3 \ ,$$

whence $\varphi \in C^\infty(\mathbb{R}^3)$. Moreover, $\varphi(x) = 0$ for $|x - x^*| > (3/4)\epsilon$ because then $\mathrm{supp}(\psi(x-\cdot))$ and $\mathrm{supp}(\chi(|\cdot|)) = B_{\epsilon/2}(0)$ are disjoint, i.e., $\varphi \in C_0^\infty(B_\epsilon(x^*))$.

Defining $\tilde{k}(x) := (\chi(|x|)-1)/(4\pi|x|)$, $x \in \mathbb{R}^3$, we know $\tilde{k} \in C^\infty(\mathbb{R}^3)$ and we can conclude from Theorem 1.10

$$\begin{aligned}(\Delta\varphi + q\varphi)(x) &= -\psi(x) + \int\limits_{B_{\epsilon/4}(x^*)} (\Delta\tilde{k})(x-y)\psi(y)dy \\ &\quad + q(x) \int\limits_{B_{\epsilon/4}(x^*)} \chi(|x-y|)\Phi_0(x,y)\psi(y)dy \ , \ x \in B_\epsilon(x^*) \ .\end{aligned}$$

Reversing the order of integration we thus have

$$\begin{aligned}0 &= \int\limits_{B_\epsilon(x^*)} (\Delta\varphi + q\varphi)u dx \\ &= \int\limits_{B_{\epsilon/4}(x^*)} \psi(y)\Big\{-u(y) + \int\limits_{B_\epsilon(x^*)} (\Delta\tilde{k})(x-y)u(x)dx \\ &\qquad + \int\limits_{B_\epsilon(x^*)} q(x)\chi(|x-y|)\Phi_0(x,y)u(x)dx\Big\}dy\end{aligned}$$

for any $\psi \in C_0^\infty(B_{\epsilon/4}(x^*))$, whence for almost every $y \in B_{\epsilon/4}(x^*)$

$$u(y) = \int\limits_{B_\epsilon(x^*)} (\Delta\tilde{k})(x-y)u(x)dx + \int\limits_{B_\epsilon(x^*)} q(x)\chi(|x-y|)\Phi_0(x,y)u(x)dx \tag{2.6}$$

holds true.

The Cauchy-Schwarz inequality yields for any function $v \in C_0^\infty(D)$

$$\begin{aligned}&\sup_{y\in B_{\epsilon/4}(x^*)} \Big| \int\limits_{B_\epsilon(x^*)} q(x)\chi(|x-y|)\Phi_0(x,y)v(x)dx\Big|^2 \\ &\leq \sup_{y\in B_{\epsilon/4}(x^*)} \int\limits_{B_\epsilon(x^*)} \frac{|q(x)\chi(|x-y|)|^2}{(4\pi|x-y|)^2} dx \, \|v\|^2_{L^2(D)} \\ &\leq M_1^2 \|v\|^2_{L^2(D)} \ .\end{aligned}$$

Approximating $u \in L^2(D)$ by elements from $C_0^\infty(D)$ we can conclude that the second integral on the right hand side of (2.6) is a continuous function on $\overline{B_{\epsilon/4}(x^*)}$ whose maximum norm is bounded by $M_1\|u\|_{L^2(D)}$. Together with the fact that $\Delta\tilde{k}$ is a smooth bounded function in $\mathbb{R}^3$ we can infer from (2.6) that u is continuous in $\overline{B_{\epsilon/4}(x^*)}$ and that $\|u\|_{\infty,B_{\epsilon/4}(x^*)} \leq M_2\|u\|_{L^2(D)}$ where the constant M_2 depends on ϵ (via χ) and on $\|q\|_{\infty,D}$.

Now we replace ϵ by $\epsilon/4$ and repeat the procedure which lead to equation (2.6) with an adjusted cut-off function χ. We arrive at

$$u(y) = \int\limits_{B_{\epsilon/4}(x^*)} (\Delta\tilde{k})(x-y)u(x)dx + \int\limits_{B_{\epsilon/4}(x^*)} q(x)\chi(|x-y|)\Phi_0(x,y)u(x)dx$$

for $y \in B_{\epsilon/16}(x^*)$. Since we already know $u \in C(\overline{B_{\epsilon/4}(x^*)})$, we obtain from Theorem 1.9 that $u \in C^{1,\gamma}(\overline{B_{\epsilon/16}(x^*)})$ and $\|u\|_{1,\gamma,\overline{B_{\epsilon/16}(x^*)}} \leq M_3\|u\|_{L^2(D)}$.

Repeating the procedure one last time in the ball $B_{\epsilon/16}(x^*)$ we finally conclude from Theorem 1.10 that $u \in C^2(B_{\epsilon/64}(x^*))$. Since $x^* \in D$ can be chosen arbitrarily, we have proved $u \in C^2(D)$. Integration by parts immediately yields for all $\varphi \in C_0^\infty(D)$

$$0 = \int\limits_D (\Delta\varphi + q\varphi)u dx = \int\limits_D (\Delta u + qu)\varphi dx \ ,$$

and thus $\Delta u + qu = 0$ in D.

For part (b) we cover the compact set $\overline{D'}$ by finitely many balls of the form $B_{\epsilon_j/64}(x_j)$ where ϵ_j is chosen sufficiently small to ensure $B_{\epsilon_j}(x_j) \subset D$. Patching together the above norm estimates for $\|u\|_{1,\gamma,\overline{B_{\epsilon_j/16}(x_j)}}$ implies the first inequality of assertion (b). In order to bound the second derivatives of u we use the formula from Theorem 1.10 (a) and relation (2.6) to compute for $y \in B_{\epsilon/64}(x^*)$

$$\begin{aligned}(\partial_j\partial_l u)(y) &= \int\limits_{B_{\epsilon/16}(x^*)} (\Delta\partial_j\partial_l\tilde{k})(x-y)u(x)dx \\ &+ \int\limits_{B_{\epsilon/16}(x^*)} \frac{\partial^2}{\partial y_j\partial y_l}\Big(\chi(|x-y|)\Phi_0(x,y)\Big)(q(x)u(x)-q(y)u(y))dx \\ &- q(y)u(y)\int\limits_{\partial B_{\epsilon/16}(x^*)} \nu_j(x)\frac{\partial}{\partial y_l}\Big(\chi(|x-y|)\Phi_0(x,y)\Big)ds(x) \ .\end{aligned}$$

Consequently, we can bound

$$\|\partial_j\partial_l u\|_{\infty,B_{\epsilon/64}(x^*)} \leq M_4\|u\|_{0,\gamma,\overline{B_{\epsilon/16}(x^*)}} \leq M_5\|u\|_{L^2(D)} \ ,$$

which implies the second estimate of the assertion (b).

□

We are now in a position to prove that G'_ξ can be regarded as a volume potential. We define the cube $C := (-R', R')^3$, the grid Γ and the orthonormal basis e_α, $\alpha \in \Gamma$, as in the first chapter and we assume $R' > R > 0$.

Lemma 2.7 *Assume $\xi = (\sqrt{t^2+\kappa^2}, it, 0) \in \mathbb{C}^3$ with $\kappa \geq 0$, $t > 0$. Then, there exist functions $g_\xi \in C^\infty(B_{2R'} \setminus \{0\})$, $\tilde{g}_\xi \in C^\infty(B_{2R'})$ with the following properties:*

(a) $(G'_\xi f)(x) = -\int\limits_{B_R} g_\xi(x-y) f(y) dy$ *for almost every $x \in B_R$ and for all functions $f \in C(\overline{B_R})$*

(f is regarded as an element of $L^2(C)$ by defining it to be zero outside of $\overline{B_R}$).

(b) $g_\xi(x) = \dfrac{e^{-i\xi\cdot x} e^{i\kappa|x|}}{4\pi|x|} + e^{-i\xi\cdot x}\tilde{g}_\xi(x) \ , \ x \in B_{2R'} \setminus \{0\}$.

(c) $\Delta\tilde{g}_\xi + \kappa^2 \tilde{g}_\xi = 0$ *in* $B_{2R'}$.

Proof: We define

$$g_\xi := \sum_{\alpha\in\Gamma} \frac{1}{\alpha\cdot\alpha + 2\xi\cdot\alpha} e_\alpha \ .$$

Since there is a constant M such that $|(\alpha\cdot\alpha + 2\xi\cdot\alpha)^{-1}| \leq M(1+\alpha\cdot\alpha)^{-1}$ for all $\alpha \in \Gamma$, we can estimate

$$\sum_{\alpha\in\Gamma} \left| \frac{1}{\alpha\cdot\alpha + 2\xi\cdot\alpha} \right|^2 < \infty \ ,$$

and obtain $g_\xi \in L^2(C)$. g_ξ satisfies $g_\xi(x_1, x_2 + 2R', x_3) = -g_\xi(x_1, x_2, x_3)$, $x \in \mathbb{R}^3$, and g_ξ has period $2R'$ with respect to the other coordinates, whence it is square-integrable on each compact subset of $\mathbb{R}^3$.

We prove assertion (a) by showing the coincidence of the Fourier coefficients of the two functions. For $y \in C$ the function $g_\xi(\cdot - y)$ has the Fourier coefficients

$$\begin{aligned}(g_\xi(\cdot - y))\hat{}(\alpha) &= \int\limits_C g_\xi(x-y)\overline{e_\alpha(x)}dx \\ &= \int\limits_C g_\xi(z)\overline{e_\alpha(z)}dz\,\overline{e_\alpha(y)} = \frac{\overline{e_\alpha(y)}}{\alpha\cdot\alpha + 2\xi\cdot\alpha}\ .\end{aligned}$$

If $f \in C(\overline{B_R})$ is extended by zero, the function $-\int\limits_{B_R} g_\xi(\cdot - y)f(y)dy \in L^2(C)$ has the Fourier coefficients

$$\begin{aligned}-\int\limits_C \Big\{ \int\limits_{B_R} g_\xi(x-y)f(y)dy\Big\}\overline{e_\alpha(x)}dx &= -\int\limits_C\Big\{\int\limits_C g_\xi(x-y)\overline{e_\alpha(x)}dx\Big\}f(y)dy \\ &= \frac{-1}{\alpha\cdot\alpha+2\xi\cdot\alpha}\int\limits_C f(y)\overline{e_\alpha(y)}dy \\ &= \frac{-\hat{f}(\alpha)}{\alpha\cdot\alpha+2\xi\cdot\alpha} \\ &= (G'_\xi f)\hat{}(\alpha)\ ,\ \ \alpha\in\Gamma\ .\end{aligned}$$

Hence, $-\int\limits_{B_R} g_\xi(\cdot - y)f(y)dy$ and $G'_\xi f$ coincide almost everywhere in $B_R \subset C$.

Before we proceed to the remaining assertions let us add one remark: if $f \in C_0^\infty(C)$, then the Fourier coefficients of f are rapidly decaying. Hence

$$G'_\xi f = -\sum_{\alpha\in\Gamma}\frac{\hat{f}(\alpha)}{(\alpha\cdot\alpha+2\xi\cdot\alpha)}e_\alpha$$

not only converges with respect to the $L^2(C)$-norm but is also absolutely and uniformly convergent. Therefore, $G'_\xi f$ is a continuous function on C and it makes sense to evaluate $(G'_\xi f)(x)$ at a point $x \in C$. Similarly, for $f \in C_0^\infty(B_{R'})$ we have

$$-\int\limits_{B_{R'}} g_\xi(\cdot - y)f(y)dy = -\int\limits_{\mathbb{R}^3} g_\xi(z)f(\cdot - z)dz \in C(\mathbb{R}^3)\ .$$

Hence, for $f \in C_0^\infty(B_{R'})$ assertion (a) holds true for all $x \in B_{R'}$.

In order to prove the regularity of g_ξ and $\tilde{g}_\xi$ and assertions (b), (c) we show that $\tilde{g}_\xi(x) := e^{i\xi\cdot x}g_\xi(x) - e^{i\kappa|x|}/(4\pi|x|)$, $x \in B_{2R'}$, satisfies

$$\int\limits_{B_{R'}(x^*)} (\Delta\varphi + \kappa^2\varphi)(x)\tilde{g}_\xi(x)dx = 0$$

for all $x^* \in B_{R'}$ and for all $\varphi \in C_0^\infty(B_{R'}(x^*))$. We can then conclude from the preceding lemma that $\tilde{g}_\xi$ is a $C^2(B_{2R'})$-smooth classical solution of the Helmholtz equation, whence it is a C^∞-smooth function. This implies the remaining assertions.

If $x^* \in B_{R'}$ and $\varphi \in C_0^\infty(B_{R'}(x^*)) \subset C_0^\infty(B_{2R'})$, the representation formula (1.10) from Theorem 1.3 yields

$$- \int_{B_{2R'}} (\Delta\varphi + \kappa^2\varphi)(x) \frac{e^{i\kappa|x|}}{4\pi|x|} dx = \varphi(0) \ . \qquad (2.7)$$

Defining $\psi(y) := e^{i\xi\cdot(x^*-y)}\varphi(x^*-y)$, $y \in B_{R'}$, we have $\psi \in C_0^\infty(B_{R'})$ and

$$((\Delta + 2i\xi\cdot\nabla)\psi)(y) = e^{i\xi\cdot(x^*-y)}(\Delta\varphi + \kappa^2\varphi)(x^*-y) \ , \ y \in B_{R'} \ .$$

Using this equation, the substitution $x = x^* - y$, part (a), Theorem 1.1 (b), and the remark after part (a) we compute

$$\begin{aligned}
&\int_{B_{2R'}} (\Delta\varphi + \kappa^2\varphi)(x) e^{i\xi\cdot x} g_\xi(x) dx \\
&\quad = \int_{B_{R'}} e^{i\xi\cdot(x^*-y)}(\Delta\varphi + \kappa^2\varphi)(x^*-y) g_\xi(x^*-y) dy \\
&\quad = \int_{B_{R'}} g_\xi(x^*-y)(\Delta\psi + 2i\xi\cdot\nabla\psi)(y) dy \\
&\quad = -(G'_\xi(\Delta\psi + 2i\xi\cdot\nabla\psi))(x^*) = -\psi(x^*) = -\varphi(0) \ . \qquad (2.8)
\end{aligned}$$

Adding the equations (2.7) and (2.8) yields the equation

$$\int_{B_{2R'}} (\Delta\varphi + \kappa^2\varphi)(x)\Big(e^{i\xi\cdot x} g_\xi(x) - \frac{e^{i\kappa|x|}}{4\pi|x|}\Big) dx = 0 \ .$$

Hence $\tilde{g}_\xi$ is a C^2-smooth solution of the Helmholtz equation in $B_{R'}(x^*)$ and then in $B_{2R'}$ because $x^* \in B_{R'}$ can be chosen arbitrarily. This ends the proof of the lemma.

□

We are now in a position to define g_ζ for arbitrary vectors $\zeta \in \mathbb{C}^3$, satisfying $\zeta \cdot \zeta = \kappa^2$ and $\Im(\zeta) \neq 0$.

Assume $\kappa \geq 0$ and $\zeta \in \mathbb{C}^3$ satisfies $\zeta \cdot \zeta = \kappa^2$, $\Im(\zeta) \neq 0$. This implies $\Re(\zeta) \cdot \Im(\zeta) = 0$ and $\Re(\zeta) \cdot \Re(\zeta) - \Im(\zeta) \cdot \Im(\zeta) = \kappa^2$, whence $\Re(\zeta) \neq 0$. We define $\xi := (|\Re(\zeta)|, i|\Im(\zeta)|, 0) \in \mathbb{C}^3$ and choose the uniquely determined unitary transformation Q of $\mathbb{R}^3$ satisfying $Q(\Re(\zeta)) = (|\Re(\zeta)|, 0, 0)$, $Q(\Im(\zeta)) = (0, |\Im(\zeta)|, 0)$ and $\det(Q) = 1$. Then, we have $Q^T(\xi) = \zeta$, where Q^T denotes the transpose of Q. Now, we define the function g_ζ by

$$g_\zeta \colon B_{2R'} \setminus \{0\} \to \mathbb{C} \qquad g_\zeta(x) := g_\xi(Qx) \ , \tag{2.9}$$

where

$$g_\xi = \sum_{\alpha \in \Gamma} \frac{1}{(\alpha \cdot \alpha + 2\xi \cdot \alpha)} e_\alpha$$

is the function which we examined in the last lemma.

In the next theorem we study some properties of volume potentials $G_\zeta f$ with the kernel g_ζ. Note that these volume potentials are defined for functions on a ball, whereas G'_ξ is defined for functions on a cube. Moreover, compared to G'_ξ we have chosen the opposite sign for G_ζ because then the function $\Psi_\zeta(x) = e^{i\zeta \cdot x} g_\zeta(x)$ inherits all the properties of the fundamental solution $e^{i\kappa|x|}/(4\pi|x|)$ later. Nevertheless, it should be clear by the preceding lemma that G_ζ and G'_ξ are closely related.

Theorem 2.8 *Assume $0 < R'' < R'$, $\kappa \geq 0$ and $\zeta \in \mathbb{C}^3$, $\zeta \cdot \zeta = \kappa^2$. Define for $f \in C(\overline{B_{R''}})$ the function $G_\zeta f$ by*

$$(G_\zeta f)(x) := \int_{B_{R''}} g_\zeta(x - y) f(y) dy \ , \ x \in \overline{B_{R'}} \ , \tag{2.10}$$

with g_ζ from (2.9). Then, the following assertions hold true:

(a) $G_\zeta f \in C^{1,\gamma}(\overline{B_{R''}})$ for all $0 < \gamma < 1$, and $\|G_\zeta f\|_{1,\gamma} \leq C_{\gamma,\zeta} \|f\|_\infty$.

(b) If $f \in C^{0,\gamma}(\overline{B_{R''}})$, then $G_\zeta f \in C^2(B_{R''})$ and $(\Delta + 2i\zeta \cdot \nabla)(G_\zeta f) = -f$.

(c) If $f \in C^1_0(B_{R''})$, then $\partial_j(G_\zeta f) = G_\zeta(\partial_j f)$.

(d) $\|G_\zeta f\|_{L^2(B_{R''})} \leq \dfrac{R'}{\pi|\Im(\zeta)|} \|f\|_{L^2(B_{R''})}$ for all $f \in C(\overline{B_{R''}})$.

Proof: By the definition of g_ζ and by Lemma 2.7 we conclude

$$\begin{aligned} g_\zeta(x) &= \frac{e^{-i\xi\cdot Qx}e^{i\kappa|Qx|}}{4\pi|Qx|} + e^{-i\xi\cdot Qx}\tilde{g}_\xi(Qx) \\ &= \frac{e^{-i\zeta\cdot x}e^{i\kappa|x|}}{4\pi|x|} + e^{-i\zeta\cdot x}\tilde{g}_\xi(Qx)\ ,\ x \in B_{2R'}\setminus\{0\}\ . \end{aligned}$$

Since $\tilde{g}_\xi$ is smooth and since the function $x \mapsto e^{-i\zeta\cdot x}e^{i\kappa|x|}/(4\pi|x|)$ satisfies the assumptions from Theorems 1.9 and 1.10, we may apply these theorems. Theorem 1.9 immediately yields assertions (a) and (c). Theorem 1.10 states that $G_\zeta f \in C^2(B_{R''})$ if $f \in C^{0,\gamma}(\overline{B_{R''}})$. Moreover, with the help of

$$[\Delta + 2i\zeta\cdot\nabla]u(x) = e^{-i\zeta\cdot x}(\Delta+\kappa^2)(e^{i\zeta\cdot x}u(x))$$

we compute

$$\begin{aligned} &([\Delta + 2i\zeta\cdot\nabla](G_\zeta f))(x) \\ &= e^{-i\zeta\cdot x}(\Delta+\kappa^2)\int\limits_{B_{R''}}\Big\{\Phi_\kappa(x,y) + \tilde{g}_\xi(Q(x-y))\Big\}(e^{i\zeta\cdot y}f(y))dy \\ &= e^{-i\zeta\cdot x}\Big\{-e^{i\zeta\cdot x}f(x) + (\Delta+\kappa^2)\int\limits_{B_{R''}}\tilde{g}_\xi(Q(x-y))(e^{i\zeta\cdot y}f(y))dy\Big\} \\ &= -f(x)\ ,\ x \in B_{R''}\ , \end{aligned}$$

where in the last step we have used that $\tilde{g}_\xi(Q\,\cdot)$ is a smooth solution of the Helmholtz equation in $B_{2R''}$. This proves assertion (b).

For part (d) we extend a function $f \in C(\overline{B_{R''}})$ by zero outside of $\overline{B_{R''}}$ and observe the relation

$$(G_\zeta f)(Q^T x) = \int\limits_{B_{R''}} g_\xi(x-y)f(Q^T y)dy\ ,\ x \in B_{R''}\ .$$

Then we employ the last lemma and the L^2-norm estimate from Theorem 1.1 to obtain

$$\begin{aligned} \|G_\zeta f\|^2_{L^2(B_{R''})} &= \int\limits_{B_{R''}} |(G_\zeta f)(x)|^2 dx \\ &= \int\limits_{B_{R''}} |(G_\zeta f)(Q^T x)|^2 dx \end{aligned}$$

$$
\begin{aligned}
&= \int\limits_{B_{R''}} |(G'_\xi(f\circ Q^T))(x)|^2 dx \\
&\leq \|G'_\xi(f\circ Q^T)\|^2_{L^2(C)} \\
&\leq \frac{R'^2}{\pi^2|\Im(\zeta)|^2}\|(f\circ Q^T)\|^2_{L^2(C)} \\
&= \frac{R'^2}{\pi^2|\Im(\zeta)|^2}\|f\|^2_{L^2(B_{R''})} \ .
\end{aligned}
$$

This ends the proof of the theorem.

□

The last theorem enables us to derive the existence of special solutions to the equation $\Delta u + \kappa^2 nu = 0$ which can be used similarly to $e^{i\zeta\cdot x}$ in Lemma 2.5.

Lemma 2.9 *Assume $\kappa \geq 0$, $0 < R'' < R'$ and $n \in C^{0,\gamma}(\overline{B_{R''}})$. Then there is a constant $c > 0$, depending only on R'', R', κ and $\|1-n\|_\infty$, with the following property:*

for all $\zeta \in \mathbb{C}^3$ satisfying $\zeta\cdot\zeta = \kappa^2$ and $|\Im(\zeta)| \geq 2\kappa^2(R'/\pi)\|1-n\|_\infty + 1$ there exists a function $v(\cdot,\zeta) \in C^2(B_{R''})$ such that

$$
u(x,\zeta) := e^{i\zeta\cdot x}(1 + v(x,\zeta)) \ , \ x \in B_{R''} \ , \tag{2.11}
$$

is a solution to $\Delta u + \kappa^2 nu = 0$ in $B_{R''}$, and such that the estimate

$$
\|v(\cdot,\zeta)\|_{L^2(B_{R''})} \leq \frac{c}{|\Im(\zeta)|} \tag{2.12}
$$

holds true.

Proof: Inserting (2.11) into $\Delta u + \kappa^2 nu = 0$ yields the differential equation

$$
(\Delta + 2i\zeta\cdot\nabla)v(\cdot,\zeta) = \kappa^2(1-n)v(\cdot,\zeta) + \kappa^2(1-n) \tag{2.13}
$$

for $v(\cdot,\zeta)$. Theorem 2.8 (b) suggests to look for a solution of the integral equation

$$
v(\cdot,\zeta) = -\kappa^2 G_\zeta((1-n)v(\cdot,\zeta)) - \kappa^2 G_\zeta(1-n) \tag{2.14}
$$

and then to proceed similarly to the existence proof of (DAP).

The mapping $T\colon C(\overline{B_{R''}}) \to C(\overline{B_{R''}})$ defined by

$$
T\varphi = -\kappa^2 G_\zeta((1-n)\varphi)
$$

is compact due to Theorem 2.8 (a). Since $\varphi - T\varphi = 0$, $\varphi \in C(\overline{B_{R''}})$, implies together with Theorem 2.8 (d)

$$\begin{aligned} \|\varphi\|_{L^2(B_{R''})} &= \|T\varphi\|_{L^2(B_{R''})} \\ &\leq \frac{\kappa^2 R'}{\pi|\Im(\zeta)|}\|1-n\|_\infty \|\varphi\|_{L^2(B_{R''})} \\ &\leq \frac{1}{2}\|\varphi\|_{L^2(B_{R''})} , \end{aligned}$$

whence $\varphi = 0$, we know from the Riesz theory that (2.14) has a unique solution $v(\cdot, \zeta) \in C(\overline{B_{R''}})$. Moreover, we can infer from the properties of G_ζ stated in Theorem 2.8 that $v(\cdot, \zeta) \in C^2(B_{R''})$ satisfies equation (2.13). Then, straightforward calculations show that $u(\cdot, \zeta)$ defined by (2.11) is a solution to $\Delta u + \kappa^2 n u = 0$ in $B_{R''}$.

In order to obtain the norm estimate we observe that

$$\begin{aligned} \|v(\cdot,\zeta)\|_{L^2(B_{R''})} &\leq \|Tv(\cdot,\zeta)\|_{L^2(B_{R''})} + \kappa^2 \|G_\zeta(1-n)\|_{L^2(B_{R''})} \\ &\leq \frac{1}{2}\|v(\cdot,\zeta)\|_{L^2(B_{R''})} + \frac{\kappa^2 R'}{\pi|\Im(\zeta)|}\|1-n\|_\infty \|1\|_{L^2(B_{R''})} . \end{aligned}$$

Hence, we arrive at

$$\|v(\cdot,\zeta)\|_{L^2(B_{R''})} \leq 2\frac{\kappa^2 R'}{\pi|\Im(\zeta)|}\|1-n\|_\infty \|1\|_{L^2(B_{R''})}$$

and we have proved the lemma.

□

Remark: Let us point out a different view of the integral equation (2.14). Multiplying both sides of (2.14) with $e^{i\zeta\cdot x}$ and then adding $e^{i\zeta\cdot x}$ on both sides yields

$$\begin{aligned} u(x,\zeta) &= e^{i\zeta\cdot x}(1 + v(x,\zeta)) \\ &= e^{i\zeta\cdot x} - \kappa^2 e^{i\zeta\cdot x}\int_{B_{R''}} g_\zeta(x-y)(1-n)(y)(1+v(y,\zeta))dy \\ &= e^{i\zeta\cdot x} - \kappa^2 \int_{B_{R''}} e^{i\zeta\cdot(x-y)} g_\zeta(x-y)(1-n)(y)u(y,\zeta)dy , \ x \in \overline{B_{R''}} . \end{aligned} \tag{2.15}$$

This is the analogue of the Lippmann-Schwinger equation with an incident wave $e^{i\zeta\cdot x}$ where the fundamental solution $\Phi_\kappa(x,y)$ is replaced by the function $e^{i\zeta\cdot(x-y)}g_\zeta(x-y)$. But due to the representation

$$\begin{aligned} g_\zeta(x) &= \frac{e^{-i\xi\cdot Qx}e^{i\kappa|Qx|}}{4\pi|Qx|} + e^{-i\xi\cdot Qx}\tilde{g}_\xi(Qx) \\ &= \frac{e^{-i\zeta\cdot x}e^{i\kappa|x|}}{4\pi|x|} + e^{-i\zeta\cdot x}\tilde{g}_\xi(Qx)\ ,\ x \in B_{2R'} \setminus \{0\}\ , \end{aligned}$$

we know that

$$e^{i\zeta\cdot(x-y)}g_\zeta(x-y) = \Phi_\kappa(x,y) + \tilde{g}_\zeta(x-y)\ ,\ x,y \in B_{R'}\ ,\ x \neq y\ ,$$

is also a fundamental solution to the Helmholtz equation in $B_{R'}$ because $\tilde{g}_\zeta := \tilde{g}_\xi \circ Q$ is a solution to the Helmholtz equation in $B_{2R'}$. Hence, we have solved a Lippmann-Schwinger equation with an unphysical incident wave and an unphysical fundamental solution in order to obtain the special solutions $u(\cdot,\zeta)$.

We also want to emphasize that the integral equation (2.15) has a unique solution if $|\Im(\zeta)| \geq 2\kappa^2(R'/\pi)\|1-n\|_\infty + 1$. This can be seen by defining $v(x) := e^{-i\zeta\cdot x}u(x)$, $x \in \overline{B_{R''}}$, for a solution u of the homogeneous equation (2.15) and by multiplying this equation by $e^{-i\zeta\cdot x}$. Then, we obtain the homogeneous equation (2.14) for v, whence $v = 0$ and $u = 0$ by the proof of Lemma 2.9.

Now, we can conclude this section with the uniqueness result for the inverse scattering problem.

Theorem 2.10 *Let $\kappa > 0$ and assume the refractive indices n and $\tilde{n}$ satisfy the assumptions made at the beginning of this section. If the far field patterns coincide for all incident plane waves, i.e., $u_\infty(\hat{x},d) = \tilde{u}_\infty(\hat{x},d)$ for all $\hat{x}$, $d \in S^2$, then $n = \tilde{n}$.*

Proof: We choose R_1 such that $R < R_1 < R'$. We know from Lemma 2.4 that for all solutions $\tilde{u} \in C^2(B_{R_1})$ to $\Delta\tilde{u} + \kappa^2\tilde{n}\tilde{u} = 0$ in B_{R_1} and for all solutions $u \in C^2(B_{R_1})$ to $\Delta u + \kappa^2 nu = 0$ in B_{R_1} the relation

$$\int_{B_R} (n(x) - \tilde{n}(x))u(x)\tilde{u}(x)dx = 0 \tag{2.16}$$

holds true. Imitating the reasoning in Lemma 2.5 we choose for a fixed vector $\alpha \in \Gamma$ the unit vectors d_1, $d_2 \in \mathbb{R}^3$ such that $d_1 \cdot d_2 = d_1 \cdot \alpha = d_2 \cdot \alpha = 0$ and define

$$\begin{aligned}\zeta_t &:= -\frac{1}{2}\alpha + i\sqrt{t^2 - \kappa^2 + \frac{|\alpha|^2}{4}}d_1 + td_2 \in \mathbb{C}^3 \ ,\\ \tilde{\zeta}_t &:= -\frac{1}{2}\alpha - i\sqrt{t^2 - \kappa^2 + \frac{|\alpha|^2}{4}}d_1 - td_2 \ \in \mathbb{C}^3\end{aligned}$$

for $t > \kappa$. Then, the relations $\zeta_t + \tilde{\zeta}_t = -\alpha$, $\zeta_t \cdot \zeta_t = \tilde{\zeta}_t \cdot \tilde{\zeta}_t = \kappa^2$ are satisfied. With the help of the preceding lemma, for sufficiently large $\sqrt{t^2 - \kappa^2 + \frac{|\alpha|^2}{4}} = |\Im(\zeta_t)| = |\Im(\tilde{\zeta}_t)|$, i.e., for sufficiently large t, we obtain solutions

$$u(x, \zeta_t) = e^{i\zeta_t \cdot x}(1 + v(x, \zeta_t)) \ , \ x \in B_{R_1} \ ,$$

to $\Delta u + \kappa^2 nu = 0$ in B_{R_1} with $\|v(\cdot, \zeta_t)\|_{L^2} \to 0$, $t \to \infty$. Similarly, we have solutions

$$\tilde{u}(x, \tilde{\zeta}_t) = e^{i\tilde{\zeta}_t \cdot x}(1 + \tilde{v}(x, \tilde{\zeta}_t)) \ , \ x \in B_{R_1} \ ,$$

to $\Delta\tilde{u} + \kappa^2\tilde{n}\tilde{u} = 0$ in B_{R_1} with $\|\tilde{v}(\cdot, \tilde{\zeta}_t)\|_{L^2} \to 0$, $t \to \infty$.

Inserting these solutions into (2.16) we arrive at

$$0 = \int_{B_R} (n(x) - \tilde{n}(x))e^{-i\alpha\cdot x}(1 + v(x, \zeta_t))(1 + \tilde{v}(x, \tilde{\zeta}_t))dx \ . \tag{2.17}$$

Taking $t \to \infty$ implies that the Fourier coefficient $(n - \tilde{n})\hat{}(\alpha)$ must vanish. Since α is arbitrary, we know that the Fourier coefficients of n and $\tilde{n}$ coincide, whence $n = \tilde{n}$. This ends the proof of the theorem.

□

Appendix

This appendix contains a few remarks concerning the preceding section which would have disturbed the logical order that lead to the uniqueness proof. First, we prove the equivalence of equation (2.2) and the coincidence of $\tilde{u}_\infty(\cdot, d)$ and $u_\infty(\cdot, d)$. Then, we give a second proof for Lemma 2.2, i.e., the completeness of the plane waves in the space of solutions to the Helmholtz equation. We also study the completeness of point sources in that space. And finally we derive another norm estimate for the operator G_ζ which allows to state that products of solutions to $\Delta u + \kappa^2 nu = 0$ are complete in $L^2(B_R)$.

Theorem 2.11 *Suppose* $0 < R < R_1$, $d \in S^2$ *and* $u(\cdot,d)$, $\tilde{u}(\cdot,d)$ *are solutions to (DAP) for the incident wave* $u^i(x) = e^{i\kappa d\cdot x}$, $x \in \mathbb{R}^3$, *and for the refractive indices* n, $\tilde{n}$, *respectively. Then the following assertions are equivalent:*

(a) $u(x,d) = \tilde{u}(x,d)$ *for all* $x \in \partial B_R$.

(b) $u(x,d) = \tilde{u}(x,d)$ *and* $\dfrac{\partial u}{\partial \nu}(x,d) = \dfrac{\partial \tilde{u}}{\partial \nu}(x,d)$ *for all* $x \in \partial B_R$.

(c) $u_\infty(\cdot,d) = \tilde{u}_\infty(\cdot,d)$ *on* S^2.

(d) $\displaystyle\int_{B_R} (n(x) - \tilde{n}(x))u(x,d)\tilde{u}(x)dx = 0$ *for all solutions* $\tilde{u} \in C^2(B_{R_1})$ *to* $\Delta\tilde{u} + \kappa^2\tilde{n}\tilde{u} = 0$ *in* B_{R_1}.

Proof: Assertion (b) follows from assertion (a) by the uniqueness for the exterior Dirichlet problem for the Helmholtz equation ([7, Theorem 3.7]).

The coincidence of the Cauchy data of $u(\cdot,d)$ and $\tilde{u}(\cdot,d)$ on ∂B_R implies $u^s(\cdot,d) = \tilde{u}^s(\cdot,d)$ and $(\partial u^s/\partial\nu)(\cdot,d) = (\partial\tilde{u}^s/\partial\nu)(\cdot,d)$ on ∂B_R. With the help of the representation (1.13) we obtain $\tilde{u}^s(\cdot,d) = u^s(\cdot,d)$ in the exterior of B_R, whence the coincidence of the far field patterns.

Assertion (d) was derived from assertion (c) in Lemma 2.1.

In order to obtain (a) from (d) we observe that $v := u(\cdot,d) - \tilde{u}(\cdot,d)$ is a radiating solution to the Helmholtz equation in $\mathbb{R}^3 \setminus B_R$. Moreover, Green's second theorem yields

$$\int_{\partial B_R} \Big(\tilde{u}(\cdot,d)\frac{\partial\tilde{u}}{\partial\nu} - \tilde{u}\frac{\partial\tilde{u}}{\partial\nu}(\cdot,d)\Big)ds = 0,$$

whence

$$\begin{aligned}\int_{\partial B_R} \Big(v\frac{\partial\tilde{u}}{\partial\nu} - \tilde{u}\frac{\partial v}{\partial\nu}\Big)ds &= \int_{\partial B_R} \Big(u(\cdot,d)\frac{\partial\tilde{u}}{\partial\nu} - \tilde{u}\frac{\partial u}{\partial\nu}(\cdot,d)\Big)ds \\ &= \kappa^2\int_{B_R} (n-\tilde{n})\tilde{u}u(\cdot,d)dx = 0 \qquad (2.18)\end{aligned}$$

for all solutions $\tilde{u} \in C^2(B_{R_1})$ to $\Delta\tilde{u} + \kappa^2\tilde{n}\tilde{u} = 0$ in B_{R_1}. For a fixed $|x| > R_1$ we choose $\tilde{u}$ to be the solution to the Lippmann-Schwinger equation

$$\tilde{u}(z) + \kappa^2\int_{B_R} \Phi_\kappa(z,y)(1-\tilde{n}(y))\tilde{u}(y)dy = \Phi_\kappa(x,z)\ ,\ z \in \overline{B_{R_1}}\ .$$

From Theorem 1.5 (b) we know the relation

$$0 = \int_{\partial B_R} \{v(z)\frac{\partial \Phi_\kappa(y,z)}{\partial \nu(z)} - \frac{\partial v}{\partial \nu}(z)\Phi_\kappa(y,z)\}ds(z)$$

for all $y \in B_R$. Hence, we obtain

$$0 = \int_{\partial B_R} \{v(z)\frac{\partial}{\partial \nu(z)} \int_{B_R} \Phi_\kappa(z,y)(1-\tilde{n}(y))\tilde{u}(y)dy$$
$$-\frac{\partial v}{\partial \nu}(z) \int_{B_R} \Phi_\kappa(z,y)(1-\tilde{n}(y))\tilde{u}(y)dy\}ds(z)$$

and inserting $\tilde{u}$ into (2.18)

$$\begin{aligned} 0 &= \int_{\partial B_R} \Big(v\frac{\partial \tilde{u}}{\partial \nu} - \tilde{u}\frac{\partial v}{\partial \nu}\Big)ds \\ &= \int_{\partial B_R} \{v(z)\frac{\partial \Phi_\kappa}{\partial \nu(z)}(x,z) - \frac{\partial v}{\partial \nu}(z)\Phi_\kappa(x,z)\}ds(z) = v(x) \ . \end{aligned}$$

Thus, we know $v(x) = 0$ for all $|x| > R_1$ and then $v = 0$ in $\mathbb{R}^3 \setminus B_R$ and we have proved the theorem.

□

In order to replace $u(\cdot, d)$ in equation (2.2) by arbitrary solutions to $\Delta u + \kappa^2 nu = 0$ we proved the denseness of span $\{u^i(\cdot,d) \colon d \in S^2\}$ in the linear space of solutions to the Helmholtz equation in Lemma 2.2 and derived an approximation result for general n with the help of the Lippmann-Schwinger equation in Lemma 2.3. The next lemma gives another proof for the case $n = 1$. We also examine the completeness of $\{\Phi_\kappa(\cdot,z) \colon |z| = R_1\}$ in the space of solutions to the Helmholtz equation with respect to $L^2(B_R)$.

Lemma 2.12 *Suppose* $0 < R < R_2$ *and let* $u^i \in C^2(B_{R_2})$ *satisfy* $\Delta u^i + \kappa^2 u^i = 0$ *in* B_{R_2}.

(a) For $R_1 > R$ *there exists a sequence* $u^i_j \in$ *span*$\{\Phi_\kappa(\cdot,z) \colon |z| = R_1\}$, $j \in \mathbb{N}$, *such that* $\|u^i - u^i_j\|_{L^2(B_R)} \to 0$, $j \to \infty$.

(b) There exists a sequence $u^i_j \in$ *span*$\{u^i(\cdot,d) \colon d \in S^2\}$, $j \in \mathbb{N}$, *such that* $\|u^i - u^i_j\|_{\infty,B_R} \to 0$, $j \to \infty$.

Proof: We define the linear subspace $\overline{X}$ as in the proof of Lemma 2.2 and assume $v_0 \in \overline{X}$ being orthogonal to span $\{\Phi_\kappa(\cdot, z) \colon |z| = R_1\}$. Defining again

$$w(x) := \int_{B_R} \overline{v_0(y)} \Phi_\kappa(x, y) dy \ , \ x \in \mathbb{R}^3 \setminus \overline{B_R} \ ,$$

we know that $w \in C^2(\mathbb{R}^3 \setminus \overline{B_R})$ is a radiating solution to the Helmholtz equation in the exterior of $\overline{B_R}$ and we compute

$$w(z) = \int_{B_R} \overline{v_0(y)} \Phi_\kappa(z, y) dy = 0 \ , \ |z| = R_1 \ .$$

Uniqueness for the exterior Dirichlet problem implies $w = 0$ and we can finish the proof of part (a) as in Lemma 2.2.

For part (b) let Y_l^k, $k = -l, \ldots, l$, $l \in \mathbb{N}_0$, denote a complete system of spherical harmonics on S^2 and let j_l, $l \in \mathbb{N}_0$, denote the spherical Bessel functions (see [7, sections 2.3 and 2.4] for a concise treatment). Inserting the addition theorem for the fundamental solution Φ_κ ([7, Theorem 2.10]) into Green's representation theorem for u^i in B_{R_1} we obtain the absolutely and uniformly convergent series expansion

$$u^i(x) = \sum_{l=0}^{\infty} \sum_{k=-l}^{l} a_{lk} j_l(\kappa |x|) Y_l^k(\hat{x}) \ , \ x \in \overline{B_R} \ ,$$

with suitable coefficients a_{lk}.

For a given positive ϵ we approximate u^i by a partial sum of the above series better than $\epsilon/2$. Next, using the Funk-Hecke formula ([7, p. 31]), we replace the terms $j_l(\kappa|x|) Y_l^k(\hat{x})$ by

$$j_l(\kappa|x|) Y_l^k(\hat{x}) = \frac{i^l}{4\pi} \int_{S^2} e^{-i\kappa x \cdot d} Y_l^k(d) ds(d) \ , \ x \in \overline{B_R} \ ,$$

and obtain

$$\left| u^i(x) - \int_{S^2} e^{-i\kappa x \cdot d} g_\epsilon(d) ds(d) \right| < \frac{\epsilon}{2} \ , \ x \in \overline{B_R} \ ,$$

with a continuous function g_ϵ on S^2. Finally, we approximate the integral by a quadrature rule, e.g. a Riemannian sum, better than $\epsilon/2$ uniformly on B_R. This is possible because the integrand is uniformly continuous on $B_R \times S^2$.

Hence, it is possible to approximate u^i uniformly on B_R by a superposition of plane waves and we have proved the lemma.

□

In Theorem 2.10 we had probed the medium with all incident plane waves and we had a knowledge of the far field patterns. We can infer from the last lemma that we may replace the set of all plane waves by point sources located on a sphere. Any set of solutions to the Helmholtz equation in B_R which is complete in the linear space of solutions to the Helmholtz equation with respect to $L^2(B_R)$ can be used as incident probing waves because the proof of Lemma 2.3 applies equally well in this situation, whence equation (2.4) holds true.

Instead of measuring far field patterns, due to Theorem 2.11 we might as well use the Cauchy data of the total waves on a sphere with radius R, i.e., u and $\partial u/\partial\nu$ on ∂B_R, or the near field $u|_{\partial B_R}$ of the total waves.

In a nutshell, the uniqueness theorem for the inverse scattering problem remains true, if point sources located on a sphere are used as incident waves and if the Cauchy data (or near field data) on a large sphere are measured instead of far field patterns.

The rest of this appendix is devoted to a more functional analytic formulation of our knowledge about the products $u\tilde{u}$ of solutions to perturbed Helmholtz equations:

The set

$$\{u\tilde{u}\colon u, \tilde{u} \in C^2(B_R) \cap C(\overline{B_R}),\ \Delta u + \kappa^2 n u = 0,\ \Delta\tilde{u} + \kappa^2\tilde{n}\tilde{u} = 0 \text{ in } B_R\}$$

is complete in $L^2(B_R)$, i.e., any function $q \in L^2(B_R)$ satisfying

$$\int_{B_R} q(x)u(x)\tilde{u}(x)dx = 0$$

for all functions u, $\tilde{u}$ as above must vanish identically.

An inspection of the proof of Theorem 2.10 shows that we have not proved this result so far, because we only know $v(\cdot,\zeta_t)\tilde{v}(\cdot,\tilde{\zeta}_t) \to 0$, $t \to \infty$, with respect to the $L^1(B_R)$-norm in equation (2.17). We have to know this with respect to the $L^2(B_R)$-norm, if we want to use the same reasoning as in Theorem 2.10. Therefore, we examine the operators G_ζ more closely in order to bound $\|v(\cdot,\zeta_t)\|_{\infty,B_R}$ uniformly in t. Then we obtain $\|v(\cdot,\zeta_t)\tilde{v}(\cdot,\tilde{\zeta}_t)\|_{L^2(B_R)} \le \|v(\cdot,\zeta_t)\|_{\infty,B_R}\|\tilde{v}(\cdot,\tilde{\zeta}_t)\|_{L^2(B_R)} \to 0$, $t \to \infty$.

The aim of the next lemma is to bound $\sum_{\alpha\in\Gamma} |\alpha\cdot\alpha + 2\xi\cdot\alpha|^{-2}$ uniformly in $t \geq 1$ where $\xi = (s, it, 0)$ and $\xi\cdot\xi = \kappa^2$. Essentially, we replace the sum by an integral

$$\int_{\mathbb{R}^3} [(|x|^2 - s^2)^2 + (\pi/R')^2 t^2]^{-1} dx \ .$$

Bounding the integral is not difficult (see part (a) of the next lemma) but the replacement of the sum by the integral is lengthy.

Lemma 2.13 *Let $\kappa \geq 0$ and $R' \geq \pi$ be fixed. Define $\gamma := \pi/R'$ and $s := \sqrt{t^2+\kappa^2}$, $\xi := (s, it, 0) \in \mathbb{C}^3$ for $t \geq 1$.*

(a) There is a constant c_1 such that

$$\int_0^\infty \frac{r^2 dr}{(r^2 - s^2)^2 + \gamma^2 t^2} \leq c_1$$

for all $t \geq 1$.

(b) There is a constant c_2 such that

$$\sum_{\alpha\in\Gamma} \Big| \frac{1}{\alpha\cdot\alpha + 2\xi\cdot\alpha} \Big|^2 \leq c_2^2$$

for all $t \geq 1$.

Proof: For part (a) we use the substitution $u = r/s$ and obtain

$$\int_0^\infty \frac{r^2 dr}{(r^2 - s^2)^2 + \gamma^2 t^2}$$

$$= \frac{1}{s}\Big\{ \int_0^2 \frac{u^2 du}{(u^2-1)^2 + \gamma^2(t^2/s^4)} + \int_2^\infty \frac{u^2 du}{(u^2-1)^2 + \gamma^2(t^2/s^4)} \Big\} \ .$$

Since $u^2 - 1 \geq u^2/2$ for all $u \geq 2$, we can bound the second integral by

$$\int_2^\infty \frac{u^2 du}{(u^2-1)^2 + \gamma^2(t^2/s^4)} \leq \int_2^\infty \frac{u^2 du}{(u^2/2)^2} = 2 \ .$$

We estimate the first integral with the help of the substitution $v = u^2 - 1$ by

$$\begin{aligned}
\int_0^2 \frac{u^2 du}{(u^2-1)^2 + \gamma^2(t^2/s^4)} &\leq \int_0^2 \frac{2u du}{(u^2-1)^2 + \gamma^2(t^2/s^4)} \\
&\leq \int_{\mathbb{R}} \frac{dv}{v^2 + \gamma^2(t^2/s^4)} \\
&= \frac{s^2\pi}{\gamma t} \ .
\end{aligned}$$

For the proof of assertion (b) we observe

$$\begin{aligned}
&|\alpha \cdot \alpha + 2\xi \cdot \alpha|^2 \\
&\quad = \ \{(\alpha_1 + s)^2 + \alpha_2^2 + \alpha_3^2 - s^2\}^2 + 4t^2\alpha_2^2 \\
&\quad \geq \ \{(\alpha_1 + s)^2 + \alpha_2^2 + \alpha_3^2 - s^2\}^2 + \gamma^2 t^2 \ , \ \alpha \in \Gamma \ .
\end{aligned}$$

Our first aim is to estimate the number of grid points in a spherical shell, to be more precise we derive the bound $\operatorname{card}(A_k) \leq M_1 k^2$ where

$$A_k := \{\alpha \in \Gamma \colon (k-1)^2 < (\alpha_1 + s)^2 + \alpha_2^2 + \alpha_3^2 \leq k^2\} \ , \ k \in \mathbb{N} \ .$$

This enables us to reduce the series over all $\alpha \in \Gamma$ to a series over $k \in \mathbb{N}$ which in turn can be estimated with the help of Maclaurin's (Cauchy's) integral test.

We define A'_k to be the grid points contained in a ball with radius k and center $(-s, 0, 0)$,

$$A'_k := \{\alpha \in \Gamma \colon (\alpha_1 + s)^2 + \alpha_2^2 + \alpha_3^2 \leq k^2\} \ , \ k \in \mathbb{N} \ .$$

In order to obtain bounds for $\operatorname{card}(A'_k)$ we use the disjoint open cubes C_α, $\alpha \in \Gamma$, having the center α and having edges of length π/R' parallel to the coordinate axes. Since $\alpha \in A'_k$ implies $C_\alpha \subset B_{k+2\gamma}((-s,0,0))$, we can conclude

$$\left(\frac{\pi}{R'}\right)^3 \operatorname{card}(A'_k) = \operatorname{vol}\Bigg(\bigcup_{\alpha \in A'_k} C_\alpha\Bigg) \leq \operatorname{vol}\Big(B_{k+2\gamma}((-s,0,0))\Big) = \frac{4\pi}{3}(k + 2\gamma)^3 \ .$$

Since any $x \in B_{k-1-2\gamma}((-s,0,0))$, $k > 1 + 2\gamma$, is lying in some C_α, there is an $\alpha \in \Gamma$ with $|\alpha - x| \leq 2\gamma$. Hence, we have

$$|(\alpha_1 + s, \alpha_2, \alpha_3)| \leq |(x_1 + s, x_2, x_3)| + |\alpha - x| \leq k - 1 \ , \quad \text{i.e., } \alpha \in A'_{k-1}.$$

This means

$$B_{k-1-2\gamma}((-s,0,0)) \subset \bigcup_{\alpha\in A'_{k-1}} C_\alpha$$

and therefore

$$\frac{4\pi}{3}(k-1-2\gamma)^3 \leq \Big(\frac{\pi}{R'}\Big)^3 \text{card}\,(A'_{k-1}) \ .$$

Then we obtain

$$\begin{aligned} \text{card}\,(A_k) &= \text{card}\,(A'_k) - \text{card}\,(A'_{k-1}) \\ &\leq \gamma^{-3}\frac{4\pi}{3}\Big((k+2\gamma)^3 - (k-1-2\gamma)^3\Big) \\ &\leq M_1 k^2 \end{aligned}$$

for all $k > 1+2\gamma$. By suitably enlarging the constant M_1 the last inequality holds true for all $k \in \mathbb{N}$.

Now, we split

$$\begin{aligned} &\sum_{\alpha\in\Gamma}\Big|\frac{1}{\alpha\cdot\alpha+2\xi\cdot\alpha}\Big|^2 \\ &\quad = \sum_{\{k\in\mathbb{N}:k\leq s\}}\sum_{\alpha\in A_k}\Big|\frac{1}{\alpha\cdot\alpha+2\xi\cdot\alpha}\Big|^2 + \sum_{\{k\in\mathbb{N}:k-1\geq s\}}\sum_{\alpha\in A_k}\Big|\frac{1}{\alpha\cdot\alpha+2\xi\cdot\alpha}\Big|^2 \\ &\quad\quad + \sum_{\{k\in\mathbb{N}:s<k<s+1\}}\sum_{\alpha\in A_k}\Big|\frac{1}{\alpha\cdot\alpha+2\xi\cdot\alpha}\Big|^2 \ . \end{aligned} \tag{2.19}$$

From $\alpha \in A_k$ and $k \leq s$ we conclude $0 \leq s^2 - k^2 \leq s^2 - |(\alpha_1+s,\alpha_2,\alpha_3)|^2$ and therefore

$$\begin{aligned} \Big|\frac{1}{\alpha\cdot\alpha+2\xi\cdot\alpha}\Big|^2 &= \frac{1}{(|(\alpha_1+s,\alpha_2,\alpha_3)|^2-s^2)^2+4t^2\alpha_2^2} \\ &\leq \frac{1}{(k^2-s^2)^2+t^2\gamma^2} \ . \end{aligned}$$

Together with the bound on card (A_k) we arrive at the following estimate for the first term on the right hand side of (2.19):

$$\begin{aligned} &\sum_{\{k\in\mathbb{N}:k\leq s\}}\sum_{\alpha\in A_k}\Big|\frac{1}{\alpha\cdot\alpha+2\xi\cdot\alpha}\Big|^2 \\ &\quad \leq \sum_{k\leq s-1}\frac{M_1k^2}{(k^2-s^2)^2+t^2\gamma^2} + \frac{M_1 s^2}{t^2\gamma^2} \end{aligned}$$

$$\leq \int_1^s \frac{M_1 r^2 dr}{(r^2-s^2)^2+t^2\gamma^2} + \frac{M_1 s^2}{t^2\gamma^2}$$
$$\leq M_1 c_1 + M_2 \leq M_3 \ , \tag{2.20}$$

with suitable constants M_j and c_1 from part (a), where in the third line we have used the fact that $r \mapsto \frac{r^2}{(r^2-s^2)^2+t^2\gamma^2}$ is an increasing function on $[1,s]$ (the numerator is an increasing function, whereas the denominator is decreasing).

Similarly, for $\alpha \in A_k$ and $k-1 \geq s$ we compute $0 \leq (k-1)^2 - s^2 \leq |(\alpha_1+s,\alpha_2,\alpha_3)|^2 - s^2$,

$$\Big|\frac{1}{\alpha\cdot\alpha+2\xi\cdot\alpha}\Big|^2 = \frac{1}{(|(\alpha_1+s,\alpha_2,\alpha_3)|^2-s^2)^2+4t^2\alpha_2^2}$$
$$\leq \frac{1}{((k-1)^2-s^2)^2+t^2\gamma^2} \ ,$$

and we estimate the second term on the right hand side of (2.19) by

$$\sum_{\{k\in\mathbb{N}:k-1\geq s\}} \sum_{\alpha\in A_k} \Big|\frac{1}{\alpha\cdot\alpha+2\xi\cdot\alpha}\Big|^2$$
$$\leq \sum_{k\geq s+3} \frac{M_1 k^2}{((k-1)^2-s^2)^2+t^2\gamma^2} + 2\frac{M_1(s+3)^2}{t^2\gamma^2}$$
$$\leq \sum_{l\geq s+2} \frac{4M_1 l^2}{(l^2-s^2)^2+t^2\gamma^2} + 2\frac{M_1(s+3)^2}{t^2\gamma^2}$$

$$\leq \int_{s+1}^{\infty} \frac{4M_1 r^2 dr}{(r^2-s^2)^2+t^2\gamma^2} + M_4$$
$$\leq 4M_1 c_1 + M_4 \leq M_5 \ . \tag{2.21}$$

Here, we have substituted $l = k-1$ in third line and employed the inequality $(l+1)^2 \leq 4l^2$. Moreover, in the fourth line we have used the fact that $r \mapsto \frac{(r^2-s^2)+s^2}{(r^2-s^2)^2+t^2\gamma^2}$ is a decreasing function on $[s+1,\infty)$. This can be seen by observing that $\rho = r^2 - s^2 > 2s > t\gamma$ for $r \geq s+1$, and that the derivative of the mapping $\rho \mapsto \frac{\rho}{\rho^2+\gamma^2 t^2}$ is negative if $\rho > t\gamma$. (Note that here we have used the assumption $R' \geq \pi$, i.e., $\gamma \leq 1$, in order to have $2s > t \geq \gamma t$.)

Finally, we bound the last term in inequality (2.19) by

$$\sum_{\{k\in\mathbb{N}:s<k<s+1\}}\sum_{\alpha\in A_k}\Big|\frac{1}{\alpha\cdot\alpha+2\xi\cdot\alpha}\Big|^2\leq\frac{M_1(s+1)^2}{\gamma^2t^2}\leq M_6\ .$$

Inserting this inequality together with inequalities (2.20) and (2.21) into (2.19) yields assertion (b). This ends the proof of the lemma.

□

The preceding lemma easily implies the next results on the operator G_ζ and on the functions $v(\cdot,\zeta)$ from Lemma 2.9.

Lemma 2.14 *Assume $\kappa\geq 0$ and $\pi<R''<R'$.*

(a) The inequality

$$\|G_\zeta f\|_{\infty,B_{R''}}=\sup_{x\in B_{R''}}\Big|\int\limits_{B_{R''}}g_\zeta(x-y)f(y)dy\Big|\leq c_2\|f\|_{L^2(B_{R''})}$$

holds true for all $f\in C(\overline{B_{R''}})$ and for all $\zeta\in\mathbb{C}^3$ satisfying $\zeta\cdot\zeta=\kappa^2$, $|\Im(\zeta)|\geq 1$ where c_2 is the constant from the preceding lemma.

(b) There is a constant c uniformly bounding the functions $v(\cdot,\zeta)$ from Lemma 2.9: $\|v(\cdot,\zeta)\|_{\infty,B_{R''}}\leq c$ for all $\zeta\in\mathbb{C}^3$ satisfying $\zeta\cdot\zeta=\kappa^2$ and $|\Im(\zeta)|\geq 2\kappa^2(R'/\pi)\|1-n\|_\infty+1$.

Proof: As in the proof of Theorem 2.8 (d) we have for $f\in C_0^\infty(B_{R''})$

$$\begin{aligned}
\|G_\zeta f\|^2_{\infty,B_{R''}} &= \|(G_\zeta f)\circ Q^T\|^2_{\infty,B_{R''}}\\
&= \|G'_\xi(f\circ Q^T)\|^2_{\infty,B_{R''}}\\
&= \Big\|\sum_{\alpha\in\Gamma}\frac{(f\circ Q^T)\hat{}(\alpha)}{\alpha\cdot\alpha+2\xi\cdot\alpha}e_\alpha\Big\|^2_{\infty,B_{R''}}\\
&\leq \sum_{\alpha\in\Gamma}\Big|\frac{1}{\alpha\cdot\alpha+2\xi\cdot\alpha}\Big|^2\sum_{\alpha\in\Gamma}|(f\circ Q^T)\hat{}(\alpha)|^2\\
&\leq c_2^2\|(f\circ Q^T)\|^2_{L^2(C)}\\
&= c_2^2\|f\|^2_{L^2(B_{R''})}\ .
\end{aligned}$$

Approximating an arbitrary continuous function f in $\overline{B_{R''}}$ by $C_0^\infty(B_{R''})$-functions with respect to the L^2-norm yields assertion (a).

We obtain from (2.14) together with part (a)

$$\begin{aligned}\|v(\cdot,\zeta)\|_{\infty,B_{R''}} &\le \kappa^2\|G_\zeta((1-n)(1+v(\cdot,\zeta))\|_{\infty,B_{R''}}\\ &\le \tilde{c}\|(1-n)(1+v(\cdot,\zeta))\|_{L^2(B_{R''})}\\ &\le c\ ,\end{aligned}$$

where we also use from Lemma 2.9 that $\|v(\cdot,\zeta)\|_{L^2(B_{R''})} \le M$ for all sufficiently large $|\Im(\zeta)|$. This proves the lemma.

□

Looking again at the proof of the uniqueness theorem for the refractive indices, we see that $u(\cdot,\zeta_t)\tilde{u}(\cdot,\tilde{\zeta}_t) \in C(\overline{B_{R''}}) \subset L^2(B_{R''})$ satisfy

$$\begin{aligned}&\|u(\cdot,\zeta_t)\tilde{u}(\cdot,\tilde{\zeta}_t) - e^{-i\alpha\cdot}\|_{L^2(B_{R''})}\\ &= \|v(\cdot,\zeta_t) + \tilde{v}(\cdot,\tilde{\zeta}_t) + v(\cdot,\zeta_t)\tilde{v}(\cdot,\tilde{\zeta}_t)\|_{L^2(B_{R''})}\\ &\le \|v(\cdot,\zeta_t) + \tilde{v}(\cdot,\tilde{\zeta}_t)\|_{L^2(B_{R''})} + \|v(\cdot,\zeta_t)\|_\infty\|\tilde{v}(\cdot,\tilde{\zeta}_t)\|_{L^2(B_{R''})} \to 0\ ,\ t\to\infty\ .\end{aligned}$$

Then, the reasoning in the proof of Theorem 2.10 implies that any function $q \in L^2(B_{R''})$ which is orthogonal to all products $u\tilde{u}$ must have vanishing Fourier coefficients. This proves the last theorem of this appendix.

Theorem 2.15 *Assume $n, \tilde{n} \in C^{0,\gamma}(\overline{B_{R''}})$ are uniformly Hölder continuous in $\overline{B_{R''}}$. Then, the set*

$$\{u\tilde{u} \colon u, \tilde{u} \in C^2(B_{R''}) \cap C(\overline{B_{R''}}),\ \Delta u + \kappa^2 n u = 0,\ \Delta\tilde{u} + \kappa^2\tilde{n}\tilde{u} = 0 \text{ in } B_{R''}\}$$

is complete in $L^2(B_{R''})$.

Though the last theorem improves our knowledge about the products $u\tilde{u}$ it is of no use in proving a uniqueness result for scattering problems when we only know $n \in L^2(B_R)$ instead of $n \in C^{0,\gamma}(\overline{B_R})$. This is due to the fact that our proof of the behavior of the special solutions $u(\cdot,\zeta)$ in Lemma 2.9 essentially uses the boundedness of $\|1-n\|_\infty$.

If we want to deal with $n \in L^2(B_R)$, we have to improve Lemma 2.13 to obtain

$$\sum_{\alpha\in\Gamma} \frac{1}{|\alpha\cdot\alpha + 2\xi\cdot\alpha|^2} \to 0\ ,\ t = |\Im(\xi)| \to \infty\ . \tag{2.22}$$

This estimate can be obtained by using the ideas from the proof of [40, III.2 Lemma 3, pp. 51–56] together with the techniques which we employed during the proof of Lemma 2.13. Inequality (2.22) implies the norm estimate $\|G_\zeta\|_{L^2(B_{R''})\to C(\overline{B_{R''}})} \to 0$, $|\Im(\zeta)| \to \infty$. We can then modify the existence and uniqueness proofs from the first chapter to arrive at unique solutions to (DAP) with $n \in L^2$. It is also possible to verify our results for the inverse problem in this case and to derive uniqueness of n from a knowledge of the far field patterns.

2.2 Stability of the Inverse Problem

This section is devoted to the continuous dependence of the refractive index n on the far field pattern. We assume throughout this section that the refractive indices n satisfy $n \in C^{0,\gamma}(\mathbb{R}^3)$, $0 < \gamma < 1$, $\Im(n) \geq 0$, and $\operatorname{supp}(1-n) \subset B_R$. For convenience we define $\tilde{C}(B_R)$ to be the set of these functions.

Let us start with an informal outline of this section. We introduce a very strong norm $\|\cdot\|_{\mathcal{F}}$ on the far field patterns by prescribing a very rapid decay of the Fourier coefficients

$$\mu_{l_1k_1l_2k_2} := \int_{S^2}\int_{S^2} u_\infty(\hat{x},d)\overline{Y_{l_1}^{k_1}(\hat{x})Y_{l_2}^{k_2}(d)}ds(\hat{x})ds(d) \ , \qquad (2.23)$$
$$l_1, l_2 = 0, 1, \ldots, \ -l_1 \leq k_1 \leq l_1 \ , \ -l_2 \leq k_2 \leq l_2 \ ,$$

of the far field patterns u_∞. Here, $Y_{l_1}^{k_1}$, $l_1 = 0, 1, \ldots$, $-l_1 \leq k_1 \leq l_1$, denote a complete orthonormal system of spherical harmonics on S^2. Our aim is to derive the estimate

$$\|n - \tilde{n}\|_\infty \leq c\Big[-\ln(\|u_{\infty,n} - u_{\infty,\tilde{n}}\|_{\mathcal{F}})\Big]^{-1/7}$$

with a constant c for all refractive indices n, $\tilde{n}$ lying in some small subset $\mathcal{O}$ of $\tilde{C}(B_R)$. The subscript n indicates the dependence of certain quantities on n if necessary. The estimate implies that the mapping $u_{\infty,n} \mapsto n$ is continuous. It is also a local uniqueness result because $u_{\infty,n}$ uniquely determines n in $\mathcal{O}$ by the above estimate. The reader should be warned that $\mathcal{O}$ is not only small with respect to the maximum norm but with respect to a C^2-norm, i.e., we need additional information in a stronger norm in order to obtain the stability result. We also want to emphasize that this result does not mean that all functions from a small $\|\cdot\|_{\mathcal{F}}$-neighborhood of $u_{\infty,n}$ are far fields originating from a refractive index. It only allows to conclude that two refractive indices from $\mathcal{O}$ are close together with respect to $\|\cdot\|_\infty$, if they produce far fields whose difference with respect to the $\|\cdot\|_{\mathcal{F}}$-norm is small.

Our reasoning follows the main ideas from the paper [42] of Stefanov and can be divided into three steps. In the first three lemmas we examine the decay of the Fourier coefficients, show that the norm $\|\cdot\|_{\mathcal{F}}$ is well defined and prove that the mapping $n \mapsto u_{\infty,n}$ is continuous.

In the second step we reconstruct the Green's function $s_n(x,y)$ for $|x| = |y| = R_2 > R$, $x \neq y$, with the help of a series expansion involving the Fourier coefficients $\mu_{l_1k_1l_2k_2}$ belonging to $u_{\infty,n}$. By the Green's function s_n

we mean the kernel $s_n(x,y)$, x, $y \in \mathbb{R}^3$, $x \neq y$, having the property that $u = -\int\limits_{\mathbb{R}^3} s_n(\cdot,y)f(y)dy$ is the radiating solution to $\Delta u + \kappa^2 nu = f$ in $\mathbb{R}^3$. Hence, we know the single-layer operator

$$(S_n\varphi)(x) = 2\int\limits_{\partial B_{R_2}} s_n(x,y)\varphi(y)ds(y)\ ,\ x \in \partial B_{R_2}\ .$$

Actually, we construct S_n directly without using s_n and prove later that S_n is an integral operator whose kernel s_n can be computed from $u_{\infty,n}$. Although we never prove that s_n is the Green's function the idea behind S_n is easier to explain with the help of s_n.

We discuss the dependence of S_n on n and on $u_{\infty,n}$. It turns out that due to the very strong norm $\|\cdot\|_{\mathcal{F}}$ the mapping $u_{\infty,n} \mapsto S_n$ is linear and bounded. However, in practice the $\|\cdot\|_{\mathcal{F}}$-norm is not appropriate for measured far field patterns. Hence, the transfer of information from infinity (the far field pattern) to the sphere of radius R_2 is severely ill-posed.

In the last step we investigate the dependence of n on S_n, insert our result from the previous step on the dependence of S_n on $u_{\infty,n}$ and arrive at our main estimate.

In [42] Stefanov uses the relation $\Lambda_n - \Lambda_1 = 2S_n^{-1} - 2S_1^{-1}$ (see [32]) between the Dirichlet-to-Neumann maps Λ_n, Λ_1, and the inverses to the single-layer operators to estimate the operator norms $\|\Lambda_n - \Lambda_{\tilde{n}}\|$ in some suitable norm by the far field norms $\|u_{\infty,n} - u_{\infty,\tilde{n}}\|_{\mathcal{F}}$. Then he can employ the ideas of Alessandrini in [2] who has studied the dependence of n on Λ_n for refractive indices n having a special form. Alessandrini's main idea is to estimate the Fourier transform of $n - \tilde{n}$ with the help of the special solutions from Lemma 2.9 and with considerations similar to those that lead to equation (2.2). Stefanov encounters difficulties because the Dirichlet-to-Neumann map is not defined for interior Dirichlet eigenvalues. We use a different relation on ∂B_{R_2} in order to avoid these difficulties. We are also able to avoid the technical Lemma 4.2 from [42] (Lemma 3 in [2]).

After this outline let us begin with the first step. In the first lemma we prove estimates about spherical Bessel functions and about spherical Hankel functions of the first kind which we will need later. The spherical Bessel function j_l of order $l \in \mathbb{N}_0$ is defined by

$$j_l(t) := \sum_{p=0}^{\infty} \frac{(-1)^p t^{l+2p}}{2^p p!\, 1 \cdot 3 \cdots (2l+2p+1)}\ ,\ t \in \mathbb{R}\ .$$

With the help of the spherical Neumann function y_l of order $l \in \mathbb{N}_0$ which is defined by

$$y_l(t) := -\frac{(2l)!}{2^l l!} \sum_{p=0}^{\infty} \frac{(-1)^p t^{2p-l-1}}{2^p p!(-2l+1)(-2l+3)\cdots(-2l+2p-1)} \ , \ t > 0 \ ,$$

we can define the spherical Hankel functions $h_l^{(1)}$ of the first kind of order $l \in \mathbb{N}_0$:

$$h_l^{(1)}(t) := j_l(t) + i y_l(t) \ , \ t > 0 \ .$$

Lemma 2.16 *Assume $\kappa > 0$.*

(a) Given $R_1 > 0$ there is a constant $M_1 > 0$ such that

$$|j_l(\kappa r)| \le M_1 \Big(\frac{e\kappa r}{2l+1}\Big)^l \frac{1}{2l+1} \ , \ 0 \le r \le R_1 \ , \ l \in \mathbb{N}_0 \ .$$

(b) For $0 < R < R_2$ there is a constant $M_2 > 0$ such that

$$\begin{aligned} |h_l^{(1)}(\kappa r)| &\le M_2 \Big(\frac{2l+1}{e\kappa r}\Big)^l \quad \text{and} \\ |(h_l^{(1)})'(\kappa r)| &\le M_2 (2l+1) \Big(\frac{2l+1}{e\kappa r}\Big)^l \ , \ R \le r \le R_2 \ , \ l \in \mathbb{N}_0 \ . \end{aligned}$$

Proof: The definition of j_l yields

$$j_l(t) = \frac{t^l}{1 \cdot 3 \cdots (2l+1)} \Big(1 + \sum_{p=1}^{\infty} \frac{(-1)^p t^{2p}}{2^p p!(2l+3)\cdots(2l+2p+1)}\Big) \ .$$

Comparing the series with the exponential series we obtain the estimate

$$|j_l(t)| \le M_3 \frac{t^l}{1 \cdot 3 \cdots (2l+1)} = M_3 \frac{t^l}{2l+1} \frac{2^l l!}{(2l)!} \ , \ 0 \le t \le \kappa R_1 \ , \ l \in \mathbb{N}_0 \ .$$

This certainly implies the assertion for $l = 0$. From Stirling's formula we know

$$M_4 l^l e^{-l} \sqrt{2\pi l} < l! < M_5 l^l e^{-l} \sqrt{2\pi l} \ , \ l \in \mathbb{N} \ ,$$

with suitable positive constants M_4, M_5. Inserting these inequalities into the above estimate and using the fact that $(\frac{2l+1}{2l})^l$, $l \in \mathbb{N}$, is a bounded sequence we arrive at

$$|j_l(\kappa r)| \le M_1 \Big(\frac{e\kappa r}{2l+1}\Big)^l \frac{1}{2l+1} \ , \ 0 \le r \le R_1 \ , \ l \in \mathbb{N} \ .$$

This completes the proof of part (a).

Part (b) follows similarly with the help of

$$\begin{aligned} h_l^{(1)}(t) &= -\frac{(2l)!}{2^l l!} t^{-l-1} \Big\{ -t^{l+1} \frac{2^l l!}{(2l)!} j_l(t) \\ &\quad + i \Big(1 + \sum_{p=1}^{\infty} \frac{(-1)^p t^{2p}}{2^p p! (-2l+1)(-2l+3) \cdots (-2l+2p-1)} \Big) \Big\} \end{aligned}$$

because the series and the first term in the curly brackets remain uniformly bounded in $l \in \mathbb{N}_0$ and $\kappa R \leq t \leq \kappa R_2$. Differentiating the last equation with respect to t and using the same techniques as before also yields the second part of assertion (b).

□

The previous lemma enables us to study the decay of the Fourier coefficients $\mu_{l_1 k_1 l_2 k_2}$ of a far field pattern. The decay determines the smoothness of u_∞ and *vice versa*. In [30] and [31, Theorems 17,18] Müller has examined the smoothness of the scattering amplitude of a radiating solution to the Helmholtz equation. However, he expresses the regularity of the scattering amplitude by the growth at infinity of an entire harmonic function which coincides with the scattering amplitude on S^2. Therefore, his results are not directly applicable to our needs. We follow the ideas of Stefanov in [42] (see [7, Theorems 2.15,2.16] for related rsults) but give weaker estimates because we are not interested in optimal results.

To simplify notation we abbreviate

$$\sum_{l_1,k_1,l_2,k_2} \qquad \text{for} \qquad \sum_{l_1=0}^{\infty} \sum_{k_1=-l_1}^{l_1} \sum_{l_2=0}^{\infty} \sum_{k_2=-l_2}^{l_2}$$

and we define

$$T_n : C(\overline{B_R}) \to C(\overline{B_R}) \quad (T_n \varphi)(x) := \kappa^2 \int_{B_R} (1 - n(y)) \Phi_\kappa(x,y) \varphi(y) dy \; , \; x \in \overline{B_R} \; .$$

Lemma 2.17 *Assume the far field pattern $u_\infty : S^2 \times S^2 \to \mathbb{C}$ originates from the refractive index $n \in \tilde{C}(B_R)$ satisfying supp$\,(1-n) \subset B_{R_1}$ for some $0 < R_1 < R$. Let $\mu_{l_1 k_1 l_2 k_2}$ denote the Fourier coefficients of u_∞ as defined in (2.23). Then, there is a constant c depending on u_∞ such that*

$$|\mu_{l_1 k_1 l_2 k_2}|^2 \leq c \Big(\frac{e \kappa R_1}{2l_1 + 1} \Big)^{2l_1+3} \Big(\frac{e \kappa R_1}{2l_2 + 1} \Big)^{2l_2+3} \; .$$

Furthermore

$$\sum_{l_1,k_1,l_2,k_2} \Big(\frac{2l_1+1}{e\kappa R}\Big)^{2l_1+3} \Big(\frac{2l_2+1}{e\kappa R}\Big)^{2l_2+3} |\mu_{l_1k_1l_2k_2}|^2 < \infty \ .$$

Proof: According to (1.25) the far field pattern corresponding to the refractive index $n \in \tilde{C}(B_R)$ with $\operatorname{supp}(1-n) \subset B_{R_1}$, $0 < R_1 < R$, is given by

$$u_\infty(\hat{x}, d) = -\frac{\kappa^2}{4\pi} \int\limits_{B_{R_1}} (1-n(y))u(y,d)e^{-i\kappa\hat{x}\cdot y}dy \ , \ \hat{x}, d \in S^2 \ . \tag{2.24}$$

The Lippmann-Schwinger equation yields

$$u(x,d) = -[T_n(u(\cdot,d))](x) + e^{i\kappa d\cdot x} \ , \ x \in B_R \ , \ d \in S^2 \ , \tag{2.25}$$

whence u depends continuously on x and d, since $e^{i\kappa d\cdot x}$ depends continuously on d with respect to $\|\cdot\|_{\infty,B_R}$ and $(I+T_n)^{-1}$ is continuous in $C(\overline{B_R})$. Then, we may interchange the order of integration and we obtain

$$\begin{aligned} \mu_{l_1k_1l_2k_2} &= \int\limits_{S^2}\int\limits_{S^2} u_\infty(\hat{x},d)\overline{Y_{l_1}^{k_1}(\hat{x})Y_{l_2}^{k_2}(d)}ds(\hat{x})ds(d) \\ &= -\frac{\kappa^2}{4\pi}\int\limits_{B_{R_1}} (1-n(y)) \int\limits_{S^2} u(y,d)\overline{Y_{l_2}^{k_2}(d)}ds(d) \int\limits_{S^2} e^{-i\kappa\hat{x}\cdot y}\overline{Y_{l_1}^{k_1}(\hat{x})}ds(\hat{x})dy \ . \end{aligned}$$

Applying the Cauchy-Schwarz inequality we conclude

$$\begin{aligned} |\mu_{l_1k_1l_2k_2}|^2 \ \leq \ & c\Big\| \int\limits_{S^2} u(\cdot,d)\overline{Y_{l_2}^{k_2}(d)}ds(d)\Big\|^2_{L^2(B_{R_1})} \cdot \\ & \cdot \Big\| \int\limits_{S^2} e^{-i\kappa d\cdot x}\overline{Y_{l_1}^{k_1}(d)}ds(d)\Big\|^2_{L^2(B_{R_1})} \ . \end{aligned} \tag{2.26}$$

Note that c will denote different constants during the proof.

With the help of the Funk-Hecke formula (see [7, (2.44)]) we compute

$$v_{l_1k_1}(y) := \int\limits_{S^2} e^{-i\kappa\hat{x}\cdot y}\overline{Y_{l_1}^{k_1}(\hat{x})}ds(\hat{x}) = \frac{4\pi}{i^{l_1}} j_{l_1}(\kappa|y|)\overline{Y_{l_1}^{k_1}(\hat{y})} \ , \ y \in \mathbb{R}^3 \ ,$$

whence with Lemma 2.16 (a)

$$\begin{aligned} \|v_{l_1k_1}\|^2_{L^2(B_{R_1})} &= 64\pi^3 \int_0^{R_1} |j_{l_1}(\kappa r)|^2 r^2 dr \\ &\le c\Big(\frac{e\kappa R_1}{2l_1+1}\Big)^{2l_1+3} . \end{aligned} \tag{2.27}$$

Reversing the order of integration and employing the Funk-Hecke formula again we obtain from (2.25)

$$\int_{S^2} u(x,d)\overline{Y^{k_2}_{l_2}(d)}ds(d) = -\Big[T_n\Big(\int_{S^2} u(\cdot,d)\overline{Y^{k_2}_{l_2}(d)}ds(d)\Big)\Big](x) + (-1)^{l_2} v_{l_2k_2}(x) ,$$

i.e.,

$$\int_{S^2} u(x,d)\overline{Y^{k_2}_{l_2}(d)}ds(d) = (-1)^{l_2}[(I+T_n)^{-1}v_{l_2k_2}](x) . \tag{2.28}$$

With the help of Theorem 1.9 (c) we can conclude that the operator T_n is compact in $(C(\overline{B_R}), \|\cdot\|_{L^2})$, whence $(I+T_n)^{-1}$ is bounded with respect to the $L^2(B_R)$-norm. Then, inserting (2.27) and

$$\begin{aligned} \Big\|\int_{S^2} u(\cdot,d)\overline{Y^{k_2}_{l_2}(d)}ds(d)\Big\|^2_{L^2(B_{R_1})} &\le c\|v_{l_2k_2}\|^2_{L^2(B_{R_1})} \\ &\le c\Big(\frac{e\kappa R_1}{2l_2+1}\Big)^{2l_2+3} \end{aligned}$$

into (2.26) we arrive at

$$|\mu_{l_1k_1l_2k_2}|^2 \le c\Big(\frac{e\kappa R_1}{2l_1+1}\Big)^{2l_1+3}\Big(\frac{e\kappa R_1}{2l_2+1}\Big)^{2l_2+3} .$$

Finally, we estimate

$$\begin{aligned} &\sum_{l_1,k_1,l_2,k_2}\Big(\frac{2l_1+1}{e\kappa R}\Big)^{2l_1+3}\Big(\frac{2l_2+1}{e\kappa R}\Big)^{2l_2+3}|\mu_{l_1k_1l_2k_2}|^2 \\ \le\; & c\sum_{l_1,k_1,l_2,k_2}\Big(\frac{2l_1+1}{e\kappa R}\Big)^{2l_1+3}\Big(\frac{2l_2+1}{e\kappa R}\Big)^{2l_2+3}\Big(\frac{e\kappa R_1}{2l_1+1}\Big)^{2l_1+3}\Big(\frac{e\kappa R_1}{2l_2+1}\Big)^{2l_2+3} \\ =\; & c\sum_{l_1,l_2}(2l_1+1)(2l_2+1)\Big(\frac{R_1}{R}\Big)^{2l_1+3}\Big(\frac{R_1}{R}\Big)^{2l_2+3} \\ <\; & \infty \end{aligned}$$

by the ratio test.

□

The last lemma asserts that the norm $\|u_{\infty,n}\|_{\mathcal{F}}$ defined by

$$\|u_{\infty,n}\|^2_{\mathcal{F}} := \sum_{l_1,k_1,l_2,k_2} \Big(\frac{2l_1+1}{e\kappa R}\Big)^{2l_1+3} \Big(\frac{2l_2+1}{e\kappa R}\Big)^{2l_2+3} |\mu_{l_1k_1l_2k_2}|^2$$

is well defined, if $n \in \tilde{C}(B_R)$ because $\operatorname{supp}(1-n) \subset B_R$ implies that there is a radius $R_1 < R$ with $\operatorname{supp}(1-n) \subset B_{R_1}$.

Next we study the continuous dependence of $u_{\infty,n}$ on n. In the end we are interested in the reverse result but we will need the dependence of $u_{\infty,n}$ on n in order to obtain the small set in which n depends continuously on $u_{\infty,n}$.

Lemma 2.18 *Let $n_0 \in \tilde{C}(B_{R_1})$, $R_1 < R$, be given. Then, there are positive constants M and ϵ such that $\|u_{\infty,n} - u_{\infty,n_0}\|_{\mathcal{F}} \leq M\|n-n_0\|_\infty$ for all $n \in \tilde{C}(B_{R_1})$ satisfying $\|n-n_0\|_\infty < \epsilon$.*

Proof: Similarly to the previous lemma we start with

$$\begin{aligned}
&u_{\infty,n}(\hat{x},d) - u_{\infty,n_0}(\hat{x},d) \\
&= -\frac{\kappa^2}{4\pi}\int_{B_{R_1}} (1-n(y))u_n(y,d)e^{-i\kappa\hat{x}\cdot y}dy \\
&\quad +\frac{\kappa^2}{4\pi}\int_{B_{R_1}} (1-n_0(y))u_{n_0}(y,d)e^{-i\kappa\hat{x}\cdot y}dy \\
&= \frac{\kappa^2}{4\pi}\int_{B_{R_1}} (n(y)-n_0(y))u_{n_0}(y,d)e^{-i\kappa\hat{x}\cdot y}dy \\
&\quad +\frac{\kappa^2}{4\pi}\int_{B_{R_1}} (1-n(y))(u_{n_0}(y,d)-u_n(y,d))e^{-i\kappa\hat{x}\cdot y}dy\ ,\ \ \hat{x},d \in S^2\ .
\end{aligned} \tag{2.29}$$

Multiplying by $\overline{Y^{k_1}_{l_1}(\hat{x})Y^{k_2}_{l_2}(d)}$ and integrating we can use the reasoning in Lemma 2.17 and bound the term originating from the first integral on the right hand side by

$$M_1\|n-n_0\|_\infty\Big(\frac{e\kappa R_1}{2l_1+1}\Big)^{(2l_1+3)/2}\Big(\frac{e\kappa R_1}{2l_2+1}\Big)^{(2l_2+3)/2}\ .$$

For the term originating from the second integral on the right hand side we observe that there is a constant M_2 such that for all $n \in \tilde{C}(B_{R_1})$ the inequality $\|T_n - T_{n_0}\|_{L^2(B_R)} \le M_2 \|n - n_0\|_\infty$ holds true. Thus we can choose $\epsilon > 0$ sufficiently small to ensure $\|T_n - T_{n_0}\|_{L^2(B_R)} \le 1/(2\|(I + T_{n_0})^{-1}\|_{L^2})$ for all $\|n - n_0\|_\infty < \epsilon$ and, with the help of a Neumann series argument, we obtain $\|(I + T_n)^{-1}\|_{L^2} \le 2\|(I + T_{n_0})^{-1}\|_{L^2}$ for those n. Using

$$\int_{S^2} u_n(x,d)\overline{Y_{l_2}^{k_2}(d)}ds(d) = (I + T_n)^{-1} \int_{S^2} e^{i\kappa d\cdot x}\overline{Y_{l_2}^{k_2}(d)}ds(d)$$

and

$$(I + T_n)^{-1} - (I + T_{n_0})^{-1} = (I + T_n)^{-1}(T_{n_0} - T_n)(I + T_{n_0})^{-1}$$

this yields

$$\begin{aligned}
&\Big\|\int_{S^2} (u_n(\cdot,d) - u_{n_0}(\cdot,d))\overline{Y_{l_2}^{k_2}(d)}ds(d)\Big\|_{L^2} \\
&\quad = \Big\|\Big[(I + T_n)^{-1} - (I + T_{n_0})^{-1}\Big] \int_{S^2} e^{i\kappa d\cdot x}\overline{Y_{l_2}^{k_2}(d)}ds(d)\Big\|_{L^2} \\
&\quad \le \Big\|(I + T_n)^{-1} - (I + T_{n_0})^{-1}\Big\|_{L^2}\Big\|\int_{S^2} e^{i\kappa d\cdot x}\overline{Y_{l_2}^{k_2}(d)}ds(d)\Big\|_{L^2} \\
&\quad \le M_3\|n - n_0\|_\infty\Big\|\int_{S^2} e^{i\kappa d\cdot x}\overline{Y_{l_2}^{k_2}(d)}ds(d)\Big\|_{L^2} .
\end{aligned}$$

Now, the term originating from the second integral in (2.29) can be bounded similarly to the preceding lemma and we arrive at

$$\begin{aligned}
&\Big|\int_{S^2} (u_{\infty,n}(\hat{x},d) - u_{\infty,n_0}(\hat{x},d))\overline{Y_{l_1}^{k_1}(\hat{x})Y_{l_2}^{k_2}(d)}ds(\hat{x})ds(d)\Big|^2 \\
&\quad \le M_4\|n - n_0\|_\infty^2\Big(\frac{e\kappa R_1}{2l_1 + 1}\Big)^{2l_1+3}\Big(\frac{e\kappa R_1}{2l_2 + 1}\Big)^{2l_2+3} .
\end{aligned}$$

This implies as in Lemma 2.17 that

$$\|u_{\infty,n} - u_{\infty,n_0}\|_{\mathcal{F}} \le M\|n - n_0\|_\infty$$

for $n \in \tilde{C}(B_{R_1})$ with $\|n - n_0\|_\infty < \epsilon$ and we have proved the lemma. □

Although we used the Green's function s_n during the motivation at the beginning, contrary to Stefanov we avoid working explicitly with it because then we circumvent proving its existence and its properties. We observe $s_1(x,y) = \Phi_\kappa(x,y)$ in the case $n = 1$. In view of the regularity properties and jump relations of the single-layer potential with kernel $\Phi_\kappa(x,y)$, and since we expect that the single-layer potential

$$2 \int_{\partial B_{R_2}} s_n(x,y) f(y) ds(y) \ , \ x \in \mathbb{R}^3 \ ,$$

satisfies analogous regularity properties and jump relations, we guess that it is a solution of the following boundary value problem (BVP):

Suppose $R_2 > R$. Given $\kappa > 0$, $n \in \tilde{C}(B_R)$ and $f \in C(\partial B_{R_2})$, find $u \in C(\mathbb{R}^3)$ such that u is C^2-smooth in B_{R_2} and in $\mathbb{R}^3 \setminus \overline{B_{R_2}}$, such that u satisfies the Sommerfeld radiation condition and such that u satisfies the following requirements:

$$\Delta u + \kappa^2 n u = 0 \ \text{ in } \mathbb{R}^3 \setminus \partial B_{R_2},$$

$$\frac{\partial u_+}{\partial \nu}(x) := \lim_{t \to 0, t > 0} \nu(x) \cdot \nabla u(x + t\nu(x))$$

and

$$\frac{\partial u_-}{\partial \nu}(x) := \lim_{t \to 0, t > 0} \nu(x) \cdot \nabla u(x - t\nu(x))$$

exist uniformly for $x \in \partial B_{R_2}$ and

$$\frac{\partial u_-}{\partial \nu} - \frac{\partial u_+}{\partial \nu} = 2f \ \text{ on } \partial B_{R_2}.$$

Here, ν denotes the unit normal vector on ∂B_{R_2} directed into the exterior of B_{R_2}.

Lemma 2.19 *For all $f \in C(\partial B_{R_2})$ the boundary value problem (BVP) has a unique solution u. u is given by*

$$u(x) := 2 \int_{\partial B_{R_2}} \Phi_\kappa(x,y) f(y) ds(y) - \kappa^2 \int_{B_R} (1 - n(y)) \Phi_\kappa(x,y) \varphi(y) dy \ , \ x \in \mathbb{R}^3 \ , \tag{2.30}$$

where $\varphi \in C(\overline{B_R})$ is the unique solution to the Lippmann-Schwinger equation

$$(\varphi + T_n \varphi)(x) = 2 \int_{\partial B_{R_2}} \Phi_\kappa(x,y) f(y) ds(y) \ , \ x \in \overline{B_R} \ .$$

Proof: In order to prove that (BVP) has at most one solution we assume that u is a solution of (BVP) with $f = 0$. Then we can follow the first part of the reasoning in Theorem 1.8 and we obtain $u = 0$ in the exterior of B_{R_2}, whence $u = \frac{\partial u_-}{\partial \nu} = 0$ on ∂B_{R_2}. Now, Green's representation formula (1.10) implies that u is a solution of the homogeneous Lippmann-Schwinger equation. Thus, we also have $u = 0$ in B_{R_2}.

It is immediately seen that u defined as in (2.30) is a solution to $\Delta u + \kappa^2 u = 0$ in the exterior of $\overline{B_{R_2}}$. Moreover, the properties of volume potentials (see Theorem 1.11) show $\varphi \in C^{0,\gamma}(\overline{B_R})$, whence $u \in C^2(B_{R_2})$ and

$$\begin{aligned}(\Delta u + \kappa^2 u)(x) &= \kappa^2(1 - n(x))\varphi(x) \\ &= \kappa^2(1 - n(x))\Big\{2 \int_{\partial B_{R_2}} \Phi_\kappa(x, y) f(y) ds(y) \\ &\qquad - \kappa^2 \int_{B_R} (1 - n(y))\Phi_\kappa(x, y)\varphi(y) dy\Big\} \\ &= \kappa^2(1 - n(x))u(x)\ ,\ x \in B_{R_2}\ .\end{aligned}$$

Finally, we can conclude from the regularity properties of the single-layer potential with kernel Φ_κ (see [6]) and of the volume potential (see Theorem 1.11) that u as defined in (2.30) satisfies the boundary conditions. Hence it is a solution of (BVP).

□

Now, we define the operator S_n by

$$S_n : C(\partial B_{R_2}) \to C(\partial B_{R_2}) \qquad (S_n f)(x) := u(x)\ ,\ x \in \partial B_{R_2}\ ,$$

with u being defined as in (2.30). Note, that S_1 is the single-layer operator defined in [6, 7].

We need some properties of S_n which we derive in the following lemma.

Lemma 2.20 *The linear operators S_n satisfy:*

(a) $S_n : C(\partial B_{R_2}) \to C(\partial B_{R_2})$ *and* $S_n : C^{0,\gamma}(\partial B_{R_2}) \to C^{1,\gamma}(\partial B_{R_2})$ *are bounded* $(0 < \gamma < 1)$.

(b) $\int_{\partial B_{R_2}} f(S_n g) ds = \int_{\partial B_{R_2}} (S_n f) g ds$ *for all* $f,\ g \in C(\partial B_{R_2})$.

(c) *The mapping* $n \mapsto S_n$, *from* $(\tilde{C}(B_R), \|\cdot\|_\infty)$ *to the space of linear and bounded operators in* $C(\partial B_{R_2})$ *equipped with the* $\|\cdot\|_\infty$*-operator norm, is continuous.*

Proof: From the continuity of $(I+T_n)^{-1}$ in $C(\overline{B_R})$ and from the properties of the single-layer operator S_1 we can conclude that S_n is a bounded linear operator in $C(\partial B_{R_2})$ and similarly a bounded operator from $C^{0,\gamma}(\partial B_{R_2})$ to $C^{1,\gamma}(\partial B_{R_2})$.

For assertion (b) we define u as in (2.30) and v analogously where we replace f by g. Then we compute

$$\begin{aligned} &\int\limits_{\partial B_{R_2}} \{f(S_n g) - (S_n f) g\} ds \\ &= \frac{1}{2} \int\limits_{\partial B_{R_2}} \Big\{ (\frac{\partial u_-}{\partial \nu} - \frac{\partial u_+}{\partial \nu})(S_n g) - (S_n f)(\frac{\partial v_-}{\partial \nu} - \frac{\partial v_+}{\partial \nu}) \Big\} ds \\ &= \frac{1}{2} \int\limits_{\partial B_{R_2}} \{ \frac{\partial u_-}{\partial \nu} v_- - u_- \frac{\partial v_-}{\partial \nu} \} ds - \frac{1}{2} \int\limits_{\partial B_{R_2}} \{ \frac{\partial u_+}{\partial \nu} v_+ - u_+ \frac{\partial v_+}{\partial \nu} \} ds \ . \end{aligned}$$

Since the first integral on the right hand side vanishes due to Green's second theorem and the second integral due to Lemma 1.4, we have proved part (b).

The proof of assertion (c) is very similar to the proof of Lemma 2.18. We fix $n_0 \in \tilde{C}(B_R)$, observe that the inequality $\|T_n - T_{n_0}\|_{\infty, B_R} \le M_1 \|n - n_0\|_\infty$ holds for all $n \in \tilde{C}(B_R)$ with a suitable constant M_1 and derive that $\|(I + T_n)^{-1}\|_{\infty, B_R}$ is uniformly bounded in a suitable set $\{n \in \tilde{C}(B_R) \colon \|n - n_0\|_\infty < \epsilon\}$. This yields the inequality

$$\|(I+T_n)^{-1} - (I + T_{n_0})^{-1}\|_\infty \le M_2 \|n - n_0\|_\infty$$

for all n from the above set.

Defining for $f \in C(\partial B_{R_2})$

$$w(x) := 2 \int\limits_{\partial B_{R_2}} \Phi_\kappa(x,y) f(y) ds(y) \ , \ x \in \overline{B_{R_2}} \ ,$$

we compute

$$\|S_n f - S_{n_0} f\|_{\infty, \partial B_{R_2}}$$

$$
\begin{aligned}
&= \kappa^2 \Big\| \int_{B_R} (1-n(y))\Phi_\kappa(\cdot,y)\Big[(I+T_n)^{-1}w\Big](y)dy \\
&\qquad - \int_{B_R} (1-n_0(y))\Phi_\kappa(\cdot,y)\Big[(I+T_{n_0})^{-1}w\Big](y)dy \Big\|_{\infty,\partial B_{R_2}} \\
&\le \kappa^2 \Big\| \int_{B_R} (1-n_0(y))\Phi_\kappa(\cdot,y)\Big[(I+T_n)^{-1}w-(I+T_{n_0})^{-1}w\Big](y)dy \Big\|_{\infty,\partial B_{R_2}} \\
&\qquad + \kappa^2 \Big\| \int_{B_R} (n(y)-n_0(y))\Phi_\kappa(\cdot,y)\Big[(I+T_n)^{-1}w\Big](y)dy \Big\|_{\infty,\partial B_{R_2}} \\
&\le M_3\|n-n_0\|_\infty \|w\|_{\infty,B_R} \\
&\le M_4\|n-n_0\|_\infty \|f\|_{\infty,\partial B_{R_2}}
\end{aligned}
$$

for all $\|n-n_0\|_\infty < \epsilon$, i.e., $\|S_n - S_{n_0}\|_\infty \le M_4\|n-n_0\|_\infty$, and we have proved the lemma.

□

Next, we study the relation between the Fourier coefficients $\mu_{l_1k_1l_2k_2}$ of the far field pattern $u_{\infty,n}$ and the Fourier coefficients of the far field pattern belonging to the function u defined by (2.30) for the special functions $f(x) = \overline{Y_{l_2}^{k_2}\left(\frac{x}{|x|}\right)}$, $x \in \partial B_{R_2}$. This allows to reconstruct S_n from $u_{\infty,n}$ and to derive continuous dependence of S_n on $u_{\infty,n}$.

Lemma 2.21 *Assume the far field pattern $u_{\infty,n}: S^2 \times S^2 \to \mathbb{C}$ originates from the refractive index $n \in \tilde{C}(B_R)$ and has the Fourier coefficients $\mu_{l_1k_1l_2k_2}$. Furthermore, define for x, $y \in \partial B_{R_2}$, $x \neq y$, the function*

$$
\begin{aligned}
s_n(x,y) &:= \Phi_\kappa(x,y) \\
&- \frac{\kappa^2}{4\pi} \sum_{l_1,k_1,l_2,k_2} i^{l_1-l_2}\mu_{l_1k_1l_2k_2} h_{l_1}^{(1)}(\kappa R_2) h_{l_2}^{(1)}(\kappa R_2) Y_{l_1}^{k_1}(\frac{x}{|x|}) Y_{l_2}^{k_2}(\frac{y}{|y|}) \ .
\end{aligned}
\tag{2.31}
$$

(a) For all $f \in C(\partial B_{R_2})$ there holds

$$
(S_n f)(x) = 2 \int_{\partial B_{R_2}} s_n(x,y)f(y)ds(y) \ , \ x \in \partial B_{R_2} \ .
$$

(b) *The mapping $u_{\infty,n} \mapsto S_n$ is continuous, to be more precise: there is a constant M such that $\|S_n - S_{\tilde{n}}\|_\infty \le M\|u_{\infty,n} - u_{\infty,\tilde{n}}\|_{\mathcal{F}}$ for all n, $\tilde{n} \in \tilde{C}(B_R)$.*

Proof: From [7, Theorem 2.8] we infer the estimate $\|Y_l^k\|_\infty^2 \le 2l+1$ for the spherical harmonics Y_l^k. Applying the Cauchy-Schwarz inequality to the series in (2.31) and using the rapid decay of the Fourier coefficients from Lemma 2.17 together with the estimate for $|h_l^{(1)}(\kappa R_2)|$ from Lemma 2.16 we see that the series is absolutely and uniformly convergent on $\partial B_{R_2} \times \partial B_{R_2}$. Hence it represents a continuous function there and s_n is well defined.

We prove assertion (a) for the special functions $f(y) = \overline{Y_{l_2}^{k_2}(\frac{y}{|y|})}$, $y \in \partial B_{R_2}$. Since the linear span of these functions is dense in $C(\partial B_{R_2})$ and since S_n and the integral operator with kernel s_n are bounded in $C(\partial B_{R_2})$, this suffices to prove part (a).

First, we compute for $x \in \partial B_{R_2}$

$$\begin{aligned} &2\int_{\partial B_{R_2}} s_n(x,y)\overline{Y_{l_2}^{k_2}(\frac{y}{|y|})}ds(y) \\ &= S_1(\overline{Y_{l_2}^{k_2}(\frac{\cdot}{|\cdot|})})(x) - \frac{\kappa^2 R_2^2}{2\pi}\sum_{l_1,k_1} i^{l_1-l_2}\mu_{l_1k_1l_2k_2}h_{l_1}^{(1)}(\kappa R_2)h_{l_2}^{(1)}(\kappa R_2)Y_{l_1}^{k_1}(\frac{x}{|x|})\ . \end{aligned}$$

Then we turn to $S_n(\overline{Y_{l_2}^{k_2}(\frac{\cdot}{|\cdot|})})$. To this end we define the functions w, φ, and u by

$$w(x) := 2\int_{\partial B_{R_2}} \Phi_\kappa(x,y)\overline{Y_{l_2}^{k_2}(\frac{y}{|y|})}ds(y)\ ,\ x \in \mathbb{R}^3\ ,$$

$$\varphi := (I+T_n)^{-1}w \ \text{ in } \overline{B_R}\ , \qquad \text{and}$$

$$u(x) := w(x) - \kappa^2\int_{B_R}(1-n(y))\Phi_\kappa(x,y)\varphi(y)dy\ ,\ x \in \mathbb{R}^3\ .$$

From [7, Theorem 2.10] together with the Funk-Hecke formula ([7, (2.44)]) for $|x| < R_2$ we obtain the relation

$$\begin{aligned} \int_{\partial B_{R_2}} \Phi_\kappa(x,y)\overline{Y_{l_2}^{k_2}(\frac{y}{|y|})}ds(y) &= i\kappa R_2^2 h_{l_2}^{(1)}(\kappa R_2)j_{l_2}(\kappa|x|)\overline{Y_{l_2}^{k_2}(\hat{x})} \\ &= i\kappa R_2^2 h_{l_2}^{(1)}(\kappa R_2)\frac{(-i)^{l_2}}{4\pi}\int_{S^2} e^{i\kappa x\cdot d}\overline{Y_{l_2}^{k_2}(d)}ds(d)\ . \end{aligned} \tag{2.32}$$

Similarly to the derivation of (2.28) we compute

$$\begin{aligned}
\varphi &= (I+T_n)^{-1}w \\
&= 2i\kappa R_2^2 h_{l_2}^{(1)}(\kappa R_2)\frac{(-i)^{l_2}}{4\pi}(I+T_n)^{-1}\Big(\int\limits_{S^2} e^{i\kappa d\cdot x}\overline{Y_{l_2}^{k_2}(d)}ds(d)\Big) \\
&= 2i\kappa R_2^2 h_{l_2}^{(1)}(\kappa R_2)\frac{(-i)^{l_2}}{4\pi}\int\limits_{S^2} u(\cdot,d)\overline{Y_{l_2}^{k_2}(d)}ds(d)\ .
\end{aligned}$$

Now, for the function $v := u - w$, the definition of u and the relation (2.24) yield the far field pattern

$$\begin{aligned}
v_\infty(\hat{x}) &= -\frac{\kappa^2}{4\pi}\int\limits_{B_R}(1-n(y))\varphi(y)e^{-i\kappa\hat{x}\cdot y}dy \\
&= 2i\kappa R_2^2 h_{l_2}^{(1)}(\kappa R_2)\frac{(-i)^{l_2}}{4\pi}\int\limits_{S^2} u_{\infty,n}(\hat{x},d)\overline{Y_{l_2}^{k_2}(d)}ds(d)\ ,
\end{aligned}$$

i.e.,

$$\int\limits_{S^2} v_\infty(\hat{x})\overline{Y_{l_1}^{k_1}(\hat{x})}ds(\hat{x}) = 2i\kappa R_2^2 h_{l_2}^{(1)}(\kappa R_2)\frac{(-i)^{l_2}}{4\pi}\mu_{l_1k_1l_2k_2}\ . \tag{2.33}$$

Since v is a radiating solution of the Helmholtz equation in the exterior of B_{R_1} if $\operatorname{supp}(1-n) \subset B_{R_1}$, $R_1 < R$, according to [7, Theorem 2.14] it has an expansion

$$v(x) = \sum_{l_1=0}^{\infty}\sum_{k_1=-l_1}^{l_1} a_{l_1k_1}h_{l_1}^{(1)}(\kappa|x|)Y_{l_1}^{k_1}(\hat{x})$$

which converges absolutely and uniformly on compact subsets of $\{|x| \geq R\}$. Therorem 2.15 in [7] states that the Fourier coefficients of the far field pattern of v satisfy

$$\int\limits_{S^2} v_\infty(\hat{x})\overline{Y_{l_1}^{k_1}(\hat{x})}ds(\hat{x}) = \frac{1}{\kappa i^{l_1+1}}a_{l_1k_1}\ .$$

Comparing with (2.33) we arrive at

$$a_{l_1k_1} = -2\kappa^2 R_2^2\frac{i^{l_1-l_2}}{4\pi}h_{l_2}^{(1)}(\kappa R_2)\mu_{l_1k_1l_2k_2}\ .$$

This implies for $|x| = R_2$

$$
\begin{aligned}
& u(x) - w(x) \\
&= -2\kappa^2 R_2^2 \sum_{l_1=0}^{\infty} \sum_{k_1=-l_1}^{l_1} \frac{i^{l_1-l_2}}{4\pi} \mu_{l_1k_1l_2k_2} h_{l_2}^{(1)}(\kappa R_2) h_{l_1}^{(1)}(\kappa R_2) Y_{l_1}^{k_1}(\frac{x}{|x|}) \ ,
\end{aligned}
$$

whence

$$
S_n(\overline{Y_{l_2}^{k_2}(\frac{\cdot}{|\cdot|})}) = u|_{\partial B_{R_2}} = 2 \int\limits_{\partial B_{R_2}} s_n(\cdot, y) \overline{Y_{l_2}^{k_2}(\frac{y}{|y|})} ds(y) \ .
$$

This completes the proof of assertion (a).

Finally, we can conclude for two refractive indices $n,\ \tilde{n} \in \tilde{C}(B_R)$ with Fourier coefficients $\mu_{l_1k_1l_2k_2},\ \tilde{\mu}_{l_1k_1l_2k_2}$:

$$
\begin{aligned}
& \|(S_n - S_{\tilde{n}})f\|_{\infty,\partial B_{R_2}}^2 \\
&= \Big\| \int\limits_{\partial B_{R_2}} (s_n(\cdot, y) - s_{\tilde{n}}(\cdot, y)) f(y) ds(y) \Big\|_{\infty,\partial B_{R_2}}^2 \\
&\leq M_1 \|f\|_\infty^2 \ \cdot \\
&\quad \cdot \Big(\sum_{l_1k_1l_2k_2} |\mu_{l_1k_1l_2k_2} - \tilde{\mu}_{l_1k_1l_2k_2}| |h_{l_2}^{(1)}(\kappa R_2) h_{l_1}^{(1)}(\kappa R_2)| \|Y_{l_1}^{k_1}\|_\infty \|Y_{l_2}^{k_2}\|_\infty \Big)^2 \\
&\leq M_2 \|f\|_\infty^2 \sum_{l_1k_1l_2k_2} \Big(\frac{2l_1+1}{e\kappa R}\Big)^{2l_1+3} \Big(\frac{2l_2+1}{e\kappa R}\Big)^{2l_2+3} |\mu_{l_1k_1l_2k_2} - \tilde{\mu}_{l_1k_1l_2k_2}|^2 \ \cdot \\
&\quad \cdot \sum_{l_1k_1l_2k_2} \Big(\frac{e\kappa R}{2l_1+1}\Big)^{2l_1+2} \Big(\frac{e\kappa R}{2l_2+1}\Big)^{2l_2+2} |h_{l_2}^{(1)}(\kappa R_2) h_{l_1}^{(1)}(\kappa R_2)|^2 \\
&\leq M_3 \|f\|_\infty^2 \|u_{\infty,n} - u_{\infty,\tilde{n}}\|_{\mathcal{F}}^2 \ .
\end{aligned}
$$

This ends the proof of the lemma.

□

Let us add a remark concerning the relation (2.31) between s_n and the Fourier coefficients of the far field pattern. In [33, (3.10)] Nachman derives for all $g \in L^2(S^2)$ and for all $\hat{x} \in S^2$ the relation

$$
\Big(\mathcal{F} S_n \mathcal{F}_-^* g\Big)(\hat{x}) = \frac{1}{2\pi} \int\limits_{S^2} \Big(u_{\infty,n}(\hat{x}, d) - u_{\infty,DP}(\hat{x}, d)\Big) g(d) ds(d) \ . \qquad (2.34)
$$

Here, $\mathcal{F}: C(\partial B_{R_2}) \to C(S^2)$, $f \mapsto u^s_\infty$, denotes the operator which maps the Dirichlet boundary values $f = u^s|_{\partial B_{R_2}}$ of a radiating solution u^s to the Helmholtz equation onto its far field pattern u^s_∞. If, for $d \in S^2$, $u_{DP}(\cdot, d) \in C^2(\mathbb{R}^3 \setminus \overline{B_{R_2}}) \cap C(\mathbb{R}^3 \setminus B_{R_2})$ satisfies $u_{DP}(y, d) = 0$, $y \in \partial B_{R_2}$, and $u_{DP}(\cdot, d) - u^i(\cdot, d)$ is a radiating solution to the Helmholtz equation, i.e., $u_{DP}(\cdot, d)$ is the total field to the exterior Dirichlet problem with incident wave $u^i(\cdot d)$, then $\mathcal{F}$ can be written as

$$(\mathcal{F}f)(\hat{x}) = \int\limits_{\partial B_{R_2}} \frac{\partial u_{DP}(y, -\hat{x})}{\partial \nu(y)} f(y) ds(y) \ , \ \hat{x} \in S^2 \ .$$

$\mathcal{F}^*_-: C(S^2) \to C(\partial B_{R_2})$ is defined by

$$(\mathcal{F}^*_- g)(y) = \int\limits_{\partial B_{R_2}} \frac{\partial u_{DP}(y, d)}{\partial \nu(y)} g(d) ds(d) \ , \ y \in \partial B_{R_2} \ .$$

Finally, $u_{\infty,DP}(\hat{x}, d)$ is the far field pattern which corresponds to the scattering at the obstacle B_{R_2} assuming Dirichlet boundary conditions on ∂B_{R_2}. Using the relations

$$\mathcal{F}\Big(Y_l^k\Big(\frac{\cdot}{|\cdot|}\Big)\Big) = \frac{1}{\kappa i^{l+1} h_l^{(1)}(\kappa R_2)} Y_l^k \ , \ \mathcal{F}^*_-(Y_l^k) = \frac{(-1)^l}{R_2^2 \kappa i^{l+1} h_l^{(1)}(\kappa R_2)} Y_l^k\Big(\frac{\cdot}{|\cdot|}\Big) \ ,$$

it can be seen that (2.31) implies (2.34) and *vice versa*. Nachman's relation (2.34) is also true, if the ball B_{R_2} is replaced by a different obstacle, whereas a definition of s_n in the spirit of (2.31) is not possible on an arbitrary surface surrounding the support of $(1-n)$ because it is not at all clear whether the series is convergent on such a surface.

Let us now turn to the third step of the stability result, namely the continuous dependence of n on S_n and the continuous dependence of n on $u_{\infty,n}$. The idea is to estimate the Fourier coefficients $|(n-\tilde{n})\hat{}(\alpha)|$ in order to bound $\|n - \tilde{n}\|_\infty$. From the proof of the Uniqueness Theorem 2.10 we know that $(n - \tilde{n})\hat{}(\alpha)$ can be computed with the help of special solutions u, $\tilde{u}$ to the perturbed Helmholtz equation and with the help of the integrals

$$\int\limits_{B_R} (n - \tilde{n}) u \tilde{u} dx \ .$$

Hence, we need a relation connecting these integrals and $\|S_n - S_{\tilde{n}}\|_{\infty,\partial B_{R_2}}$ to derive the dependence of $\|n-\tilde{n}\|_\infty$ on $\|S_n - S_{\tilde{n}}\|_{\infty,\partial B_{R_2}}$. We obtain this relation in the following lemma in which we employ the operator $K\colon C(\partial B_{R_2}) \to C(\partial B_{R_2})$ which is defined by

$$(K\varphi)(x) := 2 \int\limits_{\partial B_{R_2}} \frac{\partial \Phi_\kappa(x,y)}{\partial \nu(y)} \varphi(y) ds(y) \ , \ x \in \partial B_{R_2} \ .$$

Lemma 2.22 *Assume $R < R_2 < R''$ and $c_1 > 0$ are positive constants. Then, there exists a positive constant c such that for all n, $\tilde{n} \in \tilde{C}(B_R)$ with $\|n\|_\infty$, $\|\tilde{n}\|_\infty \le c_1$, and for all solutions u, $\tilde{u} \in C^2(B_{R''}) \cap L^2(B_{R''})$ to $\Delta u + \kappa^2 n u = 0$, $\Delta \tilde{u} + \kappa^2 \tilde{n} \tilde{u} = 0$ in $B_{R''}$ the estimate*

$$\Big| \int\limits_{B_R} (n-\tilde{n}) u \tilde{u} dx \Big| \le c \|S_n - S_{\tilde{n}}\|_{\infty,\partial B_{R_2}} \|u\|_{L^2(B_{R''})} \|\tilde{u}\|_{L^2(B_{R''})} \qquad (2.35)$$

holds true.

Proof: We first extend u outside of $\overline{B_{R_2}}$ to a radiating solution w to the Helmholtz equation with $u|_{\partial B_{R_2}} = w|_{\partial B_{R_2}}$ on ∂B_{R_2}. Then, $u|_{B_{R_2}}$ together with w is a solution of (BVP) with a certain f. This allows to connect $u|_{\partial B_{R_2}}$ and the operator S_n. We define

$$w(x) := \begin{cases} u(x) \ , & x \in \overline{B_{R_2}} \\ 2 \int\limits_{\partial B_{R_2}} \Big(\frac{\partial \Phi_\kappa(x,y)}{\partial \nu(y)} + i \Phi_\kappa(x,y) \Big) \varphi(y) ds(y) \ , & x \in \mathbb{R}^3 \setminus \overline{B_{R_2}} \ , \end{cases}$$

with $\varphi = (I + K + iS_1)^{-1} u|_{\partial B_{R_2}}$. The existence of $(I+K+iS_1)^{-1}$ is proved in [7, p. 47]. The jump relations and regularity properties of surface layers imply $w \in C(\mathbb{R}^3)$, $w \in C^2(B_{R_2}) \cap C^1(\overline{B_{R_2}})$ and $w \in C^2(\mathbb{R}^3 \setminus \overline{B_{R_2}}) \cap C^1(\mathbb{R}^3 \setminus B_{R_2})$. Furthermore, w satisfies the Sommerfeld radiation condition. Hence, we know from Lemma 2.19 that

$$w = \frac{1}{2} S_n \Big(\frac{\partial w_-}{\partial \nu} - \frac{\partial w_+}{\partial \nu} \Big)$$

on ∂B_{R_2}. Finally, we can conclude from the regularity properties of surface potentials and from Lemma 2.6 (b) that there are constants c_2 and c_3, independent of u and n, such that

$$\Big\| \frac{\partial w_-}{\partial \nu} - \frac{\partial w_+}{\partial \nu} \Big\|_{\infty,\partial B_{R_2}} \le c_2 \|u\|_{1,\gamma,\overline{B_{R_2}}} \le c_3 \|u\|_{L^2(B_{R''})} \ . \qquad (2.36)$$

We can proceed analogously and define a function $\tilde{w}$ for $\tilde{u} \in C^2(B_{R''}) \cap L^2(B_{R''})$.

Then, we use Lemma 2.20 (b) and Green's theorem to compute

$$
\begin{aligned}
&\frac{1}{2} \int\limits_{\partial B_{R_2}} \Big(\frac{\partial w_-}{\partial \nu} - \frac{\partial w_+}{\partial \nu}\Big)(S_n - S_{\tilde{n}})\Big(\frac{\partial \tilde{w}_-}{\partial \nu} - \frac{\partial \tilde{w}_+}{\partial \nu}\Big) ds \\
&= \frac{1}{2} \int\limits_{\partial B_{R_2}} \Big[S_n(\frac{\partial w_-}{\partial \nu} - \frac{\partial w_+}{\partial \nu})\Big]\Big(\frac{\partial \tilde{w}_-}{\partial \nu} - \frac{\partial \tilde{w}_+}{\partial \nu}\Big) ds \\
&\qquad -\frac{1}{2} \int\limits_{\partial B_{R_2}} (\frac{\partial w_-}{\partial \nu} - \frac{\partial w_+}{\partial \nu})\Big[S_{\tilde{n}}(\frac{\partial \tilde{w}_-}{\partial \nu} - \frac{\partial \tilde{w}_+}{\partial \nu})\Big] ds \\
&= \int\limits_{\partial B_{R_2}} \Big\{w\Big(\frac{\partial \tilde{w}_-}{\partial \nu} - \frac{\partial \tilde{w}_+}{\partial \nu}\Big) - \tilde{w}(\frac{\partial w_-}{\partial \nu} - \frac{\partial w_+}{\partial \nu})\Big\} ds \\
&= \int\limits_{\partial B_{R_2}} \{w\frac{\partial \tilde{w}_-}{\partial \nu} - \tilde{w}\frac{\partial w_-}{\partial \nu}\} ds \\
&= \int\limits_{B_{R_2}} \{w\Delta\tilde{w} - \tilde{w}\Delta w\} dx \\
&= \kappa^2 \int\limits_{B_R} (n - \tilde{n}) u\tilde{u} dx \ , \qquad (2.37)
\end{aligned}
$$

where the terms

$$\int\limits_{\partial B_{R_2}} \{w\frac{\partial \tilde{w}_+}{\partial \nu} - \tilde{w}\frac{\partial w_+}{\partial \nu}\} ds = 0$$

vanish in the fifth line because w and $\tilde{w}$ are radiating solutions to the Helmholtz equation in the exterior of B_{R_2}.

Finally, we conclude from (2.36) and (2.37)

$$
\begin{aligned}
\Big|\int\limits_{B_R} (n - \tilde{n}) u\tilde{u} dx\Big| &= \Big|\frac{1}{2\kappa^2} \int\limits_{\partial B_{R_2}} \Big(\frac{\partial w_-}{\partial \nu} - \frac{\partial w_+}{\partial \nu}\Big)(S_n - S_{\tilde{n}})\Big(\frac{\partial \tilde{w}_-}{\partial \nu} - \frac{\partial \tilde{w}_+}{\partial \nu}\Big) ds\Big| \\
&\leq c\|S_n - S_{\tilde{n}}\|_{\infty,\partial B_{R_2}} \|u\|_{L^2(B_{R''})} \|\tilde{u}\|_{L^2(B_{R''})} \ ,
\end{aligned}
$$

and we have proved the lemma.

□

We are now in a position to prove the desired stability results. By

$$\|\varphi\|_{C^2} := \|\varphi\|_{\infty,B_R} + \sup_{x\in B_R} |\nabla\varphi(x)| + \max_{j,k} \sup_{x\in B_R} |\partial_j\partial_k\varphi(x)|$$

we denote the C^2-maximum norm of a function φ in B_R having bounded derivatives up to order two.

Theorem 2.23 *Let $n_0 \in \tilde{C}(B_R) \cap C^2(B_R)$ be given. Then, there are a neighborhood $\mathcal{O}$ of n_0 of the form*

$$\mathcal{O} := \{n \in \tilde{C}(B_R) \cap C^2(B_R) \colon \|n - n_0\|_{C^2} < \epsilon\} ,$$

and a positive constant c, such that for all n, $\tilde{n} \in \mathcal{O}$ the estimate

$$\|n - \tilde{n}\|_{\infty,B_R} \le c[-\ln(\|S_n - S_{\tilde{n}}\|_{\infty,\partial B_{R_2}})]^{-1/7}$$

holds true.

Proof: The main idea is to use the previous lemma and the special solutions $u(\cdot,\zeta)$, $\tilde{u}(\cdot,\tilde{\zeta})$ from the uniqueness proof of the preceding section to derive estimates for the Fourier coefficients $(n-\tilde{n})\hat{}(\alpha)$, $\alpha \in \Gamma$.

Assume $R < R_2 < R'' < R' < 2R_2$. We have for n, $\tilde{n} \in \tilde{C}(B_R) \cap C^2(B_R)$ and any $\rho \ge 2$

$$\begin{aligned} \|n-\tilde{n}\|_\infty &= \Big\|\sum_{\alpha\in\Gamma} (n-\tilde{n})\hat{}(\alpha) e_\alpha\Big\|_\infty \\ &\le (2R')^{-3/2} \sum_{\alpha\cdot\alpha\le\rho^2} |(n-\tilde{n})\hat{}(\alpha)| + (2R')^{-3/2} \sum_{\alpha\cdot\alpha>\rho^2} |(n-\tilde{n})\hat{}(\alpha)| \, . \end{aligned} \tag{2.38}$$

The Cauchy-Schwarz inequality implies

$$\begin{aligned} &\sum_{\alpha\cdot\alpha>\rho^2} |(n-\tilde{n})\hat{}(\alpha)| \\ &\le \Big(\sum_{\alpha\cdot\alpha>\rho^2} (1+\alpha\cdot\alpha)^2 |(n-\tilde{n})\hat{}(\alpha)|^2\Big)^{1/2} \Big(\sum_{\alpha\cdot\alpha>\rho^2} \frac{1}{(1+\alpha\cdot\alpha)^2}\Big)^{1/2} \\ &\le \frac{M}{\sqrt{\rho}} \, . \end{aligned} \tag{2.39}$$

Here, with the help of Parseval's relation (1.2) we have bounded the first factor by $\|(-\Delta+1)(\tilde{n}-n)\|_{L^2(C)}$ which in turn can be estimated by a constant which is valid for all n, $\tilde{n}$ with $\|n-n_0\|_{C^2} \leq 1$ and $\|\tilde{n}-n_0\|_{C^2} \leq 1$. For the second we have employed the inequality

$$\sum_{\alpha\cdot\alpha>\rho^2} \frac{1}{(1+\alpha\cdot\alpha)^2} \leq \sum_{k>\rho} \sum_{(k-1)^2<\alpha\cdot\alpha\leq k^2} \frac{1}{(1+(k-1)^2)^2} \leq M \sum_{\rho<k} \frac{1}{k^2} \leq \frac{2M}{\rho}$$

(see the proof of Lemma 2.13 (b)). Note, that we use the same letter M for different constants during the proof.

Now, we turn to the first sum $\sum\limits_{\alpha\cdot\alpha\leq\rho^2}$ in (2.38). We will insert the solutions $u(\cdot,\zeta_t)$ and $\tilde{u}(\cdot,\tilde{\zeta}_t)$ from Theorem 2.10 into (2.35) in order to estimate $|(n-\tilde{n})\hat{\ }(\alpha)|$.

To this end we choose $t_0 := 2\kappa^2(R'/\pi)\{\|1-n_0\|_\infty + 1\} + 2\kappa + 20$ and $0 < \epsilon_1 < 1/2$ sufficiently small to ensure $(-3\ln(2\epsilon_1))/(7(4R_2+1)) > t_0$. Due to the continuous dependence of S_n on n (Lemma 2.20 (c)) we can find ϵ with $0 < \epsilon < \epsilon_1$ such that

$$\|S_n - S_{\tilde{n}}\|_{\infty,\partial B_{R_2}} \leq \|S_n - S_{n_0}\|_{\infty,\partial B_{R_2}} + \|S_{\tilde{n}} - S_{n_0}\|_{\infty,\partial B_{R_2}} \leq 2\epsilon_1$$

for all

$$n,\tilde{n} \in \mathcal{O} := \{n \in \tilde{C}(B_R) \cap C^2(B_R) \colon \|n-n_0\|_{C^2} < \epsilon\}\ .$$

For a vector $\alpha \in \Gamma$ with $\alpha\cdot\alpha \leq \rho^2$ and a real number $t \geq t_0$, we choose as in Theorem 2.10

$$\begin{aligned} \zeta_t &:= -\frac{1}{2}\alpha + i\sqrt{t^2-\kappa^2+\frac{|\alpha|^2}{4}}d_1 + td_2\ , \\ \tilde{\zeta}_t &:= -\frac{1}{2}\alpha - i\sqrt{t^2-\kappa^2+\frac{|\alpha|^2}{4}}d_1 - td_2\ . \end{aligned}$$

Then, we have for all $n \in \mathcal{O}$ that $|\Im(\zeta_t)| \geq t-\kappa \geq t/2$ and $|\Im(\zeta_t)| \geq t-\kappa \geq 2\kappa^2(R'/\pi)\|1-n\|_\infty + 1$, whence by Lemma 2.9 there exist the special solutions $u(x,\zeta_t) = e^{i\zeta_t\cdot x}(1+v(x,\zeta_t))$ and the $L^2(B_{R''})$-norms of the functions $v(\cdot,\zeta_t)$ can be bounded by M/t uniformly in $n\in\mathcal{O}$, $t \geq t_0$ and $\alpha \in \Gamma$. The analogous assertions apply to $\tilde{v}(\cdot,\tilde{\zeta}_t)$ and $\tilde{u}(x,\tilde{\zeta}_t) = e^{i\tilde{\zeta}_t\cdot x}(1+\tilde{v}(x,\tilde{\zeta}_t))$.

Using (2.35) we estimate

$$|(\tilde{n}-n)\hat{\ }(\alpha)|$$

$$
\begin{aligned}
&= (2R')^{-3/2}\Big|\int_C (\tilde n - n)(x) e^{-i\alpha\cdot x} dx\Big| \\
&= (2R')^{-3/2}\Big|\int_{B_{R_2}} (\tilde n - n)(x) u(x,\zeta_t)\tilde u(x,\tilde\zeta_t) dx \\
&\quad - \int_{B_{R_2}} (\tilde n - n)(x) e^{-i\alpha\cdot x}(v(x,\zeta_t) + \tilde v(x,\tilde\zeta_t) + v(x,\zeta_t)\tilde v(x,\tilde\zeta_t)) dx\Big| \\
&\leq M\|S_n - S_{\tilde n}\|_\infty \|u(\cdot,\zeta_t)\|_{L^2(B_{R''})}\|\tilde u(\cdot,\tilde\zeta_t)\|_{L^2(B_{R''})} + \frac{M}{t} \\
&\leq M(e^{4R_2(t+|\alpha|)}\|S_n - S_{\tilde n}\|_\infty + \frac{1}{t}) \ , \qquad (2.40)
\end{aligned}
$$

where we have used the fact that

$$
\begin{aligned}
\|u(\cdot,\zeta_t)\|_{L^2(B_{R''})} &= \|e^{i\zeta_t\cdot x}(1 + v(\cdot,\zeta_t))\|_{L^2(B_{R''})} \\
&\leq \|e^{i\zeta_t\cdot x}\|_{\infty, B_{2R_2}}\|1 + v(\cdot,\zeta_t)\|_{L^2(B_{R''})} \\
&\leq Me^{2R_2(t+|\alpha|)}
\end{aligned}
$$

for all $t \geq t_0$, $n \in \mathcal{O}$, and $\alpha \in \Gamma$, since $|\Im(\zeta_t)| \leq t + |\alpha|$.

Inequality (2.40) implies

$$
\begin{aligned}
\sum_{\alpha\cdot\alpha\leq\rho^2} |(\tilde n - n)\hat{}(\alpha)| &\leq M \sum_{\alpha\cdot\alpha\leq\rho^2} (e^{4R_2(t+|\alpha|)}\|S_n - S_{\tilde n}\|_\infty + \frac{1}{t}) \\
&\leq M\{e^{4R_2 t}e^{4R_2\rho}\rho^3\|S_n - S_{\tilde n}\|_\infty + \frac{\rho^3}{t}\} \\
&\leq M\{e^{(4R_2+1)(t+\rho)}\|S_n - S_{\tilde n}\|_\infty + \frac{\rho^3}{t}\} \ ,
\end{aligned}
$$

because of $\rho^3 \leq 6e^\rho$. If we fix $\rho := t^{2/7}$, for $t \geq t_0$ our condition $\rho \geq 2$ is satisfied and we obtain from (2.38), (2.39) and our last estimate

$$
\begin{aligned}
\|n - \tilde n\|_\infty &\leq M\{e^{(4R_2+1)(t+\rho)}\|S_n - S_{\tilde n}\|_\infty + \frac{\rho^3}{t} + \frac{1}{\sqrt{\rho}}\} \\
&\leq M\{e^{(8R_2+2)t}\|S_n - S_{\tilde n}\|_\infty + \frac{2}{t^{1/7}}\} \ , \qquad (2.41)
\end{aligned}
$$

since $\rho = t^{2/7} \leq t$.

Finally, we choose $t := -\frac{3}{7(4R_2+1)} \ln \|S_n - S_{\tilde{n}}\|_\infty$. Then, the inequalities $\|S_n - S_{\tilde{n}}\|_\infty < 1$ and $t \geq t_0$ are satisfied for all n, $\tilde{n} \in \mathcal{O}$ by the definition of ϵ and inequality (2.41) reads

$$\begin{aligned} \|n - \tilde{n}\|_\infty &\leq M\Big\{(\|S_n - S_{\tilde{n}}\|_\infty)^{1/7} + (-\ln \|S_n - S_{\tilde{n}}\|_\infty)^{-1/7}\Big\} \\ &\leq c(-\ln \|S_n - S_{\tilde{n}}\|_\infty)^{-1/7} \end{aligned}$$

for all n, $\tilde{n} \in \mathcal{O}$ because

$$-\ln(x) = \int_x^1 \frac{dy}{y} \leq \frac{1}{x}(1-x) \leq \frac{1}{x}$$

implies $x \leq (-\ln(x))^{-1}$ for $0 < x < 1$, and we have proved the theorem.

□

Theorem 2.24 *Let $n_0 \in \tilde{C}(B_{R_1}) \cap C^2(B_R)$ with $R_1 < R$ be given. Then, there are a neighborhood $\mathcal{O}$ of n_0 of the form*

$$\mathcal{O} := \{n \in \tilde{C}(B_{R_1}) \cap C^2(B_R) \colon \|n - n_0\|_{C^2} < \epsilon\} \ ,$$

and a positive constant c, such that for all n, $\tilde{n} \in \mathcal{O}$ the estimate

$$\|n - \tilde{n}\|_{\infty, B_R} \leq c[-\ln(\|u_{\infty,n} - u_{\infty,\tilde{n}}\|_{\mathcal{F}})]^{-1/7}$$

holds true.

Proof: We know from Lemma 2.18 that the mapping $n \mapsto u_{\infty,n}$ is continuous from $\tilde{C}(B_{R_1})$ to the far field patterns equipped with the norm $\|\cdot\|_{\mathcal{F}}$. Then, in the proof of Theorem 2.23 we can choose $\epsilon > 0$ sufficiently small to satisfy the additional requirements

$$(1+M)\|u_{\infty,n} - u_{\infty,\tilde{n}}\|_{\mathcal{F}} \leq 2\epsilon_1 \quad \text{and} \quad M\|u_{\infty,n} - u_{\infty,\tilde{n}}\|_{\mathcal{F}} \leq \|u_{\infty,n} - u_{\infty,\tilde{n}}\|_{\mathcal{F}}^{1/2}$$

for all n, $\tilde{n} \in \mathcal{O}$, too, where M denotes the constant from Lemma 2.21 (b). Inserting the estimate

$$\|S_n - S_{\tilde{n}}\|_\infty \leq M\|u_{\infty,n} - u_{\infty,\tilde{n}}\|_{\mathcal{F}}$$

from Lemma 2.21 (b) into Theorem 2.23 we arrive at the assertion of the theorem.

□

2.3 The Reconstruction of the Refractive Index

This section is devoted to a procedure to reconstruct the refractive index n from $u_{\infty,n}$, to be more precise: we compute the Fourier coefficients $(n-1)\hat{}(\alpha)$, $\alpha \in \Gamma$.

In the first step (Theorem 2.27) we prove that the Robin-to-Dirichlet map $\Lambda_n\colon C^{0,\gamma}(\partial B_{R_2}) \to C^{1,\gamma}(\partial B_{R_2})$, given by $f = \partial u/\partial\nu - iu \mapsto u|_{\partial B_{R_2}}$ where u is a solution to $\Delta u + \kappa^2 nu = 0$ in B_{R_2}, is well defined, and we examine how it can be computed from $u_{\infty,n}$, i.e., from the Fourier coefficients $\mu_{l_1k_1l_2k_2}$ of $u_{\infty,n}$. This can be established by essentially constructing the Green's function s_n for the differential equation $\Delta u + \kappa^2 nu = f$ and using a single-layer approach with kernel s_n. We used a similar approach when we examined the stability in the preceding section. However, we do not only need the operator S_n but also the operator K_n' which arises from the normal derivative of the single-layer potential with kernel s_n. We study its properties in Lemma 2.25. As in the previous section we avoid working with s_n directly. We also avoid working with the Dirichlet-to-Neumann map as in [32] because then we would have to worry about Dirichlet eigenvalues. Since the Robin problem has no real eigenvalues (Lemma 2.26), it is more suitable for our purposes. Nevertheless, this whole section is largely influenced by Nachman's paper [32].

In the second step we derive a uniquely solvable equation of the form

$$(I - A_{n,\zeta})(\frac{\partial u}{\partial\nu}(\cdot,\zeta) - iu(\cdot,\zeta)) = \frac{\partial e^{i\zeta\cdot\cdot}}{\partial\nu} - ie^{i\zeta\cdot\cdot} \quad \text{on } \partial B_{R_2}$$

for the Robin data $\partial u(\cdot,\zeta)/\partial\nu - iu(\cdot,\zeta)$ of the special solutions $u(\cdot,\zeta)$ which we constructed in Lemma 2.9. Here, $A_{n,\zeta}$ is a compact operator in $C^{0,\gamma}(\partial B_{R_2})$ which is composed of Λ_n and integral operators having kernels originating from the unphysical fundamental solution $\Psi_\zeta := e^{i\kappa|\cdot|}/(4\pi|\cdot|) + \tilde{g}_\zeta$. Ψ_ζ already occured in section 2.1 after Lemma 2.9 and is the composition of a Fourier series, a unitary transformation and a multiplication by $e^{i\zeta\cdot x}$, whence it is known. This means that $A_{n,\zeta}$ can be computed from $u_{\infty,n}$ and ζ. In a nutshell, given $u_{\infty,n}$, we can compute the Robin data $\partial u(\cdot,\zeta)/\partial\nu - iu(\cdot,\zeta)$ and the Dirichlet data $u(\cdot,\zeta) = \Lambda_n(\partial u(\cdot,\zeta)/\partial\nu - iu(\cdot,\zeta))$ on ∂B_{R_2}.

In the last step we apply Green's second theorem in B_{R_2} with the functions $e^{i\tilde{\zeta}\cdot x}$ and $u(x,\zeta) = e^{i\zeta\cdot x}(1 + v(x,\zeta))$ from Lemma 2.9. Here, $\zeta = \zeta_t$ and $\tilde{\zeta} = \tilde{\zeta}_t$ are chosen as in the Uniqueness Theorem 2.10 for a fixed $\alpha \in \Gamma$. We then obtain

$$\begin{aligned}
&\int\limits_{\partial B_{R_2}} \Big\{ \big(\frac{\partial e^{i\tilde{\zeta}\cdot x}}{\partial \nu} - i e^{i\tilde{\zeta}\cdot x}\big)[\Lambda_n\big(\frac{\partial u(\cdot,\zeta)}{\partial \nu} - iu(\cdot,\zeta)\big)](x) \\
&\qquad\qquad - e^{i\tilde{\zeta}\cdot x}\big(\frac{\partial u(x,\zeta)}{\partial \nu} - iu(x,\zeta)\big)\Big\} ds(x) \\
= &\int\limits_{\partial B_{R_2}} \Big\{ \frac{\partial e^{i\tilde{\zeta}\cdot x}}{\partial \nu} u(x,\zeta) - e^{i\tilde{\zeta}\cdot x}\frac{\partial u(x,\zeta)}{\partial \nu}\Big\} ds(x) \\
= &\ \kappa^2 \int\limits_{B_R} (n(x)-1) e^{i\tilde{\zeta}\cdot x} u(x,\zeta)\, dx \\
= &\ \kappa^2 \int\limits_{B_R} (n(x)-1) e^{-i\alpha\cdot x}(1 + v(x,\zeta))\, dx \ . \qquad (2.42)
\end{aligned}$$

The left hand side of this equation can be computed from $u_{\infty,n}$ according to our preceding considerations. The right hand side converges to $\kappa^2(2R')^{3/2}(n-1)\hat{\ }(\alpha)$ for $t \to \infty$, i.e., $|\Im(\zeta)| \to \infty$, because $\|v(\cdot,\zeta)\|_{L^2} \to 0$ for $|\Im(\zeta)| \to \infty$ by Lemma 2.9. Finally, having calculated all Fourier coefficients $(n-1)\hat{\ }(\alpha)$, $\alpha \in \Gamma$, we obtain n in B_{R_2} as the L^2-limit of the series

$$n = 1 + \sum_{\alpha\in\Gamma} (n-1)\hat{\ }(\alpha) e_\alpha \ .$$

After this outline let us start with the operator Λ_n. In order to motivate the following analysis we briefly sketch how to analyze Λ_1. In the case $n = 1$ uniqueness of a solution to the Robin boundary value problem $\Delta u + \kappa^2 u = 0$ in B_{R_2}, $\partial u/\partial\nu - iu = f$ on ∂B_{R_2}, can be easily inferred from Green's first theorem. Existence can be established with the help of a single-layer ansatz

$$u(x) = 2 \int\limits_{\partial B_{R_2}} \Phi_\kappa(x,y)\psi(y)\, ds(y) \ , \ x \in B_{R_2} \ .$$

If ψ is a solution to $(I + K_1' - iS_1)\psi = f$, then u defined as above solves the Robin problem. Here, $K_1'\colon C^{0,\gamma}(\partial B_{R_2}) \to C^{0,\gamma}(\partial B_{R_2})$ denotes the integral operator

$$(K_1'\psi)(x) = 2 \int\limits_{\partial B_{R_2}} \frac{\partial \Phi_\kappa(x,y)}{\partial \nu(x)} \psi(y)\, ds(y) \ , \ x \in \partial B_{R_2} \ . \qquad (2.43)$$

The existence of a solution to this integral equation is then derived via the Riesz theory and one arrives at $\Lambda_1 = S_1(I + K_1' - iS_1)^{-1}$.

For arbitrary n we want to replace the Green's function Φ_κ by s_n and use the same procedure to establish the existence of a solution. Hence we need the boundary integral operators corresponding to S_1 and K_1'. In the preceding section we have defined the operator S_n with the help of the functions u given by (2.30). Lemma 2.21 states that S_n can be computed from $u_{\infty,n}$. We would like to do the same for the operator K_n', i.e., for the operator having the normal derivative $\tilde{k}_n(x,y) := \partial s_n(x,y)/\partial\nu(x)$ of the Green's function s_n as kernel. To this end, analogously to (2.30), we define for $f \in C^{0,\gamma}(\partial B_{R_2})$:

$$(K_n' f)(x) \quad := \quad (K_1' f)(x) - \kappa^2 \int\limits_{B_R} \frac{\partial \Phi_\kappa(x,y)}{\partial \nu(x)} (1 - n(y))\varphi(y) dy \ , \ x \in \partial B_{R_2} \ , \tag{2.44}$$

where $\varphi \in C(\overline{B_R})$ is the unique solution to the Lippmann-Schwinger equation

$$(\varphi + T_n\varphi)(x) = 2 \int\limits_{\partial B_{R_2}} \Phi_\kappa(x,y) f(y) ds(y) \ , \ x \in \overline{B_R} \ .$$

Note, that if u is defined as in (2.30), that is

$$u(x) = 2 \int\limits_{\partial B_{R_2}} \Phi_\kappa(x,y) f(y) ds(y) - \kappa^2 \int\limits_{B_R} \Phi_\kappa(x,y)(1 - n(y))\varphi(y) dy \ , \ x \in \mathbb{R}^3 \ , \tag{2.45}$$

then we can infer from the properties of the single-layer potential with kernel Φ_κ:

$u|_{\overline{B_{R_2}}} \in C^{1,\gamma}(\overline{B_{R_2}})$, $u|_{\mathbb{R}^3 \setminus B_{R_2}} \in C^{1,\gamma}(\mathbb{R}^3 \setminus B_{R_2})$,

and $\partial u_-/\partial\nu = K_n' f + f$, $\partial u_+/\partial\nu = K_n' f - f$.

In the following lemma we show that K_n' is compact and how it can be computed from a knowledge of $u_{\infty,n}$.

Lemma 2.25 *Assume the far field pattern $u_{\infty,n}\colon S^2 \times S^2 \to \mathbb{C}$ originates from the refractive index $n \in \tilde{C}(B_R)$ and has the Fourier coefficients $\mu_{l_1 k_1 l_2 k_2}$.*

Furthermore, define for $x,\, y \in \partial B_{R_2}$, $x \neq y$, the function

$$\tilde{k}_n(x,y) := \frac{\partial \Phi_\kappa(x,y)}{\partial \nu(x)} - \frac{\kappa^3}{4\pi} \sum_{l_1,k_1,l_2,k_2} i^{l_1-l_2} \mu_{l_1 k_1 l_2 k_2} \Big(\frac{dh_{l_1}^{(1)}}{dt}\Big)(\kappa R_2) h_{l_2}^{(1)}(\kappa R_2) Y_{l_1}^{k_1}(\frac{x}{|x|}) Y_{l_2}^{k_2}(\frac{y}{|y|}) \ . \tag{2.46}$$

(a) $K_n' \colon C^{0,\gamma}(\partial B_{R_2}) \to C^{0,\gamma}(\partial B_{R_2})$ is compact $(0 < \gamma < 1)$.

(b) For all $\psi \in C(\partial B_{R_2})$ there holds

$$(K_n'\psi)(x) = 2 \int_{\partial B_{R_2}} \tilde{k}_n(x,y)\psi(y) ds(y) \ , \ x \in \partial B_{R_2} \ .$$

Proof: As in Lemma 2.21 we can conclude that the series in the definition of $\tilde{k}_n$ is absolutely and uniformly convergent on $\partial B_{R_2} \times \partial B_{R_2}$, whence $\tilde{k}_n$ is well defined.

For assertion (a) we observe that the mapping

$$\psi \mapsto w := 2 \int_{\partial B_{R_2}} \Phi_\kappa(\cdot, y)\psi(y) ds(y)$$

from $C^{0,\gamma}(\partial B_{R_2})$ to $C(\overline{B_R})$ is compact due to the properties of the single-layer with kernel Φ_κ. Taking into account the continuity of $(I + T_n)^{-1}$ in $C(\overline{B_R})$ and the regularity of the volume potential we obtain that the mapping

$$\psi \mapsto -\kappa^2 \int_{B_R} \frac{\partial \Phi_\kappa(\cdot, y)}{\partial \nu(\cdot)} (1 - n(y))[(I + T_n)^{-1} w](y) dy$$

is compact in $C^{0,\gamma}(\partial B_{R_2})$. Together with the compactness of K_1' this proves part (a).

The proof of part (b) closely follows the proof of Lemma 2.21 (a). It suffices to show the assertion for the functions $\psi(y) = \overline{Y_{l_2}^{k_2}(\frac{y}{|y|})}$, $y \in \partial B_{R_2}$. We compute

$$2 \int_{\partial B_{R_2}} \tilde{k}_n(x,y) \overline{Y_{l_2}^{k_2}(\frac{y}{|y|})} ds(y) = K_1'(\overline{Y_{l_2}^{k_2}(\frac{\cdot}{|\cdot|})})(x) - \frac{\kappa^3 R_2^2}{2\pi} \sum_{l_1,k_1} i^{l_1-l_2} \mu_{l_1 k_1 l_2 k_2} (\frac{dh_{l_1}^{(1)}}{dt})(\kappa R_2) h_{l_2}^{(1)}(\kappa R_2) Y_{l_1}^{k_1}(\frac{x}{|x|}) \ , \ x \in \partial B_{R_2} \ .$$

Furthermore,

$$v := -\kappa^2 \int\limits_{B_R} \Phi_\kappa(\cdot, y)(1 - n(y))\varphi(y)dy$$

has an absolutely and uniformly convergent series expansion in compact subsets of $\{|x| > R\}$:

$$v(x) = \sum_{l_1=0}^{\infty} \sum_{k_1=-l_1}^{l_1} a_{l_1 k_1} h_{l_1}^{(1)}(\kappa|x|) Y_{l_1}^{k_1}(\hat{x}) \ .$$

The series may be differentiated termwise with respect to $|x|$ because the derivatives are also absolutely and uniformly convergent. Hence, we obtain for $x \in \partial B_{R_2}$

$$\frac{\partial v}{\partial \nu}(x) = \kappa \sum_{l_1=0}^{\infty} \sum_{k_1=-l_1}^{l_1} a_{l_1 k_1} \Big(\frac{dh_{l_1}^{(1)}}{dt}\Big)(\kappa|x|) Y_{l_1}^{k_1}(\hat{x}) \ .$$

As in the proof of Lemma 2.21 we also have the relation

$$a_{l_1 k_1} = -2\kappa^2 R_2^2 \frac{i^{l_1 - l_2}}{4\pi} h_{l_2}^{(1)}(\kappa R_2) \mu_{l_1 k_1 l_2 k_2} \ .$$

Inserting this in the above series yields the assertion.

□

Our next aim is to prove that the following Robin boundary value problem (RP) has a unique solution:

given $\kappa > 0$, $n \in C^{0,\gamma}(\mathbb{R}^3)$ with $\operatorname{supp}(1 - n) \subset B_R$ and $\Im(n) \geq 0$, and given $f \in C^{0,\gamma}(\partial B_{R_2})$,

find $u \in C^2(B_{R_2}) \cap C^1(\overline{B_{R_2}})$ such that u satisfies $\Delta u + \kappa^2 n u = 0$ in B_{R_2} and the Robin boundary condition $\frac{\partial u}{\partial \nu} - iu = f$ on ∂B_{R_2}. (ν is directed into the exterior of B_{R_2})

We imitate the proof for the case $n = 1$ and start with uniqueness.

Lemma 2.26 *If u is a solution to (RP) with $f = 0$, then $u = 0$ in $\overline{B_{R_2}}$.*

Proof: Applying Green's first theorem to a solution u of the homogeneous boundary value problem (RP) yields

$$i \int\limits_{\partial B_{R_2}} |u|^2 ds = \int\limits_{\partial B_{R_2}} \frac{\partial u}{\partial \nu} \overline{u} ds = \int\limits_{B_{R_2}} (|\nabla u|^2 - \kappa^2 n |u|^2) dx \ .$$

Taking the imaginary part we arrive at

$$\int_{\partial B_{R_2}} |u|^2 ds = -\kappa^2 \int_{B_{R_2}} \Im(n)|u|^2 dx \le 0 \ ,$$

whence $u = \partial u/\partial\nu = 0$ on ∂B_{R_2}. Now, Green's representation theorem for u in B_{R_2} implies that u is a solution to the homogeneous Lippmann-Schwinger equation in B_{R_2}, thus $u = 0$.

□

Theorem 2.27 *For any given $f \in C^{0,\gamma}(\partial B_{R_2})$ there is a unique solution u to (RP). The mapping $\Lambda_n\colon C^{0,\gamma}(\partial B_{R_2}) \to C^{1,\gamma}(\partial B_{R_2})$ defined by $\Lambda_n f = u|_{\partial B_{R_2}}$ is well defined and can be computed from $u_{\infty,n}$. The linear operator $P\colon C^{0,\gamma}(\partial B_{R_2}) \to C(\overline{B_{R_2}})$ defined by $Pf = u$ is compact.*

Proof: From the considerations before Lemma 2.25 we see that, if $\psi \in C^{0,\gamma}(\partial B_{R_2})$ is a solution to

$$(I + K_n' - iS_n)\psi = f \ , \tag{2.47}$$

then $u|_{B_{R_2}}$ with u defined by

$$u(x) = 2\int_{\partial B_{R_2}} \Phi_\kappa(x,y)\psi(y)ds(y) - \kappa^2 \int_{B_R} \Phi_\kappa(x,y)(1-n(y))\varphi(y)dy \ , \ x \in \mathbb{R}^3 \ ,$$

and $\varphi \in C(\overline{B_R})$ being the solution to

$$\varphi + T_n\varphi = 2\int_{\partial B_{R_2}} \Phi_\kappa(\cdot,y)\psi(y)ds(y) \ ,$$

is a solution to (RP). As in the proof of Lemma 2.19 the differential equation is verified by applying $(\Delta + \kappa^2)$ to the definition of u.

Since the operators S_n and K_n' are compact in $C^{0,\gamma}(\partial B_{R_2})$, it suffices to prove that the integral equation (2.47) has a trivial nullspace.

Let $\psi \in C^{0,\gamma}(\partial B_{R_2})$ be a solution to $(I + K_n' - iS_n)\psi = 0$ and define u as above. Then, the uniqueness result for (RP) implies $u|_{B_{R_2}} = 0$, whence $u|_{\partial B_{R_2}} = 0$ and then $u|_{\mathbb{R}^3\setminus B_{R_2}} = 0$ because the exterior Dirichlet problem for the Helmholtz equation is uniquely solvable. Now, we use the jump relations as indicated before Lemma 2.25 and conclude $2\psi = \partial u_-/\partial\nu - \partial u_+/\partial\nu = 0$.

From our above considerations we have the relation $\Lambda_n = S_n(I + K_n' - iS_n)^{-1}$, i.e., $\Lambda_n: C^{0,\gamma}(\partial B_{R_2}) \to C^{1,\gamma}(\partial B_{R_2})$ is well defined and it can be computed from $u_{\infty,n}$ because the kernels of the integral operators S_n and K_n' can be computed from the Fourier coefficients of $u_{\infty,n}$.

Finally, the boundedness of $(I + K_n' - iS_n)^{-1}$ in $C^{0,\gamma}(\partial B_{R_2})$ together with the boundedness of $(I + T_n)^{-1}$ in $C(\overline{B_{R_2}})$ and the compactness of the single-layer potential with kernel Φ_κ from $C^{0,\gamma}(\partial B_{R_2})$ to $C(\overline{B_{R_2}})$ imply the compactness of P.

$\square$

Our next task is to construct the Robin data $\partial u(\cdot,\zeta)/\partial\nu - iu(\cdot,\zeta)$ of the special solutions used in the Uniqueness Theorem 2.10. We will derive a uniquely solvable equation for the Robin data which only contains Λ_n and integral operators built on the special fundamental solution $e^{i\zeta\cdot x}g_\zeta(x)$ for the Helmholtz equation. These unphysical fundamental solutions were introduced after Lemma 2.9.

Our reasoning after Lemma 2.9 implies that for $\zeta \in \mathbb{C}^3$ with $\Im(\zeta) \neq 0$ and $\zeta \cdot \zeta = \kappa^2$ the relation

$$e^{i\zeta\cdot(x-y)}g_\zeta(x-y) = \Phi_\kappa(x,y) + \tilde{g}_\zeta(x-y)\ ,\ x, y \in B_{R'}\ ,\ x \neq y\ ,$$

holds true where $\tilde{g}_\zeta \in C^2(B_{2R'})$ satisfies $\Delta\tilde{g}_\zeta + \kappa^2\tilde{g}_\zeta = 0$ in $B_{2R'}$. To simplify notation we define

$$\Psi_\zeta(x) := e^{i\zeta\cdot x}g_\zeta(x) = \frac{e^{i\kappa|x|}}{4\pi|x|} + \tilde{g}_\zeta(x)\ ,\ 0 < |x| < 2R'\ .$$

We choose $R < R_2 < R'' < R'$. Equation (2.15) tells us that for sufficiently large $|\Im(\zeta)|$ the special solutions $u(\cdot,\zeta)$ are the unique solutions to

$$u(x,\zeta) = e^{i\zeta\cdot x} - \kappa^2 \int\limits_{B_{R''}} \Psi_\zeta(x-y)(1-n)(y)u(y,\zeta)dy\ ,\ x \in \overline{B_{R''}}\ . \qquad (2.48)$$

The following lemma states Green's representation theorem for $u(\cdot,\zeta)$ in the spherical shell $R_2 < |x| < R''$ with the fundamental solution Φ_κ replaced by Ψ_ζ.

Lemma 2.28 *Assume $\kappa > 0$ and $\zeta \in \mathbb{C}^3$ satisfies $\Im(\zeta) \neq 0$ and $\zeta \cdot \zeta = \kappa^2$. Furthermore, let $u(\cdot, \zeta) \in C(\overline{B_{R''}})$ be a solution to the modified Lippmann-Schwinger equation (2.48). Then, for $R_2 < |x| < R''$ $u(\cdot,\zeta)$ admits the representation*

$$u(x,\zeta) = e^{i\zeta\cdot x} + \int\limits_{\partial B_{R_2}} \Big(\frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)} u(y,\zeta) - \Psi_\zeta(x-y)\frac{\partial u(y,\zeta)}{\partial \nu}\Big) ds(y)\ . \quad (2.49)$$

Proof: Taking into account the regularity properties of volume potentials and the fact that $\Psi_\zeta(x-y) = \Phi_\kappa(x,y) + \tilde{g}_\zeta(x-y)$ with a smooth function $\tilde{g}_\zeta$ satisfying $\Delta\tilde{g}_\zeta + \kappa^2\tilde{g}_\zeta = 0$ we conclude for the solution $u(\cdot,\zeta)$ of the modified Lippmann-Schwinger equation (2.48) that $u(\cdot,\zeta) \in C^2(B_{R''})$ and $\Delta u(\cdot,\zeta) + \kappa^2 n u(\cdot,\zeta) = 0$ in $B_{R''}$. Since for a fixed $x \in \mathbb{R}^3$ with $R_2 < |x| < R''$ the function $\Psi_\zeta(x-\cdot) \in C^2(\overline{B_{R_2}})$ is a solution to the Helmholtz equation in B_{R_2}, we obtain from Green's second theorem

$$\begin{aligned} &\int\limits_{\partial B_{R_2}} \Big(\frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)} u(y,\zeta) - \Psi_\zeta(x-y)\frac{\partial u(y,\zeta)}{\partial \nu}\Big) ds(y) \\ &= \ -\kappa^2 \int\limits_{B_{R_2}} \Psi_\zeta(x-y)(1-n)(y)u(y,\zeta)dy\ , \end{aligned}$$

whence the assertion.

$\square$

We now intend to consider the limit $x \to x_0 \in \partial B_{R_2}$ in (2.49) in order to arrive at the desired boundary integral equation for the Robin data. To this end we define the operators $\mathcal{S}_\zeta$, $\mathcal{K}_\zeta$, $\mathcal{K}'_\zeta$ and $\mathcal{T}_\zeta$ analogously to the known operators in classical potential theory:

$$\mathcal{S}_\zeta: C^{0,\gamma}(\partial B_{R_2}) \to C^{1,\gamma}(\partial B_{R_2}) \qquad (\mathcal{S}_\zeta f)(x) := 2\int\limits_{\partial B_{R_2}} \Psi_\zeta(x-y) f(y) ds(y),$$

$$\mathcal{K}_\zeta: C^{1,\gamma}(\partial B_{R_2}) \to C^{1,\gamma}(\partial B_{R_2}) \qquad (\mathcal{K}_\zeta f)(x) := 2\int\limits_{\partial B_{R_2}} \frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)} f(y) ds(y),$$

$$\mathcal{K}'_\zeta: C^{0,\gamma}(\partial B_{R_2}) \to C^{0,\gamma}(\partial B_{R_2}) \qquad (\mathcal{K}'_\zeta f)(x) := 2\int\limits_{\partial B_{R_2}} \frac{\partial \Psi_\zeta(x-y)}{\partial \nu(x)} f(y) ds(y)$$

for $x \in \partial B_{R_2}$ and $\mathcal{T}_\zeta : C^{1,\gamma}(\partial B_{R_2}) \to C^{0,\gamma}(\partial B_{R_2})$

$$(\mathcal{T}_\zeta f)(x) := 2\frac{\partial}{\partial\nu(x)} \int_{\partial B_{R_2}} \frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)} f(y) ds(y) \ , \ x \in \partial B_{R_2} \ .$$

Since Ψ_ζ and $e^{i\kappa|\cdot|}/(4\pi|\cdot|)$ only differ by a smooth function, the above boundary operators inherit the properties of the analogous ones with kernel Φ_κ (see [6, Chapter 2]). Similarly, the jump relations and mapping properties of single and double-layer potentials and their derivatives defined with the kernel Ψ_ζ are the same as those with the kernel Φ_κ.

Lemma 2.29 *Assume $\kappa > 0$ and $\zeta \in \mathbb{C}^3$ satisfies $\Im(\zeta) \neq 0$ and $\zeta \cdot \zeta = \kappa^2$. Furthermore, let $u(\cdot, \zeta) \in C(\overline{B_{R''}})$ be a solution to the modified Lippmann-Schwinger equation (2.48). Then, the Robin data $f = \partial u(\cdot,\zeta)/\partial\nu - iu(\cdot,\zeta)$ on ∂B_{R_2} are a solution to*

$$\begin{aligned} f(x) &= \left(\frac{\partial e^{i\zeta\cdot x}}{\partial \nu} - ie^{i\zeta\cdot x}\right) \\ &+ \frac{1}{2}\{(\mathcal{T}_\zeta - i\mathcal{K}'_\zeta)\Lambda_n f - (\mathcal{K}'_\zeta - I)f - i(\mathcal{K}_\zeta - i\mathcal{S}_\zeta)\Lambda_n f + i\mathcal{S}_\zeta f\}(x) \end{aligned} \tag{2.50}$$

on ∂B_{R_2}.

Proof: In order to have a representation of $u(\cdot,\zeta)$ containing $f = \partial u/\partial\nu - iu$ and $u = \Lambda_n(\partial u/\partial\nu - iu)$ we reformulate equation (2.49) as

$$\begin{aligned} u(x,\zeta) &= e^{i\zeta\cdot x} + \int_{\partial B_{R_2}} \Big\{ \Big(\frac{\partial \Psi_\zeta(x-y)}{\partial\nu(y)} - i\Psi_\zeta(x-y)\Big) u(y,\zeta) \\ &\qquad - \Psi_\zeta(x-y)\Big(\frac{\partial u(y,\zeta)}{\partial \nu} - iu(y,\zeta)\Big)\Big\} ds(y) \ , \ R_2 < |x| < R'' \ . \end{aligned}$$

From this we can infer together with the jump relations

$$\begin{aligned} 2u(x,\zeta) &= 2e^{i\zeta\cdot x} + \Big\{(\mathcal{K}_\zeta - i\mathcal{S}_\zeta + I)\Lambda_n f - \mathcal{S}_\zeta f\Big\}(x) \ , \\ 2\frac{\partial u(x,\zeta)}{\partial\nu} &= 2\frac{\partial e^{i\zeta\cdot x}}{\partial\nu} + \Big\{(\mathcal{T}_\zeta - i(\mathcal{K}'_\zeta - I))\Lambda_n f - (\mathcal{K}'_\zeta - I)f\Big\}(x) \end{aligned}$$

on ∂B_{R_2}. Multiplying the first equation by i and subtracting the result from the second one we finally obtain for the Robin data f of the function $u(\cdot,\zeta)$ the equation

$$\begin{aligned} f(x) &= \left(\frac{\partial e^{i\zeta\cdot x}}{\partial\nu} - ie^{i\zeta\cdot x}\right) \\ &+ \frac{1}{2}\{(\mathcal{T}_\zeta - i(\mathcal{K}'_\zeta - I))\Lambda_n f - (\mathcal{K}'_\zeta - I)f \\ &\qquad -i(\mathcal{K}_\zeta - i\mathcal{S}_\zeta + I)\Lambda_n f + i\mathcal{S}_\zeta f\}(x) \end{aligned}$$

on ∂B_{R_2} and we have proved the lemma.

□

Next, we examine the operator

$$A_{n,\zeta}: C^{0,\gamma}(\partial B_{R_2}) \to C^{0,\gamma}(\partial B_{R_2}) \ ,$$

defined by

$$A_{n,\zeta} f := \frac{1}{2}\{(\mathcal{T}_\zeta - i\mathcal{K}'_\zeta)\Lambda_n f - (\mathcal{K}'_\zeta - I)f - i(\mathcal{K}_\zeta - i\mathcal{S}_\zeta)\Lambda_n f + i\mathcal{S}_\zeta f\} \ ,$$

which occurs in (2.50).

Lemma 2.30 $A_{n,\zeta}: C^{0,\gamma}(\partial B_{R_2}) \to C^{0,\gamma}(\partial B_{R_2})$ *is compact.*

Proof: We define $v := Pf$ to be the solution of $\Delta v + \kappa^2 n v = 0$ in B_{R_2} and $\partial v/\partial\nu - iv = f$ on ∂B_{R_2} (see Theorem 2.27). Green's representation theorem applied to v in B_{R_2} reads

$$\begin{aligned} v(x) &= \int_{\partial B_{R_2}} \{\Phi_\kappa(x,y)\frac{\partial v}{\partial\nu}(y) - \frac{\partial\Phi_\kappa(x,y)}{\partial\nu(y)}v(y)\}ds(y) \\ &\quad -\kappa^2 \int_{B_{R_2}} \Phi_\kappa(x,y)(1-n)(y)v(y)dy \ , \ x \in B_{R_2} \ . \end{aligned}$$

From Green's second theorem applied to v and $\tilde{g}_\zeta(x-\cdot)$ we conclude

$$\begin{aligned} 0 &= \int_{\partial B_{R_2}} \{\tilde{g}_\zeta(x-y)\frac{\partial v}{\partial\nu}(y) - \frac{\partial\tilde{g}_\zeta(x-y)}{\partial\nu(y)}v(y)\}ds(y) \\ &\quad -\kappa^2 \int_{B_{R_2}} \tilde{g}_\zeta(x-y)(1-n)(y)v(y)dy \ , \ x \in B_{R_2} \ . \end{aligned}$$

Then, we add the two equations and arrive at

$$
\begin{aligned}
v(x) &= \int\limits_{\partial B_{R_2}} \{\Psi_\zeta(x-y)\frac{\partial v}{\partial \nu}(y) - \frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)} v(y)\} ds(y) \\
&\quad -\kappa^2 \int\limits_{B_{R_2}} \Psi_\zeta(x-y)(1-n)(y)v(y)dy \ , \ x \in B_{R_2} \ . \qquad (2.51)
\end{aligned}
$$

Reordering terms we obtain

$$
\begin{aligned}
v(x) &= \int\limits_{\partial B_{R_2}} \Big\{\Psi_\zeta(x-y)f(y) - (\frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)} - i\Psi_\zeta(x-y))(\Lambda_n f)(y)\Big\} ds(y) \\
&\quad -\kappa^2 \int\limits_{B_{R_2}} \Psi_\zeta(x-y)(1-n)(y)v(y)dy \ , \ x \in B_{R_2} \ .
\end{aligned}
$$

This yields

$$
\begin{aligned}
2f &= 2\frac{\partial v}{\partial \nu} - 2iv \\
&= \{(\mathcal{K}'_\zeta + I)f - (\mathcal{T}_\zeta - i(\mathcal{K}'_\zeta + I))\Lambda_n f - i\mathcal{S}_\zeta f + i(\mathcal{K}_\zeta - i\mathcal{S}_\zeta - I)\Lambda_n f\} \\
&\quad -2\kappa^2 \int\limits_{B_{R_2}} (\frac{\partial \Psi_\zeta(\cdot - y)}{\partial \nu(\cdot)} - i\Psi_\zeta(\cdot - y))(1-n)(y)v(y)dy
\end{aligned}
$$

on ∂B_{R_2}. Finally, subtracting on both sides the term in curly brackets and dividing by two we have

$$
A_{n,\zeta} f = -\kappa^2 \int\limits_{B_{R_2}} (\frac{\partial \Psi_\zeta(\cdot - y)}{\partial \nu(\cdot)} - i\Psi_\zeta(\cdot - y))(1-n)(y)(Pf)(y)dy \ .
$$

The compactness then follows from the compactness of P and the mapping properties of volume potentials.

□

Now, we would like to prove that the integral equation (2.50) has a trivial nullspace. To this end we state and prove a helpful assertion in the following lemma.

Lemma 2.31 *Assume* $R_2 < R'' < R'$.

(a) For all $x,\ z \in B_{R''}$ the relation

$$\int_{\partial B_{R''}} \left(\frac{\partial \Psi_\zeta(y-z)}{\partial \nu(y)} \Psi_\zeta(x-y) - \Psi_\zeta(y-z) \frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)}\right) ds(y) = 0$$

holds true.

(b) If v is defined by

$$v(y) = \int_{\partial B_{R_2}} \Psi_\zeta(y-z) f(z) ds(z)\ ,\ R_2 \le |y| \le R''\ ,$$

with $f \in C^{0,\gamma}(\partial B_{R_2})$, then for all $|x| < R''$ the relation

$$\int_{\partial B_{R''}} \left(\Psi_\zeta(x-y) \frac{\partial v}{\partial \nu}(y) - \frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)} v(y)\right) ds(y) = 0$$

holds true. This is also true, if v is defined by

$$v(y) = \int_{\partial B_{R_2}} \frac{\partial \Psi_\zeta(y-z)}{\partial \nu(z)} f(z) ds(z)\ ,\ R_2 < |y| \le R''\ ,$$

with $f \in C^{1,\gamma}(\partial B_{R_2})$.

(c) A function v defined as in part (b) admits for $R_2 < |x| < R''$ the representation

$$v(x) = \int_{\partial B_{R_2}} \left(\frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)} v(y) - \Psi_\zeta(x-y) \frac{\partial v}{\partial \nu}(y)\right) ds(y)\ .$$

Proof: Since for $|z|, |x| < R''$ $\Phi_\kappa(\cdot, z)$ and $\Phi_\kappa(x, \cdot)$ are radiating solutions to the Helmholtz equation in $\mathbb{R}^3 \setminus B_{R''}$, we have from Lemma 1.4 that

$$\int_{\partial B_{R''}} \left(\frac{\partial \Phi_\kappa(y,z)}{\partial \nu(y)} \Phi_\kappa(x,y) - \Phi_\kappa(y,z) \frac{\partial \Phi_\kappa(x,y)}{\partial \nu(y)}\right) ds(y) = 0\ .$$

Green's second theorem applied with $\tilde{g}_\zeta(\cdot - z)$ and $\tilde{g}_\zeta(x - \cdot)$ in $B_{R''}$ yields

$$\int_{\partial B_{R''}} \left(\frac{\partial \tilde{g}_\zeta(y-z)}{\partial \nu(y)} \tilde{g}_\zeta(x-y) - \tilde{g}_\zeta(y-z) \frac{\partial \tilde{g}_\zeta(x-y)}{\partial \nu(y)}\right) ds(y) = 0\ .$$

Finally, we know from Green's representation theorem

$$\int_{\partial B_{R''}} (\frac{\partial \Phi_\kappa(y,z)}{\partial \nu(y)} \tilde{g}_\zeta(x-y) - \Phi_\kappa(y,z) \frac{\partial \tilde{g}(x-y)}{\partial \nu(y)}) ds(y) = -\tilde{g}_\zeta(x-z)$$

and

$$\int_{\partial B_{R''}} (\frac{\partial \tilde{g}_\zeta(y-z)}{\partial \nu(y)} \Phi_\kappa(x,y) - \tilde{g}_\zeta(y-z) \frac{\partial \Phi_\kappa(x,y)}{\partial \nu(y)}) ds(y) = \tilde{g}_\zeta(x-z) \ .$$

Adding the last four equalities we arrive at assertion (a).

If we insert the definition of the single-layer v, defined as in assertion (b), into the integral over $\partial B_{R''}$, reverse the order of integration, and use part (a), we can conclude that the integral over $\partial B_{R''}$ vanishes for all $|x| < R''$. For the double-layer v we apply $\nu(z) \cdot \nabla_z$ to the relation from part (a) and arrive at

$$\int_{\partial B_{R''}} (\frac{\partial}{\partial \nu(y)} \frac{\partial \Psi_\zeta(y-z)}{\partial \nu(z)} \Psi_\zeta(x-y) - \frac{\partial \Psi_\zeta(y-z)}{\partial \nu(z)} \frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)}) ds(y) = 0$$

for $|z| = R_2$ and $|x| < R''$. Now, we can proceed analogously as for the single-layer and we have proved part (b) of the lemma.

For part (c) we observe that v, defined as in assertion (b), is a solution to the Helmholtz equation in $R_2 < |x| < R''$. Hence, it can be represented as

$$\begin{aligned} v(x) &= \int_{\partial B_{R_2}} (\frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)} v(y) - \Psi_\zeta(x-y) \frac{\partial v}{\partial \nu}(y)) ds(y) \\ &+ \int_{\partial B_{R''}} (\Psi_\zeta(x-y) \frac{\partial v}{\partial \nu}(y) - \frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)} v(y)) ds(y) \ . \end{aligned}$$

This can be seen by inserting $\Psi_\zeta(x-y) = \Phi_\kappa(x,y) + \tilde{g}_\zeta(x-y)$ and using Green's representation theorem for the intgrals containing Φ_κ and Green's second theorem for the remaining integrals. (Note that on both spheres ν is directed into the exterior, i.e., to infinity)

Since the integral over $\partial B_{R''}$ vanishes due to part (b), we have completed the proof of the lemma.

□

We are now in a position to prove that the operator $I - A_{n,\zeta}$ is injective for sufficiently large $|\Im(\zeta)|$. As with the normal derivatives, for a function v defined in B_{R_2} we denote by v_- the uniform limit $v(\cdot - h\nu(\cdot))$ for $h \to 0$, $h > 0$, on ∂B_{R_2}. And we define analogously the limit v_+ from the exterior if v is defined outside of $\overline{B_{R_2}}$. We summarize our results about the Robin data of the special solutions $u(\cdot, \zeta)$ in the following theorem.

Theorem 2.32 *Let $n \in C^{0,\gamma}(\mathbb{R}^3)$ with supp$(1-n) \subset B_R$, $\Im(n) \geq 0$, and $R < R_2 < R'' < R'$ be given. Assume $\kappa > 0$ and $\zeta \in \mathbb{C}^3$ satisfies $\zeta \cdot \zeta = \kappa^2$ and $|\Im(\zeta)| \geq 2\kappa^2(R'/\pi)\|1-n\|_\infty + 1$. Furthermore, let Λ_n be the Robin-to-Dirichlet map and let $u(\cdot, \zeta) \in C(\overline{B_{R''}})$ be the solution to the modified Lippmann-Schwinger equation (2.48).*

Then, the Robin boundary values $f := \partial u(\cdot, \zeta)/\partial\nu - iu(\cdot, \zeta)$ are the unique solution to the equation (2.50), i.e.,

$$f(x) = (\frac{\partial e^{i\zeta \cdot x}}{\partial \nu} - ie^{i\zeta \cdot x}) + (A_{n,\zeta} f)(x) \ , \ x \in \partial B_{R_2} \ .$$

Moreover, $A_{n,\zeta}$ is a compact operator in $C^{0,\gamma}(\partial B_{R_2})$.

Proof: The remark after Lemma 2.9 implies that the modified Lippmann-Schwinger equation (2.48) has a unique solution. Lemmas 2.29 and 2.30 show that f is in fact a solution of (2.50) and that $A_{n,\zeta}$ is compact.

It remains to prove the injectivity of equation (2.50). To this end assume $f \in C^{0,\gamma}(\partial B_{R_2})$ is a solution to $f = A_{n,\zeta} f$. We define v in B_{R_2} to be the solution of $\Delta v + \kappa^2 nv = 0$ in B_{R_2} which has the Robin data $\partial v/\partial\nu - iv = f$. For $R_2 < |x| \leq R''$ we define

$$v(x) := \int\limits_{\partial B_{R_2}} \Big\{ (\frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)} - i\Psi_\zeta(x-y))(\Lambda_n f)(y) - \Psi_\zeta(x-y) f(y) \Big\} ds(y) \ . \tag{2.52}$$

From the jump relations we conclude

$$\begin{aligned} & \frac{\partial v_+}{\partial \nu} - iv_+ \\ = \ & \frac{1}{2}\{(\mathcal{T}_\zeta - i(\mathcal{K}'_\zeta - I))\Lambda_n f - (\mathcal{K}'_\zeta - I) f - i(\mathcal{K}_\zeta - i\mathcal{S}_\zeta + I)\Lambda_n f + i\mathcal{S}_\zeta f\} \\ = \ & A_{n,\zeta} f \\ = \ & f \ . \end{aligned} \tag{2.53}$$

Our next aim is to show that $v_+ = v_-$. Lemma 2.31 (c) together with (2.53) implies for $R_2 < |x| \le R''$

$$v(x) = \int\limits_{\partial B_{R_2}} \left(\frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)} v_+(y) - \Psi_\zeta(x-y)\frac{\partial v_+}{\partial \nu}(y)\right) ds(y)$$
$$= \int\limits_{\partial B_{R_2}} \Big\{\left(\frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)} - i\Psi_\zeta(x-y)\right)v_+(y) - \Psi_\zeta(x-y)f(y)\Big\} ds(y) \ .$$

We compute the difference of (2.52) and the last equation and we obtain for $g := \Lambda_n f - v_+$ and

$$w(x) := \int\limits_{\partial B_{R_2}} \left(\frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)} - i\Psi_\zeta(x-y)\right) g(y) ds(y) \ , \ x \in B_{R''} \setminus \partial B_{R_2} \ ,$$

that $w(x) = 0$ for $R_2 < |x| < R''$. From the jump relations we conclude $-w_- = w_+ - w_- = g$, $-\partial w_-/\partial \nu = \partial w_+/\partial \nu - \partial w_-/\partial \nu = ig$ and we arrive with Green's first theorem at

$$i \int\limits_{\partial B_{R_2}} |g|^2 ds = \int\limits_{\partial B_{R_2}} \frac{\partial w_-}{\partial \nu}\overline{w_-} ds = \int\limits_{B_{R_2}} (|\nabla w|^2 - \kappa^2 |w|^2) dx \ .$$

Taking the imaginary part yields $g = 0$, whence $v_+ = \Lambda_n f = v_-$. Together with (2.53), i.e., $\partial v_-/\partial \nu - iv_- = f = \partial v_+/\partial \nu - iv_+$, we now know $v_+ = v_-$ and $\partial v_-/\partial \nu = \partial v_+/\partial \nu$. Then, as in (2.51), we can represent v with the help of the fundamental solution Ψ_ζ:

$$v(x) = \int\limits_{\partial B_{R_2}} \left(\Psi_\zeta(x-y)\frac{\partial v_+}{\partial \nu}(y) - \frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)} v_+(y)\right) ds(y)$$
$$-\kappa^2 \int\limits_{B_{R_2}} \Psi_\zeta(x-y)(1-n)(y) v(y) dy \ , \ x \in B_{R_2} \ .$$

Moreover, with Green's second theorem and Lemma 2.31 (b) we compute

$$\int\limits_{\partial B_{R_2}} \left(\Psi_\zeta(x-y)\frac{\partial v_+}{\partial \nu}(y) - \frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)} v_+(y)\right) ds(y)$$
$$= \int\limits_{\partial B_{R''}} \left(\Psi_\zeta(x-y)\frac{\partial v}{\partial \nu}(y) - \frac{\partial \Psi_\zeta(x-y)}{\partial \nu(y)} v(y)\right) ds(y)$$
$$= 0 \ , \ x \in B_{R_2} \ .$$

Hence, v is a solution to the homogeneous modified Lippmann-Schwinger equation

$$v(x) = -\kappa^2 \int_{B_{R_2}} \Psi_\zeta(x-y)(1-n)(y)v(y)dy \ , \ x \in B_{R_2} \ ,$$

and must vanish in B_{R_2} due to the remark after Lemma 2.9. This finally implies $f = \partial v_-/\partial\nu - iv_- = 0$.

□

The last step of the reconstruction of $(n-1)\hat{}(\alpha)$ is summarized in the next theorem. As in the Uniqueness Theorem 2.10 we choose for a fixed vector $\alpha \in \Gamma$ the unit vectors d_1, $d_2 \in \mathbb{R}^3$ such that $d_1 \cdot d_2 = d_1 \cdot \alpha = d_2 \cdot \alpha = 0$ and define

$$\begin{aligned} \zeta_t &:= -\frac{1}{2}\alpha + i\sqrt{t^2 - \kappa^2 + \frac{|\alpha|^2}{4}}d_1 + td_2 \in \mathbb{C}^3 \ , \\ \tilde{\zeta}_t &:= -\frac{1}{2}\alpha - i\sqrt{t^2 - \kappa^2 + \frac{|\alpha|^2}{4}}d_1 - td_2 \ \in \mathbb{C}^3 \end{aligned}$$

for $t > 2\kappa^2(R'/\pi)\|1-n\|_\infty + 1 + \kappa$.

Theorem 2.33 *Let the notation and assumptions of Theorem 2.32 be given. Moreover, define for a fixed $\alpha \in \Gamma$ and for $t > 2\kappa^2(R'/\pi)\|1-n\|_\infty + 1 + \kappa$ the vectors ζ_t, $\tilde{\zeta}_t$ as above and let $f_t \in C^{0,\gamma}(\partial B_{R_2})$ be the unique solution to*

$$f_t(x) = (\frac{\partial e^{i\zeta_t \cdot x}}{\partial\nu} - ie^{i\zeta_t\cdot x}) + (A_{n,\zeta_t}f_t)(x) \ , \ x \in \partial B_{R_2} \ .$$

Then,

$$\lim_{t\to\infty} \int_{\partial B_{R_2}} \{(\frac{\partial e^{i\tilde{\zeta}_t\cdot x}}{\partial\nu} - ie^{i\tilde{\zeta}_t\cdot x})(\Lambda_n f_t)(x) - e^{i\tilde{\zeta}_t\cdot x} f_t(x)\}ds(x)$$

$$= \kappa^2(2R')^{3/2}(n-1)\hat{}(\alpha) \ .$$

Proof: Following our considerations in (2.42) we apply Green's second theorem in B_{R_2} with the functions $e^{i\tilde{\zeta}_t\cdot x}$ and $u(x,\zeta_t) = e^{i\zeta_t\cdot x}(1 + v(x,\zeta_t))$ from

Lemma 2.9 and we arrive at

$$
\begin{aligned}
& \int\limits_{\partial B_{R_2}} \{(\frac{\partial e^{i\tilde{\zeta}_t\cdot x}}{\partial \nu} - ie^{i\tilde{\zeta}_t\cdot x})(\Lambda_n f_t)(x) - e^{i\tilde{\zeta}_t\cdot x} f_t(x)\} ds(x) \\
&= \int\limits_{\partial B_{R_2}} \Big\{\frac{\partial e^{i\tilde{\zeta}_t\cdot x}}{\partial \nu} u(x,\zeta_t) - e^{i\tilde{\zeta}_t\cdot x}\frac{\partial u(x,\zeta_t)}{\partial \nu}\Big\} ds(x) \\
&= \kappa^2 \int\limits_{B_R} (n(x)-1) e^{i\tilde{\zeta}_t\cdot x} u(x,\zeta_t)\,dx \\
&= \kappa^2 \int\limits_{B_R} (n(x)-1) e^{-i\alpha\cdot x}(1+v(x,\zeta_t))\,dx \ .
\end{aligned}
$$

The theorem now follows because $\|v(\cdot,\zeta_t)\|_{L^2} \to 0$ as $t \to \infty$ according to Lemma 2.9.

□

Let us conclude this chapter by summarizing the reconstruction procedure assuming that the far field pattern $u_{\infty,n}\colon S^2 \times S^2 \to \mathbb{C}$ originating from a refractive index $n \in \tilde{C}(B_R)$ is exactly known. We wish to emphasize that it is a theoretical reconstruction procedure. In view of the fact that at present no quadrature rules are available for surfaces in $\mathbb{R}^3$ which are comparable in quality to their counterparts for arcs in $\mathbb{R}^2$, it is not possible to solve integral equations on such surfaces with a comparable amount of work and a comparable accuracy. In addition, the kernels of our integral equation have to be computed with the help of a series expansion and show an oscillating behavior, especially for large ζ. This strongly indicates that at present one will encounter serious difficulties when attempting a numerical reconstruction of n according to this procedure.

- Compute the Fourier coefficients

$$
\mu_{l_1k_1l_2k_2} := \int\limits_{S_2}\int\limits_{S_2} u_{\infty,n}(\hat{x},d)\overline{Y_{l_1}^{k_1}(\hat{x})Y_{l_2}^{k_2}(d)}\,ds(\hat{x})\,ds(d) \ .
$$

- Compute for $x,\, y \in \partial B_{R_2}$ the kernels

$$
\begin{aligned}
& s_n(x,y) := \Phi_\kappa(x,y) \\
& \quad - \frac{\kappa^2}{4\pi}\sum_{l_1,k_1,l_2,k_2} i^{l_1-l_2}\mu_{l_1k_1l_2k_2} h_{l_1}^{(1)}(\kappa R_2) h_{l_2}^{(1)}(\kappa R_2) Y_{l_1}^{k_1}(\frac{x}{|x|}) Y_{l_2}^{k_2}(\frac{y}{|y|})
\end{aligned}
$$

and

$$\tilde{k}_n(x,y) := \frac{\partial \Phi_\kappa(x,y)}{\partial \nu(x)}$$
$$- \quad \frac{\kappa^3}{4\pi} \sum_{l_1,k_1,l_2,k_2} i^{l_1-l_2} \mu_{l_1k_1l_2k_2} \Big(\frac{dh_{l_1}^{(1)}}{dt}\Big)(\kappa R_2) h_{l_2}^{(1)}(\kappa R_2) Y_{l_1}^{k_1}(\frac{x}{|x|}) Y_{l_2}^{k_2}(\frac{y}{|y|})$$

(see (2.31) and (2.46)).

Define the integral operators

$$(S_n f)(x) = 2 \int_{\partial B_{R_2}} s_n(x,y) f(y) ds(y) \ , \ x \in \partial B_{R_2},$$

$$(K'_n f)(x) = 2 \int_{\partial B_{R_2}} \tilde{k}_n(x,y) f(y) ds(y) \ , \ x \in \partial B_{R_2},$$

and $\Lambda_n = S_n(I + K'_n - iS_n)^{-1}$ (see Theorem 2.27).

- Fix $\alpha \in \Gamma$ and choose $\zeta = \zeta_t$, $\tilde{\zeta} = \tilde{\zeta}_t$ as before Theorem 2.33; compute

$$A_{n,\zeta} := \frac{1}{2}\{(\mathcal{T}_\zeta - i\mathcal{K}'_\zeta)\Lambda_n - (\mathcal{K}'_\zeta - I) - i(\mathcal{K}_\zeta - i\mathcal{S}_\zeta)\Lambda_n + i\mathcal{S}_\zeta\} \ ,$$

  where the operators $\mathcal{S}_\zeta$, $\mathcal{K}_\zeta$, $\mathcal{K}'_\zeta$ and $\mathcal{T}_\zeta$ are defined on page 99.

- Solve the equation

$$f(x) = (\frac{\partial e^{i\zeta\cdot x}}{\partial \nu} - ie^{i\zeta\cdot x}) + (A_{n,\zeta} f)(x) \ , \ x \in \partial B_{R_2} \ .$$

  (It has a unique solution due to Theorem 2.32)

- Insert the solution f into

$$\int_{\partial B_{R_2}} \{(\frac{\partial e^{i\tilde{\zeta}\cdot x}}{\partial \nu} - ie^{i\tilde{\zeta}\cdot x})(\Lambda_n f)(x) - e^{i\tilde{\zeta}\cdot x} f(x)\} ds(x) \ ,$$

  and calculate the limit as $t \to \infty$. Divide the limit by $\kappa^2(2R')^{3/2}$ and set the result to $(n-1)\hat{}(\alpha)$.

- Repeat the last three items for all $\alpha \in \Gamma$.

- $$n = 1 + \sum_{\alpha\in\Gamma} (n-1)\hat{}(\alpha) e_\alpha \ \text{ in } L^2(B_{R_2}).$$

Chapter 3

The Direct Electromagnetic Scattering Problem

The propagation of electromagnetic waves in an inhomogeneous isotropic medium is governed by the Maxwell equations

$$\nabla \wedge \mathcal{E} + \mu_0 \frac{\partial \mathcal{H}}{\partial t} = 0 \ , \ \nabla \wedge \mathcal{H} - \epsilon \frac{\partial \mathcal{E}}{\partial t} = J \ .$$

Here, the electric field $\mathcal{E}$ and the magnetic field $\mathcal{H}$ are mappings from space and time, $\mathbb{R}^3 \times \mathbb{R}$, into $\mathbb{R}^3$. The magnetic permeability μ_0 is a positive constant, whereas the electric permittivity $\epsilon = \epsilon(x) > 0$ is assumed to be a positive function of the space variables. Finally, we assume that Ohm's law, $J(x,t) = \sigma(x)\mathcal{E}(x,t)$, relates the current density J to the electric field $\mathcal{E}$ where the conductivity $\sigma \geq 0$ is a nonnegative function.

Since we want to consider a medium whose inhomogeneity is compactly supported and which is a dielectric outside a large ball, we have $\sigma(x) = 0$ and $\epsilon(x) = \epsilon_0 > 0$ with a constant ϵ_0 for all $|x| > R$.

Suppose now that the electromagnetic wave is time-harmonic with frequency $\omega > 0$ having the form

$$\mathcal{E}(x,t) = \Re\Big(\frac{1}{\sqrt{\epsilon_0}} E(x) e^{-i\omega t}\Big) \ , \ \mathcal{H}(x,t) = \Re\Big(\frac{1}{\sqrt{\mu_0}} H(x) e^{-i\omega t}\Big) \ .$$

Then, the fields E and H must satisfy the equations

$$\nabla \wedge E - i\kappa H = 0 \ , \ \nabla \wedge H + i\kappa n E = 0 \ \text{ in } \mathbb{R}^3, \tag{3.1}$$

where $\kappa := \omega\sqrt{\epsilon_0\mu_0}$ is the wave number and

$$n(x) := \frac{1}{\epsilon_0}\Big(\epsilon(x) + i\frac{\sigma(x)}{\omega}\Big) \ , \ x \in \mathbb{R}^3 \ ,$$

is the refractive index of the medium. Henceforth we will refer to (3.1) as the perturbed Maxwell equations and to (3.1) in the special case $n = 1$ as the Maxwell equations.

The main subject of this chapter is the following direct electromagnetic scattering problem: given κ, n and an incident wave E^i, H^i, i.e., a solution to the Maxwell equations, find the scattered fields E^s and H^s such that the total fields $E := E^i + E^s$, $H := H^i + H^s$ satisfy the perturbed Maxwell equations and such that E^s, H^s satisfy a radiation condition.

In section 3.2 we establish the existence of a unique solution to this problem. As in the acoustic case, in the next section we start with a vector Green's theorem and representation theorems because these results are basic tools in our analysis of the direct and inverse scattering problem. Since solutions to the Maxwell equations are also solutions to the Helmholtz equation, some results from the acoustic case, especially Rellich's lemma, are also useful in the electromagnetic case. Uniqueness for the direct electromagnetic scattering problem is proved via Green's theorem, Rellich's lemma, and unique continuation. In order to prove existence of a solution we derive a Fredholm integral equation for the electric field E with the help of the representation theorem. We then show that this integral equation has a unique solution.

We shall employ the special unique continuation principle formulated in Theorem 1.2. Otherwise our analysis follows the treatment in [7]. The assumption that μ_0 is a constant is a good approximation for many materials. It also simplifies the analysis of the direct scattering problem and even more of the inverse scattering problem. The reader who is interested in direct scattering problems with a variable μ should consult [31, section 22] or [51], where the authors also employ integral equation techniques. By coupling integral equation techniques and variational methods Leis is dealing with anisotropic inhomogeneous media in [26].

3.1 Representation Formulas for the Maxwell Equations

As in the acoustic case we first review Green's theorems for vector valued functions and proceed to representation formulas for solutions to the Maxwell equations via the Stratton-Chu representation of vector fields. The Silver-Müller radiation condition, which is the analogue of the Sommerfeld radiation condition, allows to represent solutions to the Maxwell equations in exterior domains with the help of surface layers, too. Finally, we discuss the far field pattern of a radiating solution to the Maxwell equations.

We have included this section in order to refer to its results later. Since the proofs of the assertions are worked out in [7, Sections 6.1, 6.2], we omit the proofs.

If E, $F\colon \overline{D} \to \mathbb{C}^3$ denote $C^1(\overline{D})$-smooth vector fields in a C^2-smooth, bounded, open set $D \subset \mathbb{R}^3$, then

$$\int_{\partial D} (\nu \wedge E) \cdot F ds = \int_{D} \{(\nabla \wedge E) \cdot F - E \cdot (\nabla \wedge F)\} dx \ . \tag{3.2}$$

This follows from the identities $(\nu \wedge E) \cdot F = \nu \cdot (E \wedge F)$ and $\nabla \cdot (E \wedge F) = (\nabla \wedge E) \cdot F - E \cdot (\nabla \wedge F)$ together with Gauss' theorem. Here, we use $\nabla \wedge E$ for the curl and $\nabla \cdot E$ for the divergence of a vector field E. $a \wedge b$ denotes the vector product of two vectors a, $b \in \mathbb{C}^3$.

The regularity assumptions on E and F can be weakened. E, $F \in C^1(D) \cap C(\overline{D})$ and $\nabla \wedge E$, $\nabla \wedge F \in C(\overline{D})$ are sufficient for (3.2).

If we have four vector fields E, H and E', H' as above and use equation (3.2), we arrive at

$$\begin{aligned}
&\int_{\partial D} \{(\nu \wedge E) \cdot H' - (\nu \wedge E') \cdot H\} ds \\
&= \int_{D} \{(\nabla \wedge E - i\kappa H) \cdot H' + (\nabla \wedge H + i\kappa E) \cdot E'\} dx \\
&\quad - \int_{D} \{(\nabla \wedge E' - i\kappa H') \cdot H + (\nabla \wedge H' + i\kappa E') \cdot E\} dx \ ,
\end{aligned} \tag{3.3}$$

where $\kappa \in \mathbb{C}$ is an arbitrary constant.

The following representation theorem for vector fields due to Stratton and Chu [43] will be very useful in later sections.

Theorem 3.1 *Let $D \subset \mathbb{R}^3$ be a bounded, open, C^2-smooth set with exterior unit normal vector ν. For vector fields $E,\ H \in C^1(\overline{D})$ the Stratton-Chu formula*

$$
\begin{aligned}
E(x) \; = \; & -\nabla \wedge \int_{\partial D} \nu(y) \wedge E(y)\Phi_\kappa(x,y)ds(y) \\
& +\nabla \int_{\partial D} \nu(y) \cdot E(y)\Phi_\kappa(x,y)ds(y) \\
& -i\kappa \int_{\partial D} \nu(y) \wedge H(y)\Phi_\kappa(x,y)ds(y) \\
& +\nabla \wedge \int_D \{\nabla \wedge E(y) - i\kappa H(y)\}\Phi_\kappa(x,y)dy \\
& -\nabla \int_D \nabla \cdot E(y)\Phi_\kappa(x,y)dy \\
& +i\kappa \int_D \{\nabla \wedge H(y) + i\kappa E(y)\}\Phi_\kappa(x,y)dy \ , \ x \in D \ . \qquad (3.4)
\end{aligned}
$$

A similar formula holds with the roles of E and H interchanged.

If E and H are a solution to the Maxwell equations

$$\nabla \wedge E - i\kappa H = 0 \ , \qquad \nabla \wedge H + i\kappa E = 0 \ , \qquad (3.5)$$

the Stratton-Chu formula can be reformulated.

Theorem 3.2 *If D satisfies the assumptions of Theorem 3.1 and if $E,\ H \in C^1(D) \cap C(\overline{D})$ are a solution to (3.5) in D, then we have*

$$
\begin{aligned}
E(x) \; = \; & -\nabla \wedge \int_{\partial D} \nu(y) \wedge E(y)\Phi_\kappa(x,y)ds(y) \\
& +\frac{1}{i\kappa}\nabla \wedge \nabla \wedge \int_{\partial D} \nu(y) \wedge H(y)\Phi_\kappa(x,y)ds(y) \ , \ x \in D \ , \qquad (3.6)
\end{aligned}
$$

and

$$
\begin{aligned}
H(x) \; = \; & -\nabla \wedge \int_{\partial D} \nu(y) \wedge H(y)\Phi_\kappa(x,y)ds(y) \\
& -\frac{1}{i\kappa}\nabla \wedge \nabla \wedge \int_{\partial D} \nu(y) \wedge E(y)\Phi_\kappa(x,y)ds(y) \ , \ x \in D \ . \qquad (3.7)
\end{aligned}
$$

A consequence of the last representation is the analyticity of solutions to the Maxwell equations. Hence, we may take the divergence and the curl of the Maxwell equations (3.5) and arrive at $\nabla \cdot E = \nabla \cdot H = 0$ and $\Delta E + \kappa^2 E = 0$, $\Delta H + \kappa^2 H = 0$ in D, if E and H are solutions to the Maxwell equations in D. Here, we have used the identity $\nabla \wedge \nabla \wedge = -\Delta + \nabla\nabla\cdot$.

In order to have the analogous representation to Theorem 3.2 for solutions to the Maxwell equations in exterior domains we have to impose the Silver-Müller radiation condition on the solutions.

Let $E,\ H \in C^1(\mathbb{R}^3 \setminus B_R)$ be a solution to the Maxwell equations in $\mathbb{R}^3 \setminus \overline{B_R}$. E, H satisfy the Silver-Müller radiation condition (E, H are a radiating solution) if

$$\lim_{|x| \to \infty} (H(x) \wedge x - |x| E(x)) = 0 \tag{3.8}$$

where the limit holds uniformly in all directions $|x|^{-1}x$.

Similarly to Lemma 1.4 it is possible to derive from the Silver-Müller radiation condition

$$\int_{|y|=r} |E(y)|^2 ds(y) \le M$$

for all $r \ge R$ (see the first part of the proof of Theorem 6.6 in [7]). This implies for radiating solutions E, H and $\tilde{E}$, $\tilde{H}$ to the Maxwell equations which are defined in the exterior of a ball B_R:

$$\int_{|y|=r} \{(\nu \wedge E) \cdot \tilde{H} - (\nu \wedge \tilde{E}) \cdot H\} ds = 0 \ , \ r > R \ . \tag{3.9}$$

For a proof use the vector Green's theorem (3.3) in the spherical shell $\{r < |x| < r'\}$, i.e.,

$$0 = -\int_{|y|=r} \{(\nu \wedge E) \cdot \tilde{H} - (\nu \wedge \tilde{E}) \cdot H\} ds + \int_{|y|=r'} \{(\nu \wedge E) \cdot \tilde{H} - (\nu \wedge \tilde{E}) \cdot H\} ds \ ,$$

write

$$\int_{|y|=r'} \{(\nu \wedge E) \cdot \tilde{H} - (\nu \wedge \tilde{E}) \cdot H\} ds = \int_{|y|=r'} \{E \cdot ((\tilde{H} \wedge \nu) - \tilde{E}) - \tilde{E} \cdot ((H \wedge \nu) - E)\} ds \ ,$$

and observe that this integral converges to 0 as $r' \to \infty$ due to the Cauchy-Schwarz inequality and the radiation condition.

For a radiating solution to the Maxwell equations in the exterior of a ball Theorem 3.2 takes the form (see [6, Theorem 4.5]):

Theorem 3.3 *Let $E,\ H \in C^1(\mathbb{R}^3 \setminus B_R)$ be a radiating solution to (3.5) in $\mathbb{R}^3 \setminus \overline{B_R}$.*

(a) Then we have

$$\begin{aligned} E(x) &= \nabla \wedge \int_{\partial B_R} \nu(y) \wedge E(y)\Phi_\kappa(x,y)ds(y) \\ &\quad - \frac{1}{i\kappa}\nabla \wedge \nabla \wedge \int_{\partial B_R} \nu(y) \wedge H(y)\Phi_\kappa(x,y)ds(y)\ ,\ \ |x| > R\ , \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} H(x) &= \nabla \wedge \int_{\partial B_R} \nu(y) \wedge H(y)\Phi_\kappa(x,y)ds(y) \\ &\quad + \frac{1}{i\kappa}\nabla \wedge \nabla \wedge \int_{\partial B_R} \nu(y) \wedge E(y)\Phi_\kappa(x,y)ds(y)\ ,\ \ |x| > R\ . \end{aligned} \tag{3.11}$$

(b) For $x \in B_R$ the right hand sides of (3.10) and (3.11) vanish.

We can infer from (3.10) and (3.11) that each cartesian component of E and H is a radiating solution to the Helmholtz equation, i.e., it satisfies the Sommerfeld radiation condition. It is also possible to show the converse: solutions to the Maxwell equations for which each cartesian component satisfies the Sommerfeld radiation condition also satisfy the Silver-Müller radiation condition.

We are now in a position to define the far fields E_∞ and H_∞ as

$$E_\infty(\hat{x}) = \lim_{r\to\infty} re^{-i\kappa r}E(r\hat{x})\ ,\ \ H_\infty(\hat{x}) = \lim_{r\to\infty} re^{-i\kappa r}H(r\hat{x})\ ,\ \ \hat{x} \in S^2\ .$$

Again Rellich's lemma, Lemma 1.7, implies that the far field E_∞ uniquely determines E. Hence, by $H = (i\kappa)^{-1}\nabla \wedge E$ it also determines H. Analogously H_∞ determines E and H. A closer examination with respect to the asymptotics $|x| \to \infty$ in (3.10) and (3.11) shows $H_\infty(\hat{x}) = \hat{x} \wedge E_\infty(\hat{x})$ and $\hat{x} \cdot E_\infty(\hat{x}) = \hat{x} \cdot H_\infty(\hat{x}) = 0$, i.e., E_∞ and H_∞ are tangential fields on the unit sphere.

In the last lemma of this section we establish a transformation mapping solutions to the perturbed Maxwell equations to solutions of a perturbed Helmholtz equation. This transformation, due to Colton and Päivärinta ([8]), is a basic ingredient in the uniqueness proof of the direct scattering problem and in the analysis of the inverse problem because it allows to employ the results from previous chapters.

Lemma 3.4 *Assume* $n \in C^{2,\gamma}(B_R)$, $\Im(n) \geq 0$, $\Re(n) > 0$, *supp* $(1-n) \subset B_R$, *and let* E, $H \in C^1(\overline{B_R})$ *satisfy*

$$\nabla \wedge E - i\kappa H = 0 \ , \qquad \nabla \wedge H + i\kappa n E = 0$$

in the ball B_R. *Then,* E, $H \in C^2(B_R)$ *and* $E' := n^{1/2}E$, $H' := H$ *are a solution to*

$$(\Delta + \kappa^2)\begin{pmatrix} E' \\ H' \end{pmatrix} = \mathcal{Q}\begin{pmatrix} E' \\ H' \end{pmatrix} \ , \tag{3.12}$$

where the operator $\mathcal{Q}$ *is defined by*

$$\mathcal{Q}\begin{pmatrix} E' \\ H' \end{pmatrix} := \begin{pmatrix} \kappa^2(1-n)E' - i\kappa n^{-1/2}\nabla n \wedge H' - (E'\cdot\nabla)\left(\frac{1}{n}\nabla n\right) + (n^{-1/2}\Delta n^{1/2})E' \\ i\kappa n^{-1/2}\nabla n \wedge E' + \kappa^2(1-n)H' \end{pmatrix} . \tag{3.13}$$

Proof: First, we examine the regularity of E and H. To this end we compute for any $\varphi \in C_0^\infty(B_R)$ with the help of the second perturbed Maxwell equation and Gauss' theorem:

$$\begin{aligned} \int_{B_R} \varphi\nabla\cdot(nE)dx &= -\int_{B_R}(\nabla\varphi)\cdot(nE)dx \\ &= \frac{1}{i\kappa}\int_{B_R}(\nabla\varphi)\cdot(\nabla\wedge H)dx \\ &= \frac{1}{i\kappa}\int_{B_R}[\nabla\wedge(\nabla\varphi)]\cdot H dx \\ &= 0 \ . \end{aligned}$$

This implies $\nabla \cdot (nE) = 0$, i.e., $\nabla \cdot E = -(1/n)\nabla n \cdot E$. Now, we insert the last relation and $\nabla \wedge H + i\kappa E = i\kappa(1-n)E$ into the representation (3.4). The equality

$$\nabla \int_{B_R} \frac{1}{n(y)} \nabla n(y) \cdot E(y) \Phi_\kappa(x,y) dy = \int_{B_R} \Phi_\kappa(x,y) \nabla \Big[\frac{1}{n}\nabla n \cdot E\Big](y) dy$$

from Theorem 1.9 (b) and the regularity properties of volume potentials imply $E \in C^{1,\gamma}(B_R)$. Applying the smoothing properties of a volume potential once more we have $E \in C^2(B_R)$. Computing $H = (i\kappa)^{-1}\nabla \wedge E$ from the representation of E a similar reasoning yields $H \in C^2(B_R)$.

Next, we apply the operator $(\nabla \cdot)$ to the first Maxwell equation and arrive at $\nabla \cdot H = 0$. Taking the curl of the perturbed Maxwell equations and using the identities

$$\nabla \wedge \nabla \wedge F = -\Delta F + \nabla \nabla \cdot F \ ,$$

$$\nabla(A \cdot F) = A \wedge (\nabla \wedge F) + F \wedge (\nabla \wedge A) + (A \cdot \nabla)F + (F \cdot \nabla)A$$

for vector fields A, F we obtain

$$\Delta H + \kappa^2 H = i\kappa n^{-1/2} \nabla n \wedge (n^{1/2}E) + \kappa^2(1-n)H \tag{3.14}$$

and

$$\begin{aligned} \Delta E &= -\kappa^2 nE - \nabla\Big(\frac{1}{n}\nabla n \cdot E\Big) \\ &= -\kappa^2 nE - \frac{1}{n}\nabla n \wedge (\nabla \wedge E) - (\frac{1}{n}\nabla n \cdot \nabla)E - (E \cdot \nabla)\Big(\frac{1}{n}\nabla n\Big) \ , \end{aligned}$$

where we have also used $\nabla \wedge \{(1/n)\nabla n\} = 0$. From the last equation we derive

$$\begin{aligned} \Delta(n^{1/2}E) &= n^{1/2}\{\Delta E + (\frac{1}{n}\nabla n \cdot \nabla)E\} + (\Delta n^{1/2})E \\ &= -\kappa^2 n(n^{1/2}E) - i\kappa n^{1/2}\frac{1}{n}\nabla n \wedge H - ((n^{1/2}E)\cdot\nabla)\Big(\frac{1}{n}\nabla n\Big) \\ &\quad +(n^{-1/2}\Delta n^{1/2})(n^{1/2}E) \ . \end{aligned} \tag{3.15}$$

From (3.15) and (3.14) we can deduce

$$(\Delta + \kappa^2)\begin{pmatrix} E' \\ H' \end{pmatrix} = \mathcal{Q}\begin{pmatrix} E' \\ H' \end{pmatrix} \ .$$

This ends the proof of the lemma.

□

Note that we can associate to any $x \in \mathbb{R}^3$ a matrix $\mathcal{Q}(x) \in \mathbb{C}^{6\times 6}$ such that

$$\Big(\mathcal{Q}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}\Big)(x) = \mathcal{Q}(x)\Big(\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}(x)\Big)$$

holds true for all vector fields $(U_1, U_2)\colon \mathbb{R}^3 \to \mathbb{C}^6$. Each entry of $\mathcal{Q}$ is a uniformly γ-Hölder continuous function in $\mathbb{R}^3$ having compact support in B_R.

3.2 Existence and Uniqueness for the Direct Electromagnetic Scattering Problem

Our considerations at the beginning of this chapter lead us to the following direct electromagnetic scattering problem (DEP):

Given the wave number $\kappa > 0$, the refractive index $n \in C^{2,\gamma}(\mathbb{R}^3)$ with $\Re(n) > 0$, $\Im(n) \geq 0$ and $\operatorname{supp}(1-n) \subset B_R$, and the incoming wave E^i, $H^i \in C^1(\mathbb{R}^3)$ satisfying

$$\nabla \wedge E^i - i\kappa H^i = 0 \ , \ \nabla \wedge H^i + i\kappa E^i = 0 \ \text{ in } \mathbb{R}^3,$$

find the fields E, $H \in C^1(\mathbb{R}^3)$ which are a solution to the perturbed Maxwell equations

$$\nabla \wedge E - i\kappa H = 0 \quad , \quad \nabla \wedge H + i\kappa n E = 0 \quad \text{in } \mathbb{R}^3 \tag{3.16}$$

such that the scattered fields $E^s := E - E^i$, $H^s := H - H^i$ satisfy the Silver-Müller radiation condition $\lim\limits_{|x|\to\infty} (H^s(x) \wedge x - |x| E^s(x)) = 0$ uniformly for all directions $\hat{x} := (1/|x|)x \in S^2$.

First, we want to establish uniqueness for (DEP). As in the acoustic case we use Green's theorem and Rellich's lemma to arrive at $E = 0$ in the exterior of B_R. Then we apply the unique continuation principle.

Theorem 3.5 *If E, $H \in C^1(\mathbb{R}^3)$ are a solution to the perturbed Maxwell equations (3.16) in $\mathbb{R}^3$ and satisfy the Silver-Müller radiation condition, then $E = H = 0$ in $\mathbb{R}^3$, especially, the direct electromagnetic scattering problem (DEP) has at most one solution.*

Proof: The radiation condition implies

$$\begin{aligned} 0 &= \lim_{r\to\infty} \int_{|x|=r} |H(x) \wedge \nu(x) - E(x)|^2 ds(x) \\ &= \lim_{r\to\infty} \left\{ \int_{|x|=r} \{|\nu \wedge H|^2 + |E|^2\} ds - 2\Re\Big\{ \int_{|x|=r} (\nu \wedge \overline{E}) \cdot H ds \Big\} \right\} . \end{aligned} \tag{3.17}$$

Since we also know from Green's theorem (3.2) together with the perturbed Maxwell equations (3.16) that

$$\begin{aligned} -\Re\Big\{ \int\limits_{|x|=r} (\nu \wedge \overline{E}) \cdot H ds \Big\} &= -\Re\Big\{ \int\limits_{B_r} \{ (\nabla \wedge \overline{E}) \cdot H - \overline{E} \cdot (\nabla \wedge H) \} dx \Big\} \\ &= -\Re\Big\{ \int\limits_{B_r} \{ -i\kappa |H|^2 + i\kappa n |E|^2 \} dx \Big\} \\ &= \kappa \int\limits_{B_r} \Im(n) |E|^2 dx \geq 0 \ , \end{aligned}$$

we obtain with (3.17)

$$\int\limits_{|x|=r} |E|^2 ds \to 0 \ , \ r \to \infty \ .$$

Then, Rellich's lemma yields $E = H = 0$ in $\mathbb{R}^3 \setminus B_R$ because the cartesian components of E are radiating solutions to the Helmholtz equation in the exterior of B_R.

According to Lemma 3.4 $E' := n^{1/2}E$, $H' := H \in C_0^2(\mathbb{R}^3)$ satisfy the inequality

$$\Big| \Delta \begin{pmatrix} E' \\ H' \end{pmatrix} \Big| = \Big| -\kappa^2 \begin{pmatrix} E' \\ H' \end{pmatrix} + \mathcal{Q} \begin{pmatrix} E' \\ H' \end{pmatrix} \Big| \leq M \Big| \begin{pmatrix} E' \\ H' \end{pmatrix} \Big|$$

in $\mathbb{R}^3$ with a suitable constant M. Therefore, we can conclude $E' = H' = 0$ in B_R by Theorem 1.2, whence $E = H = 0$ in $\mathbb{R}^3$.

□

Before we proceed let us point out that we made stronger regularity assumptions on n than necessary. We need $C^{2,\gamma}$-smoothness of n in order to obtain C^2-smoothness of E and H, whence of $E' = n^{1/2}E$ and $H' = H$. With the help of a better unique continuation principle than our Theorem 1.2 it is possible to prove the existence of a unique solution to (DEP) under the weaker assumption $n \in C^{1,\gamma}(\mathbb{R}^3)$ (see [7, Chapter 9]). However, when we study the inverse electromagnetic problem we shall need $C^{2,\gamma}$-regularity of n, hence it is reasonable to work with this smoothness from the beginning.

In order to prove existence of a solution we derive a Fredholm integral equation for E with the help of the representation from Theorem 3.1. We show that a solution of the integral equation is a solution to (DEP) and that the integral equation has a trivial nullspace. By the Riesz-Fredholm theory this implies the existence of a solution to (DEP).

Let $E = E^i + E^s$, $H = H^i + H^s$ be a solution to (DEP). The equation $\nabla \wedge H + i\kappa n E = 0$ yields $\nabla \wedge H + i\kappa E = i\kappa(1-n)E$ and, by taking the divergence, $\nabla \cdot E = -\frac{1}{n}\nabla n \cdot E$. If we insert these expressions together with $\nabla \wedge E - i\kappa H = 0$ into the representation formula (3.4) for E in the ball B_r, $r > R$, we obtain

$$\begin{aligned} E(x) &= -\nabla \wedge \int_{\partial B_r} \nu(y) \wedge E(y)\Phi_\kappa(x,y)ds(y) \\ &\quad +\nabla \int_{\partial B_r} \nu(y) \cdot E(y)\Phi_\kappa(x,y)ds(y) \\ &\quad -i\kappa \int_{\partial B_r} \nu(y) \wedge H(y)\Phi_\kappa(x,y)ds(y) \\ &\quad -\kappa^2 \int_{B_R} \Phi_\kappa(x,y)(1-n(y))E(y)dy \\ &\quad +\nabla \int_{B_R} \Phi_\kappa(x,y)\frac{1}{n(y)}\nabla n(y) \cdot E(y)dy \ , \ x \in B_r \ . \end{aligned}$$

Next, we want to show that the sum of the boundary integrals is $E^i(x)$. We have

$$\int_{\partial B_r} \nu(y) \cdot \nabla_y \wedge \{H(y)\Phi_\kappa(x,y)\}ds(y) = 0$$

due to Stokes' theorem. Then, using

$$\begin{aligned} &\nabla_x \cdot \{\nu(y) \wedge H(y)\Phi_\kappa(x,y)\} \\ &\quad = \nu(y) \cdot \Big[\nabla_y \wedge \{H(y)\Phi_\kappa(x,y)\}\Big] - \Phi_\kappa(x,y)\nu(y) \cdot \Big[\nabla \wedge H(y)\Big] \end{aligned}$$

and $\nabla \wedge H(y) = -i\kappa E(y)$, $|y| = r$, we arrive at

$$\nabla \cdot \int_{\partial B_r} \nu(y) \wedge H(y)\Phi_\kappa(x,y)ds(y) = i\kappa \int_{\partial B_r} \nu(y) \cdot E(y)\Phi_\kappa(x,y)ds(y) \ .$$

Our considerations so far imply that

$$-\nabla \wedge \int_{\partial B_r} \nu(y) \wedge E(y)\Phi_\kappa(x,y)ds(y)$$

$$
\begin{aligned}
&+\frac{1}{i\kappa}\nabla \wedge \nabla \wedge \int_{\partial B_r} \nu(y) \wedge H(y)\Phi_\kappa(x,y)ds(y) \\
= \quad & -\nabla \wedge \int_{\partial B_r} \nu(y) \wedge E(y)\Phi_\kappa(x,y)ds(y) \\
&+\nabla \int_{\partial B_r} \nu(y) \cdot E(y)\Phi_\kappa(x,y)ds(y) \\
&-i\kappa \int_{\partial B_r} \nu(y) \wedge H(y)\Phi_\kappa(x,y)ds(y) \ . \qquad (3.18)
\end{aligned}
$$

Finally, we insert $E = E^i + E^s$, $H = H^i + H^s$ into the left hand side of the last equation and use Theorems 3.2 and 3.3 (b) to see that it coincides with $E^i(x)$.

Observing that r can be chosen arbitrarily we obtain the integral equation

$$
\begin{aligned}
E(x) \quad = \quad & E^i(x) - \kappa^2 \int_{B_R} \Phi_\kappa(x,y)(1-n(y))E(y)dy \\
&+\nabla \int_{B_R} \Phi_\kappa(x,y)\frac{1}{n(y)}\nabla n(y) \cdot E(y)dy \ , \ x \in \mathbb{R}^3 \ , \qquad (3.19)
\end{aligned}
$$

for the unknown field E. This is the analogue of the Lippmann-Schwinger equation in the acoustic case.

Lemma 3.6 *Let κ and n be given as in (DEP). Moreover, assume E^i, $H^i \in C^1(\mathbb{R}^3)$ are a solution to the Maxwell equations (3.5) in $\mathbb{R}^3$ and $E \in C(\overline{B_R})$ is a solution to (3.19) in $\overline{B_R}$. Then,*

$$
\begin{aligned}
E(x) \quad := \quad & E^i(x) - \kappa^2 \int_{B_R} \Phi_\kappa(x,y)(1-n(y))E(y)dy \\
&+\nabla \int_{B_R} \Phi_\kappa(x,y)\frac{1}{n(y)}\nabla n(y) \cdot E(y)dy \ , \ x \in \mathbb{R}^3 \ ,
\end{aligned}
$$

and $H(x) := (i\kappa)^{-1}\nabla \wedge E(x)$, $x \in \mathbb{R}^3$, are the solution to (DEP) for the incident wave E^i, H^i.

Proof: First note that there is no ambiguity in the definition of E in $\overline{B_R}$ because E is a solution of the integral equation. Moreover, the smoothing properties of the volume potential and the analyticity of E^i, H^i imply that $E \in C^2(\mathbb{R}^3)$, $H \in C^1(\mathbb{R}^3)$.

Our next aim is to show $\nabla \cdot (nE) = 0$, i.e., $\nabla \cdot E = -(1/n)\nabla n \cdot E$. To this end we take the divergence of both sides of the integral equation (3.19), observe $(1/n)\nabla \cdot (nE) = \nabla \cdot E + (1/n)\nabla n \cdot E$, reorder terms and arrive at

$$\frac{1}{n(x)}\nabla \cdot (nE)(x) = -\kappa^2 \int\limits_{B_r} (1 - n(y))\Phi_\kappa(x, y)\frac{1}{n(y)}\nabla \cdot (nE)(y)dy \ , \ x \in \mathbb{R}^3 \ .$$

Hence, $(1/n)\nabla \cdot (nE) = 0$, because the homogeneous Lippmann-Schwinger equation has only the trivial solution due to the proof of Theorem 1.13.

Now, we compute

$$\begin{aligned} H(x) &= \frac{1}{i\kappa}\nabla \wedge E(x) \\ &= H^i(x) + i\kappa\nabla \wedge \int\limits_{B_R} \Phi_\kappa(x, y)(1 - n(y))E(y)dy \ , \ x \in \mathbb{R}^3 \ , \end{aligned}$$

and

$$\begin{aligned} \nabla \wedge H(x) + i\kappa E(x) &= i\kappa\Big\{\nabla \wedge \nabla \wedge \int\limits_{B_R} \Phi_\kappa(x, y)(1 - n(y))E(y)dy \\ &\qquad -\kappa^2 \int\limits_{B_R} \Phi_\kappa(x, y)(1 - n(y))E(y)dy \\ &\qquad +\nabla \int\limits_{B_R} \Phi_\kappa(x, y)\frac{1}{n(y)}\nabla n(y) \cdot E(y)dy\Big\} \\ &= -i\kappa(\Delta + \kappa^2)\int\limits_{B_R} \Phi_\kappa(x, y)(1 - n(y))E(y)dy \\ &\qquad +i\kappa\nabla \int\limits_{B_R} \Phi_\kappa(x, y)\nabla \cdot \{(1 - n)E\}(y)dy \\ &\qquad +i\kappa\nabla \int\limits_{B_R} \Phi_\kappa(x, y)\frac{1}{n(y)}\nabla n(y) \cdot E(y)dy \\ &= i\kappa(1 - n(x))E(x) \ , \ x \in \mathbb{R}^3 \ , \end{aligned}$$

where we have used

$$\nabla \cdot \{(1 - n)E\} + \frac{1}{n}\nabla n \cdot E \;=\; \nabla \cdot \{(1 - n)E\} - \nabla \cdot E$$

$$
\begin{aligned}
&= -\nabla \cdot (nE) \\
&= 0
\end{aligned}
$$

for the last equation. Hence, E and H are a solution to the perturbed Maxwell equations.

For the radiation condition we use the relation $E^s = E - E^i = -(i\kappa)^{-1}\nabla \wedge (H - H^i)$ in the exterior of B_R to derive $H^s(x) \wedge \hat{x} - E^s(x) = H^s(x) \wedge \hat{x} + (i\kappa)^{-1}\nabla \wedge H^s(x)$ for $|x| > R$. From

$$
H^s(x) = i\kappa \nabla \wedge \int\limits_{B_R} \Phi_\kappa(x,y)(1 - n(y))E(y)dy \ , \ x \in \mathbb{R}^3 \ ,
$$

and

$$
\left| (\nabla_x \wedge \{p\Phi_\kappa(x,y)\}) \wedge \hat{x} + \frac{1}{i\kappa}\nabla_x \wedge \nabla_x \wedge \{p\Phi_\kappa(x,y)\} \right| \leq M \frac{|p|}{|x|^2}
$$

for all $p \in \mathbb{C}^3$, $|y| \leq R$ and $|x| \geq 2R + 1$ with a suitable constant M we can see that E^s, H^s satisfy the Silver-Müller radiation condition. This completes the proof of the lemma.

□

We are now in a position to prove the main result of this section, namely that (DEP) has a unique solution.

Theorem 3.7 *The integral equation (3.19) has a unique solution. The direct electromagnetic scattering problem* (DEP) *has a unique solution which is also the solution to the integral equation (3.19).*

Proof: The mapping properties of volume potentials imply that the integral operators from equation (3.19) are compact operators in $C(\overline{B_R})$. Hence, it suffices to show that (3.19) has a trivial nullspace in order to obtain the existence of a unique solution. Assuming that $E \in C(\overline{B_R})$ is a solution to the homogeneous equation (3.19), i.e., with $E^i = 0$, we know from the preceding lemma that E defined in $\mathbb{R}^3$ by the right hand side of (3.19) and $H := (i\kappa)^{-1}\nabla \wedge E$ are a solution to the homogeneous problem (DEP), whence vanish identically by Theorem 3.5. This proves that (3.19) has a unique solution.

Since the solution of equation (3.19) yields a solution of (DEP) by the last lemma, we have also proved the existence of a solution to (DEP), which is unique by Theorem 3.5.

□

Chapter 4

The Inverse Electromagnetic Scattering Problem

We are now turning to the inverse electromagnetic scattering problem. We assume that we know the far fields of the scattered fields E^s for sufficiently many incident plane waves (all angles of propagation, all polarizations) at a fixed wave number. The task is to reconstruct the refractive index n from these data.

As in the second chapter we start with a uniqueness result, i.e., two refractive indices producing the same far field patterns must coincide. Uniqueness for this inverse scattering problem was first proved in [8] by Colton and Päivärinta. The main difficulty lies in the fact that for the Maxwell equations special solutions cannot be obtained by simply imitating the acoustic case. This is due to the terms containing derivatives in the electromagnetic Lippmann-Schwinger equation. Colton and Päivärinta were able to transform solutions to the perturbed Maxwell equations into solutions of a perturbed Helmholtz equation which in turn could be used to construct special solutions.

Compared to the paper [8] our analysis is based on the fundamental solutions Ψ_ζ and g_ζ, i.e., on Fourier series, whereas Colton and Päivärinta use results from [32, 46] which are derived via Fourier transforms. Moreover, we give a proof for the completeness of total fields originating from plane incident waves in the space of all solutions to the perturbed Maxwell equations which is different from the proof of Lemma 2.1 in [8]. The latter seems to have a gap. Our proof is based on the idea of [20, Lemma 5.20] which we already employed in the acoustic case.

Then, we proceed to the question of stability. So far this has not been examined in the literature for the electromagnetic problem. Starting from the far field pattern belonging to the refractive index n we first construct integral operators N_n on a sphere surrounding the inhomogeneity. This construction is severely ill-posed. Then, we derive a logarithmic stability estimate for $\|n - \tilde{n}\|_\infty$ in terms of $\|N_n - N_{\tilde{n}}\|_\infty$ and in terms of the difference of the far field patterns. As in the acoustic case it is a local result and we have to assume some *a priori* knowledge on the smoothness of the refractive indices.

Finally, we show how to recover n from its far field pattern. To this end we derive a uniquely solvable integral equation of the second kind for certain boundary data belonging to the special solutions of the perturbed Maxwell equations. The operators and the right hand side of this integral equation are known or can be computed from the far field pattern. These boundary data together with Green's theorem admit to compute the Fourier coefficients $(n-1)\hat{}(\alpha)$, $\alpha \in \Gamma$.

As in the previous chapter we assume the magnetic permeability μ to be a constant. The reader who is interested in an inhomogeneous μ is referred to the two papers [38, 39] where the authors examine the reconstruction of the material parameters ϵ, μ and σ from boundary measurements of the electric and magnetic field. They obtain the Fourier transform of the right hand side of a system of semilinear elliptic equations for the searched-for parameters. This difficulty does not arise in our case because we assume μ to be a constant. However, we have to construct boundary integral operators from the far field pattern, whereas they already start with the impedance map on the boundary.

As in the acoustic case we have included a separate proof for the uniqueness of the inverse problem, though the construction implies uniqueness, because then the procedure is easier to understand.

Many technical details can be worked out similarly to the acoustic case. Occasionally, we therefore briefly mention the analogous proofs for the acoustic case and do not repeat the entire analysis. Consequently, the reader should be warned that it is necessary to know the second chapter in order to read this chapter.

4.1 Uniqueness for the Inverse Electromagnetic Scattering Problem

The aim of this section is to prove that the far field pattern uniquely determines the refractive index. To this end we first define what is meant by the notion far field pattern. In the acoustic case a plane incident wave $u^i(x,d) = e^{id\cdot x}$ was essentially given by its direction of propagation $d \in S^2$. In the electromagnetic case an incident plane wave

$$E^i(x,d,p) := d \wedge (p \wedge d) \exp(i\kappa d \cdot x) \ , \ H^i(x,d,p) := (i\kappa)^{-1} \nabla_x \wedge E^i(x,d,p) \ ,$$

$x \in \mathbb{R}^3$, is determined by its direction of propagation, $d \in S^2$, and by the vector $p \in \mathbb{C}^3$ controlling its polarization. $E^i(\cdot,d,p)$, $H^i(\cdot,d,p)$ are a solution to the Maxwell equations. Hence, given the wave number $\kappa > 0$, the refractive index $n \in C^{2,\gamma}(\mathbb{R}^3)$ with $\mathrm{supp}\,(1-n) \subset B_R$, $\Re(n) > 0$ and $\Im(n) \geq 0$, and the incident wave $E^i(\cdot,d,p)$, $H^i(\cdot,d,p)$, there exists a unique solution $E(\cdot,d,p)$, $H(\cdot,d,p)$ to the direct electromagnetic scattering problem (DEP) from the preceding chapter. Each cartesian component of the scattered electric field $E^s(\cdot,d,p) := E(\cdot,d,p) - E^i(\cdot,d,p)$ is a radiating solution to the Helmholtz equation. Consequently, $E^s(\cdot,d,p)$ satisfies for $\hat{x} \in S^2$

$$E^s(r\hat{x},d,p) = \frac{e^{i\kappa r}}{r}\{E_\infty(\hat{x},d,p) + o(1)\} \ , \ r \to \infty \ .$$

We define

$$E_\infty : S^2 \times S^2 \times \mathbb{C}^3 \to \mathbb{C}^3 \qquad (\hat{x},d,p) \mapsto E_\infty(\hat{x},d,p)$$

to be the far field pattern corresponding to the refractive index n.

For convenience we denote by $\tilde{C}(B_R)$ the set of refractive indices we are interested in:

$$\tilde{C}(B_R) := \{n \in C^{2,\gamma}(\mathbb{R}^3) : \mathrm{supp}\,(1-n) \subset B_R \ , \ \Re(n) > 0 \ , \ \Im(n) \geq 0\} \ .$$

Let the wave number $\kappa > 0$ be fixed. If $\tilde{n} \in \tilde{C}(B_R)$ is another refractive index producing the far field pattern $\tilde{E}_\infty$ and if $\tilde{E}_\infty = E_\infty$, then we want to show $n = \tilde{n}$.

The main steps of the proof closely follow the acoustic case. We first derive the relation

$$\int_{B_R} (n - \tilde{n}) E \cdot \tilde{E} \, dx = 0 \tag{4.1}$$

for all solutions E, H to the perturbed Maxwell equations

$$\nabla \wedge E - i\kappa H = 0 \ , \ \nabla \wedge H + i\kappa n E = 0 \ \text{ in } B_{R_1},$$

and for all solutions $\tilde{E}$, $\tilde{H}$ to the perturbed Maxwell equations

$$\nabla \wedge \tilde{E} - i\kappa \tilde{H} = 0 \ , \ \nabla \wedge \tilde{H} + i\kappa \tilde{n} \tilde{E} = 0 \ \text{ in } B_{R_1}, \tag{4.2}$$

where $R_1 > R$.

In the case $E = E(\cdot, d, p)$ equation (4.1) follows from the coincidence of the far field patterns with the help of Green's theorem. In order to show that (4.1) holds true for a general E we approximate E by elements from span $\{E(\cdot, d, p) \colon d \in S^2 \ , \ p \in \mathbb{C}^3\}$ with respect to the $L^2(B_R)$-norm.

The second step consists in the construction of special solutions to the perturbed Maxwell equations which depend on parameters ζ, $\eta \in \mathbb{C}^3$. Although we already derived the right fundamental solutions Ψ_ζ for this task in the acoustic case, we still need some more ideas because the Lippmann-Schwinger equation (3.19) contains derivatives of the volume potential. These derivatives do not allow a straightforward treatment along the lines of the acoustic case.

But in the end, for a fixed $\alpha \in \Gamma$, we arrive at special solutions such that $E(x) \cdot \tilde{E}(x)$ converges to $e^{-i\alpha \cdot x}$ with respect to the $L^1(B_R)$-norm as $|\Im(\zeta)| \to \infty$. Equation (4.1) then implies that the Fourier coefficients of n and $\tilde{n}$ coincide, i.e., $n = \tilde{n}$.

After this outline of the section let us start with relation (4.1) for $E = E(\cdot, d, p)$.

Lemma 4.1 *Let $0 < R < R_1$ and n, $\tilde{n} \in \tilde{C}(B_R)$. Furthermore, assume $\tilde{E}$, $\tilde{H}$ are a solution to (4.2) in B_{R_1}. If for fixed $p \in \mathbb{C}^3$, $d \in S^2$ the far field patterns $E_\infty(\cdot, d, p)$ and $\tilde{E}_\infty(\cdot, d, p)$ coincide on S^2, i.e.,*

$$E_\infty(\hat{x}, d, p) = \tilde{E}_\infty(\hat{x}, d, p) \quad \textit{for all } \hat{x} \in S^2,$$

then the relation

$$\int_{B_R} (n(x) - \tilde{n}(x)) E(x, d, p) \cdot \tilde{E}(x) dx = 0$$

holds true.

Proof: We define $E'(x) := E(x,d,p) - \tilde{E}(x,d,p) = E^s(x,d,p) - \tilde{E}^s(x,d,p)$ and $H'(x) := H(x,d,p) - \tilde{H}(x,d,p)$, $x \in \mathbb{R}^3$. From $E_\infty(\cdot,d,p) = \tilde{E}_\infty(\cdot,d,p)$ and Rellich's lemma we conclude $E'(x) = H'(x) = 0$, $|x| > R$. Moreover, we compute

$$\nabla \wedge E' - i\kappa H' = 0 \ , \ \nabla \wedge H' + i\kappa E' = i\kappa\{(1-n)E(\cdot,d,p) - (1-\tilde{n})\tilde{E}(\cdot,d,p)\}$$

and

$$\nabla \wedge \tilde{E} - i\kappa \tilde{H} = 0 \ , \ \nabla \wedge \tilde{H} + i\kappa \tilde{E} = i\kappa(1-\tilde{n})\tilde{E}$$

in $\mathbb{R}^3$. Then, Green's theorem (3.3) applied with $\tilde{E}$, $\tilde{H}$ and E', H' in the ball B_{R_1} yields

$$\begin{aligned} 0 &= \int\limits_{\partial B_{R_1}} \{(\nu \wedge \tilde{E}) \cdot H' - (\nu \wedge E') \cdot \tilde{H}\} ds \\ &= i\kappa \int\limits_{B_{R_1}} \{(1-\tilde{n})\tilde{E} \cdot (E(\cdot,d,p) - \tilde{E}(\cdot,d,p)) \\ &\qquad - ((1-n)E(\cdot,d,p) - (1-\tilde{n})\tilde{E}(\cdot,d,p)) \cdot \tilde{E}\} dx \\ &= i\kappa \int\limits_{B_{R_1}} (n-\tilde{n})E(\cdot,d,p) \cdot \tilde{E} dx \ , \end{aligned}$$

and we have proved the lemma.

□

Our next aim is to approximate the electric field of a solution to the perturbed Maxwell equations by elements from span $\{E(\cdot,d,p): d \in S^2 \ , \ p \in \mathbb{C}^3\}$ with respect to the $L^2(B_R)$-norm. We start with the special case $n = 1$ which is treated similarly to Lemma 2.2, i.e., in the spirit of [20, Lemma 5.20].

Lemma 4.2 *Assume $0 < R < R_2$ and let E^i, $H^i \in C^1(B_{R_2})$ satisfy the Maxwell equations in B_{R_2}. Then, there exists a sequence*

$$(E^i_j, H^i_j) \in \text{span}\{(E^i(\cdot,d,p), H^i(\cdot,d,p)): d \in S^2 \ , \ p \in \mathbb{C}^3\} \ , \ j \in \mathbb{N} \ ,$$

such that $\|E^i - E^i_j\|^2_{L^2(B_R)} + \|H^i - H^i_j\|^2_{L^2(B_R)} \to 0$, $j \to \infty$.

Proof: We define

$$\begin{aligned} X \;&:=\; \{(E|_{B_R}, H|_{B_R}) \colon E, H \in C^1(B_{R_2}) \\ &\qquad\qquad \text{are a solution to the Maxwell equations in } B_{R_2}\} \subset L^2(B_R) \end{aligned}$$

and $\overline{X}$ to be the completion of X in $L^2(B_R)$. It suffices to show that any $(E_0, H_0) \in \overline{X}$ which is orthogonal to all $E^i(\cdot, d, p)$, $H^i(\cdot, d, p)$, i.e.,

$$\int_{B_R} \{\overline{E_0(x)} \cdot E^i(x, d, p) + \overline{H_0(x)} \cdot H^i(x, d, p)\} dx = 0 \tag{4.3}$$

for all $d \in S^2$, $p \in \mathbb{C}^3$, must vanish in $L^2(B_R)$.

If $(E_0, H_0) \in \overline{X}$ satisfies (4.3), we define for $|x| > R$

$$\begin{aligned} V(x) \;&:=\; \nabla \wedge \nabla \wedge \int_{B_R} \Phi_\kappa(x, y)\overline{E_0(y)}dy - i\kappa\nabla \wedge \int_{B_R} \Phi_\kappa(x, y)\overline{H_0(y)}dy \\ W(x) \;&:=\; -i\kappa\nabla \wedge \int_{B_R} \Phi_\kappa(x, y)\overline{E_0(y)}dy - \nabla \wedge \nabla \wedge \int_{B_R} \Phi_\kappa(x, y)\overline{H_0(y)}dy \ . \end{aligned}$$

Then, $V, W \in C^1(\mathbb{R}^3 \setminus \overline{B_R})$ are a radiating solution to the Maxwell equations in $\mathbb{R}^3 \setminus \overline{B_R}$. Furthermore, the asymptotic behavior of the derivatives of $\Phi_\kappa(x, y)$ for large $|x|$ (see [7, formulas (6.25),(6.26)]) implies for any vector $p \in \mathbb{C}^3$ and any $d \in S^2$:

$$\begin{aligned} &4\pi p \cdot V_\infty(-d) \\ &\quad= p \cdot \int_{B_R} \kappa^2 e^{i\kappa d\cdot y} d \wedge (\overline{E_0(y)} \wedge d)dy - \kappa^2 p \cdot \int_{B_R} e^{i\kappa d\cdot y} d \wedge \overline{H_0(y)}dy \\ &\quad= \kappa^2 \int_{B_R} \{\overline{E_0(y)} \cdot E^i(y, d, p) + \overline{H_0(y)} \cdot H^i(y, d, p)\}dy \\ &\quad= 0 \ . \end{aligned}$$

Hence, we know that the far field V_∞ of V vanishes and thus $V(x) = W(x) = 0$ for all $|x| > R$.

Next, we choose a sequence $(E_l, H_l) \in X$, $l \in \mathbb{N}$, approximating (E_0, H_0),

$$\|E_l - E_0\|^2_{L^2(B_R)} + \|H_l - H_0\|^2_{L^2(B_R)} \to 0 \ , \ l \to \infty \ .$$

The representation formulas (3.6) and (3.7) read

$$\begin{aligned} E_l(x) &= -\nabla \wedge \int\limits_{\partial B_{R_3}} \nu(y) \wedge E_l(y)\Phi_\kappa(x,y)ds(y) \\ &\quad + \frac{1}{i\kappa}\nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_3}} \nu(y) \wedge H_l(y)\Phi_\kappa(x,y)ds(y) \ , \ x \in B_R \ , \end{aligned}$$

and

$$\begin{aligned} H_l(x) &= -\nabla \wedge \int\limits_{\partial B_{R_3}} \nu(y) \wedge H_l(y)\Phi_\kappa(x,y)ds(y) \\ &\quad - \frac{1}{i\kappa}\nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_3}} \nu(y) \wedge E_l(y)\Phi_\kappa(x,y)ds(y) \ , \ x \in B_R \ , \end{aligned}$$

where R_3 satisfies $R < R_3 < R_2$.

Now, we insert these expressions for E_l and H_l in

$$\int\limits_{B_R} \{E_l(x) \cdot \overline{E_0(x)} + H_l(x) \cdot \overline{H_0(x)}\}dx \ ,$$

interchange the order of integration, and arrive at

$$\begin{aligned} &\int\limits_{B_R} \{E_l(x) \cdot \overline{E_0(x)} + H_l(x) \cdot \overline{H_0(x)}\}dx \\ &= \frac{1}{i\kappa} \int\limits_{\partial B_{R_3}} \{(\nu(y) \wedge E_l(y)) \cdot W(y) + (\nu(y) \wedge H_l(y)) \cdot V(y)\}ds(y) \\ &= 0 \ , \ l \in \mathbb{N} \ , \end{aligned}$$

because V and W vanish on ∂B_{R_3}. The limit $l \to \infty$ yields $E_0 = H_0 = 0$ in $L^2(B_R)$ and we have proved the assertion.

□

In order to obtain this approximation result for general n we want to use the Lippmann-Schwinger equation (3.19). For convenience, if $n \in \tilde{C}(B_R)$ and U is a vector field, we define the integral operator $T_n: C(\overline{B_R}) \to C(\overline{B_R})$,

$$\begin{aligned} (T_n U)(x) &:= \kappa^2 \int\limits_{B_R} \Phi_\kappa(x,y)(1 - n(y))U(y)dy \\ &\quad - \nabla \int\limits_{B_R} \Phi_\kappa(x,y)\frac{1}{n(y)}\nabla n(y) \cdot U(y)dy \ , \ x \in \overline{B_R} \ . \qquad (4.4) \end{aligned}$$

Then, the Lippmann-Schwinger equation reads $(I+T_n)E = E^i$ in $\overline{B_R}$. We already know that $(I+T_n)^{-1}$ is a bounded operator in $C(\overline{B_R})$ equipped with the maximum norm. However, we need the boundedness of $(I+T_n)^{-1}$ with respect to the $L^2(B_R)$-norm. In the second chapter, a simple application of the Cauchy-Schwarz inequality implied that the analogous acoustic operator T_n is compact in $C(\overline{B_R})$ with respect to the $L^2(B_R)$-norm. This allowed to conclude that $(I+T_n)^{-1}$ is bounded in $(C(\overline{B_R}), \|\cdot\|_{L^2(B_R)})$. Due to the stronger singularity of the kernel of T_n in the electromagnetic case we have to work harder now. We use a functional analytic tool provided by Lax in [25] which allows to infer the boundedness in $(C(\overline{B_R}), \|\cdot\|_{L^2(B_R)})$ from the boundedness in $(C(\overline{B_R}), \|\cdot\|_\infty)$ (see also [7, Theorem 3.5]). Lax's theorem states for a linear operator $A: C(\overline{B_R}) \to C(\overline{B_R})$ which is bounded with respect to $\|\cdot\|_\infty$ and symmetric with respect to the $L^2(B_R)$-scalar product, that A is also bounded in $(C(\overline{B_R}), \|\cdot\|_{L^2(B_R)})$ and that $\|A\|_{L^2(B_R)} \le \|A\|_\infty$.

Lemma 4.3 *If $T_n: C(\overline{B_R}) \to C(\overline{B_R})$ is defined as in (4.4), then $(I+T_n)^{-1}$ is bounded in $C(\overline{B_R})$ equipped with the $L^2(B_R)$-norm.*

Proof: The operator $T_n^*: C(\overline{B_R}) \to C(\overline{B_R})$, defined by

$$\begin{aligned}(T_n^* V)(y) \;:=\; & \kappa^2(1-\overline{n(y)}) \int_{B_R} \overline{\Phi_\kappa(x,y)} V(x)dx \\ & + \nabla \cdot \int_{B_R} \overline{\Phi_\kappa(x,y)} V(x) dx \frac{1}{\overline{n(y)}} \nabla \overline{n(y)} \;,\; y \in \overline{B_R} \;,\end{aligned}$$

is the adjoint operator to T_n with respect to the $L^2(B_R)$-scalar product. By the mapping properties of the volume potential T_n^* is a compact operator in $(C(\overline{B_R}), \|\cdot\|_\infty)$, hence, due to the Fredholm alternative theorem and the injectivity of $I+T_n$, $(I+T_n^*)^{-1}$ exists and is bounded in $(C(\overline{B_R}), \|\cdot\|_\infty)$. Then, the symmetric operator $(I+T_n^*)^{-1}(I+T_n)^{-1}$ is bounded in $(C(\overline{B_R}), \|\cdot\|_\infty)$ and we can conclude from Lax's theorem that

$$\|(I+T_n)^{-1}\|^2_{L^2(B_R)} = \|(I+T_n^*)^{-1}(I+T_n)^{-1}\|_{L^2(B_R)} \le \|(I+T_n^*)^{-1}(I+T_n)^{-1}\|_\infty \,.$$

$\square$

We can now prove the approximation result for general n.

Lemma 4.4 *Assume $0 < R < R_1$ and let E, $H \in C^1(B_{R_1})$ satisfy the perturbed Maxwell equations (3.16) in B_{R_1}. Then, there exists a sequence*

$$E_j \in \operatorname{span}\{E(\cdot, d, p) \colon d \in S^2\ ,\ p \in \mathbb{C}^3\}\ ,\ j \in \mathbb{N}\ ,$$

such that $\|E - E_j\|_{L^2(B_R)} \to 0$, $j \to \infty$.

Proof: We fix $R < R_2 < R_1$ and define

$$\begin{aligned} E^i(x) \quad &:= \quad -\nabla \wedge \int\limits_{\partial B_{R_2}} \nu(y) \wedge E(y)\Phi_\kappa(x, y)ds(y) \\ &\qquad + \frac{1}{i\kappa}\nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} \nu(y) \wedge H(y)\Phi_\kappa(x, y)ds(y) \end{aligned}$$

and $H^i(x) := (i\kappa)^{-1}\nabla \wedge E^i(x)$ for $x \in B_{R_2}$. Then, E^i, H^i are a solution to the Maxwell equations in B_{R_2}.

Starting from the representation formula (3.4) for E in B_{R_2} and following the considerations that lead to (3.18) and (3.19) we obtain the integral equation

$$\begin{aligned} E(x) \quad &= \quad E^i(x) - \kappa^2 \int\limits_{B_R} \Phi_\kappa(x, y)(1 - n(y))E(y)dy \\ &\qquad + \nabla \int\limits_{B_R} \Phi_\kappa(x, y)\frac{1}{n(y)}\nabla n(y) \cdot E(y)dy\ ,\ x \in B_{R_2}\ , \end{aligned} \tag{4.5}$$

for the field E, i.e., $E = (I + T_n)^{-1}E^i$ in $\overline{B_R}$.

Now, according to Lemma 4.2, we choose a sequence (E_j^i, H_j^i), $j \in \mathbb{N}$, from $\operatorname{span}\{(E^i(\cdot, d, p), H^i(\cdot, d, p)) \colon d \in S^2, p \in \mathbb{C}^3\}$ which approximates (E^i, H^i) in $L^2(B_R)$, and we set E_j to be the solution to the Lippmann-Schwinger equation (4.5) with incident field E_j^i. Hence, we have $E_j \in \operatorname{span}\{E(\cdot, d, p) \colon d \in S^2, p \in \mathbb{C}^3\}$ and

$$E_j - E = (I + T_n)^{-1}(E_j^i - E^i) \qquad \text{in } \overline{B_R},$$

whence

$$\|E_j - E\|_{L^2(B_R)} \le \|(I + T_n)^{-1}\|_{L^2(B_R)}\|E_j^i - E^i\|_{L^2(B_R)} \to 0\ ,\ j \to \infty\ .$$

$\square$

Finally, if we approximate the electric field of an an arbitrary solution to the perturbed Maxwell equations in B_{R_1} by elements from

$$\text{span}\,\{E(\cdot,d,p)\colon d\in S^2\ ,\ p\in\mathbb{C}^3\}$$

with respect to the $L^2(B_R)$-norm and use Lemma 4.1, we obtain relation (4.1). This is stated in the next lemma.

Lemma 4.5 *Assume $0<R<R_1$ and that the far field patterns for the refractive indices $n,\ \tilde{n}\in\tilde{C}(B_R)$ coincide on $S^2\times S^2\times\mathbb{C}^3$, i.e., $E_\infty=\tilde{E}_\infty$. If $E,\ H\in C^1(B_{R_1})$ are a solution to*

$$\nabla\wedge E-i\kappa H=0\ ,\ \nabla\wedge H+i\kappa nH=0\ \ \text{in } B_{R_1},$$

and if $\tilde{E},\ \tilde{H}\in C^1(B_{R_1})$ are a solution to

$$\nabla\wedge\tilde{E}-i\kappa\tilde{H}=0\ ,\ \nabla\wedge\tilde{H}+i\kappa\tilde{n}\tilde{H}=0\ \ \text{in } B_{R_1},$$

then we have the relation

$$\int\limits_{B_R}(n(x)-\tilde{n}(x))E(x)\cdot\tilde{E}(x)dx=0\ .$$

Our next task is the construction of special solutions to the perturbed Maxwell equations. For a given $\alpha\in\Gamma$ we are looking for solutions $E(\cdot,\zeta,\eta)$, $H(\cdot,\zeta,\eta)$ and $\tilde{E}(\cdot,\tilde{\zeta},\tilde{\eta})$, $\tilde{H}(\cdot,\tilde{\zeta},\tilde{\eta})$ to the perturbed Maxwell equations which depend in such a way on the parameters ζ, η, $\tilde{\zeta}$, $\tilde{\eta}\in\mathbb{C}^3$ that

$$E(x,\zeta,\eta))\cdot\tilde{E}(x,\tilde{\zeta},\tilde{\eta})\to e^{-i\alpha\cdot x}$$

with respect to the $L^1(B_R)$-norm for an appropriately chosen sequence of the parameters.

Our knowledge from the acoustic case suggests to use an incident field $E^i(x)=\eta e^{i\zeta\cdot x}$ where $\zeta\in\mathbb{C}^3$ satisfies $\zeta\cdot\zeta=\kappa^2$ and $|\Im(\zeta)|$ is sufficiently large, and where $\eta\cdot\zeta=0$. The conditions on ζ and η imply that E^i and $H^i:=(i\kappa)^{-1}\nabla\wedge E^i$ are a solution to the Maxwell equations. Furthermore, the physical fundamental solution Φ_κ should be replaced by the nonphysical Ψ_ζ in the Lippmann-Schwinger equation (3.19).

We remind the reader that $\Psi_\zeta(x)=(e^{i\kappa|x|}/4\pi|x|)+\tilde{g}_\zeta(x)$ was defined on page 97 and that $\tilde{g}$ is a solution to the Helmholtz equation in $B_{2R'}$. Moreover, the properties of the volume potential G_ζ having kernel $g_\zeta(x-y)=e^{-i\zeta\cdot(x-y)}\Psi_\zeta(x-y)$ were investigated in Theorem 2.8.

We first prove the analogue to Lemma 3.6.

Lemma 4.6 *Suppose* $0 < R < R'' < R'$ *and* $\eta,\ \zeta \in \mathbb{C}^3$ *satisfy* $\zeta \cdot \zeta = \kappa^2$, $|\Im(\zeta)| \geq 2\kappa^2(R'/\pi)\|1-n\|_\infty + 1$ *and* $\eta \cdot \zeta = 0$. *Furthermore, define* $E^i(x) := \eta e^{i\zeta\cdot x}$, $H^i(x) := (i\kappa)^{-1}\nabla \wedge E^i(x)$, $x \in \mathbb{R}^3$, *and assume* $E \in C(\overline{B_{R''}})$ *is a solution to*

$$\begin{aligned} E(x) &= E^i(x) - \kappa^2 \int\limits_{B_R} \Psi_\zeta(x-y)(1-n(y))E(y)dy \\ &\quad + \nabla \int\limits_{B_R} \Psi_\zeta(x-y)\frac{1}{n(y)}\nabla n(y)\cdot E(y)dy\ ,\ x \in \overline{B_{R''}}\ . \end{aligned} \tag{4.6}$$

Then, $E \in C^2(B_{R''})$, *and* E, $H := (i\kappa)^{-1}\nabla\wedge E$ *satisfy the perturbed Maxwell equations in* $B_{R''}$.

Proof: We conclude from the smoothing properties of the volume potential and the analyticity of E^i, H^i that $E \in C^2(B_{R''})$ and $H \in C^1(B_{R''})$.

Taking the divergence of both sides of the integral equation (4.6), multiplying by $e^{-i\zeta\cdot x}$ and defining $u := (1/n)\nabla\cdot(nE) = \nabla\cdot E + (1/n)\nabla n\cdot E$, we arrive at

$$e^{-i\zeta\cdot x}u(x) = -\kappa^2\Big[G_\zeta((1-n)(e^{-i\zeta\cdot x}u))\Big](x) \qquad \text{in } B_{R''}.$$

Since the linear mapping $v \mapsto \kappa^2 G_\zeta((1-n)v)$ has an L^2-norm less than one, we obtain $e^{-i\zeta\cdot x}u(x) = 0$, $x \in B_{R''}$, i.e.,

$$\nabla\cdot E = -\frac{1}{n}\nabla n\cdot E\ .$$

Now, we compute as in the proof of Lemma 3.6

$$\begin{aligned} H(x) &= \frac{1}{i\kappa}\nabla\wedge E(x) \\ &= H^i(x) + i\kappa\nabla\wedge\int\limits_{B_R}\Psi_\zeta(x-y)(1-n(y))E(y)dy\ ,\ x \in B_{R''}\ , \end{aligned} \tag{4.7}$$

and

$$\nabla\wedge H(x) + i\kappa E(x) = i\kappa\Big\{\nabla\wedge\nabla\wedge\int\limits_{B_R}\Psi_\zeta(x-y)(1-n(y))E(y)dy$$

$$-\kappa^2 \int_{B_R} \Psi_\zeta(x-y)(1-n(y))E(y)dy$$

$$+\nabla \int_{B_R} \Psi_\zeta(x-y)\frac{1}{n(y)}\nabla n(y)\cdot E(y)dy\Big\}$$

$$\begin{aligned} &= -i\kappa(\Delta+\kappa^2)\int_{B_R} \Psi_\zeta(x-y)(1-n(y))E(y)dy \\ &\quad +i\kappa\nabla\int_{B_R} \Psi_\zeta(x-y)\nabla\cdot\{(1-n)E\}(y)dy \\ &\quad +i\kappa\nabla\int_{B_R} \Psi_\zeta(x-y)\frac{1}{n(y)}\nabla n(y)\cdot E(y)dy \\ &= i\kappa(1-n(x))E(x)\ ,\ x\in B_{R''}\ . \end{aligned}$$

This ends the proof of the lemma.

□

Equation (4.6) is a Fredholm integral equation of the second kind. Thus, analogously to Theorem 3.7, we would like to show that it has a trivial nullspace in order to ensure its unique solvability. However, we cannot proceed as in the direct electromagnetic scattering problem because $E-E^i$ and $H-H^i$ from the above lemma are not radiating solutions. Hence, we cannot apply the Silver-Müller radiation condition which implied uniqueness of E in the preceding chapter. Our reasoning from the acoustic case is not directly applicable either. If we multiply equation (4.6) by $e^{-i\zeta\cdot x}$, we obtain one term $G_\zeta((1-n)(e^{-i\zeta\cdot x}E))$ and a second term

$$(\nabla+i\zeta)G_\zeta\Big(\frac{1}{n}\nabla n\cdot(e^{-i\zeta\cdot x}E)\Big)\ .$$

This second operator does not converge to zero with respect to $L^2(B_{R''})$. Furthermore, this term causes difficulties to obtain the asymptotic behavior of $e^{-i\zeta\cdot x}(E-E^i)(x)$ for $|\Im(\zeta)|\to\infty$ from (4.6). The unique solvability can be achieved, if $\|n-1\|_{C^1}$ is sufficiently small, but the derivation of the asymptotic behavior still needs hard work which is done in [45].

The following trick to overcome these difficulties is due to Colton and Päivärinta (see [8]). They derive an integral equation with kernel g_ζ for

$e^{-i\zeta\cdot x}n^{1/2}(E-E^i)$, $e^{-i\zeta\cdot x}(H-H^i)$ where they make use of the fact that $E' := n^{1/2}E$, $H' := H$ satisfy a Helmholtz type differential equation. We already used their idea in the proof of Lemma 3.4.

Lemma 4.7 *With the assumptions as in the preceding lemma suppose E is a solution of (4.6). Then $E''(x) := e^{-i\zeta\cdot x}\,n^{1/2}(x)(E-E^i)(x)$, $H''(x) := e^{-i\zeta\cdot x}(H-H^i)(x)$, $x \in \overline{B_{R''}}$, are a solution of the equation*

$$\begin{pmatrix} E'' \\ H'' \end{pmatrix} = -G_\zeta \mathcal{Q} \begin{pmatrix} E'' \\ H'' \end{pmatrix} + \begin{pmatrix} F_1(\cdot,\zeta,\eta) \\ F_2(\cdot,\zeta,\eta) \end{pmatrix} \tag{4.8}$$

where $\mathcal{Q}$ is the operator defined in (3.13) and

$$\begin{pmatrix} F_1(\cdot,\zeta,\eta) \\ F_2(\cdot,\zeta,\eta) \end{pmatrix} := -G_\zeta \begin{pmatrix} (-in^{-1/2}\zeta\cdot\nabla n - \Delta n^{1/2})\eta \\ 0 \end{pmatrix} - G_\zeta \mathcal{Q} \begin{pmatrix} n^{1/2}\eta \\ \kappa^{-1}\zeta\wedge\eta \end{pmatrix} \quad .$$

Proof: We first establish the equality

$$\begin{pmatrix} n^{1/2}(E-E^i) \\ H-H^i \end{pmatrix} = -\int\limits_{B_R} \Psi_\zeta(\cdot - y)(\Delta+\kappa^2) \begin{pmatrix} n^{1/2}(E-E^i) \\ H-H^i \end{pmatrix}(y)dy \ . \tag{4.9}$$

To this end we note that due to (4.7) and Theorem 1.9 (b) $H-H^i$ can be written as

$$H(x) - H^i(x) = \int\limits_{B_R} \Psi_\zeta(x-y)U(y)dy \ , \ x \in \overline{B_{R''}} \ ,$$

with a vector field U. This implies

$$\begin{aligned} H(x)-H^i(x) &= \int\limits_{\partial B_{R''}} \Big(\frac{\partial(H-H^i)}{\partial\nu}(y)\Psi_\zeta(x-y) \\ &\qquad - \frac{\partial\Psi_\zeta(x-y)}{\partial\nu(y)}(H-H^i)(y)\Big)ds(y) \\ &\quad - \int\limits_{B_{R''}} ((\Delta+\kappa^2)(H-H^i))(y)\Psi_\zeta(x-y)dy \\ &= -\int\limits_{B_{R''}} ((\Delta+\kappa^2)(H-H^i))(y)\Psi_\zeta(x-y)dy \ , \ x \in B_{R''} \ . \end{aligned}$$

Here, the first equality follows from Green's representation formula (1.10) by inserting $\Psi_\zeta(x-y) = \Phi_\kappa(x,y) + \tilde{g}_\zeta(x-y)$ and using the identity (1.9) for the terms with $\tilde{g}_\zeta$. For the second equality we observe that the boundary terms vanish due to Lemma 2.31 (a) (see the proof of Lemma 2.31 (b) where surface potentials are examined instead of a volume potential). This proves the lower line of (4.9).

Due to (4.6), the same reasoning yields the analogous representation for $E - E^i$. Finally, $(n^{1/2}-1)(E-E^i) \in C_0^2(B_{R''})$ also admits this representation because the boundary terms vanish due to the compact support of $(n^{1/2} - 1)(E - E^i)$. Adding the representations for $(n^{1/2} - 1)(E - E^i)$ and $E - E^i$ we obtain the upper line of (4.9).

Finally, we compute with the help of Lemma 3.4

$$\begin{aligned} &(\Delta + \kappa^2)\begin{pmatrix} n^{1/2}(E-E^i) \\ H - H^i \end{pmatrix} \\ &= \mathcal{Q}\begin{pmatrix} n^{1/2}(E-E^i) \\ H - H^i \end{pmatrix} + \mathcal{Q}\begin{pmatrix} n^{1/2}E^i \\ H^i \end{pmatrix} - (\Delta + \kappa^2)\begin{pmatrix} n^{1/2}E^i \\ H^i \end{pmatrix} , \end{aligned}$$

and insert the result into (4.9). Multiplying both sides of the equation by $e^{-i\zeta\cdot x}$ completes the proof of the assertion.

□

Now we are in a position to prove the existence of a unique solution to (4.6), if $|\Im(\zeta)|$ is sufficiently large. We can also obtain the asymptotic behavior of the solution E as $|\Im(\zeta)| \to \infty$.

If $A\colon \mathbb{C}^6 \to \mathbb{C}^6$ is a linear operator, we define

$$\|A\|_2 := \max_{|p|=1} |Ap| \ .$$

Moreover, we denote by t_0 the positive number

$$t_0 := 2\frac{R'}{\pi}\{\max_{x\in\overline{B_R}} \|\mathcal{Q}(x)\|_2 + \kappa^2\|1-n\|_\infty\} + 1 \ , \tag{4.10}$$

which only depends on κ, R', the C^2-norm of $1-n$ and $\|1/n\|_\infty$.

Theorem 4.8 *Suppose the assumptions of Lemma 4.6 hold true. Furthermore, let ζ satisfy the additional requirement $|\Im(\zeta)| \geq t_0$, where t_0 is defined in (4.10).*

(a) *Then, the integral equations (4.6) and (4.8) both have a unique solution.*

(b) *There is a positive constant M (depending only on κ, R', $\|1/n\|_\infty + \|1-n\|_{C^2}$), such that the solution E to (4.6) satisfies*

$$E(x) = E(x,\zeta,\eta) = e^{i\zeta\cdot x}\{\eta + f(x,\zeta,\eta)\zeta + V(x,\zeta,\eta)\}\ ,\ x \in B_{R''}\ ,$$

where the L^2-norms of the vector fields $V(\cdot,\zeta,\eta)$ and of the functions $f(\cdot,\zeta,\eta)$ can be estimated by

$$\|V(\cdot,\zeta,\eta)\|_{L^2(B_{R''})} + \|f(\cdot,\zeta,\eta)\|_{L^2(B_{R''})} \le \frac{M|\eta|}{|\Im(\zeta)|}\ .$$

Proof: Since we can estimate

$$\Big\|\mathcal{Q}\begin{pmatrix}E''\\H''\end{pmatrix}\Big\|_{L^2(B_{R''})} \le \max_{x\in\overline{B_R}}\|\mathcal{Q}(x)\|_2\Big\|\begin{pmatrix}E''\\H''\end{pmatrix}\Big\|_{L^2(B_{R''})}\ ,$$

we have for $|\Im(\zeta)| \ge t_0$ and for any solution (E'',H'') of the homogeneous equation (4.8):

$$\Big\|\begin{pmatrix}E''\\H''\end{pmatrix}\Big\|_{L^2(B_{R''})} \le \Big\|G_\zeta\mathcal{Q}\begin{pmatrix}E''\\H''\end{pmatrix}\Big\|_{L^2(B_{R''})} \le \frac{1}{2}\Big\|\begin{pmatrix}E''\\H''\end{pmatrix}\Big\|_{L^2(B_{R''})}\ .$$

Therefore, equation (4.8) has a trivial nullspace and thus a unique solution by the Riesz-Fredholm theory.

With the help of Lemma 4.7 we see that any solution E to the homogeneous integral equation (4.6) yields an element $(e^{-i\zeta\cdot x}\,n^{1/2}E, (i\kappa)^{-1}\nabla\wedge E)$ in the nullspace of (4.8). Hence, the nullspace of (4.6) is also trivial and equation (4.6) has a unique solution by the Riesz-Fredholm theory. This proves part (a) of the lemma.

For part (b) we note that $E(x) = E^i(x) + e^{i\zeta\cdot x}n^{-1/2}(x)E''(x)$ by the definition of E'' and that (E'',H'') is a solution to (4.8), i.e.,

$$\begin{pmatrix}E''\\H''\end{pmatrix} = -G_\zeta\Big(\mathcal{Q}\begin{pmatrix}E''\\H''\end{pmatrix}\Big) + \begin{pmatrix}F_1(\cdot,\zeta,\eta)\\F_2(\cdot,\zeta,\eta)\end{pmatrix}\ .$$

The equation $|\zeta|^2 = |\Re(\zeta)|^2 + |\Im(\zeta)|^2 = 2|\Im(\zeta)|^2 + \kappa^2$ allows to bound $|\zeta|$ by $|\Im(\zeta)|$. Hence, by Theorem 2.8 there is a constant M_1 such that

$$\|(F_1(\cdot,\zeta,\eta),F_2(\cdot,\zeta,\eta))\|_{L^2(B_{R''})} \le M_1|\eta|\ .$$

Then, we can infer from the integral equation (4.8) that $\|(E'', H'')\|_{L^2(B_{R''})} \leq M_2|\eta|$ for all $|\Im(\zeta)| \geq t_0$ with a suitable constant M_2.

Now, we write down the equation for E'' from (4.8), define $E' := n^{1/2}\eta + E''$, reorder terms, and obtain

$$\begin{aligned} E'' &= \Big\{-G_\zeta(-in^{-1/2}\zeta \cdot \nabla n\, \eta) - G_\zeta(-in^{-1/2}\nabla n \wedge (\zeta \wedge \eta))\Big\} \\ &\quad + \Big[-G_\zeta\Big[\kappa^2(1-n)E' - i\kappa n^{-1/2}\nabla n \wedge H'' - (E' \cdot \nabla)\Big(\frac{1}{n}\nabla n\Big) \\ &\qquad + (n^{-1/2}\Delta n^{1/2})E'\Big] - G_\zeta(-\Delta n^{1/2}\eta)\Big] . \end{aligned}$$

With the help of

$$in^{-1/2}\zeta \cdot \nabla n\, \eta + in^{-1/2}\nabla n \wedge (\zeta \wedge \eta) = in^{-1/2}\nabla n \cdot \eta\, \zeta$$

we arrive at

$$E'' = G_\zeta(in^{-1/2}\nabla n \cdot \eta)\zeta + V'(\cdot, \zeta, \eta) ,$$

where V' denotes the term in large square brackets from the previous formula for E''. Finally, we define

$$\begin{aligned} f(\cdot, \zeta, \eta) &:= n^{-1/2}G_\zeta(in^{-1/2}\nabla n \cdot \eta) , \\ V(\cdot, \zeta, \eta) &:= n^{-1/2}V'(\cdot, \zeta, \eta) , \end{aligned}$$

and use the decay of $\|G_\zeta\|_{L^2}$ for large $|\Im(\zeta)|$ in order to obtain assertion (b) of the theorem.

□

With the help of these special solutions we can prove the desired uniqueness theorem for the inverse electromagnetic scattering problem similarly to the inverse acoustic scattering problem.

Theorem 4.9 *Assume $\kappa > 0$ is fixed. If the far field patterns corresponding to the refractive indices n, $\tilde{n} \in \tilde{C}(B_R)$ coincide, i.e., $E_\infty(\hat{x}, d, p) = \tilde{E}_\infty(\hat{x}, d, p)$ for all $(\hat{x}, d, p) \in S^2 \times S^2 \times \mathbb{C}^3$, then $n = \tilde{n}$.*

Proof: With $R < R_1 < R'$ we know from Lemma 4.5 that

$$\int_{B_R} (n(x) - \tilde{n}(x))E(x) \cdot \tilde{E}(x)dx = 0 \tag{4.11}$$

whenever E, $H \in C^1(B_{R_1})$ are a solution to

$$\nabla \wedge E - i\kappa H = 0 \ , \ \nabla \wedge H + i\kappa n E = 0 \ \text{ in } B_{R_1},$$

and $\tilde{E}$, $\tilde{H} \in C^1(B_{R_1})$ are a solution to

$$\nabla \wedge \tilde{E} - i\kappa \tilde{H} = 0 \ , \ \nabla \wedge \tilde{H} + i\kappa \tilde{n} \tilde{E} = 0 \ \text{ in } B_{R_1}.$$

Now, we fix a vector $\alpha \in \Gamma$ and choose the unit vectors d_1, $d_2 \in \mathbb{R}^3$ such that α, d_1 and d_2 are orthogonal. Next we define for sufficiently large $t > 0$ the vectors

$$\begin{aligned}
\zeta_t &:= -\frac{1}{2}\alpha + i\sqrt{t^2 - \kappa^2 + \frac{|\alpha|^2}{4}}d_1 + td_2 \ , \\
\tilde{\zeta}_t &:= -\frac{1}{2}\alpha - i\sqrt{t^2 - \kappa^2 + \frac{|\alpha|^2}{4}}d_1 - td_2 \ , \\
\eta_t &:= \frac{1}{|\alpha|}\alpha + \frac{|\alpha|}{2t}d_2 \ , \\
\tilde{\eta}_t &:= \frac{1}{|\alpha|}\alpha - \frac{|\alpha|}{2t}d_2 \ .
\end{aligned}$$

Note that $|\alpha| \neq 0$ for $\alpha \in \Gamma$. Straightforward computations show $\zeta_t \cdot \zeta_t = \tilde{\zeta}_t \cdot \tilde{\zeta}_t = \kappa^2$, $\tilde{\zeta}_t \cdot \tilde{\eta}_t = \zeta_t \cdot \eta_t = 0$ and $|\eta_t| = |\tilde{\eta}_t| \leq M_\alpha$ for all sufficiently large t. Therefore, from the preceding theorem we can infer the existence of special solutions $E(\cdot, \zeta_t, \eta_t)$, $H(\cdot, \zeta_t, \eta_t)$ and $\tilde{E}(\cdot, \tilde{\zeta}_t, \tilde{\eta}_t)$, $\tilde{H}(\cdot, \tilde{\zeta}_t, \tilde{\eta}_t)$ to the perturbed Maxwell equations with refractive index n, $\tilde{n}$, resp., such that

$$E(x, \zeta_t, \eta_t) = e^{i\zeta_t \cdot x}\{\eta_t + f(x, \zeta_t, \eta_t)\zeta_t + V(x, \zeta_t, \eta_t)\} \ , \ x \in B_{R_1} \ ,$$

$$\tilde{E}(x, \tilde{\zeta}_t, \tilde{\eta}_t) = e^{i\tilde{\zeta}_t \cdot x}\{\tilde{\eta}_t + \tilde{f}(x, \tilde{\zeta}_t, \tilde{\eta}_t)\tilde{\zeta}_t + \tilde{V}(x, \tilde{\zeta}_t, \tilde{\eta}_t)\} \ , \ x \in B_{R_1} \ ,$$

and

$$\begin{aligned}
&\|V(\cdot, \zeta_t, \eta_t)\|_{L^2(B_R)} + \|\tilde{V}(\cdot, \tilde{\zeta}_t, \tilde{\eta}_t)\|_{L^2(B_R)} \\
&\quad + \|f(\cdot, \zeta_t, \eta_t)\|_{L^2(B_R)} + \|\tilde{f}(\cdot, \tilde{\zeta}_t, \tilde{\eta}_t)\|_{L^2(B_R)} \leq \frac{M'_\alpha}{|\Im(\zeta_t)|} \ . \qquad (4.12)
\end{aligned}$$

Using $e^{i\zeta_t \cdot x} e^{i\tilde{\zeta}_t \cdot x} = e^{-i\alpha \cdot x}$, $\eta_t \cdot \tilde{\eta}_t = 1 - (|\alpha|^2/4t^2)$, $\eta_t \cdot \tilde{\zeta}_t = -|\alpha| = \tilde{\eta}_t \cdot \zeta_t$, $\zeta_t \cdot \tilde{\zeta}_t = (|\alpha|^2/2) - \kappa^2$ together with (4.12) we arrive at

$$E(x, \zeta_t, \eta_t) \cdot \tilde{E}(x, \tilde{\zeta}_t, \tilde{\eta}_t) = e^{-i\alpha \cdot x}(1 + h(x, t))$$

with

$$\int_{B_R} |h(x,t)|dx \to 0 \ , \ t \to \infty \ .$$

Hence, inserting the special solutions into (4.11) implies $(n - \tilde{n})\hat{}(\alpha) = 0$ as $t \to \infty$. Proceeding as above for all $\alpha \in \Gamma$ we finally arrive with the help of (1.2) at

$$\|n - \tilde{n}\|^2_{L^2(B_R)} = \sum_{\alpha \in \Gamma} |(n - \tilde{n})\hat{}(\alpha)|^2 = 0 \ ,$$

whence $n = \tilde{n}$. This ends the proof of the theorem.

□

Let us close this section with two remarks. As in the acoustic case it is possible to replace the plane incident waves by any set of solutions to the Maxwell equations which is complete in the space of all solutions to the Maxwell equations with respect to $L^2(B_R)$. Second, instead of measuring far field data one might also use near field data like the tangential components of the electric or magnetic field on a large sphere because these data uniquely determine a radiating solution to the Maxwell equations.

4.2 Stability of the Inverse Electromagnetic Problem

We will now examine the continuous dependence of the refractive index n on the far field pattern. We assume throughout this section that the refractive indices n satisfy $n \in \tilde{C}(B_R)$, i.e., $n \in C^{2,\gamma}(\mathbb{R}^3)$, $0 < \gamma < 1$, $\Re(n) > 0$, $\Im(n) \geq 0$, and $\operatorname{supp}(1-n) \subset B_R$.

If $\hat{x}$, $d \in S^2$ are fixed, the mapping

$$p \in \mathbb{C}^3 \mapsto E_\infty(\hat{x}, d, p) \in \mathbb{C}^3$$

is linear. Therefore, we regard the far field pattern as a matrix valued mapping

$$e_\infty \colon S^2 \times S^2 \to \mathbb{C}^{3\times 3} \; .$$

$e_\infty(\hat{x}, d)$ has the vector $E_\infty(\hat{x}, d, d_k)$ as its kth column where d_1, d_2, d_3 denote the usual cartesian unit vectors.

Analogously to the acoustic case we use a very strong norm $\|\cdot\|_{\mathcal{F}}$ on the far field patterns by prescribing a very rapid decay of the Fourier coefficients

$$\begin{aligned} \mu_{l_1k_1l_2k_2} \;\; &:= \;\; \int_{S^2}\int_{S^2} e_\infty(\hat{x}, d)\overline{Y_{l_1}^{k_1}(\hat{x})Y_{l_2}^{k_2}(d)}ds(\hat{x})ds(d) \; , \qquad (4.13) \\ & l_1, l_2 = 0, 1, \ldots, \; -l_1 \leq k_1 \leq l_1 \; , \; -l_2 \leq k_2 \leq l_2 \; , \end{aligned}$$

of the far field patterns e_∞. Here, the integral over a matrix valued function is defined by computing the integral for each entry of the matrix valued function. Hence, the Fourier coefficients $\mu_{l_1k_1l_2k_2} \in \mathbb{C}^{3\times 3}$ are matrices, too. Furthermore, we denote for a matrix $A = (a_{jk}) \in \mathbb{C}^{3\times 3}$ by

$$\|A\|_F := \Big(\sum_{j,k=1}^{3} |a_{jk}|^2\Big)^{1/2}$$

the Frobenius norm.

We want to derive the estimate

$$\|n - \tilde{n}\|_\infty \leq c\Big[-\ln(\|e_{\infty,n} - e_{\infty,\tilde{n}}\|_{\mathcal{F}})\Big]^{-1/15}$$

with a constant c for all refractive indices n, $\tilde{n}$ lying in some small subset $\mathcal{O}$ of $\tilde{C}(B_R)$. This means that the mapping $e_{\infty,n} \mapsto n$ is continuous and that a local uniqueness result holds. Having the acoustic case in mind we expect that $\mathcal{O}$ is not only small with respect to the maximum norm but with respect to a C^2-norm, i.e., we need additional information in a stronger norm in order to obtain the stability result.

Imitating the reasoning in the acoustic case we start by studying the decay of the Fourier coefficients and proving continuity of the mapping $n \mapsto e_{\infty,n}$.

Next, we reconstruct the kernel of a certain boundary integral operator N_n with the help of a series expansion involving the Fourier coefficients $\mu_{l_1k_1l_2k_2}$. N_n depends continuously on $e_{\infty,n}$. However, since the $\|\cdot\|_{\mathcal{F}}$-norm is a very strong norm, which is not appropriate for measured far field patterns, this mapping is severely ill-posed.

Finally, we investigate the dependence of n on N_n with the help of the special solutions from the last section and arrive at our main estimate.

For convenience we define as in (4.4) the operator $T_n\colon C(\overline{B_R}) \to C(\overline{B_R})$ by

$$\begin{aligned}(T_nU)(x) \;:=\; & \kappa^2 \int\limits_{B_R} \Phi_\kappa(x,y)(1-n(y))U(y)dy \\ & -\nabla \int\limits_{B_R} \Phi_\kappa(x,y)\frac{1}{n(y)}\nabla n(y)\cdot U(y)dy\ ,\ x \in \overline{B_R}\ .\end{aligned}$$

Lemma 4.10 *Assume the far field pattern $e_\infty\colon S^2 \times S^2 \to \mathbb{C}^{3\times 3}$ originates from the refractive index $n \in \tilde{C}(B_R)$ satisfying supp$\,(1-n) \subset B_{R_1}$ for some $0 < R_1 < R$. Let $\mu_{l_1k_1l_2k_2}$ denote the Fourier coefficients of e_∞ as defined in (4.13). Furthermore, define $R_3 := (1/2)(R+R_1)$. Then, there is a constant c depending on e_∞ such that*

$$\|\mu_{l_1k_1l_2k_2}\|_F^2 \le c\Big(\frac{e\kappa R_3}{2l_1+1}\Big)^{2l_1+3}\Big(\frac{e\kappa R_3}{2l_2+1}\Big)^{2l_2+3}\ .$$

We also have

$$\sum_{l_1,k_1,l_2,k_2}\Big(\frac{2l_1+1}{e\kappa R}\Big)^{2l_1+3}\Big(\frac{2l_2+1}{e\kappa R}\Big)^{2l_2+3}\|\mu_{l_1k_1l_2k_2}\|_F^2 < \infty\ .$$

Proof: Using the Lippmann-Schwinger equation (3.19) and the asymptotic behavior of Φ_κ for large $|x|$ we can compute the columns of e_∞ as

$$E_\infty(\hat{x},d,d_k) \;=\; -\frac{\kappa^2}{4\pi}\int\limits_{B_{R_1}}(1-n(y))E(y,d,d_k)e^{-i\kappa\hat{x}\cdot y}dy$$

$$+ \quad \frac{i\kappa}{4\pi}\int\limits_{B_{R_1}} \frac{1}{n(y)}\nabla n(y)\cdot E(y,d,d_k)e^{-i\kappa\hat{x}\cdot y}dy\,\hat{x}\ ,$$

$$\hat{x}, d \in S^2\ ,\ k=1,2,3\ . \tag{4.14}$$

Interchanging the order of integration we obtain

$$\begin{aligned}
&\int\limits_{S^2}\int\limits_{S^2} E_\infty(\hat{x},d,d_k)\overline{Y_{l_1}^{k_1}(\hat{x})Y_{l_2}^{k_2}(d)}ds(\hat{x})ds(d)\\
&= -\frac{\kappa^2}{4\pi}\int\limits_{B_{R_1}}(1-n(y))\int\limits_{S^2} E(y,d,d_k)\overline{Y_{l_2}^{k_2}(d)}ds(d)\int\limits_{S^2} e^{-i\kappa\hat{x}\cdot y}\overline{Y_{l_1}^{k_1}(\hat{x})}ds(\hat{x})dy\\
&\quad -\frac{1}{4\pi}\int\limits_{B_{R_1}}\Big\{\frac{1}{n(y)}\nabla n(y)\cdot\\
&\qquad \cdot\int\limits_{S^2} E(y,d,d_k)\overline{Y_{l_2}^{k_2}(d)}ds(d)\nabla\int\limits_{S^2} e^{-i\kappa\hat{x}\cdot y}\overline{Y_{l_1}^{k_1}(\hat{x})}ds(\hat{x})\Big\}dy\ ,
\end{aligned}$$

and then by the Cauchy-Schwarz inequality

$$\begin{aligned}
\|\mu_{l_1k_1l_2k_2}\|_F^2 &= \sum_{k=1}^{3}\left|\int\limits_{S^2}\int\limits_{S^2} E_\infty(\hat{x},d,d_k)\overline{Y_{l_1}^{k_1}(\hat{x})Y_{l_2}^{k_2}(d)}ds(\hat{x})ds(d)\right|^2\\
&\le c\sum_{k=1}^{3}\left\|\int\limits_{S^2} E(\cdot,d,d_k)\overline{Y_{l_2}^{k_2}(d)}ds(d)\right\|^2_{L^2(B_{R_1})}\cdot\\
&\qquad\cdot\left[\left\|\int\limits_{S^2} e^{-i\kappa d\cdot x}\overline{Y_{l_1}^{k_1}(d)}ds(d)\right\|^2_{L^2(B_{R_1})}\right.\\
&\qquad\qquad\left.+\left\|\nabla\int\limits_{S^2} e^{-i\kappa d\cdot x}\overline{Y_{l_1}^{k_1}(d)}ds(d)\right\|^2_{L^2(B_{R_1})}\right]\ .
\end{aligned} \tag{4.15}$$

c will denote various constants during the proof.

In formula (2.27) we have bounded the first term in $[\ldots]$ by a multiple of $\left(\frac{e\kappa R_1}{2l_1+1}\right)^{2l_1+3}$.

For the first factor and for the second term in $[\ldots]$ we note that by Lemma 2.6 (b) there is a constant c such that the inequality

$$\int\limits_{B_{R_1}}|\nabla u|^2dx+\sum_{l,m=1}^{3}\int\limits_{B_{R_1}}\left|\frac{\partial^2 u}{\partial x_l\partial x_m}\right|^2dx \le c\int\limits_{B_{R_3}}|u|^2dx$$

holds true for all $u \in C^2(\mathbb{R}^3)$ satisfying $\Delta u + \kappa^2 u = 0$ in $\mathbb{R}^3$. We can then estimate

$$\begin{aligned} \Big\| \nabla \int_{S^2} e^{-i\kappa d\cdot x} \overline{Y_{l_1}^{k_1}(d)} ds(d) \Big\|^2_{L^2(B_{R_1})} &\leq c \Big\| \int_{S^2} e^{-i\kappa d\cdot x} \overline{Y_{l_1}^{k_1}(d)} ds(d) \Big\|^2_{L^2(B_{R_3})} \\ &\leq c \Big(\frac{e\kappa R_3}{2l_1+1} \Big)^{2l_1+3} , \end{aligned}$$

where, in the second line, we have used the analogous estimate to (2.27) again. Moreover, we have

$$\begin{aligned} & \Big\| \int_{S^2} E^i(\cdot, d, d_k) \overline{Y_{l_2}^{k_2}(d)} ds(d) \Big\|^2_{L^2(B_{R_1})} \\ &= \Big\| \frac{1}{\kappa^2} \nabla \wedge \nabla \wedge \int_{S^2} e^{i\kappa d\cdot x}\, \overline{Y_{l_2}^{k_2}(d)} d_k ds(d) \Big\|^2_{L^2(B_{R_1})} \\ &\leq c \Big\| \int_{S^2} e^{i\kappa d\cdot x}\, \overline{Y_{l_2}^{k_2}(d)} ds(d) \Big\|^2_{L^2(B_{R_3})} \\ &\leq c \Big(\frac{e\kappa R_3}{2l_2+1} \Big)^{2l_2+3} , \end{aligned}$$

and finally by the boundedness of $(I+T_n)^{-1}$ in $L^2(B_{R_1})$ (see Lemma 4.3):

$$\begin{aligned} & \Big\| \int_{S^2} E(\cdot, d, d_k) \overline{Y_{l_2}^{k_2}(d)} ds(d) \Big\|^2_{L^2(B_{R_1})} \\ &= \Big\| (I+T_n)^{-1} \int_{S^2} E^i(\cdot, d, d_k) \overline{Y_{l_2}^{k_2}(d)} ds(d) \Big\|^2_{L^2(B_{R_1})} \\ &\leq c \Big(\frac{e\kappa R_3}{2l_2+1} \Big)^{2l_2+3} . \end{aligned}$$

Now, we can complete the proof analogously to Lemma 2.17.

□

By this lemma we know that the norm $\|e_{\infty,n}\|_{\mathcal{F}}$ defined by

$$\|e_{\infty,n}\|^2_{\mathcal{F}} := \sum_{l_1,k_1,l_2,k_2} \Big(\frac{2l_1+1}{e\kappa R} \Big)^{2l_1+3} \Big(\frac{2l_2+1}{e\kappa R} \Big)^{2l_2+3} \|\mu_{l_1k_1l_2k_2}\|^2_F$$

is well defined if $n \in \tilde{C}(B_R)$ because supp $(1-n) \subset B_R$ implies that there is a radius $R_1 < R$ with supp $(1-n) \subset B_{R_1}$.

Proceeding similarly to the proof of Lemma 2.18 we can prove the continuous dependence of $e_{\infty,n}$ on n.

Lemma 4.11 *Let $n_0 \in \tilde{C}(B_{R_1})$, $R_1 < R$, be given. Then, there are positive constants c and ϵ such that $\|e_{\infty,n} - e_{\infty,n_0}\|_{\mathcal{F}} \leq c\{\|n-n_0\|_\infty + \|\nabla n - \nabla n_0\|_\infty\}$ for all $n \in \tilde{C}(B_{R_1})$ satisfying $\|n-n_0\|_\infty + \|\nabla n - \nabla n_0\|_\infty < \epsilon$.*

Our next aim is to introduce a certain boundary integral operator N_n which is the electromagnetic analogue of the operator S_n in the acoustic case. To this end we must introduce some spaces of tangential fields on the sphere ∂B_{R_2}. We denote by $T(\partial B_{R_2})$ the continuous, tangential vector fields a on ∂B_{R_2}, i.e., which satisfy $x \cdot a(x) = 0$, $x \in \partial B_{R_2}$, and by $T^{0,\gamma}(\partial B_{R_2})$ the space of uniformly γ-Hölder continuous vector fields a on ∂B_{R_2} which are tangential to ∂B_{R_2}. By Grad ψ we mean the surface gradient of a function $\psi \in C^1(\partial B_{R_2})$. If for $a \in T^{0,\gamma}(\partial B_{R_2})$ there exists a function $\varphi \in C^{0,\gamma}(\partial B_{R_2})$ such that

$$\int_{\partial B_{R_2}} \operatorname{Grad} \psi \cdot a ds = -\int_{\partial B_{R_2}} \psi \varphi ds$$

holds true for all $\psi \in C^1(\partial B_{R_2})$, we define $\operatorname{Div} \mathrm{a} := \varphi$ to be the surface divergence of a. The reader can find more details in [7, p. 161]. The space of all tangential fields from $T^{0,\gamma}(\partial B_{R_2})$ possessing a γ-Hölder continuous surface divergence is denoted by $T_d^{0,\gamma}(\partial B_{R_2})$. Moreover, we introduce the norm $\|a\|_{T_d^{0,\gamma}} := \|a\|_{0,\gamma} + \|\operatorname{Div} a\|_{0,\gamma}$ for $a \in T_d^{0,\gamma}(\partial B_{R_2})$.

In [7, Theorem 6.17] the authors prove that the operator

$$N_1 : T_d^{0,\gamma}(\partial B_{R_2}) \to T_d^{0,\gamma}(\partial B_{R_2})$$

defined by

$$(N_1 a)(x) := 2\nu(x) \wedge \left\{ \nabla \wedge \nabla \wedge \int_{\partial B_{R_2}} \Phi_\kappa(x,y) a(y) ds(y) \right\}, \ x \in \partial B_{R_2},$$

is bounded with respect to $\|\cdot\|_{T_d^{0,\gamma}}$. At this point we slightly deviate from the notation in [7] where the authors define $Na := N_1(\nu \wedge a)$. $N_1 a$ is the tangential component on ∂B_{R_2} of the vector potential

$$E(x) := 2\nabla \wedge \nabla \wedge \int_{\partial B_{R_2}} \Phi_\kappa(x,y) a(y) ds(y), \ x \in \mathbb{R}^3 \setminus \partial B_{R_2}, \tag{4.16}$$

i.e., $\nu \wedge E_+ = \nu \wedge E_- = N_1 a$ on ∂B_{R_2}. The subscripts, $+$ and $-$, indicate that we approach the boundary ∂B_{R_2} from the exterior and interior, respectively. Furthermore, the fields E and $H := (i\kappa)^{-1}\nabla \wedge E$ are a solution to the Maxwell equations in $\mathbb{R}^3 \setminus \partial B_{R_2}$, satisfy the Silver-Müller radiation condition and $\nu \wedge (H_+ - H_-) = -2i\kappa a$ on ∂B_{R_2}.

In order to define an analogous operator N_n we proceed as in the acoustic case and consider the following boundary value problem (BVP):

Given $R_2 > R$, $\kappa > 0$, $n \in \tilde{C}(B_R)$, and $a \in T_d^{0,\gamma}(\partial B_{R_2})$, find E, H defined in $\mathbb{R}^3 \setminus \partial B_{R_2}$ satisfying the following requirements:

$$E|_{B_{R_2}},\ H|_{B_{R_2}} \in C^1(B_{R_2}) \cap C(\overline{B_{R_2}}),$$

$$E|_{\mathbb{R}^3\setminus\overline{B_{R_2}}},\ H|_{\mathbb{R}^3\setminus\overline{B_{R_2}}} \in C^1(\mathbb{R}^3 \setminus \overline{B_{R_2}}) \cap C(\mathbb{R}^3 \setminus B_{R_2}),$$

$$\nabla \wedge E - i\kappa H = 0\ ,\ \ \nabla \wedge H + i\kappa n E = 0 \ \text{ in } \mathbb{R}^3 \setminus \partial B_{R_2},$$

E, H satisfy the Silver-Müller radiation condition,

$$\nu \wedge (E_+ - E_-) = 0,\ \nu \wedge (H_+ - H_-) = -2i\kappa a \text{ on } \partial B_{R_2}.$$

Lemma 4.12 *For all $a \in T_d^{0,\gamma}(\partial B_{R_2})$ the boundary value problem (BVP) has a unique solution E, H. E, H are given by*

$$\begin{aligned} E(x) \;&:=\; 2\nabla \wedge \nabla \wedge \int_{\partial B_{R_2}} \Phi_\kappa(x,y) a(y) ds(y) \\ &\quad -\kappa^2 \int_{B_R} (1 - n(y))\Phi_\kappa(x,y) U(y) dy \\ &\quad +\nabla \int_{B_R} \frac{1}{n(y)} \nabla n(y) \cdot U(y) \Phi_\kappa(x,y) dy\ ,\ \ x \in \mathbb{R}^3 \setminus \partial B_{R_2}\ , \end{aligned} \tag{4.17}$$

and $H := (i\kappa)^{-1}\nabla \wedge E$, where $U \in C(\overline{B_R})$ is the unique solution to the Lippmann-Schwinger equation

$$(U + T_n U)(x) = 2\nabla \wedge \nabla \wedge \int_{\partial B_{R_2}} \Phi_\kappa(x,y) a(y) ds(y)\ ,\ \ x \in \overline{B_R}\ .$$

Proof: If we assume that E, H are a solution to (BVP) with $a = 0$, the first part of the reasoning in Theorem 3.5 implies $E = H = 0$ in the exterior of B_{R_2}, whence $\nu \wedge E_- = \nu \wedge H_- = 0$ on ∂B_{R_2}.

Now, starting from the Stratton-Chu formula (3.4) for E in B_{R_2} and following the considerations which lead to (3.18) and (3.19), we see that E is a solution of the homogeneous Lippmann-Schwinger equation. Thus, we also have $E = H = 0$ in B_{R_2}. This completes the uniqueness proof for (BVP).

In order to prove that E, H defined as in the assertion are a solution to (BVP) for an arbitrary vector field a, we imitate the proof of Lemma 3.6. Taking the divergence of both sides of the Lippmann-Schwinger equation for U implies $\nabla\cdot(nU) = 0$ in B_R. Then, computations as in Lemma 3.6 show that E, H are a radiating solution to the perturbed Maxwell equations. Finally, the properties of the vector potential (4.16) and of the volume potential yield that E, H satisfy the boundary conditions.

□

We are now in a position to define

$$N_n\colon T_d^{0,\gamma}(\partial B_{R_2}) \to T_d^{0,\gamma}(\partial B_{R_2}) \qquad (N_n a)(x) := \nu(x) \wedge E_+(x)\ ,\ x \in \partial B_{R_2}\ , \tag{4.18}$$

E, H being the solution to (BVP). Since the last two terms in the definition (4.17) are C^2-smooth in $\mathbb{R}^3$, the linear operator N_n is well defined and bounded. Furthermore, we also have $N_n a = \nu \wedge E_-$ on ∂B_{R_2}. Finally, we note that the definition of U is possible for any continuous vector field $a \in C(\partial B_{R_2})$, whence the last two integrals on the right hand side of (4.17) are still well defined for $a \in C(\partial B_{R_2})$. This allows to regard $(N_n - N_1)$ as a linear and bounded operator in the spaces $C(\partial B_{R_2})$ or $T(\partial B_{R_2})$.

The following lemma states some properties of N_n.

Lemma 4.13 *The linear operators N_n satisfy:*

(a) $\displaystyle\int_{\partial B_{R_2}} (N_n a)\cdot(b\wedge\nu)\,ds = \int_{\partial B_{R_2}} (a\wedge\nu)\cdot(N_n b)\,ds$ *for all* $a,\, b \in T_d^{0,\gamma}(\partial B_{R_2})$.

(b) The mapping $n \mapsto (N_n - N_1)$, from $(\tilde{C}(B_R), \|\cdot\|_{C^1})$ to the space of linear and bounded operators in $T(\partial B_{R_2})$ equipped with the $\|\cdot\|_\infty$-operator norm, is continuous.

Proof: For a, $b \in T_d^{0,\gamma}(\partial B_{R_2})$ we define E as in (4.17), $H := (i\kappa)^{-1}\nabla \wedge E$, and E', H' analogously where we replace a by b. Then we use the formulas (3.3) and (3.9) to compute

$$\begin{aligned}
&\int_{\partial B_{R_2}} \{(N_n a)\cdot(b\wedge\nu) - (a\wedge\nu)\cdot(N_n b)\}ds \\
&= -\frac{1}{2i\kappa}\int_{\partial B_{R_2}} \{(\nu\wedge E)\cdot([\nu\wedge(H'_+ - H'_-)]\wedge\nu) \\
&\qquad\qquad -([\nu\wedge(H_+ - H_-)]\wedge\nu)\cdot(\nu\wedge E')\}ds \\
&= -\frac{1}{2i\kappa}\int_{\partial B_{R_2}} \{((\nu\wedge E)\cdot H'_+ - (\nu\wedge E')\cdot H_+) \\
&\qquad\qquad -((\nu\wedge E)\cdot H'_- - (\nu\wedge E')\cdot H_-)\}ds \\
&= 0\ .
\end{aligned}$$

This proves part (a).

Assertion (b) can be established along the lines of Lemma 2.20 (c).

□

Now, we examine how to compute the operator N_n from a knowledge of the Fourier coefficients $\mu_{l_1k_1l_2k_2}$ of $e_{\infty,n}$. This also allows to derive continuous dependence of N_n on $e_{\infty,n}$. We remind the reader that $N_n - N_1$ is well defined in $T(\partial B_{R_2})$, hence it makes sense to examine $\|N_n - N_{\tilde n}\|_\infty = \|(N_n - N_1) - (N_{\tilde n} - N_1)\|_\infty$.

Lemma 4.14 *Let the far field pattern $e_{\infty,n}\colon S^2 \times S^2 \to \mathbb{C}^{3\times 3}$ with Fourier coefficients $\mu_{l_1k_1l_2k_2}$ originate from the refractive index $n \in \tilde C(B_R)$. For x, $y \in \partial B_{R_2}$ we define the matrix*

$$k_n(x,y) := -\frac{\kappa^4}{4\pi}\sum_{l_1,k_1,l_2,k_2} i^{l_1-l_2} h_{l_1}^{(1)}(\kappa R_2) h_{l_2}^{(1)}(\kappa R_2) Y_{l_1}^{k_1}\left(\frac{x}{|x|}\right) Y_{l_2}^{k_2}\left(\frac{y}{|y|}\right)\mu_{l_1k_1l_2k_2}\ . \tag{4.19}$$

(a) For all $a \in T_d^{0,\gamma}(\partial B_{R_2})$ there holds

$$(N_n a)(x) - (N_1 a)(x) = 2\nu(x)\wedge\int_{\partial B_{R_2}} k_n(x,y)a(y)ds(y)\ ,\ x\in\partial B_{R_2}\ .$$

(b) *There is a constant* c *such that for all* n, $\tilde{n} \in \tilde{C}(B_R)$ *the inequality* $\|N_n - N_{\tilde{n}}\|_\infty \le c\|e_{\infty,n} - e_{\infty,\tilde{n}}\|_{\mathcal{F}}$ *holds true.*

Proof: As in the proof of Lemma 2.21 (a) we can use the Cauchy-Schwarz inequality for the series in (4.19) together with the rapid decay of the Fourier coefficients (Lemma 4.10) and the estimate for $|h_l^{(1)}(\kappa R_2)|$ (Lemma 2.16) in order to see that the series is absolutely and uniformly convergent on $\partial B_{R_2} \times \partial B_{R_2}$. Therefore, k_n is a well defined continuous matrix valued function.

From the definition of N_n in (4.18) we know that for $a \in T_d^{0,\gamma}(\partial B_{R_2})$ the difference $N_n a - N_1 a$ has the form

$$\begin{aligned}(N_n a - N_1 a)(x) \\ = \quad & -\kappa^2 \nu(x) \wedge \int\limits_{B_R} (1 - n(y))\Phi_\kappa(x,y)U(y)dy \\ & + \nu(x) \wedge \nabla \int\limits_{B_R} \frac{1}{n(y)} \nabla n(y) \cdot U(y)\Phi_\kappa(x,y)dy \ , \ x \in \partial B_{R_2} \ ,\end{aligned} \tag{4.20}$$

where $U \in C(\overline{B_R})$ is the solution to

$$(U + T_n U)(x) = 2\nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} \Phi_\kappa(x,y)a(y)ds(y) \ , \ x \in \overline{B_R} \ .$$

The right hand sides of the last equation and of (4.20) are well defined for any continuous vector field $a \in C(\partial B_{R_2})$ and represent bounded linear operators with respect to the maximum norm. Since $T_d^{0,\gamma}(\partial B_{R_2}) \subset C(\partial B_{R_2})$, it suffices to prove that the right hand side of (4.20) and

$$2\nu(x) \wedge \int\limits_{\partial B_{R_2}} k_n(x,y)a(y)ds(y)$$

coincide for all $x \in \partial B_{R_2}$ and for all $a \in C(\partial B_{R_2})$. As

$$\text{span}\left\{\overline{Y_l^k(\frac{\cdot}{|\cdot|})}d_m : m = 1,2,3; l = 0,1,\ldots; -l \le k \le l\right\}$$

is dense in $C(\partial B_{R_2})$, we establish the desired equality only for the fields $\overline{Y_l^k(\frac{\cdot}{|\cdot|})}d_m$. Here, d_1, d_2, d_3 denote the usual cartesian unit vectors in $\mathbb{R}^3$.

First, we compute for $x \in \partial B_{R_2}$

$$
\begin{aligned}
& 2 \int_{\partial B_{R_2}} k_n(x,y)\overline{Y_{l_2}^{k_2}(\frac{y}{|y|})}d_m ds(y) \\
&= -\frac{\kappa^4 R_2^2}{2\pi} \sum_{l_1,k_1} i^{l_1-l_2} h_{l_1}^{(1)}(\kappa R_2) h_{l_2}^{(1)}(\kappa R_2) Y_{l_1}^{k_1}(\frac{x}{|x|}) \mu_{l_1 k_1 l_2 k_2} d_m \ .
\end{aligned}
$$

For the computation of the right hand side of (4.20) we proceed similarly to the proof of Lemma 2.21 (a) and compute

$$
\begin{aligned}
& 2\nabla \wedge \nabla \wedge \int_{\partial B_{R_2}} \Phi_\kappa(x,y)\overline{Y_{l_2}^{k_2}(\frac{y}{|y|})}d_m \, ds(y) \\
&= 2i\kappa R_2^2 h_{l_2}^{(1)}(\kappa R_2)\frac{(-i)^{l_2}}{4\pi}\nabla \wedge \nabla \wedge \int_{S^2} e^{i\kappa x\cdot d}\overline{Y_{l_2}^{k_2}(d)}d_m \, ds(d) \\
&= 2i\kappa R_2^2 h_{l_2}^{(1)}(\kappa R_2)\frac{(-i)^{l_2}}{4\pi}\kappa^2 \int_{S^2} E^i(x,d,d_m)\overline{Y_{l_2}^{k_2}(d)}ds(d) \ , \ x \in \overline{B_R} \ ,
\end{aligned}
$$

and

$$
\begin{aligned}
U &= 2(I+T_n)^{-1}\{\nabla \wedge \nabla \wedge \int_{\partial B_{R_2}} \Phi_\kappa(\cdot,y)\overline{Y_{l_2}^{k_2}(\frac{y}{|y|})}d_m \, ds(y)\} \\
&= 2i\kappa R_2^2 h_{l_2}^{(1)}(\kappa R_2)\frac{(-i)^{l_2}}{4\pi}\kappa^2 \int_{S^2} E(\cdot,d,d_m)\overline{Y_{l_2}^{k_2}(d)}ds(d) \ .
\end{aligned}
$$

Defining

$$
\begin{aligned}
V(x) &:= -\kappa^2 \int_{B_R} (1-n(y))\Phi_\kappa(x,y)U(y)dy \\
& \quad +\nabla \int_{B_R} \frac{1}{n(y)}\nabla n(y)\cdot U(y)\Phi_\kappa(x,y)dy \ , \ |x| \geq R \ ,
\end{aligned}
$$

we obtain with the help of (4.14)

$$
\begin{aligned}
V_\infty(\hat{x}) &= -\frac{\kappa^2}{4\pi}\int_{B_R} (1-n(y))e^{-i\kappa\hat{x}\cdot y}U(y)dy \\
& \quad +\frac{i\kappa}{4\pi}\int_{B_R} \frac{1}{n(y)}\nabla n(y)\cdot U(y)e^{-i\kappa\hat{x}\cdot y}dy \, \hat{x}
\end{aligned}
$$

$$= \quad 2i\kappa R_2^2 h_{l_2}^{(1)}(\kappa R_2)\frac{(-i)^{l_2}}{4\pi}\kappa^2 \int\limits_{S^2} E_\infty(\hat{x},d,d_m)\overline{Y_{l_2}^{k_2}(d)}ds(d)\ ,\ \hat{x}\in S^2\ ,$$

whence

$$\int\limits_{S^2} V_\infty(\hat{x})\overline{Y_{l_1}^{k_1}(\hat{x})}ds(\hat{x}) = 2i\kappa R_2^2 h_{l_2}^{(1)}(\kappa R_2)\frac{(-i)^{l_2}}{4\pi}\kappa^2 \mu_{l_1k_1l_2k_2} d_m\ . \tag{4.21}$$

Since V is a radiating solution to the Helmholtz equation in the exterior of B_{R_1} if $\operatorname{supp}(1-n)\subset B_{R_1}$, $R_1<R$, according to [7, Theorem 2.14] it has an expansion

$$V(x) = \sum_{l_1=0}^{\infty}\sum_{k_1=-l_1}^{l_1} a_{l_1k_1} h_{l_1}^{(1)}(\kappa|x|)Y_{l_1}^{k_1}(\hat{x})$$

which converges absolutely and uniformly on compact subsets of $\{|x|\geq R\}$. Here, the coefficients $a_{l_1k_1}$ are vectors in $\mathbb{C}^3$. Now, comparing the Fourier coefficients for the far field of the above series expansion with (4.21) we can finish the proof of assertion (a) as in Lemma 2.21 (a).

The analogous estimates to the proof of Lemma 2.21 (b) yield part (b) of the lemma.

□

As in the acoustic case we need a connection between the integrals

$$\Big|\int\limits_{B_R}(n-\tilde{n})E\cdot\tilde{E}dx\Big|$$

and the quantity $\|N_n - N_{\tilde{n}}\|_\infty$ which is established in the following lemma. During the proof of the lemma we employ two more boundary integral operators, namely $L_0: T_d^{0,\gamma}(\partial B_{R_2}) \to T_d^{0,\gamma}(\partial B_{R_2})$ and $M: T_d^{0,\gamma}(\partial B_{R_2}) \to T_d^{0,\gamma}(\partial B_{R_2})$ which are defined by

$$\begin{aligned}(L_0a)(x) \quad &:= \quad 4\nu(x)\wedge\int\limits_{\partial B_{R_2}}\Phi_0(x,y)\Big\{\int\limits_{\partial B_{R_2}}\Phi_0(y,z)a(z)ds(z)\Big\}ds(y)\ ,\\ (Ma)(x) \quad &:= \quad 2\int\limits_{\partial B_{R_2}}\nu(x)\wedge\nabla_x\wedge\{\Phi_\kappa(x,y)a(y)\}ds(y)\ ,\ x\in\partial B_{R_2}\ .\end{aligned} \tag{4.22}$$

The proof of Theorem 6.19 in [7] shows that the operator $I+M+iN_1L_0$ has a continuous inverse in $T_d^{0,\gamma}(\partial B_{R_2})$. Moreover, given a tangential field

$b \in T_d^{0,\gamma}(\partial B_{R_2})$ and defining $a := 2(I + M + iN_1 L_0)^{-1} b \in T_d^{0,\gamma}(\partial B_{R_2})$ the fields

$$
\begin{aligned}
V(x) &:= \nabla \wedge \int\limits_{\partial B_{R_2}} \Phi_\kappa(x,y) a(y) ds(y) \\
&\quad + i \nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} \Phi_\kappa(x,y)(L_0 a)(y) ds(y) \ , \\
W(x) &:= \frac{1}{i\kappa} \nabla \wedge V(x) \ , \ |x| > R_2 \ ,
\end{aligned}
$$

are a radiating solution to the Maxwell equations in $\mathbb{R}^3 \setminus \overline{B_{R_2}}$ with $\nu \wedge V_+ = b$ on ∂B_{R_2}, i.e., we can solve the exterior Dirichlet problem for the Maxwell equations. Finally, Theorem 6.20 in [7] states that $\|\nu \wedge W_+\|_{T_d^{0,\gamma}} \leq c \|b\|_{T_d^{0,\gamma}}$ for a suitable constant c which is independent of b.

Lemma 4.15 *Assume $R < R_2 < R''$ and $c_1 > 0$ are positive constants. Then, there exists a positive constant c such that for all n, $\tilde{n} \in \tilde{C}(B_R)$ with $\|n\|_{C^2}$, $\|\tilde{n}\|_{C^2}$, $\|1/n\|_\infty$, $\|1/\tilde{n}\|_\infty \leq c_1$, and for all solutions E, $H \in C^1(B_{R''}) \cap L^2(B_{R''})$ to $\nabla \wedge E - i\kappa H = 0$, $\nabla \wedge H + i\kappa n E = 0$ in $B_{R''}$ and all solutions $\tilde{E}$, $\tilde{H} \in C^1(B_{R''}) \cap L^2(B_{R''})$ to $\nabla \wedge \tilde{E} - i\kappa \tilde{H} = 0$, $\nabla \wedge \tilde{H} + i\kappa \tilde{n} \tilde{E} = 0$ in $B_{R''}$ the estimate*

$$
\Big| \int\limits_{B_R} (n - \tilde{n}) E \cdot \tilde{E} dx \Big| \leq c \|N_n - N_{\tilde{n}}\|_{\infty, \partial B_{R_2}} \|E\|_{L^2(B_{R''})} \|\tilde{E}\|_{L^2(B_{R''})} \tag{4.23}
$$

holds true.

Proof: As in the acoustic case we extend E, H outside of $\overline{B_{R_2}}$ to radiating solutions V, W to the Maxwell equations such that $\nu \wedge E_- = \nu \wedge V_+$. This allows to connect $(\nu \wedge E_-)|_{\partial B_{R_2}}$ and the operator N_n. We define $V(x) := E(x)$, $x \in B_{R_2}$, and

$$
\begin{aligned}
V(x) &:= 2\nabla \wedge \int\limits_{\partial B_{R_2}} \Phi_\kappa(x,y) a(y) ds(y) \\
&\quad + 2i \nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} \Phi_\kappa(x,y)(L_0 a)(y) ds(y) \ , \ x \in \mathbb{R}^3 \setminus \overline{B_{R_2}} \ ,
\end{aligned}
$$

with $a := (I + M + iN_1 L_0)^{-1} (\nu \wedge E_-)|_{\partial B_{R_2}}$. Moreover, we set $W := (i\kappa)^{-1} \nabla \wedge E$ in $\mathbb{R}^3 \setminus \partial B_{R_2}$. Then, we know $V|_{B_{R_2}}$, $W|_{B_{R_2}} \in C^1(B_{R_2}) \cap C(\overline{B_{R_2}})$, and

$V|_{\mathbb{R}^3\setminus\overline{B_{R_2}}}$, $W|_{\mathbb{R}^3\setminus\overline{B_{R_2}}} \in C^1(\mathbb{R}^3 \setminus \overline{B_{R_2}}) \cap C(\mathbb{R}^3 \setminus B_{R_2})$. Furthermore, V, W satisfy the Silver-Müller radiation condition and $\nu \wedge (V_+ - V_-) = 0$. Hence, we know from Lemma 4.12 that

$$\nu \wedge V_\pm = \frac{1}{2i\kappa} N_n(\nu \wedge (W_- - W_+))$$

on ∂B_{R_2}. From the remarks preceding this lemma we can conclude $\|\nu \wedge (W_- - W_+)\|_\infty \le c_2 \|E\|_{1,\gamma,\overline{B_{R_2}}}$. $\|E\|_{1,\gamma,\overline{B_{R_2}}}$ can be bounded by a multiple of $\|n^{1/2}E\|_{1,\gamma,\overline{B_{R_2}}}$. Since $(n^{1/2}E, H)$ are a solution to a perturbed Helmholtz equation (Lemma 3.4), we obtain from Lemma 2.6 (b) that there are constants c_3, c_4 independent of E, H and n (but dependent on c_1, R_2 and R'') such that

$$\|\nu \wedge (W_- - W_+)\|_\infty \le c_3 \|n^{1/2}E\|_{L^2(B_{R''})} \le c_4 \|E\|_{L^2(B_{R''})} \, . \tag{4.24}$$

We can proceed analogously and define vector fields $\tilde{V}$, $\tilde{W}$ starting with $\tilde{E}$, $\tilde{H} \in C^1(B_{R''}) \cap L^2(B_{R''})$.

Then, we use Lemma 4.13 (a), (3.9) and (3.3) to compute

$$\begin{aligned}
&\frac{1}{2i\kappa} \int_{\partial B_{R_2}} \Big([\nu \wedge (W_- - W_+)] \wedge \nu\Big) \cdot (N_n - N_{\tilde{n}})\Big(\nu \wedge (\tilde{W}_- - \tilde{W}_+)\Big) ds \\
&= \frac{1}{2i\kappa} \int_{\partial B_{R_2}} \Big[N_n(\nu \wedge (W_- - W_+))\Big] \cdot \Big([\nu \wedge (\tilde{W}_- - \tilde{W}_+)] \wedge \nu\Big) ds \\
&\qquad - \frac{1}{2i\kappa} \int_{\partial B_{R_2}} \Big([\nu \wedge (W_- - W_+)] \wedge \nu\Big) \cdot N_{\tilde{n}}\Big(\nu \wedge (\tilde{W}_- - \tilde{W}_+)\Big) ds \\
&= \int_{\partial B_{R_2}} \Big\{(\nu \wedge V_\pm) \cdot (\tilde{W}_- - \tilde{W}_+) - (\nu \wedge \tilde{V}_\pm) \cdot (W_- - W_+)\Big\} ds \\
&= \int_{\partial B_{R_2}} \Big\{(\nu \wedge V_-) \cdot \tilde{W}_- - (\nu \wedge \tilde{V}_-) \cdot W_-\Big\} ds \\
&= -i\kappa \int_{B_R} (n - \tilde{n}) E \cdot \tilde{E} dx \, .
\end{aligned} \tag{4.25}$$

Finally, we conclude from (4.24) and (4.25)

$$\Big| \int_{B_R} (n - \tilde{n}) E \tilde{E} dx \Big|$$

$$= \Big|\frac{1}{2\kappa^2}\int_{\partial B_{R_2}} \Big([\nu\wedge(W_- - W_+)]\wedge\nu\Big)\cdot(N_n - N_{\tilde n})\Big(\nu\wedge(\tilde W_- - \tilde W_+)\Big)ds\Big|$$

$$\leq c\|N_n - N_{\tilde n}\|_{\infty,\partial B_{R_2}}\|E\|_{L^2(B_{R''})}\|\tilde E\|_{L^2(B_{R''})} ,$$

and we have proved the lemma.

□

We are now in a position to prove our main estimate which implies the continuous dependence of n on N_n or $e_{\infty,n}$.

Theorem 4.16 *Let $n_0 \in \tilde C(B_R)$ be given. Then, there are a neighborhood $\mathcal{O}$ of n_0 of the form*

$$\mathcal{O} := \{n \in \tilde C(B_R) \colon \|n - n_0\|_{C^2} < \epsilon\} ,$$

and a positive constant c, such that for all n, $\tilde n \in \mathcal{O}$ the estimate

$$\|n - \tilde n\|_{\infty,B_R} \leq c[-\ln(\|N_n - N_{\tilde n}\|_{\infty,\partial B_{R_2}})]^{-1/15}$$

holds true.

Proof: We choose $R < R_2 < R'' < R' < 2R_2$. Furthermore, with $\mathcal{Q}_{n_0}$ defined as in (3.13) for the refractive index n_0 we set

$$t_1 := 2\frac{R'}{\pi}\{\max_{x\in\overline{B_R}}\|\mathcal{Q}_{n_0}(x)\|_2 + \kappa^2\|1 - n_0\|_\infty + 1\} + 2\kappa + 200 ,$$

and choose $0 < \epsilon_1 < 1/2$ sufficiently small to ensure

$$\frac{-7}{15(4R_2 + 1)}\ln(2\epsilon_1) > t_1 .$$

Due to the continuous dependence of $\mathcal{Q}_n$ and of $(N_n - N_1)$ on n (Lemma 4.13 (b)) we can find ϵ with $0 < \epsilon < \epsilon_1$ such that

$$\max_{x\in\overline{B_R}}\|\mathcal{Q}_{n_0}(x) - \mathcal{Q}_n(x)\|_2 + \kappa^2\|n - n_0\|_\infty \leq 1$$

and

$$\|N_n - N_{\tilde n}\|_{\infty,\partial B_{R_2}} \leq 2\epsilon_1$$

for all

$$n, \tilde n \in \mathcal{O} := \{n \in \tilde C(B_R) \colon \|n - n_0\|_{C^2} < \epsilon\} .$$

From (2.38) and (2.39) we know that for n, $\tilde{n} \in \mathcal{O}$ and any $\rho \geq 2$ the estimate

$$\|n - \tilde{n}\|_\infty \leq (2R')^{-3/2} \sum_{\alpha\cdot\alpha \leq \rho^2} |(n-\tilde{n})\hat{}(\alpha)| + \frac{c}{\sqrt{\rho}} \tag{4.26}$$

holds true. c may denote various constants during the proof.

As in the proof of Theorem 2.23 we have to pick a suitable ρ, depending on $\|N_n - N_{\tilde{n}}\|_{\infty,\partial B_{R_2}}$, in order to estimate the right hand side of (4.26). The Fourier coefficients $|(n-\tilde{n})\hat{}(\alpha)|$, $\alpha \cdot \alpha \leq \rho^2$, can be bounded by using the preceding lemma and the special solutions from the uniqueness proof for the inverse problem.

We choose $t := -\frac{7}{15(4R_2+1)} \ln \|N_n - N_{\tilde{n}}\|_\infty$ and $\rho := t^{2/15}$. Then, the inequalities $\|N_n - N_{\tilde{n}}\|_\infty < 1$ and $t \geq t_1$ are satisfied for all n, $\tilde{n} \in \mathcal{O}$ by the definition of ϵ and we also have $\rho \geq 2$.

For a vector $\alpha \in \Gamma$ with $\alpha \cdot \alpha \leq \rho^2$ we choose as in Theorem 4.9

$$\begin{aligned}
\zeta_t &:= -\frac{1}{2}\alpha + i\sqrt{t^2 - \kappa^2 + \frac{|\alpha|^2}{4}}d_1 + td_2 \ , \\
\tilde{\zeta}_t &:= -\frac{1}{2}\alpha - i\sqrt{t^2 - \kappa^2 + \frac{|\alpha|^2}{4}}d_1 - td_2 \ , \\
\eta_t &:= \frac{1}{|\alpha|}\alpha + \frac{|\alpha|}{2t}d_2 \ , \\
\tilde{\eta}_t &:= \frac{1}{|\alpha|}\alpha - \frac{|\alpha|}{2t}d_2 \ .
\end{aligned}$$

Then, we have $|\Im(\zeta_t)| \geq t - \kappa \geq t/2$, $|\zeta_t|/|\Im(\zeta_t)| \leq 2$, and $|\Im(\zeta_t)| \geq t - \kappa \geq t_0$ for all $n \in \mathcal{O}$ (t_0 as in (4.10)), whence by Theorem 4.8 there exist the special solutions $E(x, \zeta_t, \eta_t) = e^{i\zeta_t \cdot x}(\eta_t + f(x, \zeta_t, \eta_t) + V(x, \zeta_t, \eta_t))$ and the $L^2(B_{R''})$-norms $\|f(\cdot, \zeta_t, \eta_t)\|_{L^2} + \|V(\cdot, \zeta_t, \eta_t)\|_{L^2}$ can be bounded by $(|\eta_t|c)/|\Im(\zeta_t)|$ uniformly in $n \in \mathcal{O}$, $t \geq t_1$. The analogous assertions apply to $\tilde{E}(x, \tilde{\zeta}_t, \tilde{\eta}_t) = e^{i\tilde{\zeta}_t \cdot x}(\tilde{\eta}_t + \tilde{f}(x, \tilde{\zeta}_t, \tilde{\eta}_t) + \tilde{V}(x, \tilde{\zeta}_t, \tilde{\eta}_t))$.

Now, we estimate with the help of the preceding lemma

$$\begin{aligned}
&|(\tilde{n} - n)\hat{}(\alpha)| \\
&= (2R')^{-3/2} \Big| \int_C (\tilde{n} - n)(x) e^{-i\alpha\cdot x} dx \Big| \\
&= (2R')^{-3/2} \Big| \int_{B_R} (\tilde{n} - n)(x) E(x, \zeta_t, \eta_t) \cdot \tilde{E}(x, \tilde{\zeta}_t, \tilde{\eta}_t) dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_{B_R}(\tilde{n}-n)(x)e^{-i\alpha\cdot x}\Big\{\frac{|\alpha|^2}{4t^2}+|\alpha|(f(x,\zeta_t,\eta_t)+\tilde{f}(x,\tilde{\zeta}_t,\tilde{\eta}_t))\\
&\quad-(\frac{|\alpha|^2}{2}-\kappa^2)f(x,\zeta_t,\eta_t)\tilde{f}(x,\tilde{\zeta}_t,\tilde{\eta}_t)-V(x,\zeta_t,\eta_t)\cdot\tilde{V}(x,\tilde{\zeta}_t,\tilde{\eta}_t)\\
&\quad-(\eta_t+f(x,\zeta_t,\eta_t)\zeta_t)\cdot\tilde{V}(x,\tilde{\zeta}_t,\tilde{\eta}_t)\\
&\quad-(\tilde{\eta}_t+\tilde{f}(x,\tilde{\zeta}_t,\tilde{\eta}_t)\tilde{\zeta}_t)\cdot V(x,\zeta_t,\eta_t)\Big\}dx\Big|\\
\leq\ & c\|N_n-N_{\tilde{n}}\|_\infty\|E(\cdot,\zeta_t,\eta_t)\|_{L^2(B_{R''})}\|\tilde{E}(\cdot,\tilde{\zeta}_t,\tilde{\eta}_t)\|_{L^2(B_{R''})}+\frac{c|\alpha|^4}{t}\\
\leq\ & c(|\alpha|^2e^{4R_2(t+|\alpha|)}\|N_n-N_{\tilde{n}}\|_\infty+\frac{|\alpha|^4}{t})\ ,
\end{aligned}
\tag{4.27}
$$

where we have used the fact that

$$
\begin{aligned}
\|E(\cdot,\zeta_t,\eta_t)\|_{L^2(B_{R''})} &\leq \|e^{i\zeta_t\cdot x}\|_{\infty,B_{2R_2}}\|\eta_t+f(\cdot,\zeta_t,\eta_t)\zeta_t+V(\cdot,\zeta_t,\eta_t)\|_{L^2(B_{R''})}\\
&\leq c|\alpha|e^{2R_2(t+|\alpha|)}
\end{aligned}
$$

for all $t\geq t_1$, $n\in\mathcal{O}$, and $\alpha\in\Gamma$ because $|\Im(\zeta_t)|\leq t+|\alpha|$. Note that the terms in (4.6) originating from the integral containing derivatives of the modified fundamental solution contribute $|\alpha|^2$ in formula (4.27) and so does our choice of η_t and of $\tilde{\eta}_t$. These powers of $|\alpha|$, which did not occur in the acoustic case, imply the different exponent $-1/15$ in the assertion when compared to the exponent $-1/7$ in the acoustic case.

Inequality (4.27) implies

$$
\begin{aligned}
\sum_{\alpha\cdot\alpha\leq\rho^2}|(\tilde{n}-n)\hat{\ }(\alpha)| &\leq c\sum_{\alpha\cdot\alpha\leq\rho^2}(|\alpha|^2e^{4R_2(t+|\alpha|)}\|N_n-N_{\tilde{n}}\|_\infty+\frac{|\alpha|^4}{t})\\
&\leq c\{e^{4R_2t}e^{4R_2\rho}\rho^5\|N_n-N_{\tilde{n}}\|_\infty+\frac{\rho^7}{t}\}\\
&\leq c\{e^{(4R_2+1)(t+\rho)}\|N_n-N_{\tilde{n}}\|_\infty+\frac{\rho^7}{t}\}\ ,
\end{aligned}
$$

because of $\rho^5\leq 5!e^\rho$.

Finally, we obtain from (4.26), our last estimate, $\rho=t^{2/15}\leq t$, and the definition of t

$$
\|n-\tilde{n}\|_\infty \leq c\{e^{(4R_2+1)(t+\rho)}\|N_n-N_{\tilde{n}}\|_\infty+\frac{\rho^7}{t}+\frac{1}{\sqrt{\rho}}\}
$$

$$\begin{aligned}
&\leq \; c\{e^{(8R_2+2)t}\|N_n - N_{\tilde{n}}\|_\infty + \frac{2}{t^{1/15}}\} \\
&\leq \; c\Big\{(\|N_n - N_{\tilde{n}}\|_\infty)^{1/15} + (-\ln\|N_n - N_{\tilde{n}}\|_\infty)^{-1/15}\Big\} \\
&\leq \; c(-\ln\|N_n - N_{\tilde{n}}\|_\infty)^{-1/15}
\end{aligned}$$

for all n, $\tilde{n} \in \mathcal{O}$ because $x \leq (-\ln(x))^{-1}$ for $0 < x < 1$, and we have proved the theorem.

□

Theorem 4.17 *Let $n_0 \in \tilde{C}(B_{R_1})$ with $R_1 < R$ be given. Then, there are a neighborhood $\mathcal{O}$ of n_0 of the form*

$$\mathcal{O} := \{n \in \tilde{C}(B_{R_1}) \colon \|n - n_0\|_{C^2} < \epsilon\} \ ,$$

and a positive constant c, such that for all n, $\tilde{n} \in \mathcal{O}$ the estimate

$$\|n - \tilde{n}\|_{\infty, B_R} \leq c[-\ln(\|e_{\infty,n} - e_{\infty,\tilde{n}}\|_{\mathcal{F}})]^{-1/15}$$

holds true.

Proof: We know from Lemma 4.11 that the mapping $n \mapsto e_{\infty,n}$ is continuous from $\tilde{C}(B_{R_1})$ to the far field patterns equipped with the norm $\|\cdot\|_{\mathcal{F}}$. Then, in the proof of Theorem 4.16 we can choose $\epsilon > 0$ sufficiently small to satisfy the additional requirements

$$(1 + c')\|e_{\infty,n} - e_{\infty,\tilde{n}}\|_{\mathcal{F}} \leq 2\epsilon_1 \quad \text{and} \quad c'\|e_{\infty,n} - e_{\infty,\tilde{n}}\|_{\mathcal{F}} \leq \|e_{\infty,n} - e_{\infty,\tilde{n}}\|_{\mathcal{F}}^{1/2}$$

for all n, $\tilde{n} \in \mathcal{O}$, too, where c' denotes the constant c from Lemma 4.14 (b). Inserting the estimate

$$\|N_n - N_{\tilde{n}}\|_\infty \leq c'\|e_{\infty,n} - e_{\infty,\tilde{n}}\|_{\mathcal{F}}$$

from Lemma 4.14 (b) into Theorem 4.16 we arrive at the assertion of the theorem.

□

4.3 The Reconstruction of the Refractive Index

In this section we derive a method to reconstruct the refractive index n from a knowledge of the far field pattern $e_{\infty,n}$. We shall give a procedure to compute the Fourier coefficients $(n-1)\hat{}(\alpha)$, if the Fourier coefficients $\mu_{l_1k_1l_2k_2}$ of $e_{\infty,n}$ are known. However, although our method is theoretically satisfying it is certainly not appropriate for practical computations because we never take care of the severe ill-posedness of the problem.

The main ideas from the acoustic case are also applicable to the electromagnetic problem. There are some additional technical difficulties because replacing the Robin boundary value problem from section 2.3 by an impedance boundary value problem for the Maxwell equations leads to more complicated integral equations, even for $n = 1$, than those encountered in the acoustic case (see [7, section 9.5] and references given there). The impedance map which is considered in [45, 38, 39] can only be employed, if κ is not an eigenvalue in B_{R_2}.

Therefore, we choose an unphysical boundary value problem for the perturbed Maxwell equations by prescribing $\nu\wedge H - L_0(\nu\wedge E)$ on ∂B_{R_2} where the operator L_0 is defined in (4.22). We prove that the map $\Lambda_n : T_d^{0,\gamma}(\partial B_{R_2}) \to T_d^{0,\gamma}(\partial B_{R_2})$, given by $\nu\wedge H - L_0(\nu\wedge E) \mapsto \nu\wedge E$, is well defined for all $\kappa > 0$ and can be computed from the Fourier coefficients $\mu_{l_1k_1l_2k_2}$ of $e_{\infty,n}$.

In the second step we derive a uniquely solvable equation of the form

$$(I - A_{n,\zeta})b_{\zeta,\eta} = \nu\wedge H^i_{\zeta,\eta} - L_0(\nu\wedge E^i_{\zeta,\eta}) \quad \text{on } \partial B_{R_2}$$

for the boundary values $b_{\zeta,\eta} = \nu\wedge H(\cdot,\zeta,\eta) - L_0(\nu\wedge E(\cdot,\zeta,\eta))$ of the special solutions $E(\cdot,\zeta,\eta)$, $H(\cdot,\zeta,\eta)$ from Theorem 4.8. Here $E^i_{\zeta,\eta}(x) = \eta\cdot e^{i\zeta\cdot x}$, $H^i_{\zeta,\eta}(x) = (i\kappa)^{-1}\nabla\wedge E^i_{\zeta,\eta}(x)$ are known. $A_{n,\zeta}$ is a compact operator, which can be computed with the help of Λ_n and integral operators having kernels originating from the unphysical fundamental solution Ψ_ζ. Hence, given $e_{\infty,n}$, we can obtain $\nu\wedge H(\cdot,\zeta,\eta) - L_0(\nu\wedge E(\cdot,\zeta,\eta))$ and $\nu\wedge E(\cdot,\zeta,\eta)$ on ∂B_{R_2}.

In the last step, similarly to the Uniqueness Theorem 4.9, we obtain the Fourier coefficients $(n-1)\hat{}(\alpha)$ from the above boundary data.

Our first aim is the definition of the operator Λ_n. To this end we have to prove that the following boundary value problem (RP) has a unique solution:

Given $0 < R < R_2$, $\kappa > 0$, $n \in \tilde{C}(B_R)$ and $b \in T_d^{0,\gamma}(\partial B_{R_2})$, find $E,\ H \in C^1(B_{R_2}) \cap C(\overline{B_{R_2}})$ satisfying the perturbed Maxwell equations

$$\nabla \wedge E - i\kappa H = 0\ ,\ \nabla \wedge H + i\kappa n E = 0 \qquad \text{in } B_{R_2},$$

and

$$\nu \wedge H - L_0(\nu \wedge E) = b \qquad \text{on } \partial B_{R_2}.$$

We would like to prove that the vector fields E, defined in (4.17), together with $H := (i\kappa)^{-1}\nabla\wedge E$ are a solution to (RP), if the density $a \in T_d^{0,\gamma}(\partial B_{R_2})$ is chosen appropriately. We already know from the preceding section $\nu \wedge E_- = N_n a$. Since we also need $\nu \wedge H_-$, we must study the boundary values of H more closely. To this end we define $M_n \colon T_d^{0,\gamma}(\partial B_{R_2}) \to T_d^{0,\gamma}(\partial B_{R_2})$ by

$$\begin{aligned}(M_n a)(x) \quad := \quad & 2 \int\limits_{\partial B_{R_2}} \nu(x) \wedge [\nabla_x \wedge \{\Phi_\kappa(x,y)a(y)\}]ds(y) \\ & -\nu(x) \wedge \nabla \wedge \int\limits_{B_R} (1-n(y))\Phi_\kappa(x,y)U(y)dy\ ,\ x \in \partial B_{R_2}\ ,\end{aligned} \tag{4.28}$$

where $U \in C(\overline{B_R})$ is the unique solution to

$$U + T_n U = 2\nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} \Phi_\kappa(\cdot,y)a(y)ds(y) \qquad \text{in } \overline{B_R}.$$

Regarding the kernel k_n from (4.19) as being defined in a neighborhood of ∂B_{R_2}, $M_n - M_1$ corresponds to an integral operator having as kernel the matrix valued function $\hat{k}_n(x,y)$, where the mth column of $\hat{k}_n(x,y)$ is given by $\hat{k}_n(x,y)d_m = 2\kappa^{-2}\nu(x) \wedge \nabla_x \wedge \{k_n(x,y)d_m\}$. Hence, the relation between N_n and M_n is similar to the one between S_n and K_n' in the acoustic case.

The mapping properties of volume potentials imply that T_n is a compact operator in $C^{1,\gamma}(\overline{B_R})$. Hence, by the Riesz-Fredholm theory $(I+T_n)^{-1}$ is bounded in $C^{1,\gamma}(\overline{B_R})$ and $U \in C^{1,\gamma}(\overline{B_R})$. Then, the mapping properties of the volume potential and of the vector potential also imply that M_n is a linear and bounded operator.

If E is defined as in (4.17), i.e.,

$$E(x) \quad := \quad 2\nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} \Phi_\kappa(x,y)a(y)ds(y)$$

$$-\kappa^2 \int\limits_{B_R} (1-n(y))\Phi_\kappa(x,y)U(y)dy$$

$$+\nabla \int\limits_{B_R} \frac{1}{n(y)} \nabla n(y) \cdot U(y)\Phi_\kappa(x,y)dy \ , \ x \in \mathbb{R}^3 \setminus \partial B_{R_2} \ ,$$

with U as above, we have

$$\begin{aligned} H(x) &:= \frac{1}{i\kappa}\nabla \wedge E(x) \\ &= -2i\kappa\nabla \wedge \int\limits_{\partial B_{R_2}} \Phi_\kappa(x,y)a(y)ds(y) \\ &\quad +i\kappa\nabla \wedge \int\limits_{B_R} (1-n(y))\Phi_\kappa(x,y)U(y)dy \ , \ x \in \mathbb{R}^3 \setminus \partial B_{R_2} \ , \end{aligned}$$

whence by the jump relations

$$\nu \wedge H_- = -i\kappa(M_n - I)a \text{ and } \nu \wedge H_+ = -i\kappa(M_n + I)a \text{ on } \partial B_{R_2}.$$

The regularity properties of volume potentials and of the boundary layer potential also imply $E,\, H \in C^1(B_{R_2}) \cap C(\overline{B_{R_2}})$ and $E,\, H \in C^1(\mathbb{R}^3 \setminus \overline{B_{R_2}}) \cap C(\mathbb{R}^3 \setminus B_{R_2})$.

In the following lemma we show that M_n is compact and how it can be computed from a knowledge of $e_{\infty,n}$.

Lemma 4.18 *Assume the far field pattern $e_{\infty,n}\colon S^2 \times S^2 \to \mathbb{C}^{3\times 3}$ originates from the refractive index $n \in \tilde{C}(B_R)$ and has the Fourier coefficients $\mu_{l_1k_1l_2k_2}$. Furthermore, define the matrix valued function $\tilde{k}_n\colon \partial B_{R_2} \times \partial B_{R_2} \to \mathbb{C}^{3\times 3}$ which has as its mth column the vector*

$$\begin{aligned} &\tilde{k}_n(x,y)d_m \\ &:= -\frac{\kappa^2}{4\pi R_2} \sum_{l_1,k_1,l_2,k_2} i^{l_1-l_2} h^{(1)}_{l_2}(\kappa R_2) Y^{k_2}_{l_2}(\frac{y}{|y|}) \Big\{ \kappa \Big(\frac{dh^{(1)}_{l_1}}{dt}\Big)(\kappa R_2) Y^{k_1}_{l_1}(\frac{x}{|x|})x \\ &\quad + h^{(1)}_{l_1}(\kappa R_2)(\operatorname{Grad} Y^{k_1}_{l_1})(\frac{x}{|x|}) \Big\} \wedge (\mu_{l_1k_1l_2k_2} d_m) \ , \ x,y \in \partial B_{R_2} \qquad (4.29) \end{aligned}$$

(d_1, d_2, d_3 denoting the cartesian unit vectors).

(a) The operator $M_n\colon T^{0,\gamma}_d(\partial B_{R_2}) \to T^{0,\gamma}_d(\partial B_{R_2})$ defined in (4.28) is compact ($0 < \gamma < 1$).

(b) For all $a \in T_d^{0,\gamma}(\partial B_{R_2})$ there holds

$$(M_n a)(x) = (Ma)(x) + 2\nu(x) \wedge \int\limits_{\partial B_{R_2}} \tilde{k}_n(x,y)a(y)ds(y) \ , \ x \in \partial B_{R_2} \ ,$$

where $M = M_1$ is defined in (4.22).

Proof: The series expansion in (4.29) is absolutely and uniformly convergent on $\partial B_{R_2} \times \partial B_{R_2}$. Hence $\tilde{k}_n$ is a well defined, continuous matrix valued function.

For assertion (a) we observe that the operator M from (4.22) is compact ([7, Theorem 6.16]) and that the mapping $T_d^{0,\gamma}(\partial B_{R_2}) \to C(\overline{B_R})$, $a \mapsto U$, is compact. Since Φ_κ is a smooth function for $x \in \partial B_{R_2}$, $y \in \overline{B_R}$, we can conclude that $M_n - M$ is compact, whence M_n is compact.

Assertion (b) is proved along the lines of Lemma 4.14 (a) by checking for $a = \overline{Y_{l_2}^{k_2}(\frac{\cdot}{|\cdot|})}\, d_m$ the coincidence of

$$2\nu \wedge \int\limits_{\partial B_{R_2}} \tilde{k}_n(\cdot, y)a(y)ds(y)$$

and the term originating from the volume potential in the definition of M_n.

The latter can be represented as $\kappa^{-2}\nu(x) \wedge [\nabla \wedge V(x)]$, $x \in \partial B_{R_2}$, with V defined as in the proof of Lemma 4.14 (a). Applying $\kappa^{-2}\nabla \wedge \cdot$ to the series expansion of V given in the proof of Lemma 4.14 (a) yields the assertion.

□

Before turning to the boundary value problem (RP) let us define

$$(S_0 a)(x) := 2 \int\limits_{\partial B_{R_2}} \Phi_0(x,y)a(y)ds(y) \ , \ x \in \partial B_{R_2} \ ,$$

for a vector field $a \in C(\partial B_{R_2})$. S_0 is an injective operator (see the proof of Theorem 3.10 in [7]) and can be regarded as a bounded operator from $C(\partial B_{R_2})$ to $C^{0,\gamma}(\partial B_{R_2})$ or as a bounded operator between $C^{0,\gamma}(\partial B_{R_2})$ and $C^{1,\gamma}(\partial B_{R_2})$. Moreover, S_0 is symmetric with respect to the $L^2(\partial B_{R_2})$-scalar product. Note that the operator L_0 from (4.22) has the form $L_0 a = \nu \wedge S_0 S_0 a$. The identity $\operatorname{Div}(\nu \wedge E) = -\nu \cdot (\nabla \wedge E)$ on ∂B_{R_2} for smooth vector fields E

in $\overline{B_{R_2}}$ ([7, (6.38)]) together with the jump relation for the first derivatives of the single-layer potential ([7, Theorem 6.12]) implies the relation

$$(\mathrm{Div}(L_0 a))(x) = -2 \int_{\partial B_{R_2}} \nu(x) \cdot \nabla_x \wedge \{\Phi_0(x,y)(S_0 a)(y)\} ds(y) \ , \ x \in \partial B_{R_2} \ ,$$

the integral being a Cauchy principal value. Hence, L_0 is a bounded operator from $T(\partial B_{R_2})$ to $T_d^{0,\gamma}(\partial B_{R_2})$.

We are now in a position to prove that (RP) has a unique solution. We start with uniqueness.

Lemma 4.19 *If E, H are a solution to (RP) with $b = 0$, then $E = H = 0$ in $\overline{B_{R_2}}$.*

Proof: If E, H are a solution to the homogeneous problem (RP), we compute

$$\begin{aligned}
\int_{\partial B_{R_2}} |S_0(\nu \wedge E)|^2 ds &= \int_{\partial B_{R_2}} (\nu \wedge \overline{E}) \cdot S_0^2(\nu \wedge E) ds \\
&= -\int_{\partial B_{R_2}} \overline{E} \cdot L_0(\nu \wedge E) ds \\
&= -\int_{\partial B_{R_2}} \overline{E} \cdot (\nu \wedge H) ds \\
&= \int_{B_{R_2}} \{(\nabla \wedge \overline{E}) \cdot H - \overline{E} \cdot (\nabla \wedge H)\} dx \\
&= -i\kappa \int_{B_{R_2}} \{|H|^2 - n|E|^2\} dx \ .
\end{aligned}$$

Taking the real part of this equation we can conclude $S_0(\nu \wedge E) = 0$, whence $\nu \wedge H = 0$ by the boundary condition and $\nu \wedge E = 0$ by the injectivity of S_0. Now, the Stratton-Chu formula for E yields via (3.18) that E is a solution to the homogeneous Lippmann-Schwinger equation and therefore must vanish identically.

□

Theorem 4.20 *For any given $b \in T_d^{0,\gamma}(\partial B_{R_2})$ there is a unique solution E, H to (RP). The mapping $\Lambda_n : T_d^{0,\gamma}(\partial B_{R_2}) \to T_d^{0,\gamma}(\partial B_{R_2})$, $\Lambda_n b := \nu \wedge E$, is well defined and can be computed from $e_{\infty,n}$. The linear operator $P : T_d^{0,\gamma}(\partial B_{R_2}) \to C^{1,\gamma}(\overline{B_R})$ defined by $Pb = E|_{\overline{B_R}}$ is compact.*

Proof: Our reasoning before Lemma 4.18 implies that E defined as in (4.17) and $H := (i\kappa)^{-1}\nabla \wedge E$ satisfy the boundary conditions of (RP), if $a \in T_d^{0,\gamma}(\partial B_{R_2})$ is a solution to

$$[-i\kappa(M_n - I) - L_0 N_n]a = b\ . \tag{4.30}$$

As in the proof of Lemma 4.12 E and H also satisfy the differential equations.

Since N_n is bounded in $T_d^{0,\gamma}(\partial B_{R_2})$ (see (4.18)), since the imbedding $T_d^{0,\gamma}(\partial B_{R_2}) \subset T(\partial B_{R_2})$ is compact, and since $L_0: T(\partial B_{R_2}) \to T_d^{0,\gamma}(\partial B_{R_2})$ is bounded (see the remark before Lemma 4.19), the operator $L_0 N_n$ is a compact operator in $T_d^{0,\gamma}(\partial B_{R_2})$. Due to the compactness of the operator M_n (Lemma 4.18 (a)) in $T_d^{0,\gamma}(\partial B_{R_2})$ it suffices to prove that equation (4.30) has a trivial nullspace.

If $a \in T_d^{0,\gamma}(\partial B_{R_2})$ is a solution to the homogeneous equation (4.30), we define E as in (4.17) and $H := (i\kappa)^{-1}\nabla\wedge E$ in $\mathbb{R}^3\backslash\partial B_{R_2}$. From the uniqueness of (RP) we can conclude $E|_{B_{R_2}} = H|_{B_{R_2}} = 0$. Since E, H are a radiating solution to the Maxwell equations in the exterior of $\overline{B_{R_2}}$ with $\nu \wedge E_+ = \nu \wedge E_- = 0$, the uniqueness of the exterior Maxwell problem also implies $E = H = 0$ in the exterior of $\overline{B_{R_2}}$ ([7, Theorem 6.18]). We can now complete the existence proof by observing $0 = \nu \wedge (H_+ - H_-) = -2i\kappa a$.

The proof of the unique solvability of (RP) implies that

$$\Lambda_n = N_n[-i\kappa(M_n - I) - L_0 N_n]^{-1}\ ,$$

i.e., $\Lambda_n: T_d^{0,\gamma}(\partial B_{R_2}) \to T_d^{0,\gamma}(\partial B_{R_2})$ is well defined, bounded, and can be computed from $e_{\infty,n}$ because the kernels of the integral operators M_n and N_n can be computed from the Fourier coefficients of $e_{\infty,n}$.

Finally, the boundedness of $[-i\kappa(M_n - I) - L_0 N_n]^{-1}$ in $T_d^{0,\gamma}(\partial B_{R_2})$ together with the boundedness of $(I + T_n)^{-1}$ in $C^{1,\gamma}(\overline{B_R})$ and the compactness of the mapping $T_d^{0,\gamma}(\partial B_{R_2}) \times C^{1,\gamma}(\overline{B_R}) \to C^{1,\gamma}(\overline{B_R})$

$$\begin{aligned}(a, U) \ \mapsto\ & 2\nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} \Phi_\kappa(x, y)a(y)ds(y) \\ & -\kappa^2 \int\limits_{B_R} (1 - n(y))\Phi_\kappa(x, y)U(y)dy \\ & +\nabla \int\limits_{B_R} \frac{1}{n(y)}\nabla n(y) \cdot U(y)\Phi_\kappa(x, y)dy\ ,\ x \in \overline{B_R}\ ,\end{aligned}$$

imply the compactness of P.

□

Our next aim is to compute the boundary data $b_{\zeta,\eta} = \nu \wedge H(\cdot,\zeta,\eta) - L_0(\nu \wedge E(\cdot,\zeta,\eta))$ of the special solutions used in the Uniqueness Theorem 4.9 assuming that $e_{\infty,n}$, ζ and η are known. To this end we use a uniquely solvable Fredholm equation of the second kind for $b_{\zeta,\eta}$, which only contains Λ_n and integral operators having Ψ_ζ and its derivatives as kernels.

We choose $R < R_2 < R'' < R'$. First, we need an analogous representation to Theorem 3.3 for the special solutions $E(\cdot,\zeta,\eta)$, $H(\cdot,\zeta,\eta)$ in the spherical shell $R_2 < |x| < R''$ where the fundamental solution Φ_κ is replaced by Ψ_ζ.

Lemma 4.21 *Assume $\kappa > 0$, $n \in \tilde{C}(B_R)$, and $\zeta,\ \eta \in \mathbb{C}^3$ satisfy $|\Im(\zeta)| \geq 2\kappa^2(R'/\pi)\|1-n\|_\infty + 1$, $\eta \cdot \zeta = 0$, and $\zeta \cdot \zeta = \kappa^2$. Furthermore, define $E^i_{\zeta,\eta}(x) := \eta e^{i\zeta\cdot x}$, $H^i_{\zeta,\eta}(x) := (i\kappa)^{-1}\nabla \wedge E^i_{\zeta,\eta}(x)$ in $\mathbb{R}^3$. If $E(\cdot,\zeta,\eta) \in C(\overline{B_{R''}})$ is a solution to the modified Lippmann-Schwinger equation*

$$\begin{aligned} E(x,\zeta,\eta) &= E^i_{\zeta,\eta}(x) - \kappa^2 \int\limits_{B_R} \Psi_\zeta(x-y)(1-n(y))E(y,\zeta,\eta)dy \\ &\quad + \nabla \int\limits_{B_R} \Psi_\zeta(x-y)\frac{1}{n(y)}\nabla n(y)\cdot E(y,\zeta,\eta)dy\ ,\ \ x \in \overline{B_{R''}}\ , \end{aligned} \tag{4.31}$$

and $H(\cdot,\zeta,\eta) := (i\kappa)^{-1}\nabla \wedge E(\cdot,\zeta,\eta)$, then for $R_2 < |x| < R''$ the representation

$$\begin{aligned} E(x,\zeta,\eta) &= E^i_{\zeta,\eta}(x) + \nabla \wedge \int\limits_{\partial B_{R_2}} \nu(y) \wedge E(y,\zeta,\eta)\Psi_\zeta(x-y)ds(y) \\ &\quad - \frac{1}{i\kappa}\nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} \nu(y) \wedge H(y,\zeta,\eta)\Psi_\zeta(x-y)ds(y) \end{aligned} \tag{4.32}$$

holds true.

Proof: According to Lemma 4.6 $E(\cdot,\zeta,\eta)$ is C^2-smooth and the vector fields $E(\cdot,\zeta,\eta)$, $H(\cdot,\zeta,\eta)$ satisfy the perturbed Maxwell equations in $B_{R''}$, in particular

$$\nabla \cdot \{(1-n)E(\cdot,\zeta,\eta)\} = \nabla \cdot E(\cdot,\zeta,\eta) = -(1/n)\nabla n \cdot E(\cdot,\zeta,\eta)\ .$$

For fixed vectors $p \in \mathbb{C}^3$, $R_2 < |x| < R''$, the vector fields

$$E'(y) := -\frac{1}{i\kappa}\nabla_y \wedge \nabla_y \wedge \{\Psi_\zeta(x-y)p\}\ ,\ H'(y) := \nabla_y \wedge \{\Psi_\zeta(x-y)p\}\ ,\ y \in \overline{B_{R_2}}\ ,$$

are a solution to the Maxwell equations. Now we use (3.3) to compute

$$\begin{aligned}
& p \cdot \Big\{\nabla \wedge \int\limits_{\partial B_{R_2}} \nu(y) \wedge E(y,\zeta,\eta)\Psi_\zeta(x-y)ds(y) \\
& \qquad -\frac{1}{i\kappa}\nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} \nu(y) \wedge H(y,\zeta,\eta)\Psi_\zeta(x-y)ds(y)\Big\} \\
&= \int\limits_{\partial B_{R_2}} \{(\nu \wedge E(\cdot,\zeta,\eta)) \cdot H' - (\nu \wedge E') \cdot H(\cdot,\zeta,\eta)\}ds \\
&= -\int\limits_{B_{R_2}} (1-n)E(y,\zeta,\eta) \cdot \{\nabla_x \wedge \nabla_x \wedge (\Psi_\zeta(x-y)p)\}dy \\
&= p \cdot \Big\{-\kappa^2 \int\limits_{B_{R_2}} \Psi_\zeta(x-y)(1-n(y))E(y,\zeta,\eta)dy \\
& \qquad +\nabla \int\limits_{B_{R_2}} \frac{1}{n(y)}\nabla n(y) \cdot E(y,\zeta,\eta)\Psi_\zeta(x-y)dy\Big\}\ . \qquad (4.33)
\end{aligned}$$

Since p and x are arbitrary, the integrals on the right hand side of (4.32) and the integrals in (4.31) coincide and we have proved the lemma.

□

Analogously to the acoustic case we need boundary integral operators containing the fundamental solution Ψ_ζ instead of Φ_κ. To this end we define the operators $\mathcal{M}_\zeta$ and $\mathcal{N}_\zeta$: $T_d^{0,\gamma}(\partial B_{R_2}) \to T_d^{0,\gamma}(\partial B_{R_2})$ by

$$\begin{aligned}
(\mathcal{M}_\zeta a)(x) &:= 2\int\limits_{\partial B_{R_2}} \nu(x) \wedge \nabla_x \wedge \{\Psi_\zeta(x-y)a(y)\}ds(y), \\
(\mathcal{N}_\zeta a)(x) &:= 2\nu(x) \wedge \Big\{\nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} \Psi_\zeta(x-y)a(y)ds(y)\Big\}\ ,\ x \in \partial B_{R_2}\ .
\end{aligned}$$

Since the difference of Ψ_ζ and $e^{i\kappa|\cdot|}/(4\pi|\cdot|)$ is an analytic function, boundary potentials with kernel Ψ_ζ inherit the mapping properties and jump relations from those defined with Φ_κ, which can be found in [6, chapter 2] and [7, section 6.3]. Let us note again that our definition of $\mathcal{N}_\zeta$ and the definition of N in the above references slightly differ.

We are now in a position to derive the desired equation for the boundary data $b_{\zeta,\eta} = \nu \wedge H(\cdot,\zeta,\eta) - L_0(\nu \wedge E(\cdot,\zeta,\eta))$ of the special solutions $E(\cdot,\zeta,\eta)$, $H(\cdot,\zeta,\eta)$ from Theorem 4.8.

Lemma 4.22 *With $E(\cdot,\zeta,\eta)$, $H(\cdot,\zeta,\eta)$ being defined as in Lemma 4.21 the boundary data $b_{\zeta,\eta} = \nu \wedge H(\cdot,\zeta,\eta) - L_0(\nu \wedge E(\cdot,\zeta,\eta))$ on ∂B_{R_2} are a solution to*

$$\begin{aligned} b_{\zeta,\eta} &= \nu \wedge H^i_{\zeta,\eta} - L_0(\nu \wedge E^i_{\zeta,\eta}) \\ &+\frac{1}{2}\Big\{\mathcal{M}_\zeta\{b_{\zeta,\eta} + L_0\Lambda_n b_{\zeta,\eta}\} + b_{\zeta,\eta} + \frac{1}{i\kappa}\mathcal{N}_\zeta\Lambda_n b_{\zeta,\eta} \\ &\qquad -L_0\Big[\mathcal{M}_\zeta\Lambda_n b_{\zeta,\eta} - \frac{1}{i\kappa}\mathcal{N}_\zeta\{b_{\zeta,\eta} + L_0\Lambda_n b_{\zeta,\eta}\}\Big]\Big\} \end{aligned} \tag{4.34}$$

on ∂B_{R_2}.

Proof: With the help of $\nu \wedge E(\cdot,\zeta,\eta) = \Lambda_n b_{\zeta,\eta}$ we rewrite the representation (4.32) as

$$\begin{aligned} E(x,\zeta,\eta) &= E^i_{\zeta,\eta}(x) + \nabla \wedge \int_{\partial B_{R_2}} (\Lambda_n b_{\zeta,\eta})(y)\Psi_\zeta(x-y)ds(y) \\ &\quad -\frac{1}{i\kappa}\nabla \wedge \nabla \wedge \int_{\partial B_{R_2}} \{b_{\zeta,\eta} + L_0\Lambda_n b_{\zeta,\eta}\}(y)\Psi_\zeta(x-y)ds(y) \end{aligned}$$

for $R_2 < |x| < R''$. Applying $(i\kappa)^{-1}\nabla \wedge \cdot$ yields

$$\begin{aligned} H(x,\zeta,\eta) &= H^i_{\zeta,\eta}(x) + \nabla \wedge \int_{\partial B_{R_2}} \{b_{\zeta,\eta} + L_0\Lambda_n b_{\zeta,\eta}\}(y)\Psi_\zeta(x-y)ds(y) \\ &\quad +\frac{1}{i\kappa}\nabla \wedge \nabla \wedge \int_{\partial B_{R_2}} (\Lambda_n b_{\zeta,\eta})(y)\Psi_\zeta(x-y)ds(y)\ . \end{aligned}$$

Then, we obtain from the jump relations

$$b_{\zeta,\eta} = \nu \wedge H_+(\cdot,\zeta,\eta) - L_0(\nu \wedge E_+(\cdot,\zeta,\eta))$$

$$
\begin{aligned}
&= \nu \wedge H^i_{\zeta,\eta} - L_0(\nu \wedge E^i_{\zeta,\eta}) \\
&\quad + \frac{1}{2}\Big\{ (\mathcal{M}_\zeta + I)\{b_{\zeta,\eta} + L_0\Lambda_n b_{\zeta,\eta}\} + \frac{1}{i\kappa}\mathcal{N}_\zeta \Lambda_n b_{\zeta,\eta} \\
&\qquad - L_0\Big[(\mathcal{M}_\zeta + I)\Lambda_n b_{\zeta,\eta} - \frac{1}{i\kappa}\mathcal{N}_\zeta\{b_{\zeta,\eta} + L_0\Lambda_n b_{\zeta,\eta}\}\Big]\Big\}
\end{aligned}
$$

and we have proved the lemma.

□

Next, we prove that the operator $A_{n,\zeta}: T_d^{0,\gamma}(\partial B_{R_2}) \to T_d^{0,\gamma}(\partial B_{R_2})$,

$$
A_{n,\zeta} := \frac{1}{2}\Big\{ \mathcal{M}_\zeta(I + L_0\Lambda_n) + I + \frac{1}{i\kappa}\mathcal{N}_\zeta\Lambda_n - L_0\Big[\mathcal{M}_\zeta\Lambda_n - \frac{1}{i\kappa}\mathcal{N}_\zeta(I + L_0\Lambda_n)\Big]\Big\} ,
$$

which occurs in (4.34), is compact.

Lemma 4.23 *The operator* $A_{n,\zeta}: T_d^{0,\gamma}(\partial B_{R_2}) \to T_d^{0,\gamma}(\partial B_{R_2})$ *is compact.*

Proof: For a given tangential field $b \in T_d^{0,\gamma}(\partial B_{R_2})$ let V, W be the solution to (RP) having the boundary data $\nu \wedge W - L_0(\nu \wedge V) = b$, i.e., $\nu \wedge V = \Lambda_n b$.

By the Stratton-Chu formula for V together with (3.18) we know

$$
\begin{aligned}
V(x) &= -\nabla \wedge \int\limits_{\partial B_{R_2}} \nu(y) \wedge V(y)\Phi_\kappa(x,y)ds(y) \\
&\quad + \frac{1}{i\kappa}\nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} \nu(y) \wedge W(y)\Phi_\kappa(x,y)ds(y) \\
&\quad - \kappa^2 \int\limits_{B_R} \Phi_\kappa(x,y)(1 - n(y))V(y)dy \\
&\quad + \nabla \int\limits_{B_R} \Phi_\kappa(x,y)\frac{1}{n(y)}\nabla n(y) \cdot V(y)dy \ , \ x \in B_{R_2} \ .
\end{aligned}
$$

The analogous reasoning as in (4.33) yields

$$
\begin{aligned}
0 &= -\nabla \wedge \int\limits_{\partial B_{R_2}} \nu(y) \wedge V(y)\tilde{g}_\zeta(x-y)ds(y) \\
&\quad + \frac{1}{i\kappa}\nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} \nu(y) \wedge W(y)\tilde{g}_\zeta(x-y)ds(y)
\end{aligned}
$$

$$-\kappa^2 \int\limits_{B_R} \tilde{g}_\zeta(x-y)(1-n(y))V(y)dy$$
$$+\nabla \int\limits_{B_R} \tilde{g}_\zeta(x-y)\frac{1}{n(y)}\nabla n(y)\cdot V(y)dy \ , \ x \in B_{R_2} \ .$$

If we add these two equations, we get

$$\begin{aligned} V(x) &= -\nabla \wedge \int\limits_{\partial B_{R_2}} (\Lambda_n b)(y)\Psi_\zeta(x-y)ds(y) \\ &+\frac{1}{i\kappa}\nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} \{b + L_0\Lambda_n b\}(y)\Psi_\zeta(x-y)ds(y) \\ &-\kappa^2 \int\limits_{B_R} \Psi_\zeta(x-y)(1-n(y))V(y)dy \\ &+\nabla \int\limits_{B_R} \Psi_\zeta(x-y)\frac{1}{n(y)}\nabla n(y)\cdot V(y)dy \ , \ x \in B_{R_2} \ , \end{aligned} \tag{4.35}$$

and after applying $(i\kappa)^{-1}\nabla \wedge \cdot$ to both sides

$$\begin{aligned} W(x) &= -\nabla \wedge \int\limits_{\partial B_{R_2}} \{b + L_0\Lambda_n b\}(y)\Psi_\zeta(x-y)ds(y) \\ &-\frac{1}{i\kappa}\nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} (\Lambda_n b)(y)\Psi_\zeta(x-y)ds(y) \\ &+i\kappa\nabla \wedge \int\limits_{B_R} \Psi_\zeta(x-y)(1-n(y))V(y)dy \ , \ x \in B_{R_2} \ . \end{aligned}$$

The jump relations now imply

$$\begin{aligned} b &= \nu \wedge W_- - L_0(\nu \wedge V_-) \\ &= -\frac{1}{2}\Big\{(\mathcal{M}_\zeta - I)\{b + L_0\Lambda_n b\} + \frac{1}{i\kappa}\mathcal{N}_\zeta\Lambda_n b \\ &\qquad -L_0\Big[(\mathcal{M}_\zeta - I)\Lambda_n b - \frac{1}{i\kappa}\mathcal{N}_\zeta\{b + L_0\Lambda_n b\}\Big]\Big\} \\ &\quad +i\kappa\nu \wedge \nabla \wedge \int\limits_{B_R} \Psi_\zeta(\cdot - y)(1-n(y))V(y)dy \end{aligned}$$

$$
\begin{aligned}
&+L_0\Big(\nu \wedge \Big\{\kappa^2 \int\limits_{B_R} \Psi_\zeta(\cdot - y)(1-n(y))V(y)dy \\
&\qquad -\nabla \int\limits_{B_R} \Psi_\zeta(\cdot - y)\frac{1}{n(y)}\nabla n(y)\cdot V(y)dy\Big\}\Big) \ ,
\end{aligned}
$$

whence, reordering terms and inserting $V|_{\overline{B_R}} = Pb$, P being the solution operator from Theorem 4.20,

$$
\begin{aligned}
A_{n,\zeta}b &= i\kappa\nu \wedge \nabla \wedge \int\limits_{B_R} \Psi_\zeta(\cdot - y)(1-n(y))(Pb)(y)dy \\
&\quad +L_0\Big(\nu \wedge \Big\{\kappa^2 \int\limits_{B_R} \Psi_\zeta(\cdot - y)(1-n(y))(Pb)(y)dy \\
&\qquad -\nabla \int\limits_{B_R} \Psi_\zeta(\cdot - y)\frac{1}{n(y)}\nabla n(y)\cdot (Pb)(y)dy\Big\}\Big) \ .
\end{aligned}
$$

The compactness of $A_{n,\zeta}$ now follows from the compactness of P (Theorem 4.20) together with the mapping properties of the volume potential. □

The previous lemma admits to apply the Riesz theory in order to establish the existence of a unique solution to (4.34). Hence, our next aim is to establish the injectivity of the operator $I - A_{n,\zeta}$. The following lemma is the electromagnetic analogue to Lemma 2.31 and is needed during the injectivity proof for $I - A_{n,\zeta}$.

Lemma 4.24 *Assume $R_2 < R'' < R'$.*

(a) For all $x, z \in B_{R''}$ and $p \in \mathbb{C}^3$ the relation

$$
\begin{aligned}
0 &= \nabla_x \wedge \int\limits_{\partial B_{R''}} \nu(y) \wedge \Big[\nabla_y \wedge \{\Psi_\zeta(y-z)p\}\Big]\Psi_\zeta(x-y)ds(y) \\
&\quad -\frac{1}{i\kappa}\nabla_x \wedge \nabla_x \wedge \\
&\qquad \int\limits_{\partial B_{R''}} \nu(y) \wedge \Big[\frac{1}{i\kappa}\nabla_y \wedge \nabla_y \wedge \{\Psi_\zeta(y-z)p\}\Big]\Psi_\zeta(x-y)ds(y)
\end{aligned}
\tag{4.36}
$$

holds true.

(b) If E, H are defined by

$$E(y) := \nabla_y \wedge \int\limits_{\partial B_{R_2}} \Psi_\zeta(y-z)a(z)ds(z)\ ,\ \ R_2 < |y| \le R''\ ,$$

$H := (i\kappa)^{-1}\nabla \wedge E$ with $a \in T_d^{0,\gamma}(\partial B_{R_2})$, then for all $|x| < R''$ the relation

$$\begin{aligned} 0 &= \nabla_x \wedge \int\limits_{\partial B_{R''}} \nu(y) \wedge E(y)\Psi_\zeta(x-y)ds(y) \\ &\quad -\frac{1}{i\kappa}\nabla_x \wedge \nabla_x \wedge \int\limits_{\partial B_{R''}} \nu(y) \wedge H(y)\Psi_\zeta(x-y)ds(y) \end{aligned}$$

holds true. This is also true, if E, H are defined by

$$E(y) := \nabla_y \wedge \nabla_y \wedge \int\limits_{\partial B_{R_2}} \Psi_\zeta(y-z)a(z)ds(z)\ ,\ \ R_2 < |y| \le R''\ ,$$

$H := (i\kappa)^{-1}\nabla \wedge E$ with $a \in T_d^{0,\gamma}(\partial B_{R_2})$.

(c) If vector fields E, H are defined as in part (b), then for $R_2 < |x| < R''$ we have the formula

$$\begin{aligned} E(x) &= \nabla \wedge \int\limits_{\partial B_{R_2}} \nu(y) \wedge E(y)\Psi_\zeta(x-y)ds(y) \\ &\quad -\frac{1}{i\kappa}\nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} \nu(y) \wedge H(y)\Psi_\zeta(x-y)ds(y)\ . \end{aligned}$$

Proof: For part (a) let $q \in \mathbb{C}^3$ be an arbitrary vector. Defining

$$E(y) := \nabla_y \wedge \{\Phi_\kappa(y,z)p\}\ ,\ H(y) := \frac{1}{i\kappa}\nabla_y \wedge E(y)\ ,\ y \in \mathbb{R}^3 \setminus \{z\},$$

and

$$E'(y) := -\frac{1}{i\kappa}\nabla_y \wedge \nabla_y \wedge \{\Phi_\kappa(x,y)q\}\ ,\ H'(y) := \frac{1}{i\kappa}\nabla_y \wedge E'(y)\ ,\ y \in \mathbb{R}^3 \setminus \{x\},$$

we compute with the help of (3.9)

$$\begin{aligned}
& q \cdot \Bigg(\nabla_x \wedge \int\limits_{\partial B_{R''}} \nu(y) \wedge \Big[\nabla_y \wedge \{\Phi_\kappa(y,z)p\} \Big] \Phi_\kappa(x,y) ds(y) \\
& \qquad - \frac{1}{i\kappa} \nabla_x \wedge \nabla_x \wedge \\
& \qquad\qquad \int\limits_{\partial B_{R''}} \nu(y) \wedge \Big[\frac{1}{i\kappa} \nabla_y \wedge \nabla_y \wedge \{\Phi_\kappa(y,z)p\} \Big] \Phi_\kappa(x,y) ds(y) \Bigg) \\
& = \int\limits_{\partial B_{R''}} \Big\{ (\nu \wedge E) \cdot H' - (\nu \wedge E') \cdot H \Big\} ds \\
& = 0 \, .
\end{aligned}$$

Replacing Φ_κ by $\tilde{g}_\zeta$ in the above computation and using (3.3) yields

$$\begin{aligned}
& q \cdot \Bigg(\nabla_x \wedge \int\limits_{\partial B_{R''}} \nu(y) \wedge \Big[\nabla_y \wedge \{\tilde{g}_\zeta(y-z)p\} \Big] \tilde{g}_\zeta(x-y) ds(y) \\
& \qquad - \frac{1}{i\kappa} \nabla_x \wedge \nabla_x \wedge \\
& \qquad\qquad \int\limits_{\partial B_{R''}} \nu(y) \wedge \Big[\frac{1}{i\kappa} \nabla_y \wedge \nabla_y \wedge \{\tilde{g}_\zeta(y-z)p\} \Big] \tilde{g}_\zeta(x-y) ds(y) \Bigg) \\
& = 0 \, . \qquad (4.37)
\end{aligned}$$

From the representation in Theorem 3.2 applied to the field $\nabla_y \wedge \{\tilde{g}_\zeta(y-z)p\}$, $y \in B_{R''}$, we have

$$\begin{aligned}
& q \cdot \Bigg(\nabla_x \wedge \int\limits_{\partial B_{R''}} \nu(y) \wedge \Big[\nabla_y \wedge \{\tilde{g}_\zeta(y-z)p\} \Big] \Phi_\kappa(x,y) ds(y) \\
& \qquad - \frac{1}{i\kappa} \nabla_x \wedge \nabla_x \wedge \\
& \qquad\qquad \int\limits_{\partial B_{R''}} \nu(y) \wedge \Big[\frac{1}{i\kappa} \nabla_y \wedge \nabla_y \wedge \{\tilde{g}_\zeta(y-z)p\} \Big] \Phi_\kappa(x,y) ds(y) \Bigg) \\
& = -q \cdot \Big[\nabla_x \wedge \{\tilde{g}_\zeta(x-z)p\} \Big] \\
& = -(\nabla_x \tilde{g}_\zeta(x-z) \wedge p) \cdot q \, .
\end{aligned}$$

Finally, Theorem 3.2 also implies

$$\begin{aligned}
& q\cdot\Bigg(\nabla_x\wedge\int\limits_{\partial B_{R''}}\nu(y)\wedge\Big[\nabla_y\wedge\{\Phi_\kappa(y,z)p\}\Big]\tilde g_\zeta(x-y)ds(y) \\
&\qquad -\frac{1}{i\kappa}\nabla_x\wedge\nabla_x\wedge \\
&\qquad\qquad \int\limits_{\partial B_{R''}}\nu(y)\wedge\Big[\frac{1}{i\kappa}\nabla_y\wedge\nabla_y\wedge\{\Phi_\kappa(y,z)p\}\Big]\tilde g_\zeta(x-y)ds(y)\Bigg) \\
&= -p\cdot\Bigg(\nabla_z\wedge\int\limits_{\partial B_{R''}}\nu(y)\wedge\Big[\nabla_y\wedge\{\tilde g_\zeta(x-y)q\}\Big]\Phi_\kappa(z,y)ds(y) \\
&\qquad -\frac{1}{i\kappa}\nabla_z\wedge\nabla_z\wedge \\
&\qquad\qquad \int\limits_{\partial B_{R''}}\nu(y)\wedge\Big[\frac{1}{i\kappa}\nabla_y\wedge\nabla_y\wedge\{\tilde g_\zeta(x-y)q\}\Big]\Phi_\kappa(z,y)ds(y)\Bigg) \\
&= p\cdot(\nabla_z\wedge\{\tilde g_\zeta(x-z)q\}) \\
&= (\nabla_x\tilde g_\zeta(x-z)\wedge p)\cdot q\ .
\end{aligned}$$

Adding the last four equations we arrive at assertion (a) because $q\in\mathbb{C}^3$ is arbitrary.

For part (b) we insert the definition of E and H in the assertion, interchange the order of integration, and apply part (a). The second assertion of part (b) is obtained similarly with the help of

$$\begin{aligned}
0 &= \nabla_x\wedge\int\limits_{\partial B_{R''}}\nu(y)\wedge\Big[\nabla_y\wedge\nabla_y\wedge\{\Psi_\zeta(y-z)p\}\Big]\Psi_\zeta(x-y)ds(y) \\
&\quad -\frac{1}{i\kappa}\nabla_x\wedge\nabla_x\wedge \\
&\qquad \int\limits_{\partial B_{R''}}\nu(y)\wedge\Big[(-i\kappa)\nabla_y\wedge\{\Psi_\zeta(y-z)p\}\Big]\Psi_\zeta(x-y)ds(y)\ ,
\end{aligned}$$

which is the result of applying $\nabla\wedge\cdot$ to (4.36).

Finally, for part (c) we use (3.3) and establish analogously to (4.37) for $R_2<|x|<R''$ the relation

$$0 = -\nabla_x\wedge\int\limits_{\partial(B_{R''}\setminus B_{R_2})}\nu(y)\wedge E(y)\tilde g_\zeta(x-y)ds(y)$$

$$+\frac{1}{i\kappa}\nabla_x \wedge \nabla_x \wedge \int\limits_{\partial(B_{R''}\setminus B_{R_2})} \nu(y)\wedge H(y)\tilde{g}_\zeta(x-y)ds(y)\ .$$

If we add the representation from Theorem 3.2 applied to the field E in $\{R_2 < |x| < R''\}$, we have

$$\begin{aligned} E(x) &= -\nabla_x \wedge \int\limits_{\partial(B_{R''}\setminus B_{R_2})} \nu(y)\wedge E(y)\Psi_\zeta(x-y)ds(y) \\ &\quad +\frac{1}{i\kappa}\nabla_x \wedge \nabla_x \wedge \int\limits_{\partial(B_{R''}\setminus B_{R_2})} \nu(y)\wedge H(y)\Psi_\zeta(x-y)ds(y)\ . \end{aligned}$$

Here, ν is directed into the exterior of $\{R_2 < |x| < R''\}$. Since the integrals over $\partial B_{R''}$ cancel due to part (b), assertion (c) follows.

□

The next theorem states the injectivity of $I - A_{n,\zeta}$ and summarizes our knowledge about the boundary data $b_{\zeta,\eta} = \nu \wedge H(\cdot,\zeta,\eta) - L_0(\nu\wedge E(\cdot,\zeta,\eta))$ of the special solutions $E(\cdot,\zeta,\eta)$, $H(\cdot,\zeta,\eta)$ from Theorem 4.8.

Theorem 4.25 *Let $n \in C^{2,\gamma}(\mathbb{R}^3)$ with supp $(1-n) \subset B_R$, $\Re(n) > 0$, $\Im(n) \geq 0$, and $R < R_2 < R'' < R'$ be given. Assume $\kappa > 0$ and ζ, $\eta \in \mathbb{C}^3$ satisfy $\zeta\cdot\zeta = \kappa^2$, $\zeta\cdot\eta = 0$,*

$$|\Im(\zeta)| \geq 2\frac{R'}{\pi}\{\max_{x\in\overline{B_R}} \|\mathcal{Q}(x)\|_2 + \kappa^2\|1-n\|_\infty\} + 1\ ,$$

$\mathcal{Q}$ being defined in (3.13). Furthermore, Λ_n denotes the map $\nu\wedge H - L_0(\nu\wedge E) \mapsto \nu\wedge E$ from Theorem 4.20, $E(\cdot,\zeta,\eta)$ is the solution to (4.31), and $H(\cdot,\zeta,\eta) := (i\kappa)^{-1}\nabla\wedge E(\cdot,\zeta,\eta)$.

Then, the boundary values $b_{\zeta,\eta} := \nu\wedge H(\cdot,\zeta,\eta) - L_0(\nu\wedge E(\cdot,\zeta,\eta))$ are the unique solution to the equation (4.34), i.e.,

$$b_{\zeta,\eta}(x) = \frac{e^{i\zeta\cdot x}}{\kappa}\nu(x)\wedge(\zeta\wedge\eta) - \Big[L_0\Big(e^{i\zeta\cdot x}\nu\wedge\eta\Big)\Big](x) + \Big[A_{n,\zeta}b_{\zeta,\eta}\Big](x)\ ,\ x\in\partial B_{R_2}\ .$$

Moreover, $A_{n,\zeta}$ is a compact operator in $T_d^{0,\gamma}(\partial B_{R_2})$.

Proof: Due to Theorem 4.8 equation (4.31) has a unique C^2-smooth solution, whence $E(\cdot,\zeta,\eta)$, $H(\cdot,\zeta,\eta)$ are well defined. Then, Lemma 4.22 states that $b_{\zeta,\eta}$ is a solution to (4.34). The compactness of $A_{n,\zeta}$ was proved in Lemma 4.23.

It remains to show the injectivity of equation (4.34). To this end assume $b \in T_d^{0,\gamma}(\partial B_{R_2})$ is a solution to $b = A_{n,\zeta} b$. In B_{R_2} we define E, H to be the solution to (RP) having boundary data $\nu \wedge H - L_0(\nu \wedge E) = b$ and for $R_2 < |x| < R''$ we set

$$
\begin{aligned}
E(x) &:= \nabla \wedge \int\limits_{\partial B_{R_2}} (\Lambda_n b)(y) \Psi_\zeta(x-y) ds(y) \\
&\qquad -\frac{1}{i\kappa} \nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} \{b + L_0 \Lambda_n b\}(y) \Psi_\zeta(x-y) ds(y) \ , \\
H(x) &:= (i\kappa)^{-1} \nabla \wedge E(x) \\
&= \nabla \wedge \int\limits_{\partial B_{R_2}} \{b + L_0 \Lambda_n b\}(y) \Psi_\zeta(x-y) ds(y) \\
&\qquad + \frac{1}{i\kappa} \nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} (\Lambda_n b)(y) \Psi_\zeta(x-y) ds(y) \ .
\end{aligned}
\tag{4.38}
$$

Using the jump relations we arrive at

$$
\nu \wedge H_+ - L_0(\nu \wedge E_+) = A_{n,\zeta} b = b = \nu \wedge H_- - L_0(\nu \wedge E_-) \ . \tag{4.39}
$$

Now, we want to show that $\nu \wedge E_+ = \nu \wedge E_-$. To this end we employ Lemma 4.24 (c) together with (4.39) and represent E for $R_2 < |x| \le R''$ by

$$
\begin{aligned}
E(x) &= \nabla \wedge \int\limits_{\partial B_{R_2}} (\nu \wedge E_+)(y) \Psi_\zeta(x-y) ds(y) \\
&\qquad -\frac{1}{i\kappa} \nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} \{b + L_0(\nu \wedge E_+)\}(y) \Psi_\zeta(x-y) ds(y) \ .
\end{aligned}
$$

Computing the difference of (4.38) and the last equation we obtain for

$$
\begin{aligned}
a &:= \Lambda_n b - \nu \wedge E_+ \ , \\
V(x) &:= \nabla \wedge \int\limits_{\partial B_{R_2}} a(y) \Psi_\zeta(x-y) ds(y)
\end{aligned}
$$

$$
-\frac{1}{i\kappa}\nabla\wedge\nabla\wedge\int\limits_{\partial B_{R_2}}(L_0 a)(y)\Psi_\zeta(x-y)ds(y)\ ,\ x\in\overline{B_{R''}}\setminus\partial B_{R_2}\ ,
$$

$$
W \ := \ (i\kappa)^{-1}\nabla\wedge V\ ,
$$

that $V(x)=0$ for $R_2<|x|<R''$.

From the jump relations we conclude $-\nu\wedge V_- = \nu\wedge(V_+ - V_-) = a$, $-\nu\wedge W_- = \nu\wedge(W_+ - W_-) = L_0 a$, and the vector Green's theorem (3.2) yields

$$
\begin{aligned}
\int\limits_{\partial B_{R_2}}|S_0 a|^2 ds &= \int\limits_{\partial B_{R_2}}\overline{a}\cdot[(\nu\wedge S_0^2 a)\wedge\nu]ds \\
&= \int\limits_{\partial B_{R_2}}(\nu\wedge\overline{V_-})\cdot W_- ds \\
&= -i\kappa\int\limits_{B_{R_2}}(|W|^2 - n|V|^2)dx\ .
\end{aligned}
$$

From the real part of this equation we see $S_0 a = 0$, whence $a=0$ and $\nu\wedge E_+ = \Lambda_n b = \nu\wedge E_-$. Together with (4.39) this also implies $\nu\wedge H_+ = \nu\wedge H_-$.

Then, as in (4.35), we can represent E with the help of the fundamental solution Ψ_ζ:

$$
\begin{aligned}
E(x) &= -\nabla\wedge\int\limits_{\partial B_{R_2}}(\nu\wedge E_+)(y)\Psi_\zeta(x-y)ds(y) \\
&\quad +\frac{1}{i\kappa}\nabla\wedge\nabla\wedge\int\limits_{\partial B_{R_2}}(\nu\wedge H_+)(y)\Psi_\zeta(x-y)ds(y) \\
&\quad -\kappa^2\int\limits_{B_R}\Psi_\zeta(x-y)(1-n(y))E(y)dy \\
&\quad +\nabla\int\limits_{B_R}\Psi_\zeta(x-y)\frac{1}{n(y)}\nabla n(y)\cdot E(y)dy\ ,\ x\in B_{R_2}\ .
\end{aligned}
$$

Furthermore, with formula (3.3) and Lemma 4.24 (b) we compute for all $p\in\mathbb{C}^3$

$$
p\cdot\Bigg[-\nabla\wedge\int\limits_{\partial B_{R_2}}(\nu\wedge E_+)(y)\Psi_\zeta(x-y)ds(y)
$$

$$
\begin{aligned}
& + \frac{1}{i\kappa} \nabla \wedge \nabla \wedge \int\limits_{\partial B_{R_2}} (\nu \wedge H_+)(y) \Psi_\zeta(x-y) ds(y) \Bigg] \\
= \;& p \cdot \Bigg[-\nabla \wedge \int\limits_{\partial B_{R''}} (\nu \wedge E)(y) \Psi_\zeta(x-y) ds(y) \\
& + \frac{1}{i\kappa} \nabla \wedge \nabla \wedge \int\limits_{\partial B_{R''}} (\nu \wedge H)(y) \Psi_\zeta(x-y) ds(y) \Bigg] \\
= \;& 0 \; , \; x \in B_{R_2} \; .
\end{aligned}
$$

Hence, E is a solution to the homogeneous modified Lippmann-Schwinger equation (4.31) and must vanish in B_{R_2} according to Theorem 4.8. This finally implies $b = \nu \wedge H_- - L_0(\nu \wedge E_-) = 0$.

□

By our considerations so far we know that, given the far field pattern $e_{\infty,n}$, it is possible to compute the operator Λ_n and the boundary values $\nu \wedge H(\cdot,\zeta,\eta) - L_0(\nu \wedge E(\cdot,\zeta,\eta))$, $\nu \wedge E(\cdot,\zeta,\eta)$ of the special solutions $E(\cdot,\zeta,\eta)$, $H(\cdot,\zeta,\eta)$ from Theorem 4.8. Hence, we know the Cauchy data $\nu \wedge E(\cdot,\zeta,\eta)$ and $\nu \wedge H(\cdot,\zeta,\eta)$ of these solutions provided $|\Im(\zeta)|$ is sufficiently large. The final theorem of this section shows how to compute $(n-1)\hat{}(\alpha)$ from this information.

As in the Uniqueness Theorem 4.9 we choose for a fixed vector $\alpha \in \Gamma$ the unit vectors d_1, $d_2 \in \mathbb{R}^3$ such that $d_1 \cdot d_2 = d_1 \cdot \alpha = d_2 \cdot \alpha = 0$, and define for $t \geq 2\frac{R'}{\pi}\{\max_{x \in \overline{B_R}} \|\mathcal{Q}(x)\|_2 + \kappa^2 \|1-n\|_\infty\} + 1 + \kappa$

$$
\begin{aligned}
\zeta_t &:= -\frac{1}{2}\alpha + i\sqrt{t^2 - \kappa^2 + \frac{|\alpha|^2}{4}}\, d_1 + t d_2 \; , \\
\tilde{\zeta}_t &:= -\frac{1}{2}\alpha - i\sqrt{t^2 - \kappa^2 + \frac{|\alpha|^2}{4}}\, d_1 - t d_2 \; , \\
\eta_t &:= \frac{1}{|\alpha|}\alpha + \frac{|\alpha|}{2t} d_2 \; , \\
\tilde{\eta}_t &:= \frac{1}{|\alpha|}\alpha - \frac{|\alpha|}{2t} d_2 \; .
\end{aligned}
$$

Theorem 4.26 *With the notation and assumptions of Theorem 4.25 define for a fixed $\alpha \in \Gamma$ and for*

$$t \geq 2\frac{R'}{\pi}\{\max_{x\in \overline{B_R}} \|\mathcal{Q}(x)\|_2 + \kappa^2\|1-n\|_\infty\} + 1 + \kappa$$

the vectors ζ_t, $\tilde{\zeta}_t$, η_t, $\tilde{\eta}_t$ as above, and let $b_t \in T_d^{0,\gamma}(\partial B_{R_2})$ be the unique solution to

$$b_t(x) = \frac{e^{i\zeta_t\cdot x}}{\kappa}\nu(x)\wedge(\zeta_t\wedge\eta_t) - \Big[L_0\Big(e^{i\zeta_t\cdot x}\,\nu\wedge\eta_t\Big)\Big](x) + \Big[A_{n,\zeta_t}b_t\Big](x)\ ,\ x\in\partial B_{R_2}\ . \tag{4.40}$$

Finally, let $\tilde{E}_t$, $\tilde{H}_t$ denote the vector fields $\tilde{E}_t(x) := e^{i\tilde{\zeta}_t\cdot x}\tilde{\eta}_t$, $x \in \mathbb{R}^3$, $\tilde{H}_t := (i\kappa)^{-1}\nabla\wedge\tilde{E}_t$.

Then, $\nu\wedge E(\cdot,\zeta_t,\eta_t) = \Lambda_n b_t$, $\nu\wedge H(\cdot,\zeta_t,\eta_t) = b_t + L_0\Lambda_n b_t$ are the Cauchy data of the special solutions from Theorem 4.8 and

$$\lim_{t\to\infty}\int\limits_{\partial B_{R_2}}\{(\nu\wedge\tilde{E}_t)\cdot H(\cdot,\zeta_t,\eta_t) - (\nu\wedge E(\cdot,\zeta_t,\eta_t))\cdot\tilde{H}_t\}ds$$

$$= i\kappa(2R')^{3/2}(n-1)\hat{\ }(\alpha)\ .$$

Proof: Since $|\Im(\zeta_t)| \geq 2\frac{R'}{\pi}\{\max_{x\in\overline{B_R}}\|\mathcal{Q}(x)\|_2 + \kappa^2\|1-n\|_\infty\} + 1$, by the preceding theorem equation (4.40) has a unique solution which concides with the boundary data $\nu\wedge H(\cdot,\zeta_t,\eta_t) - L_0(\nu\wedge E(\cdot,\zeta_t,\eta_t))$ of the special solutions from Theorem 4.8. Hence, we have $\nu\wedge E(\cdot,\zeta_t,\eta_t) = \Lambda_n b_t$, $\nu\wedge H(\cdot,\zeta_t,\eta_t) = b_t + L_0\Lambda_n b_t$.

Relation (3.3) immediately yields

$$\begin{aligned}
&\int\limits_{\partial B_{R_2}}\{(\nu\wedge\tilde{E}_t)\cdot H(\cdot,\zeta_t,\eta_t) - (\nu\wedge E(\cdot,\zeta_t,\eta_t))\cdot\tilde{H}_t\}ds \\
&\quad= i\kappa\int\limits_{B_{R_2}}(n-1)\tilde{E}_t\cdot E(\cdot,\zeta_t,\eta_t)dx \\
&\quad= i\kappa\int\limits_{B_{R_2}}(n-1)(x)e^{-i\alpha\cdot x}\Big(1+ \\
&\qquad\qquad +\Big\{-\frac{|\alpha|^2}{4t^2} - f(x,\zeta_t,\eta_t)|\alpha| + \tilde{\eta}_t\cdot V(x,\zeta_t,\eta_t)\Big\}\Big)dx\ ,
\end{aligned}$$

where we have also used $\eta_t \cdot \tilde{\eta}_t = 1 - (|\alpha|^2/4t^2)$ and $\zeta_t \cdot \tilde{\eta}_t = -|\alpha|$ in the last line.

According to Theorem 4.8 the L^2-norm of the terms in curly brackets in the last line converges to 0 as $t \to \infty$. This completes the proof of the reconstruction procedure.

□

Let us close this section with a summary of the reconstruction procedure. We assume that the far field $e_{\infty,n}$ originating from the refractive index $n \in \tilde{C}(B_R)$ is exactly known.

- Compute the Fourier coefficients

$$\mu_{l_1k_1l_2k_2} := \int\limits_{S_2}\int\limits_{S_2} e_{\infty,n}(\hat{x},d)\overline{Y_{l_1}^{k_1}(\hat{x})Y_{l_2}^{k_2}(d)}ds(\hat{x})ds(d) \ .$$

- Compute for $x,\ y \in \partial B_{R_2}$ the matrices

$$\begin{aligned} & k_n(x,y) \\ & := \ -\frac{\kappa^4}{4\pi}\sum_{l_1,k_1,l_2,k_2} i^{l_1-l_2}h_{l_1}^{(1)}(\kappa R_2)h_{l_2}^{(1)}(\kappa R_2)Y_{l_1}^{k_1}(\frac{x}{|x|})Y_{l_2}^{k_2}(\frac{y}{|y|})\mu_{l_1k_1l_2k_2} \end{aligned}$$

  and $\tilde{k}_n(x,y)$ which has as its mth column the vector

$$\begin{aligned} & \tilde{k}_n(x,y)d_m \\ & := \ -\frac{\kappa^2}{4\pi R_2}\sum_{l_1,k_1,l_2,k_2} i^{l_1-l_2}h_{l_2}^{(1)}(\kappa R_2)Y_{l_2}^{k_2}(\frac{y}{|y|})\Big\{\kappa\Big(\frac{dh_{l_1}^{(1)}}{dt}\Big)(\kappa R_2)Y_{l_1}^{k_1}(\frac{x}{|x|})x \\ & \quad +h_{l_1}^{(1)}(\kappa R_2)(\operatorname{Grad} Y_{l_1}^{k_1})(\frac{x}{|x|})\Big\} \wedge (\mu_{l_1k_1l_2k_2}d_m) \ , \end{aligned}$$

  d_m being the mth cartesian unit vector (see (4.19) and (4.29)).

  Define the operators N_n, M_n and Λ_n: $T_d^{0,\gamma}(\partial B_{R_2}) \to T_d^{0,\gamma}(\partial B_{R_2})$ by

$$\begin{aligned} (N_n a)(x) \ &:= \ 2\nu(x)\wedge\nabla\wedge\nabla\wedge\int\limits_{\partial B_{R_2}}\Phi_\kappa(x,y)a(y)ds(y) \\ & \quad +2\nu(x)\wedge\int\limits_{\partial B_{R_2}}k_n(x,y)a(y)ds(y) \ , \ x\in\partial B_{R_2} \ , \end{aligned}$$

$$
\begin{aligned}
(M_n a)(x) &:= 2 \int\limits_{\partial B_{R_2}} \nu(x) \wedge \nabla_x \wedge \{\Phi_\kappa(x,y) a(y)\} ds(y) \\
&\qquad + 2\nu(x) \wedge \int\limits_{\partial B_{R_2}} \tilde{k}_n(x,y) a(y) ds(y) \ , \ x \in \partial B_{R_2} \ , \\
\Lambda_n &:= N_n[-i\kappa(M_n - I) - L_0 N_n]^{-1}
\end{aligned}
$$

(see Theorem 4.20 for Λ_n; the smoothing operator L_0, defined in (4.22), is known).

- Fix $\alpha \in \Gamma$ and choose $\zeta = \zeta_t$, $\tilde{\zeta} = \tilde{\zeta}_t$, $\eta = \eta_t$, $\tilde{\eta} = \tilde{\eta}_t$ as before Theorem 4.26; compute

$$
\begin{aligned}
A_{n,\zeta} &:= \frac{1}{2}\Big\{\mathcal{M}_\zeta\{I + L_0\Lambda_n\} + I + \frac{1}{i\kappa}\mathcal{N}_\zeta\Lambda_n \\
&\qquad - L_0\Big[\mathcal{M}_\zeta\Lambda_n - \frac{1}{i\kappa}\mathcal{N}_\zeta\{I + L_0\Lambda_n\}\Big]\Big\} \ ,
\end{aligned}
$$

where the operators $\mathcal{M}_\zeta$ and $\mathcal{N}_\zeta$ are defined on page 167.

- Solve the equation

$$
b_{\zeta,\eta}(x) = \frac{e^{i\zeta\cdot x}}{\kappa}\nu(x) \wedge (\zeta \wedge \eta) - \Big[L_0\Big(e^{i\zeta\cdot x}\nu \wedge \eta\Big)\Big](x) + \Big[A_{n,\zeta} b_{\zeta,\eta}\Big](x)
$$

on ∂B_{R_2}. (It has a unique solution due to Theorem 4.25)

- Define $a_{\zeta,\eta} := \Lambda_n b_{\zeta,\eta}$, $c_{\zeta,\eta} := b_{\zeta,\eta} + L_0 a_{\zeta,\eta}$ ($a_{\zeta,\eta}$ and $c_{\zeta,\eta}$ are the Cauchy data $\nu \wedge E(\cdot,\zeta,\eta)$ and $\nu \wedge H(\cdot,\zeta,\eta)$ of the special solutions).

- Insert $a_{\zeta,\eta}$ and $c_{\zeta,\eta}$ into

$$
\int\limits_{\partial B_{R_2}} \{e^{i\tilde{\zeta}\cdot x}(\nu(x) \wedge \tilde{\eta}) \cdot (c_{\zeta,\eta} \wedge \nu)(x) - a_{\zeta,\eta}(x) \cdot (\kappa^{-1}\tilde{\zeta} \wedge \tilde{\eta}) e^{i\tilde{\zeta}\cdot x}\} ds(x)
$$

and calculate the limit as $t \to \infty$. Divide the limit by $i\kappa(2R')^{3/2}$ and set the result to $(n-1)\hat{}(\alpha)$.

- Repeat the last four items for all $\alpha \in \Gamma$.

- $$
n = 1 + \sum_{\alpha \in \Gamma} (n-1)\hat{}(\alpha) e_\alpha \ \text{ in } B_{R_2}.
$$

Chapter 5

The Direct Scattering Problem in Elasticity

In linear elasticity theory the displacement vector

$$u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \ , \ x \in \mathbb{R}^3 \ , t \in \mathbb{R} \ ,$$

obeys the law

$$\rho(x)\frac{\partial^2}{\partial t^2}u_j(x,t) = \sum_{k=1}^{3} \frac{\partial}{\partial x_k} S_{jk}(x,t) \ , \ j = 1,2,3 \ , \ x \in \mathbb{R}^3 \ , \ t \in \mathbb{R} \ ,$$

if there are no body forces acting in the medium occupying $\mathbb{R}^3$. Here, (S_{jk}) denotes the stress tensor and ρ is the mass density of the medium. In addition, one assumes a linear relation between the stress tensor and the linear strain tensor (ϵ_{lm}) (Hooke's law)

$$S_{jk} = \sum_{l,m=1}^{3} C_{jklm}\epsilon_{lm} \ , \ j,k = 1,2,3 \ ,$$

where the linear strain tensor is defined by

$$\epsilon_{lm} = \frac{1}{2}\Big(\frac{\partial u_l}{\partial x_m} + \frac{\partial u_m}{\partial x_l}\Big) \ , \ l,m = 1,2,3 \ .$$

We will assume throughout that the medium is isotropic. Then, we can describe the medium by the two Lamé coefficients λ and μ, which determine all the coefficients C_{jklm}, and Hooke's law reads

$$S_{jk} = 2\mu\epsilon_{jk} + \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33})\delta_{jk} \ , \ j,k = 1,2,3 \ ,$$

δ_{jk} denoting the Kronecker delta. The reader can find a more detailed description in [9]. We simplify the discussion further by supposing that λ and μ are constants satisfying $\mu > 0$ and $2\mu + \lambda > 0$.

We want to consider a medium having an inhomogeneous mass density ρ which satisfies $\rho(x) = 1$ in the exterior of a large ball. Moreover, we examine time-harmonic waves, i.e., we assume $u(x,t) = \Re\{U(x)e^{-i\omega t}\}$ with a fixed frequency $\omega > 0$. Then, the vector field $U\colon \mathbb{R}^3 \to \mathbb{C}^3$ must obey

$$\Delta^* U + \omega^2 \rho U = 0 \qquad \text{in } \mathbb{R}^3. \tag{5.1}$$

The operator Δ^* is defined by

$$\begin{aligned}
[\Delta^* U]_j &= \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\Big\{\mu\Big(\frac{\partial U_j}{\partial x_k} + \frac{\partial U_k}{\partial x_j}\Big)\Big\} + \frac{\partial}{\partial x_j}[\lambda \nabla \cdot U] \\
&= \mu \Delta U_j + (\lambda + \mu)\frac{\partial}{\partial x_j}[\nabla \cdot U] \\
&= -\mu[\nabla \wedge \nabla \wedge U]_j + (\lambda + 2\mu)\frac{\partial}{\partial x_j}[\nabla \cdot U]\ ,\ j = 1,2,3\ .
\end{aligned}$$

Henceforth, we will refer to (5.1) as the elasticity equation.

Assuming $\rho(x) = 1$ in $\mathbb{R}^3$ and applying $\nabla \wedge \cdot$ or $\nabla \cdot$ to (5.1) we see that the quantities $\nabla \cdot U$ and $\nabla \wedge U$ satisfy the Helmholtz equations

$$\Delta(\nabla \cdot U) + \frac{\omega^2}{2\mu + \lambda}(\nabla \cdot U) = 0\ ,$$

$$\Delta(\nabla \wedge U) + \frac{\omega^2}{\mu}(\nabla \wedge U) = 0\ .$$

Hence, we define for a given frequency $\omega > 0$ and given Lamé constants λ, μ the wave numbers

$$\kappa_s := \frac{\omega}{\sqrt{\mu}}\ , \qquad \kappa_p := \frac{\omega}{\sqrt{2\mu + \lambda}}\ .$$

In this chapter we are interested in the following direct elastic scattering problem: given λ, μ, ω, ρ and an incident wave U^i, i.e., a solution to $\Delta^* U^i + \omega^2 U^i = 0$ in $\mathbb{R}^3$,

find the scattered field U^s, such that the total field $U := U^i + U^s$ is a solution to (5.1) and such that U^s satisfies a radiation condition.

In the next section we start as usual with Green's theorems (called Betti formulas for the elasticity equation in [23]) and representation formulas. In view of the above Helmholtz equations we expect to see the fundamental solutions Φ_{κ_s} and Φ_{κ_p} to appear in the fundamental solution for the elasticity equation. However, despite our knowledge about Φ_κ from previous chapters this section is longer than the former analogous ones because we prove some theorems for the elastic case which we accepted as proven in the acoustic and electromagnetic case.

The second section deals with uniqueness and existence for the direct elastic scattering problem. The key words for uniqueness are again Green's theorem, Rellich's lemma and unique continuation. For the elasticity equation the proof of the unique continuation principle uses the same idea as in the acoustic case. Nevertheless, it is more involved because it not only requires estimates of the L^2-norms $\|G_\zeta f\|_{L^2}$ but also of $\|\nabla(G_\zeta f)\|_{L^2}$. Existence of a solution is derived with the help of the representation theorem which leads to a Fredholm integral equation of Lippmann-Schwinger type for the displacement vector U. Although the elastic scattering problem itself is not studied in the books [23, 24] the reader can find there a rather exhaustive treatment of boundary value problems in linear elasticity by potential methods and our approach is in the spirit of these methods.

5.1 The Fundamental Solution

From now on we suppose that the given real constants ω, λ and μ satisfy

$$\omega > 0\ ,\ \ \mu > 0\ ,\ \ 2\mu + \lambda > 0\ .$$

Moreover, we define $\kappa_s := \omega/\sqrt{\mu}$, $\kappa_p := \omega/\sqrt{2\mu+\lambda}$, and, for a smooth vector field U, $\Delta^* U := \mu\Delta U + (\lambda+\mu)\nabla(\nabla\cdot U)$. If $\nu \in \mathbb{C}^3$ is a vector and U is a smooth vector field in a neighborhood of a point $x \in \mathbb{R}^3$, we denote by $[T(U,\nu)](x)$ the vector

$$[T(U,\nu)](x) := (\beta_1+\mu)\frac{\partial U}{\partial \nu}(x) + \beta_2(\nabla\cdot U)(x)\nu(x) + \beta_1 \nu(x)\wedge[\nabla\wedge U(x)]\ .$$

Here, β_1, $\beta_2 \in \mathbb{R}$ are arbitrary constants satisfying $\beta_1+\beta_2 = \lambda+\mu$. Our notation does not indicate the dependence of T on β_1 and β_2 because large parts of the subsequent analysis are independent of a special choice of these parameters. Let us note that, for $\beta_1 = \mu$ and $\beta_2 = \lambda$, $[T(U,\nu)](x)$ is the traction vector on a surface containing x with normal vector ν at x, whence it has a physical meaning. When we need a special choice of β_1 and β_2 in later sections, we indicate this choice. Moreover, we shall often suppress the dependence of T on ν, too. When we integrate on a surface having the normal vector ν, we simply write $(TU)(x)$ instead of $[T(U,\nu(x))](x)$.

If $D \subset \mathbb{R}^3$ is a C^2-smooth, bounded, open set and if U, $V\colon \overline{D} \to \mathbb{C}^3$ denote $C^2(\overline{D})$-smooth vector fields, then Gauss' theorem implies

$$\begin{aligned}
&\int_{\partial D} (TU)\cdot V\,ds \\
&= \int_D (\beta_1+\mu)\Big(\Delta U\cdot V + \sum_{j,k=1}^{3}\frac{\partial U_j}{\partial x_k}\frac{\partial V_j}{\partial x_k}\Big)dx \\
&\quad + \int_D \beta_2[\nabla(\nabla\cdot U)\cdot V + (\nabla\cdot U)(\nabla\cdot V)]dx \\
&\quad + \int_D \beta_1[(\nabla\wedge\nabla\wedge U)\cdot V - (\nabla\wedge U)\cdot(\nabla\wedge V)]dx \\
&= \int_D \Big\{(\beta_1+\mu)\sum_{j,k=1}^{3}\frac{\partial U_j}{\partial x_k}\frac{\partial V_j}{\partial x_k} + \beta_2(\nabla\cdot U)(\nabla\cdot V) \\
&\qquad - \beta_1(\nabla\wedge U)\cdot(\nabla\wedge V)\Big\}dx
\end{aligned}$$

$$+\int_D (\Delta^* U) \cdot V dx \ , \tag{5.2}$$

which we call the first Betti formula.

It is possible to weaken the regularity assumptions on U and V. U, $V \in C^1(\overline{D}) \cap C^2(D)$ and $\Delta^* U \in C(\overline{D})$ are sufficient for (5.2).

Interchanging the roles of U and V in (5.2) and subtracting yields the second Betti formula

$$\begin{aligned} &\int_{\partial D} \{(TU) \cdot V - (TV) \cdot U\} ds \\ &\quad = \int_D \{(\Delta^* U + \omega^2 U) \cdot V - (\Delta^* V + \omega^2 V) \cdot U\} dx \ , \end{aligned} \tag{5.3}$$

for any $\omega > 0$.

In order to state representation theorems we need a fundamental solution for the operator $\Delta^* + \omega^2 I$. This will be a matrix valued function $\Pi: \mathbb{R}^3 \backslash \{0\} \to \mathbb{C}^{3\times 3}$. Denoting by d_1, d_2, d_3 the cartesian unit vectors in $\mathbb{R}^3$ we define for $x \in \mathbb{R}^3$, $x \neq 0$, the jth column of $\Pi(x)$ by

$$\Pi(x) d_j := \frac{e^{i\kappa_s |x|}}{4\pi\mu|x|} d_j + \frac{1}{\omega^2} \nabla\nabla \cdot \Big\{ \frac{e^{i\kappa_s|x|} - e^{i\kappa_p|x|}}{4\pi|x|} d_j \Big\} \ , \ j = 1, 2, 3 \ .$$

This matrix is called Kupradze's matrix in [24]. We denote its entry in the jth row and kth column by Π_{jk}. From its definition we can infer that it is an even function of x satisfying $\Pi(x) = \Pi(x)^T$, i.e., it coincides with its transpose. In addition, we see with the help of $\nabla \wedge \nabla \wedge \cdot = -\Delta + \nabla(\nabla \cdot .)$ that

$$\Pi(x) d_j = \frac{1}{\omega^2} \nabla \wedge \nabla \wedge \Big\{ \frac{e^{i\kappa_s|x|}}{4\pi|x|} d_j \Big\} - \frac{1}{\omega^2} \nabla\nabla \cdot \Big\{ \frac{e^{i\kappa_p|x|}}{4\pi|x|} d_j \Big\} \ , \ j = 1, 2, 3 \ . \tag{5.4}$$

We have to study some more properties of Π, especially its behavior for $|x| \to 0$. To this end we expand $e^{i\kappa|x|}/(4\pi|x|)$ in a power series and obtain

$$\begin{aligned} \frac{e^{i\kappa|x|}}{4\pi|x|} &= \frac{\cos(\kappa|x|)}{4\pi|x|} + i \frac{\sin(\kappa|x|)}{4\pi|x|} \\ &= \frac{1}{4\pi|x|} - \frac{\kappa^2}{8\pi}|x| + \kappa^4 |x|^3 f_1(\kappa^2|x|^2) + i\kappa f_2(\kappa^2|x|^2) \end{aligned} \tag{5.5}$$

with two entire functions f_1 and f_2. Inserting these expressions in the definition of Π, collecting the terms having a $1/|x|$-singularity, and using $(\kappa_p^2 - \kappa_s^2)/\omega^2 = -(\lambda+\mu)/[\mu(2\mu+\lambda)]$ motivates the definition of

$$\Pi_{jk}^{(0)}(x) := \frac{\delta_{jk}}{4\pi\mu|x|} - \frac{\lambda+\mu}{8\pi\mu(2\mu+\lambda)} \frac{\partial^2 |x|}{\partial x_j \partial x_k} \ , \ j,k = 1,2,3 \ ,$$

and of the matrix $\Pi^{(0)}(x) := (\Pi_{jk}^{(0)}(x))$ for $x \neq 0$. For the Δ^*-operator Kelvin's matrix $\Pi^{(0)}$ has the same role that $1/(4\pi|\cdot|)$ has for the Δ-operator.

Lemma 5.1 Π *and* $\Pi^{(0)}$ *satisfy:*

(a) $\Delta^*(\Pi(x)d_j) + \omega^2((\Pi(x)d_j) = 0 \ , \ \ \Delta^*(\Pi^{(0)}(x)d_j) = 0$ *in* $\mathbb{R}^3 \setminus \{0\}$.

(b) For any constant c_1 *there exists a constant* c_2 *such that for all* $0 < |x| \leq c_1$ *and all* $j,k = 1,2,3$, $l_1, l_2, l_3 = 1,2,3$, *the estimates*

$$|\Pi_{jk}(x) - \Pi_{jk}^{(0)}(x)| \leq c_2 \ , \ \Big|\frac{\partial}{\partial x_{l_1}}(\Pi_{jk}(x) - \Pi_{jk}^{(0)}(x))\Big| \leq \frac{c_2}{|x|} \ ,$$

$$\Big|\frac{\partial^2}{\partial x_{l_1}\partial x_{l_2}}(\Pi_{jk}(x) - \Pi_{jk}^{(0)}(x))\Big| \leq \frac{c_2}{|x|^2} \ ,$$

$$\Big|\frac{\partial^3}{\partial x_{l_1}\partial x_{l_2}\partial x_{l_3}}(\Pi_{jk}(x) - \Pi_{jk}^{(0)}(x))\Big| \leq \frac{c_2}{|x|^3}$$

hold true.

(c) For any constant c_1 *there exists a constant* c_2 *such that for all* $0 < |x| \leq c_1$ *and all* $j,k = 1,2,3$, $l_1, l_2 = 1,2,3$, *the estimates*

$$|\Pi_{jk}(x)| \leq \frac{c_2}{|x|} \ , \Big|\frac{\partial}{\partial x_{l_1}}\Pi_{jk}(x)\Big| \leq \frac{c_2}{|x|^2} \ ,$$

$$\Big|\frac{\partial^2}{\partial x_{l_1}\partial x_{l_2}}\Pi_{jk}(x)\Big| \leq \frac{c_2}{|x|^3}$$

hold true. The same estimates are valid, if Π *is replaced by* $\Pi^{(0)}$.

Proof: For Π part (a) follows by straightforward calculations using (5.4) and $\Delta^* = -\mu\nabla \wedge \nabla \wedge \cdot + (\lambda + 2\mu)\nabla(\nabla \cdot .)$. For $\Pi^{(0)}$ we compute

$$\begin{aligned}
&\Delta^*(\Pi^{(0)}(x)d_j) \\
&= (\mu\Delta + (\lambda+\mu)\nabla(\nabla\cdot .))\Big[\frac{1}{4\pi\mu|x|}d_j\Big] \\
&\quad -\frac{\lambda+\mu}{8\pi\mu(2\mu+\lambda)}(-\mu\nabla\wedge\nabla\wedge\cdot+(\lambda+2\mu)\nabla(\nabla\cdot .))\Big[\nabla(\nabla\cdot(|x|d_j))\Big] \\
&= \frac{\lambda+\mu}{4\pi\mu}\Big[\nabla(\nabla\cdot(|x|^{-1}d_j))\Big] - \frac{\lambda+\mu}{8\pi\mu}\Big[\nabla(\nabla\cdot([\Delta|x|]d_j))\Big] \\
&= 0\ .
\end{aligned}$$

For part (b) we insert the expansion (5.5) into the definition of $\Pi_{jk}(x)$ and we arrive at

$$\Pi_{jk}(x) - \Pi^{(0)}_{jk}(x) = |x|f_{jk}(|x|^2) + \tilde{f}_{jk}(|x|^2) + \frac{\partial^2}{\partial x_j \partial x_k}\Big\{|x|^3 g_{jk}(|x|^2) + h_{jk}(|x|^2)\Big\}$$

with entire functions f_{jk}, $\tilde{f}_{jk}$, g_{jk}, h_{jk}. Observing for any odd integer l the relation $\partial|x|^l/\partial x_m = l x_m |x|^{l-2}$, assertion (b) follows by differentiating the above expression.

The estimates for $\Pi^{(0)}$ in assertion (c) are also a consequence of the above observation. Together with part (b) the assertions are valid for Π, too, and we have proved the lemma.

□

Before we state and prove the representation theorem for the elasticity equation we want to prove separately one technical ingredient for the representation theorem.

Lemma 5.2 *Let U be a continuous vector field in a neighborhood of $x \in \mathbb{R}^3$. Then, we have*

$$\begin{aligned}
-U_j(x) &= \lim_{\epsilon\to 0} \int_{\partial B_\epsilon(x)} U(y)\cdot T_y(\Pi(x-y)d_j)ds(y) \\
&= \lim_{\epsilon\to 0} \int_{\partial B_\epsilon(x)} U(y)\cdot T_y(\Pi^{(0)}(x-y)d_j)ds(y)\ .
\end{aligned}$$

Proof: We will prove the identity

$$\int\limits_{\partial B_\epsilon(x)} [T_y(\Pi^{(0)}(x-y)d_j)]_l ds(y) = -\delta_{jl} \ , \ \epsilon > 0 \ , \tag{5.6}$$

$[T_y(\Pi^{(0)}(x-y)d_j)]_l$ being the lth component of the vector $T_y(\Pi^{(0)}(x-y)d_j)$.

Then, using the inequalities for the first derivatives of $\Pi^{(0)}_{jk}$ from Lemma 5.1, we can conclude

$$\begin{aligned} &\Big| \int\limits_{\partial B_\epsilon(x)} [U(y)-U(x)] \cdot T_y(\Pi^{(0)}(x-y)d_j) ds(y) \Big| \\ &\leq \max_{y\in\partial B_\epsilon(x)} |U(y)-U(x)| \int\limits_{\partial B_\epsilon(x)} c\epsilon^{-2} ds(y) \to 0 \ , \ \epsilon \to 0 \ , \end{aligned}$$

whence

$$\begin{aligned} &\int\limits_{\partial B_\epsilon(x)} U(y) \cdot T_y(\Pi^{(0)}(x-y)d_j) ds(y) \\ &= \int\limits_{\partial B_\epsilon(x)} U(x) \cdot T_y(\Pi^{(0)}(x-y)d_j) ds(y) \\ &\quad + \int\limits_{\partial B_\epsilon(x)} [U(y)-U(x)] \cdot T_y(\Pi^{(0)}(x-y)d_j) ds(y) \\ &\to -U_j(x) \ , \ \epsilon \to 0 \ . \end{aligned}$$

Note that we use the same letter c for various constants during the proof.

Moreover, the inequalities

$$\Big| \frac{\partial}{\partial y_l} \Pi_{jk}(x-y) - \frac{\partial}{\partial y_l} \Pi^{(0)}_{jk}(x-y) \Big| \leq \frac{c}{|x-y|}$$

from Lemma 5.1 (b) imply

$$\Big| T_y(\{\Pi(x-y) - \Pi^{(0)}(x-y)\} d_j) \Big| \leq c\epsilon^{-1} \ , \ y \in \partial B_\epsilon(x) \ ,$$

whence

$$\lim_{\epsilon\to 0} \int\limits_{\partial B_\epsilon(x)} U(y) \cdot T_y(\{\Pi(x-y) - \Pi^{(0)}(x-y)\} d_j) ds(y) = 0$$

and

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int\limits_{\partial B_\epsilon(x)} U(y)\cdot T_y(\Pi(x-y)d_j)ds(y)\\
&= \quad -U_j(x)+\lim_{\epsilon\to 0}\int\limits_{\partial B_\epsilon(x)} U(y)\cdot T_y(\{\Pi(x-y)-\Pi^{(0)}(x-y)\}d_j)ds(y)\\
&= \quad -U_j(x)\ .
\end{aligned}$$

Let us now turn to the proof of (5.6). Without loss of generality we assume $x = 0$, whence $\nu(y) = \epsilon^{-1}y$ for $y \in \partial B_\epsilon(0)$. We start with the calculation of $[T_y((4\pi\mu|y|)^{-1}d_j)]_l$. To this end we use $\beta_1+\beta_2=\lambda+\mu$ and

$$[\nu\wedge(\nabla\wedge U)]_l = -\frac{\partial U_l}{\partial\nu}+\sum_{k=1}^{3}\nu_k\frac{\partial U_k}{\partial x_l} \tag{5.7}$$

to arrive at

$$\begin{aligned}
\Big[T_y\Big(\frac{1}{4\pi\mu|y|}d_j\Big)\Big]_l &= (\beta_1+\mu)\frac{\partial}{\partial\nu}\Big(\frac{\delta_{jl}}{4\pi\mu|y|}\Big)+\beta_2\frac{\partial}{\partial y_j}\Big(\frac{1}{4\pi\mu|y|}\Big)\nu_l\\
&\quad +\beta_1\Big[-\frac{\partial}{\partial\nu}\Big(\frac{\delta_{jl}}{4\pi\mu|y|}\Big)+\sum_{k=1}^{3}\nu_k\frac{\partial}{\partial y_l}\Big(\frac{\delta_{jk}}{4\pi\mu|y|}\Big)\Big]\\
&= -\frac{\delta_{jl}}{4\pi\epsilon^2}-\frac{\lambda+\mu}{4\pi\epsilon^4\mu}y_jy_l\ .
\end{aligned}$$

Next, we compute

$$\begin{aligned}
\Big[T_y\Big(\nabla\frac{y_j}{|y|}\Big)\Big]_l &= (\beta_1+\mu)\sum_{k=1}^{3}\nu_k\frac{\partial}{\partial y_k}\Big(\frac{\delta_{jl}}{|y|}-\frac{y_jy_l}{|y|^3}\Big)+\beta_2\Delta\Big(\frac{y_j}{|y|}\Big)\nu_l\\
&= -(\beta_1+\mu)\frac{\delta_{jl}}{\epsilon^2}+(\beta_1+\mu)\frac{y_jy_l}{\epsilon^4}-2\beta_2\frac{y_jy_l}{\epsilon^4}\ .
\end{aligned}$$

With the help of the last two equations we finally arrive at

$$\begin{aligned}
[T_y(\Pi^{(0)}(y)d_j)]_l &= -\frac{\delta_{jl}}{4\pi\epsilon^2}-\frac{\lambda+\mu}{4\pi\epsilon^4\mu}y_jy_l\\
&\quad -\frac{\lambda+\mu}{8\pi\mu(2\mu+\lambda)}\Big[-(\beta_1+\mu)\frac{\delta_{jl}}{\epsilon^2}+(\beta_1+\mu)\frac{y_jy_l}{\epsilon^4}-2\beta_2\frac{y_jy_l}{\epsilon^4}\Big]\\
&= -\frac{\delta_{jl}}{4\pi\epsilon^2}\Big[1-\frac{(\lambda+\mu)(\beta_1+\mu)}{2\mu(2\mu+\lambda)}\Big]\\
&\quad -\frac{y_jy_l}{4\pi\epsilon^4}\frac{\lambda+\mu}{\mu}\Big(1+\frac{\beta_1-2\beta_2+\mu}{2(2\mu+\lambda)}\Big)\ .
\end{aligned}$$

From Gauss' theorem we know

$$\int_{\partial B_\epsilon(0)} \frac{y_j y_l}{4\pi\epsilon^4} ds = \frac{1}{4\pi\epsilon^3} \int_{\partial B_\epsilon(0)} y_j \nu_l ds = \frac{1}{4\pi\epsilon^3} \int_{B_\epsilon(0)} \frac{\partial y_j}{\partial y_l} dy = \frac{\delta_{jl}}{3} \ ,$$

whence

$$\begin{aligned} & \int_{\partial B_\epsilon(0)} [T_y(\Pi^{(0)}(y)d_j)]_l ds \\ = & \ -\delta_{jl}\Big[1 - \frac{(\lambda+\mu)(\beta_1+\mu)}{2\mu(2\mu+\lambda)} + \frac{1}{3}\frac{\lambda+\mu}{\mu}\Big(1 + \frac{\beta_1 - 2\beta_2 + \mu}{2(2\mu+\lambda)}\Big)\Big] \\ = & \ -\delta_{jl} \ , \end{aligned}$$

and we have proved the lemma.

□

With the help of the preceding lemma we can now prove representation theorems. To this end we also need the analogues of the double-layer potentials and we define for a vector $\nu \in \mathbb{C}^3$ the matrix valued functions Ξ and $\Xi^{(0)}$ by

$$\begin{aligned} \Xi(x,y,\nu)^T d_j &:= T_y(\Pi(x-y)d_j, \nu) \ , \ x, y \in \mathbb{R}^3 \ , \ x \neq y \ , \\ \Xi^{(0)}(x,y,\nu)^T d_j &:= T_y(\Pi^{(0)}(x-y)d_j, \nu) \ , \ x, y \in \mathbb{R}^3 \ , \ x \neq y \ , \end{aligned}$$

i.e., the jth row of $\Xi(x,y,\nu)$ consists of the pseudostress vector of the jth column of Π and similarly for $\Xi^{(0)}$. Since in the sequel ν is always the unit normal vector at a point y lying on a surface, we omit the dependence on ν and write $\Xi(x,y)$ instead of $\Xi(x,y,\nu(y))$ and similarly for $\Xi^{(0)}$.

Theorem 5.3 *Let $D \subset \mathbb{R}^3$ be a bounded, open, C^2-smooth set with exterior unit normal vector ν. For a vector field $U \in C^1(\overline{D}) \cap C^2(D)$ with $\Delta^* U \in C(\overline{D})$ we have the representation formulas*

$$\begin{aligned} U(x) = & \int_{\partial D} \Big\{\Pi(x-y)(TU)(y) - \Xi(x,y)U(y)\Big\} ds(y) \\ & - \int_D \Pi(x-y)(\Delta^* U + \omega^2 U)(y) dy \ , \ x \in D \ . \qquad (5.8) \end{aligned}$$

and

$$\begin{aligned} U(x) &= \int\limits_{\partial D} \Big\{ \Pi^{(0)}(x-y)(TU)(y) - \Xi^{(0)}(x,y)U(y) \Big\} ds(y) \\ &\quad - \int\limits_{D} \Pi^{(0)}(x-y)(\Delta^* U)(y) dy \ , \ x \in D \ . \end{aligned} \tag{5.9}$$

Proof: For a given $x \in D$ we choose $\epsilon > 0$ sufficiently small to ensure $\overline{B_\epsilon(x)} \subset D$ and apply the second Betti formula in $D_\epsilon := D \setminus \overline{B_\epsilon(x)}$ with $V := \Pi(x - \cdot)d_j$. With the normal vector ν on $\partial B_\epsilon(x)$ being directed into the exterior of $B_\epsilon(x)$ we obtain

$$\begin{aligned} &\int\limits_{\partial D} d_j \cdot \Big\{ \Xi(x,y)U(y) - \Pi(x-y)(TU)(y) \Big\} ds(y) \\ &\qquad - \int\limits_{\partial B_\epsilon(x)} d_j \cdot \Big\{ \Xi(x,y)U(y) - \Pi(x-y)(TU)(y) \Big\} ds(y) \\ &= \ - \int\limits_{D_\epsilon} d_j \cdot \Pi(x-y)(\Delta^* U + \omega^2 U)(y) dy \ . \end{aligned}$$

Since the entries of $\Pi(x-y)$ are of magnitude ϵ^{-1} on $\partial B_\epsilon(x)$, these terms vanish as $\epsilon \to 0$. According to the previous lemma the terms

$$- \int\limits_{\partial B_\epsilon(x)} d_j \cdot \Xi(x,y)U(y) ds(y)$$

converge to $U_j(x)$ as $\epsilon \to 0$. Hence, the limit $\epsilon \to 0$ reveals the jth row of equation (5.8).

Equation (5.9) is proved analogously.

□

The above representation formulas imply that solutions to $\Delta^* U + \omega^2 U = 0$ or $\Delta^* U = 0$ are analytic.

Our next aim is the derivation of the corresponding representation formula, if U is a solution to $\Delta^* U + \omega^2 U = 0$ in an exterior domain. To this end we have to impose an additional requirement on U, namely a radiation condition. The radiation condition and the fundamental solution must match. There are two ways to obtain a radiation condition. One can study the behavior of the fundamental solution for large $|x|$ and then formulate a radiation condition accordingly. This is done in [23, 24]. A second possibility is to require an integral relation which corresponds to (1.14). We choose the latter approach now.

Let $U \in C^2(\mathbb{R}^3 \setminus B_R)$ be a solution to $\Delta^* U + \omega^2 U = 0$. U is a radiating solution, if for all $r > R$ and for all $|x| < r$ the identity

$$\int\limits_{|y|=r} \Big\{\Pi(x-y)(TU)(y) - \Xi(x,y)U(y)\Big\} ds(y) = 0 \tag{5.10}$$

holds true.

The radiation condition and the representation formula (5.8) applied in the spherical shell $\{R < |x| < r\}$ to a radiating solution U to $\Delta^* U + \omega^2 U = 0$ immediately yields the following theorem.

Theorem 5.4 *Let $U \in C^2(\mathbb{R}^3 \setminus B_R)$ be a radiating solution to $\Delta^* U + \omega^2 U = 0$ in $\mathbb{R}^3 \setminus \overline{B_R}$. Then we have*

$$U(x) = \int\limits_{\partial B_R} \Big\{\Xi(x,y)U(y) - \Pi(x-y)(TU)(y)\Big\} ds(y) \ , \ \ |x| > R \ . \tag{5.11}$$

We next check whether the columns of the fundamental solution $\Pi(y-z)$ regarded as vector fields of the variable y are radiating solutions.

Lemma 5.5 *Fix $z \in \mathbb{R}^3$, $k \in \{1,2,3\}$, and $R > |z|$. Then, $U := \Pi(\cdot - z)d_k$ is a radiating solution to $\Delta^* U + \omega^2 U = 0$ in $\mathbb{R}^3 \setminus \{z\}$.*

Proof: Suppose $r > R$, $|x| < r$, and $x \neq z$, and define the vector

$$I(x) := \int\limits_{|y|=r} \Big\{\Pi(x-y)(TU)(y) - \Xi(x,y)U(y)\Big\} ds(y) \ .$$

Writing for a sufficiently small $\epsilon > 0$

$$\begin{aligned} I(x) \ &= \ \Bigg[\int\limits_{|y|=r} \Big\{...\Big\} ds - \int\limits_{\partial B_\epsilon(x)} \Big\{...\Big\} ds - \int\limits_{\partial B_\epsilon(z)} \Big\{...\Big\} ds \Bigg] \\ &\quad + \int\limits_{\partial B_\epsilon(x)} \Big\{...\Big\} ds + \int\limits_{\partial B_\epsilon(z)} \Big\{...\Big\} ds \ , \end{aligned}$$

we can conclude from the second Betti formula applied in $B_r \setminus (B_\epsilon(z) \cup B_\epsilon(x))$ that the components $d_j \cdot [\ldots]$, $j = 1,2,3$, of the terms in square brackets vanish. Moreover, the representation (5.8) applied to U in $B_\epsilon(x)$ implies

$$d_j \cdot \int\limits_{\partial B_\epsilon(x)} \Big\{ ... \Big\} ds = U_j(x) = d_j \cdot \Pi(x - z) d_k \ , \ j = 1,2,3 \ .$$

Finally, using that Π is an even function and (5.8) again we compute

$$\begin{aligned} & d_j \cdot \int\limits_{\partial B_\epsilon(z)} \Big\{ \Pi(x-y)(TU)(y) - \Xi(x,y)U(y) \Big\} ds \\ = & - \int\limits_{\partial B_\epsilon(z)} \Big\{ [\Pi(z-y)d_k] \cdot [T_y(\Pi(x-y)d_j)] \\ & \qquad - [T_y(\Pi(z-y)d_k)] \cdot [\Pi(x-y)d_j] \Big\} ds \\ = & -d_j \cdot \Pi(z-x) d_k \ . \end{aligned}$$

Hence, we know $I(x) = 0$ for all $|x| < r$, $x \neq z$. Since the definition of I reveals that I is continuous in $|x| < r$, we have proved $I(x) = 0$ for all $|x| < r$, i.e., the columns of $\Pi(\cdot - z)$ satisfy the radiation condition.

□

We conclude this section by studying the mapping properties of a volume potential with kernel $\Pi(x - y)$. Lemma 5.1 (c) together with Theorem 1.9 yield that

$$(\mathcal{V}\varphi)(x) := \int\limits_{B_R} \Pi(x-y)\varphi(y) dy \ , \ x \in \mathbb{R}^3 \ ,$$

is a uniformly γ-Hölder continuous differentiable vector field on each compact subset of $\mathbb{R}^3$, if $\varphi \in C(\overline{B_R})$ is a continuous vector field ($0 < \gamma < 1$). Furthermore, we have $\|\mathcal{V}\varphi\|_{1,\gamma,B_r} \leq c \|\varphi\|_{\infty,B_R}$ with a suitable constant $c = c(r)$ and the derivatives can be computed by

$$(\partial_j(\mathcal{V}\varphi))(x) = \int\limits_{B_R} (\partial_j \Pi)(x-y)\varphi(y) dy \ , \ x \in \mathbb{R}^3 \ , \ j = 1,2,3 \ .$$

$(\partial_j \Pi)$ denotes the matrix obtained by taking the jth derivative of each entry of Π. In addition we have for $\varphi \in C_0^1(B_R)$ the identity $\partial_j(\mathcal{V}\varphi) = \mathcal{V}(\partial_j \varphi)$. Of course, the same is true, if Π is replaced by $\Pi^{(0)}$. Furthermore, the behavior of $\Pi - \Pi^{(0)}$ at $x = 0$ stated in Lemma 5.1 (b) allows to conclude that even the first derivatives of

$$\int_{B_R} \{\Pi(x-y) - \Pi^{(0)}(x-y)\}\varphi(y)dy \ , \ x \in \mathbb{R}^3 \ ,$$

can be treated as above. Therefore, in order to study the second derivatives of the volume potential with kernel $\Pi(x-y)$ it suffices to examine the volume potential with kernel $\Pi^{(0)}(x-y)$.

Theorem 5.6 *Assume $R > 0$ and define for a vector field $\varphi \in C^{0,\gamma}(\overline{B_R})$, $\gamma \in (0,1)$, the volume potential*

$$(\mathcal{V}\varphi)(x) := \int_{B_R} \Pi(x-y)\varphi(y)dy \ , \ x \in B_R \ .$$

Then $\mathcal{V}\varphi \in C^2(B_R)$ and

$$(\Delta^*(\mathcal{V}\varphi))(x) + \omega^2(\mathcal{V}\varphi)(x) = -\varphi(x) \ , \ x \in B_R \ .$$

An analogous assertion holds true, if Π is replaced by $\Pi^{(0)}$.

Proof: Lemma 5.1 together with Theorem 1.10 (a) imply $\mathcal{V}\varphi \in C^2(B_R)$. Moreover, for $\psi \in C^{0,\gamma}(\overline{B_R})$ the second derivatives of

$$u(x) = \int_{B_R} \Pi^{(0)}_{l_1 l_2}(x-y)\psi(y)dy \ , \ x \in B_R \ ,$$

are given by

$$\begin{aligned} \partial_l \partial_j u(x) &= \int_{B_R} \Big(\partial_l \partial_j \Pi^{(0)}_{l_1 l_2})(x-y)\Big)(\psi(y) - \psi(x))dy \\ &\quad - \psi(x) \int_{\partial B_R} \nu_l(y)(\partial_j \Pi^{(0)}_{l_1 l_2})(x-y)ds(y) \ , \ x \in B_R \ . \end{aligned}$$

Now, for a vector field $\varphi = \psi d_k$ we compute with the help of the above formula

$$\Big[\Delta^*(\mathcal{V}\varphi)\Big](x) \ = \ \Big[(\mu+\beta_1)\Delta(\mathcal{V}\varphi) + \beta_2 \nabla\nabla\cdot(\mathcal{V}\varphi) + \beta_1 \nabla\wedge\nabla\wedge(\mathcal{V}\varphi)\Big](x)$$

$$
\begin{aligned}
&= \int\limits_{B_R} \Delta_x^*\Big(\Pi^{(0)}(x-y)d_k\Big)(\psi(y)-\psi(x))dy \\
&\quad + \psi(x)\int\limits_{\partial B_R}\Big\{(\mu+\beta_1)\frac{\partial}{\partial\nu(y)}[\Pi^{(0)}(x-y)d_k] \\
&\qquad\qquad + \beta_2\sum_{m=1}^{3}\frac{\partial}{\partial y_m}[\Pi^{(0)}(x-y)d_k]_m\nu(y) \\
&\qquad\qquad + \beta_1\nu(y)\wedge\nabla_y\wedge[\Pi^{(0)}(x-y)d_k]\Big\}ds(y) \\
&= \psi(x)\int\limits_{\partial B_R} T_y(\Pi^{(0)}(x-y)d_k)ds(y) \\
&= -\psi(x)d_k \ , \ x\in B_R \ .
\end{aligned}
$$

The last equality follows by applying the representation formula (5.9) to an arbitrary constant vector field $U(x)=p\in\mathbb{C}^3$, $x\in\mathbb{R}^3$, which yields

$$
p = -\int\limits_{\partial B_R} \Xi^{(0)}(x,y)pds(y) \ , \ x\in B_R \ ,
$$

and thus

$$
\begin{aligned}
d_k\cdot p &= -\int\limits_{\partial B_R}\Big\{[\Xi^{(0)}(x,y)]^T d_k\Big\}\cdot pds(y) \\
&= -\int\limits_{\partial B_R} T_y(\Pi^{(0)}(x-y)d_k)ds(y)\cdot p \ , \ x\in B_R \ .
\end{aligned}
$$

Since any vector field $\varphi=(\varphi_1,\varphi_2,\varphi_3)$ can be decomposed into $\varphi=\sum\varphi_k d_k$, we have proved the assertions for the kernel $\Pi^{(0)}$.

Due to the behavior of $\Pi-\Pi^{(0)}$ near $x=0$ (Lemma 5.1 (b)) and due to the relation $\Delta^*\Big\{[\Pi(x)-\Pi^{(0)}(x)]d_k\Big\}=-\omega^2\Pi(x)d_k$, $x\neq 0$, our results are also true for the volume potential with kernel $\Pi(x-y)$ by Theorem 1.9.

□

5.2 Unique Solvability of the Direct Elastic Scattering Problem

This section is devoted to the following scattering problem for elastic waves:

Given real constants ω, λ and μ satisfying $\omega > 0$, $\mu > 0$, $\lambda + 2\mu > 0$, and given a real valued function $\rho \in C^{1,\gamma}(\mathbb{R}^3)$ $(0 < \gamma < 1)$ with $\operatorname{supp}(1-\rho) \subset B_R$, and given an incident wave U^i, i.e., the vector field $U^i \in C^2(\mathbb{R}^3)$ is a solution to $\Delta^* U^i + \omega^2 U^i = 0$ in $\mathbb{R}^3$,

find the vector field $U^s \in C^2(\mathbb{R}^3)$ such that the total field $U := U^i + U^s$ satisfies $\Delta^* U + \omega^2 \rho U = 0$ in $\mathbb{R}^3$, and such that U^s satisfies the radiation condition (5.10).

We start by proving that a solution of the above scattering problem is also a solution to a Lippmann-Schwinger type integral equation and *vice versa*.

Lemma 5.7 *If $U^s \in C^2(\mathbb{R}^3)$ is a solution to the above elastic scattering problem, then $U = U^i + U^s$ is a solution to*

$$U(x) = U^i(x) - \omega^2 \int_{B_R} (1-\rho(y))\Pi(x-y)U(y)dy\ ,\ \ x \in \mathbb{R}^3\ . \tag{5.12}$$

If $\varphi \in C(\overline{B_R})$ is a solution to (5.12) in $\overline{B_R}$ and if U^s is defined by

$$U^s(x) := -\omega^2 \int_{B_R} (1-\rho(y))\Pi(x-y)\varphi(y)dy\ ,\ \ x \in \mathbb{R}^3\ ,$$

then, U^s is a solution to the elastic scattering problem with incident wave U^i.

Proof: Let U^s be a solution to the elastic scattering problem with incident wave U^i. Applying the representation formula (5.8) to $U := U^i + U^s$ in the ball B_r, $r > R$, we get

$$\begin{aligned} U(x) &= \int_{\partial B_r} \Big\{\Pi(x-y)(TU)(y) - \Xi(x,y)U(y)\Big\}ds(y) \\ &\quad - \int_{B_r} \Pi(x-y)(\Delta^* U + \omega^2 U)(y)dy \\ &= \int_{\partial B_r} \Big\{\Pi(x-y)(TU^i)(y) - \Xi(x,y)U^i(y)\Big\}ds(y) \end{aligned}$$

$$+\int_{\partial B_r}\Big\{\Pi(x-y)(TU^s)(y)-\Xi(x,y)U^s(y)\Big\}ds(y)$$
$$-\omega^2\int_{B_r}(1-\rho(y))\Pi(x-y)U(y)dy\ ,\ x\in B_r\ .$$

The radiation condition states that the integrals on ∂B_r containing U^s vanish, whereas the integrals on ∂B_r containing U^i can be replaced by $U^i(x)$ due to Theorem 5.3 applied to U^i. This proves the first part of the lemma.

Suppose now that $\varphi \in C(\overline{B_R})$ is a solution to

$$\varphi(x)=U^i(x)-\omega^2\int_{B_R}(1-\rho(y))\Pi(x-y)\varphi(y)dy\ ,\ x\in\overline{B_R}\ .$$

Applying the smoothing properties of volume potentials we obtain $\varphi \in C^{1,\gamma}(\overline{B_R})$, whence $(1-\rho)\varphi \in C_0^1(B_R)$ and $U^s \in C^2(\mathbb{R}^3)$. Since the columns of $\Pi(x-y)$, considered as a function of x, satisfy the radiation condition by Lemma 5.5, U^s satisfies the radiation condition. Finally, we compute for $U = U^i + U^s$: $U|_{\overline{B_R}} = \varphi$ by the integral equation and

$$\begin{aligned}
&\Delta^* U+\omega^2 U\\
&\quad= -\omega^2(\Delta^*+\omega^2)\int_{B_R}(1-\rho(y))\Pi(\cdot-y)U(y)dy\\
&\quad= \omega^2(1-\rho)U
\end{aligned}$$

by Theorem 5.6. This completes the proof of the lemma.

□

Since the integral operator in (5.12) is compact, it now suffices to prove that the elastic scattering problem has at most one solution in order to establish the existence of a unique solution. Although the main ideas for the uniqueness proof are the same as for the previous scattering problems, it becomes longer than the former ones due to some additional technical difficulties.

The first part of the uniqueness proof is the following lemma. We show that a radiating solution $U \in C^2(\mathbb{R}^3 \setminus \overline{B_R})$ to $\Delta^* U + \omega^2 U = 0$ must vanish in the exterior of B_R, if an additional condition on the sign of

$$\Im\Big(\int_{\partial B_{R_1}} U\cdot\{-\mu\nu\wedge\nabla\wedge\overline{U}+(\lambda+2\mu)(\nabla\cdot\overline{U})\nu\}ds\Big)\ ,\ R_1>R\ ,$$

is satisfied. Note that the integrand is equal to $U \cdot T\overline{U}$ with the special choice $\beta_1 = -\mu$, $\beta_2 = \lambda + 2\mu$. The idea is to apply Rellich's lemma to $\nabla \cdot U$ and $\nabla \wedge U$ which are both solutions to a Helmholtz equation in $\mathbb{R}^3 \setminus B_R$. Then, we can infer from the differential equation

$$-\mu \nabla \wedge \nabla \wedge U + (\lambda + 2\mu)\nabla(\nabla \cdot U) + \omega^2 U = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B_R}$$

that

$$U = \frac{\mu}{\omega^2}\nabla \wedge \nabla \wedge U - \frac{\lambda + 2\mu}{\omega^2}\nabla(\nabla \cdot U) \ , \tag{5.13}$$

whence $U = 0$.

Lemma 5.8 *If $U \in C^2(\mathbb{R}^3 \setminus \overline{B_R})$ satisfies $\Delta^* U + \omega^2 U = 0$ in $\mathbb{R}^3 \setminus \overline{B_R}$, the radiation condition (5.10), and*

$$\Im\Big(\int\limits_{\partial B_{R_1}} U \cdot \{-\mu\nu \wedge \nabla \wedge \overline{U} + (\lambda + 2\mu)(\nabla \cdot \overline{U})\nu\} ds\Big) \geq 0 \ ,$$

for a fixed $R_1 > R$, then $U(x) = 0$ for all $|x| > R$.

Proof: We choose a function $\chi \in C_0^\infty(\mathbb{R}^3 \setminus \overline{B_R})$ such that $\chi(x) = 1$ for all $|x| \geq (R + R_1)/2$ and define $U' := \chi U \in C^\infty(\mathbb{R}^3)$. U' is a solution to $\Delta^* U' + \omega^2 U' = \omega^2 F$ in $\mathbb{R}^3$ with a vector field $F \in C_0^\infty(B_{R_1})$ depending on χ and U. U' coincides with U in the exterior of B_{R_1}, whence it is also a radiating solution and we know from the representation (5.8), the radiation condition (5.10), and by (5.4)

$$\begin{aligned} U(x) = U'(x) &= -\omega^2 \int\limits_{B_{R_1}} \Pi(x - y)F(y)dy \\ &= -\nabla \wedge \nabla \wedge \int\limits_{B_{R_1}} \Phi_{\kappa_s}(x, y)F(y)dy \\ &\quad + \nabla\nabla \cdot \int\limits_{B_{R_1}} \Phi_{\kappa_p}(x, y)F(y)dy \ , \ x \in \mathbb{R}^3 \setminus \overline{B_{R_1}} \ . \end{aligned}$$

Hence,

$$\begin{aligned} \nabla \cdot U(x) &= -\kappa_p^2 \nabla \cdot \int\limits_{B_{R_1}} \Phi_{\kappa_p}(x, y)F(y)dy \\ &= -\kappa_p^2 \int\limits_{B_{R_1}} \Phi_{\kappa_p}(x, y)\nabla \cdot F(y)dy \ , \ x \in \mathbb{R}^3 \setminus \overline{B_{R_1}} \ , \end{aligned}$$

and

$$\begin{aligned}\nabla \wedge U(x) &= -\kappa_s^2 \nabla \wedge \int\limits_{B_{R_1}} \Phi_{\kappa_s}(x,y) F(y) dy \\ &= -\kappa_s^2 \int\limits_{B_{R_1}} \Phi_{\kappa_s}(x,y) \nabla \wedge F(y) dy \ , \ x \in \mathbb{R}^3 \setminus \overline{B_{R_1}} \ ,\end{aligned}$$

are radiating solutions to a Helmholtz equation with wave number κ_p, κ_s, respectively. This implies for large $|x|$ and $\hat{x} := |x|^{-1} x$ the estimate

$$|\nabla \cdot U(x)| + |\nabla \wedge U(x)| \leq \frac{c}{|x|} \ , \tag{5.14}$$

and the radiation conditions

$$|\hat{x} \cdot \nabla(\nabla \cdot U)(x) - i\kappa_p (\nabla \cdot U)(x)| \leq \frac{c}{|x|^2} \ , \tag{5.15}$$

$$|(\nabla \wedge \nabla \wedge U)(x) \wedge \hat{x} - i\kappa_s (\nabla \wedge U)(x)| \leq \frac{c}{|x|^2} \ . \tag{5.16}$$

Furthermore, the behavior

$$\nabla_x \Phi_{\kappa_p}(x,y) \wedge \hat{x} = O\Big(\frac{1}{|x|^2}\Big) \ , \ \Big(\nabla_x \wedge \{\Phi_{\kappa_s}(x,y) q\}\Big) \cdot \hat{x} = O\Big(\frac{|q|}{|x|^2}\Big) \ ,$$

for large $|x|$, uniformly in $y \in B_{R_1}$, reveals

$$|\nabla(\nabla \cdot U)(x) \wedge \hat{x}| + |(\nabla \wedge \nabla \wedge U(x)) \cdot \hat{x}| \leq \frac{c}{|x|^2} \ .$$

Hence, we obtain from (5.13)

$$(\nabla \wedge \nabla \wedge U)(x) \wedge \hat{x} = \frac{\omega^2}{\mu} U(x) \wedge \hat{x} + O\Big(\frac{1}{|x|^2}\Big) \ , \ |x| \to \infty \ , \tag{5.17}$$

$$\nabla(\nabla \cdot U)(x) \cdot \hat{x} = -\frac{\omega^2}{2\mu + \lambda} U(x) \cdot \hat{x} + O\Big(\frac{1}{|x|^2}\Big) \ , \ |x| \to \infty \ . \tag{5.18}$$

Employing the inequalities (5.15), (5.16) we compute

$$0 = \lim_{r \to \infty} \int\limits_{\partial B_r} \Big\{ \frac{\mu^2}{\kappa_s} \Big| (\nabla \wedge \nabla \wedge U) \wedge \nu - i\kappa_s \nabla \wedge U \Big|^2$$

$$
\begin{aligned}
& \qquad +\frac{(2\mu+\lambda)^2}{\kappa_p}\Big|\frac{\partial}{\partial\nu}(\nabla\cdot U) - i\kappa_p(\nabla\cdot U)\Big|^2\Big\}ds \\
= \; & \lim_{r\to\infty}\Big[\int_{\partial B_r}\Big\{\frac{\mu^2}{\kappa_s}|(\nabla\wedge\nabla\wedge U)\wedge\nu|^2 + \mu^2\kappa_s|\nabla\wedge U|^2 \\
& \qquad +\frac{(2\mu+\lambda)^2}{\kappa_p}\Big|\frac{\partial}{\partial\nu}(\nabla\cdot U)\Big|^2 + (2\mu+\lambda)^2\kappa_p|\nabla\cdot U|^2\Big\}ds \\
& +2\Re\Big(i\mu^2\int_{\partial B_r}\{(\nabla\wedge\nabla\wedge U)\wedge\nu\}\cdot\{\nabla\wedge\overline{U}\}ds \\
& \qquad +i(2\mu+\lambda)^2\int_{\partial B_r}\frac{\partial}{\partial\nu}(\nabla\cdot U)(\nabla\cdot\overline{U})ds\Big)\Big] .
\end{aligned}
\tag{5.19}
$$

We will now show that $\liminf_{r\to\infty}\Re(\ldots)\geq 0$. Then, we can conclude from (5.19) that

$$
\int_{\partial B_r}|\nabla\wedge U|^2ds + \int_{\partial B_r}|\nabla\cdot U|^2ds \to 0 \ , \ r\to\infty \ ,
$$

whence, by Rellich's lemma, $\nabla\wedge U = 0$ and $\nabla\cdot U = 0$ in the exterior of B_{R_1}. Finally, formula (5.13) shows $U = 0$ in the exterior of B_{R_1} and then $U = 0$ in the exterior of B_R by the analyticity of U.

In order to compute $\Re(\ldots)$ we insert the right hand side of (5.17) for $(\nabla\wedge\nabla\wedge U)\wedge\nu$ and the right hand side of (5.18) for $(\partial/\partial\nu)(\nabla\cdot U) = \hat{x}\cdot\nabla(\nabla\cdot U)$. With the help of inequality (5.14) we thus arrive at

$$
\begin{aligned}
& \Re\Big(\ldots\Big) \\
= \; & \omega^2\Re\Big(i\mu\int_{\partial B_r}(U\wedge\nu)\cdot(\nabla\wedge\overline{U})ds - i(2\mu+\lambda)\int_{\partial B_r}(\nu\cdot U)(\nabla\cdot\overline{U})ds\Big) \\
& +O\Big(\frac{1}{r}\Big) \\
= \; & \omega^2\Im\Big(\int_{\partial(B_r\setminus B_{R_1})}U\cdot\{-\mu\nu\wedge\nabla\wedge\overline{U} + (\lambda+2\mu)(\nabla\cdot\overline{U})\nu\}ds\Big) \\
& +\omega^2\Im\Big(\int_{\partial B_{R_1}}U\cdot\{-\mu\nu\wedge\nabla\wedge\overline{U} + (\lambda+2\mu)(\nabla\cdot\overline{U})\nu\}ds\Big) + O\Big(\frac{1}{r}\Big) \\
= \; & \omega^2\Im\Big(\mu\int_{B_r\setminus B_{R_1}}|\nabla\wedge U|^2dx - \mu\int_{B_r\setminus B_{R_1}}U\cdot(\nabla\wedge\nabla\wedge\overline{U})dx
\end{aligned}
$$

$$
\begin{aligned}
&\quad +(2\mu+\lambda) \int\limits_{B_r \setminus B_{R_1}} |\nabla \cdot U|^2 dx + (2\mu+\lambda) \int\limits_{B_r \setminus B_{R_1}} U \cdot \nabla(\nabla \cdot \overline{U}) dx \Big) \\
&\quad +\omega^2 \Im \Big(\int\limits_{\partial B_{R_1}} U \cdot \{-\mu\nu \wedge \nabla \wedge \overline{U} + (\lambda+2\mu)(\nabla \cdot \overline{U})\nu\} ds \Big) + O\Big(\frac{1}{r}\Big) \\
= &\; \omega^2 \Im \Big(\int\limits_{B_r \setminus B_{R_1}} U \cdot \Delta^* \overline{U} dx \Big) \\
&\quad +\omega^2 \Im \Big(\int\limits_{\partial B_{R_1}} U \cdot \{-\mu\nu \wedge \nabla \wedge \overline{U} + (\lambda+2\mu)(\nabla \cdot \overline{U})\nu\} ds \Big) + O\Big(\frac{1}{r}\Big) \\
= &\; \omega^2 \Im \Big(\int\limits_{\partial B_{R_1}} U \cdot \{-\mu\nu \wedge \nabla \wedge \overline{U} + (\lambda+2\mu)(\nabla \cdot \overline{U})\nu\} ds \Big) + O\Big(\frac{1}{r}\Big)
\end{aligned}
$$

which completes the proof of the lemma.

□

The next step of the uniqueness proof is a unique continuation result which allows to conclude $U = 0$ in B_R. Unique continuation principles for the elasticity equation are proved in [19, 48]. However, we stay with our way to prove a weak form of a unique continuation result by employing the solution operator G_ζ. To this end we examine the L^2-norms of the first derivatives of $G_\zeta \varphi$ in the next lemma.

Lemma 5.9 *Suppose $\kappa \geq 0$, $0 < R < R'$ and $\zeta \in \mathbb{C}^3$ satisfies $\zeta \cdot \zeta = \kappa^2$, $|\Im(\zeta)| \geq 1 + \kappa$. Then, there exists a constant $c > 0$ such that*

$$
\Big\| \nabla \int\limits_{B_R} g_\zeta(\cdot - y)\varphi(y) dy \Big\|_{L^2(B_R)} \leq c \|\varphi\|_{L^2(B_R)}
$$

for all $\varphi \in C_0(B_R)$. Here, g_ζ denotes the function defined in (2.9).

Proof: Let Q be the unitary transformation with $Q(\Re(\zeta)) = (|\Re(\zeta)|, 0, 0)$, $Q(\Im(\zeta)) = (0, |\Im(\zeta)|, 0)$ and define $\xi := (|\Re(\zeta)|, i|\Im(\zeta)|, 0)$. For a function $\varphi \in C_0^\infty(B_R)$ we have $\psi := \varphi \circ Q^T \in C_0^\infty(B_R)$, hence the Fourier coefficients $\hat{\psi}(\alpha)$, $\alpha \in \Gamma$, are rapidly decaying. Moreover, we know fom the definition of g_ζ that

$$
\int\limits_{B_R} \Big| \nabla \int\limits_{B_R} g_\zeta(x-y)\varphi(y) dy \Big|^2 dx \;=\; \int\limits_{B_R} \Big| \nabla \int\limits_{B_R} g_\zeta(Q^T x - y)\varphi(y) dy \Big|^2 dx
$$

$$= \int_{B_R} \Big|\nabla \int_{B_R} g_\xi(x-y)\psi(y)dy\Big|^2 dx \ .$$

Now, we conclude from Lemma 2.7 (a) that

$$\int_{B_R} g_\xi(x-y)\psi(y)dy = \sum_{\alpha\in\Gamma} \frac{\hat{\psi}(\alpha)}{\alpha\cdot\alpha + 2\xi\cdot\alpha} e_\alpha(x)$$

and

$$\nabla \int_{B_R} g_\xi(x-y)\psi(y)dy = \sum_{\alpha\in\Gamma} i\alpha \frac{\hat{\psi}(\alpha)}{\alpha\cdot\alpha + 2\xi\cdot\alpha} e_\alpha(x)$$

in B_R where both series are absolutely and uniformly convergent due to the rapid decay of the $\hat{\psi}(\alpha)$. Therefore, we arrive at

$$\begin{aligned}
\Big\|\nabla \int_{B_R} g_\zeta(\cdot - y)\varphi(y)dy\Big\|^2_{L^2(B_R)} &\leq \sum_{\alpha\in\Gamma} \frac{\alpha\cdot\alpha}{|\alpha\cdot\alpha + 2\xi\cdot\alpha|^2} |\hat{\psi}(\alpha)|^2 \\
&\leq \sup_{\alpha\in\Gamma} \frac{\alpha\cdot\alpha}{|\alpha\cdot\alpha + 2\xi\cdot\alpha|^2} \|\varphi\|^2_{L^2(B_R)} \ .
\end{aligned}$$

Estimating for $\alpha\in\Gamma$, $|\alpha| \leq 5|\Im(\zeta)|$,

$$\begin{aligned}
|\alpha\cdot\alpha + 2\xi\cdot\alpha| &\geq |\Im(\alpha\cdot\alpha + 2\xi\cdot\alpha)| \\
&= 2|\alpha_2||\Im(\zeta)| \\
&\geq \frac{\pi}{5R'}|\alpha| \ ,
\end{aligned}$$

and for $\alpha\in\Gamma$, $|\alpha| \geq 5|\Im(\zeta)|$,

$$\begin{aligned}
|\alpha\cdot\alpha + 2\xi\cdot\alpha| &\geq |\Re(\alpha\cdot\alpha + 2\xi\cdot\alpha)| \\
&\geq |\alpha|(|\alpha| - 4|\Im(\zeta)|) \\
&\geq |\alpha| \ ,
\end{aligned}$$

we see that

$$\sup_{\alpha\in\Gamma} \frac{\alpha\cdot\alpha}{|\alpha\cdot\alpha + 2\xi\cdot\alpha|^2} \leq c^2 \tag{5.20}$$

for a suitable constant c. This proves the assertion for densities $\varphi\in C_0^\infty(B_R)$. The lemma also holds for a general density $\varphi\in C_0(B_R)$ because φ can be approximated by $C_0^\infty(B_R)$-functions with respect to the $\|\cdot\|_\infty$-norm and because the volume potential with kernel $g_\zeta(x-y)$ is bounded from $C(\overline{B_R})$ to $C^{1,\gamma}(\overline{B_R})$.

□

We are now in a position to establish the existence of a unique solution to the elastic scattering problem.

Theorem 5.10 *For any incident wave $U^i \in C^2(\mathbb{R}^3)$, i.e., $\Delta^* U^i + \omega^2 U^i = 0$ in $\mathbb{R}^3$, the integral equation (5.12) and the direct elastic scattering problem both have the same unique solution.*

Proof: The equivalence of the scattering problem and the integral equation (5.12) stated in Lemma 5.7 implies that it suffices to show that the scattering problem has at most one solution in order to establish the existence of a solution.

Let U be a solution to the scattering problem with incident wave $U^i = 0$. We pick $R_1 > R$ and compute

$$\begin{aligned}
&\Im\Big(\int\limits_{\partial B_{R_1}} U \cdot \{-\mu\nu \wedge \nabla \wedge \overline{U} + (\lambda + 2\mu)(\nabla \cdot \overline{U})\nu\} ds\Big) \\
&= \Im\Big(\int\limits_{B_{R_1}} U \cdot \Delta^* \overline{U} dx + \int\limits_{B_{R_1}} \{\mu |\nabla \wedge U|^2 + (\lambda + 2\mu)|\nabla \cdot U|^2\} dx\Big) \\
&= 0
\end{aligned}$$

Hence, U vanishes in the exterior of B_R by Lemma 5.8.

By Lemma 5.7 we can represent U as

$$U(x) = -\omega^2 \int\limits_{B_R} (1-\rho(y))\Pi(x-y)U(y)dy \ , \ x \in \mathbb{R}^3 \ .$$

Then, the mapping properties of the volume potential imply that U is C^3-smooth, i.e., $U \in C_0^3(B_R)$. Next, we define $v := \nabla \cdot U \in C_0^2(B_R)$ and we obtain the following system of differential equations for U and v:

$$\begin{aligned}
\Delta U + \frac{\lambda + \mu}{\mu} \nabla v + \kappa_s^2 \rho U &= 0 \ , \\
\Delta v + \kappa_p^2 \rho v + \kappa_p^2 \nabla \rho \cdot U &= 0 \ .
\end{aligned}$$

The first equation is the elasticity equation $\mu\Delta U + (\lambda+\mu)\nabla(\nabla\cdot U) + \omega^2\rho U = 0$ and the second one arises when taking the divergence of the elasticity equation.

Hence, there is a constant c_1 such that

$$\begin{aligned} |\Delta U(x)| &\leq c_1\Big(|U(x)|^2 + |\nabla v(x)|^2\Big)^{1/2} , \\ |\Delta v(x)| &\leq c_1\Big(|U(x)|^2 + |v(x)|^2\Big)^{1/2} , \ x \in \mathbb{R}^3 . \end{aligned}$$

We choose $R' > R$ and $t \geq 1$ sufficiently large to ensure

$$\frac{c_1^2 R'^2}{\pi^2 t^2}\Big[2c^2c_1^2 + 4\frac{c_1^2 R'^2}{\pi^2} + 2\Big] < 1 . \tag{5.21}$$

Here, c denotes the constant from the previous lemma. Furthermore, we define $\xi := (t, it, 0) \in \mathbb{C}^3$ and

$$\begin{aligned} V(x) &= (V_1(x), V_2(x), V_3(x), V_4(x)) \\ &= e^{-i\xi\cdot x}(U_1(x), U_2(x), U_3(x), v(x)) , \ x \in C := (-R', R')^3 . \end{aligned}$$

If we are able to show $V = 0$, we can conclude $U(x) = 0$, $x \in C$, whence U vanishes identically in $\mathbb{R}^3$.

Now, we estimate

$$\begin{aligned} &\sum_{j=1}^{3} |(\Delta + 2i\xi\cdot\nabla)V_j(x)|^2 \\ &\quad = |e^{-i\xi\cdot x}\Delta U(x)|^2 \\ &\quad \leq c_1^2\Big(\sum_{j=1}^{3}|V_j(x)|^2 + |e^{-i\xi\cdot x}\nabla v(x)|^2\Big) \\ &\quad \leq c_1^2\Big(\sum_{j=1}^{3}|V_j(x)|^2 + 2|\nabla V_4(x)|^2 + 4t^2|V_4(x)|^2\Big) , \end{aligned} \tag{5.22}$$

where we have used $e^{-i\xi\cdot x}\nabla v(x) = \nabla V_4(x) + iV_4(x)\xi$ in the last line, and similarly

$$|(\Delta + 2i\xi\cdot\nabla)V_4(x)|^2 = |e^{-i\xi\cdot x}\Delta v(x)|^2 \leq c_1^2\Big(\sum_{j=1}^{4}|V_j(x)|^2\Big) . \tag{5.23}$$

Theorem 1.1 (b) applied to $V_4 \in C_0^2(C)$ reveals $V_4 = -G_\xi((\Delta + 2i\xi \cdot \nabla)V_4)$, whence by (5.23) and Theorem 2.8 (d) (or Theorem 1.1 (a))

$$\|V_4\|_{L^2}^2 \le \frac{c_1^2 R'^2}{\pi^2 t^2} \sum_{j=1}^4 \|V_j\|_{L^2}^2 \ . \tag{5.24}$$

Due to the preceding lemma we also know

$$\|\nabla V_4\|_{L^2}^2 \le c^2 c_1^2 \sum_{j=1}^4 \|V_j\|_{L^2}^2 \ . \tag{5.25}$$

The same reasoning applied to V_1, V_2, $V_3 \in C_0^2(C)$ leads to

$$\begin{aligned}
&\sum_{j=1}^3 \|V_j\|_{L^2}^2 \\
&\quad = \sum_{j=1}^3 \|G_\xi((\Delta + 2i\xi \cdot \nabla)V_j)\|_{L^2}^2 \\
&\quad \le \frac{c_1^2 R'^2}{\pi^2 t^2}\Big(\sum_{j=1}^3 \|V_j\|_{L^2}^2 + 2\|\nabla V_4\|_{L^2}^2 + 4t^2\|V_4\|_{L^2}^2\Big) \\
&\quad \le \frac{c_1^2 R'^2}{\pi^2 t^2}\Big(\sum_{j=1}^3 \|V_j\|_{L^2}^2 + 2c^2 c_1^2 \sum_{j=1}^4 \|V_j\|_{L^2}^2 + 4t^2 \frac{c_1^2 R'^2}{\pi^2 t^2} \sum_{j=1}^4 \|V_j\|_{L^2}^2\Big) \\
&\quad \le \frac{c_1^2 R'^2}{\pi^2 t^2}\Big(\Big[2c^2 c_1^2 + 4\frac{c_1^2 R'^2}{\pi^2} + 1\Big] \sum_{j=1}^4 \|V_j\|_{L^2}^2\Big) \ .
\end{aligned} \tag{5.26}$$

Here, we have used (5.22) in the third line and we have inserted (5.24), (5.25) in the fourth line. Adding (5.24) and (5.26) finally yields the inequality

$$\sum_{j=1}^4 \|V_j\|_{L^2}^2 \le \frac{c_1^2 R'^2}{\pi^2 t^2}\Big[2c^2 c_1^2 + 4\frac{c_1^2 R'^2}{\pi^2} + 2\Big] \sum_{j=1}^4 \|V_j\|_{L^2}^2 \ ,$$

whence $V = 0$ because of (5.21).

This means that the scattering problem with $U^i = 0$ only has the trivial solution and the proof of the theorem is complete.

□

We conclude this section with a discussion of the asymptotic behavior of U^s. Since the solution U^s of the elastic scattering problem has the form

$$\begin{aligned} U^s(x) &= -\omega^2 \int\limits_{B_R} (1-\rho(y))\Pi(x-y)U(y)dy \\ &= -\nabla \wedge \nabla \wedge \int\limits_{B_R} \Phi_{\kappa_s}(x,y)(1-\rho(y))U(y)dy \\ &\quad +\nabla\nabla \cdot \int\limits_{B_R} \Phi_{\kappa_p}(x,y)(1-\rho(y))U(y)dy \ , \end{aligned}$$

we obtain from the asymptotic behavior of Φ_κ that

$$\begin{aligned} U^s(x) &= -\frac{\kappa_s^2 e^{i\kappa_s|x|}}{4\pi|x|} \int\limits_{B_R} e^{-i\kappa_s \hat{x}\cdot y}(1-\rho(y))\hat{x} \wedge (U(y)\wedge \hat{x})dy \\ &\quad -\frac{\kappa_p^2 e^{i\kappa_p|x|}}{4\pi|x|} \int\limits_{B_R} e^{-i\kappa_p \hat{x}\cdot y}(1-\rho(y))\hat{x}\cdot U(y)dy\,\hat{x} \\ &\quad +O\Big(\frac{1}{|x|^2}\Big) \ , \ |x| \to \infty \ . \end{aligned} \tag{5.27}$$

Hence, we know

$$U^s(x) = \frac{e^{i\kappa_s|x|}}{|x|}a(\hat{x}) + \frac{e^{i\kappa_p|x|}}{|x|}u(\hat{x})\,\hat{x} + O\Big(\frac{1}{|x|^2}\Big) \ , \ |x| \to \infty \ ,$$

with a smooth function u and a smooth tangential vector field a on S^2. We call $U^s_\infty(\hat{x}) := a(\hat{x}) + u(\hat{x})\,\hat{x}$, $\hat{x} \in S^2$, the far field of U^s.

The formulas

$$\begin{aligned} \nabla \wedge U^s(x) &= -\kappa_s^2 \nabla \wedge \int\limits_{B_R} \Phi_{\kappa_s}(x,y)(1-\rho(y))U(y)dy \ , \\ \nabla \cdot U^s(x) &= -\kappa_p^2 \nabla \cdot \int\limits_{B_R} \Phi_{\kappa_p}(x,y)(1-\rho(y))U(y)dy \ , \end{aligned}$$

show that $\nabla \wedge U^s$ and $\nabla \cdot U^s$ are both radiating solutions to a Helmholtz equation in the exterior of B_R. Furthermore, using the asymptotic behavior of Φ_κ again, we can compute their far field patterns and compare them with (5.27). This yields $[\nabla \wedge U^s]_\infty(\hat{x}) = i\kappa_s \hat{x} \wedge a(\hat{x})$ and $(\nabla \cdot U)_\infty(\hat{x}) = i\kappa_p u(\hat{x})$,

$\hat{x} \in S^2$. We can now infer from the one-to-one correspondence between far field patterns and radiating solutions to the Helmholtz equation, i.e., from Rellich's lemma, that any solution U^s to the elastic scattering problem, which has a vanishing far field $U^s_\infty = 0$, must vanish identically in the exterior of B_R. This follows immediately from $(\nabla \cdot U^s)_\infty(\hat{x}) = i\kappa_p \hat{x} \cdot U^s_\infty(\hat{x}) = 0$, $[\nabla \wedge U^s]_\infty(\hat{x}) = i\kappa_s \hat{x} \wedge U^s_\infty(\hat{x}) = 0$, whence $\nabla \wedge U^s = 0$ and $\nabla \cdot U^s = 0$ in $\mathbb{R}^3 \setminus B_R$. Relation (5.13)

$$U^s = \frac{\mu}{\omega^2} \nabla \wedge \nabla \wedge U^s - \frac{\lambda + 2\mu}{\omega^2} \nabla(\nabla \cdot U^s)$$

now implies $U^s = 0$. Let us summarize this one-to-one correspondence between radiating solutions which are the scattered part of a solution to the elastic scattering problem and its far field patterns in the following theorem.

Theorem 5.11 *Let U^s be the scattered part of a solution $U = U^i + U^s$ to the elastic scattering problem. Then, the far field U^s_∞ of U^s uniquely determines U^s in the exterior of B_R.*

In elastic scattering a plane incident wave is defined by

$$U^i(x, d, p) = -\frac{1}{\omega^2} \nabla_x (\nabla_x \cdot [p e^{i\kappa_p d \cdot x}]) + \frac{1}{\omega^2} \nabla_x \wedge \nabla_x \wedge [p e^{i\kappa_s d \cdot x}] \ , \ x \in \mathbb{R}^3 \ ,$$

where $d \in S^2$ is its direction of propagation and $p \in \mathbb{C}^3$ controls its polarization. Straightforward calculations show that U^i is a solution to $\Delta^* U^i + \omega^2 U^i = 0$. Note, that for $d \cdot p = 0$ the first term vanishes and we have a pure shear wave, whereas for $d \wedge p = 0$ the second term vanishes and we have a pure pressure wave.

Denoting by $U^s(\cdot, d, p)$, $U(\cdot, d, p)$ the scattered wave and the total wave belonging to the elastic scattering problem with incident wave $U^i(\cdot, d, p)$, we define the far field pattern belonging to the density ρ to be the matrix valued function $U_\infty \colon S^2 \times S^2 \to \mathbb{C}^{3\times 3}$, having as its jth column

$$U_\infty(\hat{x}, d) d_j = [U^s(\cdot, d, d_j)]_\infty(\hat{x}) \ , \ j = 1, 2, 3 \ , \ \hat{x}, d \in S^2 \ .$$

Our considerations from above imply that

$$\begin{aligned} U_\infty(\hat{x}, d) d_j \quad &= \quad -\frac{\kappa_s^2}{4\pi} \int_{B_R} e^{-i\kappa_s \hat{x} \cdot y} (1 - \rho(y)) \hat{x} \wedge (U(y, d, d_j) \wedge \hat{x}) dy \\ &\quad -\frac{\kappa_p^2}{4\pi} \int_{B_R} e^{-i\kappa_p \hat{x} \cdot y} (1 - \rho(y)) \hat{x} \cdot U(y, d, d_j) dy \, \hat{x} \ . \end{aligned} \tag{5.28}$$

Chapter 6

The Inverse Elastic Scattering Problem

The last chapter of this thesis deals with the inverse elastic scattering problem. We assume the far field pattern corresponding to the density ρ to be known and we want to obtain information about ρ from this data. Deviating from our results of the acoustic and the electromagnetic case we shall only give a uniqueness proof in the first section and a stability result in the second section.

In [10] uniqueness is proved, if the far field pattern is known for an interval of frequencies. We will improve this result by only using the far field pattern at one fixed frequency ω as data. The main idea is the same as for the previous scattering problems. The coincidence of two far field patterns originating from two densities allows to prove an analogous orthogonality relation to (2.1) or to (4.1). Then, we construct solutions to the elasticity equation which depend in an appropriate way on parameters η and ζ, insert these solutions into the orthogonality relation, and can conclude that the Fourier coefficients of the densities coincide, hence that the densities must coincide. The reader who is interested in a global uniqueness theorem for the (nonconstant) Lamé coefficients should consult the paper [36] by Nakamura and Uhlmann.

As in the acoustic case the special solutions to the elasticity equation can be used to obtain bounds for the Fourier coefficients of the difference of two densities $|(\tilde{\rho} - \rho)\hat{\ }(\alpha)|$, which in turn allow to estimate $\|\tilde{\rho} - \rho\|_\infty$ by the difference of certain boundary integral operators or by the difference of the far field patterns belonging to the densities ρ and $\tilde{\rho}$. To this end, in the acoustic case, we had to solve certain boundary value problems. We used boundary integral operators whose properties are investigated in [6]. The analogous integral operators for the elastic case are examined in [23, 24]. However, since our aim is to keep this thesis as self-contained as possible, we have included an analysis of the elastic single-layer potential in an appendix to the second section where we only take results from [6] for granted. Nevertheless, in this chapter we sometimes briefly refer to the analogous acoustic or electromagnetic results instead of repeating a technical proof.

A reconstruction proof, which follows the lines of the reconstruction in the acoustic case, would require an analysis of the elastic double-layer potential, too, and we consider this to be beyond the scope of this thesis. We have thus omitted the reconstruction procedure of the density from the far field pattern although we believe that a treatment as in the acoustic case is possible.

6.1 Uniqueness for the Inverse Elastic Scattering Problem

We assume two densities $\rho,\, \tilde{\rho} \in \tilde{C}(B_R)$ are given, where we denote again by

$$\tilde{C}(B_R) := \{\rho \in C^{1,\gamma}(\mathbb{R}^3) : \mathrm{supp}\,(1-\rho) \subset B_R\ ,\ \rho \text{ real valued}\}$$

the set of densities we are interested in.

Moreover, we suppose that the frequency $\omega > 0$ and the Lamé constants $\mu > 0$ and λ with $2\mu + \lambda > 0$ are fixed and known. It is our aim to prove that the coincidence of the far field patterns U_∞ and $\tilde{U}_\infty$ belonging to ρ and $\tilde{\rho}$, respectively, imply the equality $\rho = \tilde{\rho}$.

Following the reasoning of the acoustic and electromagnetic case we start with the relation

$$\int_{B_R} (\rho - \tilde{\rho}) U \cdot \tilde{U} dx = 0 \tag{6.1}$$

for all solutions U, $\tilde{U}$ to the elasticity equations

$$\Delta^* U + \omega^2 \rho U = 0 \quad , \quad \Delta^* \tilde{U} + \omega^2 \tilde{\rho} \tilde{U} = 0 \quad \text{in } B_{R_1},$$

respectively, where $R_1 > R$. This relation will be established first in the case $U = U(\cdot, d, p)$ and then via an approximation argument for a general U.

Lemma 6.1 *Assume $0 < R < R_1$ and $\rho,\, \tilde{\rho} \in \tilde{C}(B_R)$. Furthermore, assume $\tilde{U}$ is a solution to $\Delta^* \tilde{U} + \omega^2 \tilde{\rho} \tilde{U} = 0$ in B_{R_1}. If the far field patterns U_∞ and $\tilde{U}_\infty$ coincide on $S^2 \times S^2$, i.e.,*

$$\Big[U^s(\cdot, d, p)\Big]_\infty(\hat{x}) = \Big[\tilde{U}^s(\cdot, d, p)\Big]_\infty(\hat{x}) \quad \textit{for all } \hat{x},\ d \in S^2,\ p \in \mathbb{C}^3,$$

then the relation

$$\int_{B_R} (\rho(x) - \tilde{\rho}(x)) U(x, d, p) \cdot \tilde{U}(x) dx = 0$$

holds true for all $d \in S^2$, $p \in \mathbb{C}^3$.

Proof: For fixed $d \in S^2$, $p \in \mathbb{C}^3$ we have $U(x, d, p) = \tilde{U}(x, d, p)$, $|x| \geq R$, by the coincidence of the far fields and the one-to-one relation between far fields and scattering solutions to the elasticity equation, which we derived at the end of the last chapter. Then the second Betti formula (5.3) implies

$$\begin{aligned}
0 &= \int_{\partial B_R} \Big\{[T(U(\cdot,d,p)-\tilde U(\cdot,d,p))]\cdot \tilde U - (T\tilde U)\cdot (U(\cdot,d,p)-\tilde U(\cdot,d,p))\Big\} ds \\
&= \omega^2 \int_{B_R} \Bigg\{\Big[(1-\rho(x))U(x,d,p) - (1-\tilde\rho(x))\tilde U(x,d,p)\Big]\cdot \tilde U(x) \\
&\qquad\qquad -(1-\tilde\rho(x))\tilde U(x)\cdot (U(x,d,p)-\tilde U(x,d,p))\Bigg\} dx \\
&= \omega^2 \int_{B_R} (\tilde\rho(x)-\rho(x))U(x,d,p)\cdot \tilde U(x) dx \ ,
\end{aligned}$$

and we have proved the lemma.

□

Next we turn to the approximation of a solution to the elasticity equation in B_{R_1}, $R_1 > R$, by elements from $\operatorname{span}\{U(\cdot,d,p) : d \in S^2\ ,\ p \in \mathbb{C}^3\}$ with respect to the $L^2(B_R)$-norm. Once more we first use the idea from [20, Lemma 5.20] for the special case $\rho = 1$.

Lemma 6.2 *Assume* $0 < R < R_2$ *and let* $U^i \in C^2(B_{R_2})$ *satisfy* $\Delta^* U^i + \omega^2 U^i = 0$ *in* B_{R_2}. *Then, there exists a sequence*

$$U^i_j \in \operatorname{span}\{U^i(\cdot,d,p) : d \in S^2\ ,\ p \in \mathbb{C}^3\}\ ,\ j \in \mathbb{N}\ ,$$

such that $\|U^i - U^i_j\|^2_{L^2(B_R)} \to 0$, $j \to \infty$.

Proof: With

$$X \ := \ \{U|_{B_R} : U \in C^2(B_{R_2}) \text{ and } \Delta^* U^i + \omega^2 U^i = 0 \text{ in } B_{R_2}\} \subset L^2(B_R)$$

and $\overline{X}$ being the completion of X in $L^2(B_R)$ we assume that $U_0 \in \overline{X}$ satisfies

$$\int_{B_R} \overline{U_0(x)} \cdot U^i(x,d,p) dx = 0$$

for all $d \in S^2$, $p \in \mathbb{C}^3$. Then, we must show that U_0 vanishes in $L^2(B_R)$.

For $|x| > R$ we define

$$W(x) := \int_{B_R} \Pi(x-y)\overline{U_0(y)}dy \ .$$

$W \in C^2(\mathbb{R}^3 \setminus \overline{B_R})$ is a radiating solution to the elasticity equation in $\mathbb{R}^3 \setminus \overline{B_R}$. Using the asymptotic behavior of the derivatives of $\Phi_\kappa(x,y)$ for large $|x|$ (see [7, formulas (6.25),(6.26)]) we compute for any vector $p \in \mathbb{C}^3$ and any $d \in S^2$:

$$\begin{aligned} 4\pi p \cdot W_\infty(-d) &= -\kappa_s^2 p \cdot \int_{B_R} e^{i\kappa_s d\cdot y} d \wedge (\overline{U_0(y)} \wedge d) dy \\ &\quad -\kappa_p^2 p \cdot d \int_{B_R} e^{i\kappa_p d\cdot y} d \cdot \overline{U_0(y)} dy \\ &= -\omega^2 \int_{B_R} \overline{U_0(y)} \cdot U^i(y,d,p) dy \\ &= 0 \ . \end{aligned}$$

Hence, the far field W_∞ of W vanishes and $W(x) = 0$ for all $|x| > R$.

Now, let $U_l \in X$, $l \in \mathbb{N}$, be a sequence approximating U_0,

$$\|U_l - U_0\|^2_{L^2(B_R)} \to 0 \ , \ l \to \infty \ .$$

By the representation formula (5.8) we can write U_l, $l \in \mathbb{N}$, as

$$U_l(x) = \int_{\partial B_{R_3}} \Big\{ \Pi(x-y)(TU_l)(y) - \Xi(x,y)U_l(y) \Big\} ds(y) \ , \ x \in B_R \ ,$$

where R_3 satisfies $R < R_3 < R_2$.

Inserting this representation for U_l and interchanging the order of integration we conclude

$$\begin{aligned} &\int_{B_R} U_l(x) \cdot \overline{U_0(x)} dx \\ &= \int_{\partial B_{R_3}} (TU_l)(y) \cdot \int_{B_R} \Pi(x-y)^T \overline{U_0(x)} dx \, ds(y) \\ &\quad - \int_{\partial B_{R_3}} U_l(y) \cdot \int_{B_R} \Xi(x,y)^T \overline{U_0(x)} dx \, ds(y) \\ &= \int_{\partial B_{R_3}} \{ (TU_l)(y) \cdot W(y) - (TW)(y) \cdot U_l(y) \} ds(y) \\ &= 0 \ , \ l \in \mathbb{N} \ , \end{aligned}$$

since W vanishes on ∂B_{R_3}. As $l \to \infty$ we arrive at $U_0 = 0$ in $L^2(B_R)$ and we have proved the assertion.

$\square$

The approximation result for a general ρ can now be derived with the help of the Lippmann-Schwinger equation (5.12). For convenience we define the operator $\mathcal{V}_\rho : C(\overline{B_R}) \to C(\overline{B_R})$ by

$$(\mathcal{V}_\rho U)(x) := \omega^2 \int_{B_R} (1 - \rho(y))\Pi(x-y)U(y)dy \ , \ x \in \overline{B_R} \ .$$

Lemma 6.3 *Assume* $0 < R < R_1$ *and let* $U \in C^2(B_{R_1})$ *satisfy* $\Delta^* U + \omega^2 \rho U = 0$ *in* B_{R_1}. *Then, there exists a sequence*

$$U_j \in \operatorname{span}\{U(\cdot, d, p) : d \in S^2 \ , \ p \in \mathbb{C}^3\} \ , \ j \in \mathbb{N} \ ,$$

such that $\|U - U_j\|_{L^2(B_R)} \to 0$, $j \to \infty$.

Proof: We fix $R < R_2 < R_1$ and define

$$U^i(x) := \int_{\partial B_{R_2}} \Big\{ \Pi(x-y)(TU)(y) - \Xi(x,y)U(y) \Big\} ds(y) \ , \ x \in B_{R_2} \ .$$

The representation formula (5.8) together with the elasticity equation imply the integral equation

$$U(x) = U^i(x) - \omega^2 \int_{B_R} (1 - \rho(y))\Pi(x-y)U(y)dy \ , \ x \in B_{R_2} \ , \qquad (6.2)$$

for the field U, i.e., $U = (I + \mathcal{V}_\rho)^{-1} U^i$. U^i is a solution to $\Delta^* U^i + \omega^2 U^i = 0$ in B_{R_2}. This can be seen by applying $\Delta^* + \omega^2 I$ to both sides of (6.2).
(At first sight it seems reasonable to obtain $\Delta^* U^i + \omega^2 U^i = 0$ directly from the definition of U^i. But this requires an examination of the elastic double-layer potential, and it is not obvious (as in the acoustic case) why the double-layer potential is a solution to $\Delta^* U + \omega^2 U = 0$.)

Now, according to Lemma 6.2, there exists a sequence U^i_j, $j \in \mathbb{N}$, from $\operatorname{span}\{U^i(\cdot, d, p) : d \in S^2, p \in \mathbb{C}^3\}$ approximating U^i in $L^2(B_R)$, and we set U_j to be the solution to the Lippmann-Schwinger equation (6.2) with incident field U^i_j. This implies $U_j \in \operatorname{span}\{U(\cdot, d, p) : d \in S^2, p \in \mathbb{C}^3\}$ and

$$U_j - U = (I + \mathcal{V}_\rho)^{-1}(U^i_j - U^i) \qquad \text{in } \overline{B_R}.$$

Since $\mathcal{V}_\rho \colon (C(\overline{B_R}), \|\cdot\|_{L^2(B_R)}) \to (C(\overline{B_R}), \|\cdot\|_{L^2(B_R)})$ is a compact operator by Lemma 5.1 (c) and Theorem 1.9 (c), we finally employ the Riesz theory to conclude

$$\|U_j - U\|_{L^2(B_R)} \le \|(I+\mathcal{V}_\rho)^{-1}\|_{L^2(B_R)} \|U_j^i - U^i\|_{L^2(B_R)} \to 0 \ , \ j \to \infty \ .$$

This ends the proof of the lemma.

□

We are now in a position to prove relation (6.1) by approximating an arbitrary solution to $\Delta^* U + \omega^2 \rho U = 0$ in B_{R_1} by elements from

$$\text{span}\,\{U(\cdot, d, p) \colon d \in S^2 \ , \ p \in \mathbb{C}^3\}$$

with respect to the $L^2(B_R)$-norm and by using Lemma 6.1. This is stated in the next lemma.

Lemma 6.4 *Assume $0 < R < R_1$ and that the far field patterns for the densities ρ, $\tilde{\rho} \in \tilde{C}(B_R)$ coincide on $S^2 \times S^2$, i.e., $U_\infty = \tilde{U}_\infty$. If $U \in C^2(B_{R_1})$ is a solution to $\Delta^* U + \omega^2 \rho U = 0$ and $\tilde{U} \in C^2(B_{R_1})$ is a solution to $\Delta^* \tilde{U} + \omega^2 \tilde{\rho} \tilde{U} = 0$ in B_{R_1}, then we have the relation*

$$\int_{B_R} (\rho(x) - \tilde{\rho}(x)) U(x) \cdot \tilde{U}(x) dx = 0 \ .$$

Next, we want to exploit the above relation for the proof that all Fourier coefficients of $\tilde{\rho}$ and ρ must coincide. To this end we have to construct special solutions to the elasticity equation. For a given $\alpha \in \Gamma$ the solutions $U(\cdot, \zeta_s, \eta)$ and $\tilde{U}(\cdot, \tilde{\zeta}_s, \tilde{\eta})$ should depend in such a way on the parameters ζ_s, η, $\tilde{\zeta}_s$, $\tilde{\eta} \in \mathbb{C}^3$ that

$$U(x, \zeta_s, \eta)) \cdot \tilde{U}(x, \tilde{\zeta}_s, \tilde{\eta}) \to e^{-i\alpha \cdot x}$$

with respect to $L^1(B_R)$ for an appropriately chosen sequence of the parameters.

Of course we will imitate the procedure used in the acoustic and the electromagnetic case, i.e., we use an incident field $U^i(x) = \eta e^{i\zeta_s \cdot x}$, where $\zeta_s \in \mathbb{C}^3$ satisfies $\zeta_s \cdot \zeta_s = \kappa_s^2$, and where $\eta \cdot \zeta_s = 0$. These conditions on ζ_s

and η imply that U^i is a solution to $\Delta^* U^i + \omega^2 U^i = 0$. Moreover, in the Lippmann-Schwinger equation (5.12) we replace the fundamental solutions Φ_{κ_s}, Φ_{κ_p}, which occur in the definition of Π, by Ψ_{ζ_s} and Ψ_{ζ_p}.

The reader can find the definition of $\Psi_\zeta(x) = (e^{i\kappa|x|}/4\pi|x|) + \tilde{g}_\zeta(x)$ on page 97. $\tilde{g}_\zeta$ is a solution to the Helmholtz equation in $B_{2R'}$. The properties of the volume potential operator G_ζ having kernel $g_\zeta(x-y) = e^{-i\zeta\cdot(x-y)}\Psi_\zeta(x-y)$ were investigated in Theorem 2.8.

We still have some freedom in the choice of the parameter $\zeta_p \in \mathbb{C}^3$. It turns out that ζ_p with $\zeta_p \cdot \zeta_p = \kappa_p^2$, $\Im(\zeta_p) = \Im(\zeta_s)$ and $\Re(\zeta_p)$ being a positive multiple of $\Re(\zeta_s)$ is appropriate for our purpose.

Let us first introduce some notation. If $\zeta_s \in \mathbb{C}^3$ is a vector satisfying $\zeta_s \cdot \zeta_s = \kappa_s^2$ and $|\Im(\zeta_s)| > 0$, we denote by $Q \in \mathbb{R}^{3\times 3}$ the unitary transformation with $\det(Q) = 1$, which maps ζ_s to $\xi_s := (|\Re(\zeta_s)|, i|\Im(\zeta_s)|, 0)$, i.e., $Q\zeta_s = \xi_s$. Note, that $\xi_s \cdot \xi_s = \kappa_s^2$ implies $|\Re(\zeta_s)| = \sqrt{|\Im(\zeta_s)|^2 + \kappa_s^2}$. Next, we define $\xi_p := (\sqrt{|\Im(\zeta_s)|^2 + \kappa_p^2}, i|\Im(\zeta_s)|, 0)$ and $\zeta_p := Q^T \xi_p$. Hence, we have $Q\zeta_p = \xi_p$.

Finally, we define the modified fundamental solution Ω_{ζ_s}, which replaces the fundamental solution Π, by

$$\Omega_{\zeta_s}(x) d_j := \frac{1}{\mu}\Psi_{\zeta_s}(x) d_j + \frac{1}{\omega^2}\nabla\nabla\cdot\Big\{(\Psi_{\zeta_s} - \Psi_{\zeta_p}) d_j\Big\}(x)\ ,\ |x| > 0\ ,\ j = 1,2,3,$$

where d_1, d_2, d_3 denote the cartesian unit vectors, i.e., $\Omega_{\zeta_s} : \mathbb{R}^3 \setminus \{0\} \to \mathbb{C}^{3\times 3}$ is a matrix valued function. Inserting $\Psi_\zeta = (e^{i\kappa|\cdot|}/4\pi|\cdot|) + \tilde{g}_\zeta$ into this definition, we see that $\Omega_{\zeta_s} = \Pi + \tilde{\Omega}_{\zeta_s}$ where the columns of the matrix $\tilde{\Omega}_{\zeta_s}$ are analytic solutions to $\Delta^* U + \omega^2 U = 0$ in $B_{2R'}$.

We start with the assertion that a solution to a modified Lippmann-Schwinger equation is also a solution to the elasticity equation.

Lemma 6.5 *Suppose $0 < R < R'' < R'$, and η, $\zeta_s \in \mathbb{C}^3$ satisfy $\zeta_s \cdot \zeta_s = \kappa_s^2$, $|\Im(\zeta_s)| > 0$, and $\eta \cdot \zeta_s = 0$. Let $\zeta_p \in \mathbb{C}^3$ be defined as above. Furthermore, define $U^i(x) := \eta e^{i\zeta_s\cdot x}$, $x \in \mathbb{R}^3$, and assume $U \in C(\overline{B_{R''}})$ is a solution to*

$$U(x) = U^i(x) - \omega^2 \int_{B_R} (1-\rho(y))\Omega_{\zeta_s}(x-y) U(y) dy\ ,\ x \in \overline{B_{R''}}\ . \tag{6.3}$$

Then, $U \in C^2(B_{R''})$ is a solution to $\Delta^* U + \omega^2 \rho U = 0$ in $B_{R''}$.

The proof follows immediately by applying $\Delta^* + \omega^2 I$ to equation (6.3) and observing that $\Delta^* U^i + \omega^2 U^i = 0$,

$$(\Delta^* + \omega^2)\int_{B_R} (1-\rho(y))\Pi(\cdot - y) U(y) dy = -(1-\rho) U\ ,$$

$$(\Delta^* + \omega^2) \int\limits_{B_R} (1-\rho(y))\tilde{\Omega}_{\zeta_s}(\cdot - y)U(y)dy = 0 \ .$$

Equation (6.3) is a Fredholm integral equation of the second kind. Thus, we have to inspect its nullspace. Furthermore, we need the asymptotic behavior of its solutions as $|\Im(\zeta_s)| \to \infty$. The next remarks and the next lemma prepare this inspection which will be carried out in Theorem 6.7.

We rewrite the jth column of Ω_{ζ_s} as

$$\Omega_{\zeta_s}(x)d_j = \frac{1}{\omega^2}\nabla\wedge\nabla\wedge\Big\{\Psi_{\zeta_s}(x)d_j\Big\} - \frac{1}{\omega^2}\nabla\nabla\cdot\Big\{\Psi_{\zeta_p}(x)d_j\Big\} \ , \ |x|>0 \ , \ j=1,2,3 \ ,$$

take the divergence of (6.3), and use integration by parts to arrive at

$$(\nabla\cdot U)(x) = -\kappa_p^2 \int\limits_{B_R} \Psi_{\zeta_p}(x-y)\{(1-\rho)\nabla\cdot U - \nabla\rho\cdot U\}(y)dy \ , \ x\in\overline{B_{R''}} \ .$$

Moreover, using the definition of Ω_{ζ_s} and integration by parts we find from (6.3) for $x\in\overline{B_{R''}}$

$$\begin{aligned} U(x) &= U^i(x) - \frac{\omega^2}{\mu}\int\limits_{B_R}(1-\rho(y))\Psi_{\zeta_s}(x-y)U(y)dy \\ &\quad -\nabla\int\limits_{B_R}(\Psi_{\zeta_s}-\Psi_{\zeta_p})(x-y)\{(1-\rho)\nabla\cdot U - \nabla\rho\cdot U\}(y)dy \ . \end{aligned}$$

Now, we define $W(x) := e^{-i\zeta_s\cdot x}U(x)$, $w(x) := e^{-i\zeta_s\cdot x}(\nabla\cdot U)(x)$, $x\in\overline{B_{R''}}$. Multiplying both sides of the last two equations by $e^{-i\zeta_s\cdot x}$ we finally obtain

$$w(x) = -\kappa_p^2\int\limits_{B_R} g_{\zeta_p}(x-y)e^{i(\zeta_p-\zeta_s)\cdot(x-y)}\{(1-\rho)w - \nabla\rho\cdot W\}(y)dy \ , \ x\in\overline{B_{R''}} \ , \tag{6.4}$$

and

$$\begin{aligned} W(x) &= \eta - \frac{\omega^2}{\mu}\int\limits_{B_R}(1-\rho(y))g_{\zeta_s}(x-y)W(y)dy \\ &\quad -(\nabla + i\zeta_s)\int\limits_{B_R}(g_{\zeta_s}(x-y) - e^{i(\zeta_p-\zeta_s)\cdot(x-y)}g_{\zeta_p}(x-y)) \\ &\qquad\qquad \{(1-\rho)w - \nabla\rho\cdot W\}(y)dy \ , \ x\in\overline{B_{R''}} \ . \end{aligned} \tag{6.5}$$

If we can show that $W = 0$ for $\eta = 0$, we have proved injectivity of the integral equation (6.3). Moreover, if we know the behavior of W for $|\Im(\zeta_s)| \to \infty$, we also know the behavior of U. To this end we study more closely the operator $A_{\zeta_s}: C(\overline{B_{R''}}) \to C(\overline{B_{R''}})$,

$$(A_{\zeta_s}\varphi)(x) := (\nabla + i\zeta_s) \int\limits_{B_{R''}} (g_{\zeta_s}(x-y) - e^{i(\zeta_p-\zeta_s)\cdot(x-y)} g_{\zeta_p}(x-y))\varphi(y)dy$$

which occurs as the second integral in (6.5).

Lemma 6.6 *There is a constant c depending on R'', R', κ_s, and κ_p such that the inequality*

$$\|A_{\zeta_s}\varphi\|_{L^2(B_{R''})} \leq \frac{c}{|\Im(\zeta_s)|} \|\varphi\|_{L^2(B_{R''})}$$

holds true for all $\varphi \in C_0(B_{R''})$ and for all $|\Im(\zeta_s)| \geq 1 + \kappa_p + \kappa_s$.

Proof: We denote by Q the unitary transformation with $\det(Q) = 1$ mapping ζ_s to $\xi_s := (|\Re(\zeta_s)|, i|\Im(\zeta_s)|, 0)$. ζ_p and ξ_p are defined as on page 218. A simple computation shows that

$$\begin{aligned} |\xi_p - \xi_s| &= \left|\sqrt{|\Im(\zeta_s)|^2 + \kappa_p^2} - \sqrt{|\Im(\zeta_s)|^2 + \kappa_s^2}\right| \\ &= \frac{|\kappa_p^2 - \kappa_s^2|}{\sqrt{|\Im(\zeta_s)|^2 + \kappa_p^2} + \sqrt{|\Im(\zeta_s)|^2 + \kappa_s^2}} \\ &\leq \frac{c_1}{|\Im(\zeta_s)|} \ . \end{aligned} \tag{6.6}$$

We split the integrand in the definition of A_{ζ_s} into

$$\begin{aligned} &(g_{\zeta_s}(x-y) - e^{i(\zeta_p-\zeta_s)\cdot(x-y)} g_{\zeta_p}(x-y))\varphi(y) \\ &\quad = (g_{\zeta_s}(x-y) - g_{\zeta_p}(x-y))\varphi(y) \\ &\qquad + (1 - e^{i(\zeta_p-\zeta_s)\cdot(x-y)}) g_{\zeta_p}(x-y)\varphi(y) \end{aligned}$$

and proceed as in the proof of Lemma 5.9, i.e., we show the assertion for $\varphi \in C_0^\infty(B_{R''})$ and define $\psi := \varphi \circ Q^T \in C_0^\infty(B_{R''})$. The definition of g_ζ

yields

$$\begin{aligned}
&\int\limits_{B_{R''}} \Big|\int\limits_{B_{R''}} (g_{\zeta_s}(x-y) - g_{\zeta_p}(x-y))\varphi(y)dy\Big|^2 dx \\
&= \int\limits_{B_{R''}} \Big|\int\limits_{B_{R''}} (g_{\zeta_s} - g_{\zeta_p})(Q^T x - y)\varphi(y)dy\Big|^2 dx \\
&= \int\limits_{B_{R''}} \Big|\int\limits_{B_{R''}} (g_{\xi_s} - g_{\xi_p})(x - y)\psi(y)dy\Big|^2 dx \\
&\leq \sum_{\alpha\in\Gamma} \Big|\frac{1}{\alpha\cdot\alpha + 2\xi_s\cdot\alpha} - \frac{1}{\alpha\cdot\alpha + 2\xi_p\cdot\alpha}\Big|^2 |\hat{\psi}(\alpha)|^2 \\
&= \sum_{\alpha\in\Gamma} \Big|\frac{2(\xi_p - \xi_s)\cdot\alpha}{(\alpha\cdot\alpha + 2\xi_s\cdot\alpha)(\alpha\cdot\alpha + 2\xi_p\cdot\alpha)}\Big|^2 |\hat{\psi}(\alpha)|^2 \\
&\leq \sum_{\alpha\in\Gamma} \frac{4|\xi_p - \xi_s|^2}{|\alpha\cdot\alpha + 2\xi_s\cdot\alpha|^2} \frac{|\alpha|^2}{|\alpha\cdot\alpha + 2\xi_p\cdot\alpha|^2} |\hat{\psi}(\alpha)|^2 \\
&\leq \frac{c_2}{|\Im(\zeta_s)|^4} \|\varphi\|^2_{L^2(B_{R''})} .
\end{aligned}$$

Here we have used the estimate (6.6), and (1.5), (5.20) from the proofs of Theorem 1.1 and Lemma 5.9.

Moreover, a reasoning as above and as in the proof of Lemma 5.9 implies

$$\Big\|\nabla \int\limits_{B_{R''}} (g_{\zeta_s} - g_{\zeta_p})(\cdot - y)\varphi(y)dy\Big\|^2_{L^2(B_{R''})} \leq \frac{c_3}{|\Im(\zeta_s)|^2} \|\varphi\|^2_{L^2(B_{R''})} .$$

Together with the inequality $|\zeta_s| \leq 2|\Im(\zeta_s)|$ for $\zeta_s \cdot \zeta_s = \kappa_s^2$ with $|\Im(\zeta_s)| \geq \kappa_s$ we arrive at

$$\Big\|(\nabla + i\zeta_s) \int\limits_{B_{R''}} (g_{\zeta_s} - g_{\zeta_p})(\cdot - y)\varphi(y)dy\Big\|^2_{L^2(B_{R''})} \leq \frac{c_4}{|\Im(\zeta_s)|^2} \|\varphi\|^2_{L^2(B_{R''})} .$$

Hence, it remains to estimate

$$(\nabla + i\zeta_s) \int\limits_{B_{R''}} (1 - e^{i(\zeta_p - \zeta_s)\cdot(x-y)}) g_{\zeta_p}(x - y)\varphi(y)dy .$$

Splitting

$$\begin{aligned}
&(1 - e^{i(\zeta_p - \zeta_s)\cdot(x-y)}) g_{\zeta_p}(x - y)\varphi(y) \\
&\quad = g_{\zeta_p}(x - y)(1 - e^{-i(\zeta_p - \zeta_s)\cdot y})\varphi(y) \\
&\qquad + (1 - e^{i(\zeta_p - \zeta_s)\cdot x}) g_{\zeta_p}(x - y) e^{-i(\zeta_p - \zeta_s)\cdot y}\varphi(y)
\end{aligned}$$

we note that

$$\sup_{y\in\overline{B_{R''}}} |1 - e^{-i(\zeta_p-\zeta_s)\cdot y}| \leq \frac{c_5}{|\Im(\zeta_s)|}$$

and obtain with the help of Theorem 2.8 and Lemma 5.9 that

$$\left\|(\nabla + i\zeta_s) \int_{B_{R''}} g_{\zeta_p}(\cdot - y)(1 - e^{-i(\zeta_p-\zeta_s)\cdot y})\varphi(y)dy\right\|^2_{L^2(B_{R''})} \leq \frac{c_6}{|\Im(\zeta_s)|^2}\|\varphi\|^2_{L^2(B_{R''})} .$$

Finally, the uniform boundedness of $|e^{-i(\zeta_p-\zeta_s)\cdot y}|$ in $\overline{B_{R''}}$ and an analogous reasoning as above leads to

$$\begin{aligned}&\left\|(\nabla + i\zeta_s) \int_{B_{R''}} (1 - e^{i(\zeta_p-\zeta_s)\cdot x}) g_{\zeta_p}(x-y) e^{-i(\zeta_p-\zeta_s)\cdot y}\varphi(y)dy\right\|^2_{L^2(B_{R''})} \\ &\leq \frac{c_7}{|\Im(\zeta_s)|^2}\|\varphi\|^2_{L^2(B_{R''})} .\end{aligned}$$

Plugging all estimates together we have proved the assertion of the lemma.

□

We are now in a position to prove that the modified Lippmann-Schwinger equation (6.3) has a unique solution, provided $|\Im(\zeta_s)|$ is sufficiently large, and to discuss its behavior.

Theorem 6.7 *There is a constant t_0, depending only on R'', R', ω, κ_s, κ_p and $\|1-\rho\|_{1,\gamma}$, such that the modified Lippmann-Schwinger equation (6.3) has a unique solution if $|\Im(\zeta_s)| \geq t_0$.*

Furthermore, there is a positive constant c (depending only on R', R'', ω, κ_s, κ_p and $\|1-\rho\|_{1,\gamma}$) such that the solution U to (6.3) satisfies

$$U(x) = U(x,\zeta_s,\eta) = e^{i\zeta_s\cdot x}\{\eta + F(x,\zeta_s,\eta)\} \ , \ x \in B_{R''} \ ,$$

where the L^2-norms of the vector fields $F(\cdot,\zeta_s,\eta)$ can be estimated by

$$\|F(\cdot,\zeta_s,\eta)\|_{L^2(B_{R''})} \leq \frac{c|\eta|}{|\Im(\zeta_s)|}$$

for all $|\Im(\zeta_s)| \geq t_0$.

Proof: In order to prove that equation (6.3) has a trivial nullspace we set $W(x) := e^{-i\zeta_s\cdot x}U(x)$, $w(x) := e^{-i\zeta_s\cdot x}(\nabla\cdot U)(x)$, $x \in \overline{B_{R''}}$, for a solution U of the homogeneous equation. The reasoning before (6.4) and (6.5) now leads to the equations

$$W(x) = -\frac{\omega^2}{\mu}\int\limits_{B_R}(1-\rho(y))g_{\zeta_s}(x-y)W(y)dy - \Big[A_{\zeta_s}\Big((1-\rho)w - \nabla\rho\cdot W\Big)\Big](x)$$

and

$$w(x) = -\kappa_p^2\int\limits_{B_R} g_{\zeta_p}(x-y)e^{i(\zeta_p-\zeta_s)\cdot(x-y)}\{(1-\rho)w - \nabla\rho\cdot W\}(y)dy\ ,\ x\in\overline{B_{R''}}\ .$$

This is a fixed point equation in $C(\overline{B_{R''}})\times C(\overline{B_{R''}})$. We equip this linear space with the norm $\|(W,w)\|^2 := \|W\|_{L^2}^2 + \|w\|_{L^2}^2$. By the last lemma and our knowledge about G_{ζ_s} we know that the linear mapping in this space which is defined by the right hand sides of the above equations has an operator norm bounded by $\tilde{c}/|\Im(\zeta_s)|$, i.e., for a sufficiently large $t_0 > 0$ the operator norm is bounded by $1/2$ provided $|\Im(\zeta_s)| \geq t_0$. Hence, Banach's fixed point theorem states $W = 0$, $w = 0$, and therefore $U = 0$. The Riesz theory now implies that equation (6.3) has a unique solution for all right hand sides U^i.

An analogous reasoning to the scalar case immediately gives the asymptotic behavior

$$U(x,\zeta_s,\eta) = e^{i\zeta_s\cdot x}\{\eta + F(x,\zeta_s,\eta)\}\ ,\ x\in B_{R''}\ ,$$

with $\|F(\cdot,\zeta_s,\eta)\|_{L^2(B_{R''})} \leq c|\eta|/|\Im(\zeta_s)|$.

□

The uniqueness proof for the inverse elastic scattering problem is now an easy consequence of these special solutions and the orthogonality relation stated in Lemma 6.4.

Theorem 6.8 *Let the Lamé constants λ and μ of the elasticity equation be given and let $\omega > 0$ be fixed. If the far field patterns corresponding to the densities ρ, $\tilde{\rho} \in \tilde{C}(B_R)$ coincide, i.e., $U_\infty(\hat{x},d) = \tilde{U}_\infty(\hat{x},d)$ for all $(\hat{x},d) \in S^2\times S^2$, then $\rho = \tilde{\rho}$.*

Proof: We fix R_1 with $R < R_1 < R'$. Then, for a vector $\alpha \in \Gamma$ we choose the unit vectors d_1, $d_2 \in \mathbb{R}^3$ such that α, d_1, and d_2 are orthogonal, and we define for sufficiently large $t > 0$ the vectors

$$\begin{aligned}
\zeta_s(t) &:= -\frac{1}{2}\alpha + i\sqrt{t^2 - \kappa_s^2 + \frac{|\alpha|^2}{4}}d_1 + td_2 \ , \\
\tilde{\zeta}_s(t) &:= -\frac{1}{2}\alpha - i\sqrt{t^2 - \kappa_s^2 + \frac{|\alpha|^2}{4}}d_1 - td_2 \ , \\
\eta(t) &:= \frac{1}{|\alpha|}\alpha + \frac{|\alpha|}{2t}d_2 \ , \\
\tilde{\eta}(t) &:= \frac{1}{|\alpha|}\alpha - \frac{|\alpha|}{2t}d_2 \ .
\end{aligned}$$

This is possible because $|\alpha| \neq 0$ for $\alpha \in \Gamma$. As in the proof of Theorem 4.9 we have $\zeta_s(t) \cdot \zeta_s(t) = \tilde{\zeta}_s(t) \cdot \tilde{\zeta}_s(t) = \kappa_s^2$, $\tilde{\zeta}_s(t) \cdot \tilde{\eta}(t) = \zeta_s(t) \cdot \eta(t) = 0$ and $|\eta(t)| = |\tilde{\eta}(t)| \leq c_\alpha$ for all sufficiently large t. Therefore, by the preceding theorem there exist special solutions $U(\cdot, \zeta_s(t), \eta(t))$ and $\tilde{U}(\cdot, \tilde{\zeta}_s(t), \tilde{\eta}(t))$ to the elasticity equations with densities ρ, $\tilde{\rho}$, resp., such that

$$U(x, \zeta_s(t), \eta(t)) = e^{i\zeta_s(t)\cdot x}\{\eta(t) + F(x, \zeta_s(t), \eta(t))\} \ , \ x \in B_{R_1} \ ,$$

$$\tilde{U}(x, \tilde{\zeta}_s(t), \tilde{\eta}(t)) = e^{i\tilde{\zeta}_s(t)\cdot x}\{\tilde{\eta}(t) + \tilde{F}(x, \tilde{\zeta}_s(t), \tilde{\eta}(t))\} \ , \ x \in B_{R_1} \ ,$$

and

$$\|F(\cdot, \zeta_s(t), \eta(t))\|_{L^2(B_R)} + \|\tilde{F}(\cdot, \tilde{\zeta}_s(t), \tilde{\eta}(t))\|_{L^2(B_R)} \leq \frac{c'_\alpha}{|\Im(\zeta_s(t))|} \ .$$

Using $e^{i\zeta_s(t)\cdot x}e^{i\tilde{\zeta}_s(t)\cdot x} = e^{-i\alpha\cdot x}$ and $\eta(t) \cdot \tilde{\eta}(t) = 1 - (|\alpha|^2/4t^2)$, we arrive at

$$U(x, \zeta_s(t), \eta(t)) \cdot \tilde{U}(x, \tilde{\zeta}_s(t), \tilde{\eta}(t)) = e^{-i\alpha\cdot x}(1 + h(x,t))$$

with

$$\int_{B_R} |h(x,t)|dx \to 0 \ , \ t \to \infty \ .$$

We insert these special solutions into the orthogonality relation from Lemma 6.4 and obtain $(\rho - \tilde{\rho})\hat{\ }(\alpha) = 0$ as $t \to \infty$. The coincidence of the Fourier coefficients yields the desired coincidence of the densities and we have proved the theorem. $\square$

As usual we remark that it is possible to replace the plane incident waves by any set of solutions to $\Delta^* U + \omega^2 U = 0$ which is complete in the space of all solutions to this equation with respect to $L^2(B_R)$. Second, instead of measuring far field data one might also use near field data like the displacement vector on a large sphere because these data uniquely determine a radiating solution to $\Delta^* U + \omega^2 U = 0$.

6.2 Stability of the Inverse Elastic Problem

In this section we want to establish a result which essentially states that on a sufficiently small set the densities depend continuously on their corresponding far field patterns. We assume throughout this section that the real valued densities ρ satisfy $\rho \in \tilde{C}(B_R)$, i.e., $\rho \in C^{1,\gamma}(\mathbb{R}^3)$, $0 < \gamma < 1$, and $\operatorname{supp}(1-\rho) \subset B_R$.

For fixed $\hat{x}$, $d \in S^2$ the mapping

$$p \in \mathbb{C}^3 \mapsto \Big[U^s(\cdot, d, p)\Big]_\infty(\hat{x}) \in \mathbb{C}^3$$

is linear. Therefore, as at the end of chapter 5 we regard the far field pattern as a matrix valued mapping

$$U_\infty \colon S^2 \times S^2 \to \mathbb{C}^{3\times 3} \ .$$

$U_\infty(\hat{x}, d)$ has the vector $U_\infty(\hat{x}, d, d_k)$ as its kth column where d_1, d_2, d_3 denote the usual cartesian unit vectors.

As usual we employ a very strong norm $\|\cdot\|_{\mathcal{F}}$ on the far field patterns. Due to the two wave numbers κ_p and κ_s corresponding to pure pressure waves and pure shear waves we split the far field U_∞ into four parts, namely the normal and the tangential components of the far field corresponding to a pure incident plane pressure wave and to a pure incident plane shear wave. We define a plane incident pressure wave by

$$U^{i,press}(x, d, p) := -\frac{1}{\omega^2}\nabla(\nabla \cdot [pe^{i\kappa_p d\cdot x}]) \ , \ x \in \mathbb{R}^3 \ ,$$

and a plane incident shear wave by

$$U^{i,shear}(x, d, p) := \frac{1}{\omega^2}\nabla \wedge \nabla \wedge [pe^{i\kappa_s d\cdot x}] \ , \ x \in \mathbb{R}^3 \ .$$

The waves are propagating into the direction $d \in S^2$, whereas $p \in \mathbb{C}^3$ controls their amplitude and polarization. By $U^{press}(\cdot, d, p)$ we denote the total field corresponding to the incident wave $U^{i,press}(\cdot, d, p)$ and by $U_\infty^{press}(\hat{x}, d, p)$, $\hat{x} \in S^2$, we mean the far field of the scattered wave corresponding to the incident wave $U^{i,press}(\cdot, d, p)$. Due to the linearity of the map $p \mapsto U_\infty^{press}(\hat{x}, d, p)$ there is a matrix $U_\infty^{press}(\hat{x}, d) \in \mathbb{C}^{3\times 3}$ such that $U_\infty^{press}(\hat{x}, d, p) = U_\infty^{press}(\hat{x}, d)p$ for all $p \in \mathbb{C}^3$. We use the analogous notation $U^{shear}(\cdot, d, p)$, $U_\infty^{shear}(\hat{x}, d, p)$ and $U_\infty^{shear}(\hat{x}, d)$, if the incident wave $U^{i,press}(\cdot, d, p)$ is replaced by $U^{i,shear}(\cdot, d, p)$.

Furthermore, let $A(\hat{x}) \in \mathbb{R}^{3\times 3}$ denote the matrix having the entry $\hat{x}_j\hat{x}_k$ in the jth row and kth column where $\hat{x} \in S^2$ is a unit vector . Then the relation

$$U^{i,press}(x,d,p) = U^i(x,d,(p\cdot d)d) = U^i(x,d,A(d)p)$$

yields $U_\infty^{press}(\hat{x},d) = U_\infty(\hat{x},d)A(d)$. The normal component of the far field $U_\infty^{press}(\cdot,d,p)$ on S^2 is given by $A(\hat{x})U_\infty^{press}(\hat{x},d,p)$, $\hat{x}\in S^2$, whereas the tangential components are given by $(I-A(\hat{x}))U_\infty^{press}(\hat{x},d,p)$, $\hat{x}\in S^2$, where $I\in\mathbb{C}^{3\times 3}$ is the identity matrix. Similarly, we can compute $U^{i,shear}(x,d,p) = U^i(x,d,(I-A(d))p)$, whence $U_\infty^{shear}(\hat{x},d) = U_\infty(\hat{x},d)(I-A(d))$, and then split $U_\infty^{shear}(\hat{x},d)$ into its normal and tangential part.

Summarizing we can write

$$\begin{aligned} U_\infty(\hat{x},d) &= (I-A(\hat{x}))U_\infty(\hat{x},d)(I-A(d)) + A(\hat{x})U_\infty(\hat{x},d)(I-A(d)) \\ &\quad +(I-A(\hat{x}))U_\infty(\hat{x},d)A(d) + A(\hat{x})U_\infty(\hat{x},d)A(d)\ ,\ \hat{x},d\in S^2\ . \end{aligned} \tag{6.7}$$

Especially, the knowledge of U_∞ allows to compute the normal and tangential components of the far fields from pure incident plane shear and pressure waves.

We need this splitting because the Fourier coefficients of the above terms show a different behavior. We define these Fourier coefficients by

$$\begin{aligned} \mu^{(1)}_{l_1k_1l_2k_2} &:= \int_{S^2}\int_{S^2}(I-A(\hat{x}))U_\infty(\hat{x},d)(I-A(d))\overline{Y^{k_1}_{l_1}(\hat{x})Y^{k_2}_{l_2}(d)}ds(\hat{x})ds(d)\ , \\ \mu^{(2)}_{l_1k_1l_2k_2} &:= \int_{S^2}\int_{S^2} A(\hat{x})U_\infty(\hat{x},d)(I-A(d))\overline{Y^{k_1}_{l_1}(\hat{x})Y^{k_2}_{l_2}(d)}ds(\hat{x})ds(d)\ , \\ \mu^{(3)}_{l_1k_1l_2k_2} &:= \int_{S^2}\int_{S^2}(I-A(\hat{x}))U_\infty(\hat{x},d)A(d)\overline{Y^{k_1}_{l_1}(\hat{x})Y^{k_2}_{l_2}(d)}ds(\hat{x})ds(d)\ , \\ \mu^{(4)}_{l_1k_1l_2k_2} &:= \int_{S^2}\int_{S^2} A(\hat{x})U_\infty(\hat{x},d)A(d)\overline{Y^{k_1}_{l_1}(\hat{x})Y^{k_2}_{l_2}(d)}ds(\hat{x})ds(d)\ , \\ & \qquad l_1,l_2 = 0,1,\ldots,\ -l_1\le k_1\le l_1\ ,\ -l_2\le k_2\le l_2\ . \end{aligned} \tag{6.8}$$

Note, that the Fourier coefficients $\mu^{(m)}_{l_1k_1l_2k_2}\in\mathbb{C}^{3\times 3}$ are matrices.

The norm $\|\cdot\|_{\mathcal{F}}$ on the far fields will be defined by prescribing a rapid decay for each of the Fourier coefficients $\mu^{(m)}_{l_1k_1l_2k_2}$, $m=1,\ldots,4$.

The main estimate of this section reads

$$\|\rho-\tilde{\rho}\|_\infty \le c\Big|\ln(\|U_{\infty,\rho}-U_{\infty,\tilde{\rho}}\|_{\mathcal{F}})\Big|^{-1/11}$$

with a constant c for all densities ρ, $\tilde{\rho}$ lying in some small subset $\mathcal{O}$ of $\tilde{C}(B_R)$. Hence, the mapping $U_{\infty,\rho} \mapsto \rho$ is continuous and we have local uniqueness. Of course, $\mathcal{O}$ is not only small with respect to the maximum norm but with respect to a C^2-norm, i.e., stability is only obtained with the help of an *a priori* information.

Imitating the reasoning in the acoustic case we begin with the decay of the Fourier coefficients and prove continuity of the mapping $\rho \mapsto U_{\infty,\rho}$.

Then, we show how to reconstruct the kernel of the Green's operator for the elasticity equation on a large sphere with the help of a series expansion involving the Fourier coefficients which originate from U_∞ . Since the $\|\cdot\|_{\mathcal{F}}$-norm is a very strong norm, which is not appropriate for measured far field patterns, this mapping is severely ill-posed.

Finally, employing the special solutions from the last section, we investigate the dependence of ρ on the Green's operator and arrive at our main estimate.

We remind the reader that

$$\|A\|_F := \Big(\sum_{j,k=1}^{3} |a_{jk}|^2\Big)^{1/2}$$

denotes the Frobenius norm for a matrix $A = (a_{jk}) \in \mathbb{C}^{3\times 3}$.

Furthermore, for convenience we define the operator $\mathcal{V}_\rho\colon C(\overline{B_R}) \to C(\overline{B_R})$ by

$$(\mathcal{V}_\rho U)(x) := \omega^2 \int\limits_{B_R} (1-\rho(y))\Pi(x-y)U(y)dy \ , \ x \in \overline{B_R} \ .$$

Lemma 6.9 *Assume the far field pattern $U_\infty\colon S^2 \times S^2 \to \mathbb{C}^{3\times 3}$ originates from the density $\rho \in \tilde{C}(B_R)$ satisfying supp$\,(1-\rho) \subset B_{R_1}$ for some $0 < R_1 < R$. Let $\mu^{(m)}_{l_1k_1l_2k_2}$, $m = 1,\ldots,4$, denote the Fourier coefficients of U_∞ as defined in (6.8). Furthermore, define $R_3 := (R+R_1)/2$. Then, there is a constant c depending on U_∞ such that*

$$\begin{aligned}
\|\mu^{(1)}_{l_1k_1l_2k_2}\|_F^2 &\le c\Big(\frac{e\kappa_s R_3}{2l_1+1}\Big)^{2l_1+3}\Big(\frac{e\kappa_s R_3}{2l_2+1}\Big)^{2l_2+3} ,\\
\|\mu^{(2)}_{l_1k_1l_2k_2}\|_F^2 &\le c\Big(\frac{e\kappa_p R_3}{2l_1+1}\Big)^{2l_1+3}\Big(\frac{e\kappa_s R_3}{2l_2+1}\Big)^{2l_2+3} ,\\
\|\mu^{(3)}_{l_1k_1l_2k_2}\|_F^2 &\le c\Big(\frac{e\kappa_s R_3}{2l_1+1}\Big)^{2l_1+3}\Big(\frac{e\kappa_p R_3}{2l_2+1}\Big)^{2l_2+3} ,
\end{aligned}$$

$$\|\mu^{(4)}_{l_1k_1l_2k_2}\|^2_F \quad \leq \quad c\Big(\frac{e\kappa_p R_3}{2l_1+1}\Big)^{2l_1+3}\Big(\frac{e\kappa_p R_3}{2l_2+1}\Big)^{2l_2+3} \ .$$

Furthermore, we have

$$\sum_{l_1,k_1,l_2,k_2}\Big(\frac{2l_1+1}{e\kappa_s R}\Big)^{2l_1+3}\Big(\frac{2l_2+1}{e\kappa_s R}\Big)^{2l_2+3}\|\mu^{(1)}_{l_1k_1l_2k_2}\|^2_F < \infty \ ,$$

and analogous inequalities are true for the other Fourier coefficients $\mu^{(m)}_{l_1k_1l_2k_2}$*,* $m = 2,3,4$*.*

Proof: Let us examine the behavior of $\mu^{(2)}_{l_1k_1l_2k_2}$. The other Fourier coefficients $\mu^{(j)}_{l_1k_1l_2k_2}$ can be estimated analogously.

From the far field representation (5.28) and our considerations that lead to the splitting of the far field we obtain

$$A(\hat{x})U_\infty(\hat{x},d)(I-A(d))d_j = -\frac{\kappa_p^2}{4\pi}\int\limits_{B_R} e^{-i\kappa_p\hat{x}\cdot y}(1-\rho(y))\hat{x}\cdot U^{shear}(y,d,d_j)dy\,\hat{x} \ ,$$

for the jth column of $A(\hat{x})U_\infty(\hat{x},d)(I-A(d)) = A(\hat{x})U^{shear}_\infty(\hat{x},d)$. Interchanging the order of integration we obtain

$$\begin{aligned} \mu^{(2)}_{l_1k_1l_2k_2}d_j &= \int\limits_{S^2}\int\limits_{S^2} A(\hat{x})U^{shear}_\infty(\hat{x},d,d_j)\overline{Y^{k_1}_{l_1}(\hat{x})Y^{k_2}_{l_2}(d)}ds(\hat{x})ds(d) \\ &= -\frac{\kappa_p^2}{4\pi}\int\limits_{B_R}\Big\{(1-\rho(y)) \\ &\quad \int\limits_{S^2} e^{-i\kappa_p\hat{x}\cdot y}\overline{Y^{k_1}_{l_1}(\hat{x})}A(\hat{x})ds(\hat{x})\int\limits_{S^2} U^{shear}(y,d,d_j)\overline{Y^{k_2}_{l_2}(d)}ds(d)\Big\}dy \ . \end{aligned}$$

Now, the Cauchy-Schwarz inequality and an analogous reasoning to the proof of Lemma 4.10 complete the proof of the lemma.

□

By this lemma we know that the norm $\|U_{\infty,\rho}\|_{\mathcal{F}}$ defined by

$$\|U_{\infty,\rho}\|^2_{\mathcal{F}} \quad := \quad \sum_{l_1,k_1,l_2,k_2}\Big(\frac{2l_1+1}{e\kappa_s R}\Big)^{2l_1+3}\Big(\frac{2l_2+1}{e\kappa_s R}\Big)^{2l_2+3}\|\mu^{(1)}_{l_1k_1l_2k_2}\|^2_F$$

$$+\sum_{l_1,k_1,l_2,k_2}\Big(\frac{2l_1+1}{e\kappa_p R}\Big)^{2l_1+3}\Big(\frac{2l_2+1}{e\kappa_s R}\Big)^{2l_2+3}\|\mu^{(2)}_{l_1k_1l_2k_2}\|_F^2$$
$$+\sum_{l_1,k_1,l_2,k_2}\Big(\frac{2l_1+1}{e\kappa_s R}\Big)^{2l_1+3}\Big(\frac{2l_2+1}{e\kappa_p R}\Big)^{2l_2+3}\|\mu^{(3)}_{l_1k_1l_2k_2}\|_F^2$$
$$+\sum_{l_1,k_1,l_2,k_2}\Big(\frac{2l_1+1}{e\kappa_p R}\Big)^{2l_1+3}\Big(\frac{2l_2+1}{e\kappa_p R}\Big)^{2l_2+3}\|\mu^{(4)}_{l_1k_1l_2k_2}\|_F^2$$

is well defined, if $\rho \in \tilde{C}(B_R)$ because $\operatorname{supp}(1-\rho) \subset B_R$ implies that there is a radius $R_1 < R$ with $\operatorname{supp}(1-\rho) \subset B_{R_1}$.

Proceeding similarly to the proof of Lemma 2.18 we can prove the continuous dependence of $U_{\infty,\rho}$ on ρ.

Lemma 6.10 *Let $\rho_0 \in \tilde{C}(B_{R_1})$, $R_1 < R$, be given. Then, there are positive constants c and ϵ such that $\|U_{\infty,\rho} - U_{\infty,\rho_0}\|_{\mathcal{F}} \le c\|\rho-\rho_0\|_\infty$ for all $\rho \in \tilde{C}(B_{R_1})$ satisfying $\|\rho-\rho_0\|_\infty < \epsilon$.*

Now, we want to study the Green's operator for the elasticity equation, i.e., an integral operator having a matrix valued kernel Π_ρ such that for any smooth, compactly supported vector field F the vector field $U(x) = \int \Pi_\rho(x,y)F(y)dy$, $x \in \mathbb{R}^3$, satisfies $\Delta^* U + \omega^2\rho U = -F$ and the radiation condition. For $\rho = 1$ we know this operator because then we have $\Pi_1(x,y) = \Pi(x-y)$. Since we are merely interested in the operator $S_\rho: C^{0,\gamma}(\partial B_{R_2}) \to C^{0,\gamma}(\partial B_{R_2})$

$$(S_\rho\varphi)(x) := 2\int_{\partial B_{R_2}} \Pi_\rho(x,y)\varphi(y)ds(y)\ ,\ x \in \partial B_{R_2}\ ,$$

we deduce its properties from S_1 with the help of the Lippmann-Schwinger equation without ever studying the kernel Π_ρ. The properties of S_1 correspond to the single-layer in the acoustic case. The reader can find the necessary regularity results in the appendix of this chapter (see Theorem 6.23 and Lemma 6.24):
$S_1: C^{0,\gamma}(\partial B_{R_2}) \to C^{1,\gamma}(\partial B_{R_2})$ is bounded. The potential

$$U(x) := 2\int_{\partial B_{R_2}} \Pi(x-y)\varphi(y)ds(y)\ ,\ x \in \mathbb{R}^3 \setminus \partial B_{R_2}\ , \tag{6.9}$$

with density $\varphi \in C^{0,\gamma}(\partial B_{R_2})$ satisfies

$U|_{B_{R_2}} \in C^{1,\gamma}(\overline{B_{R_2}})$, $U|_{\mathbb{R}^3\setminus\overline{B_{R_2}}} \in C^{1,\gamma}(\mathbb{R}^3 \setminus B_{R_2})$,

$U_+ = U_- = S_1\varphi$ and $TU_- - TU_+ = 2\varphi$ on ∂B_{R_2} (here, T can be defined with any choice of β_1, $\beta_2 \in \mathbb{R}$, $\beta_1 + \beta_2 = \lambda + \mu$).

As usual the subscripts, $+$ and $-$, indicate whether we approach the boundary ∂B_{R_2} from the exterior and interior, respectively. Furthermore, the field U satisfies $\Delta^* U + \omega^2 U = 0$ in $\mathbb{R}^3 \setminus \partial B_{R_2}$ and the radiation condition.

Now we define the operator S_ρ with the help of the following boundary value problem (BVP):

Given $R_2 > R$, $\omega > 0$, $\mu > 0$, $\lambda \in \mathbb{R}$ $(2\mu + \lambda > 0)$, $\rho \in \tilde{C}(B_R)$ and $\varphi \in C^{0,\gamma}(\partial B_{R_2})$, find $U \in C^2(\mathbb{R}^3 \setminus \partial B_{R_2})$ satisfying the following requirements:

$U_- = U|_{B_{R_2}} \in C^1(\overline{B_{R_2}})$, $U_+ = U|_{\mathbb{R}^3\setminus\overline{B_{R_2}}} \in C^1(\mathbb{R}^3 \setminus B_{R_2})$,

$$\Delta^* U + \omega^2 \rho U = 0 \quad \text{in } \mathbb{R}^3 \setminus \partial B_{R_2},$$

U satisfies the radiation condition,

$U_- - U_+ = 0$ and

$-\mu\nu \wedge \nabla \wedge [U_- - U_+] + (\lambda + 2\mu)(\nabla \cdot [U_- - U_+])\nu = 2\varphi$ on ∂B_{R_2}.

Note that the last requirement means $TU_- - TU_+ = 2\varphi$, where the traction operator T is defined with $\beta_1 = -\mu$, $\beta_2 = 2\mu + \lambda$.

Lemma 6.11 *For all $\varphi \in C^{0,\gamma}(\partial B_{R_2})$ the boundary value problem (BVP) has a unique solution U. U is given by*

$$\begin{aligned} U(x) \;&:=\; 2 \int\limits_{\partial B_{R_2}} \Pi(x-y)\varphi(y)ds(y) \\ &\quad -\omega^2 \int\limits_{B_R} (1-\rho(y))\Pi(x-y)W(y)dy \;,\; x \in \mathbb{R}^3 \;, \end{aligned} \tag{6.10}$$

where $W \in C(\overline{B_R})$ denotes the unique solution to the Lippmann-Schwinger equation

$$(W + \mathcal{V}_\rho W)(x) = 2 \int\limits_{\partial B_{R_2}} \Pi(x-y)\varphi(y)ds(y) \;,\; x \in \overline{B_R} \;.$$

Proof: Assuming U is a solution to (BVP) with $\varphi = 0$ we choose $R_1 > R_2$ and compute with Betti's first formula (5.2) ($\beta_1 = -\mu$, $\beta_2 = \lambda + 2\mu$)

$$\begin{aligned}
&\Im\Big(\int_{\partial B_{R_1}} U \cdot \{-\mu\nu \wedge \nabla \wedge \overline{U} + (\lambda + 2\mu)(\nabla \cdot \overline{U})\nu\} ds\Big) \\
&= \Im\Big(\int_{\partial B_{R_2}} U_+ \cdot \{-\mu\nu \wedge \nabla \wedge \overline{U_+} + (\lambda + 2\mu)(\nabla \cdot \overline{U_+})\nu\} ds\Big) \\
&= \Im\Big(\int_{\partial B_{R_2}} U_- \cdot \{-\mu\nu \wedge \nabla \wedge \overline{U_-} + (\lambda + 2\mu)(\nabla \cdot \overline{U_-})\nu\} ds\Big) \\
&= \Im\Big(\int_{B_{R_2}} U_- \cdot \Delta^* \overline{U_-} dx\Big) \\
&= 0 \; .
\end{aligned}$$

Then, we can conclude from Lemma 5.8 that U vanishes in $\mathbb{R}^3 \setminus B_{R_2}$, whence $U_- = 0$ and $-\mu\nu\wedge\nabla\wedge\overline{U_-}+(\lambda+2\mu)(\nabla\cdot\overline{U_-})\nu = 0$ on ∂B_{R_2}. The representation formula from Theorem 5.3 ($\beta_1 = -\mu$, $\beta_2 = \lambda+2\mu$) applied to U_- implies that U_- is a solution of the homogeneous Lippmann-Schwinger equation (5.12). Thus U_- must vanish, too, and we have proved uniqueness for (BVP).

In order to show that U defined as in (6.10) is a solution to (BVP) we follow the proof of the second part of Lemma 5.7 to obtain that U satisfies the elasticity equation, due to the Lippmann-Schwinger equation for W, and the radiation condition. For the boundary conditions we observe that the volume potential in the definition of U is a C^2-smooth vector field in $\mathbb{R}^3$ and that the single-layer potential satisfies the needed regularity conditions and jump relations at the boundary.

□

We are now in a position to define

$$S_\rho : C^{0,\gamma}(\partial B_{R_2}) \to C^{1,\gamma}(\partial B_{R_2}) \qquad (S_\rho\varphi)(x) := U_+(x) \; , \; x \in \partial B_{R_2} \; , \quad (6.11)$$

where U is the unique solution to (BVP). S_ρ is well defined, bounded, and $S_\rho\varphi = U_-$ on ∂B_{R_2}. Note, that the definition of $S_\rho\varphi$ makes sense, too, if we suppose φ to be continuous instead of Hölder continuous. We need more properties of S_ρ.

Lemma 6.12 *The linear operators S_ρ satisfy:*

(a) $\displaystyle\int_{\partial B_{R_2}} (S_\rho\varphi)\cdot\psi ds = \int_{\partial B_{R_2}} \varphi\cdot(S_\rho\psi)ds$ *for all* $\varphi,\ \psi\in C^{0,\gamma}(\partial B_{R_2})$.

(b) *The mapping* $\rho\mapsto S_\rho$, *from* $(\tilde{C}(B_R),\|\cdot\|_\infty)$ *to the space of linear and bounded operators in* $C(\partial B_{R_2})$ *equipped with the* $\|\cdot\|_\infty$*-operator norm, is continuous.*

Proof: For $\varphi,\ \psi\in C^{0,\gamma}(\partial B_{R_2})$ we define U as in (6.10) and U' analogously, where we replace φ by ψ. Then we compute

$$\begin{aligned}&\int_{\partial B_{R_2}} \{(S_\rho\varphi)\cdot\psi-\varphi\cdot(S_\rho\psi)\}ds\\ &= \frac{1}{2}\int_{\partial B_{R_2}} \{U\cdot(TU'_- - TU'_+)-(TU_- - TU_+)\cdot U'\}ds\\ &= 0\end{aligned}$$

because the integrals containing TU_- and TU'_- vanish by the second Betti formula. Moreover, the integrals involving TU_+ and TU'_+ can be seen to be zero by the radiation condition after replacing the integral over ∂B_{R_2} by an integral over ∂B_r, $r>R_2$, inserting the definition of U, and interchanging the order of integration. This proves part (a).

The proof of assertion (b) follows the proof of Lemma 2.20 (c).

□

Our next goal is the computation of the operator S_ρ from a knowledge of the Fourier coefficients $\mu^{(m)}_{l_1k_1l_2k_2}$ of $U_{\infty,\rho}$. A consequence of this computation is the continuous dependence of S_ρ on $U_{\infty,\rho}$.

Lemma 6.13 *Let the far field pattern* $U_{\infty,\rho}\colon S^2\times S^2\to\mathbb{C}^{3\times 3}$ *originate from the density* $\rho\in\tilde{C}(B_R)$. *Let* $\mu^{(m)}_{l_1k_1l_2k_2}$, $m=1,\ldots,4$, *denote the Fourier coefficients as defined in (6.8).*

For $x,\ y\in\partial B_{R_2}$, $x\neq y$, *we define the matrix*

$$\begin{aligned}s_\rho(x,y) &:= \Pi(x-y)\\ &-\sum_{l_1k_1l_2k_2}\frac{i^{l_1-l_2}}{4\pi}\Big\{\kappa_s^2R_2^2h^{(1)}_{l_1}(\kappa_sR_2)h^{(1)}_{l_2}(\kappa_sR_2)Y^{k_1}_{l_1}\Big(\frac{x}{|x|}\Big)Y^{k_2}_{l_2}\Big(\frac{y}{|y|}\Big)\mu^{(1)}_{l_1k_1l_2k_2}\\ &\qquad+\kappa_p\kappa_sR_2^2h^{(1)}_{l_1}(\kappa_sR_2)h^{(1)}_{l_2}(\kappa_pR_2)Y^{k_1}_{l_1}\Big(\frac{x}{|x|}\Big)Y^{k_2}_{l_2}\Big(\frac{y}{|y|}\Big)\mu^{(3)}_{l_1k_1l_2k_2}\Big\}\end{aligned}$$

$$- \sum_{l_1 k_1} \frac{i^{l_1-l_2}}{4\pi} \Big\{ \kappa_s \kappa_p R_2^2 h_{l_1}^{(1)}(\kappa_p R_2) h_{l_2}^{(1)}(\kappa_s R_2) Y_{l_1}^{k_1}\Big(\frac{x}{|x|}\Big) Y_{l_2}^{k_2}\Big(\frac{y}{|y|}\Big) \mu_{l_1 k_1 l_2 k_2}^{(2)}$$
$$+ \kappa_p^2 R_2^2 h_{l_1}^{(1)}(\kappa_p R_2) h_{l_2}^{(1)}(\kappa_p R_2) Y_{l_1}^{k_1}\Big(\frac{x}{|x|}\Big) Y_{l_2}^{k_2}\Big(\frac{y}{|y|}\Big) \mu_{l_1 k_1 l_2 k_2}^{(4)} \Big\} \ . \tag{6.12}$$

(a) For all $\varphi \in C(\partial B_{R_2})$ *there holds*

$$(S_\rho \varphi)(x) = 2 \int_{\partial B_{R_2}} s_\rho(x,y)\varphi(y)ds(y) \ , \ x \in \partial B_{R_2} \ .$$

(b) There is a constant c such that for all ρ, $\tilde{\rho} \in \tilde{C}(B_R)$ *the inequality* $\|S_\rho - S_{\tilde{\rho}}\|_\infty \le c\|U_{\infty,\rho} - U_{\infty,\tilde{\rho}}\|_{\mathcal{F}}$ *holds true.*

Proof: The Cauchy-Schwarz inequality, the rapid decay of the Fourier coefficients (Lemma 6.9), and the estimate for $|h_l^{(1)}(\kappa R_2)|$ (Lemma 2.16) imply that the series in (6.12) are absolutely and uniformly convergent on $\partial B_{R_2} \times \partial B_{R_2}$. Therefore, s_ρ is a well defined continuous, matrix valued function for $x \neq y$.

From the definition of S_ρ in (6.11) we see that for $\varphi \in C(\partial B_{R_2})$ the difference $S_\rho\varphi - S_1\varphi$ has the form

$$\begin{aligned} &(S_\rho\varphi - S_1\varphi)(x) \\ &= \ -\nabla \wedge \nabla \wedge \int_{B_R} (1-\rho(y))\Phi_{\kappa_s}(x,y)W(y)dy \\ &\quad + \nabla\nabla \cdot \int_{B_R} (1-\rho(y))\Phi_{\kappa_p}(x,y)W(y)dy \ , \ x \in \partial B_{R_2} \ , \end{aligned} \tag{6.13}$$

$W \in C(\overline{B_R})$ being the solution to

$$(W + \mathcal{V}_\rho W)(x) = 2 \int_{\partial B_{R_2}} \Pi(x-y)\varphi(y)ds(y) \ , \ x \in \overline{B_R} \ .$$

It suffices to prove that the right hand side of (6.13) and

$$2 \int_{\partial B_{R_2}} (s_\rho(x,y) - \Pi(x-y))\varphi(y)ds(y)$$

coincide for all $x \in \partial B_{R_2}$ and for all $\varphi \in C(\partial B_{R_2})$ having the special form $\varphi = \overline{Y_l^k(\frac{\cdot}{|\cdot|})} d_m$. Here, d_1, d_2, d_3 denote the usual cartesian unit vectors in $\mathbb{R}^3$.

First, we compute for $x \in \partial B_{R_2}$

$$
\begin{aligned}
2 \int\limits_{\partial B_{R_2}} & (s_\rho(x,y) - \Pi(x-y)) \overline{Y_{l_2}^{k_2}(\frac{y}{|y|})} d_m ds(y) \\
= & -2 \sum_{l_1 k_1} \Big\{ \kappa_s^2 R_2^2 h_{l_2}^{(1)}(\kappa_s R_2) h_{l_1}^{(1)}(\kappa_s R_2) \frac{i^{l_1 - l_2}}{4\pi} \mu^{(1)}_{l_1 k_1 l_2 k_2} d_m \\
& \qquad + \kappa_p \kappa_s R_2^2 h_{l_2}^{(1)}(\kappa_p R_2) h_{l_1}^{(1)}(\kappa_s R_2) \frac{i^{l_1 - l_2}}{4\pi} \mu^{(3)}_{l_1 k_1 l_2 k_2} d_m \Big\} Y_{l_1}^{k_1}\Big(\frac{x}{|x|}\Big) \\
& -2 \sum_{l_1 k_1} \Big\{ \kappa_s \kappa_p R_2^2 h_{l_2}^{(1)}(\kappa_s R_2) h_{l_1}^{(1)}(\kappa_p R_2) \frac{i^{l_1 - l_2}}{4\pi} \mu^{(2)}_{l_1 k_1 l_2 k_2} d_m \\
& \qquad + \kappa_p^2 R_2^2 h_{l_2}^{(1)}(\kappa_p R_2) h_{l_1}^{(1)}(\kappa_p R_2) \frac{i^{l_1 - l_2}}{4\pi} \mu^{(4)}_{l_1 k_1 l_2 k_2} d_m \Big\} Y_{l_1}^{k_1}\Big(\frac{x}{|x|}\Big) \ .
\end{aligned}
\tag{6.14}
$$

In order to compute the right hand side of (6.13) we proceed similarly to the proof of Lemma 2.21 (a) and obtain

$$
\begin{aligned}
2 \int\limits_{\partial B_{R_2}} & \Pi(x-y) \overline{Y_{l_2}^{k_2}\Big(\frac{y}{|y|}\Big)} d_m \, ds(y) \\
= \ & 2i\kappa_s R_2^2 h_{l_2}^{(1)}(\kappa_s R_2) \frac{(-i)^{l_2}}{4\pi} \int\limits_{S^2} U^{i,shear}(x,d,d_m) \overline{Y_{l_2}^{k_2}(d)} ds(d) \\
& + 2i\kappa_p R_2^2 h_{l_2}^{(1)}(\kappa_p R_2) \frac{(-i)^{l_2}}{4\pi} \int\limits_{S^2} U^{i,press}(x,d,d_m) \overline{Y_{l_2}^{k_2}(d)} ds(d) \ , \ x \in \overline{B_R} \ ,
\end{aligned}
$$

and

$$
\begin{aligned}
W(x) \ = \ & 2i\kappa_s R_2^2 h_{l_2}^{(1)}(\kappa_s R_2) \frac{(-i)^{l_2}}{4\pi} \int\limits_{S^2} U^{shear}(x,d,d_m) \overline{Y_{l_2}^{k_2}(d)} ds(d) \\
& + 2i\kappa_p R_2^2 h_{l_2}^{(1)}(\kappa_p R_2) \frac{(-i)^{l_2}}{4\pi} \int\limits_{S^2} U^{press}(x,d,d_m) \overline{Y_{l_2}^{k_2}(d)} ds(d)
\end{aligned}
$$

for $x \in \overline{B_R}$.

Now, we compute the normal and the tangential components of the far field of

$$W'(x) := -\nabla \wedge \nabla \wedge \int_{B_R} (1-\rho(y))\Phi_{\kappa_s}(x,y)W(y)dy$$
$$+\nabla\nabla \cdot \int_{B_R} (1-\rho(y))\Phi_{\kappa_p}(x,y)W(y)dy \ , \ |x| \geq R \ ,$$

(note that $W'|_{\partial B_{R_2}}$ is the right hand side of (6.13)): the tangential components originate from the vector field

$$-\nabla \wedge \nabla \wedge \int_{B_R} (1-\rho(y))\Phi_{\kappa_s}(x,y)W(y)dy \ , \ |x| \geq R \ ,$$

which is a solution to the Helmholtz equation with wave number κ_s, and has the far field pattern

$$(I - A(\hat{x}))W'_\infty(\hat{x})$$
$$= -\frac{\kappa_s^2}{4\pi}\int_{B_R} (1-\rho(y))e^{-i\kappa_s \hat{x}\cdot y}\hat{x} \wedge (W(y) \wedge \hat{x})dy$$
$$= 2i\kappa_s R_2^2 h_{l_2}^{(1)}(\kappa_s R_2)\frac{(-i)^{l_2}}{4\pi}\int_{S^2} (I - A(\hat{x}))U_\infty^{shear}(\hat{x},d,d_m)\overline{Y_{l_2}^{k_2}(d)}ds(d)$$
$$+2i\kappa_p R_2^2 h_{l_2}^{(1)}(\kappa_p R_2)\frac{(-i)^{l_2}}{4\pi}\int_{S^2} (I - A(\hat{x}))U_\infty^{press}(\hat{x},d,d_m)\overline{Y_{l_2}^{k_2}(d)}ds(d) \ .$$

Thus we have

$$\int_{S^2} (I - A(\hat{x}))W'_\infty(\hat{x})\overline{Y_{l_1}^{k_1}(\hat{x})}ds(\hat{x}) = 2i\kappa_s R_2^2 h_{l_2}^{(1)}(\kappa_s R_2)\frac{(-i)^{l_2}}{4\pi}\mu^{(1)}_{l_1k_1l_2k_2}d_m$$
$$+2i\kappa_p R_2^2 h_{l_2}^{(1)}(\kappa_p R_2)\frac{(-i)^{l_2}}{4\pi}\mu^{(3)}_{l_1k_1l_2k_2}d_m \ . \quad (6.15)$$

On the other hand according to [7, Theorems 2.14 and 2.15] we have for $|x| > R$ a series expansion

$$-\nabla \wedge \nabla \wedge \int_{B_R} (1-\rho(y))\Phi_{\kappa_s}(x,y)W(y)dy = \sum_{l_1k_1} a_{l_1k_1} h_{l_1}^{(1)}(\kappa_s|x|)Y_{l_1}^{k_1}\Big(\frac{x}{|x|}\Big)$$

which converges absolutely and uniformly on ∂B_{R_2}. The coefficients $a_{l_1k_1}$ are vectors in $\mathbb{C}^3$ and the Fourier coefficients of the far field of this series are given by

$$\frac{1}{\kappa_s}\frac{1}{i^{l_1+1}}a_{l_1k_1}\ .$$

Comparing this expression with the equation for the Fourier coefficients derived in (6.15) yields

$$\begin{aligned} a_{l_1k_1} &= -2\kappa_s^2R_2^2h_{l_2}^{(1)}(\kappa_sR_2)\frac{i^{l_1-l_2}}{4\pi}\mu^{(1)}_{l_1k_1l_2k_2}d_m \\ &\quad -2\kappa_p\kappa_sR_2^2h_{l_2}^{(1)}(\kappa_pR_2)\frac{i^{l_1-l_2}}{4\pi}\mu^{(3)}_{l_1k_1l_2k_2}d_m\ . \end{aligned}$$

Similarly, the normal components of W'_∞ which originate from

$$\nabla\nabla\cdot\int_{B_R}(1-\rho(y))\Phi_{\kappa_p}(x,y)W(y)dy\ ,\ |x|\geq R\ ,$$

a vector valued solution to the Helmholtz equation with wave number κ_p, have the Fourier coefficients

$$\begin{aligned} \int_{S^2}A(\hat{x})W'_\infty(\hat{x})\overline{Y_{l_1}^{k_1}(\hat{x})}ds(\hat{x}) &= 2i\kappa_sR_2^2h_{l_2}^{(1)}(\kappa_sR_2)\frac{(-i)^{l_2}}{4\pi}\mu^{(2)}_{l_1k_1l_2k_2}d_m \\ &\quad +2i\kappa_pR_2^2h_{l_2}^{(1)}(\kappa_pR_2)\frac{(-i)^{l_2}}{4\pi}\mu^{(4)}_{l_1k_1l_2k_2}d_m\ . \end{aligned} \tag{6.16}$$

A series expansion of the above vector field as

$$\sum_{l_1k_1}b_{l_1k_1}h_{l_1}^{(1)}(\kappa_p|x|)Y_{l_1}^{k_1}\Big(\frac{x}{|x|}\Big)\ ,$$

computing the Fourier coefficients of its far field, and comparing them with (6.16) then yields

$$\begin{aligned} b_{l_1k_1} &= -2\kappa_s\kappa_pR_2^2h_{l_2}^{(1)}(\kappa_sR_2)\frac{i^{l_1-l_2}}{4\pi}\mu^{(2)}_{l_1k_1l_2k_2}d_m \\ &\quad -2\kappa_p^2R_2^2h_{l_2}^{(1)}(\kappa_pR_2)\frac{i^{l_1-l_2}}{4\pi}\mu^{(4)}_{l_1k_1l_2k_2}d_m\ . \end{aligned}$$

Therefore, on ∂B_{R_2} we have the expansion

$$\begin{aligned} W'(x) &= -2\sum_{l_1k_1}\Big\{\kappa_s^2R_2^2h_{l_2}^{(1)}(\kappa_sR_2)h_{l_1}^{(1)}(\kappa_sR_2)\frac{i^{l_1-l_2}}{4\pi}\mu^{(1)}_{l_1k_1l_2k_2}d_m \\ &\qquad +\kappa_p\kappa_sR_2^2h_{l_2}^{(1)}(\kappa_pR_2)h_{l_1}^{(1)}(\kappa_sR_2)\frac{i^{l_1-l_2}}{4\pi}\mu^{(3)}_{l_1k_1l_2k_2}d_m\Big\}Y_{l_1}^{k_1}\Big(\frac{x}{|x|}\Big) \\ &\quad -2\sum_{l_1k_1}\Big\{\kappa_s\kappa_pR_2^2h_{l_2}^{(1)}(\kappa_sR_2)h_{l_1}^{(1)}(\kappa_pR_2)\frac{i^{l_1-l_2}}{4\pi}\mu^{(2)}_{l_1k_1l_2k_2}d_m \\ &\qquad +\kappa_p^2R_2^2h_{l_2}^{(1)}(\kappa_pR_2)h_{l_1}^{(1)}(\kappa_pR_2)\frac{i^{l_1-l_2}}{4\pi}\mu^{(4)}_{l_1k_1l_2k_2}d_m\Big\}Y_{l_1}^{k_1}\Big(\frac{x}{|x|}\Big) \ . \end{aligned}$$

Since this coincides with (6.14), we have proved assertion (a).

The analogous estimates to the proof of Lemma 2.21 (b) yield part (b) of the lemma.

□

Next, we wish to establish the estimate

$$\Big|\int_{B_R}(\rho-\tilde\rho)U\cdot\tilde U\,dx\Big| \le c\|S_\rho - S_{\tilde\rho}\|_{\infty,\partial B_{R_2}}\|U\|_{L^2(B_{R''})}\|\tilde U\|_{L^2(B_{R''})} \qquad (6.17)$$

for solutions U, $\tilde U \in C^2(B_{R''})\cap L^2(B_{R''})$ to the elasticity equation ($R < R_2 < R''$).

In the acoustic case the proof needed two ingredients. For a given solution u to the perturbed Helmholtz equation in $B_{R''}$ we constructed a radiating solution to the Helmholtz equation in the exterior of B_{R_2}, whose Dirichlet boundary values coincided with the values of u on ∂B_{R_2}. This allowed to represent u as a single-layer having the Green's function s_n as kernel. The second ingredient was the estimate $\|u\|_{1,\gamma,\overline{B_{R_2}}} \le c\|u\|_{L^2(B_{R''})}$.

Since the solution to the exterior Dirichlet problem requires a thorough analysis of the elastic double-layer potential [23, 24, 12] (a task that we want to avoid), we use a different approach in order to represent a solution U to the elasticity equation as a single-layer with the Green's kernel s_ρ, namely we solve a Robin boundary value problem in the following lemma.

For the second ingredient, the *a priori* estimate, we use the analogous approach to the proof of Weyl's lemma, Lemma 2.6. Finally, we establish the desired estimate (6.17) in Lemma 6.16 .

During the proof of the following lemma we shall use the boundary integral operator $K'_\rho : C^{0,\gamma}(\partial B_{R_2}) \to C^{0,\gamma}(\partial B_{R_2})$

$$
\begin{aligned}
(K'_\rho \varphi)(x) \quad &:= \quad 2 \int\limits_{\partial B_{R_2}} T_x[\Pi(x-y)\varphi(y)]ds(y) \ , \\
& \qquad -\omega^2 \int\limits_{B_R} (1-\rho(y))T_x[\Pi(x-y)W(y)]dy \ , \ x \in \partial B_{R_2} \ .
\end{aligned}
$$

Here, W is defined as the unique solution to

$$
(I + \mathcal{V}_\rho)W = 2 \int\limits_{\partial B_{R_2}} \Pi(\cdot - y)\varphi(y)ds(y)
$$

in B_R, and T_x is the unphysical traction operator defined with the parameters

$$
\beta_1 = \frac{\mu(\lambda+\mu)}{\lambda+3\mu} \ , \ \beta_2 = \frac{(\lambda+\mu)(\lambda+2\mu)}{\lambda+3\mu} \ . \tag{6.18}
$$

The choice of the constants β_1 and β_2 is now important because it yields a compact operator K'_1 with a weakly singular kernel (see Lemma 6.25 in the appendix). Therefore, K'_ρ is a compact operator.

Note, that in linear elasticity the traction operator T_x corresponds to the the operator $\partial/\partial\nu(x)$ in acoustic scattering. We have therefore chosen the same name K' for the normal derivative of the acoustic single-layer and the traction of the elastic single-layer.

The jump relations for the single-layer with kernel Π and the properties of the volume potential imply for the vector field U defined in (6.10): $TU_- + iU_- = (I + K'_\rho + iS_\rho)\varphi$ on ∂B_{R_2}. This motivates to consider the following boundary value problem with Robin data ($RBVP$):

Given $R_2 > R$, $\omega > 0$, $\mu > 0$, $\lambda \in \mathbb{R}$ ($2\mu + \lambda > 0$), $\rho \in \tilde{C}(B_R)$ and $F \in C^{0,\gamma}(\partial B_{R_2})$,

find $U \in C^2(B_{R_2}) \cap C^1(\overline{B_{R_2}})$ satisfying $\Delta^* U + \omega^2 \rho U = 0$ in B_{R_2} and $TU_- + iU_- = F$ on ∂B_{R_2}. Here, the parameters β_1 and β_2 in the definition of T are chosen according to (6.18).

Lemma 6.14

(a) ($RBVP$) has at most one solution.

(b) *For all* $F \in C^{0,\gamma}(\partial B_{R_2})$ *the equation* $(I + K'_\rho + iS_\rho)\varphi = F$ *has a unique solution* $\varphi \in C^{0,\gamma}(\partial B_{R_2})$ *depending continuously on* F*. If* φ *is a solution to this equation, then, the vector field* U *defined in (6.10) is the unique solution to* $(RBVP)$*.*

Proof: For assertion (a) we compute with Betti's first formula, the homogeneous boundary condition, and the elasticity equation

$$\begin{aligned} -i \int\limits_{\partial B_{R_2}} |U|^2 ds & \\ = \int\limits_{\partial B_{R_2}} \overline{U} \cdot TU ds & \\ = - \int\limits_{B_{R_2}} \overline{U} \cdot (\omega^2 \rho U) dx & \\ + \int\limits_{B_{R_2}} \Big\{ (\beta_1 + \mu) \sum_{k=1}^{3} |\partial_k U|^2 + \beta_2 |\nabla \cdot U|^2 - \beta_1 |\nabla \wedge U|^2 \Big\} dx \ . & \end{aligned}$$

Hence, taking the imaginary part we have $U|_{\partial B_{R_2}} = 0$, and $TU = 0$ on ∂B_{R_2} by the boundary condition. Now, the representation formula from Theorem 5.3 applied to U implies that U is a solution to the homogeneous Lippmann-Schwinger equation (5.12). Thus U must vanish and we have proved uniqueness for $(RBVP)$.

For part (b) we note that, if φ is a solution to the equation $(I + K'_\rho + iS_\rho)\varphi = F$, and if U is defined by

$$\begin{aligned} U(x) \ := \ & 2 \int\limits_{\partial B_{R_2}} \Pi(x-y)\varphi(y) ds(y) \\ & - \omega^2 \int\limits_{B_R} (1-\rho(y)) \Pi(x-y) W(y) dy \ , \ x \in \mathbb{R}^3 \ , \end{aligned}$$

$W \in C(\overline{B_R})$ being the unique solution to the Lippmann-Schwinger equation

$$(W + \mathcal{V}_\rho W)(x) = 2 \int\limits_{\partial B_{R_2}} \Pi(x-y)\varphi(y) ds(y) \ , \ x \in \overline{B_R},$$

then U is a solution to the boundary value problem $(RBVP)$. The differential equation follows from the Lippmann-Schwinger equation for W, whereas the boundary behavior of U is a consequence of the elastic single-layer.

If φ is a solution for $F = 0$, we know from part (a) that U vanishes in B_{R_2}. Furthermore, U is a radiating solution to $\Delta^* U + \omega^2 U = 0$ in the exterior of B_{R_2} with vanishing Dirichlet boundary values and we can compute for $R_1 > R_2$ with Betti's first formula (5.2) (now $\beta_1 = -\mu$, $\beta_2 = \lambda + 2\mu$)

$$\begin{aligned} &\Im\Big(\int_{\partial B_{R_1}} U \cdot \{-\mu\nu \wedge \nabla \wedge \overline{U} + (\lambda+2\mu)(\nabla \cdot \overline{U})\nu\} ds \Big) \\ &= \Im\Big(\int_{\partial B_{R_2}} U_+ \cdot \{-\mu\nu \wedge \nabla \wedge \overline{U_+} + (\lambda+2\mu)(\nabla \cdot \overline{U_+})\nu\} ds \Big) \\ &= 0 \ . \end{aligned}$$

Lemma 5.8 then implies that U vanishes in $\mathbb{R}^3 \setminus B_{R_2}$, whence $2\varphi = TU_- - TU_+ = 0$ on ∂B_{R_2}. Since the operators in the integral equation are compact, existence and continuous dependence of a solution follows by the Riesz theory. This completes the proof of the lemma.

□

Now we prove the *a priori* estimate which corresponds to Weyl's lemma.

Lemma 6.15 *Assume* $0 < R_2 < R''$ *and* $c_1 > 0$ *are constants. Then, there is a constant* $c_2 > 0$ *such that for all* $\rho \in C(\overline{B_{R''}})$ *with* $\|\rho\|_\infty \le c_1$ *and for all* $U \in C^2(B_{R''}) \cap L^2(B_{R''})$ *satisfying* $\Delta^* U + \rho U = 0$ *in* $B_{R''}$ *the inequality* $\|U\|_{1,\gamma,\overline{B_{R_2}}} \le c_2 \|U\|_{L^2(B_{R''})}$ *holds true.*

Proof: We first construct suitable test vector fields $\varphi \in C_0^\infty(B_{R''})$. Let $B_\epsilon(x^*) \subset B_{R''}$, $\epsilon > 0$, be a ball and let $\chi \in C_0^\infty(\mathbb{R})$ be a cut-off function satisfying $\chi(t) = 0$, if $|t| \ge \epsilon/2$, and $\chi(t) = 1$, if $|t| \le \epsilon/4$. For a vector field $\psi \in C_0^\infty(B_{\epsilon/4}(x^*))$ we define

$$\varphi(x) := \int_{\mathbb{R}^3} \chi(|x-y|)\Pi^{(0)}(x-y)\psi(y)dy \ , \ x \in \mathbb{R}^3 \ .$$

As in Lemma 2.6 we have $\varphi \in C_0^\infty(B_\epsilon(x^*))$.

The matrix valued function $\tilde{k}(x) := (\chi(|x|) - 1)\Pi^{(0)}(x)$, $x \in \mathbb{R}^3$, satisfies $\tilde{k} \in C^\infty(\mathbb{R}^3)$ and we obtain with the help of Theorem 5.6 in $B_\epsilon(x^*)$

$$\begin{aligned}(\Delta^*\varphi + \rho\varphi)(x) &= -\psi(x) + \int\limits_{B_{\epsilon/4}(x^*)} \Delta_x^*(\tilde{k}(x-y)\psi(y))dy \\ &\quad + \rho(x) \int\limits_{B_{\epsilon/4}(x^*)} \chi(|x-y|)\Pi^{(0)}(x-y)\psi(y)dy \ .\end{aligned}$$

We denote by $\tilde{M}\colon \mathbb{R}^3 \to \mathbb{C}^{3\times 3}$ the C^∞-smooth matrix valued function satisfying $\Delta_x^*(\tilde{k}(x-y)\psi(y)) = \tilde{M}(x-y)\psi(y)$. Using integration by parts and reversing the order of integration we arrive at

$$\begin{aligned}0 &= \int\limits_{B_\epsilon(x^*)} \varphi \cdot (\Delta^* U + \rho U)dx \\ &= \int\limits_{B_\epsilon(x^*)} (\Delta^*\varphi + \rho\varphi) \cdot U dx \\ &= \int\limits_{B_{\epsilon/4}(x^*)} \psi(y) \cdot \Big\{ -U(y) + \int\limits_{B_\epsilon(x^*)} \tilde{M}^T(x-y)U(x)dx \\ &\qquad + \int\limits_{B_\epsilon(x^*)} \rho(x)\chi(|x-y|)\Pi^{(0)}(x-y)U(x)dx \Big\} dy\end{aligned}$$

for any $\psi \in C_0^\infty(B_{\epsilon/4}(x^*))$. Hence, we have for all $y \in B_{\epsilon/4}(x^*)$

$$\begin{aligned}U(y) &= \int\limits_{B_\epsilon(x^*)} \tilde{M}^T(x-y)U(x)dx \\ &\quad + \int\limits_{B_\epsilon(x^*)} \rho(x)\chi(|x-y|)\Pi^{(0)}(x-y)U(x)dx \ . \end{aligned} \tag{6.19}$$

The Cauchy-Schwarz inequality yields $\|U\|_{\infty, B_{\epsilon/4}(x^*)} \le c\|U\|_{L^2(B_{R''})}$, with a constant c depending on ϵ (via χ) and $\|\rho\|_{\infty, B_{R''}}$.

Replacing ϵ by $\epsilon/4$ and repeating the procedure, which lead to equation (6.19), with an adjusted cut-off function χ we obtain

$$\begin{aligned}U(y) &= \int\limits_{B_{\epsilon/4}(x^*)} \tilde{M}^T(x-y)U(x)dx \\ &\quad + \int\limits_{B_{\epsilon/4}(x^*)} \rho(x)\chi(|x-y|)\Pi^{(0)}(x-y)U(x)dx\end{aligned}$$

for all $y \in B_{\epsilon/16}(x^*)$. Now, the estimates from Theorem 1.9 imply $\|U\|_{1,\gamma,\overline{B_{\epsilon/16}(x^*)}} \leq c\|U\|_{L^2(B_{R''})}$.

Finally, we complete the proof by covering the compact set $\overline{B_{R_2}}$ by finitely many balls of the form $B_{\epsilon_j/32}(x_j)$, where ϵ_j is chosen sufficiently small to ensure $B_{\epsilon_j}(x_j) \subset B_{R''}$, and by patching together the above norm estimates for $\|U\|_{1,\gamma,\overline{B_{\epsilon_j/16}(x_j)}}$.

□

The desired estimate (6.17) is now a consequence of the two previous lemmas and Betti's second formula.

Lemma 6.16 *Assume $R < R_2 < R''$ and ω, $c_1 > 0$ are positive constants. Then, there exists a positive constant c such that for all ρ, $\tilde{\rho} \in \tilde{C}(B_R)$ with $\|\rho\|_{1,\gamma,\mathbb{R}^3}$, $\|\tilde{\rho}\|_{1,\gamma,\mathbb{R}^3} \leq c_1$, and for all solutions $U \in C^2(B_{R''}) \cap L^2(B_{R''})$ to $\Delta^* U + \omega^2 \rho U = 0$ in $B_{R''}$ and all solutions $\tilde{U} \in C^2(B_{R''}) \cap L^2(B_{R''})$ to $\Delta^* \tilde{U} + \omega^2 \tilde{\rho} \tilde{U} = 0$ in $B_{R''}$ the estimate*

$$\Big| \int_{B_R} (\rho - \tilde{\rho}) U \cdot \tilde{U} dx \Big| \leq c \|S_\rho - S_{\tilde{\rho}}\|_{\infty, \partial B_{R_2}} \|U\|_{L^2(B_{R''})} \|\tilde{U}\|_{L^2(B_{R''})} \tag{6.20}$$

holds true.

Proof: Let U satisfy the assumptions of the assertion. Due to Lemma 6.14 we have a representation of $U|_{B_{R_2}}$ as in (6.10), where the density $\varphi \in C^{0,\gamma}(\partial B_{R_2})$ satisfies the integral equation

$$(I + K'_\rho + iS_\rho)\varphi = TU_- + iU_- \qquad \text{on } \partial B_{R_2}.$$

Hence, we know

$$\|\varphi\|_\infty \leq \|\varphi\|_{0,\gamma} \leq c\|U\|_{1,\gamma,\overline{B_{R_2}}} \,. \tag{6.21}$$

The above inequality used the fact that $(I + K'_\rho + iS_\rho)^{-1}$ is bounded in $C^{0,\gamma}(\partial B_{R_2})$. In order to have the same inequality with a constant c, which holds uniformly for all densities ρ as in the assertion, we have to prove that the bound can be chosen uniformly for all $\rho \in \tilde{C}(B_R)$ with $\|\rho\|_{1,\gamma,\mathbb{R}^3} \leq c_1$. This can be seen by the following reasoning:

We choose $R_2 > R_1 > R$ and assume that $\rho_j \in \tilde{C}(B_R)$, $j \in \mathbb{N}$, is a sequence with $\|\rho_j\|_{1,\gamma,\mathbb{R}^3} \leq c_1$ and $\|(I + K'_{\rho_j} + iS_{\rho_j})^{-1}\|_{C^{0,\gamma}} \to \infty$, $j \to \infty$. Since the imbedding $C^{1,\gamma}(\overline{B_{R_1}}) \subset C^{1,\gamma'}(\overline{B_{R_1}})$ is compact for $0 < \gamma' < \gamma$, we assume without loss of generality that $\|\rho_j - \rho_0\|_{1,\gamma',\mathbb{R}^3} \to 0$, $j \to \infty$, for a suitable real valued function $\rho_0 \in C^{1,\gamma'}(\mathbb{R}^3)$ with $\mathrm{supp}(1-\rho_0) \subset B_{R_1}$. Due to the continuous dependence of the mapping $\rho \mapsto (I + K'_\rho + iS_\rho)^{-1}$ from

$$\{\rho \in C^{1,\gamma'}(\mathbb{R}^3)\colon \mathrm{supp}(1-\rho) \subset B_{R_1}\}$$

to the space of bounded linear operators on ∂B_{R_2} (the volume potentials in the definitions of S_ρ and K'_ρ depend continuously on ρ, see Lemma 6.12 (b)) we obtain the contradiction

$$\|(I + K'_{\rho_j} + iS_{\rho_j})^{-1}\|_{C^{0,\gamma}} \to \|(I + K'_{\rho_0} + iS_{\rho_0})^{-1}\|_{C^{0,\gamma}} < \infty \ .$$

Of course, we can apply the analogous reasoning to $\tilde{U}$, whence we have represented U and $\tilde{U}$ as a combination of a single-layer potential and a volume potential. This representation is also defined in $\mathbb{R}^3 \setminus B_{R_2}$. We thus have a continuous continuation of U and $\tilde{U}$ as radiating solutions to $\Delta^* W + \omega^2 W = 0$ in the exterior of $\overline{B_{R_2}}$.

Then, we use Lemma 6.12 (a), the jump relation, and (5.3) to compute

$$\begin{aligned}
&\int\limits_{\partial B_{R_2}} \varphi \cdot (S_{\tilde{\rho}} - S_\rho)\tilde{\varphi} ds \\
&= \frac{1}{2} \int\limits_{\partial B_{R_2}} \Big((TU_- - TU_+) \cdot \tilde{U} - (T\tilde{U}_- - T\tilde{U}_+) \cdot U\Big) ds \\
&= \frac{1}{2} \int\limits_{\partial B_{R_2}} \big(TU_- \cdot \tilde{U} - T\tilde{U}_- \cdot U\Big) ds \\
&= \frac{\omega^2}{2} \int\limits_{B_R} (\tilde{\rho} - \rho) U \cdot \tilde{U} dx \ ,
\end{aligned} \tag{6.22}$$

because, as in the proof of Lemma 6.12 (a), the integrals involving TU_+ and $T\tilde{U}_+$ vanish.

Using Lemma 6.15, (6.22), and (6.21) we complete the proof of the lemma and estimate

$$\begin{aligned}
&\Big|\int\limits_{B_R} (\rho - \tilde{\rho}) U \tilde{U} dx\Big| \\
&= \Big|\frac{2}{\omega^2} \int\limits_{\partial B_{R_2}} \varphi \cdot (S_{\tilde{\rho}} - S_\rho)\tilde{\varphi} ds\Big| \\
&\leq c\|S_\rho - S_{\tilde{\rho}}\|_{\infty,\partial B_{R_2}} \|U\|_{L^2(B_{R''})} \|\tilde{U}\|_{L^2(B_{R''})} \ . \qquad \square
\end{aligned}$$

Finally, we prove the main estimate, which implies the continuous dependence of ρ on S_ρ or on $U_{\infty,\rho}$.

Theorem 6.17 *Let $\rho_0 \in \tilde{C}(B_R) \cap C^2(\mathbb{R}^3)$ be given. Then, there are a neighborhood $\mathcal{O}$ of ρ_0,*

$$\mathcal{O} := \{\rho \in \tilde{C}(B_R) \cap C^2(\mathbb{R}^3) \colon \|\rho - \rho_0\|_{C^2} < \epsilon\} \ ,$$

and a positive constant c, such that for all ρ, $\tilde{\rho} \in \mathcal{O}$ the estimate

$$\|\rho - \tilde{\rho}\|_{\infty,B_R} \leq c[-\ln(\|S_\rho - S_{\tilde{\rho}}\|_{\infty,\partial B_{R_2}})]^{-1/11}$$

holds true.

Proof: We choose $R < R_2 < R'' < R' < 2R_2$. Furthermore, we choose $t_0 > 0$ from Theorem 6.7 sufficiently large to ensure the existence of the special solutions and the estimates stated for them in this theorem holding uniformly for all $\|\rho - \rho_0\|_{C^2} \leq 1$ and for all $|\Im(\zeta_s)| \geq t_0$. Then, we set

$$t_1 := t_0 + 2\kappa_s + 100 \ .$$

Finally, we choose $0 < \epsilon_1 < 1/2$ sufficiently small to have

$$\frac{-5}{11(4R_2 + 1)} \ln(2\epsilon_1) > t_1 \ .$$

Due to the continuous dependence of S_ρ on ρ (Lemma 6.12 (b)) we can find ϵ with $0 < \epsilon < \epsilon_1$ such that

$$\|S_\rho - S_{\tilde{\rho}}\|_{\infty,\partial B_{R_2}} \leq 2\epsilon_1$$

for all

$$\rho, \tilde{\rho} \in \mathcal{O} := \{\rho \in \tilde{C}(B_R) \cap C^2(\mathbb{R}^3) \colon \|\rho - \rho_0\|_{C^2} < \epsilon\} \ .$$

The inequalities (2.38) and (2.39) imply

$$\|\rho - \tilde{\rho}\|_\infty \leq c \sum_{\alpha \cdot \alpha \leq \delta^2} |(\rho - \tilde{\rho})\hat{}(\alpha)| + \frac{c}{\sqrt{\delta}} \qquad (6.23)$$

for ρ, $\tilde{\rho} \in \mathcal{O}$ and any $\delta \geq 2$.

As in the proof of Theorem 2.23 we wish to pick a suitable δ, depending on $\|S_\rho - S_{\tilde{\rho}}\|_{\infty,\partial B_{R_2}}$, in order to estimate the right hand side of (6.23). We bound the Fourier coefficients $|(\rho-\tilde\rho)\hat{\ }(\alpha)|$, $\alpha\cdot\alpha\le\delta^2$, with the help of the preceding lemma and the special solutions from Theorem 6.8.

To this end we set $t:=-\frac{5}{11(4R_2+1)}\ln\|S_\rho-S_{\tilde\rho}\|_\infty$ and $\delta:=t^{2/11}$. By the definition of ϵ, the inequalities $\|S_\rho-S_{\tilde\rho}\|_\infty<1$ and $t\ge t_1$ are satisfied for all ρ, $\tilde\rho\in\mathcal{O}$, and we also have $\delta\ge 2$.

For a vector $\alpha\in\Gamma$ with $\alpha\cdot\alpha\le\delta^2$ we choose as in Theorem 6.8

$$\begin{aligned}
\zeta_s(t) &:= -\frac{1}{2}\alpha+i\sqrt{t^2-\kappa_s^2+\frac{|\alpha|^2}{4}}d_1+td_2\ ,\\
\tilde\zeta_s(t) &:= -\frac{1}{2}\alpha-i\sqrt{t^2-\kappa_s^2+\frac{|\alpha|^2}{4}}d_1-td_2\ ,\\
\eta(t) &:= \frac{1}{|\alpha|}\alpha+\frac{|\alpha|}{2t}d_2\ ,\\
\tilde\eta(t) &:= \frac{1}{|\alpha|}\alpha-\frac{|\alpha|}{2t}d_2\ .
\end{aligned}$$

Then, we have $|\Im(\zeta_s(t))|\ge t-\kappa_s\ge t/2$, $|\zeta_s(t)|/|\Im(\zeta_s(t))|\le 2$, and $|\Im(\zeta_s(t))|\ge t-\kappa_s\ge t_0$ for all $\rho\in\mathcal{O}$, whence by Theorem 6.7 there exist the special solutions $U(x,\zeta_s(t),\eta(t))=e^{i\zeta_s(t)\cdot x}[\eta(t)+F(x,\zeta_s(t),\eta(t))]$ satisfying the inequality $\|F(\cdot,\zeta_s(t),\eta(t))\|_{L^2}\le(c|\eta(t)|)/|\Im(\zeta_s(t))|$ uniformly in $\rho\in\mathcal{O}$, $t\ge t_1$. The analogous assertion applies to $\tilde U(x,\tilde\zeta_s(t),\tilde\eta(t))=e^{i\tilde\zeta_s(t)\cdot x}[\tilde\eta(t)+\tilde F(x,\tilde\zeta_s(t),\tilde\eta(t))]$.

With the help of the preceding lemma we compute

$$\begin{aligned}
&|(\tilde\rho-\rho)\hat{\ }(\alpha)|\\
&= (2R')^{-3/2}\Big|\int_C(\tilde\rho-\rho)(x)e^{-i\alpha\cdot x}dx\Big|\\
&= (2R')^{-3/2}\Big|\int_{B_R}(\tilde\rho-\rho)(x)U(x,\zeta_s(t),\eta(t))\cdot\tilde U(x,\tilde\zeta_s(t),\tilde\eta(t))dx\\
&\qquad+\int_{B_R}(\tilde\rho-\rho)(x)e^{-i\alpha\cdot x}\Big\{\frac{|\alpha|^2}{4t^2}-\tilde\eta(t)\cdot F(x,\zeta_s(t),\eta(t))\\
&\qquad\qquad-\eta(t)\cdot\tilde F(x,\tilde\zeta_s(t),\tilde\eta(t))\\
&\qquad\qquad-F(x,\zeta_s(t),\eta(t))\cdot\tilde F(x,\tilde\zeta_s(t),\tilde\eta(t))\Big\}dx\Big|\\
&\le c\|S_\rho-S_{\tilde\rho}\|_\infty\|U(\cdot,\zeta_s(t),\eta(t))\|_{L^2(B_{R''})}\|\tilde U(\cdot,\tilde\zeta_s(t),\tilde\eta(t))\|_{L^2(B_{R''})}+\frac{c|\alpha|^2}{t}
\end{aligned}$$

$$\leq \quad c(|\alpha|^2 e^{4R_2(t+|\alpha|)}\|S_\rho - S_{\tilde{\rho}}\|_\infty + \frac{|\alpha|^2}{t}) \ , \tag{6.24}$$

where we have used the fact that

$$\begin{aligned} \|U(\cdot,\zeta_s(t),\eta(t))\|_{L^2(B_{R''})} &\leq \|e^{i\zeta_s(t)\cdot x}\|_{\infty,B_{2R_2}}\|\eta(t)+F(\cdot,\zeta_s(t),\eta(t))\|_{L^2(B_{R''})} \\ &\leq c|\alpha|e^{2R_2(t+|\alpha|)} \end{aligned}$$

for all $t \geq t_1$, $\rho \in \mathcal{O}$, and $\alpha \in \Gamma$ because $|\Im(\zeta_s(t))| \leq t + |\alpha|$.

Note that contrary to the acoustic case we have a factor $|\alpha|^2$ from our choice of $\eta(t)$, $\tilde{\eta}(t)$, which can only be bounded by $c|\alpha|$. These terms $|\alpha|^2$ are responsible for the different exponents occurring in the elastic and acoustic stability estimates.

Inequality (6.24) implies

$$\begin{aligned} \sum_{\alpha\cdot\alpha\leq\delta^2} |(\tilde{\rho}-\rho)\hat{}(\alpha)| &\leq c \sum_{\alpha\cdot\alpha\leq\delta^2} (|\alpha|^2 e^{4R_2(t+|\alpha|)}\|S_\rho - S_{\tilde{\rho}}\|_\infty + \frac{|\alpha|^2}{t}) \\ &\leq c\{e^{4R_2 t}e^{4R_2\delta}\delta^5\|S_\rho - S_{\tilde{\rho}}\|_\infty + \frac{\delta^5}{t}\} \\ &\leq c\{e^{(4R_2+1)(t+\delta)}\|S_\rho - S_{\tilde{\rho}}\|_\infty + \frac{\delta^5}{t}\} \ , \end{aligned}$$

because of $\delta^5 \leq 5!e^\delta$.

Finally, we obtain from (6.23), our last estimate, $\delta = t^{2/11} \leq t$, and the definition of t

$$\begin{aligned} \|\rho-\tilde{\rho}\|_\infty &\leq c\{e^{(4R_2+1)(t+\delta)}\|S_\rho - S_{\tilde{\rho}}\|_\infty + \frac{\delta^5}{t} + \frac{1}{\sqrt{\delta}}\} \\ &\leq c\{e^{(8R_2+2)t}\|S_\rho - S_{\tilde{\rho}}\|_\infty + \frac{2}{t^{1/11}}\} \\ &\leq c\Big\{(\|S_\rho - S_{\tilde{\rho}}\|_\infty)^{1/11} + (-\ln\|S_\rho - S_{\tilde{\rho}}\|_\infty)^{-1/11}\Big\} \\ &\leq c(-\ln\|S_\rho - S_{\tilde{\rho}}\|_\infty)^{-1/11} \end{aligned}$$

for all ρ, $\tilde{\rho} \in \mathcal{O}$ because $x \leq (-\ln(x))^{-1}$ for $0 < x < 1$, and we have proved the theorem.

□

Replacing $\|S_\rho - S_{\tilde{\rho}}\|$ by $\|U_{\infty,\rho} - U_{\infty,\tilde{\rho}}\|_{\mathcal{F}}$ we obtain the final estimate.

Theorem 6.18 *Let $\rho_0 \in \tilde{C}(B_{R_1}) \cap C^2(\mathbb{R}^3)$ with $R_1 < R$ be given. Then, there are a neighborhood*

$$\mathcal{O} := \{\rho \in \tilde{C}(B_{R_1}) \cap C^2(\mathbb{R}^3) \colon \|\rho - \rho_0\|_{C^2} < \epsilon\} \ ,$$

and a positive constant c, such that for all $\rho,\ \tilde{\rho} \in \mathcal{O}$ the estimate

$$\|\rho - \tilde{\rho}\|_{\infty, B_R} \leq c[-\ln(\|U_{\infty,\rho} - U_{\infty,\tilde{\rho}}\|_{\mathcal{F}})]^{-1/11}$$

holds true.

Proof: We know from Lemma 6.10 that the mapping $\rho \mapsto U_{\infty,\rho}$ is continuous from $\tilde{C}(B_{R_1})$ to the far field patterns equipped with the norm $\|\cdot\|_{\mathcal{F}}$. We choose $\epsilon > 0$ sufficiently small in the proof of Theorem 6.17 such that

$$(1+c)\|U_{\infty,\rho} - U_{\infty,\tilde{\rho}}\|_{\mathcal{F}} \leq 2\epsilon_1 \ \text{ and } \ c\|U_{\infty,\rho} - U_{\infty,\tilde{\rho}}\|_{\mathcal{F}} \leq \|U_{\infty,\rho} - U_{\infty,\tilde{\rho}}\|_{\mathcal{F}}^{1/2}$$

are satisfied for all $\rho,\ \tilde{\rho} \in \mathcal{O}$, too (here c denotes the constant from Lemma 6.13 (b)). Inserting the estimate

$$\|S_\rho - S_{\tilde{\rho}}\|_\infty \leq c\|U_{\infty,\rho} - U_{\infty,\tilde{\rho}}\|_{\mathcal{F}}$$

from Lemma 6.13 (b) into Theorem 6.17 we complete the proof of the theorem.

□

6.3 Appendix: The Elastic Single-Layer Potential

This appendix contains the regularity results and the jump relations for the single-layer potential

$$U(x) = \int_{\partial D} \Pi(x-y)\varphi(y)ds(y) \ , \ x \in \mathbb{R}^3 \setminus \partial D \ ,$$

where D is a C^2-smooth bounded domain in $\mathbb{R}^3$, φ is a uniformly Hölder continuous vector field on ∂D, and $\Pi\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{C}^{3\times 3}$ denotes the fundamental solution to the operator $(\Delta^* + \omega^2 I)$ from section 5.1.

We closely follow the reasoning in [6, chapter 2] for the acoustic single-layer potential and we base our analysis on the next two theorems whose proofs can be found in [6, Theorem 2.7, Remark 2.8, Lemma 2.10, Remark 2.11].

Theorem 6.19 *Let $D \subset \mathbb{R}^3$ be a C^2-smooth, bounded, open set, and suppose G is a closed domain containing ∂D in its interior. Assume the function K is defined and continuous for all $x \in G$, $y \in \partial D$, $x \neq y$, and assume there exists a positive constant M such that for all $x \in G$, $y \in \partial D$, $x \neq y$, we have*

$$|K(x,y)| \le M|x-y|^{-1} \ . \tag{6.25}$$

Assume further that there exists $m \in \mathbb{N}$ such that

$$|K(x_1,y) - K(x_2,y)| \le M \sum_{j=1}^{m} |x_1 - y|^{-1-j}|x_1 - x_2|^j \tag{6.26}$$

for all x_1, $x_2 \in G$, $y \in \partial D$ with $2|x_1 - x_2| \le |x_1 - y|$. Then the generalized potential u defined by

$$u(x) := \int_{\partial D} K(x,y)\varphi(y)ds(y) \ , \ x \in G \ ,$$

with density $\varphi \in C(\partial D)$ belongs to the Hölder space $C^{0,\gamma}(G)$ for all $\gamma \in (0,1)$ and

$$\|u\|_{0,\gamma,G} \le C_\gamma \|\varphi\|_{\infty,\partial D}$$

for some constant C_γ depending on γ.

The analogous assertion holds true, if G is replaced by ∂D, i.e., the kernel K is only defined for x, $y \in \partial D$, $x \neq y$, and $u(x)$ is only defined for $x \in \partial D$.

Remark: If a function K defined for $x \in B_R$, $y \in \partial D$, $x \neq y$, has first partial derivatives with respect to x for all $y \in \partial D$, and if these partial derivatives are continuous functions in x for each fixed y and satisfy $|\nabla_x K(x,y)| \leq M|x-y|^{-m}$ for all $x \neq y$ for some $m \in \mathbb{N}$, then the inequality (compare (6.26))

$$|K(x_1,y) - K(x_2,y)| \leq 2^m M |x_1 - y|^{-m} |x_1 - x_2|$$

is satisfied for all x_1, $x_2 \in B_R$, $y \in \partial D$ with $2|x_1 - x_2| \leq |x_1 - y|$, $y \neq x_1, x_2$.

This can be inferred from

$$\begin{aligned} |K(x_2,y) - K(x_1,y)| &= \Big| \int_0^1 [\nabla_x K(x_1 + t(x_2 - x_1), y)] \cdot (x_2 - x_1) dt \Big| \\ &\leq \sup_{t \in [0,1]} |\nabla_x K(x_1 + t(x_2 - x_1), y)| |x_2 - x_1| \\ &\leq \frac{M}{(|x_1 - y|/2)^m} |x_2 - x_1| \end{aligned}$$

because $|x_1 + t(x_2 - x_1) - y| \geq |x_1 - y| - |x_2 - x_1| \geq |x_1 - y|/2$ for all $t \in [0,1]$, if $2|x_1 - x_2| \leq |x_1 - y|$.

In regard of Theorem 6.19, the above remark, and the estimates from Lemma 5.1 (b) we know that the vector field

$$U(x) = \int_{\partial D} (\Pi(x-y) - \Pi^{(0)}(x-y)) \varphi(y) ds(y) \ , \ x \in \mathbb{R}^3 \ ,$$

belongs to $C^{1,\gamma}(\mathbb{R}^3)$, if $\varphi \in C(\partial D)$ is a continuous vector field. Moreover, we have $\|U\|_{1,\gamma,\mathbb{R}^3} \leq C_\gamma \|\varphi\|_\infty$ and

$$\frac{\partial}{\partial x_j} U(x) = \int_{\partial D} \Big[\frac{\partial}{\partial x_j} (\Pi(x-y) - \Pi^{(0)}(x-y)) \Big] \varphi(y) ds(y) \ , \ x \in \mathbb{R}^3 \ .$$

It is therefore sufficient to carry out the analysis for the single-layer potential having Kelvin's matrix $\Pi^{(0)}$ as kernel.

Theorem 6.19 together with the estimates from Lemma 5.1 (c) immediately imply that the single-layer potential

$$U(x) = \int_{\partial D} \Pi^{(0)}(x-y) \varphi(y) ds(y) \ , \ x \in \mathbb{R}^3 \ , \tag{6.27}$$

with a continuous density φ is uniformly Hölder continuous in $\mathbb{R}^3$ and that $\|U\|_{\gamma,\mathbb{R}^3} \leq C_\gamma \|\varphi\|_\infty$. Especially, this implies that $S_1: C(\partial D) \to C^{0,\gamma}(\partial D)$ is bounded and $S_1: C^{0,\gamma}(\partial D) \to C^{0,\gamma}(\partial D)$ is compact. Here, S_1 denotes the operator defined by

$$(S_1\varphi)(x) = 2 \int_{\partial D} \Pi(x-y)\varphi(y)ds(y)\ ,\ x \in \partial D\ ,$$

which occurred in the previous section.

Hence, it remains to study the first derivatives of the potential in (6.27). To this end another result [6, Lemma 2.10] is useful. In order to understand its notation we note the following facts:

For a C^2-smooth domain D there is number $h_0 > 0$ such that to each point x from the closed neighborhood

$$D_{h_0} := \{x = z + h\nu(z) : z \in \partial D\ ,\ |h| \leq h_0\ \}$$

of ∂D there corresponds a unique point $z \in \partial D$ such that $x = z + h\nu(z)$. Moreover, there is a small constant $R > 0$ such that for all $z_1,\ z_2 \in \partial D$, $x_1 = z_1 + h\nu(z_1)$, $x_2 = z_2 + h\nu(z_2) \in D_{h_0}$ with $0 < |x_1 - x_2| < R/4$ the estimates

$$\frac{1}{2}|x_1 - x_2| \leq |z_1 - z_2| \leq 2|x_1 - x_2|\ ,\ |x_1 - z_2|^2 \geq \frac{|z_1 - z_2|^2 + h^2}{2} \qquad (6.28)$$

hold true. We also assume R sufficiently small to ensure the following two requirements:

$$S_{z,R} := \{y \in \partial D : |y - z| < R\}$$

is connected for each $z \in \partial D$; $\nu(z) \cdot \nu(y) \geq 1/2$ for all $z,\ y \in \partial D$ with $|z - y| \leq R$.

Theorem 6.20 *Let $D \subset \mathbb{R}^3$ be a C^2-smooth, bounded, open set. Assume the function K to be defined and continuous for all $x \in D_{h_0}$, $y \in \partial D$, $x \neq y$, and assume that there exists a positive constant M such that for all $x \in D_{h_0}$, $y \in \partial D$, $x \neq y$, we have*

$$|K(x,y)| \leq M|x - y|^{-2}\ . \qquad (6.29)$$

Furthermore, assume there exists $m \in \mathbb{N}$ such that

$$|K(x_1,y) - K(x_2,y)| \leq M \sum_{j=1}^{m} |x_1 - y|^{-2-j}|x_1 - x_2|^j \qquad (6.30)$$

for all x_1, $x_2 \in D_{h_0}$, $y \in \partial D$ *with* $2|x_1 - x_2| \le |x_1 - y|$, *and that*

$$\Big| \int\limits_{\partial D \setminus S_{z,r}} K(x,y) ds(y) \Big| \le M \tag{6.31}$$

for all $z \in \partial D$ *and* $x = z + h\nu(z) \in D_{h_0}$ *and all* $0 < r < R$. *Now define*

$$u(x) := \int\limits_{\partial D} K(x,y)[\varphi(y) - \varphi(z)] ds(y) \ , \ x \in D_{h_0} \ ,$$

with density $\varphi \in C^{0,\gamma}(\partial D)$, $0 < \gamma < 1$. *Then* u *belongs to* $C^{0,\gamma}(D_{h_0})$ *and*

$$\|u\|_{0,\gamma,D_{h_0}} \le C \|\varphi\|_{0,\gamma,\partial D}$$

for some constant C.

The analogous assertion holds true, if D_{h_0} *is replaced by* ∂D, *i.e., the kernel* K *is only defined for* x, $y \in \partial D$, $x \ne y$, *and* $u(x)$ *is only defined for* $x \in \partial D$.

These two theorems are employed by the authors in [6, Theorems 2.12 and 2.17] to analyze the potential with kernel

$$K(x,y) = \frac{1}{4\pi |x-y|} \ , \ x \ne y \ .$$

They obtain that, if $\varphi \in C(\partial D)$ is continuous, then

$$u(x) = \int\limits_{\partial D} \frac{1}{4\pi|x-y|} \varphi(y) ds(y) \ , \ x \in \mathbb{R}^3 \ ,$$

is uniformly Hölder continuous in $\mathbb{R}^3$ and

$$\|u\|_{0,\gamma,\mathbb{R}^3} \le C_\gamma \|\varphi\|_{\infty,\partial D}$$

for all $0 < \gamma < 1$ and some constant C_γ depending on ∂D and γ. Moreover, if $\varphi \in C^{0,\gamma}(\partial D)$, $0 < \gamma < 1$, is uniformly Hölder continuous, then the first derivatives of the potential u can be uniformly extended in a Hölder continuous fashion from $\mathbb{R}^3 \setminus \overline{D}$ into $\mathbb{R}^3 \setminus D$ and from D into $\overline{D}$ with limiting values

$$\nabla u_\pm(x) = \int\limits_{\partial D} \nabla_x \Big(\frac{1}{4\pi|x-y|} \Big) \varphi(y) ds(y) \mp \frac{1}{2} \varphi(x) \nu(x) \ , \ x \in \partial D \ , \tag{6.32}$$

where the integral exists as a Cauchy principal value. They also have the estimates

$$\begin{aligned} \|\nabla u\|_{0,\gamma,\mathbb{R}^3\setminus D} &\leq C_\gamma \|\varphi\|_{0,\gamma,\partial D} \,, \\ \|\nabla u\|_{0,\gamma,\overline{D}} &\leq C_\gamma \|\varphi\|_{0,\gamma,\partial D} \end{aligned} \tag{6.33}$$

for some constant C_γ depending on ∂D and γ.

Since Kelvin's matrix $\Pi^{(0)}$ has the entries

$$\Pi^{(0)}_{jk}(x) := \frac{\delta_{jk}}{4\pi\mu|x|} - \frac{\lambda+\mu}{8\pi\mu(2\mu+\lambda)} \frac{\partial^2 |x|}{\partial x_j \partial x_k} \ , \ x \neq 0 \ , \ j,k = 1,2,3 \ ,$$

in view of the above results it remains to show the analogous results for the second term in the definition of $\Pi^{(0)}_{jk}(x)$, i.e., for the kernels

$$K(x,y) = \frac{\partial^2 |x-y|}{\partial x_j \partial x_k} \ , \ x \neq y \ , \ j,k = 1,2,3 \ .$$

We will pursue the same strategy that is used in [6] for the harmonic single-layer potential. Therefore, our first aim is to extend in a Hölder continuous fashion the first derivatives of the potential with constant density

$$u(x) = \int\limits_{\partial D} \frac{\partial^2 |x-y|}{\partial x_j \partial x_k} ds(y) \ , \ x \in \mathbb{R}^3 \ ,$$

from D into $\overline{D}$ and from $\mathbb{R}^3 \setminus \overline{D}$ into $\mathbb{R}^3 \setminus D$. To this end we split

$$\begin{aligned} \nabla u(x) &= -\int\limits_{\partial D} \operatorname{Grad}_y \frac{\partial^2 |x-y|}{\partial x_j \partial x_k} ds(y) \\ &\quad - \int\limits_{\partial D} \frac{\partial}{\partial \nu(y)} \frac{\partial^2 |x-y|}{\partial x_j \partial x_k} \nu(y) ds(y) \ , \ x \in \mathbb{R}^3 \setminus \partial D \ , \end{aligned} \tag{6.34}$$

where Grad denotes the surface gradient on ∂D. We are thus lead to consider potentials with kernel

$$K(x,y) := \frac{\partial}{\partial \nu(y)} \frac{\partial^2 |x-y|}{\partial x_j \partial x_k} \ , \ y \in \partial D \ , \ x \in \mathbb{R}^3 \ , \ x \neq y \ , \tag{6.35}$$

and with Hölder continuous density.

Lemma 6.21 *Assume j, $k \in \{1,2,3\}$ to be fixed and define $K(x,y)$, $y \in \partial D$, $x \in \mathbb{R}^3$, $x \neq y$, as in (6.35). Then, the generalized potential*

$$u(x) := \int_{\partial D} K(x,y)\varphi(y)ds(y) \ , \ x \in \mathbb{R}^3 \setminus \partial D \ ,$$

with Hölder continuous density $\varphi \in C^{0,\gamma}(\partial D)$ $(0 < \gamma < 1)$ can be extended in a Hölder continuous fashion from $\mathbb{R}^3 \setminus \overline{D}$ into $\mathbb{R}^3 \setminus D$ and from D into $\overline{D}$ with limiting values

$$u_\pm(x) = \int_{\partial D} K(x,y)\varphi(y)ds(y) \pm 4\pi\varphi(x)\nu_k(x)\nu_j(x) \ , \ x \in \partial D \ , \tag{6.36}$$

where the integral exists as a Cauchy principal value. Furthermore, the estimates

$$\begin{aligned} \|u\|_{0,\gamma,\mathbb{R}^3\setminus D} &\leq C_\gamma \|\varphi\|_{0,\gamma,\partial D} \ , \\ \|u\|_{0,\gamma,\overline{D}} &\leq C_\gamma \|\varphi\|_{0,\gamma,\partial D} \end{aligned} \tag{6.37}$$

hold true for some constant C_γ depending on ∂D and γ.

Proof: We first prove the assertion for the density $\varphi = 1$, i.e., we examine

$$w(x) := \int_{\partial D} \frac{\partial}{\partial \nu(y)} \frac{\partial^2}{\partial x_j \partial x_k} |x-y| ds(y) \ , \ x \in \mathbb{R}^3 \setminus \partial D \ .$$

For $x \in D$ we choose $\epsilon > 0$ sufficiently small to ensure $\overline{B_\epsilon(x)} \subset D$. Then we apply Green's second theorem (1.9) and Gauss' theorem (1.7) in $D \setminus \overline{B_\epsilon(x)}$, use $\Delta_y |x-y| = 2/|x-y|$, and obtain

$$\begin{aligned} w(x) &= \int_{|y-x|=\epsilon} \frac{\partial}{\partial \nu(y)} \frac{\partial^2}{\partial x_j \partial x_k} |x-y| ds(y) + 2 \int_{D\setminus B_\epsilon(x)} \frac{\partial^2}{\partial x_j \partial x_k} \frac{1}{|x-y|} dy \\ &= \int_{|y-x|=\epsilon} \frac{\partial}{\partial \nu(y)} \frac{\partial^2}{\partial x_j \partial x_k} |x-y| ds(y) + 2 \int_{\partial D} \nu_j(y) \frac{\partial}{\partial y_k} \frac{1}{|x-y|} ds(y) \\ &\quad -2 \int_{|y-x|=\epsilon} \nu_j(y) \frac{\partial}{\partial y_k} \frac{1}{|x-y|} ds(y) \ . \end{aligned} \tag{6.38}$$

We calculate

$$
\begin{aligned}
&\frac{\partial}{\partial\nu(y)}\frac{\partial^2}{\partial x_j\partial x_k}|x-y| \\
&= \frac{\partial}{\partial\nu(y)}\Big[\frac{\delta_{jk}}{|x-y|}-\frac{(x_k-y_k)(x_j-y_j)}{|x-y|^3}\Big] \\
&= \frac{\partial}{\partial\nu(y)}\frac{\delta_{jk}}{|x-y|}+\frac{\nu_k(y)(x_j-y_j)}{|x-y|^3} \\
&\quad +\frac{\nu_j(y)(x_k-y_k)}{|x-y|^3}-3\frac{(x_k-y_k)(x_j-y_j)\{\nu(y)\cdot(x-y)\}}{|x-y|^5}\ ,
\end{aligned}
$$

and note

$$
\frac{\nu_k(y)(x_j-y_j)}{|x-y|^3}+\frac{\nu_j(y)(x_k-y_k)}{|x-y|^3}=2\nu_j(y)\frac{\partial}{\partial y_k}\frac{1}{|x-y|}
$$

on $\partial B_\epsilon(x)$ due to $\nu(y)=(y-x)/|x-y|$. Moreover, we have

$$
\int\limits_{|y-x|=\epsilon}\frac{\partial}{\partial\nu(y)}\frac{\delta_{jk}}{|x-y|}ds(y)=-\int\limits_{|y-x|=\epsilon}\frac{\delta_{jk}}{\epsilon^2}ds(y)=-4\pi\delta_{jk}
$$

and

$$
\begin{aligned}
&-3\int\limits_{|y-x|=\epsilon}\frac{(x_k-y_k)(x_j-y_j)\{\nu(y)\cdot(x-y)\}}{|x-y|^5}ds(y) \\
&= \frac{3}{\epsilon^3}\int\limits_{\partial B_\epsilon(0)} y_k\nu_j(y)ds(y) \\
&= 4\pi\delta_{jk}\ .
\end{aligned}
$$

Hence, (6.38) reads

$$
w(x)=-2\frac{\partial}{\partial x_k}\int\limits_{\partial D}\frac{\nu_j(y)}{|x-y|}ds(y)\ ,\ x\in D\ .
$$

For $x\in \mathbb{R}^3\setminus\overline{D}$ it is not necessary to cut out the ball $B_\epsilon(x)$ and a similar reasoning yields

$$
w(x)=-2\frac{\partial}{\partial x_k}\int\limits_{\partial D}\frac{\nu_j(y)}{|x-y|}ds(y)\ .
$$

Finally, for $x \in \partial D$, we obtain in an analogous manner that

$$\begin{aligned}
&\int\limits_{\partial D\setminus S_{x,\epsilon}} K(x,y)ds(y) \\
&\quad = \int\limits_{\partial(D\setminus B_\epsilon(x))} K(x,y)ds(y) + \int\limits_{\partial B_\epsilon(x)\cap D} K(x,y)ds(y) \\
&\quad = -2\frac{\partial}{\partial x_k}\int\limits_{\partial(D\setminus B_\epsilon(x))} \frac{\nu_j(y)}{|x-y|}ds(y) + \int\limits_{\partial B_\epsilon(x)\cap D} K(x,y)ds(y) \\
&\quad \to -2\frac{\partial}{\partial x_k}\int\limits_{\partial D}\frac{\nu_j(y)}{|x-y|}ds(y)\ ,\ \epsilon\to 0\ ,
\end{aligned}$$

uniformly for all $x \in \partial D$, where the integral is to be understood as a Cauchy principal value. In the limit the integrals over $\partial B_\epsilon(x) \cap D$ of

$$\frac{\partial}{\partial\nu(y)}\frac{\delta_{jk}}{|x-y|} \text{ and } -3\frac{(x_k-y_k)(x_j-y_j)\{\nu(y)\cdot(x-y)\}}{|x-y|^5}$$

can be replaced by integrals over $\partial B_\epsilon(x) \cap \{y \in \mathbb{R}^3 \colon (y-x)\cdot\nu(x) \le 0\}$. By symmetry the integrals over the half sphere can be evaluated as one half of the corresponding integrals over the entire sphere, whence they cancel each other as they did in the case $x \in D$.

Using the jump relations for the derivatives of the harmonic single-layer potential we have proved the assertion for $\varphi = 1$.

Our computation of $K(x,y)$ reveals that K satisfies assumption (6.29). Condition (6.30) can be verified with the help of the remark after Theorem 6.19. In order to verify condition (6.31) we choose $x = z \pm h\nu(z) \in D_{h_0}$, $z \in \partial D$, $h \ge 0$. If $x = z + h\nu(z)$, we work in the domain $D \setminus B_\epsilon(z)$, if $x = z - h\nu(z)$, we work in the domain $D \cup B_\epsilon(z)$. Then we proceed as above for the case $x \in \partial D$. With the help of the estimate

$$\begin{aligned}
|z+h\nu(z)-y|^2 &\ge \min_{y\in\partial B_\epsilon(z)\cap\partial D}\{|z-y|^2+h^2+2h\nu(z)\cdot(y-z)\} \\
&= \min_{y\in\partial B_\epsilon(z)\cap\partial D}|z+h\nu(z)-y|^2 \\
&\ge \frac{1}{2}\epsilon^2
\end{aligned}$$

for all $y \in \partial B_\epsilon(z) \cap \overline{D}$, where we have used (6.28) for the second inequality, we can bound

$$
\begin{aligned}
\Big| \int\limits_{\partial D \setminus S_{z,\epsilon}} & K(x,y) ds(y) \Big| \\
&\leq \Big| \int\limits_{\partial (D \setminus B_\epsilon(z))} K(x,y) ds(y) \Big| + \int\limits_{\partial B_\epsilon(z) \cap D} |K(x,y)| ds(y) \\
&\leq \Big| 2 \frac{\partial}{\partial x_j} \int\limits_{\partial (D \setminus B_\epsilon(z))} \frac{\nu_j(y)}{|x-y|} ds(y) \Big| + \int\limits_{\partial B_\epsilon(z)} \frac{c}{\epsilon^2} ds(y) \\
&\leq M \ ,
\end{aligned}
$$

and similarly for $x = z - h\nu(z)$.

If we now have an arbitrary uniformly Hölder continuous density φ, we can split

$$
\begin{aligned}
u(x) &= \int\limits_{\partial D} K(x,y) \varphi(y) ds(y) \\
&= \int\limits_{\partial D} K(x,y)(\varphi(y) - \varphi(z)) ds(y) + w(x)\varphi(z) \ , \ x \in D_{h_0} \ ,
\end{aligned}
$$

where $z \in \partial D$ is the point with $x = z + h\nu(z)$. Since K satisfies the conditions of Theorem 6.20, the integral is uniformly Hölder continuous in D_{h_0}, whereas the behavior of the second term and the jump relation follow from our reasoning for $\varphi = 1$. This ends the proof of the lemma.

□

Next, we turn to the first integral that appeared in (6.34).

Lemma 6.22 *Assume j, $k \in \{1,2,3\}$ to be fixed and define the vector field*

$$
U(x) := \int\limits_{\partial D} \mathrm{Grad}_y \frac{\partial^2}{\partial x_j \partial x_k} |x-y| ds(y) \ , \ x \in \mathbb{R}^3 \setminus \partial D \ .
$$

Then, U can be extended in a Hölder continuous fashion into $\mathbb{R}^3$ with limiting values

$$
U(x) = \int\limits_{\partial D} \mathrm{Grad}_y \frac{\partial^2}{\partial x_j \partial x_k} |x-y| ds(y) \ , \ x \in \partial D \ , \tag{6.39}
$$

where the integral exists as a Cauchy principal value.

Proof: Integration by parts (compare [6, Theorem 2.1]) reveals

$$U(x) = -2 \int_{\partial D} \frac{\partial^2}{\partial x_j \partial x_k} |x-y| H(y) \nu(y) ds(y) \ , \ x \in \mathbb{R}^3 \setminus \partial D \ ,$$

where $H(y)$ denotes the mean curvature of ∂D at the point $y \in \partial D$. Since the kernel

$$\frac{\partial^2}{\partial x_j \partial x_k} |x-y| = \frac{\delta_{jk}}{|x-y|} - \frac{(x_k - y_k)(x_j - y_j)}{|x-y|^3} \ , \ x, y \in \mathbb{R}^3 \ , \ x \neq y \ ,$$

satisfies the assumptions of Theorem 6.19, we infer that U can be extended in a uniformly Hölder continuous fashion into $\mathbb{R}^3$ with limiting values

$$U(x) = -2 \int_{\partial D} \frac{\partial^2}{\partial x_j \partial x_k} |x-y| H(y) \nu(y) ds(y) \ , \ x \in \partial D \ ,$$

and it remains to prove for $x \in \partial D$ that

$$\lim_{\epsilon \to 0} \int_{\partial D \setminus S_{x,\epsilon}} \mathrm{Grad}_y \frac{\partial^2}{\partial x_j \partial x_k} |x-y| ds(y) = -2 \int_{\partial D} \frac{\partial^2}{\partial x_j \partial x_k} |x-y| H(y) \nu(y) ds(y) \ .$$

This can be achieved by using integration by parts again [6, Theorem 2.1],

$$\begin{aligned} \int_{\partial D \setminus S_{x,\epsilon}} \mathrm{Grad}_y \frac{\partial^2 |x-y|}{\partial x_j \partial x_k} ds(y) &= -2 \int_{\partial D \setminus S_{x,\epsilon}} \frac{\partial^2 |x-y|}{\partial x_j \partial x_k} H(y) \nu(y) ds(y) \\ &\quad - \int_{\partial D \cap \partial B_\epsilon(x)} \frac{\partial^2 |x-y|}{\partial x_j \partial x_k} \nu_0(y) dt(y) \ . \end{aligned}$$

Here, $\nu_0(y)$ denotes the unit normal vector to the curve $\partial D \cap \partial B_\epsilon(x)$ which is orthogonal to the normal $\nu(y)$ of ∂D and which is directed into the exterior of $B_\epsilon(x)$, and dt is the line element. Next we compute

$$\begin{aligned} &\int_{\partial D \cap \partial B_\epsilon(x)} \frac{\partial^2 |x-y|}{\partial x_j \partial x_k} \nu_0(y) dt(y) \\ &= \frac{\delta_{jk}}{\epsilon} \int_{\partial D \cap \partial B_\epsilon(x)} \nu_0(y) dt(y) - \frac{1}{\epsilon^3} \int_{\partial D \cap \partial B_\epsilon(x)} (x_k - y_k)(x_j - y_j) \nu_0(y) dt(y) \end{aligned}$$

$$
\begin{aligned}
&= 2\frac{\delta_{jk}}{\epsilon}\int\limits_{S_{x,\epsilon}} H(y)\nu(y)ds(y) - \frac{1}{\epsilon^3}\int\limits_{S_{x,\epsilon}} 2(x_k-y_k)(x_j-y_j)H(y)\nu(y)ds(y) \\
&\quad -\frac{1}{\epsilon^3}\int\limits_{S_{x,\epsilon}} \mathrm{Grad}_y[(x_k-y_k)(x_j-y_j)]ds(y) \\
&= O(\epsilon) - \frac{1}{\epsilon^3}\int\limits_{S_{x,\epsilon}} \mathrm{Grad}_y[(x_k-y_k)(x_j-y_j)]ds(y)\ ,
\end{aligned}
$$

where we made use of the estimates

$$
\int\limits_{S_{x,\epsilon}} |x-y|^m ds(y) \le c\epsilon^{m+2}
$$

for a suitable constant c depending only on ∂D. From

$$
\begin{aligned}
&\mathrm{Grad}_y[(x_k-y_k)(x_j-y_j)] \\
&\quad = (1-\nu(y)\nu(y)^T)\nabla_y[(x_k-y_k)(x_j-y_j)] \\
&\quad = (1-\nu(x)\nu(x)^T)\nabla_y[(x_k-y_k)(x_j-y_j)] + O(|x-y|^2)
\end{aligned}
$$

we conclude that the last term behaves like

$$
\begin{aligned}
&\frac{1}{\epsilon^3}\int\limits_{S_{x,\epsilon}} (1-\nu(x)\nu(x)^T)\nabla_y[(x_k-y_k)(x_j-y_j)]ds(y) + O(\epsilon) \\
&\quad = \frac{1}{\epsilon^3}\int\limits_{E_{x,\epsilon}} (1-\nu(x)\nu(x)^T)\nabla_y[(x_k-y_k)(x_j-y_j)]ds(y) + O(\epsilon) \\
&\quad = \frac{1}{\epsilon^3}\int\limits_{B_\epsilon(x)\cap E} (1-\nu(x)\nu(x)^T)\nabla_y[(x_k-y_k)(x_j-y_j)]ds(y) + O(\epsilon) \\
&\quad = O(\epsilon)\ ,\ \epsilon \to 0\ .
\end{aligned}
$$

Here, E denotes the tangential plane at $x \in \partial D$, and $E_{x,\epsilon}$ is the orthogonal projection of $S_{x,\epsilon}$ into E. The integral over $B_\epsilon(x) \cap E$ vanishes by symmetry because linear functions like $z = (z_1, z_2, z_3) \in \mathbb{R}^3 \mapsto z_k$ are integrated over a two-dimensional disk centered at the origin. This completes the proof of the lemma.

□

We are now prepared to study the single-layer potential having Kelvin's matrix or Kupradze's matrix as kernel.

Theorem 6.23 *Let $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in C^{0,\gamma}(\partial D)$ $(0 < \gamma < 1)$ be a uniformly Hölder continuous vector field. Then, the single-layer potential*

$$U(x) := \int_{\partial D} \Pi^{(0)}(x-y)\varphi(y)ds(y) \ , \ x \in \mathbb{R}^3 \setminus \partial D \ ,$$

is uniformly Hölder continuous in $\mathbb{R}^3$. The first derivatives of the components U_j can be extended in a Hölder continuous fashion from D into $\overline{D}$ and from $\mathbb{R}^3 \setminus \overline{D}$ into $\mathbb{R}^3 \setminus D$ with limiting values

$$\begin{aligned} \partial_l U_{j\pm}(x) &= d_j \cdot \int_{\partial D} (\partial_l \Pi^{(0)})(x-y)\varphi(y)ds(y) \\ &\quad \mp \frac{1}{2\mu}\Big\{\nu_l(x)\varphi_j(x) - \frac{\lambda+\mu}{2\mu+\lambda}[\nu(x)\cdot\varphi(x)]\nu_l(x)\nu_j(x)\Big\} \end{aligned} \tag{6.40}$$

for $x \in \partial D$, j, $l = 1,2,3$, where the integral exists as a Cauchy principal value. d_1, d_2, d_3 denote the usual cartesian unit vectors. Furthermore, the estimates

$$\begin{aligned} \|U\|_{1,\gamma,\mathbb{R}^3\setminus D} &\leq C_\gamma \|\varphi\|_{0,\gamma,\partial D} \ , \\ \|U\|_{1,\gamma,\overline{D}} &\leq C_\gamma \|\varphi\|_{0,\gamma,\partial D} \end{aligned} \tag{6.41}$$

hold true for some constant C_γ depending on ∂D and γ. The analogous assertion is true, if $\Pi^{(0)}$ is replaced by Π.

Proof: We already observed before Theorem 6.20 that it suffices to carry out the analysis for Kelvin's matrix because the difference of Kelvin's and Kupradze's matrix is sufficiently regular to apply Theorem 6.19. This theorem also implied the Hölder continuity of U. Moreover, since the single-layer potential for the Laplace equation and its derivatives are rigorously analyzed in [6] (see the remarks after Theorem 6.20), we confine ourselves to the first derivatives of the functions

$$u(x) = \int_{\partial D} \frac{\partial^2 |x-y|}{\partial x_j \partial x_k}\psi(y)ds(y) \ , \ x \in \mathbb{R}^3 \setminus \partial D \ ,$$

with a uniformly Hölder continuous density ψ and j, k fixed. To this end we split in D_{h_0}

$$\partial_l u(x) = \int_{\partial D} \frac{\partial^3 |x-y|}{\partial x_j \partial x_k \partial x_l}(\psi(y) - \psi(z))ds(y)$$

$$+\psi(z)\frac{\partial}{\partial x_l}\int\limits_{\partial D}\frac{\partial^2|x-y|}{\partial x_j\partial x_k}ds(y)\ ,\ x\in D_{h_0}\setminus\partial D\ ,$$

with $x = z + h\nu(z)$.

Our reasoning during the proofs of the two previous lemmas implies that the kernel

$$K(x,y)=\frac{\partial^3|x-y|}{\partial x_j\partial x_k\partial x_l}$$

satisfies the assumption (6.31). It also satisfies the other assumptions of Theorem 6.20, whence the first integral represents a uniformly Hölder continuous function in D_{h_0}. Our analysis of the second term during the previous lemmas implies the Hölder continuity of this term in $\mathbb{R}^3\setminus D$ and in $\overline{D}$. Furthermore, the Hölder norms of $\partial_l u|_{\overline{D}}$ and of $\partial_l u|_{\mathbb{R}^3\setminus D}$ depend continuously on $\|\psi\|_{0,\gamma,\partial D}$. Finally, the relations (6.36) and (6.39) from Lemmas 6.21 and 6.22 allow to compute the values on the boundary:

$$\partial_l u_\pm(x)=\int\limits_{\partial D}\frac{\partial^3|x-y|}{\partial x_j\partial x_k\partial x_l}\psi(y)ds(y)\mp 4\pi\nu_j(x)\nu_k(x)\nu_l(x)\psi(x)\ ,\ x\in\partial D\ ,$$

the integral being a Cauchy principal value.

Now, the regularity result and the jump relation for derivatives of the harmonic single-layer potential (6.32) together with

$$\Pi^{(0)}_{jk}(x):=\frac{\delta_{jk}}{4\pi\mu|x|}-\frac{\lambda+\mu}{8\pi\mu(2\mu+\lambda)}\frac{\partial^2|x|}{\partial x_j\partial x_k}\ ,\ x\neq 0\ ,\ j,k=1,2,3\ ,$$

imply the assertion of the theorem.

□

We needed the above regularity result during the derivation of the stability result for the inverse elastic scattering problem, but we did not use the values of $\partial_j U_l$ of the single-layer potential U on the boundary ∂D in general. However, we had to know the values of a certain linear combination of the values $\partial_j U_l$ on ∂D, namely the values of the traction of U. We remind the reader that the traction was defined as

$$[T(U,\nu)](x):=(\beta_1+\mu)\frac{\partial U}{\partial\nu}(x)+\beta_2(\nabla\cdot U)(x)\nu(x)+\beta_1\nu(x)\wedge(\nabla\wedge U(x))$$

for $x \in \partial D$, where the real constants β_1, β_2 satisfy $\beta_1 + \beta_2 = \lambda + \mu$. Relation (5.7) allows to rewrite the lth component of the above vector as

$$[T(U,\nu)]_l(x) = \mu \frac{\partial U_l}{\partial \nu}(x) + \beta_2 (\nabla \cdot U)(x)\nu_l(x) + \beta_1 \sum_{m=1}^{3} \nu_m(x) \frac{\partial U_m}{\partial x_l}(x) \ . \quad (6.42)$$

This relation, together with the jump relation of the preceding theorem allows to derive the following jump relation for the traction of the elastic single-layer potential. By $\Xi(y,x)^T$ we denote the matrix having as columns

$$\Xi(y,x)^T d_j = T_x(\Pi(x-y)d_j, \nu(x)) \ , \ x, y \in \partial D \ ,$$

i.e., the jth column of this matrix consists of the traction (with respect to x) applied to the jth column of the fundamental solution $\Pi(x-y)$. This actually is the transpose of the matrix $\Xi(x,y)$ defined in section 5.1 with its arguments x and y interchanged.

Lemma 6.24 *Let $\varphi \in C^{0,\gamma}(\partial D)$ $(0 < \gamma < 1)$ be a uniformly Hölder continuous vector field. Then, the traction from the interior and exterior of the single-layer potential*

$$U(x) := \int_{\partial D} \Pi(x-y)\varphi(y)ds(y) \ , \ x \in \mathbb{R}^3 \setminus \partial D \ ,$$

can be computed on ∂D as the uniformly Hölder continuous vector fields

$$TU_\pm(x) = \int_{\partial D} \Xi^T(y,x)\varphi(y)ds(y) \mp \frac{1}{2}\varphi(x) \ , \ x \in \partial D \ , \quad (6.43)$$

where the integral exists as a Cauchy principal value. The analogous assertion is true, if Π is replaced by $\Pi^{(0)}$.

Our last result states that the integral operator appearing in (6.43) is compact in $C^{0,\gamma}(\partial D)$ provided β_1 and β_2 are chosen in a special way.

Lemma 6.25 *For*

$$\beta_1 := \frac{\mu(\lambda+\mu)}{\lambda+3\mu} \ , \ \beta_2 := \frac{(\lambda+\mu)(\lambda+2\mu)}{\lambda+3\mu} \quad (6.44)$$

the operator $K_1': C^{0,\gamma}(\partial D) \to C^{0,\gamma}(\partial D)$

$$(K_1'\varphi)(x) := 2 \int\limits_{\partial D} \Xi^T(y,x)\varphi(y)ds(y) \ , \ x \in \partial D \ , \tag{6.45}$$

has a weakly singular kernel and is a compact operator. The analogous assertion is true, if Π *is replaced by* $\Pi^{(0)}$.

Proof: Theorem 6.19 and the estimates from Lemma 5.1 (b) imply that the operator

$$\varphi \mapsto 2 \int\limits_{\partial D} (\Xi^T(y,\cdot) - [\Xi^{(0)}(y,\cdot)]^T)\varphi(y)ds(y)$$

is bounded and linear from $C(\partial D)$ to $C^{0,\gamma}(\partial D)$ for each choice of β_1 and β_2. Hence, we have to analyze the entries of the matrix $[\Xi^{(0)}(y,x)]^T$, if β_1 and β_2 are chosen as in (6.44), i.e., we have to compute the kth entry of the vector $T_x(\Pi^{(0)}(x-y)d_j, \nu(x))$. In regard of relation (6.42) we start with

$$\begin{aligned} &\Big[T_x\Big(\frac{1}{2\pi\mu|x-y|}d_j, \nu(x)\Big)\Big]_k \\ &= \ -\frac{\delta_{jk}}{2\pi}\frac{\nu(x)\cdot(x-y)}{|x-y|^3} - \frac{\beta_2}{2\pi\mu}\frac{x_j-y_j}{|x-y|^3}\nu_k(x) - \frac{\beta_1}{2\pi\mu}\frac{x_k-y_k}{|x-y|^3}\nu_j(x) \ . \end{aligned} \tag{6.46}$$

In order to compute the traction of $\nabla\partial_j|x-y|$ we use the original definition of the traction operator, the relation $\Delta_x|x-y| = 2/|x-y|$, and obtain for the kth entry

$$\begin{aligned} &\Big[T_x\Big(\nabla_x\frac{\partial}{\partial x_j}|x-y|, \nu(x)\Big)\Big]_k \\ &= \ (\beta_1+\mu)\Big(\frac{\partial}{\partial\nu(x)}\frac{\partial^2}{\partial x_j\partial x_k}|x-y|\Big)_k - 2\beta_2\frac{x_j-y_j}{|x-y|^3}\nu_k(x) \\ &= \ (\beta_1+\mu)\Big[-\delta_{jk}\frac{\nu(x)\cdot(x-y)}{|x-y|^3} - \frac{x_j-y_j}{|x-y|^3}\nu_k(x) - \frac{x_k-y_k}{|x-y|^3}\nu_j(x) \\ &\qquad +3\frac{(x_k-y_k)(x_j-y_j)[\nu(x)\cdot(x-y)]}{|x-y|^5}\Big] \\ &\quad -2\beta_2\frac{x_j-y_j}{|x-y|^3}\nu_k(x) \ . \end{aligned} \tag{6.47}$$

Multiplying (6.47) by $-(\lambda+\mu)/\{4\pi\mu(2\mu+\lambda)\}$ and adding the result to (6.46) finally yields

$$\begin{aligned}
&2\Big[T_x\Big(\Pi^{(0)}(x-y)d_j,\nu(x)\Big)\Big]_k \\
&= -\frac{\delta_{jk}}{2\pi}\Big[1-\frac{(\beta_1+\mu)(\lambda+\mu)}{2\mu(2\mu+\lambda)}\Big]\frac{\nu(x)\cdot(x-y)}{|x-y|^3} \\
&\quad -3\frac{(\beta_1+\mu)(\lambda+\mu)}{4\pi\mu(2\mu+\lambda)}\,\frac{(x_k-y_k)(x_j-y_j)[\nu(x)\cdot(x-y)]}{|x-y|^5} \\
&\quad +\Big[-\frac{\beta_2}{2\pi\mu}+\frac{2\beta_2(\lambda+\mu)}{4\pi\mu(2\mu+\lambda)}+\frac{(\beta_1+\mu)(\lambda+\mu)}{4\pi\mu(2\mu+\lambda)}\Big]\frac{x_j-y_j}{|x-y|^3}\nu_k(x) \\
&\quad +\Big[-\frac{\beta_1}{2\pi\mu}+\frac{(\beta_1+\mu)(\lambda+\mu)}{4\pi\mu(2\mu+\lambda)}\Big]\frac{x_k-y_k}{|x-y|^3}\nu_j(x) \\
&= \delta_{jk}c_1\frac{\nu(x)\cdot(x-y)}{|x-y|^3}+c_2\frac{(x_k-y_k)(x_j-y_j)[\nu(x)\cdot(x-y)]}{|x-y|^5}
\end{aligned} \tag{6.48}$$

with two constants c_1 and c_2 because the coefficients in front of the strongly singular terms $\nu_k(x)(x_j-y_j)/|x-y|^3$ and $\nu_j(x)(x_k-y_k)/|x-y|^3$ vanish due to the special choice of β_1 and β_2.

The integral operator with kernel $\nu(x)\cdot(x-y)/(2\pi|x-y|^3)$ is studied in [6, Theorem 2.30] and is seen to be compact in $C^{0,\gamma}(\partial D)$ in view of Theorem 6.19. Let us point out that the estimates (see [6, Theorem 2.2])

$$\begin{aligned}
|\nu(x)\cdot(x-y)| &\le c|x-y|^2\ ,\ x,y\in\partial D\ ,\\
|\nu(x)-\nu(y)| &\le c|x-y|\ ,\ x,y\in\partial D\ ,
\end{aligned} \tag{6.49}$$

for a C^2-smooth boundary ∂D are crucial in order to verify the assumptions of Theorem 6.19. We proceed analogously for the kernel

$$K(x,y)=\frac{(x_k-y_k)(x_j-y_j)[\nu(x)\cdot(x-y)]}{|x-y|^5}\ ,\ x,y\in\partial D\ ,\ x\neq y\ .$$

It is weakly singular due to (6.49). Moreover, due to (6.49) we have the inequality

$$\begin{aligned}
&\Big|\nu(x)\cdot(x-y)-\nu(z)\cdot(z-y)\Big| \\
&\le \Big|(\nu(x)-\nu(z))\cdot(x-y)\Big|+\Big|\nu(z)\cdot(x-z)\Big| \\
&\le c|x-y||x-z|+c|x-z|^2\ ,\ x,y,z\in\partial D\ ,
\end{aligned}$$

which together with the remark after Theorem 6.19 immediately yields assumption (6.26) for K with $m = 2$. This completes the proof of the lemma. $\square$

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Basic Notation

$\mathbb{Z}$ integers

$\mathbb{N}$ positive integers

$\mathbb{N}_0$ nonnegative integers

$\mathbb{R}$ real numbers

$\mathbb{C}$ complex numbers, $i := \sqrt{-1}$, $z = x + iy$, $x, y \in \mathbb{R}$, $\overline{z} := x - iy$ complex conjugate

$\mathbb{R}^3$ threedimensional Euclidean space with points $x = (x_1, x_2, x_3)$, $x_j \in \mathbb{R}$, $|x| := (\sum_{j=1}^{3} x_j^2)^{1/2}$

$\mathbb{C}^3 := \{(z_1, z_2, z_3) \colon z_j \in \mathbb{C}\}$, $|z| := (\sum_{j=1}^{3} |z_j|^2)^{1/2}$

$a \cdot b := \sum_{j=1}^{3} a_j b_j$ for $a, b \in \mathbb{C}^3$ bilinear form

$a \wedge b := (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$ vector product for $a, b \in \mathbb{C}^3$

$\overline{D}$ closure of a set $D \subset \mathbb{R}^3$, ∂D boundary of D

∂D smooth, $\nu(y)$ unit normal at $y \in \partial D$ (if D is bounded $\nu(y)$ is directed into the exterior of D), ds 2-dimensional area element in ∂D p. 16

$D \setminus D' := \{x \in D \colon x \notin D'\}$

$B_r(x^*) := \{x \in \mathbb{R}^3 \colon |x - x^*| < r\}$ open ball with center $x^* \in \mathbb{R}^3$ and radius $r > 0$, $B_r := B_r(0)$

$S^2 := \{x \in \mathbb{R}^3 \colon |x| = 1\}$

$\partial_j u = \dfrac{\partial u}{\partial x_j} = \partial u / \partial x_j$ partial derivative of the function u

$\nabla u := (\partial_1 u, \partial_2 u, \partial_3 u)$ gradient of u

$\Delta u := \partial_1^2 u + \partial_2^2 u + \partial_3^2 u$ Laplacian of u

$\nabla \cdot U := \partial_1 U_1 + \partial_2 U_2 + \partial_3 U_3$ divergence of a vector field $U = (U_1, U_2, U_3)$

$\nabla \wedge U := (\partial_2 U_3 - \partial_3 U_2, \partial_3 U_1 - \partial_1 U_3, \partial_1 U_2 - \partial_2 U_1)$ curl of a vector field $U = (U_1, U_2, U_3)$

$\Delta^* U := \mu \Delta U + (\lambda + \mu) \nabla \nabla \cdot U$ elasticity operator applied to a vector field $U = (U_1, U_2, U_3)$, μ, λ are the Lamé constants

Function spaces
$D \subset \mathbb{R}^3$

$C(D)$, $C(\overline{D})$ set of continuous functions on D, $\overline{D}$ p. 12

$C^{0,\gamma}(D)$ set of bounded and uniformly Hölder continuous functions on D with exponent $0 < \gamma < 1$ p. 25

$D \subset \mathbb{R}^3$ open

$C^k(D)$ set of functions on D having all derivatives up to order $k \in \mathbb{N}$ continuous in D p. 12

$C^\infty(D) := \bigcap_{k=1}^{\infty} C^k(D)$

$C^k(\overline{D})$ set of functions in $C^k(D)$ all of whose derivatives up to order $k \in \mathbb{N}$ have continuous extensions to $\overline{D}$ p. 12

$C^{1,\gamma}(\overline{D})$ p. 25

$C^{1,\gamma}(\partial D)$ p. 25

$\mathrm{supp}(u)$ support of a function u, closure of the set $\{x : u(x) \neq 0\}$

$C_0^k(D)$ set of functions in $C^k(D)$ with compact support in D ($k \in \mathbb{N}$ or $k = \infty$) p. 12

$L^2(D)$ set of functions on D which are measurable and square-integrable on D with respect to the Lebesgue measure p. 11

$T^{0,\gamma}(\partial B_{R_2})$ set of γ-Hölder continuous tangential fields on ∂B_{R_2} p. 147

$T_d^{0,\gamma}(\partial B_{R_2})$ set of γ-Hölder continuous tangential fields on ∂B_{R_2} posessing a γ-Hölder continuous surface divergence p. 147

Note: If $E = (E_1, E_2, E_3): D \to \mathbb{C}^3$ is a vector-valued function we write $E \in C(D)$ for $E_1, E_2, E_3 \in C(D)$, and similarly for all other function spaces

Γ grid in $\mathbb{R}^3$ p. 11

$\xi = (s, it, 0) \in \mathbb{C}^3$, $s \in \mathbb{R}$, $t > 0$ p. 12

Φ_κ , Ψ_ζ fundamental solutions of the Helmholtz equation p. 17, p. 97

g_ζ fundamental solution of $\Delta + 2i\zeta \cdot \nabla$ p. 52 , G_ζ corresponding solution operator p. 52