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Abstract

We derive the fourth order q-difference equation satisfied by the first associated of the q-classical orthogonal polynomials.

The coefficients of this equation are given in terms of the polynomials σ and τ which appear in the q-Pearson difference equation $D_q(\sigma\,\rho) = \tau\,\rho$ defining the weight ρ of the q-classical orthogonal polynomials inside the q-Hahn tableau.

Keywords: q-Orthogonal polynomials, Fourth order q-difference equation.
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1 Introduction

The fourth order difference equation for the associated polynomials of all classical discrete polynomials were given for all integers r (order of association) in [5], using the properties of the Stieltjes functions of the associated linear forms.
On the other hand, the equation for the first associated ($r = 1$) of all classical discrete polynomials was obtained in [13] using a useful relation proved in [2].
In this work, mimicking the approach used in [13] we give a single fourth order q-difference equation which is valid for the first associated of all q-classical orthogonal polynomials. This equation is important for some connection coefficient problems [10], and also in order to represent finite modifications inside the Jacobi matrices of the q-classical starting family [15].
q-classical orthogonal polynomials involved in this work belong to the q-Hahn class as introduced by Hahn [8]. They are represented by the basic hypergeometric series appearing

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at the level ${}_3\phi_2$ and not at the level ${}_4\phi_3$ of the Askey-Wilson orthogonal polynomials.
The orthogonality weight ρ (defined in the interval I) for q-classical orthogonal polynomials is defined by a Pearson-type q-difference equation

$$D_q\,(\sigma\,\rho) = \tau\,\rho, \tag{1}$$

where the q-difference operator D_q is defined [8] by

$$D_q\,f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0, \quad 0 < q < 1, \tag{2}$$

and $D_q f(0) := f'(0)$ by continuity, provided that $f'(0)$ exists. σ is a polynomial of degree at most two and τ is polynomial of degree one.
The monic polynomials $P_n(x;q)$, orthogonal with respect to ρ satisfy the second order q-difference equation

$$\mathcal{Q}_{2,n}\,[y(x)] \equiv \Big[\sigma(x) D_q\, D_{\frac{1}{q}} + \tau(x) D_q + \lambda_{q,n}\mathcal{I}_d\Big]\, y(x) = 0, \tag{3}$$

an equation which can be written in the q-shifted form

$$\left[(\sigma_1 + \tau_1\, t_1)\mathcal{T}_q^2 - ((1+q)\sigma_1 + \tau_1\, t_1 - \lambda_{q,n}\, t_1^2)\mathcal{T}_q + q\sigma_1\,\mathcal{I}_d\right] y(x) = 0, \tag{4}$$

with

$$\begin{aligned} &\lambda_{q,n} = -[n]_q\{\tau' + [n-1]_{\frac{1}{q}}\frac{\sigma''}{2q}\}, \quad [n]_q = \frac{1-q^n}{1-q}, \\ &\quad \sigma_i \equiv \sigma(q^i x), \quad \tau_i \equiv \tau(q^i x), \quad t_i \equiv t(q^i x), \quad t(x) = (q-1)x \end{aligned} \tag{5}$$

and the geometric shift $\mathcal{T}_q$ defined by

$$\mathcal{T}_q^i\, f(x) = f(q^i\, x), \quad \mathcal{T}_q^0 \equiv \mathcal{I}_d\,(\equiv \text{identity operator}). \tag{6}$$

2 Fourth order q-difference equation for the first associated $P_{n-1}^{(1)}(x;q)$ of the q-classical orthogonal polynomial

The first associated of $P_n(x;q)$ is a monic polynomial of degree $n-1$, denoted by $P_{n-1}^{(1)}(x;q)$, and defined by

$$P_{n-1}^{(1)}(x;q) = \frac{1}{\gamma_0}\int\limits_I \frac{P_n(s;q) - P_n(x;q)}{s-x}\rho(s) d_q s, \tag{7}$$

where γ_0 is given by $\gamma_0 = \int\limits_I \rho(s) d_q s$ and the q-integral is defined in [7].
The polynomials $P_n(x;q) \equiv P_n^{(0)}(x;q)$ and $P_n^{(1)}(x;q)$ satisfy also the following three-term recurrence relation [4] for $r=0$ and $r=1$ respectively

$$\begin{cases} P_{n+1}^{(r)}(x;q) = (x - \beta_{n+r})\, P_n^{(r)}(x;q) - \gamma_{n+r}\, P_{n-1}^{(r)}(x;q), \quad n \geq 1, \\ P_0^{(r)}(x;q) = 1, P_1^{(r)}(x;q) = x - \beta_r. \end{cases} \tag{8}$$

Relation (7) can be written as

$$P_{n-1}^{(1)}(x;q) = \rho(x)\, Q_n(x;q) - P_n(x;q)\rho(x)\, Q_0(x;q), \tag{9}$$

where

$$Q_n(x;q) = \frac{1}{\gamma_0\, \rho(x)} \int_I \frac{P_n(s;q)}{s-x}\rho(s) d_q s.$$

It is well-known [14] that $Q_n(x;q)$ also satisfies Equation (3); hence by (9)

$$\mathcal{Q}_{2,n}\left[\frac{P_{n-1}^{(1)}(x;q)}{\rho(x)} + P_n(x;q)Q_0(x;q)\right] = 0. \tag{10}$$

In a first step, we eliminate $\rho(x)$ and $Q_0(x;q)$ in Equation (10) using Equation (1) and Equation (3) for $P_n(x;q)$. This can be easily carried out using a computer algebra system—we used Maple V Release 4 [3]—and gives the relation

$$(\sigma_1 + \tau_1\, t_1)\, \mathcal{Q}_{2,n-1}^{*}\left[P_{n-1}^{(1)}(x;q)\right] = \left[e\, \mathcal{T}_q + f\, \mathcal{I}_d\right] P_n(x;q), \tag{11}$$

with

$$\begin{aligned} \mathcal{Q}_{2,n-1}^{*} &= \sigma_2\, \mathcal{T}_q^2 - ((1+q)\sigma_1 + \tau_1\, t_1 - \lambda_{q,n}\, t_1^2)\mathcal{T}_q + q\, (\sigma + \tau\, t)\, \mathcal{I}_d, \\ e &= (\frac{\sigma''}{2} - \tau')\, ((1+q)\sigma_1 + \tau_1\, t_1 - \lambda_{q,n}\, t_1^2)\, t_1, \\ f &= -(\frac{\sigma''}{2} - \tau')\, ((q+1)\sigma_1 + \tau_1\, t_1)\, t_1. \end{aligned} \tag{12}$$

In a second step, we use Equations (11), (12) and the fact that the polynomials $P_n(x;q)$ satisfy Equation (3), again. This gives—after some computations with Maple V.4—the operator $\mathcal{Q}_{2,n-1}^{**}$ annihilating the right-hand side of Equation (11),

$$\begin{aligned} \mathcal{Q}_{2,n-1}^{**} = (\sigma_3 + \tau_3\, t_3)&[q^2\, A_1 + (1+q)\, \sigma_2 + \tau_2\, t_2]\mathcal{T}_q^2 \\ &-[q^3\, A_1\, (\sigma_2 + \tau_2\, t_2) + A_3\, (\sigma_2 + q\, A_1)]\mathcal{T}_q \\ &+q\, \sigma_1\, [q^2\, A_2 + (1+q)\, \sigma_3 + \tau_3\, t_3)]\, \mathcal{I}_d, \end{aligned} \tag{13}$$

where $A(x) = (1+q)\sigma(x) + \tau(x)\, t(x) - \lambda_{q,n}\, t(x)^2$ and $A_j \equiv A_j(x) \equiv A(q^j x),\ j = 1,2,3.$ We therefore obtain the factorized form of the fourth order q-difference equation satisfied by each $P_{n-1}^{(1)}(x;q)$,

$$\mathcal{Q}_{2,n-1}^{**}\, \mathcal{Q}_{2,n-1}^{*}\left[P_{n-1}^{(1)}(x;q)\right] = 0. \tag{14}$$

3 Limiting situations, Comments and Example

(i) Since $\lim\limits_{q\to 1} D_q = \frac{d}{dx}$, from Equations (12) and (13), we recover by a limit process the factorized form of the fourth order differential equation satisfied by the first associated $P_{n-1}^{(1)}(x)$ of the (continuous) classical orthogonal polynomials P_n [12],

$$\mathcal{Q}_{2,n-1}^{**c}\, \mathcal{Q}_{2,n-1}^{*c}\left[P_{n-1}^{(1)}(x)\right] = 0, \tag{15}$$

with

$$\mathcal{Q}^{*c}_{2,n-1} = \lim_{q\to 1} \frac{\mathcal{Q}^{*}_{2,n-1}}{q^2(q-1)^2x^2} = \sigma\frac{d^2}{dx^2} + (2\sigma' - \tau)\frac{d}{dx} + (\sigma'' - \tau' + \lambda_n)\mathcal{I}_d,$$
$$\mathcal{Q}^{**c}_{2,n-1} = \frac{1}{4\sigma(x)} \lim_{q\to 1} \frac{\mathcal{Q}^{**}_{2,n-1}}{q^2(q-1)^2x^2} = \sigma\frac{d^2}{dx^2} + (\sigma' + \tau)\frac{d}{dx} + (\tau' + \lambda_n)\mathcal{I}_d,$$

where $\lambda_n \equiv \lim_{q\to 1} \lambda_{q,n} = -n\,[(n-1)\frac{\sigma''}{2} + \tau']$.

(ii) If the polynomials σ and τ are such that $\sigma'' = 2\,\tau'$ [12,13,15], then the right-hand side of Equation (11) is equal to zero, and the first associated $P^{(1)}_{n-1}$ satisfies the second (instead of fourth) order difference equation

$$\mathcal{Q}^{*}_{2,n-1}\left[P^{(1)}_{n-1}(x;q)\right] = 0.$$

For the little q-Jacobi polynomials $p_n(x;a,b|q)$ [1,9]

$$\sigma(x) = \frac{x\,(x-1)}{q},\ \tau(x) = \frac{1 - a\,q + (abq^2 - 1)\,x}{q(q-1)},$$

and for the big q-Jacobi polynomials $P_n(x;a,b,c;q)$ [1,9]

$$\sigma(x) = acq - (a+c)x + \frac{x^2}{q},\ \tau(x) = \frac{cq + aq(1-(b+c)q) + (abq^2-1)x}{q(q-1)},$$

the constant $\sigma'' - 2\tau'$ is equal to $\frac{2\,(1-a\,b\,q)}{q-1}$. Therefore, the first associated of the little q-Jacobi polynomials (resp. big q-Jacobi polynomials) is still in the little q-Jacobi (resp. big q-Jacobi) family when $a\,b\,q = 1$.
Computations involving the coefficients β_n and γ_n (see Equation (8)) given in [1,6,11] and uses of Maple V.4 generate the following relations between the monic little q-Jacobi (resp. monic big q-Jacobi) polynomials and their respective first associated

$$p^{(1)}_n(x;a,\frac{1}{q\,a}\,|q) = (a\,q)^n\,p_n(\frac{x}{a\,q};\frac{1}{a},a\,q\,|q), \tag{16}$$
$$P^{(1)}_n(x;a,\frac{1}{q\,a,},c;q) = (a)^n\,P_n(\frac{x}{a};\frac{1}{a},a\,q,c\,q;q). \tag{17}$$

(iii) The results given in this paper (see Equations (11) and (13)), which agree with the ones obtained using the Stieltjes properties of the associated linear form [6], can be used for connection problems, expanding the first associated $P^{(1)}_{n-1}$ in terms of P_n, in the same spirit as in [10]. We have also computed the coefficients of the fourth order q-difference equation satisfied by the first associated of the q-classical orthogonal polynomials appearing in the q-Hahn tableau. In particular from the big q-Jacobi polynomials, we derive by limit processes [9] the fourth order differential (resp. q-difference) equation satisfied by the first associated of the classical (resp. q-classical) orthogonal polynomials.

(iv) For the little q-Jacobi polynomials for example, the operators $\mathcal{Q}^{*}_{2,n-1}$ and $\mathcal{Q}^{**}_{2,n-1}$ are given below, with the notation: $\nu = q^n$.

$$\mathcal{Q}^{*}_{2,n-1} = q\,x\left[(q^2\,x-1)\,\mathcal{T}_q^2 - \nu^{-1}\,(-\nu - a\,\nu + q^2\,x\,a\,b\,\nu^2 + q\,x)\,\mathcal{T}_q\right.$$
$$\left. +\ a\,(-1 + b\,q\,x)\,\mathcal{I}_d\right],$$
$$\mathcal{Q}^{**}_{2,n-1} = \nu^{-1}\,q^4x^2\left[qa(-1+bq^4x)\ \times\right.$$

$$
\begin{aligned}
&(q^3xab\nu + q^3xab\nu^2 + q^2x\nu + q^2x - q\nu - qa\nu - \nu - a\nu)\mathcal{T}_q^2 \\
&- \nu^{-1}(q^5x^2 + a\nu^2 + q\nu^2 - q^2x\nu^2 - q^3xab\nu^3 + q^7x^2a^2b^2\nu^3 \\
&- q^3xa^2b\nu^3 - q^5xab\nu^3 + q^2a^2\nu^2 - q^5xab\nu^2 - q^5xa^2b\nu^2 + q^2a\nu^2 \\
&- q^5xa^2b\nu^3 - q^2xa\nu - q^4xa\nu - q^2x\nu - q^4x\nu - q^3xa\nu + q^5x^2\nu \\
&- q^3x\nu + q^7x^2a^2b^2\nu^4 + q^6x^2ab\nu - q^4xa^2b\nu^3 + qa^2\nu^2 - q^2xa\nu^2 \\
&+ 2q^6x^2ab\nu^2 + q^6x^2ab\nu^3 + 2qa\nu^2 + \nu^2 - q^4xab\nu^3)\mathcal{T}_q \\
&+ (-1 + qx)(q^4xab\nu + q^4xab\nu^2 + q^3x\nu + q^3x - q\nu \\
&- qa\nu - \nu - a\nu)\mathcal{I}_d] \,.
\end{aligned}
$$

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