Konrad-Zuse-Zentrum
für Informationstechnik Berlin

Takustraße 7
D-14195 Berlin-Dahlem
Germany

DIMITRIOS I. DAIS, UTZ-UWE HAUS AND MARTIN HENK

# On crepant resolutions of 2-parameter series of Gorenstein cyclic quotient singularities

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# On crepant resolutions of 2-parameter series of Gorenstein cyclic quotient singularities

**Dimitrios I. Dais, Utz-Uwe Haus and Martin Henk**

**Abstract**

An immediate generalization of the classical McKay correspondence for Gorenstein quotient spaces $\mathbb{C}^r/G$ in dimensions $r \geq 4$ would primarily demand the existence of projective, crepant, full desingularizations. Since this is not always possible, it is natural to ask about special classes of such quotient spaces which would satisfy the above property. In this paper we give explicit necessary and sufficient conditions under which 2-parameter series of Gorenstein cyclic quotient singularities have torus-equivariant resolutions of this specific sort in all dimensions.

## 1. Introduction

Let $Y$ be a Calabi-Yau threefold and $G$ a finite group of analytic automorphisms of $Y$, such that for all isotropy groups $G_y$, $y \in Y$, we have $G_y \subset \mathrm{SL}(\mathsf{T}_y^{\mathrm{h}}(Y))$, where $\mathsf{T}_y^{\mathrm{h}}(Y)$ denotes the holomorphic tangent space of $Y$ at $y$. In the framework of the study of the "index" of the physical orbifold theory [11] Dixon, Harvey, Vafa and Witten introduced for the orbit-space $Y/G$ a "stringy" analogue $\chi_{\mathrm{str}}(Y,G)$ of the Euler-Poincaré characteristic. Working out some concrete examples, they verified that $\chi_{\mathrm{str}}(Y,G)$ is equal to the usual Euler-Poincaré characteristic $\chi(W)$ of a new Calabi-Yau threefold $W$, which is nothing but the overlying space of a projective desingularization $f: W \to Y/G$ of $Y/G$, as long as $f$ does not affect the triviality of the dualizing sheaf (in other words, $f$ preserves the "holomorphic volume form" or is "crepant"). Historically, this was the starting-point for several mathematical investigations about crepant resolutions of Gorenstein quotient singularities, because, in its local version, the above property can be seen to be implied by a generalization of the so-called "McKay-correspondence" in dimension $r = 3$ (cf. Hirzebruch-Höfer [27]). Meanwhile there are lots of proposed very promising approaches to the problem of generalizing McKay's 2-dimensional bijections for arbitrary $r \geq 3$ (cf. Reid [55]), though the list of closely related open questions seems to remain rather long.

• In dimension 2 the classification of quotient singularities $(\mathbb{C}^2/G, [\mathbf{0}])$ (for $G$ a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$), as well as the minimal resolution of $[\mathbf{0}]$ by a tree of $\mathbb{P}^1_{\mathbb{C}}$'s is well-known (Klein [33], Du Val [12]); in fact, these rational, Gorenstein singularities can be alternatively characterized as the *A-D-E* hypersurface singularities being embedded in $\mathbb{C}^3$ (see e.g. Lamotke [41]). Roughly formulated, the *classical* 2-dimensional McKay-correspondence for the quotient space $X = \mathbb{C}^2/G$ provides a bijection between the set of irreducible representations of $G$ and a basis of $H^*\left(\widehat{X},\mathbb{Z}\right)$ (or dually, between the set of conjugacy classes of $G$ and a basis of $H_*\left(\widehat{X},\mathbb{Z}\right)$), where $f: \widehat{X} \to X$ is the minimal (= crepant) desingularization of $X$. (See McKay [42], Gonzalez-Sprinberg, Verdier [19], Knörrer [37] and Ito-Nakamura [28]).

• The most important aspects of the *three-dimensional* generalization of McKay's bijections for $\mathbb{C}^3/G$'s, $G \subset \mathrm{SL}(3,\mathbb{C})$, were only recently clarified by the paper [29] of Ito and Reid; considering a canonical grading on the Tate-twist of the acting $G$'s by the so-called "ages", they proved that for any projective crepant resolution $f: \widehat{X} \to X = \mathbb{C}^3/G$, there are one-to-one correspondences between the elements of $G$ of age 1 and the exceptional prime divisors of $f$, and between them and the members of a basis of $H^2\left(\widehat{X},\mathbb{Q}\right)$, respectively. On the other hand, the *existence* of crepant $f$'s was proved by Markushevich, Ito and Roan:

**Theorem 1.1 (Existence-Theorem in Dimension 3).** *The underlying spaces of all 3-dimensional Gorenstein quotient singularities possess crepant resolutions.*

(The $f$'s are unique only up to "isomorphism in codimension 1", and to win projectivity, one has to make particular choices). From the point of view of birational geometry, Ito and Reid [29] proved, in addition, the following theorem working in all dimensions.

**Theorem 1.2 (Ito-Reid Correspondence).** *Let $G$ be a finite subgroup of $\mathrm{SL}(r,\mathbb{C})$ acting linearly on $\mathbb{C}^r$, $r \geq 2$, and $X = \mathbb{C}^r/G$. Then there is a canonical one-to-one correspondence between the junior conjugacy classes in $G$ and the crepant discrete valuations of $X$.*

In dimensions $r \geq 4$, however, there are already from the very beginning certain qualitative obstructions to generalize thm. 1.1, and Reid's question ([54], [29], § 4.5, [55], 5.4) still remains unanswered:

• **Main question : Under which conditions on the acting groups $G \subset \mathbf{SL}(r,\mathbb{C})$ , $r \geq 4$, do the quotient spaces $\mathbb{C}^r/G$ have projective crepant desingularizations?**

• An immediate generalization of the classical McKay correspondence for $r \geq 4$ cannot avoid the use of a satisfactory answer to the above question! The reason is simple. Using, for instance, only *partial* crepant desingularizations, we obtain overlying spaces of uncontrollable cohomology dimensions even in trivial examples. (In contrast to this, Batyrev [2] & Kontsevich [38] announced proofs of the invariance of *all* cohomology dimensions (over $\mathbb{Q}$) of the overlying spaces of all "*full*" crepant desingularizations for arbitrary $r \geq 2$. For the case of abelian acting groups, see [3], 5.4).

• It is worth mentioning that the existence of *terminal* Gorenstein singularities implies automatically that *not all* Gorenstein quotient spaces $\mathbb{C}^r/G$, $r \geq 4$, can have such desingularizations (see Morrison-Stevens [46]).

• On the other hand, as it was proved in [8] by making use of Watanabe's classification of all abelian quotient singularities $(\mathbb{C}^r/G,[\mathbf{0}])$, $G \subset \mathrm{SL}(r,\mathbb{C})$, (up to analytic isomorphism) whose underlying spaces are embeddable as complete intersections ("c.i.'s") of hypersurfaces into an affine complex space, and methods of toric and discrete geometry,

**Theorem 1.3.** *The underlying spaces of all abelian quotient c.i.-singularities admit torus-equivariant projective, crepant resolutions* (*and therefore* smooth minimal models) *in all dimensions.*

Hence, the expected answer(s) to the main question will be surely of special nature, depending crucially on the generators of the acting groups or at least on the properties of the ring $\mathbb{C}[x_1,\ldots,x_r]^G$ of invariants.

• In view of theorem 1.3, it is natural to ask which would be the behaviour of abelian Gorenstein *non-c.i.* quotient singularities with respect to existence-problem of these specific resolutions. In [7] we studied the Gorenstein 1-parameter cyclic quotient singularities, i.e., those of type

$$\frac{1}{l}\left(\underbrace{1,\ldots,1,}_{(r-1)\text{-times}}\ l-(r-1)\right), \qquad l = |G| \geq r \geq 4 \ ,$$

generalizing the classical example of the affine cone which lies over the $r$-tuple Veronese embedding of $\mathbb{P}^{r-1}_{\mathbb{C}}$. Using toric geometry, this means that the (pure) junior elements lie on a *straight line.* In the present paper we shall extend our results in the case in which also 2 "free" parameters are allowed. Exploiting the *coplanarity* of the corresponding junior lattice points, it is possible to give again a definitive answer to the above main question by an explicit arithmetical criterion involving only the weights of the defining types. As expected from the algorithmic point of view, this criterion can be regarded as a somewhat more complicated "Hirzebruch-Jung-procedure" working in all dimensions. It should be finally pointed out, that this kind of criteria seem to be again interesting for mathematical physics, this time in the framework of the theory of "D-branes" (see Mohri [45] and example 5.18 below).

• The paper is organized as follows: In §2 we recall some basic concepts from toric geometry and fix our notation. A detailed study of 2-dimensional rational s.c.p.cones and a method for the determination of the vertices of the corresponding support polygons by Kleinian approximations are presented in §3. In section 4 we explain how the underlying spaces of abelian quotient singularities are to be regarded as affine toric varieties and recall the geometric condition under which the quotient spaces $\mathbb{C}^r/G$ are Gorenstein. The first part of section 5 is devoted to the hierarchy of lattice triangulations and to the simple combinatorial mechanism leading to the hierarchy of partial (resp. full) crepant desingularizations of $\mathbb{C}^r/G$. Our main theorems on the existence of projective, crepant, full resolutions are formulated and discussed in the second part of §5 and proved in §6. Finally, in section 7 we give concrete formulae for the computation of the dimensions of the cohomology groups of the spaces resolving fully the 2-parameter Gorenstein cyclic quotient singularities by any crepant birational morphism.

*General terminology.* We always work with normal complex varieties, i.e., with normal, integral, separated schemes over $\mathbb{C}$. $\mathrm{Sing}(X)$ denotes the *singular locus* of such a variety $X$. (By the word *singularity* we intimate either a singular point or the germ of a singular point, but the meaning will be in each case clear from the context). As in [3], [7], [8], by a *desingularization* (or *resolution of singularities*) $f : \widehat{X} \to X$ of a non-smooth $X$, we mean a "full" or "overall" desingularization (if not mentioned), i.e., $\mathrm{Sing}\left(\widehat{X}\right) = \varnothing$. When we refer to *partial* desingularizations, we mention it explicitly. A partial desingularization $f : X' \to X$ of a $\mathbb{Q}$-Gorenstein complex variety $X$ (with global index $j$) is called *non-discrepant* or simply *crepant* if $\omega_X^{[j]} \cong f_*\left(\omega_{X'}^{\otimes j}\right)$, or, in other words, if the (up to rational equivalence uniquely determined) difference $jK_{X'} - f^*(jK_X)$ contains exceptional prime divisors which have vanishing multiplicities. ($\omega_X, K_X$ and $\omega_{X'}, K_{X'}$ denote here the dualizing sheaves and the canonical divisors of $X$ and $Y$ respectively). Furthermore, $f : X' \to X$ is *projective* if $X'$ admits an $f$-ample Cartier divisor. The terms *canonical* and *terminal singularity* are to be understood in the usual sense (see Reid [52], [53]).

## 2. Preliminaries from the theory of toric varieties

At first we recall some basic facts from the theory of toric varieties. We shall mainly use the same notation as in [7], [8], [9]. Our standard references on toric geometry are the textbooks of Oda [48], Fulton [18] and Ewald [13], and the lecture notes [31].

(a) The *linear hull,* the *affine hull*, the *positive hull* and *the convex hull* of a set $B \subset \mathbb{R}^r$, $r \geq 1$, will be denoted by $\mathrm{lin}(B)$, $\mathrm{aff}(B)$, $\mathrm{pos}(B)$ (or $\mathbb{R}_{\geq 0} B$) and $\mathrm{conv}(B)$ respectively. The *dimension* $\dim(B)$ of a $B \subset \mathbb{R}^r$ is defined to be the dimension of its affine hull.

(b) Let $N \cong \mathbb{Z}^r$ be a free $\mathbb{Z}$-module of rank $r \geq 1$. $N$ can be regarded as a *lattice* in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$. (For fixed identification, we shall represent the elements of $N_{\mathbb{R}}$ by column-vectors in $\mathbb{R}^r$). If $\{n_1, \ldots, n_r\}$ is a $\mathbb{Z}$-basis of $N$, then

$$\det(N) := |\det(n_1, \ldots, n_r)|$$

is the *lattice determinant.* An $n \in N$ is called *primitive* if $\mathrm{conv}(\{\mathbf{0}, n\}) \cap N$ contains no other points except $\mathbf{0}$ and $n$.

Let $N \cong \mathbb{Z}^r$ be as above, $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual lattice, $N_{\mathbb{R}}, M_{\mathbb{R}}$ their real scalar extensions, and $\langle .,. \rangle : N_{\mathbb{R}} \times M_{\mathbb{R}} \to \mathbb{R}$ the natural $\mathbb{R}$-bilinear pairing. A subset $\sigma$ of $N_{\mathbb{R}}$ is called *strongly convex polyhedral cone* (*s.c.p.cone*, for short), if there exist $n_1, .., n_k \in N_{\mathbb{R}}$, such that $\sigma = \mathrm{pos}(\{n_1, \ldots, n_k\})$ and $\sigma \cap (-\sigma) = \{\mathbf{0}\}$. Its *relative interior* $\mathrm{int}(\sigma)$ (resp. its *relative boundary* $\partial\sigma$) is the usual topological interior (resp. the usual topological boundary) of it, considered as subset of $\mathrm{lin}(\sigma)$. The *dual cone* of $\sigma$ is defined by

$$\sigma^{\vee} := \{\mathbf{x} \in M_{\mathbb{R}} \mid \langle \mathbf{x}, \mathbf{y} \rangle \geq 0, \ \forall \mathbf{y}, \ \mathbf{y} \in \sigma\}$$

and satisfies: $\sigma^{\vee} + (-\sigma^{\vee}) = M_{\mathbb{R}}$ and $\dim(\sigma^{\vee}) = r$. A subset $\tau$ of a s.c.p. cone $\sigma$ is called a *face* of $\sigma$ (notation: $\tau \prec \sigma$), if $\tau = \{\mathbf{y} \in \sigma \mid \langle m_0, \mathbf{y} \rangle = 0\}$, for some $m_0 \in \sigma^{\vee}$. A s.c.p. $\sigma = \mathrm{pos}(\{n_1, \ldots, n_k\})$ is called *simplicial* (resp. *rational*) if $n_1, \ldots, n_k$ are $\mathbb{R}$-linearly independent (resp. if $n_1, \ldots, n_k \in N_{\mathbb{Q}}$, where $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$).

(c) If $\sigma \subset N_{\mathbb{R}}$ is a rational s.c.p. cone, then $\sigma$ has $\mathbf{0}$ as its apex and the subsemigroup $\sigma \cap N$ of $N$ is a monoid. The following two propositions describe the fundamental properties of this monoid $\sigma \cap N$ and their proofs go essentially back to Gordan [20], Hilbert [24] and van der Corput [62], [63].

**Proposition 2.1 (Gordan's lemma).** *$\sigma \cap N$ is finitely generated as additive semigroup, i.e., there exist*

$$n_1, n_2, \ldots, n_\nu \in \sigma \cap N \ , \quad \textit{such that} \quad \sigma \cap N = \mathbb{Z}_{\geq 0}\, n_1 + \mathbb{Z}_{\geq 0}\, n_2 + \cdots + \mathbb{Z}_{\geq 0}\, n_\nu \ .$$

**Proposition 2.2 (Minimal generating system).** *Among all the systems of generators of $\sigma \cap N$, there is a system* $\mathbf{Hlb}_N(\sigma)$ *of* minimal cardinality, *which is uniquely determined* (*up to the ordering of its elements*) *by the following characterization* :

$$\mathbf{Hlb}_N(\sigma) = \left\{ n \in \sigma \cap (N \smallsetminus \{\mathbf{0}\}) \ \middle| \ \begin{array}{c} n \text{ cannot be expressed as the sum} \\ \text{of two other vectors belonging} \\ \text{to } \ \sigma \cap (N \smallsetminus \{\mathbf{0}\}) \end{array} \right\} \tag{2.1}$$

*In particular, if $\sigma = \mathrm{pos}(\{n_1, \ldots, n_k\})$, then*

$$\mathbf{Hlb}_N(\sigma) \subseteq \left\{ n \in \sigma \cap (N \smallsetminus \{\mathbf{0}\}) \ \middle| \ n = \textstyle\sum_{i=1}^{k} \delta_i n_i \ , \text{ with } \ 0 \leq \delta_i < 1, \ \forall i, \ 1 \leq i \leq k \right\}. \tag{2.2}$$

$\mathbf{Hlb}_N(\sigma)$ is called *the Hilbert basis of* $\sigma$ w.r.t. $N$. About *algorithms* for the determination of Hibert bases of pointed rational cones, we refer to Henk-Weismantel [22], and to the other references therein.

(d) For a lattice $N \cong \mathbb{Z}^r$ having $M$ as its dual, we define an $r$-dimensional *algebraic torus* $T_N \cong (\mathbb{C}^*)^r$ by :

$$T_N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^* .$$

Every $m \in M$ assigns a character $\mathbf{e}(m) : T_N \to \mathbb{C}^*$. Moreover, each $n \in N$ determines an 1-parameter subgroup

$$\gamma_n : \mathbb{C}^* \to T_N \quad \text{with} \quad \gamma_n(\lambda)(m) := \lambda^{\langle m,n \rangle}, \quad \text{for} \quad \lambda \in \mathbb{C}^*, \ m \in M \ .$$

We can therefore identify $M$ with the character group of $T_N$ and $N$ with the group of 1-parameter subgroups of $T_N$. On the other hand, for a rational s.c.p.c. $\sigma$ with

$$M \cap \sigma^\vee = \mathbb{Z}_{\geq 0}\, m_1 + \mathbb{Z}_{\geq 0}\, m_2 + \cdots + \mathbb{Z}_{\geq 0}\, m_k,$$

we associate to the finitely generated, normal, monoidal $\mathbb{C}$-subalgebra $\mathbb{C}[M \cap \sigma^\vee]$ of $\mathbb{C}[M]$ an affine complex variety

$$U_\sigma := \text{Max-Spec}\,(\mathbb{C}[M \cap \sigma^\vee]) ,$$

which can be identified with the set of semigroup homomorphisms :

$$U_\sigma = \left\{ u : M \cap \sigma^\vee \ \to \mathbb{C} \ \middle| \ \begin{array}{c} u(\mathbf{0}) = 1, \ u(m + m') = u(m) \cdot u(m') , \\ \text{for all} \quad m, m' \in M \cap \sigma^\vee \end{array} \right\} ,$$

where $\mathbf{e}(m)(u) := u(m)$, $\forall m, \ m \in M \cap \sigma^\vee$ and $\forall u, \ u \in U_\sigma$. In the analytic category, $U_\sigma$, identified with its image under the injective map $(\mathbf{e}(m_1), \ldots, \mathbf{e}(m_k)) : U_\sigma \hookrightarrow \mathbb{C}^k$, can be regarded as an analytic set determined by a system of equations of the form: (monomial) = (monomial). This analytic structure induced on $U_\sigma$ is independent of the semigroup generators $\{m_1, \ldots, m_k\}$ and each map $\mathbf{e}(m)$ on $U_\sigma$ is holomorphic w.r.t. it. In particular, for $\tau \prec \sigma$, $U_\tau$ is an open subset of $U_\sigma$. Moreover, if we define $d := \#(\mathbf{Hlb}_M(\sigma^\vee))$ $(\leq k)$, then $d$ is nothing but the *embedding dimension* of $U_\sigma$, i.e. the *minimal* number of generators of the maximal ideal of the local $\mathbb{C}$-algebra $\mathcal{O}_{U_\sigma, \, (\mathbf{0} \in \mathbb{C}^d)}$ (cf. [48], 1.2 & 1.3).

(e) A *fan* w.r.t. $N \cong \mathbb{Z}^r$ is a finite collection $\Delta$ of rational s.c.p. cones in $N_{\mathbb{R}}$, such that :

(i) any face $\tau$ of $\sigma \in \Delta$ belongs to $\Delta$, and

(ii) for $\sigma_1, \sigma_2 \in \Delta$, the intersection $\sigma_1 \cap \sigma_2$ is a face of both $\sigma_1$ and $\sigma_2$.

The union $|\Delta| := \cup \{\sigma \mid \sigma \in \Delta\}$ is called the *support* of $\Delta$. Furthermore, we define

$$\Delta(i) := \{\sigma \in \Delta \ \mid \dim(\sigma) = i\} \ , \text{ for } \quad 0 \leq i \leq r \ .$$

If $\varrho \in \Delta(1)$, then there exists a unique primitive vector $n(\varrho) \in N \cap \varrho$ with $\varrho = \mathbb{R}_{\geq 0}\, n(\varrho)$ and each cone $\sigma \in \Delta$ can be therefore written as

$$\sigma = \sum_{\substack{\varrho \in \Delta(1) \\ \varrho \prec \sigma}} \mathbb{R}_{\geq 0}\, n(\varrho) \quad .$$

The set $\mathrm{Gen}(\sigma) := \{n(\varrho) \mid \varrho \in \Delta(1), \varrho \prec \sigma\}$ is called the *set of minimal generators* (within the pure first skeleton) of $\sigma$. For $\Delta$ itself one defines analogously

$$\mathrm{Gen}(\Delta) := \bigcup_{\sigma \in \Delta} \mathrm{Gen}(\sigma) \quad .$$

(f) The *toric variety* $X(N, \Delta)$ associated to a fan $\Delta$ w.r.t. the lattice $N$ is by definition the identification space

$$X(N, \Delta) := \left( \left( \bigcup_{\sigma \in \Delta} U_\sigma \right) \;/\; \sim \right)$$

with $U_{\sigma_1} \ni u_1 \sim u_2 \in U_{\sigma_2}$ if and only if there is a $\tau \in \Delta$, such that $\tau \prec \sigma_1 \cap \sigma_2$ and $u_1 = u_2$ within $U_\tau$. As complex variety, $X(N, \Delta)$ turns out to be irreducible, normal, Cohen-Macaulay and to have at most rational singularities (cf. [18], p. 76, and [48], thm. 1.4, p. 7, and cor. 3.9, p. 125). $X(N, \Delta)$ is called *simplicial* if all cones of $\Delta$ are simplicial.

(g) $X(N, \Delta)$ admits a canonical $T_N$-action which extends the group multiplication of $T_N = U_{\{\mathbf{0}\}}$ :

$$T_N \times X(N, \Delta) \ni (t, u) \longmapsto t \cdot u \in X(N, \Delta) \tag{2.3}$$

where, for $u \in U_\sigma$, $(t \cdot u)(m) := t(m) \cdot u(m)$, $\forall m,\ m \in M \cap \sigma^\vee$ . The orbits w.r.t. the action (2.3) are parametrized by the set of all the cones belonging to $\Delta$. For a $\tau \in \Delta$, we denote by $\mathrm{orb}(\tau)$ (resp. by $V(\tau)$) the orbit (resp. the closure of the orbit) which is associated to $\tau$. If $\tau \in \Delta$, then $V(\tau) = X(N(\tau), \mathrm{Star}(\tau; \Delta))$ is itself a toric variety w.r.t.

$$N(\tau) := N/N_\tau \ , \qquad \mathrm{Star}(\tau; \Delta) := \{\overline{\sigma} \mid \sigma \in \Delta,\ \tau \prec \sigma\} \ ,$$

where $N_\tau$ is the sublattice of $N$ generated (as subgroup) by $N \cap \mathrm{lin}(\tau)$ and $\overline{\sigma} = (\sigma + (N_\tau)_{\mathbb{R}}) / (N_\tau)_{\mathbb{R}}$ denotes the image of $\sigma$ in $N(\tau)_{\mathbb{R}} = N_{\mathbb{R}} / (N_\tau)_{\mathbb{R}}$.

(h) A *map of fans* $\varpi : (N', \Delta') \to (N, \Delta)$ is a $\mathbb{Z}$-linear homomorphism $\varpi : N' \to N$ whose scalar extension $\varpi = \varpi_{\mathbb{R}} : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ satisfies the property:

$$\forall \sigma',\ \sigma' \in \Delta' \quad \exists\, \sigma,\ \sigma \in \Delta \quad \text{with} \quad \varpi(\sigma') \subset \sigma \, .$$

$\varpi \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{C}^*} : T_{N'} = N' \otimes_{\mathbb{Z}} \mathbb{C}^* \to T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ is a homomorphism from $T_{N'}$ to $T_N$ and the scalar extension $\varpi^\vee : M_{\mathbb{R}} \to M'_{\mathbb{R}}$ of the dual $\mathbb{Z}$-linear map $\varpi^\vee : M \to M'$ induces canonically an *equivariant holomorphic map* $\varpi_* : X(N', \Delta') \to X(N, \Delta)$. This map is *proper* if and only if $\varpi^{-1}(|\Delta|) = |\Delta'|$. In particular, if $N = N'$ and $\Delta'$ is a refinement of $\Delta$, then $\mathrm{id}_* : X(N, \Delta') \to X(N, \Delta)$ is *proper* and *birational* (cf. [48], thm. 1.15 and cor. 1.18).

(i) Let $N \cong \mathbb{Z}^r$ be a lattice of rank $r$ and $\sigma \subset N_{\mathbb{R}}$ a simplicial, rational s.c.p.c. of dimension $k \leq r$. $\sigma$ can be obviously written as $\sigma = \varrho_1 + \cdots + \varrho_k$, for distinct 1-dimensional cones $\varrho_1, \ldots, \varrho_k$. We denote by

$$\mathbf{Par}(\sigma) := \left\{ \mathbf{y} \in (N_\sigma)_{\mathbb{R}} \;\middle|\; \mathbf{y} = \sum_{j=1}^{k} \varepsilon_j\, n(\varrho_j)\,,\ \text{with}\ \ 0 \leq \varepsilon_j < 1,\ \forall j,\ 1 \leq j \leq k \right\}$$

the *fundamental* (*half-open*) *parallelotope* which is associated to $\sigma$. The *multiplicity* $\mathrm{mult}(\sigma; N)$ of $\sigma$ with respect to $N$ is defined as

$$\mathrm{mult}(\sigma; N) := \#\left(\mathbf{Par}(\sigma) \cap N_\sigma\right) = \mathrm{Vol}\left(\mathbf{Par}(\sigma); N_\sigma\right) \ ,$$

where $\mathrm{Vol}(\mathbf{Par}(\sigma))$ denotes the usual volume (Lebesgue measure) of $\mathbf{Par}(\sigma)$ and

$$\mathrm{Vol}\left(\mathbf{Par}(\sigma); N_\sigma\right) := \frac{\mathrm{Vol}\left(\mathbf{Par}(\sigma)\right)}{\det(N_\sigma)} = \frac{\det\left(\mathbb{Z}\, n(\varrho_1) \oplus \cdots \oplus \mathbb{Z}\, n(\varrho_k)\right)}{\det(N_\sigma)}$$

its the relative volume w.r.t. $N_\sigma$. If $\mathrm{mult}(\sigma; N) = 1$, then $\sigma$ is called a *basic cone w.r.t. $N$*.

**Proposition 2.3 (Smoothness Criterion).** *The affine toric variety $U_\sigma$ is smooth iff $\sigma$ is basic w.r.t. $N$. (Correspondingly, an arbitrary toric variety $X(N,\Delta)$ is smooth if and only if it is simplicial and each s.c.p. cone $\sigma \in \Delta$ is basic.)*

*Proof.* It follows from [48], thm. 1.10, p. 15. □

By Carathéodory's theorem concerning convex polyhedral cones (cf. Ewald [13], III 2.6, p. 75, & V 4.2, p. 158) one can choose a refinement $\Delta'$ of any given fan $\Delta$, so that $\Delta'$ becomes simplicial. Since further subdivisions of $\Delta'$ reduce the multiplicities of its cones, we may arrive (after finitely many subdivisions) at a fan $\widetilde{\Delta}$ having only basic cones.

**Theorem 2.4 (Existence of Desingularizations).** *For every toric variety $X(N,\Delta)$ there exists a refinement $\widetilde{\Delta}$ of $\Delta$ consisting of exclusively basic cones w.r.t. $N$, i.e., such that*

$$f = \mathrm{id}_* : X\left(N, \widetilde{\Delta}\right) \longrightarrow X(N,\Delta) = U_\sigma$$

*is a (full) desingularization.*

Though this theorem can be also treated in terms of blow-ups of (not necessarily reduced) subschemes of $X(N,\Delta)$ supporting $\mathrm{Sing}(X(N,\Delta))$ (cf. [31], §I.2), it is a theorem of purely "existential nature", because it does not provide any intrinsic characterization of a "canonical" way for the construction of the required subdivisions. On the other hand, there is another theorem due to Sebö ([57], thm. 2.2) which informs us that such an intrinsically geometric choice of subdivisions is indeed possible in **low** dimensions by considering the elements of the Hilbert basis of a cone as the set of minimal generators of the desingularizing fan.

**Theorem 2.5 (Sebö's Theorem).** *Let $N$ be a lattice of rank $r$ and $\sigma \subset N_{\mathbb{R}}$ a rational, simplicial s.c.p. cone of dimension $r$. Moreover, let $\Delta$ denote the fan consisting of $\sigma$ together with all its faces. For $r \leq 3$ there exists a (full) desingularization $f = \mathrm{id}_* : X\left(N, \widetilde{\Delta}\right) \longrightarrow X(N,\Delta)$, such that*

$$\mathrm{Gen}\left(\widetilde{\Delta}\right) = \mathbf{Hlb}_N(\sigma) \quad .$$

This theorem was reproved independently by Aguzzoli & Mundizi ( [1], prop. 2.4) and by Bouvier & Gonzalez-Sprinberg ([4], th. 1, [5], th. 2.9) by using slightly different methods. Moreover, the latter authors constructed two concrete counterexamples showing that the statement in 2.5 is not true (in general) for dimensions $r \geq 4$. (See also rem. 5.10 below).

## 3. Finite continued fractions and two-dimensional rational cones

As we have already pointed out in the introduction, for the study of the existence of torus-equivariant, crepant, full resolutions of 2-parameter-series of Gorenstein cyclic quotient singularities we shall exploit the particular property of the corresponding lattice points of the junior simplex to be *coplanar* in an essential way. Consequently we shall reduce the whole problem to a 2-dimensional one (after appropriate lattice transformations), and we shall therefore make use of the precise structure of 2-dimensional rational s.c.p.cones. That's why we take a closer look, in this section, at the interplay between the "lattice-geometry" of two-dimensional rational s.c.p. cones and the continued fraction expansions of rational numbers realized by their "parametrizing integers" $p$ and $q$.

• It was Hirzebruch [25] in the early fifties who first described the minimal resolution of *any* 2-dimensional cyclic quotient singularity by means of the "modified" euclidean division algorithm and $(-)$-sign continued fractions, by blowing up points and by generalizing some previous partial results of Jung [30]. Later Cohn [6] proposed the use of support polygons with lattice points as vertices (instead of complex coordinate charts), in order to simplify considerably the resolution-procedure. Around the same time, Saint-Donat (cf. [31], pp. 16-19 & 35-38) gave the first proofs exclusively in terms of toric geometry. A much more detailed explanation of the role of negative-regular continued fractions in the study of 2-dimensional rational cones (via polar polyhedra) is contained in Oda's book [48], §1.6.

• Though we shall mostly adopt here Oda's notation and basic concepts (so that the proofs remain valid for arbitrary lattices of rank 2), our purpose is not to say anything about the resolution itself, which is well-known and not crucial for later use, but to give an explicit method for the determination of the *vertices* of the support polygons. We are mainly motivated by the original works of Klein [34], [35], [36], written over hundred years ago, in which his "Umrißpolygone" (also known as *Kleinian polygons*, cf. Finkel'shtein [14]) were used to approximate real (not necessarily rational) numbers by only *regular* continued fractions. From the algorithmic point of view, Kleinian approximations are more "economic" (see remarks 3.3, 3.5 and 3.8 below). Technically, it seems to be more convenient to work simultaneously with both primary and dual cones $\sigma$ and $\sigma^\vee$ and to interchange their combinatorial data by specific transfer-rules.

*Notation.* We shall henceforth use the following extra notation. For $\nu \in \mathbb{N}$, $\mu \in \mathbb{Z}$, we denote by $[\,\mu\,]_\nu$ the (uniquely determined) integer for which

$$0 \leq [\,\mu\,]_\nu < \nu, \quad \mu \equiv [\,\mu\,]_\nu \,(\mathrm{mod}\ \nu)\,.$$

"gcd" will be abbreviation for greatest common divisor. If $q \in \mathbb{Q}$, we define $\lfloor q \rfloor$ (resp. $\lceil q \rceil$) to be the greatest integer number $\leq q$ (resp. the smallest integer $\geq q$).

(a) Let $\kappa$ and $\lambda$ be two given relatively prime positive integers. Suppose that $\frac{\kappa}{\lambda}$ can be written as

$$\frac{\kappa}{\lambda} = a_1 + \cfrac{\varepsilon_1}{a_2 + \cfrac{\varepsilon_2}{a_3 + \cfrac{\varepsilon_3}{\ddots \, a_{\nu-1} + \cfrac{\varepsilon_{\nu-1}}{a_\nu}}}} \tag{3.1}$$

The right-hand side of (3.1) is called *semi-regular continued fraction* for $\frac{\kappa}{\lambda}$ (and $\nu$ its *length*) if it has the following properties:

(i) $a_j$ is an integer for all $j$, $1 \leq j \leq \nu$,
(ii) $\varepsilon_j \in \{-1, 1\}$ for all $j$, $1 \leq j \leq \nu - 1$,
(iii) $a_j \geq 1$ for $j \geq 2$ and $a_\nu \geq 2\,(!)$, and
(iv) if $a_j = 1$ for some $j$, $1 < j < \nu$, then $\varepsilon_j = 1$.

In particular, if $\varepsilon_j = 1$ (resp. $\varepsilon_j = -1$) for all $j$, $1 \leq j < \nu$, then we write $\frac{\kappa}{\lambda} = [a_1, a_2, \ldots, a_\nu]$ (resp. $\frac{\kappa}{\lambda} = [\![a_1, a_2, \ldots, a_\nu]\!]$). This is the *regular* (resp. the *negative-regular*) *continued fraction expansion* of $\frac{\kappa}{\lambda}$. These two expansions are unique (in this form) and can be obtained by the usual and the modified euclidean algorithm, respectively, depending on the choice of the kind of the associated remainders. The next two lemmas outline their main properties.

**Lemma 3.1.** *Let $\lambda, \kappa$ be two integers with $0 < \lambda < \kappa$, $\gcd(\lambda, \kappa) = 1$. Then*:
(i) *There exists always a uniquely determined regular continued fraction expansion*

$$\frac{\kappa}{\lambda} = [a_1, a_2, \ldots, a_\nu] \tag{3.2}$$

*of $\frac{\kappa}{\lambda}$ (with $a_j \geq 1$ for all $j \geq 1$ and $a_\nu \geq 2$).*
(ii) *Defining the finite sequences $(\mathsf{P}_i)_{-1 \leq i \leq \nu}$ and $(\mathsf{Q}_i)_{-1 \leq i \leq \nu}$ by*

$$\begin{cases} \mathsf{P}_{-1} = 0, \quad \mathsf{P}_0 = 1, \quad \mathsf{P}_i = a_i\ \mathsf{P}_{i-1} + \mathsf{P}_{i-2},\ \forall i,\ \ 1 \leq i \leq \nu, \\[2ex] \mathsf{Q}_{-1} = 1, \quad \mathsf{Q}_0 = 0, \quad \mathsf{Q}_i = a_i\ \mathsf{Q}_{i-1} + \mathsf{Q}_{i-2},\ \forall i,\ \ 1 \leq i \leq \nu, \end{cases} \tag{3.3}$$

*we obtain for all $i$, $1 \leq i \leq \nu$,*

$$\frac{\mathsf{P}_i}{\mathsf{Q}_i} = \begin{cases} [a_1, a_2, \ldots, a_i]\,, & \text{if } \ a_i \geq 2 \\ [a_1, a_2, \ldots, a_{i-2}, a_{i-1} + 1]\,, & \text{if } \ a_i = 1 \end{cases}$$

(This is the so-called *$i$-th convergent* of the given continued fraction (3.2)).
(iii) *The above sequences satisfy the following conditions*:
(a) *For all* $i$, $1 \leq i \leq \nu$ (resp. $2 \leq i \leq \nu$),

$$\frac{\mathsf{P}_i}{\mathsf{P}_{i-1}} = \begin{cases} [a_i, a_{i-1}, \ldots, a_2, a_1], & \text{if } a_1 \geq 2 \\ [a_i, a_{i-1}, \ldots, a_3, a_2 + 1], & \text{if } a_1 = 1 \end{cases} \tag{3.4}$$

resp.

$$\frac{\mathsf{Q}_i}{\mathsf{Q}_{i-1}} = \begin{cases} [a_i, a_{i-1}, \ldots, a_3, a_2], & \text{if } a_2 \geq 2 \\ [a_i, a_{i-1}, \ldots, a_4, a_3 + 1], & \text{if } a_2 = 1 \end{cases} \tag{3.5}$$

(b) *For all* $i$, $0 \leq i \leq \nu$,

$$\mathsf{P}_i\ \mathsf{Q}_{i-1} - \mathsf{P}_{i-1}\ \mathsf{Q}_i = (-1)^i \tag{3.6}$$

*In particular, for* $i$, $1 \leq i \leq \nu$, *we have* $\gcd(\mathsf{P}_i\ , \mathsf{Q}_i) = 1$, *and*

$$\frac{\mathsf{P}_1}{\mathsf{Q}_1} < \frac{\mathsf{P}_3}{\mathsf{Q}_3} < \cdots \leq \frac{\mathsf{P}_\nu}{\mathsf{Q}_\nu} = \frac{\kappa}{\lambda} \leq \cdots < \frac{\mathsf{P}_4}{\mathsf{Q}_4} < \frac{\mathsf{P}_2}{\mathsf{Q}_2} \tag{3.7}$$

(iv) *For* $a \in \mathbb{Z}$, *setting*

$$\mathcal{M}^+(a) := \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

*we get*

$$\begin{pmatrix} \mathsf{P}_i & \mathsf{P}_{i-1} \\ \mathsf{Q}_i & \mathsf{Q}_{i-1} \end{pmatrix} = \mathcal{M}^+(a_1) \cdot \mathcal{M}^+(a_2) \cdot \mathcal{M}^+(a_3)\ \cdots\ \mathcal{M}^+(a_i) \tag{3.8}$$

*Proof.* For (i) one uses the standard euclidean division algorithm by defining successively

$$y_1 := \frac{\kappa}{\lambda}, \quad y_{i+1} := (y_i - \lfloor y_i \rfloor)^{-1}, \quad a_i := \lfloor y_i \rfloor, \quad i = 1, 2, \ldots$$

(ii) is obvious by the definition of the recurrence relation (3.3).
(iii) For the proof of (3.4) write this quotient as

$$\frac{\mathsf{P}_i}{\mathsf{P}_{i-1}} = a_i + \frac{\mathsf{P}_{i-2}}{\mathsf{P}_{i-1}} = a_i +\ 1\ /\ \left(\frac{\mathsf{P}_{i-1}}{\mathsf{P}_{i-2}}\right).$$

(3.5) is similar. (3.6) can be shown by induction on $i$ and implies $\gcd(\mathsf{P}_i\ , \mathsf{Q}_i) = 1$ and (3.7).
(iv) Write

$$\begin{pmatrix} \mathsf{P}_i & \mathsf{P}_{i-1} \\ \mathsf{Q}_i & \mathsf{Q}_{i-1} \end{pmatrix} = \begin{pmatrix} \mathsf{P}_{i-1} & \mathsf{P}_{i-2} \\ \mathsf{Q}_{i-1} & \mathsf{Q}_{i-2} \end{pmatrix} \cdot \mathcal{M}^+(a_i)$$

and use induction. $\square$

**Remark 3.2.** For $\lambda, \kappa$ as in 3.1, all integer solutions of the linear diophantine equation

$$\kappa \cdot \mathfrak{x} + \lambda \cdot \mathfrak{x}' = 1$$

can be read off directly from the regular continued fraction expansion (3.2) of $\frac{\kappa}{\lambda}$. They are of the form

$$\mathfrak{x} = \mathfrak{x}_0 + \xi \cdot \lambda, \quad \mathfrak{x}' = \mathfrak{x}_0' - \xi \cdot \kappa, \quad \xi \in \mathbb{Z},$$

where

$$\mathfrak{x}_0 = \varepsilon\ \mathsf{Q}_{\nu-1} = \begin{cases} \frac{\varepsilon \cdot \lambda}{[a_\nu, a_{\nu-1}, \ldots, a_3, a_2]} & \text{if } a_2 \geq 2 \\ \\ \frac{\varepsilon \cdot \lambda}{[a_\nu, a_{\nu-1}, \ldots, a_4, a_3 + 1]} & \text{if } a_2 = 1 \end{cases}$$

and

$$\mathfrak{x}_0' = -\varepsilon\ \mathsf{P}_{\nu-1} = \begin{cases} \frac{-\varepsilon\cdot\kappa}{[a_\nu, a_{\nu-1},\ldots,a_2,a_1]} & \text{if } \ a_1 \geq 2 \\ \\ \frac{-\varepsilon\cdot\kappa}{[a_\nu, a_{\nu-1},\ldots,a_3,a_2+1]} & \text{if } \ a_1 = 1 \end{cases}$$

with

$$\varepsilon := \begin{cases} 1, & \text{if } \ \nu \text{ is even} \\ -1, & \text{if } \ \nu \text{ is odd} \end{cases}$$

(This is an immediate consequence of the equalities (3.4), (3.5) and (3.6) for $i = \nu$).

**Remark 3.3.** As it follows from a theorem of Lamé [40], the length $\nu$ of (3.2) is smaller than 5 multiplied by the number of digits in the decimal expansion of $\lambda$. Fixing $\kappa$, we may more precisely say that, since the smallest positive integer $\lambda$ for which the regular continued fraction expansion (3.2) of $\frac{\kappa}{\lambda}$ takes a given value $\nu = j$, is the $(j+1)$-Fibonacci number $\mathsf{Fib}\,(j+1)$,

$$\mathsf{Fib}\,(j) := \frac{\mathfrak{k}^{\,j} - (-\mathfrak{k})^{-j}}{\mathfrak{k} + \mathfrak{k}^{-1}}, \quad \mathfrak{k} := \frac{1+\sqrt{5}}{2},$$

the length $\nu$ can be bounded from above by

$$\nu \leq \frac{1 + \log_{10}(\lambda)}{\log_{10}(\mathfrak{k})} = (2.0780...)\,(1 + \log_{10}(\lambda)) < 5 \cdot \log_{10}(\lambda) \tag{3.9}$$

(For the early history on the estimations of $\nu$ the reader is referred to Shallit [58]. For better approximations of $\nu$ in certain number-regions, see Dixon [10] and Kilian [32].)

The analogue of lemma 3.1 for the *negative*-regular continued fractions is formulated as follows.

**Lemma 3.4.** *Let $\lambda, \kappa$ be two integers with $0 < \lambda < \kappa$, $\gcd(\lambda,\kappa) = 1$. Then:*
(i) *There exists always a uniquely determined negative-regular continued fraction expansion*

$$\frac{\kappa}{\lambda} = [\![c_1, c_2, \ldots, c_\rho]\!] \tag{3.10}$$

*of $\frac{\kappa}{\lambda}$ (with $c_j \geq 2$, $\forall j$, $1 \leq j \leq \rho$).*
(ii) *Defining the finite sequences $(\mathsf{R}_i)_{-1\leq i\leq\rho}$ and $(\mathsf{S}_i)_{-1\leq i\leq\rho}$ by*

$$\begin{cases} \mathsf{R}_{-1} = 0, \quad \mathsf{R}_0 = 1, \quad \mathsf{R}_1 = c_1, \quad \mathsf{R}_i = c_i\ \mathsf{R}_{i-1} - \mathsf{R}_{i-2},\ \forall i,\ \ 2 \leq i \leq \rho, \\ \\ \mathsf{S}_{-1} = -1, \quad \mathsf{S}_0 = 0, \quad \mathsf{S}_1 = 1, \quad \mathsf{S}_i = c_i\ \mathsf{S}_{i-1} - \mathsf{S}_{i-2},\ \forall i,\ \ 2 \leq i \leq \rho, \end{cases} \tag{3.11}$$

*we obtain*

$$\frac{\mathsf{R}_i}{\mathsf{S}_i} = [\![c_1, c_2, \ldots, c_i]\!],\ \forall i,\ \ 1 \leq i \leq \rho.$$

(This is the corresponding *$i$-th convergent* of the continued fraction (3.10)).
(iii) *The above sequences satisfy the following conditions*:
(a) *For all $i$, $1 \leq i \leq \rho$* (resp. $2 \leq i \leq \rho$),

$$\frac{\mathsf{R}_i}{\mathsf{R}_{i-1}} = [\![c_i, c_{i-1}, \ldots, c_2, c_1]\!] \tag{3.12}$$

resp.

$$\frac{\mathsf{S}_i}{\mathsf{S}_{i-1}} = [\![c_i, c_{i-1}, \ldots, c_3, c_2]\!] \tag{3.13}$$

(b) *For all $i$, $0 \leq i \leq \rho$,*

$$\mathsf{R}_{i-1}\ \mathsf{S}_i - \mathsf{R}_i\ \mathsf{S}_{i-1} = 1. \tag{3.14}$$

*In particular, for $i$, $1 \leq i \leq \rho$, we have $\gcd(\mathsf{R}_i\ , \mathsf{S}_i) = 1$, and*

$$\frac{\kappa}{\lambda} = \frac{\mathsf{R}_\rho}{\mathsf{S}_\rho} < \frac{\mathsf{R}_{\rho-1}}{\mathsf{S}_{\rho-1}} < \cdots < \frac{\mathsf{R}_3}{\mathsf{S}_3} < \frac{\mathsf{R}_2}{\mathsf{S}_2} < \frac{\mathsf{R}_1}{\mathsf{S}_1} \tag{3.15}$$

(iv) *For $c \in \mathbb{Z}$, setting*

$$\mathcal{M}^-(c) := \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

*we get for all $i$, $1 \leq i \leq \rho$,*

$$\begin{pmatrix} \mathsf{R}_i & -\mathsf{R}_{i-1} \\ \mathsf{S}_i & -\mathsf{S}_{i-1} \end{pmatrix} = \mathcal{M}^-(c_1) \cdot \mathcal{M}^-(c_2) \cdot \mathcal{M}^-(c_3) \cdots \mathcal{M}^-(c_i) \tag{3.16}$$

*Proof.* For (i) use the modified euclidean division algorithm and define successively

$$y_1 := \frac{\kappa}{\lambda}, \quad y_{i+1} := (\lceil y_i \rceil - y_i)^{-1}, \quad c_i := \lceil y_i \rceil, \quad i = 1, 2, \ldots \tag{3.17}$$

The proofs of the remaining assertions (ii)-(iv) are similar to those of 3.1. □

**Remark 3.5.** It is easy to show that the length $\rho$ of (3.10) equals $\kappa - 1$ if and only if $c_i = 2$, for *all* $i$, $1 \leq i \leq \rho$, and $\rho \leq \lfloor \frac{k-1}{2} \rfloor$ otherwise. (For $\kappa \gg 10 \cdot \log_{10}(\lambda)$, (3.9) implies that $\rho$ might take values $\gg \nu$; that this happens in the most "generic" case will become clear in rem. 3.8).

(b) Already Lagrange [39] and Möbius [44] knew for a wide palette of examples of "non-standard" continued fraction expansions how one should modify them to get regular, i.e., "usual" continued fractions by special substitutions. However, a systematic "mechanical method" for expressing a given semi-regular continued fraction via a regular one was first developed by Minkowski (cf. [43], pp. 116-118) and was further elucidated in Perron's classical book [50], Kap. V, §40. In fact, in order to write down (3.1) as a regular continued fraction, one has to define $\varepsilon_0 := 1, \varepsilon_\nu := 1$, to add a $\frac{1}{1+\cdots}$ in front of every negative $\varepsilon_i$, to replace all minus signs by plus signs, and finally to substitute $a_i - \frac{1}{2}((1 - \varepsilon_{i-1}) + (1 - \varepsilon_i))$ for each $a_i$, for all $i$, $1 \leq i \leq \nu$. Since this method demands some extra restrictive operations, like the elimination of the zeros and the short-cutting of the probably occuring 1's at the last step, it seems to be rather laborious. Moreover, since it is important for our later arguments to provide a detailed relationship between the regular and the negative-regular continued fraction expansion (by assuming the first one as given), we shall prefer to use another, slightly different approach. Proposition 3.6 describes the transfer process explicitly. This is actually a "folklore-type-result" and can be found (without proof) in Myerson's paper [47], p. 424, who examines the average length of negative-regular expansions, as well as in works of Hirzebruch and Zagier (cf. [26], p. 215; [64], p. 131) in a version concerning infinite continued fractions and serves as auxiliary tool for studying positive definite binary forms. Our proof is relatively short (compared with the above mentioned repetitive substitutions) as it makes use only of formal multiplications of certain $2 \times 2$ unimodular matrices.

**Proposition 3.6.** *If $\lambda, \kappa \in \mathbb{Z}$ with $0 < \lambda < \kappa$,* $\gcd(\lambda, \kappa) = 1$, *and*

$$\frac{\kappa}{\lambda} = [a_1, a_2, \ldots, a_\nu] = [\![c_1, c_2, \ldots, c_\rho]\!]$$

*are the regular and negative-regular continued fraction expansions of $\frac{\kappa}{\lambda}$, respectively, then $(c_1, c_2, \ldots, c_\rho)$, as ordered $\rho$-tuple, equals*

$$(c_1, c_2, \ldots, c_\rho) =$$

$$= \begin{cases} \Big(a_1 + 1, \underbrace{2, \ldots, 2}_{(a_2-1)\text{-times}}, a_3 + 2, \underbrace{2, \ldots, 2}_{(a_4-1)\text{-times}}, a_5 + 2, \ldots, a_{\nu-1} + 2, \underbrace{2, \ldots, 2}_{(a_\nu-1)\text{-times}}\Big), & \text{if } \nu \text{ even} \\[2ex] \Big(a_1 + 1, \underbrace{2, \ldots, 2}_{(a_2-1)\text{-times}}, a_3 + 2, \underbrace{2, \ldots, 2}_{(a_4-1)\text{-times}}, a_5 + 2, \ldots, \underbrace{2, \ldots, 2}_{(a_{\nu-1}-1)\text{-times}} a_\nu + 1\Big), & \text{if } \nu \text{ odd.} \end{cases}$$

*Proof.* At first define the matrices

$$\mathcal{U} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{V} := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \mathcal{W} := \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

from $\mathrm{SL}(2,\mathbb{Z})$. Obviously, $\mathcal{W}^2 = \mathrm{Id}$, and for all $a$, $a \in \mathbb{N}$, we have

$$\mathcal{M}^+(a) = \mathcal{W} \cdot \mathcal{U} \cdot \mathcal{V}^{a-1} \tag{3.18}$$

and

$$\left(\mathcal{M}^-(2)\right)^a = \mathcal{U} \cdot \mathcal{V}^a \cdot \mathcal{U}^{-1} \tag{3.19}$$

Using (3.18) and (3.19) we obtain, in addition,

$$\mathcal{M}^+(a) = \mathcal{M}^-(a+1) \cdot \mathcal{W} = \mathcal{W} \cdot \left(\mathcal{M}^-(2)\right)^{a-1} \cdot \mathcal{U} \tag{3.20}$$

and

$$\mathcal{U} \cdot \mathcal{M}^-(a) = \mathcal{M}^-(a+1) \tag{3.21}$$

for all $a$, $a \in \mathbb{N}$. If $\nu$ is even, then (3.20) implies

$$\mathcal{M}^+(a_1) \cdot \mathcal{M}^+(a_2) \;\cdots\; \mathcal{M}^+(a_{\nu-1}) \cdot \mathcal{M}^+(a_\nu) = \prod_{j=1}^{\frac{\nu}{2}} \left(\mathcal{M}^+(a_{2j-1}) \cdot \mathcal{M}^+(a_{2j})\right) =$$

$$\begin{aligned}
&= \prod_{j=1}^{\frac{\nu}{2}} \left(\left(\mathcal{M}^-(a_{2j-1}+1) \cdot \mathcal{W}\right) \cdot \left(\mathcal{W} \cdot \left(\mathcal{M}^-(2)\right)^{a_{2j}-1} \cdot \mathcal{U}\right)\right) = \\
&= \prod_{j=1}^{\frac{\nu}{2}} \left(\mathcal{M}^-(a_{2j-1}+1) \cdot \left(\left(\mathcal{M}^-(2)\right)^{a_{2j}-1} \cdot \mathcal{U}\right)\right) \overset{\text{by } (3.21)}{=} \\
&= \left(\left(\mathcal{M}^-(a_1+1) \cdot \left(\mathcal{M}^-(2)\right)^{a_2-1}\right) \cdot \prod_{j=2}^{\frac{\nu}{2}} \left(\mathcal{M}^-(a_{2j-1}+2) \cdot \left(\left(\mathcal{M}^-(2)\right)^{a_{2j}-1}\right)\right)\right) \cdot \mathcal{U}
\end{aligned}$$

and the desired relation follows directly from the equalities (3.8) and (3.16) of the previous lemmas because $\frac{\kappa}{\lambda} = \frac{\mathsf{R}_\rho}{\mathsf{S}_\rho} = \frac{\mathsf{P}_\nu}{\mathsf{Q}_\nu}$. If $\nu$ is odd, then using the above argument for the first $\nu - 1$ factors of the product of the corresponding $\mathcal{M}^+$-matrices, we get

$$\mathcal{M}^+(a_1) \cdot \mathcal{M}^+(a_2) \;\cdots\; \mathcal{M}^+(a_{\nu-1}) \cdot \mathcal{M}^+(a_\nu) =$$

$$\begin{aligned}
&= \left(\left(\mathcal{M}^-(a_1+1) \cdot \left(\mathcal{M}^-(2)\right)^{a_2-1}\right) \cdot \prod_{j=2}^{\frac{\nu-1}{2}} \left(\mathcal{M}^-(a_{2j-1}+2) \cdot \left(\left(\mathcal{M}^-(2)\right)^{a_{2j}-1}\right)\right)\right) \cdot \mathcal{U} \cdot \mathcal{M}^+(a_\nu) = \\
&= \mathcal{M}^-(a_1+1) \cdot \left(\mathcal{M}^-(2)\right)^{a_2-1} \cdot \left(\prod_{j=2}^{\frac{\nu-1}{2}} \mathcal{M}^-(a_{2j-1}+2) \cdot \left(\mathcal{M}^-(2)\right)^{a_{2j}-1}\right) \cdot \left(\mathcal{U} \cdot \mathcal{M}^-(a_\nu+1) \cdot \mathcal{W}\right).
\end{aligned}$$

Since $\mathcal{U} \cdot \mathcal{M}^-(a_\nu+1) \cdot \mathcal{W} = \mathcal{M}^+(a_\nu+1)$ has the same first column-vector as $\mathcal{M}^-(a_\nu+1)$ the conclusion follows again from (3.8) and (3.16). □

**Remark 3.7.** From the proof of 3.6 one easily verifies that for all indices $i$, $1 \le i \le \lfloor \frac{\nu}{2} \rfloor$, the convergents of these two continued fraction expansions of $\frac{\kappa}{\lambda}$ are connected by the following "matrix-transfer rule":

$$\begin{pmatrix} \mathsf{P}_{2i} & \mathsf{P}_{2i-1} \\ \mathsf{Q}_{2i} & \mathsf{Q}_{2i-1} \end{pmatrix} = \begin{pmatrix} \mathsf{R}_{a_2+a_4+\cdots+a_{2i}} & -\mathsf{R}_{a_2+a_4+\cdots+a_{2i}-1} \\ \mathsf{S}_{a_2+a_4+\cdots+a_{2i}} & -\mathsf{S}_{a_2+a_4+\cdots+a_{2i}-1} \end{pmatrix} \cdot \mathcal{U}$$

**Remark 3.8.** By prop. 3.6 the length $\rho$ of the negative-regular continued fraction expansion (3.10) of $\frac{\kappa}{\lambda}$ equals

$$\rho = \begin{cases} \sum_{i=1}^{\frac{\nu}{2}} a_{2i}, & \text{if } \nu \text{ even} \\ \left(\sum_{i=1}^{\frac{\nu-1}{2}} a_{2i}\right) + 1, & \text{if } \nu \text{ odd} \end{cases}$$

If for at least one $i$, $1 \leq i \leq \rho$, we have $c_i \geq 3$ (cf. rem. 3.5), we get

$$1 \leq \left\lfloor \frac{\nu+1}{2} \right\rfloor \leq \rho \leq \left\lfloor \frac{\kappa-1}{2} \right\rfloor .$$

Obviously, if all (or allmost all) $a_{2i}$'s are $\gg 2$ (assumption which expresses the "generic" case), then $\rho \gg \nu$, and the number $\rho - \nu$ of the extra modified euclidean division algorithms (3.17) (coming from the extra "twos") which one needs in order to determine *directly* the $(-)$-sign continued fraction expansion of $\frac{\kappa}{\lambda}$ (i.e., without using prop. 3.6) may become tremendously large!

(c) Up to unimodular transformations, the "lattice-geometry" of two-dimensional rational s.c.p. cones is completely describable by means of just two (relatively prime) integers ("parameters").

**Lemma 3.9.** *Let $N$ be a lattice of rank 2 and $\sigma \subset N_{\mathbb{R}}$ a two-dimensional rational s.c.polyhedral cone with* $\mathrm{Gen}(\sigma) = \{n_1, n_2\}$. *Then there exist a $\mathbb{Z}$-basis $\{\mathfrak{y}_1, \mathfrak{y}_2\}$ of $N$ and two integers $p = p_\sigma,\ q = q_\sigma \in \mathbb{Z}_{\geq 0}$ with $0 \leq p < q$,* $\gcd(p,q) = 1$, *such that*

$$n_1 = \mathfrak{y}_1, \quad n_2 = p\,\mathfrak{y}_1 + q\,\mathfrak{y}_2, \quad q = \mathrm{mult}\,(\sigma; N) = \frac{\det\left(\mathbb{Z}\; n_1 \oplus \mathbb{Z}\; n_2\right)}{\det\left(N\right)} .$$

*Moreover, if $\Phi$ is a $\mathbb{Z}$-module isomorphism $\Phi : N \to N$, then the above property is preserved by the same numbers $p = p_{\Phi_{\mathbb{R}}(\sigma)},\ q = q_{\Phi_{\mathbb{R}}(\sigma)}$ for the cone $\Phi_{\mathbb{R}}(\sigma) \subset N_{\mathbb{R}}$ with respect to the $\mathbb{Z}$-basis $\{\Phi(\mathfrak{y}_1), \Phi(\mathfrak{y}_2)\}$ of the lattice $N$.*

*Proof.* Choose an arbitrary $\mathbb{Z}$-basis $\{\mathfrak{n}_1, \mathfrak{n}_2\}$ of $N$ with $\mathfrak{n}_1 = n_1$. Since $\sigma$ is 2-dimensional, it is also simplicial; this means that $n_2$ may be expressed as a linear combination of the members of this $\mathbb{Z}$-basis having the form:

$$n_2 = \lambda_1\; \mathfrak{n}_1 + \lambda_2\; \mathfrak{n}_2, \quad \text{with} \quad \lambda_1 \in \mathbb{Z}, \quad \lambda_2 \in (\mathbb{Z} \smallsetminus \{0\}) .$$

Now define $q := |\lambda_2|$ and $p := [\lambda_1]_q$. Since $0 \leq p < q$ and

$$n_2 = p\; \mathfrak{n}_1 + q\; \left(\mathrm{sgn}\,(\lambda_2)\; \mathfrak{n}_2 + \frac{\lambda_1 - p}{q}\; \mathfrak{n}_1\right) ,$$

it suffices to consider $\mathfrak{y}_1 := \mathfrak{n}_1$ and $\mathfrak{y}_2 := \mathrm{sgn}(\lambda_2)\,\mathfrak{n}_2 + \frac{\lambda_1 - p}{q}\mathfrak{n}_1$. Furthermore, $\gcd(p,q) = 1$ and $q = \mathrm{mult}(\sigma; N)$, because $n_1, n_2$ are primitive. The last assertion is obvious. $\square$

**Definition 3.10.** If $N$ is a lattice of rank 2 and $\sigma \subset N_{\mathbb{R}}$ a two-dimensional rational s.c.polyhedral cone with $\mathrm{Gen}(\sigma) = \{n_1, n_2\}$, then we call $\sigma$ a *$(p,q)$-cone w.r.t. the basis* $\{\mathfrak{y}_1, \mathfrak{y}_2\}$, if $p = p_\sigma,\ q = q_\sigma$ as in lemma 3.9. (To avoid confusion, we should stress at this point that saying "w.r.t. the basis $\{\mathfrak{y}_1, \mathfrak{y}_2\}$" we just indicate the choice of one suitable $\mathbb{Z}$-basis of $N$ among *all* its $\mathbb{Z}$-bases in order to apply lemma 3.9 for $\sigma$; but, of course, if $\{\mathfrak{y}_1, \widehat{\mathfrak{y}}_2\}$ were a $\mathbb{Z}$-basis of $N$ having the *same* property, i.e., $n_2 = \widehat{p}\,\mathfrak{y}_1 + \widehat{q}\,\widehat{\mathfrak{y}}_2$, $0 \leq \widehat{p} < \widehat{q}$, $\gcd(\widehat{p}, \widehat{q}) = 1$, then obviously $p = \widehat{p}$ and $q = \widehat{q}$, i.e., $\widehat{\mathfrak{y}}_2 = \mathfrak{y}_2$!)

**Remark 3.11.** For $p, q$ as in lemma 3.9, there is a uniquely determined integer $p' = p'_\sigma$, $0 \leq p' < q$, such that

$$pp' \equiv 1 (\mathrm{mod}\ q), \quad (\text{i.e., } [\, pp']_q = 1) .$$

$p'$ is often called *the socius* of $p$. If $p \neq 0$ (which means that $q \neq 1$), then using a formula due to Voronoi (see [61], p. 183), $p'$ can be written as

$$p' = \left[ 3 - 2p + 6\; \left( \sum_{j=1}^{p-1}\; \left( \left\lfloor \frac{j\; q}{p} \right\rfloor \right)^2 \right) \right]_q .$$

**Proposition 3.12.** *Let $N$ be a lattice of rank 2 and $\sigma, \tau \subset N_{\mathbb{R}}$ two 2-dimensional rational s.c.p. cones with* $\mathrm{Gen}(\sigma) = \{n_1, n_2\}$, $\mathrm{Gen}(\tau) = \{u_1, u_2\}$. *Then the following conditions are equivalent*:

(i) *There exists an isomorphism of germs*: $(U_\sigma, \mathrm{orb}\,(\sigma)) \cong (U_\tau, \mathrm{orb}\,(\tau))$.

(ii) *There exists a $\mathbb{Z}$-module isomorphism $\varpi : N \to N$, whose scalar extension $\varpi = \varpi_{\mathbb{R}} : N_{\mathbb{R}} \to N_{\mathbb{R}}$ has the property*: $\varpi(\sigma) = \tau$.

(iii) *For the numbers $p_\sigma$, $p_\tau$, $q_\sigma$, $q_\tau$ associated to $\sigma, \tau$ w.r.t. a $\mathbb{Z}$-basis $\{\mathfrak{y}_1, \mathfrak{y}_2\}$ of $N$ (as in 3.9) we have*

$$q_\tau = q_\sigma \quad \text{and} \quad \left\{ \begin{array}{ll} \text{either} & p_\tau = p_\sigma \\ \text{or} & p_\tau = p'_\sigma \end{array} \right. \tag{3.22}$$

*Proof.* For the equivalence (i)$\Leftrightarrow$(ii) see Ewald [13], Ch. VI, thm. 2.11, pp. 222-223.
(ii)$\Rightarrow$(iii): Since $\{\varpi(\mathfrak{y}_1), \varpi(\mathfrak{y}_2)\}$ is a $\mathbb{Z}$-basis of $N$ too, there exists a matrix $\mathcal{A} \in \mathrm{SL}(2, \mathbb{Z})$ such that

$$(\varpi(\mathfrak{y}_1), \varpi(\mathfrak{y}_2)) = (\mathfrak{y}_1, \mathfrak{y}_2) \cdot \mathcal{A} \tag{3.23}$$

Now $\varpi(\sigma) = \tau$ implies $\varpi(\mathrm{Gen}\,(\sigma)) = \mathrm{Gen}(\tau)$, i.e.,

$$\text{either} \quad (\varpi(n_1) = u_1 \ \ \& \ \ \varpi(n_2) = u_2) \quad \text{or} \quad (\varpi(n_1) = u_2 \ \ \& \ \ \varpi(n_2) = u_1)\ .$$

In the first case we obtain

$$(\varpi(n_1), \varpi(n_2)) = (\varpi(\mathfrak{y}_1), \varpi(\mathfrak{y}_2)) \begin{pmatrix} 1 & p_\sigma \\ 0 & q_\sigma \end{pmatrix} = (\mathfrak{y}_1, \mathfrak{y}_2) \begin{pmatrix} 1 & p_\tau \\ 0 & q_\tau \end{pmatrix} \tag{3.24}$$

In the second case:

$$(\varpi(n_1), \varpi(n_2)) = (\varpi(\mathfrak{y}_1), \varpi(\mathfrak{y}_2)) \begin{pmatrix} 1 & p_\sigma \\ 0 & q_\sigma \end{pmatrix} = (\mathfrak{y}_1, \mathfrak{y}_2) \begin{pmatrix} p_\tau & 1 \\ q_\tau & 0 \end{pmatrix} \tag{3.25}$$

Thus, by (3.23), (3.24) and (3.25) we get

$$\mathcal{A} = \begin{pmatrix} 1 & \dfrac{p_\tau - p_\sigma}{q_\sigma} \\ 0 & \dfrac{q_\tau}{q_\sigma} \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} p_\tau & \dfrac{1 - p_\sigma\, p_\tau}{q_\sigma} \\ q_\tau & -\dfrac{p_\sigma\, q_\tau}{q_\sigma} \end{pmatrix},$$

respectively. In the first case $\det(\mathcal{A})$ has to be equal to 1, which means that $q_\sigma = q_\tau$ and $p_\tau - p_\sigma \equiv 0$ (mod $q_\sigma$), i.e., $p_\tau = p_\sigma$ (because $0 \le p_\sigma, p_\tau \le q_\sigma = q_\tau$). In the second case, $\det(\mathcal{A}) = -1$; hence, $q_\sigma = q_\tau$ and $1 - p_\sigma\, p_\tau \equiv 0$ (mod $q_\sigma$), i.e., $p_\tau = p'_\sigma$.
(ii)$\Leftarrow$(iii): If $q_\sigma = q_\tau$ and $p_\sigma = p_\tau$, we define $\varpi := \mathrm{id}_{N_{\mathbb{R}}}$. Otherwise, $q_\sigma = q_\tau$ and $p_\tau = p'_\sigma$, and for an $\mathbf{x} \in N$ with $\mathbf{x} = \lambda_1 \mathfrak{y}_1 + \lambda_2 \mathfrak{y}_2$, ($\lambda_1$, $\lambda_2 \in \mathbb{Z}$), we set

$$\varpi(\mathbf{x}) := \frac{1}{q_\sigma} (\lambda_2\ u_1 + (\lambda_1 q_\sigma - p_\sigma \lambda_2)\ u_2)\ .$$

Its scalar extension $\varpi = \varpi_{\mathbb{R}} : N_{\mathbb{R}} \to N_{\mathbb{R}}$ is the $\mathbb{R}$-vector space isomorphism with the desired property. $\square$

**Remark 3.13.** Up to replacement of $p$ by its socius $p'$ (which corresponds just to the interchange of the analytic coordinates), these two numbers $p$ and $q$ parametrize uniquely the analytic isomorphism class of the germ $(U_\sigma, \mathrm{orb}\,(\sigma))$. (In the terminology which will be introduced in the next section, if $q \neq 1$, then $(U_\sigma, \mathrm{orb}\,(\sigma))$ is nothing but a cyclic quotient singularity of "type" $\frac{1}{q}(q - p, 1)$, cf. Oda [48], 1.24; prop. 3.12 can be therefore regarded as a special case of the isomorphism criterion 4.6).

**Lemma 3.14.** *Let $N$ be a lattice of rank 2, $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual and $\sigma \subset N_{\mathbb{R}}$ a two-dimensional $(p, q)$-cone w.r.t. a $\mathbb{Z}$-basis $\{\mathfrak{y}_1, \mathfrak{y}_2\}$ of $N$. If we denote by $\{\mathfrak{m}_1, \mathfrak{m}_2\}$ the dual $\mathbb{Z}$-basis of $\{\mathfrak{y}_1, \mathfrak{y}_2\}$ in $M$, then the cone $\sigma^\vee \subset M_{\mathbb{R}}$ is a $(q - p, q)$-cone w.r.t. $\{\mathfrak{m}_2, \mathfrak{m}_1 - \mathfrak{m}_2\}$.*

*Proof.* Let $\mathrm{Gen}(\sigma) = \{n_1, n_2\}$ be the two minimal generators of $\sigma$ with $n_1 = \mathfrak{y}_1$, $n_2 = p\,\mathfrak{y}_1 + q\,\mathfrak{y}_2$. Then

$$\sigma^\vee = \ \mathrm{pos}\left(\{\mathfrak{m}_2, q\mathfrak{m}_1 - p\mathfrak{m}_2\}\right) = \ \mathrm{pos}\left(\{\mathfrak{m}_2, (q-p)\ \mathfrak{m}_2 + q\ (\mathfrak{m}_1 - \mathfrak{m}_2)\}\right)$$

and $\mathrm{Gen}(\sigma^\vee) = \{\mathfrak{m}_2, (q-p)\ \mathfrak{m}_2 + q\ (\mathfrak{m}_1 - \mathfrak{m}_2)\}$. Since $\{\mathfrak{m}_2, \mathfrak{m}_1 - \mathfrak{m}_2\}$ is a $\mathbb{Z}$-basis of $M$ and

$$0 < q - p < q, \quad \gcd(q-p, q) = 1,$$

we are done. □

(d) From now on, and for the rest of the present section, we fix a lattice $N$ of rank 2, its dual $M$, a *non-basic* two-dimensional $(p,q)$-cone $\sigma \subset N_{\mathbb{R}}$ w.r.t. a $\mathbb{Z}$-basis $\{\mathfrak{y}_1, \mathfrak{y}_2\}$ of $N$, the dual basis $\{\mathfrak{m}_1, \mathfrak{m}_2\}$ of $\{\mathfrak{y}_1, \mathfrak{y}_2\}$ in $M$, and the dual cone $\sigma^\vee \subset M_{\mathbb{R}}$ of $\sigma$. Moreover, we consider *both* (+ & −)-sign continued fraction expansions of both rationals $\frac{q}{q-p}$ and $\frac{q}{p}$:

$$\boxed{\begin{array}{c} \dfrac{q}{q-p} = [a_1^\vee, a_2^\vee, \ldots, a_\nu^\vee] = [\![b_1, b_2, \ldots, b_\rho]\!] \\[2ex] \dfrac{q}{p} = \dfrac{q}{q-(q-p)} = [a_1, a_2, \ldots, a_k] = [\![b_1^\vee, b_2^\vee, \ldots, b_t^\vee]\!] \end{array}} \tag{3.26}$$

and

$$\left(\frac{\mathsf{P}_i^\vee}{\mathsf{Q}_i^\vee}\right)_{-1 \le i \le \nu}, \quad \left(\frac{\mathsf{R}_i}{\mathsf{S}_i}\right)_{-1 \le i \le \rho}, \quad \left(\frac{\mathsf{P}_i}{\mathsf{Q}_i}\right)_{-1 \le i \le k}, \quad \left(\frac{\mathsf{R}_i^\vee}{\mathsf{S}_i^\vee}\right)_{-1 \le i \le t}$$

the corresponding finite sequences of their convergents. It is well-known (cf. [56], p. 223, [48], p. 29) that

$$(b_1 + b_2 + \cdots + b_\rho) - \rho = (b_1^\vee + b_2^\vee + \cdots + b_t^\vee) - t = \rho + t - 1\,.$$

On the other hand, examining only the two (+)-sign expansions, the relationship between their entries can be written (strangely enough) in a *very simple* form:

**Lemma 3.15.** *For the ordered pair of the first two entries of the regular continued fraction expansions* (3.26) *of* $\frac{q}{q-p}$ *and* $\frac{q}{p}$ *we obtain*

$$(a_1^\vee, a_1) \notin \{(1,1)\} \cup (\mathbb{Z}_{\ge 2})^2 \tag{3.27}$$

*In particular, there are only two possibilities; namely either*

$$(a_1^\vee = 1 \ \ \& \ \ a_1 \ne 1) \Longleftrightarrow k = \nu - 1 \tag{3.28}$$

*or*

$$(a_1^\vee \ne 1 \ \ \& \ \ a_1 = 1) \Longleftrightarrow k = \nu + 1 \tag{3.29}$$

*In the first case, we have*

$$\boxed{a_i^\vee = \left\{ \begin{array}{ll} a_1 - 1, & \text{for} \ \ i = 2 \\ a_{i-1}, & \text{for} \ \ 3 \le i \le \nu \ \ (= k+1) \end{array} \right.} \tag{3.30}$$

*In the second case,*

$$\boxed{a_i^\vee = \left\{ \begin{array}{ll} a_2 + 1, & \text{for} \ \ i = 1 \\ a_{i+1}, & \text{for} \ \ 2 \le i \le \nu \ \ (= k-1) \end{array} \right.} \tag{3.31}$$

*Moreover, in case* (3.28) *we get*:

$$\mathsf{P}_i = \mathsf{P}^{\vee}_{i+1}, \;\; \mathsf{Q}_i = \mathsf{P}^{\vee}_{i+1} - \mathsf{Q}^{\vee}_{i+1}, \; \forall i, \; 1 \le i \le k \;\; (= \nu - 1) \tag{3.32}$$

*while in case* (3.29):

$$\mathsf{P}_i = \mathsf{P}^{\vee}_{i-1}, \;\; \mathsf{Q}_i = \mathsf{P}^{\vee}_{i-1} - \mathsf{Q}^{\vee}_{i-1}, \; \forall i, \; 1 \le i \le k \;\; (= \nu + 1) \tag{3.33}$$

*Proof.* For $p = 1$, we have

$$\frac{q}{q-1} = \;[1, q-1] = \underbrace{[\![2, 2, \ldots, 2, 2]\!]}_{(q-1)\text{-times}}$$

and the assertion is trivially true. From now on, suppose $p \ge 2$. To pass from the ordered $t$-tuple $(b_1^{\vee}, \ldots, b_t^{\vee})$ to the ordered $\rho$-tuple $(b_1, \ldots, b_\rho)$ we use the so-called Riemenschneider's *point diagram* (cf. [56], pp. 222-223). This is a plane arrangement of points

$$\mathbf{PD} \subset \{1, \ldots, t\} \times \{1, \ldots, \rho\}$$

with $t$ rows and $\rho$ columns, which is depicted as follows:

$$\begin{array}{llll}
\underbrace{\bullet\bullet\bullet\cdots\bullet\bullet\bullet}_{(b_1^{\vee}-1)\text{-times}} & & & \\
& \underbrace{\bullet\bullet\bullet\cdots\bullet\bullet\bullet}_{(b_2^{\vee}-1)\text{-times}} & & \\
& & \underbrace{\bullet\bullet\bullet\cdots\bullet\bullet\bullet}_{(b_3^{\vee}-1)\text{-times}} & \\
& & & \ddots
\end{array}$$

and has the property that

$$\# \; \{j\text{-th column points}\} = b_j - 1, \;\; \forall j, \;\; 1 \le j \le \rho \;.$$

Separating the columns with exactly one entry, we consider the first column whose number of points is $\ge 2$. (Such a column exists always for $p \ge 2$). Obviously, this number of points has to be equal to

$$\# \left\{ \begin{array}{c} \text{“twos” occuring after the first position} \\ \text{in the ordered } t\text{-tuple } (b_1^{\vee}, b_2^{\vee}, \ldots, b_t^{\vee}) \end{array} \right\} = a_2 + 1 \;.$$

Repeating the same procedure for the second, third, ... etc., column of $\mathbf{PD}$ whose numbers of points are $\ge 2$, and using the same argument, we may rewrite the point diagram as:

$$\begin{array}{lll}
\overbrace{\bullet\bullet\bullet\cdots\bullet\bullet\bullet}^{(a_1-1)\text{-times}} & \bullet & \\
& \left.\begin{array}{c}\bullet\\ \bullet\\ \vdots\\ \bullet\end{array}\right\}(a_2-1)\text{-times} & \\
& \overbrace{\bullet\bullet\bullet\cdots\bullet\bullet\bullet}^{a_3\text{-times}} & \bullet \\
& & \left.\begin{array}{c}\bullet\\ \bullet\\ \vdots\\ \bullet\end{array}\right\}(a_4-1)\text{-times} \\
& & \bullet\cdots \qquad\qquad \ddots
\end{array}$$

Inductively it is easy to show that

$$(b_1, b_2, \ldots, b_\rho) =$$

$$= \begin{cases} \left( \underbrace{2,\ldots,2}_{(a_1-1)\text{-times}}, a_2+2, \underbrace{2,\ldots,2}_{(a_3-1)\text{-times}}, a_4+2, \ldots, a_{k-2}+2, \underbrace{2,\ldots,2}_{(a_{k-1}-1)\text{-times}}, a_k+1 \right), & \text{if } k \text{ even} \\ \left( \underbrace{2,\ldots,2}_{(a_1-1)\text{-times}}, a_2+2, \underbrace{2,\ldots,2}_{(a_3-1)\text{-times}}, a_4+2, \ldots, \underbrace{2,\ldots,2}_{(a_{k-2}-1)\text{-times}}, a_{k-1}+2, \underbrace{2,\ldots,2}_{(a_k-1)\text{-times}} \right), & \text{if } k \text{ odd} \end{cases}$$

On the other hand, proposition 3.6 implies

$$(b_1, b_2, \ldots, b_\rho) =$$

$$= \begin{cases} \left( a_1^\vee+1, \underbrace{2,\ldots,2}_{(a_2^\vee-1)\text{-times}}, a_3^\vee+2, \underbrace{2,\ldots,2}_{(a_4^\vee-1)\text{-times}}, a_5^\vee+2, \ldots, a_{\nu-1}^\vee+2, \underbrace{2,\ldots,2}_{(a_\nu^\vee-1)\text{-times}} \right), & \text{if } \nu \text{ even} \\ \left( a_1^\vee+1, \underbrace{2,\ldots,2}_{(a_2^\vee-1)\text{-times}}, a_3^\vee+2, \underbrace{2,\ldots,2}_{(a_4^\vee-1)\text{-times}}, a_5^\vee+2, \ldots, \underbrace{2,\ldots,2}_{(a_{\nu-1}^\vee-1)\text{-times}} a_\nu^\vee+1 \right), & \text{if } \nu \text{ odd} \end{cases}$$

Since the "twos" must be placed at the same positions, we obtain (3.27), (3.28), (3.29); moreover, since $a_\nu^\vee$ and $a_k$ are $\geq 2$, direct comparison gives (3.30), (3.31). Finally, the equality (3.32) (resp. (3.33)) is an immediate consequence of (3.30) (resp. (3.31)). $\square$

We define now

$$d_0^\vee := 1, \quad d_i^\vee := 1 + a_2^\vee + a_4^\vee + \cdots + a_{2i}^\vee, \quad \forall i, \quad 1 \leq i \leq \left\lfloor \tfrac{\nu-1}{2} \right\rfloor,$$

and

$$d_0 := 1, \quad d_i := 1 + a_2 + a_4 + \cdots + a_{2i}, \quad \forall i, \quad 1 \leq i \leq \left\lfloor \tfrac{k-1}{2} \right\rfloor,$$

as well as

$$J := \begin{cases} \left\{ d_1^\vee, d_2^\vee, \ldots, d^\vee_{\lfloor \frac{\nu-1}{2} \rfloor} \right\}, & \text{in case (3.28)} \\ \left\{ d_0^\vee, d_1^\vee, \ldots, d^\vee_{\lfloor \frac{\nu-1}{2} \rfloor} \right\}, & \text{in case (3.29)} \end{cases}$$

and in the dual sense

$$J^\vee := \begin{cases} \left\{ d_0, d_2, \ldots, d_{\lfloor \frac{k-1}{2} \rfloor} \right\}, & \text{in case (3.28)} \\ \left\{ d_1, d_1, \ldots, d_{\lfloor \frac{k-1}{2} \rfloor} \right\}, & \text{in case (3.29)} \end{cases}$$

To formulate the main theorem of this section, let us further define

$$\Theta_\sigma := \operatorname{conv}(\sigma \cap (N \setminus \{\mathbf{0}\})) \subset N_{\mathbb{R}}, \quad \text{resp.} \quad \Theta_{\sigma^\vee} := \operatorname{conv}(\sigma^\vee \cap (M \setminus \{\mathbf{0}\})) \subset M_{\mathbb{R}},$$

denote by $\partial\Theta_\sigma^{\mathbf{cp}}$ (resp. $\partial\Theta_{\sigma^\vee}^{\mathbf{cp}}$) the part of the boundary polygon $\partial\Theta_\sigma$ (resp. $\partial\Theta_{\sigma^\vee}$) containing only its compact edges, and write $\operatorname{vert}(\partial\Theta_\sigma^{\mathbf{cp}})$ (resp. $\operatorname{vert}(\partial\Theta_{\sigma^\vee}^{\mathbf{cp}})$) for the set of the vertices of $\partial\Theta_\sigma^{\mathbf{cp}}$ (resp. $\partial\Theta_{\sigma^\vee}^{\mathbf{cp}}$).

**Theorem 3.16 (Determination of the vertices by Kleinian approximations).** *Consider lattice points* $(\mathbf{u}_j)_{0\leq j\leq \rho+1}$, $(\mathbf{v}_i)_{0\leq i\leq k+1}$ *of* $N$ *and lattice points* $\left(\mathbf{u}_j^{\vee}\right)_{0\leq j\leq t+1}$, $(\mathbf{v}_i^{\vee})_{0\leq i\leq \nu+1}$ *of* $M$ *defined by the vectorial recurrence relations*:

$$\left\{\begin{array}{c} \mathbf{u}_0 = n_1 = \mathfrak{y}_1, \quad \mathbf{u}_1 = \mathfrak{y}_1 + \mathfrak{y}_2, \\ \mathbf{u}_j = b_{j-1}\ \mathbf{u}_{j-1} - \mathbf{u}_{i-2},\ \forall j,\ \ 2\leq j\leq \rho+1;\ [\mathbf{u}_{\rho+1} = n_2] \end{array}\right. \tag{3.34}$$

*and*

$$\left\{\begin{array}{c} \mathbf{u}_0^{\vee} = \mathfrak{m}_2, \quad \mathbf{u}_1^{\vee} = \mathfrak{m}_1, \\ \mathbf{u}_j^{\vee} = b_{j-1}^{\vee}\ \mathbf{u}_{j-1}^{\vee} - \mathbf{u}_{j-2}^{\vee},\ \forall j,\ \ 2\leq j\leq t+1;\ \left[\mathbf{u}_{t+1}^{\vee} = (q-p)\,\mathfrak{m}_2 + q\,(\mathfrak{m}_1 - \mathfrak{m}_2)\right] \end{array}\right. \tag{3.35}$$

*and by the* "Kleinian recurrence relations" [34], [35], [36]:

$$\left\{\begin{array}{ll} \mathbf{v}_i = \mathsf{Q}_{i-1}\,\mathfrak{y}_1 + \mathsf{P}_{i-1}\,\mathfrak{y}_2, & \forall i,\ \ 0\leq i\leq k+1 \\ \mathbf{v}_i^{\vee} = \mathsf{Q}_{i-1}^{\vee}\,\mathfrak{m}_2 + \mathsf{P}_{i-1}^{\vee}\ (\mathfrak{m}_1 - \mathfrak{m}_2)\,, & \forall i,\ \ 0\leq i\leq \nu+1 \end{array}\right. \tag{3.36}$$

*respectively. Then*

$$\partial\Theta_{\sigma}^{\mathbf{cp}} \cap N = \{\mathbf{u}_j\ \mid 0\leq j\leq \rho+1\}\,,\quad \partial\Theta_{\sigma^{\vee}}^{\mathbf{cp}} \cap M = \left\{\mathbf{u}_j^{\vee}\ \mid 0\leq j\leq t+1\right\} \tag{3.37}$$

*and*

$$\mathrm{vert}\,(\partial\Theta_{\sigma}^{\mathbf{cp}}) = \{\mathbf{u}_j\ \mid j\in J\cup\{0,\rho+1\}\}\,,\quad \mathrm{vert}\,(\partial\Theta_{\sigma^{\vee}}^{\mathbf{cp}}) = \left\{\mathbf{u}_j^{\vee}\ \mid j\in J^{\vee}\cup\{0,t+1\}\right\} \tag{3.38}$$

*In particular,*

$$\boxed{\forall i,\ 1\leq i\leq \left\lfloor\tfrac{k}{2}\right\rfloor:\quad \mathbf{v}_{2i} = \left\{\begin{array}{ll} \mathbf{u}_{d_i^{\vee}}\ , & \text{in case (3.28)} \\ \mathbf{u}_{d_{i-1}^{\vee}}\ , & \text{in case (3.29)} \end{array}\right.} \tag{3.39}$$

*and*

$$\boxed{\forall i,\ 1\leq i\leq \left\lfloor\tfrac{\nu}{2}\right\rfloor:\quad \mathbf{v}_{2i}^{\vee} = \left\{\begin{array}{ll} \mathbf{u}_{d_i}^{\vee}\ , & \text{in case (3.29)} \\ \mathbf{u}_{d_{i-1}}^{\vee}\ , & \text{in case (3.28)} \end{array}\right.} \tag{3.40}$$

*and therefore*

$$\boxed{\mathrm{vert}\,(\partial\Theta_{\sigma}^{\mathbf{cp}}) = \{\mathbf{v}_0\}\cup\left\{\mathbf{v}_{2i}\ \middle|\ 1\leq i\leq \left\lfloor\tfrac{k}{2}\right\rfloor\right\}\cup\{\mathbf{v}_{k+1}\}} \tag{3.41}$$

*and*

$$\boxed{\mathrm{vert}\,(\partial\Theta_{\sigma^{\vee}}^{\mathbf{cp}}) = \{\mathbf{v}_0^{\vee}\}\cup\left\{\mathbf{v}_{2i}^{\vee}\ \middle|\ 1\leq i\leq \left\lfloor\tfrac{\nu}{2}\right\rfloor\right\}\cup\left\{\mathbf{v}_{\nu+1}^{\vee}\right\}} \tag{3.42}$$

*respectively. This means that the vertices of* $\partial\Theta_{\sigma}^{\mathbf{cp}}$ *and* $\partial\Theta_{\sigma^{\vee}}^{\mathbf{cp}}$ *can be read off just from the entries* (*with* even *and* final *indices*) *of the regular continued fraction expansions* (3.26), *because* (3.36) *are uniquely determined by the vectorial recurrence relations*:

$$\boxed{\left\{\begin{array}{lll} \mathbf{v}_0 = n_1 = \mathfrak{y}_1,\ \mathbf{v}_1 = \mathfrak{y}_2, & \mathbf{v}_i = a_{i-1}\ \mathbf{v}_{i-1} + \mathbf{v}_{i-2},\ \forall i,\ \ 2\leq i\leq k+1 \\ \mathbf{v}_0^{\vee} = \mathfrak{m}_2,\ \mathbf{v}_1^{\vee} = \mathfrak{m}_1 - \mathfrak{m}_2, & \mathbf{v}_i^{\vee} = a_{i-1}^{\vee}\ \mathbf{v}_{i-1}^{\vee} + \mathbf{v}_{i-2}^{\vee},\ \forall i,\ \ 2\leq i\leq \nu+1 \end{array}\right.} \tag{3.43}$$

*Proof.* At first notice that $\mathbf{v}_0 = \mathbf{u}_0 = n_1 = \mathfrak{y}_1$, $\mathbf{v}_{k+1} = \mathbf{u}_{\rho+1} = n_2$, and that the lattice points $\mathbf{u}_j$ defined by (3.34) can be determined by the vectorial matrix multiplication:

$$\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \\ \vdots \\ \mathbf{u}_{\rho-1} \\ \mathbf{u}_\rho \end{pmatrix} = \begin{pmatrix} b_1 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & b_2 & -1 & \cdots & \cdots & 0 \\ 0 & -1 & b_3 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & -1 & b_\rho \end{pmatrix}^{-1} \begin{pmatrix} n_1 \\ 0 \\ \vdots \\ \\ \vdots \\ 0 \\ n_2 \end{pmatrix}$$

(That this matrix is indeed invertible, is well-known from the theory of continuants, cf. [50], I, §4. Furthermore, its determinant equals $q = \mathsf{R}_\rho$). Computing the corresponding adjoint matrix and performing the multiplication, we obtain for all $j$, $0 \le j \le \rho + 1$,

$$\mathbf{u}_j = \left(\frac{\widetilde{\mathsf{R}}_{j-1}}{q}\right) n_1 + \left(\frac{\mathsf{R}_{j-1}}{q}\right) n_2 = \frac{1}{q}\left(\widetilde{\mathsf{R}}_{j-1} + p\,\mathsf{R}_{j-1}\right) \mathfrak{y}_1 + \mathsf{R}_{j-1}\,\mathfrak{y}_2 \tag{3.44}$$

where $\left(\widetilde{\mathsf{R}}_j\right)_{-1\le j\le\rho}$ is a finite sequence of integers defined by the recurrence relations:

$$\widetilde{\mathsf{R}}_{-1} = q, \quad \widetilde{\mathsf{R}}_0 = q - p, \quad \widetilde{\mathsf{R}}_j = b_j\,\widetilde{\mathsf{R}}_{j-1} - \widetilde{\mathsf{R}}_{j-2}, \ \ \forall j, \ \ 1 \le j \le \rho$$

(with $\widetilde{\mathsf{R}}_{\rho-2} = b_\rho$, $\widetilde{\mathsf{R}}_{\rho-1} = 1$, $\widetilde{\mathsf{R}}_\rho = 0$). If we define, in addition, another finite sequence $\left(\widetilde{\widetilde{\mathsf{R}}}_j\right)_{-1\le j\le\rho}$ of integers via

$$\widetilde{\widetilde{\mathsf{R}}}_{-1} = \widetilde{\widetilde{\mathsf{R}}}_0 = 1, \quad \widetilde{\widetilde{\mathsf{R}}}_1 = b_1 - 1, \quad \widetilde{\widetilde{\mathsf{R}}}_j = b_j\,\widetilde{\widetilde{\mathsf{R}}}_{j-1} - \widetilde{\widetilde{\mathsf{R}}}_{j-2}, \ \ \forall j, \ \ 1 \le j \le \rho$$

(with $\widetilde{\widetilde{\mathsf{R}}}_\rho = p$), then for all $j$, $0 \le j \le \rho$, we get the equalities:

$$\begin{cases} \widetilde{\mathsf{R}}_{j-1}\,\mathsf{R}_j - \widetilde{\mathsf{R}}_j\,\mathsf{R}_{j-1} = q \\ \mathsf{R}_j\,\widetilde{\widetilde{\mathsf{R}}}_{j-1} - \mathsf{R}_{j-1}\,\widetilde{\widetilde{\mathsf{R}}}_j = 1 \\ \widetilde{\mathsf{R}}_j + p\,\mathsf{R}_j - q\,\widetilde{\widetilde{\mathsf{R}}}_j = 0 \end{cases} \tag{3.45}$$

On the other hand, using the notation introduced in the proof of 3.6, we deduce:

$$\begin{pmatrix} \mathsf{R}_j & -\mathsf{R}_{j-1} \\ \widetilde{\widetilde{\mathsf{R}}}_j & -\widetilde{\widetilde{\mathsf{R}}}_{j-1} \end{pmatrix} = \mathcal{W} \cdot \left(\prod_{s=1}^{j} \mathcal{M}^-(b_s)\right) \overset{\text{by (3.16)}}{=} \mathcal{W} \cdot \begin{pmatrix} \mathsf{R}_j & -\mathsf{R}_{j-1} \\ \mathsf{S}_j & -\mathsf{S}_{j-1} \end{pmatrix}.$$

Hence, $\widetilde{\widetilde{\mathsf{R}}}_j = \mathsf{R}_j - \mathsf{S}_j$, and for all $j$, $0 \le j \le \rho + 1$, (3.44), (3.45) give:

$$\mathbf{u}_j = \widetilde{\widetilde{\mathsf{R}}}_{j-1}\,\mathfrak{y}_1 + \mathsf{R}_{j-1}\,\mathfrak{y}_2 = (\mathsf{R}_{j-1} - \mathsf{S}_{j-1})\,\mathfrak{y}_1 + \mathsf{R}_{j-1}\,\mathfrak{y}_2 \tag{3.46}$$

Clearly, $\mathbf{u}_j$'s belong to $\Theta_\sigma \cap N$ (cf. (3.15)). That these points are exactly the lattice points of $\partial\Theta_\sigma^{\mathbf{cp}} \cap N$ was proved by Oda in [48], lemma 1.20, pp. 25-26, by making use of the fact that $\{\mathbf{u}_0, \mathbf{u}_1\}$ form a $\mathbb{Z}$-basis of $N$ and induction on $j$. Since for all $j$, $2 \le j \le \rho + 1$,

$$(\mathbf{u}_{j-2}, \ \mathbf{u}_{j-1}, \ \mathbf{u}_j \quad \text{are collinear}) \Longleftrightarrow b_{j-1} = 2,$$

the first equality in (3.38) follows from proposition 3.6. To prove (3.39), note that in case (3.28) the equality (3.32) and the matrix-transfer rule of rem. 3.7 imply

$$\mathsf{P}_{2i-1} = \mathsf{P}^\vee_{2i} = \mathsf{R}_{d^\vee_i - 1}, \quad \mathsf{Q}_{2i-1} = \mathsf{Q}^\vee_{2i} = \mathsf{R}_{d^\vee_i - 1} - \mathsf{S}_{d^\vee_i - 1}$$

for all $i$, $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$. This means that

$$\mathbf{v}_{2i} = \mathsf{Q}_{2i-1}\,\mathfrak{n}_1 + \mathsf{P}_{2i-1}\,\mathfrak{n}_2 = \mathsf{Q}^{\vee}_{2i}\,\mathfrak{n}_1 + \mathsf{P}^{\vee}_{2i}\,\mathfrak{n}_2 = \left(\mathsf{R}_{d_i^{\vee}-1} - \mathsf{S}_{d_i^{\vee}-1}\right)\mathfrak{n}_1 + \mathsf{R}_{d_i^{\vee}-1}\,\mathfrak{n}_2 \overset{\text{by (3.46)}}{=} \mathbf{u}_{d_i^{\vee}}\ .$$

Case (3.29) can be treated similarly. Finally, all assertions (3.37), (3.38), (3.40), (3.42) concerning the dual cone $\sigma^{\vee}$ are proved by the same method (i.e., just by replacing $k$ and $\rho$ by $\nu$ and $t$, respectively, interchanging the "checks" in the dual sense, and using the same arguments for $\mathrm{Gen}(\sigma^{\vee})$). $\square$

**Remark 3.17.** (i) As in the case of $\mathbf{u}_j$'s, one may determine $\mathbf{v}_i$'s in (3.43) by the following vectorial matrix multiplication:

$$\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \\ \vdots \\ \mathbf{v}_{k-1} \\ \mathbf{v}_k \end{pmatrix} = \begin{pmatrix} a_1 & -1 & 0 & \cdots & \cdots & 0 \\ 1 & a_2 & -1 & \cdots & \cdots & 0 \\ 0 & 1 & a_3 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & a_k \end{pmatrix}^{-1} \begin{pmatrix} -n_1 \\ 0 \\ \vdots \\ \\ \vdots \\ 0 \\ n_2 \end{pmatrix}$$

(ii) It should be particularly mentioned that

$$\#\left(\mathrm{vert}\left(\partial\Theta_{\sigma}^{\mathbf{cp}}\right)\right) - \#\left(\mathrm{vert}\left(\partial\Theta_{\sigma^{\vee}}^{\mathbf{cp}}\right)\right) = \left\lfloor \frac{k}{2} \right\rfloor - \left\lfloor \frac{\nu}{2} \right\rfloor = \begin{cases} 1, & \text{if} \quad \nu \notin 2\,\mathbb{Z} \ \ \& \ \ k = \nu + 1 \\ \\ 0, & \text{if} \quad \begin{cases} \text{either} & \nu \notin 2\,\mathbb{Z} \ \ \& \ \ k = \nu - 1 \\ \text{or} & \nu \in 2\,\mathbb{Z} \ \ \& \ \ k = \nu + 1 \end{cases} \\ \\ -1, & \text{if} \quad \nu \in 2\,\mathbb{Z} \ \ \& \ \ k = \nu - 1 \end{cases}$$

(iii) The $\mathbf{v}_i$'s with odd indices (and final indices $1, k+1$) are the vertices of $\partial\Theta_{\tau}^{\mathbf{cp}}$, i.e.,

$$\mathrm{vert}\left(\partial\Theta_{\tau}^{\mathbf{cp}}\right) = \{\mathbf{v}_1\} \cup \left\{ \mathbf{v}_{2i+1} \;\middle|\; 1 \leq i \leq \left\lfloor \frac{k-1}{2} \right\rfloor \right\} \cup \{\mathbf{v}_{k+1}\}\ ,$$

where $\tau := \ \mathrm{pos}\,(\mathfrak{n}_2, n_2) \subset N_{\mathbb{R}}$. Geometrically, the fact that $\mathbf{v}_{2i+1}$'s belong to $\tau$ follows directly from (3.36) and the inequalities (3.7). Moreover, $\tau$ is a $\left([q]_p\,, p\right)$-cone w.r.t. $\{\mathfrak{n}_1, \mathfrak{n}_2\}$. (See figure **1**, where the illustrated cone $\sigma$ is a $(4,7)$-cone and $\tau$ is a $(3,4)$-cone).

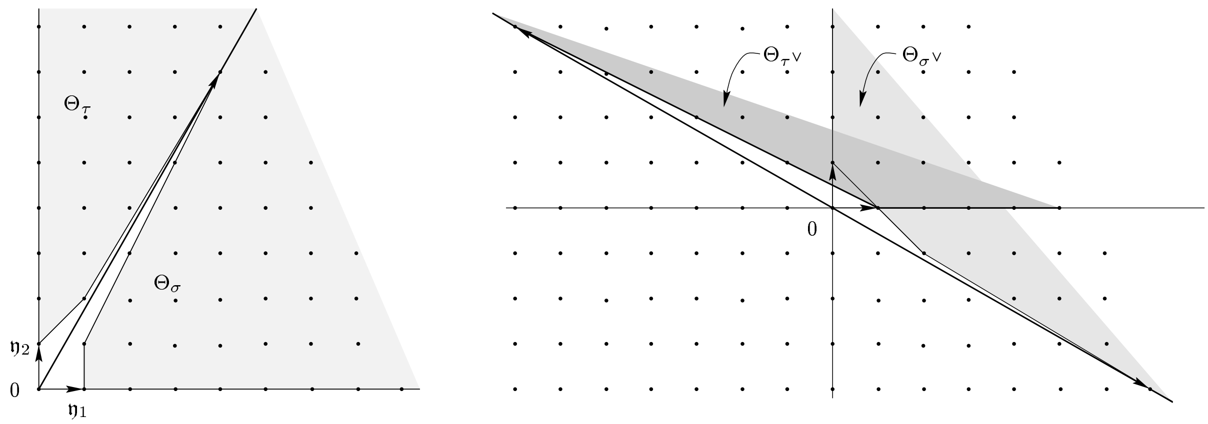


Figure **1**

## 4. The underlying spaces of abelian quotient singularities as affine toric varieties

Abelian quotient singularities can be directly investigated by means of the theory of toric varieties. If $G$ is a finite subgroup of $\mathrm{GL}(r,\mathbb{C})$, then $(\mathbb{C}^*)^r/G$ is automatically an algebraic torus embedded in $\mathbb{C}^r/G$.

(a) Let $G$ be a finite subgroup of $\mathrm{GL}(r,\mathbb{C})$ which is *small*, i.e. with no pseudoreflections, acting linearly on $\mathbb{C}^r$, and $\mathsf{p}:\mathbb{C}^r\to\mathbb{C}^r/G$ the quotient map. Denote by $(\mathbb{C}^r/G,[\mathbf{0}])$ the (germ of the) corresponding quotient singularity with $[\mathbf{0}]:=\mathsf{p}(\mathbf{0})$.

**Proposition 4.1 (Singular locus).** *If $G$ is a small finite subgroup of $\mathrm{GL}(r,\mathbb{C})$, then*

$$\mathrm{Sing}(\mathbb{C}^r/G)=\mathsf{p}\left(\{\mathbf{z}\in\mathbb{C}^r \mid G_{\mathbf{z}}\neq\{\mathrm{Id}\}\}\right)$$

*where $G_{\mathbf{z}}:=\{g\in G \mid g\cdot\mathbf{z}=\mathbf{z}\}$ is the isotropy group of $\mathbf{z}=(z_1,\ldots,z_r)\in\mathbb{C}^r$.*

**Theorem 4.2 (Prill's group-theoretic isomorphism criterion).** *Let $G_1$, $G_2$ be two small finite subgroups of $\mathrm{GL}(r,\mathbb{C})$. Then there exists an analytic isomorphism*

$$(\mathbb{C}^r/G_1,[\mathbf{0}])\cong(\mathbb{C}^r/G_2,[\mathbf{0}])$$

*if and only if $G_1$ and $G_2$ are conjugate to each other within $\mathrm{GL}(r,\mathbb{C})$.*

*Proof.* See Prill [51], thm. 2, p. 382. □

(b) Let $G$ be a finite, small, *abelian* subgroup of $\mathrm{GL}(r,\mathbb{C})$, $r\geq 2$, having order $l=|G|\geq 2$, and let

$$\{e_1=(1,0,\ldots,0,0)^{\intercal},\ldots,e_r=(0,0,\ldots,0,1)^{\intercal}\}$$

denote the standard basis of $\mathbb{Z}^r$, $N_0:=\sum_{i=1}^r\mathbb{Z}e_i$ the standard rectangular lattice, $M_0$ its dual, and

$$T_{N_0}:=\text{Max-Spec}\left(\mathbb{C}\left[\mathfrak{x}_1^{\pm1},\ldots,\mathfrak{x}_r^{\pm1}\right]\right)=(\mathbb{C}^*)^r\ .$$

Clearly,

$$T_{N_G}:=\text{Max-Spec}\left(\mathbb{C}\left[\mathfrak{x}_1^{\pm1},\ldots,\mathfrak{x}_r^{\pm1}\right]^G\right)=(\mathbb{C}^*)^r/G$$

is an $r$-dimensional algebraic torus with 1-parameter group $N_G$ and with group of characters $M_G$. Using the exponential map

$$(N_0)_{\mathbb{R}}\ni(y_1,\ldots,y_r)^{\intercal}=\mathbf{y}\longmapsto\exp(\mathbf{y}):=\left(e^{(2\pi\sqrt{-1})y_1},\ldots,e^{(2\pi\sqrt{-1})y_r}\right)^{\intercal}\in T_{N_0}$$

and the injection $\boldsymbol{\iota}:T_{N_0}\hookrightarrow\mathrm{GL}(r,\mathbb{C})$ defined by

$$T_{N_0}\ni(t_1,\ldots,t_r)^{\intercal}=\mathbf{t}\hookrightarrow\ \boldsymbol{\iota}(\mathbf{t}):=\mathrm{diag}(t_1,\ldots,t_r)\in\ \mathrm{GL}(r,\mathbb{C})\ ,$$

we have obviously

$$N_G=(\boldsymbol{\iota}\circ\exp)^{-1}(G)\qquad(\text{ and determinant }\ \det(N_G)=\frac{1}{l}\ )$$

(as long as we make a concrete choice of eigencoordinates to diagonalize the action of the elements of $G$ on $\mathbb{C}^r$) with

$$M_G=\left\{m\in M_0\ \middle|\ \begin{array}{c}\mathfrak{x}^m=\mathfrak{x}_1^{\mu_1}\cdots\mathfrak{x}_r^{\mu_r}\quad\text{is a }G\text{-invariant}\\ \text{Laurent monomial }(m=(\mu_1,\ldots,\mu_r))\end{array}\right\}\ (\text{and }\det(M_G)=l).$$

• If we define

$$\boxed{\sigma_0:=\ \mathrm{pos}(\{e_1,..,e_r\})}$$

to be the $r$-dimensional positive orthant, and $\Delta_G$ to be the fan

$$\Delta_G := \{\sigma_0 \text{ together with its faces}\}$$

then by the exact sequence $0 \to G \cong N_G/N_0 \to T_{N_0} \to T_{N_G} \to 0$ induced by the canonical duality pairing

$$M_0/M_G \times N_G/N_0 \to \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^*$$

(cf. [18], p. 34, and [48], pp. 22-23), we get as projection map: $\mathbb{C}^r = X(N_0, \Delta_G) \to X(N_G, \Delta_G)$, where

$$\boxed{X(N_G, \Delta_G) = U_{\sigma_0} = \mathbb{C}^r/G = \text{Max-Spec}\left(\mathbb{C}[\mathfrak{x}_1, \ldots, \mathfrak{x}_r]^G\right) \hookleftarrow T_{N_G}}$$

• Formally, we identify $[\mathbf{0}]$ with $\text{orb}(\sigma_0)$. Moreover, using the notation introduced in §2(g), the singular locus of $X(N_G, \Delta_G)$ can be written (by 4.1 and 2.3) as the union

$$\text{Sing}(X(N_G, \Delta_G)) = \text{ orb}(\sigma_0) \cup \left( \bigcup \left\{ \text{Max-Spec}\left(\mathbb{C}[\overline{\sigma}_0^\vee \cap M_G(\tau)]\right) \;\middle|\; \begin{array}{c} \tau \precneqq \sigma_0, \dim(\tau) \geq 2 \\ \text{and } \text{mult}(\tau; N_G) \geq 2 \end{array} \right\} \right) .$$

• In particular, if the acting group $G$ is *cyclic*, then, fixing diagonalization of the action on $\mathbb{C}^r$, we may assume that $G$ is generated by the element

$$\text{diag}(\zeta_l^{\alpha_1}, \ldots, \zeta_l^{\alpha_r})$$

(with $\zeta_l := e^{\frac{2\pi\sqrt{-1}}{l}}$) for $r$ integers $\alpha_1, \ldots, \alpha_r \in \{0, 1, \ldots, l-1\}$, at least 2 of which are $\neq 0$. This $r$-tuple $(\alpha_1, \ldots, \alpha_r)$ of *weights* is unique only up to the usual conjugacy relations (see 4.6 below), and $N_G$ is to be identified with the so-called *lattice of weights*

$$N_G = N_0 + \mathbb{Z}\left(\frac{1}{l}(\alpha_1, \ldots, \alpha_r)^\intercal\right)$$

containing all lattice points representing the elements of

$$G = \left\{ \text{diag}\left(\zeta_l^{[\lambda\alpha_1]_l}, \ldots, \zeta_l^{[\lambda\alpha_r]_l}\right) \;\middle|\; \lambda \in \mathbb{Z},\; 0 \leq \lambda \leq l-1 \right\} .$$

**Definition 4.3.** Under these conditions, we say that the quotient singularity $(X(N_G, \Delta_G), \text{orb}(\sigma_0))$ is *of type*

$$\boxed{\frac{1}{l}(\alpha_1, \ldots, \alpha_r)} \tag{4.1}$$

• Note that, since $G$ is small, $\gcd(l, \alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_r) = 1$, for all $i$, $1 \leq i \leq r$. (The symbol $\widehat{\alpha_i}$ means here that $\alpha_i$ is omitted.)

**Definition 4.4.** The *splitting codimension* of the underlying space $U_{\sigma_0} = \mathbb{C}^r/G$ of an abelian quotient singularity is defined to be the number

$$\text{splcod}(U_{\sigma_0}) := \max \left\{ \varkappa \in \{2, \ldots, r\} \;\middle|\; \begin{array}{c} U_{\sigma_0} \cong U_\tau \times \mathbb{C}^{r-\varkappa}, \text{ s.t.} \\ \tau \precneqq \sigma_0, \dim(\tau) = \varkappa \\ \text{and } \text{Sing}(U_\tau) \neq \varnothing \end{array} \right\}$$

If $\text{splcod}(U_{\sigma_0}) = r$, then $\text{orb}(\sigma_0)$ is called an *msc-singularity*, i.e., a singularity having the maximum splitting codimension.

**Lemma 4.5.** (i) *A cyclic quotient singularity of type* (4.1) *has splitting codimension* $\varkappa \in \{2, \ldots, r-1\}$ *if and only if there exists an index-subset* $\{\nu_1, \nu_2, \ldots, \nu_{r-\varkappa}\} \subset \{1, \ldots, r\}$, *such that*

$$\alpha_{\nu_1} = \alpha_{\nu_2} = \cdots = \alpha_{\nu_{r-\varkappa}} = 0 ,$$

*which is, in addition,* maximal *w.r.t. this property.*
(ii) *A cyclic quotient msc-singularity of type* (4.1) *is isolated if and only if*

$$\gcd(\alpha_i, l) = 1, \;\forall i, \;\; 1 \leq i \leq r .$$

*Proof.* It is immediate by the way we let $G$ act on $\mathbb{C}^r$. $\square$

(c) For two integers $l,\ r \geq 2$, we define

$$\Lambda(l;r) := \left\{ (\alpha_1, .., \alpha_r) \in \{0,1,2,..,l-1\}^r \;\middle|\; \begin{array}{c} \gcd(l, \alpha_1, .., \widehat{\alpha_i}, .., \alpha_r) = 1, \\ \text{for all } i,\ \ 1 \leq i \leq r \end{array} \right\}$$

and for $((\alpha_1, \ldots, \alpha_r), (\alpha'_1, \ldots, \alpha'_r)) \in \Lambda(l;r) \times \Lambda(l;r)$ the relation

$$(\alpha_1, \ldots, \alpha_r) \backsim (\alpha'_1, \ldots, \alpha'_r) :\Longleftrightarrow \left\{ \begin{array}{l} \text{there exists a permutation} \\ \theta : \{1, \ldots, r\} \to \{1, \ldots, r\} \\ \text{and an integer } \lambda,\ 1 \leq \lambda \leq l-1, \\ \text{with } \gcd(\lambda, l) = 1, \text{ such that} \\ \alpha'_{\theta(i)} = [\lambda \cdot \alpha_i]_l\,,\ \forall i,\ 1 \leq i \leq r \end{array} \right\}$$

It is easy to see that $\backsim$ is an equivalence relation on $\Lambda(l;r) \times \Lambda(l;r)$.

**Corollary 4.6 (Isomorphism criterion for cyclic acting groups).**
*Let $G$, $G'$ be two small, cyclic finite subgroups of $\mathrm{GL}(r, \mathbb{C})$ acting on $\mathbb{C}^r$, and let the corresponding quotient singularities be of type $\frac{1}{l}(\alpha_1, \ldots, \alpha_r)$ and $\frac{1}{l'}(\alpha'_1, \ldots, \alpha'_r)$ respectively. Then there exists an analytic (torus-equivariant) isomorphism*

$$(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0)) \cong (X(N_{G'}, \Delta_{G'}), \mathrm{orb}(\sigma_0))$$

*if and only if $l = l'$ and $(\alpha_1, \ldots, \alpha_r) \backsim (\alpha'_1, \ldots, \alpha'_r)$ within $\Lambda(l;r)$.*

*Proof.* It follows from 4.2 (cf. Fujiki [17], lemma 2, p. 296). $\square$

**Proposition 4.7 (Gorenstein-condition).**
*Let $(\mathbb{C}^r/G, [\mathbf{0}]) = (X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ be an abelian quotient singularity. Then the following conditions are equivalent :*

(i) *$X(N_G, \Delta_G) = U_{\sigma_0} = \mathbb{C}^r/G$ is Gorenstein,*

(ii) *$G \subset \mathrm{SL}(r, \mathbb{C})$,*

(iii) *$\langle (1,1,\ldots.1,1), n \rangle \geq 1$, for all $n$, $n \in \sigma_0 \cap (N_G \smallsetminus \{\mathbf{0}\})$,*

(iv) *$(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ is a canonical singularity of index 1.*

*In particular, if $(\mathbb{C}^r/G, [\mathbf{0}])$ is cyclic of type $\frac{1}{l}(\alpha_1, \ldots, \alpha_r)$, then (i)-(iv) are equivalent to*

$$\sum_{j=1}^{r} \alpha_j \equiv 0 \ (\mathrm{mod}\ l)$$

*Proof.* See e.g. Reid [52], thm. 3.1. $\square$

• If $X(N_G, \Delta_G)$ is Gorenstein, then the cone $\sigma_0 = \mathrm{pos}(\mathfrak{s}_G)$ is supported by the so-called *junior lattice simplex*

$$\boxed{\mathfrak{s}_G = \mathrm{conv}(\{e_1, .., e_r\})}$$

(w.r.t. $N_G$, cf. [29], [3]). Note that up to $\mathbf{0}$ there is no other lattice point of $\sigma_0 \cap N_G$ lying "under" the affine hyperplane of $\mathbb{R}^r$ containing $\mathfrak{s}_G$. Moreover, the lattice points representing the $l-1$ non-trivial group elements are exactly those belonging to the intersection of a dilation $\lambda\,\mathfrak{s}_G$ of $\mathfrak{s}_G$ with $\mathbf{Par}(\sigma_0)$, for some integer $\lambda$, $1 \leq \lambda \leq r-1$.

## 5. Lattice triangulations, crepant projective resolutions and main theorems

In this section we first briefly recall some general theorems concerning the projective, crepant resolutions of Gorenstein abelian quotient singularities in terms of appropriate lattice triangulations of the junior simplex. (For detailed expositions we refer to [7], [8], [9]). After that we formulate our main theorems.

(a) By vert($\mathcal{S}$) we denote the set of vertices of a polyhedral complex $\mathcal{S}$. By a *triangulation* $\mathcal{T}$ of a polyhedral complex $\mathcal{S}$ we mean a geometric simplicial subdivision of $\mathcal{S}$ with vert($\mathcal{S}$) $\subset$ vert($\mathcal{T}$). A polytope $P$ will be, as usual, identified with the polyhedral complex consisting of $P$ itself together with all its faces. If $\mathcal{S}_1$, $\mathcal{S}_2$ are two *simplicial* complexes, then we denote by $\mathcal{S}_1 * \mathcal{S}_2$ their *join*.

(b) A triangulation $\mathcal{T}$ of an $r$-dimensional polyhedral complex $\mathcal{S}$ is called *coherent* (or *regular*) if there exists a strictly upper convex $\mathcal{T}$-support function $\psi : |\mathcal{T}| \to \mathbb{R}$, i.e., a piecewise-linear real function defined on the underlying space $|\mathcal{T}|$ of $\mathcal{T}$, for which

$$\psi\left(\delta\ \mathbf{x} + (1-\delta)\ \mathbf{y}\right) \geq \delta\ \psi\left(\mathbf{x}\right) + (1-\delta)\ \ \psi\left(\mathbf{y}\right),\ \text{ for all }\ \mathbf{x}, \mathbf{y} \in |\mathcal{T}|\,,\ \text{ and }\ \ \delta \in [0,1]\ ,$$

so that for every maximal simplex $\mathbf{s}$ of $\mathcal{T}$, there is a linear function $\eta_{\mathbf{s}} : |\mathbf{s}| \to \mathbb{R}$ satisfying $\psi\left(\mathbf{x}\right) \leq \eta_{\mathbf{s}}\left(\mathbf{x}\right)$, for all $\mathbf{x} \in |\mathcal{T}|$, with equality being valid only for those $\mathbf{x}$ belonging to $\mathbf{s}$. The set of all strictly upper convex $\mathcal{T}$-support functions will be denoted by $\mathrm{SUCSF}_{\mathbb{R}}\left(\mathcal{T}\right)$.

(c) Let $N$ denote an $r$-dimensional lattice. By a *lattice polytope* (w.r.t. $N$) is meant a polytope in $N_{\mathbb{R}} \cong \mathbb{R}^r$ with vertices belonging to $N$. If $\{n_0, n_1, \ldots, n_k\}$ is a set of $k \leq r$ affinely independent lattice points, $\mathbf{s}$ the lattice $k$-dimensional simplex $\mathbf{s} = \mathrm{conv}(\{n_0, n_1, n_2, \ldots, n_k\})$, and $N_{\mathbf{s}} := \mathrm{lin}(\{n_1 - n_0, \ldots, n_k - n_0\}) \cap N$, then

• we say that $\mathbf{s}$ is an *elementary simplex* if

$$\{\mathbf{y} - n_0\ \mid \mathbf{y} \in \mathbf{s}\} \cap N_{\mathbf{s}} = \{\mathbf{0},\, n_1 - n_0, \ldots,\, n_k - n_0\}\,.$$

• $\mathbf{s}$ is *basic* if it has anyone of the following equivalent properties:

(i) $\{n_1 - n_0, n_2 - n_0, \ldots, n_k - n_0\}$ is a $\mathbb{Z}$-basis of $N_{\mathbf{s}}$,

(ii) $\mathbf{s}$ has relative volume $\mathrm{Vol}(\mathbf{s}; N_{\mathbf{s}}) = \dfrac{\mathrm{Vol}\left(\mathbf{s}\right)}{\det\left(N_{\mathbf{s}}\right)} = \dfrac{1}{k!}$ (w.r.t. $N_{\mathbf{s}}$) .

**Lemma 5.1.** (i) *Every basic lattice simplex is elementary.*
(ii) *Elementary lattice simplices of dimension* $\leq 2$ *are basic.*

*Proof.* See [7], lemma 6.2. □

**Example 5.2.** The lattice $r$-simplex

$$\mathbf{s} = \ \mathrm{conv}\left(\{\mathbf{0}, e_1, e_2, \ldots, e_{r-2}, e_{r-1}, (1, 1, \ldots, 1, 1, k)^{\intercal}\}\right) \subset \mathbb{R}^r,\ \ r \geq 3,\ k \geq 2,$$

(w.r.t. $\mathbb{Z}^r$) serves as example of an elementary but non-basic simplex because $\mathbf{s} \cap \mathbb{Z}^r = \mathrm{vert}(\mathbf{s})$ and

$$r!\,\mathrm{Vol}\left(\mathbf{s}; \mathbb{Z}^r\right) = \left|\det\left(e_1, \ldots, e_{r-1}, (1, 1, \ldots, 1, 1, k)^{\intercal}\right)\right| = k \neq 1\ .$$

**Definition 5.3.** A triangulation $\mathcal{T}$ of a lattice polytope $P \subset N_{\mathbb{R}} \cong \mathbb{R}^r$ (w.r.t. $N$) is called *lattice triangulation* if vert($P$) $\subset$ vert($\mathcal{T}$) $\subset N$. The set of all lattice triangulations of a lattice polytope $P$ (w.r.t. $N$) will be denoted by $\mathbf{LTR}_N\left(P\right)$.

**Definition 5.4.** A lattice triangulation $\mathcal{T}$ of $P \subset N_{\mathbb{R}} \cong \mathbb{R}^r$ (w.r.t. $N$) is called *maximal triangulation* if vert($\mathcal{T}$) $= N \cap P$. A lattice triangulation $\mathcal{T}$ of $P$ is obviously maximal if and only if each simplex $\mathbf{s}$ of $\mathcal{T}$ is elementary. A lattice triangulation $\mathcal{T}$ of $P$ is said to be *basic* if $\mathcal{T}$ consists of exclusively basic simplices. We define :

$$\mathbf{LTR}_N^{\max}\left(P\right) := \{\mathcal{T} \in \mathbf{LTR}_N\left(P\right)\ \mid \mathcal{T}\ \text{ is a maximal triangulation of }\ P\}\ ,$$

$$\mathbf{LTR}_N^{\mathrm{basic}}\left(P\right) := \{\mathcal{T} \in \mathbf{LTR}_N^{\max}\left(P\right)\ \mid \mathcal{T}\ \text{ is a basic triangulation of }\ P\}\ .$$

(Moreover, adding the prefix **Coh-** to anyone of the above sets, we shall mean the subsets of their elements which are coherent). The *hierarchy of lattice triangulations* of a $P$ (as above) is given by the following inclusion-diagram:

$$\begin{array}{ccccc} \mathbf{LTR}_N^{\mathrm{basic}}(P) & \subset & \mathbf{LTR}_N^{\max}(P) & \subset & \mathbf{LTR}_N(P) \\ \bigcup & & \bigcup & & \bigcup \\ \mathbf{Coh}\text{-}\mathbf{LTR}_N^{\mathrm{basic}}(P) & \subset & \mathbf{Coh}\text{-}\mathbf{LTR}_N^{\max}(P) & \subset & \mathbf{Coh}\text{-}\mathbf{LTR}_N(P) \end{array}$$

**Proposition 5.5.** *For any lattice polytope $P \subset N_{\mathbb{R}} \cong \mathbb{R}^r$ (w.r.t. $N$) the set of maximal coherent triangulations* $\mathbf{Coh}\text{-}\mathbf{LTR}_N^{\max}(P) \subset \mathbf{Coh}\text{-}\mathbf{LTR}_N(P)$ *of $P$ is non-empty.*

*Proof.* Consider the s.c.p.cone supported by $P$ in $N_{\mathbb{R}} \oplus \mathbb{R} \cong \mathbb{R}^{r+1}$, and then use [49], cor. 3.8, p. 394. □

**Remark 5.6 ("Pathologies").** (i) Already for $r = 2$ there exist lots of examples of $P$'s admitting basic, non-coherent triangulations ("whirlpool phenomenon").

(ii) A more remarkable pathological counterexample which was constructed recently by Hibi and Ohsugi [23] is a 9-dimensional 0/1-polytope (with 15 vertices) which possesses basic triangulations, but none of whose coherent triangulations is basic. Hence, for high-dimensional $P$'s, $\mathbf{LTR}_N^{\mathrm{basic}}(P) \neq \varnothing$ does not necessarily imply $\mathbf{Coh}\text{-}\mathbf{LTR}_N^{\mathrm{basic}}(P) \neq \varnothing$. (To the best of our knowledge, it is not as yet clear if there exists any counterexample of this kind when we restrict ourselves to the class of *lattice simplices* or not.)

(d) To pass from triangulations to desingularizations we need to introduce some extra notation.

**Definition 5.7.** Let $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ be an $r$-dimensional abelian Gorenstein quotient singularity ($r \geq 2$), and $\mathfrak{s}_G$ the $(r-1)$-dimensional junior simplex. For any simplex $\mathbf{s}$ of a lattice triangulation $\mathcal{T}$ of $\mathfrak{s}_G$ let $\sigma_{\mathbf{s}}$ denote the s.c.p. cone

$$\sigma_{\mathbf{s}} := \{\lambda\, \mathbf{y} \in (N_G)_{\mathbb{R}} \mid \lambda \in \mathbb{R}_{\geq 0}\,,\ \mathbf{y} \in \mathbf{s}\} \quad (= \mathrm{pos}(\mathbf{s}) \text{ within } (N_G)_{\mathbb{R}})$$

supporting $\mathbf{s}$. We define the fan

$$\widehat{\Delta_G}(\mathcal{T}) := \{\sigma_{\mathbf{s}} \mid \mathbf{s} \in \mathcal{T}\}$$

of s.c.p. cones in $(N_G)_{\mathbb{R}} \cong \mathbb{R}^r$, and

$$\mathbf{PCDES}(X(N_G, \Delta_G)) := \left\{ \begin{array}{c} \textit{partial crepant } T_{N_G}\text{-equivariant} \\ \text{desingularizations of } X(N_G, \Delta_G) \\ \text{with overlying spaces having} \\ \text{at most } (\mathbb{Q}\text{-factorial}) \textit{ canonical} \\ \text{singularities (of index 1)} \end{array} \right\},$$

$$\mathbf{PCDES}^{\max}(X(N_G, \Delta_G)) := \left\{ \begin{array}{c} \textit{partial crepant } T_{N_G}\text{-equivariant} \\ \text{desingularizations of } X(N_G, \Delta_G) \\ \text{with overlying spaces having} \\ \text{at most } (\mathbb{Q}\text{-factorial}) \textit{ terminal} \\ \text{singularities (of index 1)} \end{array} \right\},$$

$$\mathbf{CDES}(X(N_G, \Delta_G)) := \left\{ \begin{array}{c} \textit{crepant } T_{N_G}\text{-equivariant (full)} \\ \text{desingularizations of } X(N_G, \Delta_G) \end{array} \right\}.$$

(Setting the prefix **QP-** in the front of anyone of them, we shall mean the corresponding subsets of them consisting of those desingularizations whose overlying spaces are *quasiprojective*.)

**Theorem 5.8 (Desingularizing by triangulations).**
*Let $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ be an $r$-dimensional abelian Gorenstein quotient singularity ($r \geq 2$). Then there exist one-to-one correspondences :*

$$\boxed{\begin{array}{ccc} (\textbf{Coh-})\,\mathbf{LTR}^{\text{basic}}_{N_G}(\mathfrak{s}_G) & \overset{1:1}{\longleftrightarrow} & (\textbf{QP-})\,\mathbf{CDES}\,(X(N_G, \Delta_G)) \\ \cap & & \cap \\ (\textbf{Coh-})\,\mathbf{LTR}^{\max}_{N_G}(\mathfrak{s}_G) & \overset{1:1}{\longleftrightarrow} & (\textbf{QP-})\,\mathbf{PCDES}^{\max}\,(X(N_G, \Delta_G)) \\ \cap & & \cap \\ (\textbf{Coh-})\,\mathbf{LTR}_{N_G}(\mathfrak{s}_G) & \overset{1:1}{\longleftrightarrow} & (\textbf{QP-})\,\mathbf{PCDES}\,(X(N_G, \Delta_G)) \end{array}}$$

*which are realized by crepant $T_{N_G}$-equivariant birational morphism of the form*

$$f_{\mathcal{T}} = \mathrm{id}_* : X\left(N_G,\ \widehat{\Delta_G}(\mathcal{T})\right) \longrightarrow X(N_G, \Delta_G) \tag{5.1}$$

*induced by mapping*

$$\mathcal{T} \longmapsto \widehat{\Delta_G}(\mathcal{T}), \qquad \widehat{\Delta_G}(\mathcal{T}) \longmapsto X\left(N_G,\ \widehat{\Delta_G}(\mathcal{T})\right) .$$

*Proof.* See [8], § 4 and [7], thm. 6.9. □

(e) It is now clear by theorem 5.8 that the main question (formulated in §1), restricted to the category of torus-equivariant desingularizations of $X(N_G, \Delta_G)$'s, is equivalent to the following:

• **Question: For a Gorenstein abelian quotient singularity** $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ **with junior simplex $\mathfrak{s}_G$, under which conditions do we have Coh-LTR$_{N_G}(\mathfrak{s}_G) \neq \varnothing$ ?**

We shall answer this question in full generality for 2-parameter Gorenstein cyclic quotient singularitites below in thm. 5.13. (Further techniques being applicable to other families of singularities will be discussed in [9]). Our starting-point is the following simple, but very useful necessary criterion for the existence of basic triangulations of $\mathfrak{s}_G$.

**Theorem 5.9 (Necessary Existence-Criterion).**
*Let $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ be a Gorenstein abelian quotient singularity. If $\mathfrak{s}_G$ admits a basic triangulation $\mathcal{T}$, then*

$$\boxed{\mathbf{Hlb}_{N_G}(\sigma_0) = \mathfrak{s}_G \cap N_G} \tag{5.2}$$

*Proof.* See [7] thm. 6.15. □

**Remark 5.10.** (i) The well-known counterexamples (due to Bouvier and Gonzalez-Sprinberg [5]) of the two Gorenstein cyclic singularities proving that theorem 2.5 cannot be in general true for $r \geq 4$ (i.e., in our terminology, those with types $\frac{1}{7}(1,3,4,6)$ and $\frac{1}{16}(1,7,11,13)$), indicated the first technical difficulties for desigularizing in higher dimensions (i.e., by *only* using the members of the Hilbert basis of the corresponding monoid as the set of the minimal cone-generators). Nevertheless, since they are both *terminal singularities*, i.e., they do not possess junior elements up to the vertices of $\mathfrak{s}_G$, they *cannot* serve as counterexamples to the inverse implication of thm. 5.9 (because $\mathfrak{s}_G$ does not admit basic triangulations by construction and $\mathfrak{s}_G \cap N_G \subsetneqq \mathbf{Hlb}_{N_G}(\sigma_0)$); in fact, for a long time it was completely unknown if condition (5.2) might be sufficient or not for the existence of a basic triangulation of $\mathfrak{s}_G$. Only recently Firla and Ziegler ([15], §4.2 & [16]) discovered (by computer testing) 10 appropriate *counterexamples in dimension* 4! Among them, the counterexample of the Gorenstein cyclic quotient singularity with the smallest possible acting group-order, fulfilling property (5.2) and admitting no crepant, torus-equivariant resolutions, is that of type $\frac{1}{39}$ $(1, 5, 8, 25)$.

(ii) Our intention in the present paper is to show that the property of all the above mentioned counterexamples to have 3 or more "parameters" (i.e., freely chosen weights) is not a pure chance! As we shall see in theorems 5.11, 5.13 and 5.15, for 1- and 2-parameter series of Gorenstein cyclic quotient singularities condition (5.2) **is** indeed sufficient for the existence of crepant, $T_{N_G}$-equivariant, full resolutions in **all** dimensions.

Let us first recall what happens in the 1-parameter case. (By lemma 4.5 (i) it is enough to consider only msc-singularities. Otherwise the problem can be reduced to a lower-dimensional one).

**Theorem 5.11 (On 1-parameter singularity series).**
*Let $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ be an $r$-dimensional Gorenstein cyclic quotient msc-singularity (with $l = |G| \geq r \geq 4$). Suppose that its type contains at least $r-1$ equal weights. Then $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ is analytically isomorphic to Gorenstein cyclic quotient 1-parameter singularity of type*

$$\frac{1}{l}\left(\underbrace{1, 1, \ldots, 1, 1}_{(r-1)\text{-times}},\ l-(r-1)\right) \tag{5.3}$$

*and $X(N_G, \Delta_G)$ admits a* ***unique*** *crepant, projective, $T_{N_G}$-equivariant, full desingularization*

$$f = \mathrm{id}_* : X\left(N_G, \widehat{\Delta_G}\right) \to X(N_G, \Delta_G)$$

***iff*** either $l \equiv 0 \bmod (r-1)$ or $l \equiv 1 \bmod (r-1)$. (*These conditions are actually equivalent to condition* (5.2) *of* thm. 5.9).

*Moreover, the dimensions of the non-trivial cohomology groups of $X\left(N_G, \widehat{\Delta}_G\right)$ are given by the formulae:*

$$\dim_{\mathbb{Q}} H^{2i}\left(X\left(N_G, \widehat{\Delta}_G\right); \mathbb{Q}\right) = \begin{cases} 1 & , \text{ for } \quad i = 0 \\ \left\lfloor \frac{l}{r-1} \right\rfloor & , \text{ for } \quad i \in \{1, 2, \ldots, r-2\} \\ \left\lfloor \frac{l-1}{r-1} \right\rfloor & , \text{ for } \quad i = r-1 \end{cases} \tag{5.4}$$

*Proof.* See Dais-Henk [7], thm. 8.2. □

**Remark 5.12.** In fact, there are exactly $\left\lfloor \frac{l}{r-1} \right\rfloor$ exceptional prime divisors supported by the preimage of $\mathrm{Sing}(X(N_G, \Delta_G))$ via $f$; $\left\lfloor \frac{l}{r-1} \right\rfloor - 1$ of them are analytically isomorphic to the projectivization of certain decomposable bundles over $\mathbb{P}_{\mathbb{C}}^{r-2}$ (having only twisted hyperplane bundles as summands), and the last one is isomorphic $\mathbb{P}_{\mathbb{C}}^{r-1}$ (resp. to $\mathbb{P}_{\mathbb{C}}^{r-2} \times \mathbb{C}$) for $l \equiv 1 \bmod (r-1)$ (resp. $l \equiv 0 \bmod (r-1)$) [cf. [7], thm. 8.4].

**Theorem 5.13 (Main Theorem I: On 2-parameter singularity series).**
*Let $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ be a Gorenstein cyclic quotient msc-singularity of type $\dfrac{1}{l}(\alpha_1, \ldots, \alpha_r)$ with $l = |G| \geq r \geq 4$, for which at least $r-2$ of its defining weights are equal. Then $X(N_G, \Delta_G)$ admits crepant, $T_{N_G}$-equivariant, full desingularizations* ***if and only if*** *condition* (5.2) *is satisfied. Moreover, at least one of these desingularizations is projective.*

**Remark 5.14.** To examine the validity of condition (5.2) in practice, one has first to determine all the elements of the Hilbert basis $\mathbf{Hlb}_{N_G}(\sigma_0)$ and then to test if all of them belong to the junior simplex or not.

On the other hand, there is another, more direct method for working with condition (5.2); namely to translate the geometric properties of $\mathfrak{s}_G$ into number-theoretic conditions fulfilled by the weights of the defining "type" of the singularity. In the next theorem we apply this method in the case in which $r-2$

weights are equal to 1 and the sum of all weights equals $l$. As it turns out, since all "pure" junior elements of $s_G$ have to be coplanar, these number-theoretic conditions involve only linear congruences, restrictions for certain gcd's and regular continued fraction expansions. Hence, we have just to perform the euclidean division algorithm, which is a polynomial-time-procedure.

• Up to a very first step (involving the determination of suitable Hermitian normal forms), and up to some extra numerological conditions, we do not lose so much in generality (at least from the algorithmic point of view) by restricting ourselves to the study of cyclic singularities of type (5.5) (see rem. 6.22 below).

• Note that the problem of computing the elements of the Hilbert basis of a *general* pointed rational cone is "*NP-hard*" (cf. Henk-Weismantel [22], § 3).

• The formulae giving the dimensions of the non-trivial cohomology groups of the fully resolvable 2-parameter singularities are much more complicated than (5.4) and will be treated separately by means of Ehrhart polynomials in section 7.

**Theorem 5.15 (Main Theorem II).** *Let $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ be a Gorenstein cyclic quotient singularity of type*

$$\boxed{\frac{1}{l}\left(\underbrace{1,1,\ldots,1,1}_{(r-2)\text{-times}},\ \alpha,\beta\right), \quad l \geq r \geq 4, \quad \alpha,\beta \in \mathbb{N}, \quad \alpha+\beta = l-(r-2)} \tag{5.5}$$

• *Define*

$$\mathfrak{t}_1 := \gcd(\alpha, l), \quad \mathfrak{t}_2 := \gcd(\beta, l) = \gcd(\alpha+(r-2), l) \tag{5.6}$$

*and*

$$\mathfrak{z}_1 := \frac{l}{\mathfrak{t}_2}, \qquad \mathfrak{z}_2 := \frac{\alpha+(r-2)}{\mathfrak{t}_2} \tag{5.7}$$

• *After that, if $\mathfrak{z}_2 \neq 1$, express $\mathfrak{z}_1/\mathfrak{z}_2$ as regular continued fraction*

$$\boxed{\frac{\mathfrak{z}_1}{\mathfrak{z}_2} = [a_1, a_2, \ldots, a_{\nu-1}, a_\nu]} \tag{5.8}$$

*and define $\mathfrak{c}_1 \in \mathbb{Z}_{<0}$, $\mathfrak{c}_2 \in \mathbb{N}$, by the formulae:*

$$\mathfrak{c}_1 = \begin{cases} \dfrac{-\mathfrak{z}_2}{[a_\nu, a_{\nu-1}, \ldots, a_3, a_2]} & \text{if } a_2 \geq 2 \text{ and } \nu \text{ odd} \\[2ex] \dfrac{-\mathfrak{z}_2}{[a_\nu, a_{\nu-1}, \ldots, a_4, a_3+1]} & \text{if } a_2 = 1 \text{ and } \nu \text{ odd} \\[2ex] \left(\dfrac{1}{[a_\nu, a_{\nu-1}, \ldots, a_3, a_2]} - 1\right)\mathfrak{z}_2 & \text{if } a_2 \geq 2 \text{ and } \nu \text{ even} \\[2ex] \left(\dfrac{1}{[a_\nu, a_{\nu-1}, \ldots, a_4, a_3+1]} - 1\right)\mathfrak{z}_2 & \text{if } a_2 = 1 \text{ and } \nu \text{ even} \end{cases}$$

*and*

$$\mathfrak{c}_2 = \begin{cases} \dfrac{\mathfrak{z}_1}{[a_\nu, a_{\nu-1}, \ldots, a_2, a_1]} & \text{if } \ a_1 \geq 2 \ \text{ and } \ \nu \text{ odd} \\[2ex] \dfrac{\mathfrak{z}_1}{[a_\nu, a_{\nu-1}, \ldots, a_3, a_2+1]} & \text{if } \ a_1 = 1 \ \text{ and } \ \nu \text{ odd} \\[2ex] \left(1 - \dfrac{1}{[a_\nu, a_{\nu-1}, \ldots, a_2, a_1]}\right)\mathfrak{z}_1 & \text{if } \ a_1 \geq 2 \ \text{ and } \ \nu \text{ even} \\[2ex] \left(1 - \dfrac{1}{[a_\nu, a_{\nu-1}, \ldots, a_3, a_2+1]}\right)\mathfrak{z}_1 & \text{if } \ a_1 = 1 \ \text{ and } \ \nu \text{ even} \end{cases}$$

*(For $\mathfrak{z}_2 = 1$, set $\mathfrak{c}_1 := -1, \mathfrak{c}_2 := \mathfrak{z}_1 + 1$).*

• *Finally define*

$$\boxed{\breve{p} := \frac{\mathfrak{c}_1 \cdot l + \mathfrak{c}_2 \cdot \alpha}{\mathfrak{t}_1}, \qquad q := \frac{l}{\mathfrak{t}_1 \cdot \mathfrak{t}_2}, \qquad p := [\,\breve{p}\,]_q} \tag{5.9}$$

*and if $p \neq 0$, write $q/p$ as regular continued fraction*

$$\boxed{\frac{q}{p} = [\lambda_1, \lambda_2, \ldots, \lambda_{\kappa-1}, \lambda_\kappa]} \tag{5.10}$$

► *Then $X(N_G, \Delta_G)$ admits crepant, $T_{N_G}$-equivariant, full desingularizations (i.e., (5.2) is satisfied, as in* thm. 5.13*), and at least one of them is projective,* ***if and only if*** *one of the following (mutually exclusive) conditions* (i), (ii) *is fulfilled:*

(i) *The greatest common divisor of $\alpha, \beta$ and $l$ equals*

$$\boxed{\gcd(\alpha, \beta, l) = r - 2} \tag{5.11}$$

(ii) *The greatest common divisor of $\alpha, \beta$ and $l$ equals 1, $[\mathfrak{t}_1]_{r-2} = [\mathfrak{t}_2]_{r-2} = 1$, and* ***either*** $p = 0$ *(and consequently $q = 1$)* ***or*** *the above defined characteristic numbers satisfy the following relations:*

$$\boxed{\begin{cases} \dfrac{\breve{p} - p}{q} \equiv 0 \bmod (r-2), \\[2ex] \lambda_{2j} \equiv 0 \bmod (r-2), \ \forall j, \ \ j \in \left\{1, 2, \ldots, \left\lfloor \dfrac{\kappa - 1}{2} \right\rfloor \right\}, \\ \text{whenever } \kappa \geq 3, \text{and} \\[1ex] \lambda_\kappa \equiv 1 \bmod (r-2) \\ \text{in the case in which the length } \kappa \ (\geq 2) \text{ is } \textbf{even}. \end{cases}} \tag{5.12}$$

Though conditions (i), (ii) of theorem 5.15 are fairly restrictive, it is remarkable that they are fulfilled by several subseries of 2-parameter Gorenstein cyclic singularities having *infinitely many* members in *each dimension.*

**Example 5.16.** The subseries of non-isolated singularities with defining types

$$\frac{1}{(\xi+\xi'+1)\cdot(r-2)}\left(\underbrace{1,1,\ldots,1,1}_{(r-2)\text{-times}},\xi\cdot(r-2)\,,\xi'\cdot(r-2)\right)$$

and $\xi,\,\xi'\in\mathbb{N}$, $\gcd(\xi,\xi')=1$, $r\geq 4$, satisfies obviously (5.11).

**Example 5.17.** The subseries of isolated singularities with defining types

$$\frac{1}{2\,(r-1)^i+r-2}\left(\underbrace{1,1,\ldots,1,1}_{(r-2)\text{-times}},(r-1)^i\,,(r-1)^i\right)$$

and $i\in\mathbb{N}$, $r\geq 4$, satisfies (5.12) because $\breve{p}=2\,(r-1)^i+(r-1)-2$ and

$$q=2\,(r-1)^i+(r-1)-1,\quad p=q-1,\quad \frac{q}{p}=[1,q-1]\ .$$

**Example 5.18.** The example of 4-dimensional subseries due to Mohri [45]:

$$\frac{1}{4\,\xi}\,(1,1,2\,\xi-1,2\,\xi-1)\,,\quad \xi\in\mathbb{N}\ ,$$

satisfies (5.12) and contains only isolated singularities because $\gcd(4\xi,2\xi-1)=1$ and

$$\frac{\mathfrak{z}_1}{\mathfrak{z}_2}=\frac{4\,\xi}{2\xi+1}=[1,1,\xi-1,2]\,,\quad \mathfrak{c}_1=-\,(\xi+1)\,,\quad \mathfrak{c}_2=2\xi+1,$$

i.e.,

$$\breve{p}=-\,(4\xi+1)\,,\ q=4\,\xi,\ p=4\,\xi-1,\ \frac{q}{p}=[1,4\,\xi-1]\,,\ \text{with } 4\,\xi-1\equiv 1\pmod 2\,.$$

Note that also the single suitably resolvable cyclic singularity $\frac{1}{11}\,(1,1,3,6)$ found in [45] belongs to the subseries of isolated cyclic quotient singularities with type

$$\frac{1}{4r-5}\left(\underbrace{1,1,\ldots,1,1}_{(r-2)\text{-times}},r-1,2r-2\right)$$

satisfying (5.12). Moreover, there are examples like $1/28\,(1,1,1,4,21)$ for which $p=0,q=1$.

**Remark 5.19.** For those readers who would like to test rapidly if one of the above conditions (i), (ii) of theorem 5.15 is fulfilled for a concrete 2-parameter cyclic quotient (just by giving $\alpha$, $\beta$ and $r$ as input), we refer to the www-page [21] of the second author.

## 6. Proof of main theorems I, II

In this section we prove theorems 5.13 and 5.15.

(a) Let $(X\,(N_G,\Delta_G)\,,\mathrm{orb}\,(\sigma_0))$ be an $r$-dimensional Gorenstein cyclic quotient msc-singularity (with $l=|G|\geq r\geq 4$). Suppose that its type contains at least $r-2$ equal weights. Without loss of generality (i.e., up to analytic isomorphism, cf. cor. 4.6 and prop. 4.7), we may assume that $(X\,(N_G,\Delta_G)\,,\mathrm{orb}\,(\sigma_0))$ is of type

$$\boxed{\frac{1}{l}\left(\underbrace{k,\ldots,k}_{(r-2)\text{-times}},\alpha,\beta\right), \quad \text{with} \left\{ \begin{array}{c} \alpha+\beta+k\ (r-2)\equiv 0\ (\operatorname{mod}\ l)\,, \\ \gcd(k,\alpha,l)=\gcd(k,\beta,l)=1 \end{array} \right.} \tag{6.1}$$

$(\alpha, \beta, k \in \mathbb{N})$. Obviously,

$$\mathbf{Hlb}_{N_G}(\sigma_0) \subset \{e_1,\ldots,e_r\} \cup \left\{ \frac{1}{l}\left(\underbrace{[j\cdot k]_l,\ldots,[j\cdot k]_l}_{(r-2)\text{-times}}, [j\cdot\alpha]_l, [j\cdot\beta]_l\right)^{\intercal} \;\middle|\; 1\leq j\leq l-1 \right\}.$$

(by prop. 2.2). Define the hyperplane

$$\mathcal{H} := \left\{ \mathbf{x}=(x_1,\ldots,x_r)^{\intercal} \in (N_G)_{\mathbb{R}} \;\middle|\; \sum_{i=1}^{r} x_i = 1 \right\},$$

and the 3-dimensional linear $\mathfrak{L}$ subspace of $(N_G)_{\mathbb{R}}$

$$\mathfrak{L} := \{\mathbf{x}=(x_1,\ldots,x_r)^{\intercal} \in (N_G)_{\mathbb{R}} \mid x_1 - x_i = 0,\ \forall i,\ \ 2\leq i\leq r-2\} \ .$$

Next consider the 3-dimensional s.c.p.cone

$$\overline{\sigma_0} := \sigma_0 \cap \mathfrak{L} = \ \text{pos}\left(\left\{ \frac{1}{(r-2)} \sum_{i=1}^{r-2} e_i,\ e_{r-1},\ e_r \right\}\right) \subset \left(\overline{N_G}\right)_{\mathbb{R}} \subset (N_G)_{\mathbb{R}}$$

supporting the triangle

$$\overline{\mathfrak{s}_G} := \mathfrak{s}_G \cap \mathfrak{L} = \sigma_0 \cap \mathcal{H} \cap \mathfrak{L} = \overline{\sigma_0} \cap \mathcal{H} = \ \text{conv}\left(\left\{ \frac{1}{(r-2)} \sum_{i=1}^{r-2} e_i,\ e_{r-1},\ e_r \right\}\right) \ ,$$

where $\mathfrak{s}_G$ denotes, as usual, the corresponding junior lattice simplex (w.r.t. $N_G$), and

$$\overline{N_G} := \left\{ \begin{array}{c} \text{the sublattice of } N_G \text{ generated} \\ \text{(as subgroup) by } N_G \cap \mathfrak{L} \end{array} \right\} \ .$$

Note that if

$$\mathfrak{n}_G := n\left(\mathbb{R}_{\geq 0}\left(\frac{1}{(r-2)}\sum_{i=1}^{r-2} e_i\right)\right)$$

is the first primitive lattice point $\mathfrak{n}_G$ of $\overline{N_G} \smallsetminus \{\mathbf{0}\}$ belonging to the ray which is defined by $\frac{1}{(r-2)}\sum_{i=1}^{r-2} e_i$ (cf. §2 (e)) and

$$\mu_G := \min\left\{ \varkappa \in \mathbb{Q}_{>0} \;\middle|\; \varkappa \cdot \left(\frac{1}{(r-2)}\sum_{i=1}^{r-2} e_i\right) \in \left(\overline{N_G} \smallsetminus \{\mathbf{0}\}\right) \right\},$$

then

$$\text{Gen}\left(\overline{\sigma_0}\right) = \{\mathfrak{n}_G,\ e_{r-1},\ e_r\}$$

and $\overline{\mathfrak{s}_G}$ is a *lattice triangle* w.r.t. $\overline{N_G}$ if and only if $\mu_G = 1(!)$.

Now define the lattice polygon

$$\mathfrak{Q}_G := \ \text{conv}\left(\mathfrak{s}_G \cap \overline{N_G}\right) \subset \left(\overline{N_G}\right)_{\mathbb{R}}.$$

It should be mentioned that if $\mu_G = 1$, then $\mathfrak{Q}_G = \overline{\mathfrak{s}_G}$ and if $\mu_G \neq 1$, we have $\mathfrak{Q}_G \subsetneqq \overline{\mathfrak{s}_G}$.

**Definition 6.1.** In case in which $\mu_G \neq 1$ we denote by $\mathfrak{w}$ (resp. $\mathfrak{w}'$) the unique lattice point belonging to

$$\mathfrak{Q}_G \cap \mathrm{conv}\left(\frac{\mathfrak{n}_G}{\mu_G}, e_{r-1}\right) \cap \overline{N_G} \quad (\text{resp. to } \mathfrak{Q}_G \cap \mathrm{conv}\left(\frac{\mathfrak{n}_G}{\mu_G}, e_r\right) \cap \overline{N_G})$$

so that

$$\mathrm{conv}\left(\frac{\mathfrak{n}_G}{\mu_G}, \mathfrak{w}\right) \cap \overline{N_G} = \{\mathfrak{w}\} \quad (\text{resp. } \mathrm{conv}\left(\frac{\mathfrak{n}_G}{\mu_G}, \mathfrak{w}'\right) \cap \overline{N_G} = \{\mathfrak{w}'\}).$$

Moreover, if $\mathrm{conv}(e_{r-1}, e_r) \subsetneqq \mathfrak{Q}_G$, we fix the (clockwise ordered, uniquely determined) enumeration

$$\mathfrak{w}_0 = \mathfrak{w},\ \mathfrak{w}_1,\ \mathfrak{w}_2,\ \ldots\ , \mathfrak{w}_\rho, \mathfrak{w}_{\rho+1} = \mathfrak{w}'$$

of all lattice points of $\overline{N_G}$ lying on

$$((\mathfrak{Q}_G \smallsetminus \mathrm{conv}(\{\mathfrak{w}, \mathfrak{w}', e_{r-1}, e_r\})) \cap \partial \mathfrak{Q}_G) \cup \{\mathfrak{w}, \mathfrak{w}'\}\ .$$

(For $r = 4$, $\mu_G \neq 1$, figure **2** illustrates these lattice points within the junior tetrahedron. Note that the singularity is isolated if and only if $\mathfrak{w} = e_{r-1}$ and $\mathfrak{w}' = e_r$).

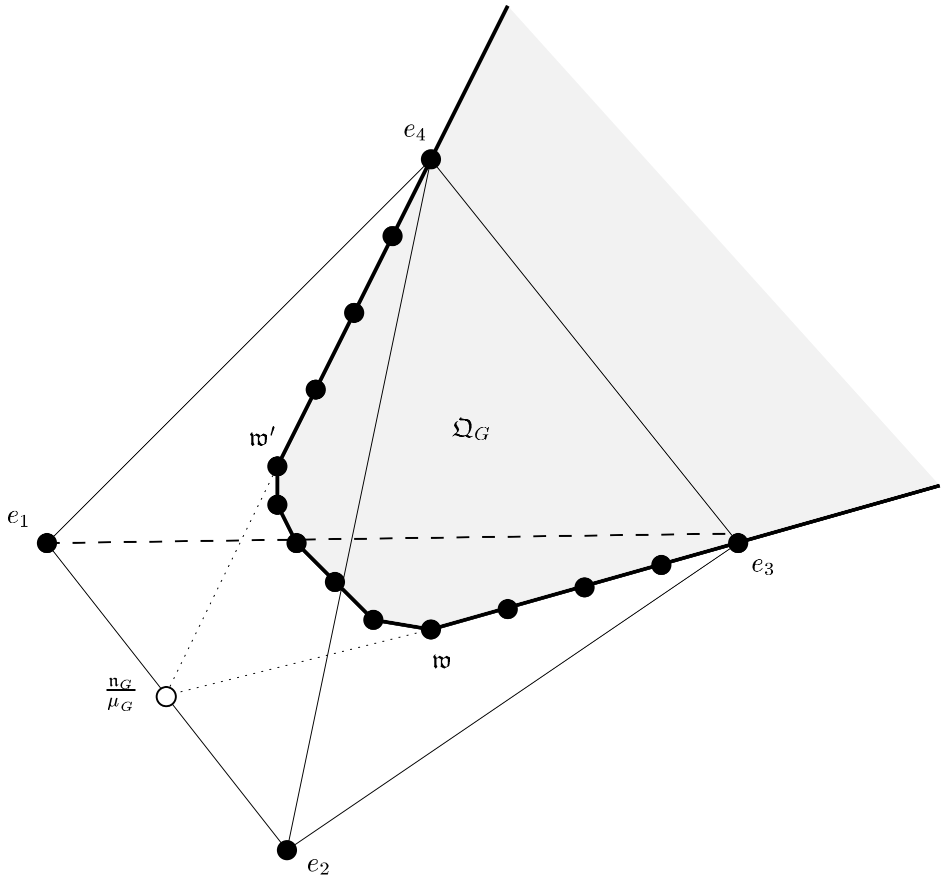


Figure **2**

**Definition 6.2.** For the given $r \geq 4$ we define $\Xi_r$ to be the set

$$\Xi_r := \left\{ (\xi_1, \xi_2, \ldots, \xi_{r-3}) \in (\{1, 2, \ldots, r-2\})^{r-3} \ \middle|\ 1 \leq \xi_1 < \xi_2 < \cdots \cdots < \xi_{r-3} \leq r-2 \right\}.$$

**Definition 6.3 (Maximal triangulations constructed by "joins").**
An auxiliary subclass of maximal lattice triangulations of the junior simplex $s_G$ which can be described easily and used efficiently for several geometric arguments is that consisting of tringulations of the form $\mathcal{T} = \mathcal{T}[\mathfrak{T}]$, with

$$\mathcal{T}[\mathfrak{T}] := \begin{cases} \mathfrak{E}_{\mathfrak{T}} \cup \left\{ \begin{array}{c} \operatorname{conv}(\{\mathfrak{w}_i, \mathfrak{w}_{i+1}, e_1, e_2, \ldots, e_{r-2}\}),\ 0 \leq i \leq \rho, \\ \text{together with their faces} \end{array} \right\} & , \text{ if } \mu_G \neq 1 \\ \mathfrak{E}_{\mathfrak{T}} & , \text{ if } \mu_G = 1 \end{cases}$$

where

$$\mathfrak{T} \in \mathbf{LTR}^{\max}_{N_G}(\mathfrak{Q}_G) \ \left(= \mathbf{LTR}^{basic}_{N_G}(\mathfrak{Q}_G)\right)$$

and

$$\mathfrak{E}_{\mathfrak{T}} := \left\{ \begin{array}{c} \left\{ \operatorname{conv}\left(\left\{n_1, n_2, n_3, e_{\xi_1}, e_{\xi_2}, \ldots, e_{\xi_{r-3}}\right\}\right) \;\middle|\; \begin{array}{c} \text{for all } (\xi_1, \xi_2, \ldots, \xi_{r-3}) \in \Xi_r \text{ and} \\ \text{all triangles } \operatorname{conv}(\{n_1, n_2, n_3\}) \text{ of } \mathfrak{T} \end{array} \right\} \\ \text{together with their faces} \end{array} \right\}.$$

• One easily verifies that all simplices of the above constructed triangulations $\mathcal{T}[\mathfrak{T}]$ are in fact representable as joins of smaller simplices; in particular we have

$$\operatorname{conv}\left(\left\{n_1, n_2, n_3, e_{\xi_1}, e_{\xi_2}, \ldots, e_{\xi_{r-3}}\right\}\right) = \operatorname{conv}(\{n_1, n_2, n_3\}) * \operatorname{conv}\left(\left\{e_{\xi_1}, e_{\xi_2}, \ldots, e_{\xi_{r-3}}\right\}\right)$$

and

$$\operatorname{conv}(\{\mathfrak{w}_i, \mathfrak{w}_{i+1}, e_1, e_2, \ldots, e_{r-2}\}) = \operatorname{conv}(\{\mathfrak{w}_i, \mathfrak{w}_{i+1}\}) * \operatorname{conv}(\{e_1, e_2, \ldots, e_{r-2}\}), \forall i,\ 0 \leq i \leq \rho,$$

respectively. Hence, we may alternatively describe $\mathfrak{E}_{\mathfrak{T}}$ and $\mathcal{T}[\mathfrak{T}]$ as

$$\mathfrak{E}_{\mathfrak{T}} = \left\{ \mathfrak{T} * \operatorname{conv}\left(\left\{e_{\xi_1}, e_{\xi_2}, \ldots, e_{\xi_{r-3}}\right\}\right) \;\middle|\; \text{for all } (\xi_1, \xi_2, \ldots, \xi_{r-3}) \in \Xi_r \right\}$$

and

$$\mathcal{T}[\mathfrak{T}] = \begin{cases} \mathfrak{E}_{\mathfrak{T}} \cup \left[ \left( \bigcup_{i=0}^{\rho} \operatorname{conv}(\{\mathfrak{w}_i, \mathfrak{w}_{i+1}\}) \right) * \operatorname{conv}(\{e_1, e_2, \ldots, e_{r-2}\}) \right] & , \text{ if } \mu_G \neq 1 \\ \mathfrak{E}_{\mathfrak{T}} & , \text{ if } \mu_G = 1 \end{cases}$$

**Remark 6.4.** (i) If $(X(N_G, \Delta_G), \operatorname{orb}(\sigma_0))$ as in thm. 5.13 is a *pure* 2-parameter singularity, i.e., if not more than $r-2$ of the weights of its defining type are equal, then there is no *unique* full (resp. maximal partial) crepant resolution of $X(N_G, \Delta_G)$, but at least such resolutions due to triangulations of the form $\mathcal{T}[\mathfrak{T}]$ are induced by maximal (and therefore basic) triangulations $\mathfrak{T}$ of the lattice polygon $\mathfrak{Q}_G$. Hence, the ***flops*** connecting any two of them are induced by classical "elementary transformations" (cf. Oda [48], prop. 1.30, p. 49).

(ii) If a singularity $(X(N_G, \Delta_G), \operatorname{orb}(\sigma_0))$ as in thm. 5.13 has a crepant, full resolution coming from a triangulation of the form $\mathcal{T}[\mathfrak{T}]$, then applying techniques similar to those of [7] (by considering the stars of the vertices $n$ of $\mathcal{T}$ and the corresponding closures $V(\mathbb{R}_{\geq 0} n)$, cf. §2(g)) it is possible to specify the structure of the exceptional prime divisors up to analytic isomorphism. For instance, all compactly supported exceptional prime divisors w.r.t. $f_{\mathcal{T}[\mathfrak{T}]}$ are the total spaces of fibrations having basis $\mathbb{P}^{r-3}_{\mathbb{C}}$ and typical fiber isomorphic either to $\mathbb{P}^1_{\mathbb{C}}$ or to a 2-dimensional compact toric variety (i.e. to a $\mathbb{P}^2_{\mathbb{C}}$ or $\mathbb{F}_{\varkappa} = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\varkappa)\right)$, probably blown up at finitely many points, cf. [48], thm. 1.28, p. 42).

**Proposition 6.5.** *The set* $\mathbf{Coh\text{-}LTR}^{\max}_{N_G}(\mathfrak{Q}_G)$ *is non-empty. Moreover, if* $\mathfrak{T} \in \mathbf{Coh\text{-}LTR}^{\max}_{N_G}(\mathfrak{Q}_G)$, *then*

$$\mathcal{T}[\mathfrak{T}] \in \mathbf{Coh\text{-}LTR}^{\max}_{N_G}(\mathfrak{s}_G) \ .$$

*Proof.* $\mathbf{Coh\text{-}LTR}^{\max}_{N_G}(\mathfrak{Q}_G) \neq \varnothing$ follows from prop. 5.5. We shall henceforth fix a coherent, strictly upper convex support funtion $\theta : |\mathfrak{T}| \to \mathbb{R}$. Now since $X(N_G, \Delta_G)$ itself is an affine toric variety, it is quasiprojective. Hence, besides $\theta$, there is also another coherent, strictly upper convex support function $\phi : \mathfrak{s}_G \to \mathbb{R}$. For any index-subset $\{i_1, \ldots, i_\kappa\} \subset \{1, 2, \ldots, r\}$ let $\phi|_{i_1,\ldots,i_\kappa}$ denote the restriction of $\phi$ onto $\mathrm{conv}(e_{i_1}, e_{i_2}, \ldots, e_{i_\kappa})$. We define a support function

$$\psi : |\,\mathcal{T}[\mathfrak{T}]\,| \longrightarrow \mathbb{R}$$

by setting

$$\psi\left(\delta \cdot \mathbf{x} + (1-\delta) \cdot \mathbf{y}\right) := \delta \cdot \theta(\mathbf{x}) + (1-\delta) \cdot \phi|_{\xi_1,\ldots,\xi_{r-3}}(\mathbf{y})$$

for all $\mathbf{x} \in |\mathfrak{T}|$, $\mathbf{y} \in \mathrm{conv}\left(\left\{e_{\xi_1}, \ldots, e_{\xi_{r-3}}\right\}\right)$, $\delta \in [0,1]$ and all $\left(\xi_1, \xi_2, \ldots, \xi_{r-3}\right) \in \Xi_r$, and by

$$\psi\left(\delta \cdot \mathbf{x} + (1-\delta) \cdot \mathbf{y}\right) := \delta \cdot \theta(\mathbf{x}) + (1-\delta) \cdot \phi|_{1,2,\ldots,r-2}(\mathbf{y})$$

for all $\mathbf{x} \in \mathrm{conv}(\{\mathfrak{w}_i, \mathfrak{w}_{i+1}\})$, $\mathbf{y} \in \mathrm{conv}(\{e_1, \ldots, e_{r-2}\})$, $\delta \in [0,1]$ and for all $i$, $0 \leq i \leq \rho$ (whenever $\mu_G \neq 1$). It is easy to verify that $\psi \in \mathrm{SUCSF}_{\mathbb{R}}(\mathcal{T}[\mathfrak{T}])$. $\square$

**Lemma 6.6.** *Let* $n_1, n_2, n_3$ *be three lattice points of* $\overline{N_G} \smallsetminus \{\mathbf{0}\}$, *such that* $\mathrm{pos}(\{n_1, n_2, n_3\})$ *is a 3-dimensional basic cone w.r.t.* $\overline{N_G}$. *Then the cone*

$$\mathrm{pos}\left(\left\{n_1, n_2, n_3, e_{\xi_1}, e_{\xi_2}, \ldots, e_{\xi_{r-3}}\right\}\right) \subset \left(\overline{N_G}\right)_{\mathbb{R}}$$

*is basic w.r.t. the lattice* $N_G$ *for every* $(r-3)$*-tuple* $\left(\xi_1, \xi_2, \ldots, \xi_{r-3}\right) \in \Xi_r$.

*Proof.* For a $u \in N_G$ there exist $\gamma_1, \gamma_2, \ldots, \gamma_r, \gamma_{r+1} \in \mathbb{Z}$, such that

$$u = \sum_{i=1}^{r} \gamma_i \ e_i + \gamma_{r+1}\left(\frac{1}{l}\,(k, \ldots, k, \alpha, \beta)^{\intercal}\right) \ .$$

On the other hand, there exist $\kappa_1, \kappa_2, \ldots, \kappa_{r-3} \in \mathbb{Z}$, such that

$$u - \sum_{i=1}^{r-3} \kappa_i \ e_{\xi_i} \in \overline{N_G}, \quad \overline{N_G} = \mathbb{Z}\,n_1 \oplus \mathbb{Z}\,n_2 \oplus \mathbb{Z}\,n_3 \ ,$$

which means that the above cone is indeed basic. $\square$

**Proposition 6.7.** *For a Gorenstein cyclic quotient singularity* $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ *of type* (6.1) *the following conditions are equivalent:*

(i) *There exists a crepant,* $T_{N_G}$*-equivariant, full desingularization*

$$f = \mathrm{id}_* : X\left(N_G, \widehat{\Delta_G}\right) \longrightarrow X(N_G, \Delta_G)$$

*of the quotient space* $X(N_G, \Delta_G)$.

(ii) *There exists a refinement* $\widehat{\Delta_G}$ *of the fan* $\Delta_G$ *consisting of basic cones.*

(iii) $\mathbf{Hlb}_{N_G}(\sigma_0) = \mathfrak{s}_G \cap N_G$.

(iv) $\overline{\mathfrak{s}_G} \cap \overline{N_G} = \begin{cases} \mathbf{Hlb}_{\overline{N_G}}(\overline{\sigma_0}) \smallsetminus \{\mathfrak{n}_G\} & , \ \text{if} \ \mu_G \neq 1 \\ \mathbf{Hlb}_{\overline{N_G}}(\overline{\sigma_0}) & , \ \text{if} \ \mu_G = 1 \ . \end{cases}$

*Proof.* (i)$\Leftrightarrow$(ii) follows from thm. 5.8. The implication (i)$\Rightarrow$(iii) follows from thm. 5.9. (iii)$\Leftrightarrow$(iv) is obvious.

(iii)$\Rightarrow$(ii): Note that $\mathbf{Hlb}_{N_G}(\sigma_0) \smallsetminus \{e_1, \ldots, e_r\} \neq \varnothing$, because otherwise $\sigma_0$ would be basic w.r.t. $N_G$ (which is impossible). More precisely,

$$\mathbf{Hlb}_{N_G}(\sigma_0) \smallsetminus \{e_1, e_2, \ldots, e_{r-2}\} = \begin{cases} \mathbf{Hlb}_{\overline{N_G}}(\overline{\sigma_0}) \smallsetminus \{\mathfrak{n}_G\} & , \text{ if } \mu_G \neq 1 \\ \mathbf{Hlb}_{\overline{N_G}}(\overline{\sigma_0}) & , \text{ if } \mu_G = 1 \end{cases}$$

Now by Szebö's theorem 2.5 there exists a proper subdivision of $\overline{\sigma_0}$ into basic subcones w.r.t. $\overline{N_G}$,

$$\overline{\sigma_0} = \bigcup_{j \in J} \operatorname{pos}\left(n_1^{(j)}, n_2^{(j)}, n_3^{(j)}\right),$$

such that $\operatorname{Gen}(\overline{\sigma_0}) = \mathbf{Hlb}_{\overline{N_G}}(\overline{\sigma_0})$. This subdivision induces a refinement

$$\widehat{\Delta_G} = \left\{\operatorname{pos}\left(\left\{n_1^{(j)}, n_2^{(j)}, n_3^{(j)}, e_{\xi_1}, e_{\xi_2}, \ldots, e_{\xi_{r-3}}\right\}\right) \mid \text{for all } (\xi_1, \xi_2, \ldots, \xi_{r-3}) \in \Xi_r \text{ and } j \in J\right\}$$

of $\Delta_G$ with $\operatorname{Gen}\left(\widehat{\Delta_G}\right) \subset \mathbf{Hlb}_{N_G}(\sigma_0)$ $(= \mathfrak{s}_G \cap N_G)$. In particular, if for some index $j \in J$ one of the lattice points $n_1^{(j)}$, $n_2^{(j)}$, $n_3^{(j)}$, say $n_3^{(j)}$, equals $\mathfrak{n}_G$, then

$$\operatorname{pos}\left(\left\{n_1^{(j)}, n_2^{(j)}, n_3^{(j)}, e_{\xi_1}, e_{\xi_2}, \ldots, e_{\xi_{r-3}}\right\}\right) = \operatorname{pos}\left(\left\{n_1^{(j)}, n_2^{(j)}, e_1, e_2, \ldots, e_{r-3}, e_{r-2}\right\}\right).$$

Applying lemma 6.6 we deduce that all cones of $\widehat{\Delta_G}$ are basic w.r.t. $N_G$. $\square$

To win projectivity we shall use a consequence of this proposition based on the following lemma.

**Lemma 6.8.** *Let $N$ be a lattice of rank 3, $\sigma$ a s.c.p. cone $\subset N_{\mathbb{R}}$, and let $F_1, F_2, \ldots, F_\varkappa$ denote the compact facets of the lower convex hull* $\operatorname{conv}(\sigma \cap (N \smallsetminus \{\mathbf{0}\}))$. *Suppose that the Hilbert basis of $\sigma$ w.r.t. $N$ equals*

$$\mathbf{Hlb}_N(\sigma) = \left(\bigcup_{i=1}^{\varkappa} F_i\right) \cap N. \tag{6.2}$$

*If $\{\mathbf{s}_{i,1}, \mathbf{s}_{i,2}, \ldots, \mathbf{s}_{i,\pi_i}\}$ is an arbitrary triangulation of $F_i$ into elementary (and therefore basic) lattice triangles, then*

$$\bigcup_{i=1}^{\varkappa} \bigcup_{j=1}^{\pi_i} \operatorname{pos}(\mathbf{s}_{i,j})$$

*constitutes a subdivision of $\sigma$ into basic subcones w.r.t. $N$, such that for all $j$, $1 \leq j \leq \pi_i$ and all $i$, $1 \leq i \leq \varkappa$,*

$$\operatorname{Gen}(\operatorname{pos}(\mathbf{s}_{i,j})) \subset \mathbf{Hlb}_N(\sigma).$$

*Proof.* It suffices to show that all $\operatorname{pos}(\mathbf{s}_{i,j})$ are basic w.r.t. $N$. If there were indices $j = j_\bullet \in \{1, .., \pi_i\}$ and $i = i_\bullet \in \{1, .., \varkappa\}$, such that $\operatorname{vert}(\mathbf{s}_{i_\bullet, j_\bullet}) = \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ and $\operatorname{pos}(\mathbf{s}_{i_\bullet, j_\bullet})$ a non-basic cone, then we would find a point $\mathbf{n}_\bullet \in \mathbf{Hlb}_N(\operatorname{pos}(\mathbf{s}_{i_\bullet, j_\bullet})) \smallsetminus \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$. Writing the affine hull of $F_{i_\bullet}$ as a hyperplane

$$\operatorname{aff}(F_{i_\bullet}) = \{\mathbf{x} \in N_{\mathbb{R}} \mid \langle \mathbf{m}_\bullet, \mathbf{x} \rangle = \gamma\}, \text{ with } \gamma \in \mathbb{N}, \quad \mathbf{m}_\bullet \in \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z}),$$

we would have $\langle \mathbf{m}_\bullet, \mathbf{n}_1 \rangle = \langle \mathbf{m}_\bullet, \mathbf{n}_2 \rangle = \langle \mathbf{m}_\bullet, \mathbf{n}_3 \rangle = \gamma$ on the one hand, and

$$\langle \mathbf{m}_\bullet, \mathbf{x} \rangle \geq \gamma, \quad \forall \mathbf{x}, \quad \mathbf{x} \in \sigma \cap (N \smallsetminus \{\mathbf{0}\}) \tag{6.3}$$

on the other. Expressing $\mathbf{n}_\bullet$ as linear combination of the form (cf. (2.2))

$$\mathbf{n}_\bullet = \delta_1 \mathbf{n}_1 + \delta_2 \mathbf{n}_2 + \delta_3 \mathbf{n}_3, \quad \text{with} \quad 0 \leq \delta_1, \delta_2, \delta_3 < 1,$$

(where at least two of $\delta_1, \delta_2, \delta_3$ are $\neq 0$), and assuming that $\langle \mathbf{m}_\bullet, \mathbf{n}_\bullet \rangle \geq 2\,\gamma$, we would obtain

$$\langle \mathbf{m}_\bullet, \mathbf{n_1} + \mathbf{n_2} + \mathbf{n_3} - \mathbf{n}_\bullet \rangle \leq \gamma \Longrightarrow \mathbf{n_1} + \mathbf{n_2} + \mathbf{n_3} - \mathbf{n}_\bullet \in (\mathbf{s}_{i_\bullet, j_\bullet} \smallsetminus \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\})$$

contradicting the fact that $\mathbf{s}_{i_\bullet, j_\bullet}$ itself is basic by construction. Thus, $\langle \mathbf{m}_\bullet, \mathbf{n}_\bullet \rangle < 2\,\gamma$. If there were two lattice points $\mathbf{n}'_\bullet, \mathbf{n}''_\bullet$ belonging to $\sigma \cap (N \smallsetminus \{\mathbf{0}\})$, such that $\mathbf{n}_\bullet = \mathbf{n}'_\bullet + \mathbf{n}''_\bullet$, then by (6.3):

$$2\,\gamma > \langle \mathbf{m}_\bullet, \mathbf{n}_\bullet \rangle = \langle \mathbf{m}_\bullet, \mathbf{n}'_\bullet \rangle + \langle \mathbf{m}_\bullet, \mathbf{n}''_\bullet \rangle \geq 2\,\gamma,$$

which is impossible. This means that $\mathbf{n}_\bullet$ necessarily belongs to $\mathbf{Hlb}_N(\sigma)$ and by (6.2):

$$\mathbf{n}_\bullet \in (\mathbf{s}_{i_\bullet, j_\bullet} \smallsetminus \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}),$$

which again contradicts our hypothesis. □

**Proposition 6.9.** *If the conditions of prop. 6.7 for the quotient singularity* $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ *of type* (6.1) *are satisfied, then all partial crepant,* $T_{N_G}$*-equivariant, desingularizations*

$$f = \mathrm{id}_* : X\left(N_G, \widehat{\Delta_G}(\mathcal{T}[\mathfrak{T}])\right) \longrightarrow X(N_G, \Delta_G)$$

*of* $X(N_G, \Delta_G)$ *induced by maximal triangulations of the junior simplex* $\mathfrak{s}_G$ *of the form* $\mathcal{T}[\mathfrak{T}]$ (*as in* 6.3) *are in particular "full"* (*and hence all* $\mathcal{T}[\mathfrak{T}]$*'s basic w.r.t.* $N_G$).

*Proof.* For the 3-dimensional s.c.p. cone $\overline{\sigma_0} = \mathrm{pos}(\{\mathfrak{n}_G, e_{r-1}, e_r\})$ condition (iv) of proposition 6.7 implies:

$$\mathbf{Hlb}_{\overline{N_G}}(\overline{\sigma_0}) = \{\mathfrak{n}_G\} \cup \left(\overline{\mathfrak{s}_G} \cap \overline{N_G}\right) = \{\mathfrak{n}_G\} \cup \left(\mathfrak{Q}_G \cap \overline{N_G}\right) .$$

Hence,

$$\left\{ \begin{array}{c} \text{compact facets of the lower} \\ \text{convex hull conv}\left(\overline{\sigma_0} \cap \left(\overline{N_G} \smallsetminus \{\mathbf{0}\}\right)\right) \end{array} \right\} = \begin{cases} \mathfrak{Q}_G, & \text{for } \mu_G = 1 \\ \mathfrak{Q}_G \cup \bigcup\limits_{j=0}^{\rho} \mathrm{conv}\left(\{\mathfrak{n}_G, \mathfrak{w}_j, \mathfrak{w}_{j+1}\}\right), & \text{for } \mu_G \neq 1. \end{cases}$$

Applying lemma 6.8 for any $\mathfrak{T} \in \mathbf{LTR}^{\mathrm{basic}}_{\overline{N_G}}(\mathfrak{Q}_G)$ (for $\mu_G = 1$, resp. $\mathfrak{T} \cup \bigcup\limits_{j=0}^{\rho} \mathrm{conv}(\{\mathfrak{w}_j, \mathfrak{w}_{j+1}, \mathfrak{n}_G\})$, for $\mu_G \neq 1$) we obtain a subdivision of $\overline{\sigma_0}$ into basic subcones w.r.t. $\overline{N_G}$ whose set of minimal generators belongs to the Hilbert basis of $\overline{\sigma_0}$. By lemma 6.6 all triangulations $\mathcal{T}[\mathfrak{T}]$ have to be basic too (w.r.t. $N_G$). □

**Corollary 6.10.** *If the conditions of prop. 6.7 are satisfied, then there exists a crepant,* $T_{N_G}$*-equivariant, full,* ***projective*** *desingularization of the quotient space* $X(N_G, \Delta_G)$.

*Proof.* It follows from 6.5 and 6.9. □

• **Proof of thm. 5.13:** It follows straightforwardly from 5.9, 6.7 and 6.10. □

(b) Our strategy to prove thm. 5.15 is based on the reduction of the problem by lattice transformations to a 2-dimensional one, and on the application of the techniques of §3. Hereafter let $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ denote a Gorenstein cyclic quotient singularity of type (5.5). Using the above introduced notation we obtain

$$\overline{N_G} = \mathbb{Z}\,\left(\frac{1}{l}\,(1, \ldots, 1, \alpha, \beta)^{\intercal}\right) + \mathbb{Z}\,e_{r-1} + \mathbb{Z}\,e_r \tag{6.4}$$

and the corresponding $\mathfrak{n}_G, \mu_G$ are given as follows:

**Lemma 6.11.** *The first primitive lattice point* $\mathfrak{n}_G$ *of* $\overline{N_G} \smallsetminus \{\mathbf{0}\}$ *belonging to the ray which is defined by* $\frac{1}{(r-2)} \sum_{i=1}^{r-2} e_i$ *equals*

$$\mathfrak{n}_G = \frac{1}{\gcd(\alpha, \beta, l)} \sum_{i=1}^{r-2} e_i,$$

*i.e.,*

$$\mathrm{Gen}\,(\overline{\sigma_0}) = \{\mathfrak{n}_G,\ e_{r-1},\ e_r\},$$

*because*

$$\mu_G := \min \left\{ \varkappa \in \mathbb{Q}_{>0} \ \middle|\ \varkappa \cdot \left( \frac{1}{(r-2)} \sum_{i=1}^{r-2} e_i \right) \in \left( \overline{N_G} \smallsetminus \{\mathbf{0}\} \right) \right\} = \frac{r-2}{\gcd(\alpha, \beta, l)} \ .$$

*Proof.* By (6.4) every lattice point $n \in \overline{N_G}$ can be written as a linear combination

$$n = \xi_1 \left( \frac{1}{l} \left( \underbrace{1, 1, \ldots, 1, 1}_{(r-2)\text{-times}}, \alpha, \beta \right)^{\intercal} \right) + \xi_2\ e_{r-1} + \xi_3\ e_r, \quad \xi_1, \xi_2, \xi_3 \in \mathbb{Z} \ .$$

We define the set

$$\mathfrak{A}_G := \left\{ \begin{array}{c} \text{all lattice vectors of } \overline{N_G} \text{ whose} \\ \text{last two coordinates are } = 0 \end{array} \right\} \ .$$

For an $n \in \overline{N_G}$ to belong to $\mathfrak{A}_G$ means that $\xi_1\,\alpha + \xi_2\,l = \xi_1\,\beta + \xi_3\,l = 0$, i.e.,

$$\xi_2 \in \left( \frac{-\alpha}{\gcd(\alpha, l)} \right) \mathbb{Z}, \quad \xi_3 \in \left( \frac{-\beta}{\gcd(\alpha, l)} \right) \mathbb{Z},$$

and

$$\xi_1 \in \left( \frac{l}{\gcd(\alpha, l)} \right) \mathbb{Z} \cap \left( \frac{l}{\gcd(\alpha, l)} \right) \mathbb{Z} = \left( \frac{l}{\gcd(\alpha, \beta, l)} \right) \mathbb{Z} \ .$$

Hence, $\mathfrak{A}_G \subset \overline{N_G}$ can be alternatively expressed as

$$\mathfrak{A}_G = \mathbb{Z} \left( \frac{1}{\gcd(\alpha, \beta, l)} \left( \underbrace{1, 1, \ldots, 1, 1}_{(r-2)\text{-times}}, 0, 0 \right)^{\intercal} \right) \ .$$

Since the last two coordinates of the vector $\frac{1}{(r-2)} \sum_{i=1}^{r-2} e_i$ are $= 0$, we obtain

$$\mu_G = \min \left\{ \varkappa \in \mathbb{Q}_{>0} \ \middle|\ \varkappa \cdot \left( \frac{1}{(r-2)} \sum_{i=1}^{r-2} e_i \right) \in \mathfrak{A}_G \right\} = \frac{r-2}{\gcd(\alpha, \beta, l)}$$

as we asserted. $\square$

**Remark 6.12.** In case $\mu_G = 1$, i.e., if $\gcd(\alpha, \beta, l) = r - 2$, it is clear by 6.9 and 6.10 that there will be a crepant (resp. crepant and projective), $T_{N_G}$-equivariant, full desingularization of the quotient space $X(N_G, \Delta_G)$. This is the reason for which, from now on, we shall focus our attention to the case $\mu_G \neq 1$, and find out which direct arithmetical conditions are equivalent to the geometric ones of prop. 6.7 (and involve exclusively the two given parameters $\alpha$ and $\beta$). This will be done in five steps and requires several lemmas.

- **First step**. Define the linear transformation

$$\Phi : (N_G)_{\mathbb{R}} \longrightarrow (N_G)_{\mathbb{R}}, \quad \Phi(\mathbf{x}) = A \cdot \mathbf{x},$$

with

$$A := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -\beta & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -\alpha & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & l & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \underbrace{-1}_{(r-2)\text{-pos.}} & 0 & 0 \end{pmatrix} \in \ \mathrm{GL}\,(r, \mathbb{Q})$$

Since

$$N_G = \mathbb{Z}\ e_1 + \mathbb{Z}\ e_2 + \cdots + +\mathbb{Z}\ e_{r-3} + \mathbb{Z}\ \left(\frac{1}{l}\,(1, \ldots, 1, \alpha, \beta)^{\intercal}\right) + \mathbb{Z}\ e_{r-1} + \mathbb{Z}\ e_r$$

and $\Phi\,(e_i) = e_{r-i+1}$, $\forall i$, $i \in \{1, \ldots, r-3, r-1, r\}$, $\Phi\left(\frac{1}{l}\,(1, \ldots, 1, \alpha, \beta)^{\intercal}\right) = e_3$, we have

$$\Lambda_G := \Phi\,(N_G) = \mathbb{Z}\,e_1 + \mathbb{Z}\,e_2 + \cdots + \mathbb{Z}\,e_r, \quad \overline{\Lambda_G} := \Phi\left(\overline{N_G}\right) = \mathbb{Z}\,e_1 + \mathbb{Z}\,e_2 + \mathbb{Z}\,e_3.$$

We can therefore work within $\left(\ \overline{\Lambda_G}\right)_{\mathbb{R}} \cong \mathbb{R}^3$ with the unit vectors

$$\mathrm{e}_1 = (1, 0, 0)^{\intercal}\,, \mathrm{e}_2 = (0, 1, 0)^{\intercal}\,, \mathrm{e}_3 = (0, 0, 1)^{\intercal}$$

and write

$$\Phi\,(\overline{\sigma_0}) = \ \mathrm{pos}\,(\{\mathrm{e}_2, \mathrm{e}_1, \eta_G\}) = \ \mathrm{pos}\left(\left\{\mathrm{e}_2, \mathrm{e}_1, \frac{\eta_G}{\mu_G}\right\}\right), \quad \Phi\,(\overline{\mathfrak{s}_G}) = \ \mathrm{conv}\left(\left(\left\{\mathrm{e}_2, \mathrm{e}_1, \frac{\eta_G}{\mu_G}\right\}\right)\right)$$

where

$$\eta_G := \Phi\,(\mathfrak{n}_G) = \frac{1}{\gcd\,(\alpha, \beta, l)}\ \begin{pmatrix} -\beta \\ -\alpha \\ l \end{pmatrix}\ .$$

Furthermore, setting

$$\mathfrak{u}_0 := \Phi\,(\mathfrak{w})\,,\ \mathfrak{u}_i := \Phi\,(\mathfrak{w}_i)\,,\ \forall i, \quad 1 \le i \le \rho, \quad \text{and} \quad \mathfrak{u}_{\rho+1} := \Phi\,(\mathfrak{w}')\,,$$

we obtain

$$\Phi\,(\mathfrak{Q}_G) = \mathrm{conv}\left(\Phi\,(\overline{\mathfrak{s}_G}) \cap \overline{N_G}\right) = \mathrm{conv}\,(\{\mathrm{e}_2, \mathfrak{u}_0, \mathfrak{u}_1, \ldots, \mathfrak{u}_\rho, \mathfrak{u}_{\rho+1}, \mathrm{e}_1\}) \subset \left(\ \overline{\Lambda_G}\right)_{\mathbb{R}}$$

(see figure **3**).

**Lemma 6.13.** *If $\mu_G \neq 1$, then each of the conditions of prop. 6.7 is equivalent to*

$$\mathbf{Hlb}_{\overline{\Lambda_G}}\,(\Phi\,(\overline{\sigma_0})) \smallsetminus \{\eta_G\} = \Phi\,(\overline{\mathfrak{s}_G}) \cap \overline{\Lambda_G}\ .$$

*Proof.* Since $\Phi\left(\mathbf{Hlb}_{\overline{N_G}}\,(\overline{\sigma_0})\right) = \mathbf{Hlb}_{\overline{\Lambda_G}}\,(\Phi\,(\overline{\sigma_0}))$, this follows from 6.7 (iv). □

**Lemma 6.14.** *If $\mu_G \neq 1$, then each of the conditions of prop. 6.7 is equivalent to the following:*

$$\mathrm{pos}\,(\{\mathfrak{u}_{i-1}, \mathfrak{u}_i, \eta_G\}) \quad \text{is basic w.r.t.} \quad \overline{\Lambda_G}, \quad \text{for all i,} \quad 1 \le i \le \rho + 1.$$

*Proof.* "$\Rightarrow$": This implication follows from lemmas 6.8 and 6.13 applied to the 3-dimensional cone $\Phi\,(\overline{\sigma_0})$.
"$\Leftarrow$": Consider a maximal triangulation $\{\mathbf{s}_j \mid j \in J\}$ of the lattice polygon $\Phi\,(\mathfrak{Q}_G)$. By lemma 5.1(i) this triangulation has to be basic w.r.t. the sublattice of $\overline{\Lambda_G}$ generated (as subgroup) by aff($\mathfrak{Q}_G$). Consequently

$$\{\mathrm{pos}\,(\mathbf{s}_j) \mid j \in J\} \cup \{\mathrm{pos}\,(\{\mathfrak{u}_{i-1}, \mathfrak{u}_i, \eta_G\}) \mid 1 \le i \le \rho + 1\}$$

constitutes a subdivision of the entire $\Phi(\overline{\sigma_0})$ into basic cones w.r.t. $\overline{\Lambda_G}$ whose set of minimal generators is exactly $\left(\Phi(\mathfrak{s}_G)\cap\overline{\Lambda_G}\right)\cup\{\eta_G\}$. It remains to use the inverse implication in the statement of lemma 6.13. $\square$

• **Second step**. We define the unimodular transformation

$$\Psi:\left(\overline{\Lambda_G}\right)_{\mathbb{R}}\longrightarrow\left(\overline{\Lambda_G}\right)_{\mathbb{R}},\quad \Psi\left(\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right)=\begin{pmatrix}1&0&1\\1&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}.$$

Obviously,

$$\Psi(\mathrm{e}_1)=\begin{pmatrix}1\\1\\0\end{pmatrix},\ \Psi(\mathrm{e}_2)=\begin{pmatrix}0\\1\\0\end{pmatrix},\ \Psi\left(\frac{\eta_G}{\mu_G}\right)=\frac{1}{r-2}\begin{pmatrix}-\beta+l\\r-2\\l\end{pmatrix}=\frac{1}{r-2}\begin{pmatrix}\alpha+(r-2)\\r-2\\l\end{pmatrix}.$$

Using the embedding

$$\mathbb{R}^2\ni\begin{pmatrix}x_1\\x_2\end{pmatrix}\underset{\iota}{\hookrightarrow}\begin{pmatrix}x_1\\1\\x_2\end{pmatrix}\in\left(\overline{\Lambda_G}\right)_{\mathbb{R}}\cong\mathbb{R}^3$$

we may work with the lattice $\widetilde{\Lambda_G}$ of rank 2 defined by

$$\widetilde{\Lambda_G}:=\iota^{-1}\left(\left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\in\left(\overline{\Lambda_G}\right)_{\mathbb{R}}\cong\mathbb{R}^3\ \middle|\ x_2=1\right\}\cap\overline{\Lambda_G}\right)$$

and having $\{\mathfrak{e}_1,\mathfrak{e}_2\}$ as a $\mathbb{Z}$-basis (where now $\mathfrak{e}_1,\mathfrak{e}_2$ denote here the unit vectors $(1,0)^{\intercal}$and $(0,1)^{\intercal}$ of $\left(\widetilde{\Lambda_G}\right)_{\mathbb{R}}\cong\mathbb{R}^2$ (!) respectively). In particular,

$$\iota^{-1}\left(\Psi\left(\Phi\left(\mathfrak{s}_G\right)\right)\right)=\ \mathrm{conv}\left(\left(\{\mathbf{0},\mathfrak{e}_1,\mathfrak{v}_G\}\right)\right)$$

where $\iota^{-1}\left(\Psi(\mathrm{e}_1)\right)=\mathfrak{e}_1,\ \ \iota^{-1}\left(\Psi(\mathrm{e}_2)\right)=\mathbf{0}$, and

$$\mathfrak{v}_G:=\iota^{-1}\left(\Psi\left(\frac{\eta_G}{\mu_G}\right)\right)=\frac{1}{r-2}\begin{pmatrix}a+(r-2)\\l\end{pmatrix}.$$

Furthermore, setting

$$\widetilde{\mathfrak{u}}_i:=\iota^{-1}\left(\Psi(\mathfrak{u}_i)\right),\ \forall i,\ 0\le i\le\rho+1,$$

we transform $\mathfrak{Q}_G$ onto the polygon

$$\widetilde{\mathfrak{Q}}_G:=\iota^{-1}\left(\Psi\left(\Phi\left(\mathfrak{Q}_G\right)\right)\right)=\mathrm{conv}\left(\{\mathbf{0},\widetilde{\mathfrak{u}}_0,\widetilde{\mathfrak{u}}_1,\widetilde{\mathfrak{u}}_2,\ldots,\widetilde{\mathfrak{u}}_\rho,\widetilde{\mathfrak{u}}_{\rho+1},\mathfrak{e}_1\}\right)\subset\left(\widetilde{\Lambda_G}\right)_{\mathbb{R}}$$

(see figure **3**).

**Lemma 6.15.** *If $\mu_G\neq1$, then each of the conditions of prop.* 6.7 *is equivalent to each of the following*:
(i) *The volumes of the triangles* $\mathrm{conv}(\{\widetilde{\mathfrak{u}}_{i-1},\widetilde{\mathfrak{u}}_i,\mathfrak{v}_G\})$ *are equal to*

$$\mathrm{Vol}\left(\mathrm{conv}\left(\{\widetilde{\mathfrak{u}}_{i-1},\widetilde{\mathfrak{u}}_i,\mathfrak{v}_G\}\right)\right)=\frac{1}{2\,\mu_G}=\frac{\gcd(\alpha,\beta,l)}{2\,(r-2)},\quad \textit{for all } i,\ 1\le i\le\rho+1.$$

(ii) *For all $i$, $1\le i\le\rho+1$,* the triangle $\mathrm{conv}(\{\widetilde{\mathfrak{u}}_{i-1},\widetilde{\mathfrak{u}}_i,\mathfrak{v}_G\})$ *is basic w.r.t. the extended lattice*

$$\widetilde{\Lambda_G^{\mathrm{ext}}}:=\widetilde{\Lambda_G}+\mathbb{Z}\,\mathfrak{v}_G\ .$$

*Proof.* That (i) is equivalent to the conditions of 6.7 follows from the equalities

$$\left|\det\left(\mathfrak{u}_{i-1},\mathfrak{u}_i,\eta_G\right)\right| = \mu_G \ \left|\det\left(\mathfrak{u}_{i-1},\mathfrak{u}_i,\frac{\eta_G}{\mu_G}\right)\right| = \mu_G \ (3!) \ \ \mathrm{Vol}\left(\mathrm{conv}\left(\left\{\mathbf{0},\mathfrak{u}_{i-1},\mathfrak{u}_i,\frac{\eta_G}{\mu_G}\right\}\right)\right) =$$

$$= \mu_G \ (3!) \ \ \mathrm{Vol}\left(\mathrm{conv}\left(\left\{\mathbf{0},\Psi\left(\mathfrak{u}_{i-1}\right),\Psi\left(\mathfrak{u}_i\right),\Psi\left(\frac{\eta_G}{\mu_G}\right)\right\}\right)\right) = \mu_G \ (3!) \ \frac{1}{3} \ \mathrm{Vol}\left(\mathrm{conv}\left(\left\{\widetilde{\mathfrak{u}}_{i-1},\widetilde{\mathfrak{u}}_i,\mathfrak{v}_G\right\}\right)\right)$$

and lemma 6.14. Now since

$$\det\left(\widetilde{\Lambda_G^{\mathrm{ext}}}\right) = \frac{1}{\#\left(\left\{\left([j\cdot\alpha]_{(r-2)},[j\cdot l]_{(r-2)}\right) \mid 1\leq j\leq r-2\right\}\right)} = \frac{1}{\mu_G}$$

the equivalence (i)⇔(ii) becomes obvious. $\square$

• **Third step**. We define the affine integral transformation (w.r.t. $\widetilde{\Lambda_G^{\mathrm{ext}}}$) :

$$\Upsilon : \left(\widetilde{\Lambda_G^{\mathrm{ext}}}\right)_{\mathbb{R}} \longrightarrow \left(\widetilde{\Lambda_G^{\mathrm{ext}}}\right)_{\mathbb{R}}, \quad \Upsilon\left(\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right)\right) = \left(\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array}\right) \left(\begin{array}{c} x_1 \\ x_2 \end{array}\right) + \mathfrak{v}_G$$

being the composite of a unimodular transformation (w.r.t. $\widetilde{\Lambda_G}$) and a lattice translation (w.r.t. $\widetilde{\Lambda_G^{\mathrm{ext}}}$). Obviously,

$$\Upsilon\left(\mathrm{conv}\left(\left\{\widetilde{\mathfrak{u}}_{i-1},\widetilde{\mathfrak{u}}_i,\mathfrak{v}_G\right\}\right)\right) = \mathrm{conv}\left(\left\{\mathbf{0},\mathfrak{v}_G-\widetilde{\mathfrak{u}}_{i-1},\mathfrak{v}_G-\widetilde{\mathfrak{u}}_i\right\}\right) \ .$$

Next let us consider the 2-dimensional s.c.p. cone

$$\boldsymbol{\tau}_G := \ \mathrm{pos}\left(\left\{\mathfrak{v}_G-\widetilde{\mathfrak{u}}_0,\mathfrak{v}_G-\widetilde{\mathfrak{u}}_{\rho+1}\right\}\right) \subset \left(\Upsilon\left(\widetilde{\Lambda_G^{\mathrm{ext}}}\right)\right)_{\mathbb{R}} = \left(\widetilde{\Lambda_G^{\mathrm{ext}}}\right)_{\mathbb{R}} \cong \mathbb{R}^2$$

and, as in §3, define

$$\Theta_{\boldsymbol{\tau}_G} := \ \mathrm{conv}\left(\boldsymbol{\tau}_G \cap \left(\left(\widetilde{\Lambda_G^{\mathrm{ext}}}\right) \smallsetminus \{\mathbf{0}\}\right)\right) \subset \left(\widetilde{\Lambda_G^{\mathrm{ext}}}\right)_{\mathbb{R}} \cong \mathbb{R}^2$$

and denote by $\partial\Theta_{\boldsymbol{\tau}_G}^{\mathbf{cp}}$ the part of the boundary $\partial\Theta_{\boldsymbol{\tau}_G}$ of $\Theta_{\boldsymbol{\tau}_G}$ containing only its compact edges (see fig. **3**). Furthermore, let us denote by

$$\partial\Theta_{\boldsymbol{\tau}_G}^{\mathbf{cp}} \cap \widetilde{\Lambda_G^{\mathrm{ext}}} = \left\{\mathfrak{l}_0 = \mathfrak{v}_G-\widetilde{\mathfrak{u}}_0,\mathfrak{l}_1,\ldots,\mathfrak{l}_{\rho'},\mathfrak{l}_{\rho'+1} = \mathfrak{v}_G-\widetilde{\mathfrak{u}}_{\rho+1}\right\}$$

the (clockwise ordered, uniquely determined) enumeration of the lattice points of $\partial\Theta_{\boldsymbol{\tau}_G}^{\mathbf{cp}}$ belonging to the extended lattice $\widetilde{\Lambda_G^{\mathrm{ext}}}$. Since we work with two *different* lattices of rank 2 it might happen that $\rho \neq \rho'$ or even that

$$\left(\partial\Theta_{\boldsymbol{\tau}_G}^{\mathbf{cp}} \cap \widetilde{\Lambda_G^{\mathrm{ext}}}\right) \cap \left\{\mathfrak{v}_G-\widetilde{\mathfrak{u}}_1,\mathfrak{v}_G-\widetilde{\mathfrak{u}}_2,\ldots,\mathfrak{v}_G-\widetilde{\mathfrak{u}}_\rho\right\} = \varnothing \ .$$

Nevertheless, in our particular situation we have:

**Lemma 6.16.** *If* $\mu_G \neq 1$, *then the conditions of lemma* 6.15 *are equivalent to each of the following*:

(i) $\mathrm{conv}(\{\mathbf{0},\mathfrak{v}_G-\widetilde{\mathfrak{u}}_{i-1},\mathfrak{v}_G-\widetilde{\mathfrak{u}}_i\})$ *is basic w.r.t.* $\widetilde{\Lambda_G^{\mathrm{ext}}}$, *for all* $i$, $1\leq i\leq \rho+1$.

(ii) $\rho=\rho'$ *and* $\mathfrak{l}_i = \mathfrak{v}_G-\widetilde{\mathfrak{u}}_i$, *for all* $i$, $0\leq i\leq \rho+1$.

(iii) $\rho=\rho'$ *and* $\mathfrak{v}_G-\mathfrak{l}_i \in \widetilde{\Lambda_G} = \mathbb{Z}\,\mathfrak{e}_1+\mathbb{Z}\,\mathfrak{e}_2$, *for all* $i$, $0\leq i\leq \rho+1$.

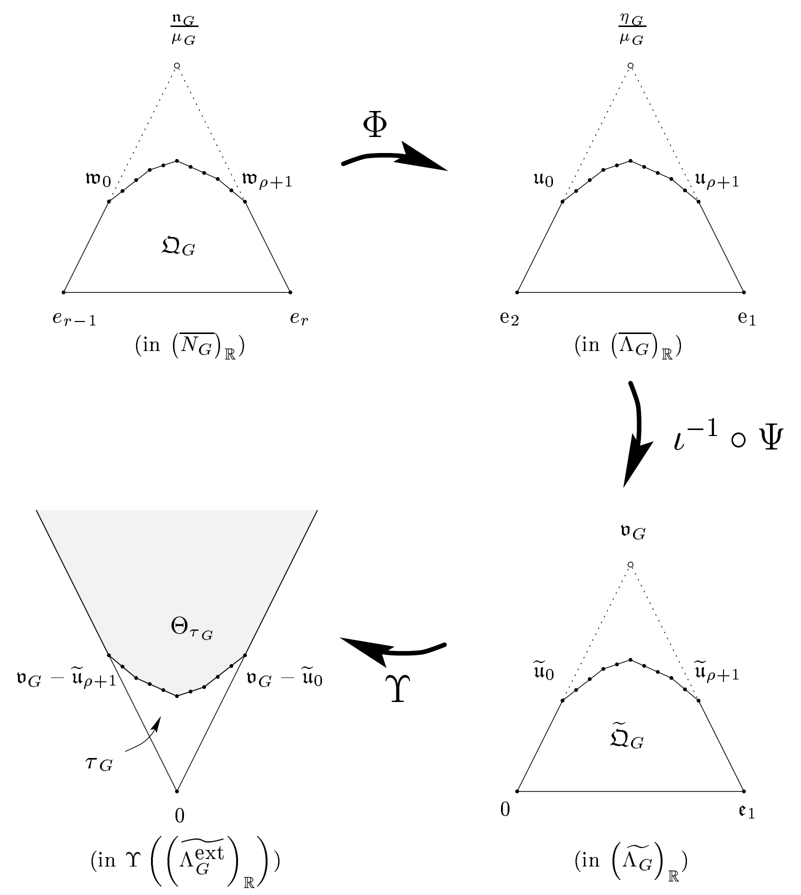


Figure **3**

*Proof.* Since $\Upsilon$ is an affine integral transformation, it is clear that (i) is equivalent to condition (ii) of lemma 6.15.

(i)$\Rightarrow$(ii)$\Rightarrow$(iii): The above mentioned possibilies are in each case to be excluded because they would for at least one index $i$ contradict the fact that both $\mathrm{conv}(\{\mathbf{0}, \mathfrak{v}_G - \widetilde{\mathfrak{u}}_{i-1}, \mathfrak{v}_G - \widetilde{\mathfrak{u}}_i\})$ and $\mathrm{conv}(\{\mathbf{0}, \mathfrak{l}_{i-1}, \mathfrak{l}_i\})$ are basic w.r.t. $\widetilde{\Lambda_G^{\mathrm{ext}}}$. Thus, $\rho = \rho'$ and $\widetilde{\mathfrak{u}}_i = \mathfrak{v}_G - \mathfrak{l}_i \in \widetilde{\Lambda_G}$, for all $i$, $0 \leq i \leq \rho + 1$.

(iii)$\Rightarrow$(ii)$\Rightarrow$(i): Since $\mathrm{conv}(\{\mathbf{0}, \mathfrak{l}_{i-1}, \mathfrak{l}_i\})$ are basic w.r.t. $\widetilde{\Lambda_G^{\mathrm{ext}}}$, $\mathrm{conv}(\{\mathfrak{v}_G, \mathfrak{v}_G - \mathfrak{l}_{i-1}, \mathfrak{v}_G - \mathfrak{l}_i\})$ will be basic too, but this time with respect to the "smaller" lattice $\widetilde{\Lambda_G}$. Since $\{\mathfrak{v}_G - \mathfrak{l}_0, \mathfrak{v}_G - \mathfrak{l}_1, \ldots, \mathfrak{v}_G - \mathfrak{l}_{\rho+1}\}$ determines again a "lower convex hull" by means of points belonging to $\widetilde{\Lambda_G}$, using similar arguments, one shows that necessarily $\mathfrak{l}_i = \mathfrak{v}_G - \widetilde{\mathfrak{u}}_i$ and $\mathrm{conv}(\{\mathbf{0}, \mathfrak{v}_G - \widetilde{\mathfrak{u}}_{i-1}, \mathfrak{v}_G - \widetilde{\mathfrak{u}}_i\})$ are basic w.r.t. $\widetilde{\Lambda_G^{\mathrm{ext}}}$. $\square$

• **Fourth step**. Maintaining the assumption $\mu_G \neq 1$, consider a $\mathbb{Z}$-basis $\{\mathfrak{v}_G - \widetilde{\mathfrak{u}}_0, \mathfrak{y}\}$ of the lattice $\widetilde{\Lambda_G^{\mathrm{ext}}}$ (cf. lemma 3.9), such that

$$\mathfrak{v}_G - \widetilde{\mathfrak{u}}_{\rho+1} = p \cdot (\mathfrak{v}_G - \widetilde{\mathfrak{u}}_0) + q \cdot \mathfrak{y}$$

for two positive integers $p$, $q$ with $0 \leq p < q$, $\gcd(p, q) = 1$, i.e., so that $\boldsymbol{\tau}_G$ becomes a $(p, q)$-cone w.r.t. $\{\mathfrak{v}_G - \widetilde{\mathfrak{u}}_0, \mathfrak{y}\}$ (in the sense of 3.10).

**Lemma 6.17.** *The multiplicity of the cone $\boldsymbol{\tau}_G$ w.r.t. $\widetilde{\Lambda_G^{\mathrm{ext}}}$ equals*

$$q = \mathrm{mult}\left(\boldsymbol{\tau}_G; \widetilde{\Lambda_G^{\mathrm{ext}}}\right) = \frac{[\mathfrak{t}_1]_{(r-2)} \cdot [\mathfrak{t}_2]_{(r-2)}}{\mathfrak{t}_1 \cdot \mathfrak{t}_2} \cdot \frac{l}{\gcd(\alpha, \beta, l)} \tag{6.5}$$

*where (as in* (5.6)*), we use the abbreviations*:

$$\mathfrak{t}_1 := \gcd(\alpha, l), \quad \mathfrak{t}_2 := \gcd(\beta, l) = \gcd(\alpha + (r-2), l).$$

*Proof.* Since $\frac{r-2}{\mathfrak{t}_2}\cdot \mathfrak{v}_G$ (resp. $\frac{r-2}{\mathfrak{t}_1}\cdot (\mathfrak{v}_G-\mathfrak{e}_1)$) is the first primitive lattice point of $(\mathbb{R}_{\geq 0}\,\mathfrak{v}_G)\cap\widetilde{\Lambda_G}$ (resp. of $(\mathbb{R}_{\geq 0}\,(\mathfrak{v}_G-\mathfrak{e}_1))\cap\widetilde{\Lambda_G}$), we obtain

$$\widetilde{\mathfrak{u}}_0=\left\lfloor\frac{\mathfrak{t}_2}{r-2}\right\rfloor\cdot\frac{r-2}{\mathfrak{t}_2}\cdot\mathfrak{v}_G\Rightarrow\mathfrak{v}_G-\widetilde{\mathfrak{u}}_0=\frac{[\mathfrak{t}_2]_{(r-2)}}{\mathfrak{t}_2}\cdot\mathfrak{v}_G$$

and

$$\widetilde{\mathfrak{u}}_{\rho+1}=\left\lfloor\frac{\mathfrak{t}_1}{r-2}\right\rfloor\cdot\frac{r-2}{\mathfrak{t}_1}\cdot(\mathfrak{v}_G-\mathfrak{e}_1)\Rightarrow\mathfrak{v}_G-\widetilde{\mathfrak{u}}_{\rho+1}=\frac{[\mathfrak{t}_1]_{(r-2)}}{\mathfrak{t}_1}\cdot(\mathfrak{v}_G-\mathfrak{e}_1)\ .$$

Therefore,

$$q=\frac{|\det(\mathfrak{v}_G-\widetilde{\mathfrak{u}}_0,\mathfrak{v}_G-\widetilde{\mathfrak{u}}_{\rho+1})|}{\det\left(\widetilde{\Lambda_G^{\mathrm{ext}}}\right)}=\mu_G\cdot\frac{[\mathfrak{t}_1]_{(r-2)}}{\mathfrak{t}_1}\cdot\frac{[\mathfrak{t}_2]_{(r-2)}}{\mathfrak{t}_2}\cdot|\det(\mathfrak{v}_G,\mathfrak{v}_G-\mathfrak{e}_1)|$$

and (6.5) follows from the equality $|\det(\mathfrak{v}_G,\mathfrak{v}_G-\mathfrak{e}_1)|=|\det(\mathfrak{e}_1,\mathfrak{v}_G)|=l\,/\,(r-2)$. □

**Lemma 6.18.** *Suppose* $\gcd(\alpha,\beta,l)=1$ *and* $[\mathfrak{t}_1]_{(r-2)}=[\mathfrak{t}_2]_{(r-2)}=1$. *If we define*

$$\mathfrak{z}_1:=\frac{l}{\mathfrak{t}_2},\qquad\mathfrak{z}_2:=\frac{\alpha+(r-2)}{\mathfrak{t}_2}$$

(*as in* (5.7)) *and if we consider two integers* $\mathfrak{c}_1\in\mathbb{Z}_{<0}$, $\mathfrak{c}_2\in\mathbb{N}$, *such that*

$$\mathfrak{c}_1\cdot\mathfrak{z}_1+\mathfrak{c}_2\cdot\mathfrak{z}_2=1\tag{6.6}$$

*then we get*

$$q=\frac{l}{\mathfrak{t}_1\cdot\mathfrak{t}_2},\qquad p=[\breve{p}]_q\tag{6.7}$$

*and* $\mathfrak{y}$ *can be taken to be*

$$\mathfrak{y}=\left(\frac{\breve{p}-p}{q}\right)(\mathfrak{v}_G-\widetilde{\mathfrak{u}}_0)+(-\mathfrak{c}_1\cdot\mathfrak{e}_1+\mathfrak{c}_2\cdot\mathfrak{e}_2)\tag{6.8}$$

*where*

$$\breve{p}:=\frac{\mathfrak{c}_1\cdot l+\mathfrak{c}_2\cdot\alpha}{\mathfrak{t}_1}$$

(*as in* (5.9)).

*Proof.* Under the above assumption we have obviously

$$\det\left(\widetilde{\Lambda_G^{\mathrm{ext}}}\right)=\frac{1}{r-2},\qquad q=\frac{l}{\mathfrak{t}_1\cdot\mathfrak{t}_2},$$

and

$$|\det(\mathfrak{v}_G-\widetilde{\mathfrak{u}}_0,\mathfrak{c}_1\cdot\mathfrak{e}_1+\mathfrak{c}_2\cdot\mathfrak{e}_2)|=\frac{1}{\mathfrak{t}_2}\ |\det(\mathfrak{v}_G-\widetilde{\mathfrak{u}}_0,\mathfrak{c}_1\cdot\mathfrak{e}_1+\mathfrak{c}_2\cdot\mathfrak{e}_2)|=\frac{1}{r-2}\ .$$

Therefore, $\{\mathfrak{v}_G-\widetilde{\mathfrak{u}}_0,\mathfrak{c}_1\cdot\mathfrak{e}_1+\mathfrak{c}_2\cdot\mathfrak{e}_2\}$ is a $\mathbb{Z}$-basis of the extended lattice $\widetilde{\Lambda_G^{\mathrm{ext}}}$. Now since

$$\mathfrak{v}_G-\widetilde{\mathfrak{u}}_{\rho+1}=\frac{1}{\mathfrak{t}_1\,(r-2)}\,(\alpha\cdot\mathfrak{e}_1+l\cdot\mathfrak{e}_2)$$

and

$$\mathfrak{v}_G-\widetilde{\mathfrak{u}}_0=\frac{1}{\mathfrak{t}_2\,(r-2)}\,((\alpha+(r-2))\cdot\mathfrak{e}_1+l\cdot\mathfrak{e}_2)=\frac{1}{r-2}\,(\mathfrak{z}_2\cdot\mathfrak{e}_1+\mathfrak{z}_1\cdot\mathfrak{e}_2)\,,$$

using (6.6) we deduce:

$$\mathfrak{v}_G-\widetilde{\mathfrak{u}}_{\rho+1}=\breve{p}\cdot(\mathfrak{v}_G-\widetilde{\mathfrak{u}}_0)+q\cdot(-\mathfrak{c}_1\cdot\mathfrak{e}_1+\mathfrak{c}_2\cdot\mathfrak{e}_2)\ .$$

Hence, it suffices to take $p$ and $\mathfrak{y}$ to be given by the formulae (6.7) and (6.8), respectively. □

**Remark 6.19.** Since $p$ is equal to the non negative remainder $[\check{p}]_q$ modulo $q$, it does not depend on the particular choice of solutions $\mathfrak{c}_1 \in \mathbb{Z}_{<0}$, $\mathfrak{c}_2 \in \mathbb{N}$ of the linear diophantine equation (6.6). Nevertheless, in view of what was discussed in remark 3.2, we give in the formulation of thm. 5.15 a specific pair $\{\mathfrak{c}_1, \mathfrak{c}_2\}$ for one of the most convenient solutions of (6.6) which can be read off directly from the regular continued fraction expansion of $\mathfrak{z}_1/\mathfrak{z}_2$.

Next lemma shows that our special assumption in lemma 6.18 is included as a part of a necessary condition for the existence of $T_{N_G}$-equivariant, crepant, full resolutions.

**Lemma 6.20.** *Let $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ be a Gorenstein cyclic quotient singularity of type (5.5). If this singularity admits a $T_{N_G}$-equivariant, crepant, full resolution, then*

$$\gcd(\alpha, \beta, l) \in \{1, r-2\} .$$

*Moreover, in the case in which* $\gcd(\alpha,\beta,l) = 1$, *we have* $[\mathfrak{t}_1]_{(r-2)} = [\mathfrak{t}_2]_{(r-2)} = 1$.

*Proof.* By definition, $1 \leq \gcd(\alpha,\beta,l) \leq r-2$. Suppose $r \geq 5$ and $\gcd(\alpha,\beta,l) \in \{2, 3, \ldots, r-3\}$. Obviously,

$$\frac{1}{l}\left(\underbrace{\left[\frac{l}{\gcd(\alpha,\beta,l)}\right]_l, \ldots, \left[\frac{l}{\gcd(\alpha,\beta,l)}\right]_l}_{(r-2)\text{-times}}, \left[\frac{l}{\gcd(\alpha,\beta,l)} \cdot \alpha\right]_l, \left[\frac{l}{\gcd(\alpha,\beta,l)} \cdot \beta\right]_l\right)^{\intercal}$$

equals

$$\frac{1}{l}\left(\underbrace{\frac{l}{\gcd(\alpha,\beta,l)}, \ldots, \frac{l}{\gcd(\alpha,\beta,l)}}_{(r-2)\text{-times}}, 0, 0\right)^{\intercal}$$

and hence it is a lattice point belonging to $\mathbf{Hlb}_{N_G}(\sigma_0)$, because it cannot be written as the sum of two other elements of $N_G \smallsetminus \{\mathbf{0}\}$ (cf. (2.1)). On the other hand,

$$\frac{1}{l}\underbrace{\left(\frac{l}{\gcd(\alpha,\beta,l)} + \cdots + \frac{l}{\gcd(\alpha,\beta,l)}\right)}_{(r-2)\text{-times}} = \frac{r-2}{\gcd(\alpha,\beta,l)} > 1,$$

contradicting thm. 5.9. Hence, $\gcd(\alpha,\beta,l) \in \{1, r-2\}$. Now if $\gcd(\alpha,\beta,l) = 1$, using lemma 6.16 and the fact that the triangles $\mathrm{conv}(\{\mathfrak{v}_G, \widetilde{\mathfrak{u}}_0, \widetilde{\mathfrak{u}}_1\})$ and $\mathrm{conv}(\{\mathfrak{v}_G, \widetilde{\mathfrak{u}}_\rho, \widetilde{\mathfrak{u}}_{\rho+1}\})$ have to be basic w.r.t. $\widetilde{\Lambda_G^{\mathrm{ext}}}$ (with $\widetilde{\mathfrak{u}}_0, \widetilde{\mathfrak{u}}_1, \widetilde{\mathfrak{u}}_\rho, \widetilde{\mathfrak{u}}_{\rho+1} \in \widetilde{\Lambda_G}$), we obtain

$$|\det(\mathfrak{v}_G - \widetilde{\mathfrak{u}}_0, \widetilde{\mathfrak{u}}_1 - \widetilde{\mathfrak{u}}_0)| = |\det(\mathfrak{v}_G - \widetilde{\mathfrak{u}}_\rho, \widetilde{\mathfrak{u}}_{\rho+1} - \widetilde{\mathfrak{u}}_\rho)| = \det\left(\widetilde{\Lambda_G^{\mathrm{ext}}}\right) = \frac{1}{r-2}$$

if and only if

$$\frac{[\mathfrak{t}_2]_{(r-2)}}{\mathfrak{t}_2} |\det(\mathfrak{v}_G, \widetilde{\mathfrak{u}}_1 - \widetilde{\mathfrak{u}}_0)| = \frac{[\mathfrak{t}_1]_{(r-2)}}{\mathfrak{t}_1} |\det(\mathfrak{v}_G, \widetilde{\mathfrak{u}}_{\rho+1} - \widetilde{\mathfrak{u}}_\rho)| = \frac{1}{r-2},$$

which is equivalent to the existence of integers $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, such that

$$\frac{[\mathfrak{t}_2]_{(r-2)}}{\mathfrak{t}_2} |\gamma_1 l + \gamma_2(\alpha + (r-2))| = \frac{[\mathfrak{t}_1]_{(r-2)}}{\mathfrak{t}_1} |\gamma_3 l + \gamma_4(\alpha + (r-2))| = 1.$$

Since necessarily

$$\gamma_1 l + \gamma_2(\alpha + (r-2)) \equiv 0 \pmod{\mathfrak{t}_2}, \quad \gamma_3 l + \gamma_4(\alpha + (r-2)) \equiv 0 \pmod{\mathfrak{t}_1},$$

we get $[\mathfrak{t}_1]_{(r-2)} = [\mathfrak{t}_2]_{(r-2)} = 1$. $\square$

• **Fifth step**. Now we revert again to the general setting and make essential use of the statements of §3 for the $(p,q)$-cone $\boldsymbol{\tau}_G$. If $\boldsymbol{\tau}_G$ is *non-basic* (i.e., if $p \neq 0$), then we expand $q/p$ as regular continued fraction

$$\frac{q}{p} = [\lambda_1, \lambda_2, \ldots, \lambda_{\kappa-1}, \lambda_\kappa] \ ,$$

($\kappa \geq 2$, $\lambda_\kappa \geq 2$). As we explained in theorem 3.16, Kleinian approximations enable us to control completely the vertices of both cones $\boldsymbol{\tau}_G$ and $\mathbb{R}_{\geq 0}\ \mathfrak{y}+\mathbb{R}_{\geq 0}\ (\mathfrak{v}_G - \widetilde{\mathfrak{u}}_{\rho+1})$ (having the ray $\mathbb{R}_{\geq 0}\ (\mathfrak{v}_G - \widetilde{\mathfrak{u}}_{\rho+1})$ as common face, cf. rem. 3.17) by introducing the recurrence relations

$$\mathbf{v}_0 := \mathfrak{v}_G - \widetilde{\mathfrak{u}}_0,\ \mathbf{v}_1 := \mathfrak{y}, \qquad \mathbf{v}_i := \lambda_{i-1}\ \mathbf{v}_{i-1} + \mathbf{v}_{i-2},\ \forall i,\ \ 2 \leq i \leq \kappa+1. \tag{6.9}$$

**Lemma 6.21.** *If $\mu_G \neq 1$ and $\boldsymbol{\tau}_G$ non-basic, then each of the conditions of lemma 6.16 is equivalent to each of the following conditions*:

(i) $\mathfrak{y} \in \widetilde{\Lambda_G}$ *and all points* $\left\{\mathbf{v}_{2i+1}\ \middle|\ 1 \leq i \leq \frac{\kappa-1}{2}\right\}$, *for $\kappa$ odd $\geq 3$,*

(*resp. all points* $\left\{\mathbf{v}_{2i+1}\ \middle|\ 1 \leq i \leq \frac{\kappa}{2}-1,\, \kappa \neq 2\right\} \cup \{\mathbf{v}_{\kappa+1} - \mathbf{v}_\kappa\}$, *for $\kappa$ even $\geq 2$*)

*of the extended lattice $\widetilde{\Lambda_G^{\mathrm{ext}}}$ belong already to $\widetilde{\Lambda_G}$.*

(ii) $\mathfrak{y} \in \widetilde{\Lambda_G}$ *and all points* $\left\{\lambda_{2i}\,(\mathfrak{v}_G - \widetilde{\mathfrak{u}}_0)\ \middle|\ 1 \leq i \leq \frac{\kappa-1}{2}\right\}$, *for $\kappa$ odd $\geq 3$,*

(*resp. all points* $\left\{\lambda_{2i}\,(\mathfrak{v}_G - \widetilde{\mathfrak{u}}_0)\ \middle|\ 1 \leq i \leq \frac{\kappa}{2}-1,\, \kappa \neq 2\right\} \cup \{(\lambda_\kappa - 1)\ (\mathfrak{v}_G - \widetilde{\mathfrak{u}}_0)\}$, *for $\kappa$ even $\geq 2$*)

*of $\widetilde{\Lambda_G^{\mathrm{ext}}}$ belong already to $\widetilde{\Lambda_G}$.*

*Proof.* Since $\mathfrak{l}_1 = (\mathfrak{v}_G - \widetilde{\mathfrak{u}}_0) + \mathfrak{y}$, we have

$$\mathfrak{v}_G - \mathfrak{l}_1 \in \widetilde{\Lambda_G} \Longleftrightarrow \mathfrak{y} = \mathbf{v}_1 \in \widetilde{\Lambda_G} \Longleftrightarrow \operatorname{conv}(\mathbf{v}_0, \mathbf{v}_2) \cap \widetilde{\Lambda_G^{\mathrm{ext}}} \subset \widetilde{\Lambda_G},$$

where $\operatorname{conv}(\mathbf{v}_0, \mathbf{v}_2)$ is the "first" edge of $\partial\Theta^{\mathbf{cp}}_{\boldsymbol{\tau}_G}$ (containing only lattice points which are multiples of $\mathbf{v}_1$). Applying the analogous argument also for the next coming edges of $\partial\Theta^{\mathbf{cp}}_{\boldsymbol{\tau}_G}$, we obtain

$$\left[\begin{array}{c} \mathfrak{v}_G - \mathfrak{l}_j \in \widetilde{\Lambda_G}, \\ \forall\ j,\ 2 \leq j \leq \rho+1 \end{array}\right] \Longleftrightarrow \left\{\begin{array}{ll} \left\{\mathbf{v}_{2i+1}\ \middle|\ 1 \leq i \leq \frac{\kappa-1}{2}\right\} \in \widetilde{\Lambda_G}, & \text{for } \ \kappa \notin 2\mathbb{Z} \\ \left\{\mathbf{v}_{2i+1}\ \middle|\ 1 \leq i \leq \frac{\kappa}{2}-1,\, \kappa \neq 2\right\} \cup \{\mathbf{v}_{\kappa+1} - \mathbf{v}_\kappa\} \in \widetilde{\Lambda_G}, & \text{for } \ \kappa \in 2\mathbb{Z} \end{array}\right.$$

Hence, condition (i) is equivalent to condition (iii) of lemma 6.16. (Note that these $\mathbf{v}_{2i+1}$'s cover the vertex-set of the interior of the compact part of the support polytope which approximates the ray $\mathbb{R}_{\geq 0}\ (\mathfrak{v}_G - \widetilde{\mathfrak{u}}_{\rho+1})$ "from above" and is determined by the cone $\mathbb{R}_{\geq 0}\ \mathfrak{y}+\mathbb{R}_{\geq 0}\ (\mathfrak{v}_G - \widetilde{\mathfrak{u}}_{\rho+1})$).

(i)$\Leftrightarrow$(ii) For $\kappa$ odd $\geq 3$ or $\kappa$ even $\geq 4$, and $i \in \left\{1, \ldots, \left\lfloor\frac{\kappa-1}{2}\right\rfloor\right\}$, one easily shows via (6.9) that

$$\mathbf{v}_{2i+1} = \left(\sum_{j=0}^{i-1} \xi_j \cdot \mathbf{v}_{2j+1}\right) + \lambda_{2i} \cdot \mathbf{v}_0$$

for suitable positive integers $\xi_0, \xi_1, \ldots, \xi_{i-1}$. Analogously for $\kappa$ even $\geq 2$ it is always possible to express $\mathbf{v}_{\kappa+1} - \mathbf{v}_\kappa$ as a positive integer linear combination

$$\mathbf{v}_{\kappa+1} - \mathbf{v}_\kappa = \left(\sum_{j=0}^{\frac{\kappa}{2}-1} \xi_j \cdot \mathbf{v}_{2j+1}\right) + (\lambda_\kappa - 1) \cdot \mathbf{v}_0$$

of the $\mathbf{v}_{2j+1}$'s and $\mathbf{v}_0$. This means that it is enough for examining if $\mathbf{v}_{2i+1}$'s (resp. $\mathbf{v}_{\kappa+1} - \mathbf{v}_\kappa$) belong to $\widetilde{\Lambda_G}$ or not to restrict ourselves to the consideration of $\lambda_{2i} \cdot \mathbf{v}_0$'s (resp. $(\lambda_\kappa - 1) \cdot \mathbf{v}_0$). $\square$

• **Proof of thm. 5.15:** Let $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ denote a Gorenstein cyclic quotient singularity of type (5.5).

($\Rightarrow$): If this singularity admits $T_{N_G}$-equivariant, crepant, full resolutions, then at least one of them is necessarily projective (by cor. 6.10), and by lemma 6.20 there are only two possibilities: **either** $\mu_G = 1$, i.e., $\gcd(\alpha, \beta, l) = r - 2$ **or** $\gcd(\alpha, \beta, l) = 1$ and $[\mathfrak{t}_1]_{(r-2)} = [\mathfrak{t}_2]_{(r-2)} = 1$. In the latter case, the 2-dimensional cone $\boldsymbol{\tau}_G \subset \left(\widetilde{\Lambda_G^{\mathrm{ext}}}\right)_{\mathbb{R}}$ is a $(p,q)$-cone with

$$q = \frac{l}{\mathfrak{t}_1 \cdot \mathfrak{t}_2}, \quad p = [\breve{p}]_q \quad \text{and} \quad \mathfrak{y} = \left(\frac{\breve{p} - p}{q}\right)(\mathfrak{v}_G - \widetilde{\mathfrak{u}}_0) + (-\mathfrak{c}_1 \cdot \mathfrak{e}_1 + \mathfrak{c}_2 \cdot \mathfrak{e}_2)$$

(by lemma 6.18). If $\boldsymbol{\tau}_G$ is non-basic, then using the one direction of lemma 6.21, we have: $\mathfrak{y} \in \widetilde{\Lambda_G}$ and all points

$$\left\{ \lambda_{2i}(\mathfrak{v}_G - \widetilde{\mathfrak{u}}_0) \;\middle|\; 1 \leq i \leq \frac{\kappa - 1}{2} \right\}, \quad \text{for } \kappa \text{ odd } \geq 3,$$

$$\text{(resp. all points } \left\{ \lambda_{2i}(\mathfrak{v}_G - \widetilde{\mathfrak{u}}_0) \;\middle|\; 1 \leq i \leq \frac{\kappa}{2} - 1, \, \kappa \neq 2 \right\} \cup \{(\lambda_\kappa - 1)(\mathfrak{v}_G - \widetilde{\mathfrak{u}}_0)\}, \text{ for } \kappa \text{ even } \geq 2)$$

of $\widetilde{\Lambda_G^{\mathrm{ext}}}$ belong already to $\widetilde{\Lambda_G}$. Since $(-\mathfrak{c}_1 \cdot \mathfrak{e}_1 + \mathfrak{c}_2 \cdot \mathfrak{e}_2) \in \widetilde{\Lambda_G}$, condition $\mathfrak{y} \in \widetilde{\Lambda_G}$ is equivalent to

$$\left(\frac{\breve{p} - p}{q}\right)(\mathfrak{v}_G - \widetilde{\mathfrak{u}}_0) = \frac{\breve{p} - p}{q(r-2)} \cdot \underbrace{(\mathfrak{z}_2 \cdot \mathfrak{e}_1 + \mathfrak{z}_1 \cdot \mathfrak{e}_2)}_{\text{primitive lattice point}} \in \widetilde{\Lambda_G} \Longleftrightarrow \frac{\breve{p} - p}{q} \equiv 0 \,(\mathrm{mod}\ (r-2)).$$

On the other hand, for $\kappa$ odd $\geq 3$ or $\kappa$ even $\geq 4$, and $i \in \{1, \ldots, \lfloor \frac{\kappa - 1}{2} \rfloor\}$, we have

$$\lambda_{2i}(\mathfrak{v}_G - \widetilde{\mathfrak{u}}_0) = \frac{\lambda_{2i}}{r-2} \cdot (\mathfrak{z}_2 \cdot \mathfrak{e}_1 + \mathfrak{z}_1 \cdot \mathfrak{e}_2) \in \widetilde{\Lambda_G} \Longleftrightarrow \lambda_{2i} \equiv 0 \,(\mathrm{mod}\ (r-2)),$$

while for $\kappa$ even $\geq 2$,

$$(\lambda_\kappa - 1)(\mathfrak{v}_G - \widetilde{\mathfrak{u}}_0) = \frac{(\lambda_\kappa - 1)}{r-2} \cdot (\mathfrak{z}_2 \cdot \mathfrak{e}_1 + \mathfrak{z}_1 \cdot \mathfrak{e}_2) \in \widetilde{\Lambda_G} \Longleftrightarrow \lambda_\kappa \equiv 1 \,(\mathrm{mod}\ (r-2)).$$

Hence, all conditions (5.12) are satisfied, as we have assserted.

($\Leftarrow$): Converserly, if *either* $\gcd(\alpha, \beta, l) = 1$ and $\boldsymbol{\tau}_G$ is basic, *or* $\gcd(\alpha, \beta, l) = r - 2$, then there is nothing to be said (cf. lemma 6.14, remark 6.12). Furthermore, in the case in which $\gcd(\alpha, \beta, l) = 1$, $\boldsymbol{\tau}_G$ is non-basic, and conditions (5.12) are fulfilled, we show again by the above arguments that each of the conditions of lemma 6.21 is true too, and then make use of the "backtracking-method" for the reverse logical implications of the conditions of our previous lemmas:

$$\begin{array}{ccccccc}
\text{lemma 6.21} & & & & & & \text{proposition 6.7} \\
\Downarrow & & & & & & \Uparrow \\
\text{lemma 6.16} & \Longrightarrow & \text{lemma 6.15} & \Longrightarrow & \text{lemma 6.14} & \Longrightarrow & \text{lemma 6.13}
\end{array}$$

Thus, also in this case the existence of $T_{N_G}$-equivariant, crepant, full resolutions (and of at least one projective, cf. 6.10) for the singularity $(X(N_G, \Delta_G), \mathrm{orb}(\sigma_0))$ is indeed ensured. $\square$

**Remark 6.22.** Note that considering 2-parameter singularities of (the most general) type (6.1), the whole "reduction-procedure" presented in the second part of this section can be again applied with minor modifications. First of all one has to determine an appropriate $\mathbb{Z}$-basis of the lattice $\overline{N_G}$; namely the "nice" elements (6.4) generating $\overline{N_G}$ in the case of type (5.5) have to be changed. This is always possible by using Hermitian normal forms. On the other hand, the corresponding lattice transformations

$\Phi$, $\Psi$ and $\Upsilon$ are also definenable, though they become a little bit more complicated. Finally, the $(p,q)$-cone $\boldsymbol{\tau}_G$ provides again the extra (somewhat "wilder") arithmetical conditions for the existence of $T_{N_G}$-equivariant, crepant, full resolutions. This is why it does not seem to be of conceptual theoretical interest to deal directly with (6.1), as this work can be done by a simple computer-program. The main and more interesting conclusion in this context is that also in this most general situation, it is enough to perform a *polynomial-time-algorithm* in order to determine all the above mentioned extra arithmetical conditions, which, in other words, is an obvious strengthening of theorem 5.13 from the purely computational point of view.

## 7. Computing cohomology group dimensions

To compute the cohomology group dimensions of the overlying spaces of the crepant resolutions of our 2-parameter series of Gorenstein cyclic quotient singularities we need some concepts from enumerative combinatorics (see e.g. Stanley [60], §4.6).

• For a lattice $d$-dimensional polytope $P \subset N_{\mathbb{R}}$, i.e., for $P$ an "integral" polytope w.r.t. a lattice $N \cong \mathbb{Z}^{d'}$ of rank $d' \geq d$, and $\kappa$ a positive integer, let

$$\mathbf{Ehr}_N\left(P,\nu\right) = \mathbf{a}_0\left(P\right) + \mathbf{a}_1\left(P\right)\ \nu + \cdots + \mathbf{a}_{d-1}\left(P\right)\ \nu^{d-1} + \mathbf{a}_d\left(P\right)\ \nu^d \ \in\ \mathbb{Q}[\nu]$$

denote the *Ehrhart polynomial* of $P$ (w.r.t. $N$), where

$$\mathbf{Ehr}_N\left(P,\nu\right) := \#\left(\nu\, P \cap \left\{ \begin{array}{c} \text{the sublattice of } N \text{ of rank } d \\ \text{spanned by } \mathrm{aff}\left(P\right) \cap N \end{array} \right\}\right)$$

and

$$\mathbf{Ehr}_N\left(P;\mathfrak{x}\right) := 1 + \sum_{\nu=1}^{\infty} \mathbf{Ehr}_N\left(P,\nu\right)\ \mathfrak{x}^{\nu} \in \mathbb{Q}\left[\left[\mathfrak{x}\right]\right]$$

the corresponding *Ehrhart series*. Writing $\mathbf{Ehr}_N\left(P;\mathfrak{x}\right)$ as

$$\mathbf{Ehr}_N\left(P;\mathfrak{x}\right) = \frac{\boldsymbol{\delta}_0\left(P\right) + \ \boldsymbol{\delta}_1\left(P\right)\ \mathfrak{x} + \cdots + \ \boldsymbol{\delta}_{d-1}\left(P\right)\mathfrak{x}^{d-1} + \ \boldsymbol{\delta}_d\left(P\right)\ \mathfrak{x}^d}{\left(1-\mathfrak{x}\right)^{d+1}}$$

we obtain the so-called $\boldsymbol{\delta}$*-vector* $\boldsymbol{\delta}\left(P\right) = \left(\boldsymbol{\delta}_0\left(P\right), \boldsymbol{\delta}_1\left(P\right), \ldots, \boldsymbol{\delta}_{d-1}\left(P\right), \boldsymbol{\delta}_d\left(P\right)\right)$ of $P$.

**Definition 7.1.** For any $d \in \mathbb{Z}_{\geq 0}$ we introduce the *transfer* $\mathbf{a}$-$\boldsymbol{\delta}$*-matrix* $\mathcal{M}_d \in \mathrm{GL}(d+1,\mathbb{Q})$ (depending only on $d$) to be defined as

$$\mathcal{M}_d := \left(\mathfrak{R}_{i,j}\right)_{0 \leq i,j \leq d} \qquad \text{with} \qquad \mathfrak{R}_{i,j} := \frac{1}{d!}\left\{\sum_{\xi=i}^{d} \begin{bmatrix} d \\ \xi \end{bmatrix} \binom{\xi}{i}\ (d-j)^{\xi-i}\right\}$$

where $\begin{bmatrix} d \\ \xi \end{bmatrix}$ denotes the Stirling number (of the first kind) of $d$ over $\xi$.

The following lemma can be proved easily.

**Lemma 7.2.** *For a lattice $d$-polytope $P \subset N_{\mathbb{R}}$ w.r.t. an $N \cong \mathbb{Z}^{d'}$, $d' \geq d$, we have*

$$\left(\boldsymbol{\delta}_0\left(P\right), \boldsymbol{\delta}_1\left(P\right), \ldots, \boldsymbol{\delta}_{d-1}\left(P\right), \boldsymbol{\delta}_d\left(P\right)\right) = \left(\mathbf{a}_0\left(P\right), \mathbf{a}_1\left(P\right), \ldots, \mathbf{a}_{d-1}\left(P\right), \mathbf{a}_d\left(P\right)\right) \cdot \left(\left(\mathcal{M}_d\right)^{\intercal}\right)^{-1} \qquad (7.1)$$

**Definition 7.3.** The $\mathbf{f}$-vector of $\mathbf{f}\left(\mathcal{S}\right) = \left(\mathbf{f}_0\left(\mathcal{S}\right), \mathbf{f}_1\left(\mathcal{S}\right), \ldots, \mathbf{f}_{d-1}\left(\mathcal{S}\right), \mathbf{f}_d\left(\mathcal{S}\right)\right)$ of a pure $d$-dimensional simplicial complex $\mathcal{S}$ is defined by

$$\mathbf{f}_j\left(\mathcal{S}\right) := \#\left\{j\text{-dimensional simplices of } \mathcal{S}\right\}\ .$$

**Lemma 7.4.** *If $P \subset N_{\mathbb{R}}$ is a lattice d-polytope w.r.t. an $N \cong \mathbb{Z}^{d'}$, $d' \geq d$, admitting a basic triangulation $\mathcal{T}$, then*

$$\mathbf{Ehr}_N\left(P,\nu\right) = \sum_{j=0}^{d} \binom{\nu-1}{j} \mathbf{f}_j\left(\mathcal{T}\right)$$

*Proof.* See Stanley [59], cor. 2.5, p. 338. □

**Theorem 7.5 (Cohomology group dimensions).**
*Let $\left(X\left(N_G,\Delta_G\right), \mathrm{orb}\left(\sigma_0\right)\right)$ be a Gorenstein abelian quotient msc-singularity. If $X\left(N_G,\Delta_G\right)$ admits crepant, $T_{N_G}$-equivariant, full desingularizations, then only the* ***even****-dimensional cohomology groups of the desingularized spaces $X\left(N_G,\widehat{\Delta}_G\right)$ can be non-trivial, and*

$$\boxed{\dim_{\mathbb{Q}} H^{2i}\left(X\left(N_G,\widehat{\Delta}_G\right);\mathbb{Q}\right) = \boldsymbol{\delta}_i\left(\mathfrak{s}_G\right)} \tag{7.2}$$

*$\forall i$, $0 \leq i \leq r-1$, where $\boldsymbol{\delta}_i\left(\mathfrak{s}_G\right)$'s are the components of the $\boldsymbol{\delta}$-vector $\boldsymbol{\delta}\left(\mathfrak{s}_G\right)$ of the $(r-1)$-dimensional junior simplex $\mathfrak{s}_G$ (w.r.t. $N_G$).*

• *By (7.2) and (7.1) it is now clear that these cohomology group dimensions* ***do not*** *depend on the basic triangulations by means of which one constructs the subdivisions $\widehat{\Delta}_G$ of $\sigma_0$, but* ***only*** *on the coefficients of the Ehrhart polynomial of $\mathfrak{s}_G$.*

• *In particular, if $\left(X\left(N_G,\Delta_G\right), \mathrm{orb}\left(\sigma_0\right)\right)$ denotes a Gorenstein cyclic quotient msc-singularity having type $\frac{1}{l}\left(\alpha_1,\ldots,\alpha_r\right)$ with $l = |G| \geq r \geq 4$, for which at least $r-2$ of its defining weights are equal, and if $X\left(N_G,\Delta_G\right)$ admits crepant, $T_{N_G}$-equivariant, full desingularizations, then the Ehrhart polynomial of $\mathfrak{s}_G$ (whose coefficients lead by (7.1) to the computation of the corresponding non-trivial cohomology group dimensions of $X\left(N_G,\widehat{\Delta}_G\right)$'s) can be determined as follows:*

• *At first we define the polynomial*

$$\mathfrak{B}_G\left(i;\nu\right) :=$$

$$\sum_{j=0}^{i+2} \binom{\nu-1}{j}\left\{\binom{i}{j+1} + \#\left(\overline{\mathfrak{s}_G} \cap \overline{N_G}\right)\binom{i}{j} + \left(3\,\mathrm{Vol}\left(\overline{\mathfrak{s}_G}\right) + \tfrac{1}{2}\#\left(\partial\left(\overline{\mathfrak{s}_G}\right) \cap \overline{N_G}\right)\right)\binom{i}{j-1} + 2\,\mathrm{Vol}\left(\overline{\mathfrak{s}_G}\right)\binom{i}{j-2}\right\} \tag{7.3}$$

*for all $i$, $1 \leq i \leq r-3$ (where $\mathfrak{Q}_G = \mathrm{conv}\left(\mathfrak{s}_G \cap \overline{N_G}\right) \subseteqq \overline{\mathfrak{s}_G} = \mathfrak{s}_G \cap \mathfrak{L} \subset \left(\overline{N_G}\right)_{\mathbb{R}}$ as in §6(a)). In addition, in the case in which $\mu_G \neq 1$, we define the polynomial*

$$\mathfrak{D}\left(i;\nu\right) := \sum_{j=0}^{i+1} \binom{\nu-1}{j}\left\{\binom{i}{j+1} + (\rho+2)\binom{i}{j} + (\rho+1)\binom{i}{j-1}\right\} \tag{7.4}$$

*for all $i$, $1 \leq i \leq r-2$ (with $\rho$ as in 6.1). Using $\mathfrak{B}_G\left(i;\nu\right)$ and $\mathfrak{D}_G\left(i;\nu\right)$ we obtain:*

(i) *If $\mu_G = 1$,*

$$\mathbf{Ehr}_{N_G}\left(\mathfrak{s}_G,\nu\right) = \sum_{i=1}^{r-3} (-1)^{r-3-i}\binom{r-2}{i}\mathfrak{B}_G\left(i;\nu\right) + (-1)^{r-3}\ \mathbf{Ehr}_{\overline{N_G}}\left(\mathfrak{Q}_G,\nu\right) \tag{7.5}$$

(ii) *If $\mu_G \neq 1$,*

$$\mathbf{Ehr}_{N_G}\left(\mathfrak{s}_G,\nu\right) = \sum_{i=1}^{r-3} (-1)^{r-3-i}\binom{r-2}{i}\mathfrak{B}_G\left(i;\nu\right) + (-1)^{r-3}\ \mathbf{Ehr}_{\overline{N_G}}\left(\mathfrak{Q}_G,\nu\right) +$$

$$+\sum_{i=1}^{r-2} (-1)^{r-2-i}\binom{r-2}{i}\mathfrak{D}_G\left(i;\nu\right) + (-1)^{r-2}\ \left((\rho+1)\ \nu+1\right) \tag{7.6}$$

*Moreover, in both cases we have*

$$\mathbf{Ehr}_{\overline{N_G}}\left(\mathfrak{Q}_G,\nu\right) = \#\left(\nu\,\mathfrak{Q}_G \cap \overline{N_G}\right) = \mathrm{Vol}\left(\mathfrak{Q}_G\right)\,\nu^2 + \frac{1}{2}\#\left(\partial\mathfrak{Q}_G \cap \overline{N_G}\right)\,\nu + 1 \tag{7.7}$$

*Proof.* For the proof of the first statement see Batyrev-Dais [3], thm. 4.4, p. 909. Now let

$$\left(X\left(N_G, \Delta_G\right), \operatorname{orb}\left(\sigma_0\right)\right)$$

be a 2-parameter Gorenstein cyclic quotient msc-singularity admitting torus-equivariant crepant resolutions. Since the dimension of $H^{2i}\left(X\left(N_G, \widehat{\Delta}_G\right); \mathbb{Q}\right)$ does not depend on the choice of the basic triangulations constructing $\widehat{\Delta}_G$'s, it is enough for its computation to consider just *one* triangulation of this sort. Hereafter take a $\widehat{\Delta}_G = \widehat{\Delta}_G\left(\mathcal{T}[\mathfrak{T}]\right)$ being induced by a fixed basic triangulation of the special form $\mathcal{T}[\mathfrak{T}]$ (see 6.3, 6.9). Formula (7.7) is nothing but Pick's theorem applied to the lattice polygon $\mathfrak{Q}_G$. Next we shall treat the two possible cases separately:

(i) If $\mu_G = 1$, then $\overline{\mathfrak{s}_G} = \mathfrak{Q}_G$ and $\mathfrak{s}_G$ can be written as

$$\mathfrak{s}_G = \bigcup_{(\xi_1, \xi_2, \ldots, \xi_{r-3}) \in \Xi_r} \overline{\mathfrak{s}_G} * \operatorname{conv}\left(\left\{e_{\xi_1}, e_{\xi_2}, \ldots, e_{\xi_{r-3}}\right\}\right)$$

Using the principle of inclusion-exclusion we deduce

$$\mathbf{Ehr}_{N_G}\left(\mathfrak{s}_G, \nu\right) =$$

$$\sum_{i=1}^{r-3} (-1)^{r-3-i} \sum_{1 \leq \xi_1 < \xi_2 < \cdots < \xi_i \leq r-2} \mathbf{Ehr}_{N_G}\left(\overline{\mathfrak{s}_G} * \operatorname{conv}\left(e_{\xi_1}, \ldots, e_{\xi_i}\right), \nu\right) + (-1)^{r-3}\ \mathbf{Ehr}_{\overline{N_G}}\left(\overline{\mathfrak{s}_G}, \nu\right) \quad (7.8)$$

By lemma 7.4 we obtain for the join $\overline{\mathfrak{s}_G} * \operatorname{conv}\left(e_{\xi_1}, \ldots, e_{\xi_i}\right)$ of dimension $2 + (i-1) + 1 = i+2$,

$$\mathbf{Ehr}_{N_G}\left(\overline{\mathfrak{s}_G} * \operatorname{conv}\left(e_{\xi_1}, \ldots, e_{\xi_i}\right), \nu\right) = \sum_{j=0}^{i+2} \tbinom{\nu-1}{j}\ \mathbf{f}_j\left(\mathcal{T}[\mathfrak{T}]\,|_{\ \overline{\mathfrak{s}_G} * \operatorname{conv}\left(e_{\xi_1}, \ldots, e_{\xi_i}\right)}\right) \quad (7.9)$$

with

$$\begin{aligned}
\mathbf{f}_j\left(\mathcal{T}[\mathfrak{T}]\,|_{\ \overline{\mathfrak{s}_G} * \operatorname{conv}\left(e_{\xi_1}, \ldots, e_{\xi_i}\right)}\right) &= \mathbf{f}_j\left(\mathfrak{T} * \operatorname{conv}\left(\left\{e_{\xi_1}, \ldots, e_{\xi_i}\right\}\right)\right) = \\
&= \mathbf{f}_j\left(\operatorname{conv}\left(\left\{e_{\xi_1}, .., e_{\xi_i}\right\}\right)\right) + \mathbf{f}_0(\mathfrak{T}) \cdot \mathbf{f}_{j-1}\left(\operatorname{conv}\left(\left\{e_{\xi_1}, .., e_{\xi_i}\right\}\right)\right) + \\
&+ \mathbf{f}_1(\mathfrak{T}) \cdot \mathbf{f}_{j-2}\left(\operatorname{conv}\left(\left\{e_{\xi_1}, .., e_{\xi_i}\right\}\right)\right) + \mathbf{f}_2(\mathfrak{T}) \cdot \mathbf{f}_{j-3}\left(\operatorname{conv}\left(\left\{e_{\xi_1}, .., e_{\xi_i}\right\}\right)\right),
\end{aligned}$$

where

$$\mathbf{f}_j\left(\operatorname{conv}\left(\left\{e_{\xi_1}, .., e_{\xi_i}\right\}\right)\right) = \binom{i}{j+1} \quad (7.10)$$

and

$$\mathbf{f}_0(\mathfrak{T}) = \#\left(\overline{\mathfrak{s}_G} \cap \overline{N_G}\right), \quad \mathbf{f}_1(\mathfrak{T}) = 3\operatorname{Vol}\left(\overline{\mathfrak{s}_G}\right) + \tfrac{1}{2}\#\left(\partial\left(\overline{\mathfrak{s}_G}\right) \cap \overline{N_G}\right), \quad \mathbf{f}_2(\mathfrak{T}) = 2\operatorname{Vol}\left(\overline{\mathfrak{s}_G}\right). \quad (7.11)$$

Since (7.10) is valid for *all* $i$, $1 \leq i \leq r-3$, and *all* $\binom{r-2}{i}$ $i$-tuples $1 \leq \xi_1 < \cdots < \xi_i \leq r-2$, the formulae (7.9), (7.10) and (7.11) imply

$$\mathbf{Ehr}_{N_G}\left(\overline{\mathfrak{s}_G} * \operatorname{conv}\left(e_{\xi_1}, \ldots, e_{\xi_i}\right), \nu\right) = \mathfrak{B}_G(i; \nu) \quad (7.12)$$

and (7.5) follows from (7.8) and (7.12).

(ii) If $\mu_G \neq 1$, then

$$\mathfrak{s}_G = \bigcup_{(\xi_1, \ldots, \xi_{r-3}) \in \Xi_r} \left(\mathfrak{Q}_G * \operatorname{conv}\left(\left\{e_{\xi_1}, .., e_{\xi_{r-3}}\right\}\right)\right) \cup \left[\left(\bigcup_{j=0}^{\rho} \operatorname{conv}\left(\left\{\mathfrak{w}_j, \mathfrak{w}_{j+1}\right\}\right)\right) * \operatorname{conv}\left(\left\{e_1, e_2, .., e_{r-2}\right\}\right)\right]$$

and applying again inclusion-exclusion-principle we get

$$\mathbf{Ehr}_{N_G}\left(\mathfrak{s}_G, \nu\right) =$$

$$= \sum_{i=1}^{r-3} (-1)^{r-3-i} \sum_{1\leq \xi_1<\xi_2<\cdots<\xi_i\leq r-2} \mathbf{Ehr}_{N_G}\left(\mathfrak{Q}_G * \mathrm{conv}\left(e_{\xi_1},\ldots,e_{\xi_i}\right),\nu\right) + (-1)^{r-3}\ \mathbf{Ehr}_{\overline{N_G}}\left(\mathfrak{Q}_G,\nu\right) +$$

$$+\,\mathbf{Ehr}_{N_G}\left(\left(\left(\bigcup_{j=0}^{\rho}\mathrm{conv}\left(\{\mathfrak{w}_j,\mathfrak{w}_{j+1}\}\right)\right) * \mathrm{conv}\left(\{e_1,e_2,..,e_{r-2}\}\right),\nu\right) -$$

$$-\,\mathbf{Ehr}_{N_G}\left(\bigcup_{(\xi_1,\xi_2,\ldots,\xi_{r-3})\in\Xi_r}\left(\left(\bigcup_{j=0}^{\rho}\mathrm{conv}\left(\{\mathfrak{w}_j,\mathfrak{w}_{j+1}\}\right)\right) * \mathrm{conv}\left(\left\{e_{\xi_1},..,e_{\xi_{r-3}}\right\}\right)\right),\nu\right) \tag{7.13}$$

By similar arguments to thosed used in (i) we conclude

$$\mathbf{Ehr}_{N_G}\left(\left(\bigcup_{j=0}^{\rho}\mathrm{conv}\left(\{\mathfrak{w}_j,\mathfrak{w}_{j+1}\}\right)\right) * \mathrm{conv}\left(\{e_1,e_2,..,e_{r-2}\}\right),\nu\right) = \mathfrak{D}_G\left(r-2;\nu\right) \tag{7.14}$$

and

$$\mathbf{Ehr}_{N_G}\left(\left(\left(\bigcup_{j=0}^{\rho}\mathrm{conv}\left(\{\mathfrak{w}_j,\mathfrak{w}_{j+1}\}\right)\right) * \mathrm{conv}\left(\left\{e_{\xi_1},..,e_{\xi_i}\right\}\right)\right),\nu\right) = \mathfrak{D}_G\left(i;\nu\right) \tag{7.15}$$

for *all* $i$, $1\leq i\leq r-3$, and *all* $\binom{r-2}{i}$ $i$-tuples $1\leq \xi_1<\cdots<\xi_i\leq r-2$. Furthermore,

$$\mathbf{Ehr}_{N_G}\left(\left(\bigcup_{j=0}^{\rho}\mathrm{conv}\left(\{\mathfrak{w}_j,\mathfrak{w}_{j+1}\}\right)\right),\nu\right) = (\rho+1)\ \nu+1 \tag{7.16}$$

and formula (7.6) follows easily from (7.13), (7.14), (7.15) and (7.16). $\square$

**Corollary 7.6.** *Let $\left(X\left(N_G,\Delta_G\right),\mathrm{orb}\left(\sigma_0\right)\right)$ be a Gorenstein cyclic quotient singularity of type (5.5). If one of the conditions* (i), (ii) *of thm.* 5.15 *is fulfilled, then the dimensions of the non-trivial cohomology groups of all spaces $X\left(N_G,\widehat{\Delta}_G\right)$ desingularizing fully $X\left(N_G,\Delta_G\right)$ by $T_{N_G}$-equivariant crepant morphisms are given by the formulae* (7.2), (7.1), (7.5),(7.6) *and* (7.7) *which depend* ***only*** *on $\alpha,\beta,r$ because*:

(i) *If $\mu_G=1$, i.e., if* $\gcd(\alpha,\beta,l)=r-2$, *then*

$$\mathrm{Vol}\left(\mathfrak{Q}_G\right) = \frac{l}{2\left(r-2\right)} = \frac{1}{2}+\frac{\alpha+\beta}{r-2} \tag{7.17}$$

*and*

$$\#\left(\partial\mathfrak{Q}_G\cap\overline{N_G}\right) = \gcd\left(\frac{\alpha}{r-2}+1,\frac{\alpha+\beta}{r-2}+1\right)+\gcd\left(\frac{\alpha}{r-2},\frac{\alpha+\beta}{r-2}+1\right)+1 \tag{7.18}$$

(ii) *If $\mu_G\neq 1$, i.e., if* $\gcd(\alpha,\beta,l)=1$, *and $p\neq 0$, then using the continued fraction expansion*

$$\frac{q}{p} = \left[\lambda_1,\lambda_2,\ldots,\lambda_{\kappa-1},\lambda_\kappa\right]$$

*(as defined in thm.* 5.15) *we obtain*

$$\mathrm{Vol}\left(\mathfrak{Q}_G\right) = \frac{1}{2\left(r-2\right)}\left(l-\left\lfloor\frac{\kappa}{2}\right\rfloor-1\right) \tag{7.19}$$

*and*

$$\#\left(\partial\mathfrak{Q}_G\cap\overline{N_G}\right) = \begin{cases} \sum_{i=1}^{\frac{\kappa}{2}}\lambda_{2i}+\left\lfloor\frac{\gcd(\alpha,l)}{r-2}\right\rfloor+\left\lfloor\frac{\gcd(\beta,l)}{r-2}\right\rfloor+2, & \text{for } \kappa \text{ even} \\ \sum_{i=1}^{\frac{\kappa-1}{2}}\lambda_{2i}+\left\lfloor\frac{\gcd(\alpha,l)}{r-2}\right\rfloor+\left\lfloor\frac{\gcd(\beta,l)}{r-2}\right\rfloor+3, & \text{for } \kappa \text{ odd} \end{cases} \tag{7.20}$$

*(If $p$ happens to be $=0$, then just delete the term $-\left\lfloor\frac{\kappa}{2}\right\rfloor$ in* (7.19), *and take the first formula of* (7.20) *for counting the boundary lattice points of $\mathfrak{Q}_G$ after having deleted the first summand.)*

*Proof.* The formula (7.17) in case (i) is obvious. (7.18) follows from

$$\#\left(\mathrm{conv}\left(\mathfrak{w}', e_r\right) \cap \overline{N_G}\right) = \gcd\left(\frac{\alpha}{r-2}, \frac{l}{r-2}\right) + 1, \quad \#\left(\mathrm{conv}\left(\mathfrak{w}, e_{r-1}\right) \cap \overline{N_G}\right) = \gcd\left(\frac{\alpha}{r-2} + 1, \frac{l}{r-2}\right) + 1.$$

In case (ii) we obtain (7.19) by

$$\mathrm{Vol}\left(\mathfrak{Q}_G\right) = \mathrm{Vol}\left(\mathfrak{s}_G\right) - \frac{1}{2\,(r-2)}\left(\#\left(\begin{array}{c}\textbf{edges} \text{ of the polygonal} \\ \text{line } \bigcup_{j=0}^{\rho} \mathrm{conv}\left(\{\mathfrak{w}_j, \mathfrak{w}_{j+1}\}\right)\end{array}\right)\right) = \frac{1}{2\,(r-2)}\left(l - \left\lfloor\frac{\kappa}{2}\right\rfloor - 1\right).$$

Finally, the number (7.20) of the lattice points lying on the boundary of $\mathfrak{Q}_G$ equals

$$\#\left(\partial\mathfrak{Q}_G \cap \overline{N_G}\right) = \rho + \#\left(\mathrm{conv}\left(\mathfrak{w}', e_r\right) \cap \overline{N_G}\right) + \#\left(\mathrm{conv}\left(\mathfrak{w}, e_{r-1}\right) \cap \overline{N_G}\right),$$

where

$$\#\left(\mathrm{conv}\left(\mathfrak{w}', e_r\right) \cap \overline{N_G}\right) = \left\lfloor\frac{\gcd(\beta, l)}{r-2}\right\rfloor + 1, \quad \#\left(\mathrm{conv}\left(\mathfrak{w}, e_{r-1}\right) \cap \overline{N_G}\right) = \left\lfloor\frac{\gcd(\alpha, l)}{r-2}\right\rfloor + 1,$$

and the proof is completed by expressing $\rho$ by the entries of the above regular continued fraction (cf. rem. 3.8). □

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Dimitrios I. Dais
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen, Germany
dais@wolga.mathematik.uni-tuebingen.de

Utz-Uwe Haus, Martin Henk
Konrad-Zuse-Zentrum Berlin
Takustraße 7
14195 Berlin, Germany
haus{henk}@zib.de