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An Alternative Derivation of the Exact DtN-Map on a Circle

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Abstract

The paper supplies an alternative derivation of the exact boundary conditions needed for the solution of time-harmonic acoustic scattering problems modeled by the Helmholtz equation. The main idea is to consider the exterior domain problem as an initial value problem with initial data given on the boundary of a disc or sphere. The solution of the exterior domain problem is obtained via Laplace transformation techniques, where the asymptotic Sommerfeld radiation condition is reformulated accordingly.

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1 Introduction

The solution of problems of acoustic or electro-magnetic radiation and scattering, based on the Helmholtz equation, is an essential ingredient in the modeling of complex physical situations or technical devices. Moreover, even in quantum mechanics the solution of time–dependent scattering problems is often closely linked to the solution of the Helmholtz equation in exterior domains. Most work spent on this subject has been concentrated on the development and analysis of fast and reliable numerical schemes combining effective methods for the interior problem with methods constructed to solve problems in exterior domains. The methods most often used are presented and analyzed in the books by Givoli [4] and by Ihlenburg [8] and can be grouped roughly into the following classes:

1. Combination of interior methods like the finite element method with boundary element methods to solve both the interior and the exterior problem.
2. Combination of interior methods with an infinite element method to solve both the interior and the exterior problem.
3. Combination of interior methods with an Dirichlet-to-Neumann (DtN) operator on an artificial boundary to solve the interior problem only.
4. Combination of interior methods with exact or approximate absorbing boundary conditions to solve the interior problem only.

The first class is applicable, if a fundamental solution is available, which satisfies Sommerfeld's radiation condition. Extensive literature on this subject exists [15], especially concerning the case of the potential non-uniqueness if the standard Helmholtz integral representation formulas are used, see [1] and the literature cited therein.

The second class uses a discretization of the exterior domain with shape functions obtained from an asymptotic expansion in radial direction of the exterior solution, see [3]. Again, a representation of the exterior solution *must* be available.

The third class utilizes separable coordinate systems to construct exterior solutions which are used to construct a mapping from Dirichlet data given on an artificial boundary separating the interior and the exterior domain to Neumann data on the same boundary. The method is closely related to the first one, except that the exterior solution is not represented by a convolution formula but by a superposition of separated solutions. This idea overcomes the potential difficulties of the first method (with respect to the elementary representation formulas) and goes back to Keller and Givoli [9]. However, again we need a tool to decide, in case when all separable solutions are known, which of them are outgoing solutions, i. e. , which of them fulfill Sommerfeld's radiation condition.

The last class of methods does not aim to solve the Helmholtz equation on an unbounded domain. Instead, absorbing boundary conditions are constructed which have to guaranty that the boundary of the computational domain does not cause artificial reflections. Most prominent here are the perfectly matched layers of Bérenger [2].

The crucial prerequisite for the first three methods is to have a representation of the exterior solution *and* a tool like the Sommerfeld condition at hand, in order to decide, what is outgoing and what is incoming. However, in some practical relevant situations such a tool is not available, for example in cases with non-constant coefficients or even in some cases with piecewise constant coefficients. In [5] Goldstein investigated the case of an waveguide type inhomogeneity. In examples like this the Sommerfeld condition is not applicable.

The purpose of this paper is to give a new view on Sommerfeld's radiation condition. We do not aim to give a new boundary conditions or a new numerical method but a new understanding of the Sommerfeld condition. In fact, in §4.1 we replace this condition by a different one. Our formulation of the radiation condition is equivalent to Sommerfeld's condition in case of the Helmholtz equation. The new condition, however, is not restricted to the special case of the Helmholtz equation with a constant Helmholtz term. In fact, we applied the same basic idea originally to the time-dependent Schrödinger equation [11], [12] in one space dimension. Besides its generality, the point is that the new formulation is constructive. It allows to derive the radiation condition directly from the differential equation and allows therefore to construct a DtN-map based directly on the given equation. Thus the use of any special functions, which are the core of the first three methods, is avoided.

We present the concept here in correspondence to the derivation of the classic DtN-map, i. .e. , based on a separation ansatz. This is not crucial for the method, but it simplifies the representation and allows a complete analytic treatment of the subject. Thus, both similarity and difference with respect to the classic approach becomes apparent.

2 Formulation of the scattering problem

Let $\mathcal{R}$ be a bounded domain in two dimensions, and let Γ be its piecewise smooth boundary. Further, let $\mathcal{B}_{r_0}$ be a disc of of radius r_0 with boundary $\partial\mathcal{B}_{r_0}$ which contains $\mathcal{R}$. We seek a solution $u(x), x \in \bar{\Omega}$, where the domain Ω is the region inside the disc and outside the scatterer, $\Omega = \mathcal{B}_{r_0} \cap \mathcal{R}$, which satisfies the Helmholtz equation

$$\begin{aligned} \Delta u + k^2 u &= f && \text{outside } \mathcal{R} \\ \alpha u + \beta \partial_n u &= g && \text{on } \Gamma \\ \partial_n u - iku &= o\left(\frac{1}{r^{1/2}}\right) . && \end{aligned} \tag{1}$$

Here k is the real wavenumber, f is the source term with compact

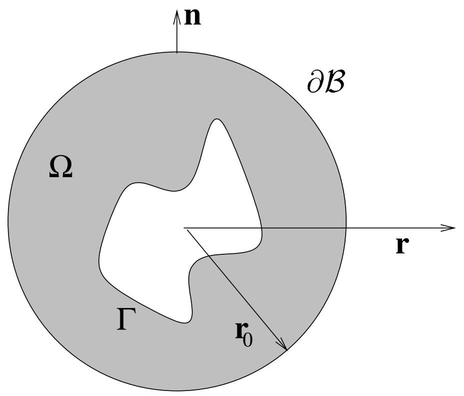

Figure 1: The computational domain Ω with the artificial boundary $\partial\mathcal{B}$

support in Ω, α, β, g are given functions and (1) is the famous Sommerfeld radiation condition. The purpose of the radiation condition is to allow only outgoing waves at infinity. Sommerfeld formulated the condition in a 1912 paper [13] and dealt with it also in his famous lecture notes [14]. To derive the condition, he argues as follows. In the potential theoretic limit ($k \to 0$) we know that the knowledge of the sources supplemented with zero boundary conditions at infinity is enough to derive a unique solution. In the case of the time-harmonic wave equation, where the time-dependence is assumed to be $\exp(-i\omega t)$, however, we have fundamental solutions of the type

$\exp(\pm ikr)/r$. Hence, we can superpose any given solution with functions, even with regular functions, which satisfy the equation and vanish at infinity. Sommerfeld concluded that, different to potential theory, not the decay behavior of the modulus of a function is the decisive property but one has to the distinguish between *incoming* and *outgoing* waves. Incoming and outgoing waves may have the same decay behavior with respect to their modulus, however, they are characterized by a different phase increment. In fact, we identify functions of type $\exp(ikr)/r$, for example, with outgoing waves.

The main idea of solution techniques to solve Helmholtz equations based on DtN maps is to decompose the unbounded exterior to $\mathcal{R}$ into the regions inside and outside the disc $\mathcal{B}_{r_0}$ and relate them to each other via the DtN-operator M as

$$\partial_n u = Mu \quad \text{on } \partial\mathcal{B}_{r_0}\,,$$

where continuity of both $\partial_n u$ and u across the artificial boundary $\partial\mathcal{B}_{r_0}$ is required. Now our problem reads

$$\text{exterior problem } P_E: \qquad \begin{aligned} \Delta u + k^2 u &= 0 && \text{outside } \mathcal{B}_{r_0} \\ \partial_n u &= M u_o && \text{on } \partial\mathcal{B}_{r_0} \end{aligned} \tag{2}$$

and

$$\text{interior problem } P_I: \qquad \begin{aligned} \Delta u + k^2 u &= f && \text{inside } \mathcal{B}_{r_0} \\ \alpha u + \beta\partial_n u &= g && \text{on } \Gamma \\ \partial_n u &= Mu && \text{on } \partial\mathcal{B}_{r_0} \end{aligned} \tag{3}$$

Here, the Sommerfeld condition is hidden behind the DtN-operator M. The function $u_0(x), x \in \partial\mathcal{B}_{r_0}$, is assumed to be given and the DtN-operator supplies, as it will be shown later, a bijective mapping, in a suited function space, between Dirichlet and Neumann data, such that for every function u_0 on $\partial\mathcal{B}_{r_0}$ the exterior problem can be solved uniquely and satisfies Sommerfeld's radiation condition. The algorithmic procedure then is as follows

1. solve problem P_I
2. set $u_0(x) := u(x), x \in \partial\mathcal{B}_{r_0}$
3. solve problem P_E.

In many practical situations we can drop the third step, because we are interested in $u(x), x \in \Omega$, only. The DtN-operator M can be computed by means of an integral representation formula based on a Green's function approach or by an explicit solution of the exterior problem by means of an separation ansatz. Following [9], we describe the latter technique.

3 Classical derivation of the DtN-map

Every function $u(x) \in L^2(\partial\mathcal{B}_{r_0})$ can be expressed via its Fourier series expansion

$$u(r_0,\phi) = \sum_{\nu=-\infty}^{\infty} a_\nu \mathrm{e}^{i\nu\phi}, \quad a_\nu = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-i\nu\phi} u(r_0,\phi)\, d\phi\,.$$

A separation ansatz in cylindric coordinates $u(r,\phi) = \sum_\nu a_\nu \mathrm{e}^{i\nu\phi} u_\nu(r)$ shows that $u_\nu(r)$ must obey Bessel's differential equation

$$\partial_r^2 u_\nu + \frac{1}{r}\partial_r u_\nu + \left(k^2 - \frac{\nu^2}{r^2}\right) u_\nu = 0\,, \quad r \geq r_0.$$

The fundamental solution and its derivative are

$$\begin{aligned} u_\nu(r) &= c_1 H_\nu^{(1)}(kr) + c_2 H_\nu^{(2)}(kr) \\ v_\nu'(r) &= c_1 {H_\nu}^{(1)'}(kr) + c_2 H_\nu^{(2)'}(kr), \end{aligned}$$

where $H_\nu^{(1)}, H_\nu^{(2)}$ are Hankel's functions of the first and the second kind, respectively, and the prime denotes the derivative with respect to the argument. From an asymptotic study of Hankel's functions we know that only the first kind obeys the Sommerfeld condition. In order to drop the Hankel function of the second kind, we must establish an additional condition, namely,

$$u_\nu'(r) = k\frac{H_\nu^{(1)'}(kr)}{H_\nu^{(1)}(kr)} u_\nu(r),$$

or, equivalently,

$$u_\nu'(r) = \left(\frac{\nu}{r} - k\frac{H_{\nu+1}^{(1)}(kr)}{H_\nu^{(1)}(kr)}\right) u_\nu(r).$$

Taking the angular-dependent part into account, we obtain the DtN-map

$$\partial_n u(r,\phi)|_{r=r_0} = \sum_{\nu=-\infty}^{\infty} \left(\frac{\nu}{r} - k\frac{H_{\nu+1}^{(1)}(kr)}{H_\nu^{(1)}(kr)}\right) \frac{\mathrm{e}^{i\nu\phi}}{2\pi} \left(\mathrm{e}^{i\nu\phi}, u(r_0,\phi)\right)_{L^2}\,.$$

The main ingredients, with respect to this classical approach, have been the expression of the fundamental solution in terms of a superposition of Hankel's functions and the decision, which of them fulfill Sommerfeld's radiation condition. For example, if we would start with another pair of fundamental solutions, say with Bessel- and Weber functions, none of them would satisfy the Sommerfeld condition and we must try to find a superposition of both, namely the Hankel functions, which obey the radiation condition. There the question arises, is there an approach which allows the construction of the DtN-operator in a canonical way, without an explicit construction of fundamental solutions given by, sometimes complicated, special functions?

4 Technique of Laplace transform

The alternative way of deriving a DtN-map is based on the technique of Laplace transformation. The key point is that we will be able to state Sommerfeld' s radiation condition (1) as an property of Laplace-transformed solutions of the exterior problem P_E. To become familiar with this point of view, we first give the alternative formulation of the Sommerfeld condition, and consider then two introductory examples.

4.1 The alternative formulation of Sommerfeld's radiation condition

Let us write the solution $u(x)$ of the exterior problem (2) with x given in polar coordinates r, ϕ, $r \geq r_0, -\pi < \phi \leq \pi$, as $u(r,\phi)$. Then the Sommerfeld condition (1) reads

$$\partial_r u(r,\phi) - iku(r,\phi) = o\left(\frac{1}{r^{1/2}}\right) . \tag{4}$$

We will replace this by the equivalent formulation

$$\oint_\gamma \hat{u}(s,\phi)\, ds = 0\,, \tag{5}$$

where $\hat{u}(s,\phi)$ denotes the Laplace-transformed counterpart of $u(r,\phi)$ with respect to the radial variable r,

$$\hat{u}(s,\phi) = \mathcal{L}u(r,\phi) = \int_0^\infty \mathrm{e}^{-sx} u(r_0 + x, \phi) dx\,.$$

Condition (5) must hold for every closed contour γ which lies in the lower half of the complex plane, and for every $-\pi < \phi \leq \pi$. In other words, we require $\hat{u}(s,\phi)$ must be holomorphic in the lower half of the complex plane.

4.2 Examples

Example 1 (1D). Let us consider the Helmholtz equation in the semi-infinite interval $[r_0, \infty)$ and let us introduce the shifted coordinate-system with the independent variable x via $r = r_0 + x, x \geq 0$,

$$\frac{d^2}{dx^2}u(x) + u(x) = 0, \quad x \geq 0.$$

Laplace-transformation yields

$$\hat{u}(s) = \frac{su(0) + \partial_x u|_{x=0}}{s^2 + 1}.$$

The function $\hat{u} : \mathcal{U} \subset \mathbb{C} \to \mathbb{C}$ is holomorphic in $\mathcal{U} = \mathbb{C} \setminus \{\pm i\}$. A partial fraction decomposition supplies

$$\hat{u}(s) = \frac{1}{2}\frac{u_0 + iu_x}{s + i} + \frac{1}{2}\frac{u_0 - iu_x}{s - i}.$$

For convenience, we introduced here the abbreviations $u_0 := u(0)$ and $u_x := \partial_x u(x)|_{x=0}$. Obviously, the pole at $s = -i$ vanishes, and condition (5) is fulfilled, if the 1D Sommerfeld condition holds: $u_0 + iu_x = 0$. Thus, the Laplace-technique supplies two essential informations:

1. It gives a natural splitting of a given function into asymptotically incoming and outgoing waves, because we can identify a complex function $\hat{u}_-(s) := \frac{1}{s+i}$ with an incoming wave $u_-(x) = \exp(-ix)$ and a function $\hat{u}_+(s) := \frac{1}{s-i}$ with an outgoing wave $u_+(x) = \exp(ix)$.
2. It supplies automatically the relation between Neumann data $u_x = \partial_x u(x)$ and the Dirichlet data $u_0 = u(x)$ at the position $x = 0$ needed to drop one of them.

Example 2 (2D). The following example is essentially for the understanding of the full 2D problem. Let us consider the complex function $\hat{u}(s)$

$$\hat{u} : \mathcal{U} \subset \mathbb{C} \to \mathbb{C}, \quad \hat{u}(s) := \frac{g(s)}{\sqrt{s^2 + 1}}. \tag{6}$$

Here, the square-root is understood as the square root on the principal plane, i. e. , for $a \in \mathbb{C}$ we define $\sqrt{a} = |a|^{1/2}\mathrm{e}^{i\phi/2}, \phi = \arg(a), \phi \in (-\pi, \pi]$. With respect to the definition of $\hat{u}(s)$ this means, that we define

$$\mathcal{U} = \mathbb{C} \setminus \{\pm i(1 + \mathbb{R}_0)\} \quad \text{with} \quad \mathbb{R}_0 = \{x \in \mathbb{R} \mid 0 \leq x < \infty\} \tag{7}$$

(see Fig. 2). The function $g(s)$ in the numerator of (6) is defined as follows: $g : \mathcal{U} \to \mathbb{C}$, $g(s)$ is holomorphic in $\mathcal{U}$ and $\lim_{|s|\to\infty} |g(s)| =$ const. The two singularities $\pm i$ and the infinite point are branch points of $\hat{u}(s)$ of order 1. The infinite point is a zero of $\hat{u}$, and the derivative $\hat{u}'(s)$ vanishes for $|s| \to \infty$, too. Hence, the infinite point is a regular point. In the following, we want to analyze the function $\hat{u}(s)$ in the same way as we analyzed the function $1/(s^2+1)$ occurring in the 1D example. The following lemma supplies the key for this investigation.

Lemma 1. (Partial fraction decomposition). *The complex function $\hat{u}(s)$ defined by* (6) *and* (7) *possesses a partial fraction decomposition*

$$\hat{u}(s) = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{g(\zeta)}{\sqrt{\zeta^2+1}\,(s-\zeta)}\, d\zeta + \frac{1}{2\pi i} \oint_{\gamma_2} \frac{g(\zeta)}{\sqrt{\zeta^2+1}\,(s-\zeta)}\, d\zeta,$$

where the paths γ_1 and γ_2 are closed contours in $\mathcal{U}$ enclosing the singularities (see Fig. 2).

Proof. For a small disc with radius ϵ we have, according to Cauchy's integral formula,

$$\hat{u}(s) = \frac{1}{2\pi i} \oint_{|\zeta - s| = \epsilon} \frac{\hat{u}(\zeta)}{\zeta - z}\, d\zeta\,.$$

Without to change the result, we deform the path, as it is shown in Fig. 2, and obtain

$$\hat{u}(s) = \frac{1}{2\pi i} \oint_{\gamma} \frac{\hat{u}(\zeta)}{\zeta - s}\, d\zeta\,.$$

Here, γ denotes the path composed from the pieces $\gamma_1, \ldots, \gamma_4$. Choosing for γ_3, γ_4 half-circles with radius r, we have only to show that the integrals along these paths vanish for $r \to \infty$. In polar coordinates r, ϕ, $\frac{\pi}{2} \le \phi \le \frac{3\pi}{2}$, we obtain on γ_3

$$\begin{aligned} \left| \int_{\gamma_3} \frac{\hat{u}(\zeta)}{\zeta - s}\, d\zeta \right| &\le \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left| \frac{|g(\zeta)|}{\sqrt{r^2 \mathrm{e}^{2i\phi} + 1}\,(r\mathrm{e}^{i\phi} - s)} i r \mathrm{e}^{i\phi} \right| d\phi \\ &\le \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{r\,|g|}{\sqrt{|r^2 - 1|}\,\big|r - |s|\big|}\, d\phi\,. \end{aligned}$$

Hence, the integral tends to zero for $r \to \infty$. The same applies to the integral on γ_4. □

We set $g(s) := 1$ and consider the first term of the partial fraction decomposition

$$\hat{u}_+(s) := \frac{1}{2\pi i} \oint_{\gamma_1} \frac{d\zeta}{\sqrt{\zeta^2+1}\,(s-\zeta)}\,.$$

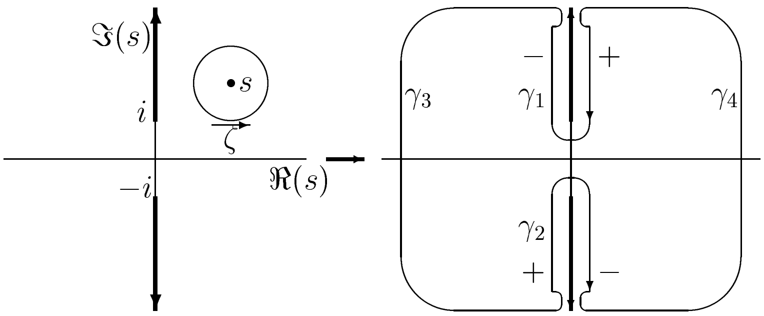

Figure 2: Definition of the Riemann sheet and the contour γ. The $+$ and $-$-signs mark the positive and the negative boundaries.

Now we split the contour γ_1 into two parts $\gamma_1 = \gamma_1' + \gamma_1''$, where γ_1' lies to the left and γ_2'' to the right of the imaginary axis and $\zeta_1 \in \gamma_1'$ and $\zeta_2 \in \gamma_1''$

$$\hat{u}_+(s) = \frac{1}{2\pi i}\int_i^{i\infty} \frac{d\zeta_1}{\sqrt{\zeta_1^2+1}\,(s-\zeta_1)} + \frac{1}{2\pi i}\int_{i\infty}^{i} \frac{d\zeta_2}{\sqrt{\zeta_2^2+1}\,(s-\zeta_2)}\,.$$

Next, we let both contours get arbitrary close to the imaginary axis. Taking the definition of the square root into account and performing the limiting process $\zeta_{1,2} \to \zeta$, such that $\Re(\zeta_{1,2}) \to 0$, with

$$\begin{array}{lll} \zeta_1 \to \zeta & \text{and} & \sqrt{\zeta_1^2+1} \to -\sqrt{\zeta^2+1} \\ \zeta_2 \to \zeta & \text{and} & \sqrt{\zeta_2^2+1} \to \sqrt{\zeta^2+1} \end{array}$$

we obtain

$$\hat{u}_+(s) = -\frac{1}{\pi i}\int_i^{i\infty} \frac{d\zeta}{\sqrt{\zeta^2+1}\,\big(s-\zeta\big)}\,. \tag{8}$$

In order to retrieve the standard form of the integral along the real axis, we compute

$$\hat{u}_+(s) = \frac{1}{\pi i}\int_{i-\infty}^{i-0} \frac{d\zeta}{\sqrt{\zeta^2+1}\big(s-\zeta\big)} \tag{9}$$

$$= \frac{1}{\pi i}\int_0^{\infty} \frac{d\tau}{\sqrt{\tau^2-2i\tau}\,\big(s-(i-\tau)\big)}\,. \tag{10}$$

In (9) we applied Cauchy's residue theorem in the left upper part of the complex plane and in (10) we introduced $\zeta = i+\tau, \tau \in \mathbb{R}$ and reversed the direction of integration. Both equations (8) and (10) possess an interesting interpretation. Using $\zeta = i\tau$ and applying $\mathcal{L}^{-1}\left(\frac{1}{s+\tau}\right) = \exp(-\tau x)$ to (8), we find

$$u_+(x) = \frac{1}{\pi}\int_1^\infty \frac{\mathrm{e}^{i\tau x}\, d\tau}{\sqrt{\tau^2-1}}, \tag{11}$$

i. e. , we can represent $u(x)$ as a superposition of Fourier modes with positive frequencies (i. e. wavenumbers) $\tau \geq 1$. On the other hand, (10) allows a further representation of the same function $u(x)$, namely,

$$\begin{aligned} u_+(x) &= \frac{1}{i\pi}\int_0^\infty \frac{\mathrm{e}^{(i-\tau)x}\, d\tau}{\sqrt{\tau^2-2i\tau}}\,, \\ &= \frac{\mathrm{e}^{i(x-\frac{\pi}{4})}}{\pi\sqrt{2x}}\int_0^\infty \frac{\mathrm{e}^{-\tau}\, d\tau}{\sqrt{\tau}\,\sqrt{1-\frac{\tau}{2ix}}}\,. \end{aligned} \tag{12}$$

That is, we have equally valid $u(x)$ as a superposition of *damped* Fourier modes of a single frequency, due to the factor $\exp(ix)$, and with positive damping parts $\tau \geq 0$. Observe, that for $x \to \infty$ the integral converges to $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, which in turn supplies the well-known asymptotic formula for the Hankel function

$$u_+(x) = H_0^{(1)}(x) = \frac{\mathrm{e}^{i(x-\pi/4)}}{\sqrt{2\pi x}}\left(1+\mathcal{O}\left(\frac{1}{x}\right)\right).$$

Differentiating (12) with respect to x and expanding the square-root term into a Taylor series supplies the asymptotic behavior

$$\frac{\partial_r u_+(r)}{u(r)} = i + \mathcal{O}\left(\frac{1}{r}\right) \quad \text{for} \quad r\to\infty\,. \tag{13}$$

Obviously, (13) tells us that u_+ behaves asymptotically and locally like an outgoing plane wave.

5 Alternative derivation of the DtN-map

The starting point of the classical derivation was to find proper solutions of the Bessel equation

$$u''(r) + \frac{1}{r}u'(r) + \left(1-\frac{\nu^2}{r^2}\right)u = 0\,, \quad r \geq r_0\,,$$

exploiting the known properties of the Hankel functions at infinity. For the alternative derivation, we start again from the Bessel equation. We introduce a new coordinate system with the independent variable x via $r = x + r_0, x \geq 0$ and denote, for convenience, the shifted function $u(x + r_0)$ again by $u(x)$. This way, we obtain our basic differential equation

$$u''(x) + \frac{1}{x + r_0} u'(x) + \left(1 - \frac{\nu^2}{(x + r_0)^2}\right) u = 0\,, \quad x \geq 0\,. \tag{14}$$

Now our concept is the following:

1. Laplace transformation of (14), which yields again a differential equation, this time in the dual domain.
2. Solve the new differential equation by means of the variation of constant formula.
3. Identify the singularities which correspond to incoming waves.
4. Remove them by a proper choice of the initial conditions at $x = 0$.

In contrast to the classical approach, we can carry out the whole procedure without to fall back upon properties of Hankel's functions.

Step 1. We multiply (14) with $x + r_0$, introduce the new variable $v(x) = u(x)/(x + r_0)$, and take into account the rule $\mathcal{L}(xu(x)) = -\hat{u}'(s)$, where the prime denotes the derivative with respect to the complex argument s, to obtain

$$\frac{d}{ds}\left[\begin{pmatrix} s^2+1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}\right] = \begin{pmatrix} r_0(s^2+1)+s & -\nu^2 \\ -1 & r_0 \end{pmatrix}\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} - r_0 \begin{pmatrix} su_0 + u_x \\ 0 \end{pmatrix}.$$

Equivalently, we have

$$\frac{d}{ds}\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} r_0 - \frac{s}{s^2+1} & -\frac{\nu^2}{s^2+1} \\ -1 & r_0 \end{pmatrix}\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} - r_0 \begin{pmatrix} \frac{su_0+u_x}{s^2+1} \\ 0 \end{pmatrix}. \tag{15}$$

Here, u_0 and u_x denote the function $u(x)$ and its normal derivative $\partial_x u(x)$, respectively, at $x = 0$. In view of the solution procedure, it is convenient, to extract the translation part and to denote the remaining matrix with $\mathbf{A}$

$$\begin{aligned}
&\frac{d}{ds}\begin{pmatrix}\hat{u}\\ \hat{v}\end{pmatrix} = r_0\mathbf{I}\begin{pmatrix}\hat{u}\\ \hat{v}\end{pmatrix} + \mathbf{A}\begin{pmatrix}\hat{u}\\ \hat{v}\end{pmatrix} - r_0\begin{pmatrix}\frac{su_0+u_x}{s^2+1}\\ 0\end{pmatrix}\\
&\text{where } \mathbf{I} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \text{and } \mathbf{A} = \begin{pmatrix}-\frac{s}{s^2+1} & -\frac{\nu^2}{s^2+1}\\ -1 & 0\end{pmatrix}
\end{aligned} \tag{16}$$

Step 2. Our goal is a solution formula of the type

$$\begin{pmatrix}\hat{u}(s)\\ \hat{v}(s)\end{pmatrix} = \Phi\int_s^\infty \Phi^{-1}\mathbf{r}(u_0,u_x)ds'\,. \tag{17}$$

Φ is the matrix of fundamental solutions and must obey the matrix differential equation

$$\Phi'(s) = (x_0\mathbf{I} + \mathbf{A}(s))\,\Phi(s)\,, \tag{18}$$

and $\mathbf{r}(u_0,u_x)$ is the source vector of (16) which depends on the initial data. The determination of Φ evolves in three sub-steps: (i) remove the translation part, (ii) factorize the solution to drop the part which corresponds to $\nu = 0$, (iii) transform the variable $s \to \tau$, to arrive at a problem with constant coefficients.
Step (i). We introduce

$$\Phi = \Phi^{(1)}\mathrm{e}^{r_0 s}\,,$$

which results in

$$\Phi^{(1)\prime}(s) = \mathbf{A}(s)\Phi^{(1)}(s)\,.$$

This way, a column $(u,v)^T$ of Φ, where we have dropped here the hat used so far to mark a Laplace transformed quantity, is transformed into a corresponding column of $\Phi^{(1)}$

$$\begin{pmatrix}u\\ v\end{pmatrix} \mapsto \begin{pmatrix}u^{(1)}\\ v^{(1)}\end{pmatrix}\,.$$

Step (ii). We factorize the first components of every column of $\Phi^{(1)}$

$$\begin{aligned}
u^{(1)} &= u^{(2)}w, \quad w = \frac{1}{\sqrt{s^2+1}}\\
v^{(1)} &= v^{(2)}\,,
\end{aligned}$$

which results in

$$\Phi^{(2)\prime}(s) = -\frac{1}{\sqrt{s^2+1}}\begin{pmatrix}0 & \nu^2\\ 1 & 0\end{pmatrix}\Phi^{(2)}(s)\,.$$

Step (iii). Finally, we transform $\Phi^{(2)}(s(\tau)) = \Phi^{(3)}(\tau)$ with

$$\frac{d\tau}{ds} = \frac{1}{\sqrt{s^2+1}}, \quad \tau = -\log\left(\sqrt{s^2+1} - s\right)$$

to obtain

$$\Phi^{(3)\prime}(s) = -\begin{pmatrix} 0 & \nu^2 \\ 1 & 0 \end{pmatrix} \Phi^{(3)}(s)\,,$$

which in turn has a solution

$$\Phi^{(3)}(s) = \begin{pmatrix} \mathrm{e}^{\nu\tau} & \mathrm{e}^{-\nu\tau} \\ -\frac{1}{\nu}\mathrm{e}^{\nu\tau} & \frac{1}{\nu}\mathrm{e}^{-\nu\tau} \end{pmatrix}\,.$$

Tracing the whole procedure backwards, we find

$$\Phi^{(3)} \mapsto \Phi^{(2)} \mapsto \Phi^{(1)} \mapsto \Phi = \\ \begin{pmatrix} \frac{(\sqrt{s^2+1}-s)^{-\nu}}{\sqrt{s^2+1}} & \frac{(\sqrt{s^2+1}-s)^{\nu}}{\sqrt{s^2+1}} \\ -\frac{1}{\nu}(\sqrt{s^2+1}-s)^{-\nu} & \frac{1}{\nu}(\sqrt{s^2+1}-s)^{\nu} \end{pmatrix} \mathrm{e}^{r_0 s}\,. \tag{19}$$

Applying (17), we obtain finally the desired representation of $\hat{u}(s)$

$$\hat{u}(s) = \\ \frac{r_0}{2}\frac{(\sqrt{s^2+1}-s)^{-\nu}}{\sqrt{s^2+1}}\mathrm{e}^{r_0 s}\int_s^\infty \mathrm{e}^{-r_0 s'}\frac{(\sqrt{s'^2+1}-s')^{\nu}}{\sqrt{s'^2+1}}(s'u_0+u_x)\,ds' + \\ \frac{r_0}{2}\frac{(\sqrt{s^2+1}-s)^{\nu}}{\sqrt{s^2+1}}\mathrm{e}^{r_0 s}\int_s^\infty \mathrm{e}^{-r_0 s'}\frac{(\sqrt{s'^2+1}-s')^{-\nu}}{\sqrt{s'^2+1}}(s'u_0+u_x)\,ds' \tag{20}$$

Step 3. According to (20), we can represent $\hat{u}(s)$ in factorized form

$$\hat{u}(s) = \frac{1}{\sqrt{s^2+1}}g(s)\,,$$

with

$$g(s) = \\ \frac{r_0}{2}(\sqrt{s^2+1}-s)^{-\nu}\mathrm{e}^{r_0 s}\int_s^\infty \mathrm{e}^{-r_0 s'}\frac{(\sqrt{s'^2+1}-s')^{\nu}}{\sqrt{s'^2+1}}(s'u_0+u_x)\,ds' + \\ \frac{r_0}{2}(\sqrt{s^2+1}-s)^{\nu}\mathrm{e}^{r_0 s}\int_s^\infty \mathrm{e}^{-r_0 s'}\frac{(\sqrt{s'^2+1}-s')^{-\nu}}{\sqrt{s'^2+1}}(s'u_0+u_x)\,ds'\,. \tag{21}$$

Consequently, $\hat{u}(s)$ may have singularities at $s = \pm i$. In order to drop the singularity at $s = -i$, the necessary condition

$$g(-i) = 0 \tag{22}$$

must be fulfilled.

Step 4. This, in turn, supplies the necessary condition

$$\begin{aligned} u_x &= b_\nu u_0 \\ b_\nu &:= -\frac{\int_{-i}^{\infty} ds\, \mathrm{e}^{-r_0 s} s \dfrac{i^{-\nu}(\sqrt{s^2+1}-s)^\nu + i^\nu(\sqrt{s^2+1}-s)^{-\nu}}{\sqrt{s^2+1}}}{\int_{-i}^{\infty} ds\, \mathrm{e}^{-r_0 s} \dfrac{i^{-\nu}(\sqrt{s^2+1}-s)^\nu + i^\nu(\sqrt{s^2+1}-s)^{-\nu}}{\sqrt{s^2+1}}} \,. \end{aligned} \tag{23}$$

In the next section we will show that the factor b_ν defined this way, is bounded away from zero and infinity. Therefore a one-to-one relation between u_x and u_0 has been established.

The following is the main statement of the paper.

Theorem 1. *The solution $\hat{u}(s)$ of the system* (15)*, given by* (20)*, together with with the boundary condition* (23) *is holomorphic in the complex plane $\mathcal{U}_- = \{s \in \mathbb{C} \mid \Im(s) < 0\}$. Moreover, the boundary condition* (5) *(analyticity of $\hat{u}(s)$ in $\mathcal{U}_-$) possesses the equivalent form*

$$u_x - iu_0 = \frac{i}{r_0}\hat{u}(-i)\,.$$

Proof. We investigate the behavior of $\hat{u}(s)$ in a close surrounding of $s = -i$. To this end, we introduce the variable ζ by $\zeta := s + i$. Let $p_k(\zeta)$ denote a Taylor series expansion with leading term larger or equal k and $P_k(\zeta)$ and $\mathbf{p}_k(\zeta)$ matrices and vectors, respectively, with elements $p_k(\zeta)$.

We rewrite the square root expressions appearing in (19) by

$$\begin{aligned} \sqrt{s^2+1} &= \sqrt{\zeta}p_0(\zeta) \\ -i\left(\sqrt{s^2+1}-s\right) &= 1 + \sqrt{\zeta}p_0(\zeta) - \zeta \\ &= 1 + \sqrt{\zeta}p_0(\zeta) + p_1(\zeta) \\ \left[-i\left(\sqrt{s^2+1}-s\right)\right]^\nu &= 1 + \sqrt{\zeta}p_0(\zeta) + p_1(\zeta) \\ \left[i\left(\sqrt{s^2+1}+s\right)\right]^\nu &= 1 + \sqrt{\zeta}p_0(\zeta) + p_1(\zeta)\,. \end{aligned} \tag{24}$$

The symbols $p_0(\zeta), p_1(\zeta)$ appearing at different places denote different expressions. The convergence radius of the series expansion of the square root terms is 2 because the convergence radius of the Taylor expansion of $\sqrt{1-\zeta/(2i)}$ is 2.

Our goal is to rewrite the convolution representation (17) for ζ small. The idea is to show that the critical terms, causing the singularities and branch points inside and outside the integrals are of the type "$\sqrt{\zeta}/\sqrt{\zeta}$" and cancel each other, if the boundary condition (23) is satisfied. To this end, we start with the integral

$$\begin{aligned} I &:= \int_s^\infty \Phi^{-1}\mathbf{r}\, ds' \\ &= \int_s^{-i} \Phi^{-1}\mathbf{r} ds' + \int_{-i}^\infty \Phi^{-1}\mathbf{r}\, ds'\,. \end{aligned}$$

The latter integral-term vanishes, due to the boundary condition (22), and it remains

$$I = -\int_0^\zeta \Phi^{-1}(s(\zeta'))\mathbf{r}\, d\zeta'\,.$$

It follows from (19) that $\Phi(s)$ can be factored

$$\Phi(s) = \begin{pmatrix} \frac{1}{\sqrt{s^2+1}} & 0 \\ 0 & 1 \end{pmatrix} \Phi_0(s),$$

where Φ_0 is holomorphic in $\pm i$. Both factors can be expanded with respect to ζ

$$\begin{aligned} \Phi(\zeta) &= \begin{pmatrix} \frac{1}{\sqrt{-2i}}\frac{1}{\sqrt{\zeta}} + \sqrt{\zeta}p_0(\zeta) & \\ & 1 \end{pmatrix} \left(\Phi_0(-i) + P_1(\zeta)\right) \\ &= \begin{pmatrix} \frac{1}{\sqrt{-2i}}\frac{1}{\sqrt{\zeta}} & \\ & 1 \end{pmatrix} \Phi_0(-i) + \sqrt{\zeta}P_0(\zeta)\,. \end{aligned}$$

Observe that there is no additional term of type $P_k(\zeta)$. In the same way, we find for the inverse

$$\Phi^{-1}(\zeta) = \Phi_0^{-1}(-i) \begin{pmatrix} \sqrt{-2i}\sqrt{\zeta} & \\ & 1 \end{pmatrix} + \sqrt{\zeta}P_1(\zeta)\,.$$

Hence, the integral expression becomes

$$\begin{aligned} I &= -r_0 \int_0^\zeta d\zeta' \left[\Phi_0^{-1}(-i) \begin{pmatrix} \sqrt{-2i}\sqrt{\zeta'} & \\ & 1 \end{pmatrix} + \sqrt{\zeta'}P_1(\zeta') \right] \begin{pmatrix} \frac{(\zeta'-i)u_0+u_x}{\zeta'(\zeta'-2i)} \\ 0 \end{pmatrix} \\ &= -r_0 \int_0^\zeta d\zeta' \left[\Phi_0^{-1}(-i) \begin{pmatrix} \frac{1}{\sqrt{-2i}} \frac{-iu_0+u_x}{\sqrt{\zeta'}} \\ 0 \end{pmatrix} + \sqrt{\zeta'}\mathbf{p}_0(\zeta') \right] \\ &= -r_0\Phi_0^{-1}(-i) \begin{pmatrix} \frac{2}{\sqrt{-2i}}\sqrt{\zeta}(-iu_0+u_x) \\ 0 \end{pmatrix} + \sqrt{\zeta}\mathbf{p}_1(\zeta)\,. \end{aligned}$$

Together with the expansion of Φ this gives

$$\begin{pmatrix} \hat{u}(s(\zeta)) \\ \hat{v}(s(\zeta)) \end{pmatrix} = \begin{pmatrix} \frac{r_0}{i}(-iu_0 + u_x) \\ 0 \end{pmatrix} + \mathbf{p}_1(\zeta)\,. \tag{25}$$

This shows, that the derivative of $\hat{u}(s)$ for $s \to -i$ remains bounded. Further, it follows

$$\hat{u}(-i) = \frac{r_0}{i}(-iu_0 + u_x)\,.$$

The Taylor series of the square root terms converges uniformly within the radius of convergence. Thus, the above operations are legitimate. Moreover, $\hat{u}(s(\zeta))$ possesses a polynomial representation within the radius of convergence, hence it is holomorphic there and we can add the part of the former branch cut, which lies in this domain of convergence, to the domain of definition of $\hat{u}$. Proceeding further this way with an analytic continuation along the negative imaginary axis, we find that $\widehat{u(s)}$ is holomorpic in all points $s = -i\tau$, $\tau \geq 1$. $\square$

5.1 Contour integral representation

Lemma 1 supplies another possibility to represent $\hat{u}$ and leads to further insight into the structure of the solution. First, in order to meet the assumptions of Lemma 1, we must show that $g(s)$, given by (21), is holomorphic in $\mathcal{U}$ and converges to a constant for $|s| \to \infty$. In fact, we have

Lemma 2. *The complex function $g : \mathcal{U} \subset \mathbb{C} \to C$, is holomorphic in $\mathcal{U}$. Further, it holds*

$$\lim_{|s|\to\infty} g(s) = u_0$$

$$\lim_{|s|\to\infty} \frac{d}{ds}\, g(s) = 0$$

for any given complex numbers u_0 and u_x.

Proof. The only point in question is the infinite point. A direct computation shows

$$\begin{aligned} \left(\sqrt{s^2+1} - s\right)^{-\nu} &= \left(\sqrt{s^2+1} + s\right)^{\nu} \\ &= (2s)^{\nu} + \mathcal{O}\left(s^{\nu-2}\right) \\ \left(\sqrt{s^2+1} - s\right)^{\nu} &= \frac{1}{(2s)^{\nu}} + \mathcal{O}\left(\frac{1}{s^{\nu+2}}\right) \\ \frac{1}{\sqrt{s^2+1}} &= \frac{1}{s} + \mathcal{O}\left(\frac{1}{s^3}\right)\,. \end{aligned}$$

Next, we split (21) into two parts and investigate the asymptotic behavior of the functions

$$f_1(s) := (\sqrt{s^2+1}-s)^{-\nu} \mathrm{e}^{r_0 s} \int_s^\infty \mathrm{e}^{-r_0 s'} \frac{(\sqrt{s'^2+1}-s')^{\nu}}{\sqrt{s'^2+1}} (s' u_0 + u_x)\, ds' \tag{26}$$

$$f_2(s) := (\sqrt{s^2+1}-s)^{\nu} \mathrm{e}^{r_0 s} \int_s^\infty \mathrm{e}^{-r_0 s'} \frac{(\sqrt{s'^2+1}-s')^{-\nu}}{\sqrt{s'^2+1}} (s' u_0 + u_x)\, ds' \,. \tag{27}$$

It follows

$$\begin{aligned} f_1(s) = (2s)^{\nu} \mathrm{e}^{r_0 s} \int_s^\infty \mathrm{e}^{-r_0 s'} \frac{u_0}{(2s')^{\nu}}\, ds' \\ + \mathcal{O}(s^{\nu}) \mathrm{e}^{r_0 s} \int_s^\infty \mathrm{e}^{-r_0 s'} \mathcal{O}\left(\frac{1}{s'^{\nu+1}}\right) ds' \,. \end{aligned}$$

An integration by parts yields

$$f_1(s) = \frac{u_0}{r_0} + \mathcal{O}(s^{\nu}) \mathrm{e}^{r_0 s} \int_s^\infty \mathrm{e}^{-r_0 s'} \mathcal{O}\left(\frac{1}{s'^{\nu+1}}\right) ds' \,.$$

The last term is, for every $r_0 > 0$, a function $\mathcal{O}(1/s)$. Hence

$$\lim_{|s|\to\infty} f_1(s) = \frac{u_0}{r_0} \,.$$

The same procedure applied to $f_2(s)$ gives

$$\begin{aligned} f_2(s) &= \frac{1}{(2s)^{\nu}} \mathrm{e}^{r_0 s} \int_s^\infty \mathrm{e}^{-r_0 s'} u_0 (2s')^{\nu}\, ds' \\ &\qquad + \mathcal{O}\left(\frac{1}{s^{\nu}}\right) \mathrm{e}^{r_0 s} \int_s^\infty \mathrm{e}^{-r_0 s'} \mathcal{O}\left(s'^{\nu-1}\right) ds' \\ &= \frac{u_0}{r_0} + \mathcal{O}\left(\frac{1}{s^{\nu}}\right) \mathrm{e}^{r_0 s} \int_s^\infty \mathrm{e}^{-r_0 s'} \mathcal{O}\left(s'^{\nu-1}\right) ds' \,. \end{aligned}$$

Again, the last term is a function $\mathcal{O}(1/s)$ for $r_0 > 0$, and we obtain the same limiting value as for $f_1(s)$

$$\lim_{|s|\to\infty} f_2(s) = \frac{u_0}{r_0} \,.$$

It follows

$$\begin{aligned} \lim_{|s|\to\infty} \hat{u}(s) &= \lim_{|s|\to\infty} \frac{r_0}{2} \left(f_1(s) + f_2(s)\right) \\ &= u_0 . \end{aligned}$$

Differentiation of $\hat{u}(s)$ and application of the same procedure results in $\lim_{|s|\to\infty} f_1'(s) = 0$ and $\lim_{|s|\to\infty} f_2'(s) = 0$, which finally yields $\lim_{|s|\to\infty} \hat{u}(s) = 0$. □

Let us assume that u_0, u_x satisfy the boundary condition (23). Then $\hat{u}(s)$ is holomorphic in $\mathcal{U}_-$, hence the partial fraction decomposition of Lemma 1 shows that we can represent $\hat{u}(s)$ as contour integral

$$\hat{u}_+(s) = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{g(\zeta)}{\sqrt{\zeta^2+1}\,(s-\zeta)}\, d\zeta, \tag{28}$$

and we choose γ_1 according to Fig. 2. The function $g(\zeta)$ itself is an integral expression. In order to to simplify (28), let us consider the two points s_-, s_+, located to the left and to the right of the cut (see Fig. 3), such that $\Im(s_-) = \Im(s_+)$ and $\Re(s_-) \to -0$, and $\Re(s_+) \to +0$.

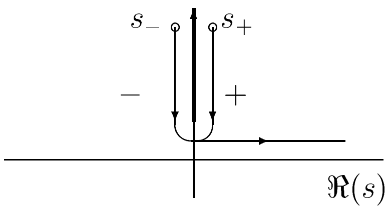

Figure 3: Definition of the paths to compute $g(s_-)$ and $g(s_+)$. The $+$ and $-$-signs mark the positive and the negative boundaries.

The values of $g(s)$ at these points can be written as, using the definitions (26) and (27),

$$\begin{aligned}
\frac{2}{r_0} g(s_-) &= \left[f_1(s_-) - f_1(i)\right] + f_1(i) + \left[f_2(s_-) - f_2(i)\right] + f_2(i) \\
\frac{2}{r_0} g(s_+) &= \left[f_1(s_+) - f_1(i)\right] + f_1(i) + \left[f_2(s_+) - f_2(i)\right] + f_2(i)\,.
\end{aligned}$$

Taking the signs of the square roots, involved in $g(s)$, with respect to the signs of the boundaries into account, we find the following symmetry

$$\begin{aligned}
\left[f_1(s_-) - f_1(i)\right] &= -\left[f_2(s_+) - f_2(i)\right] \\
\left[f_1(s_+) - f_1(i)\right] &= -\left[f_2(s_-) - f_2(i)\right],
\end{aligned}$$

i. e. , these terms cancel each other under the contour integral and it remains simply

$$\hat{u}_+(s) = \frac{g(i)}{2\pi i} \oint_{\gamma_1} \frac{1}{\sqrt{\zeta^2+1}\,(s-\zeta)}\, d\zeta. \tag{29}$$

Remark. If u_0, u_x do not satisfy the boundary condition (23), we have, instead of (29), the complete partial fraction decomposition

$$\hat{u}(s) = \frac{g(i)}{2\pi i}\oint_{\gamma_1} \frac{1}{\sqrt{\zeta^2+1}\,(s-\zeta)}\,d\zeta + \frac{g(-i)}{2\pi i}\oint_{\gamma_2} \frac{1}{\sqrt{\zeta^2+1}\,(s-\zeta)}\,d\zeta\,.$$

First, this shows again that $g(-i) = 0$ is the condition needed to realize analyticity in $\mathcal{U}_-$. Second, this equation corresponds exactly to the partial fraction decomposition of the 1D-example in §4.2. Hence, we are able to apply the same Laplace transformation techniques often applied to differential equations with constant coefficients to a large class of differential equations with non-constant coefficients.

From (29), along with our 2D-example of §4.2, we get directly the asymptotic behavior

$$u_+(x) = \frac{\mathrm{e}^{i(x-\pi/4)}g(i)}{\sqrt{2\pi x}}\left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right)\,. \tag{30}$$

The factor $g(i)$ cannot vanish. Otherwise $\hat{u}_+(s)$ would vanish identically and $u_+(x)$ would be the trivial solution. Therefore, (30) tells us that for $x^* < x < \infty$, with x^* sufficiently large, $u_+(x)$ does not possess asymptotically any resonance node. As the only difference between the general outgoing solution of Bessel's equation (29) and our 2D example is the factor $g(i)$, we obtain asymptotically again (13)

$$\frac{\partial_r u_+(r)}{u_+(r)} = i + \mathcal{O}\left(\frac{1}{r}\right) \quad \text{for} \quad r \to \infty, \quad \nu \geq 0, \text{ fixed.}$$

In fact, this shows that our alternate contour integral formulation (5) and the classic Sommerfeld radiation condition (4) are equivalent. Further, we see by (30) that any wave, which fulfills the radiation condition, must decays asymptotically. This fact that the Sommerfeld condition implies an asymptotic decay was first shown by Rellich [10].

6 Asymptotic properties

We investigate the asymptotic and spectral properties of the boundary condition (23). It is our goal to derive an analytic expression, which allows a study of the boundary condition for large ν and r_0, respectively. The technique is the following:

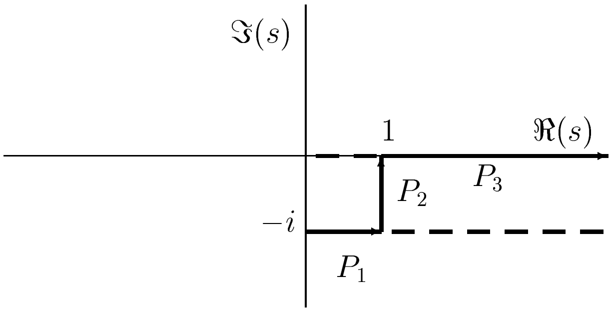

Figure 4: Definition of the integration paths. The thick solid line represents the chosen path consisting of the elements P_1, P_2, P_3. The dashed lines mark the continuation of the paths P_1, P_3 needed to obtain analytic approximations.

1. We choose a special path and decompose it into intervals, for which a uniform convergent Taylor series expansion is known.
2. We extract the leading term of the series expansion and compute the corresponding integral separately.
3. A bound for the remainder is determined.
4. The results corresponding to the different paths are summed up.

First, we choose the path consisting of the segments P_1, P_2, P_3, as it is shown in Fig. 4. The reason for doing this decomposition is simply that these different intervals contribute in a different sense to the integral. The integral along P_1 dominates the integral for $r_0 \to \infty$ and ν fixed. On the other hand, the integral on P_3 dominates the result for r_0 fixed and $\nu \to \infty$. The path P_2 does not play a decisive role.

Path P_1. Let us compute the integral in the numerator of (23) along P_1:

$$I_{1,N} = \int_{-i}^{1-i} ds\, \mathrm{e}^{-r_0 s} s \frac{\left[-i\left(\sqrt{s^2+1}-s\right)\right]^\nu + \left[i\left(\sqrt{s^2+1}+s\right)\right]^\nu}{\sqrt{s^2+1}} .$$

We proceed exactly as in the proof of Theorem 1, i. e. , we introduce local coordinates in the surrounding of the singularity by $\zeta := s + i$ and use the notation $p_k(\zeta)$ as in (24).

We rewrite the integral $I_{1,N}$

$$
\begin{aligned}
I_{1,N} &= \frac{e^{r_0 i}}{\sqrt{-2i}} \int_0^1 d\zeta e^{-r_0\zeta} \frac{\zeta - i}{\sqrt{\zeta}} \left(2 + \sqrt{\zeta} p_0(\zeta) + p_1(\zeta)\right) \\
&= \frac{-2i}{\sqrt{-2i}} e^{r_0 i} \int_0^1 d\zeta \frac{e^{-r_0\zeta}}{\sqrt{\zeta}} + R_{1,N} \,.
\end{aligned}
$$

The residual integral $R_{1,N}$ possesses a continuous integrand whose absolute value is bounded, for $0 \leq \zeta \leq 1$, by $K 3^\nu \exp(-r_0\zeta)$ with a constant K independent of ν. The factor 3^ν results from the fact that $\max|\sqrt{s^2+1} + s| < 3$ for $s \in P_1$. Therefore $R_{1,N}$ is bounded by

$$
\begin{aligned}
|R_{1,N}| &< K \int_0^1 d\zeta \, e^{-r_0\zeta} 3^\nu \\
&= K \frac{3^\nu}{r_0} \left(1 - e^{-r_0}\right) . \qquad (31)
\end{aligned}
$$

For the corresponding denominator integral we obtain

$$
I_{1,D} = \frac{2}{\sqrt{-2i}} e^{r_0 i} \int_0^1 d\zeta \frac{e^{-r_0\zeta}}{\sqrt{\zeta}} + R_{1,D},
$$

with the same bound (31) with respect to the residual integral $R_{1,D}$. In order to obtain an analytic result for the leading coefficient of the integrals $I_{1,N}$ and $I_{1,D}$, we enlarge the interval of integration from $(1,\infty)$ to the interval $(0,\infty)$ and subtract the additional term

$$
I_{1,N} = \frac{-2i}{\sqrt{-2i}} e^{r_0 i} \int_0^\infty d\zeta \frac{e^{-r_0\zeta}}{\sqrt{\zeta}} + \frac{2i}{\sqrt{-2i}} e^{r_0 i} \int_1^\infty d\zeta \frac{e^{-r_0\zeta}}{\sqrt{\zeta}} + R_{1,N}.
$$

The first integral corresponds to the Gamma–function, scaled with the factor r_0 in the argument, the second is bounded by $\sqrt{2}/r_0$, which we add to the residual expression $R_{1,N}$ to obtain

$$
I_{1,N} = \frac{-2i}{\sqrt{-2i}} e^{r_0 i} \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{r_0}} + R_{1,N} \,,
$$

where R_N is bounded by

$$
|R_{1,N}| < R = K \frac{3^\nu}{r_0} \qquad (32)
$$

with a constant K independent of ν and r_0. Accordingly, the result for the corresponding denominator integral is

$$
I_{1,D} = \frac{2}{\sqrt{-2i}} e^{r_0 i} \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{r_0}} + R_1 D \,,
$$

where R_D satisfies the same bound (32) with a different constant K. Thus, the computation of the integrals on the first segment is complete.

Path P_2. Because the integrands on P_2 are bounded, and the interval is finite, we can apply the same technique used to bound the residual integral $R_{1,N}$. The contributions of the integrals on the path P_2 are bounded again by (32).

Path P_3. The path P_3 lies on the real axis. According to (23), we have to compute the integral

$$I_{3,N} = \int_1^\infty ds\, \mathrm{e}^{-r_0 s} s \frac{\left[-i\left(\sqrt{s^2+1}-s\right)\right]^\nu + \left[i\left(\sqrt{s^2+1}+s\right)\right]^\nu}{\sqrt{s^2+1}}$$

and the corresponding integral in the denominator of (23). Analogously to P_1, we decompose the integrand into its leading term with respect to a Taylor series expansion, this time for $s \geq 1$, and the remainder, and enlarge the domain of integration

$$\begin{aligned}
I_{3,N} &= \int_0^\infty ds\, \mathrm{e}^{-r_0 s} s \left(i^\nu 2^\nu s^{\nu-1}\right) - \int_0^1 ds\, \mathrm{e}^{-r_0 s} s \left(i^\nu 2^\nu s^{\nu-1}\right) + R_N \\
R_{3,N} &:= \int_1^\infty ds\, \mathrm{e}^{-r_0 s} s \frac{\left[-i\left(\sqrt{s^2+1}-s\right)\right]^\nu + \left[i\left(\sqrt{s^2+1}+s\right)\right]^\nu}{\sqrt{s^2+1}} \\
&\quad - \int_1^\infty ds\, \mathrm{e}^{-r_0 s} s \left(i^\nu 2^\nu s^{\nu-1}\right) .
\end{aligned}$$

The integral on the interval $[0,1]$ has a bound

$$\left| \int_0^1 ds\, \mathrm{e}^{-r_0 s} s \left(i^\nu 2^\nu s^{\nu-1}\right) \right| < \frac{2^\nu}{r_0} K,$$

where K is independent of r_0 and ν. Next, we derive a bound for $R_{3,N}$. It holds

$$0 \leq \sqrt{s^2+1} - s < 1 \quad \text{for } s \geq 1, s \in \mathbb{R}.$$

This yields

$$\begin{aligned}
(\sqrt{s^2+1}-s)^\nu &< 1 \leq \frac{\sqrt{s^2+1}}{s} \\
\frac{(\sqrt{s^2+1}-s)^\nu}{\sqrt{s^2+1}} &< \frac{1}{s}.
\end{aligned}$$

We do not need to compute the corresponding integral explicitly, because this term will be dominated by the second term, for which we find from

$$2s \leq \sqrt{s^2+1} + s < 2s + 1$$

the bounds

$$\frac{(2s)^\nu}{\sqrt{s^2+1}} - 2^\nu s^{\nu-1} \le \frac{\left(\sqrt{s^2+1}+s\right)^\nu}{\sqrt{s^2+1}} - 2^\nu s^{\nu-1} < \frac{(2s+1)^\nu}{s} - 2^\nu s^{\nu-1}. \tag{33}$$

The left-most expression is always negative, the rightmost term always positive. In order to find a bound for the middle expression, we rewrite the right inequality

$$\frac{(2s+1)^\nu}{s} - 2^\nu s^{\nu-1} = 2^\nu s^{\nu-2}\left(\frac{1}{s}\left[\frac{s+1/2}{s}\right]^\nu - \frac{1}{s}\right).$$

The term in parenthesis on the right-hand side tends to zero for $s \to \infty$ and has no local maximum in the interval $1 \le s < \infty$. Hence it attains its maximum on the boundary $s = 1$. This shows

$$\frac{\left(\sqrt{s^2+1}+s\right)^\nu}{\sqrt{s^2+1}} - 2^\nu s^{\nu-1} < 2^\nu s^{\nu-2}\left(1 - \frac{1}{\sqrt{2}}\right).$$

The left-hand side of (33) is discussed in the same way. We rewrite

$$2^\nu s^{\nu-1} - \frac{(2s)^\nu}{\sqrt{s^2+1}} = 2^\nu s^{\nu-2}\left(s - \frac{s^2}{\sqrt{s^2+1}}\right).$$

Again, the term in parenthesis attains its maximum value at $s = 1$. We find, all together, the bound

$$\begin{aligned}\left|\frac{\left(\sqrt{s^2+1}+s\right)^\nu}{\sqrt{s^2+1}} - 2^\nu s^{\nu-1}\right| &\le s^{\nu-2}2^\nu \max\left\{\left(\frac{3}{2}\right)^\nu - 1, 1 - \frac{1}{\sqrt{2}}\right\} \\ &\le s^{\nu-2}3^\nu K \end{aligned} \tag{34}$$

with a constant K independent of ν. The leading term of of the numerator integral results in

$$\int_0^\infty ds\, \mathrm{e}^{-r_0 s} i^\nu s^\nu = 2^\nu \mathrm{e}^{i\frac{\pi}{2}\nu}\frac{\Gamma(\nu+1)}{r_0^\nu},$$

and the corresponding denominator integral becomes

$$\int_0^\infty ds\, \mathrm{e}^{-r_0 s} i^\nu s^{\nu-1} = 2^\nu \mathrm{e}^{i\frac{\pi}{2}\nu}\frac{\Gamma(\nu)}{r_0^{\nu-1}}.$$

By the bound (34), the residual function $R_{3,N}$ can be bounded by

$$\begin{aligned} |R_{3,N}| &< K \int_0^\infty ds\, \mathrm{e}^{-r_0 s} s^{\nu-1} 3^\nu \\ &= K 3^\nu \frac{\Gamma(\nu)}{r_0^\nu}, \end{aligned}$$

and the corresponding expression for the denominator becomes

$$\begin{aligned} |R_{3,D}| &< K \int_0^\infty ds\, \mathrm{e}^{-r_0 s} s^{\nu-2} 3^\nu \\ &= K 3^\nu \frac{\Gamma(\nu-1)}{r_0^{\nu-1}}. \end{aligned}$$

Adding all terms, we find the following expression for the boundary condition (23)

$$u_x = b_\nu u_0$$

$$b_\nu = -\frac{-i\,\mathrm{e}^{i(r_0+\frac{\pi}{4})}\sqrt{\dfrac{2\pi}{r_0}} + 2^\nu \mathrm{e}^{i\frac{\pi}{2}\nu} \dfrac{\Gamma(\nu+1)}{r_0^{\nu+1}} + R_N(r_0,\nu)}{\mathrm{e}^{i(r_0+\frac{\pi}{4})}\sqrt{\dfrac{2\pi}{r_0}} + 2^\nu \mathrm{e}^{i\frac{\pi}{2}\nu} \dfrac{\Gamma(\nu)}{r_0^{\nu}} + R_D(r_0,\nu)} \tag{35}$$

$$\begin{aligned} \text{with} \quad & |R_N(r_0,\nu)| < K_1 3^\nu \left(\frac{\Gamma(\nu)}{r_0^\nu} + \frac{1}{r_0} \right) \\ \text{and} \quad & |R_D(r_0,\nu)| < K_2 3^\nu \left(\frac{\Gamma(\nu-1)}{r_0^{\nu-1}} + \frac{1}{r_0} \right). \end{aligned} \tag{36}$$

Thus, (35) allows us to study the behavior of the boundary condition for large r_0 and ν from a single equation. First, we consider the case ν fixed and $r_0 \to \infty$. The integrals on P_1 dominate the expression, i. e. , the singularity at $s = -i$ determines the character of the solution. We get the Sommerfeld condition at infinity

$$b_\nu = i \quad \text{for } r_0 \to \infty. \tag{37}$$

The boundary condition is bounded and becomes imaginary. Next, let r_0 fixed and $\nu \to \infty$. Then, for ν sufficiently large, we find that the integrals on the infinite interval P_3 dominate the result. We obtain

$$b_\nu \to -\frac{\nu}{r_0} \quad \text{for } \nu \to \infty. \tag{38}$$

Hence, the boundary condition becomes real and unbounded.

Definiteness of the energy flux $\Im(r_0\bar{u}_0u_x)$. For the uniqueness of the solution of the interior problem (3), it is crucial to have the definiteness property $\Im(u, Mu)_{L^2} > 0$ for all $u \in H^{1/2}(\partial\mathcal{B}_{r_0})$, see Grote and Keller [6, Theorem 3.1], and Ihlenburg [8, Chapter 3.1.3]. In the following, we show that data $u_0 \neq 0$ and u_x, which satisfy the boundary condition (23), fulfill the corresponding inequality

$$\Im(r_0\bar{u}_0u_x) > 0 \quad \text{for all } r_0 > 0 \text{ and } 0 \leq \nu < \infty. \tag{39}$$

We already know two extremal cases: For ν fixed and $r_0 \to \infty$ we read from (37) $\Im(\bar{u}_0u_x) = \bar{u}_0u_0 > 0$, and for r_0 fixed and $\nu \to \infty$ we know $\Im(\bar{u}_0u_x) \to 0$ owing to (38). The key idea to verify (39) for intermediate values of r_0 and ν is to use the conservation of energy flux in the exterior to the disc, $\Im(\oint_\gamma \bar{u}\nabla u\, d\mathbf{r}) = \mathit{const}$, which holds for every closed contour enclosing the disc $\mathcal{B}_{r_0}$. Together with the asymptotic non–resonance property (30) we can derive (39). To be consistent with our approach via Bessel's differential equation, we derive the conservation law directly from (14). First, we multiply (14) with a function $v(x)$. Then we integrate the equation on an interval $x_1 \leq x \leq x_2$, perform an integration by parts and set $v(x) = (r_0 + x)\bar{u}(x)$. Taking the imaginary part of the whole expression, we find

$$\Im\big((r_0 + x)\bar{u}(x)\partial_x u(x)\big)_{x=x_1} = \Im\big((r_0 + x)\bar{u}(x)\partial_x u(x)\big)_{x=x_2}. \tag{40}$$

Let $x_1 = 0$ and x_2 a large distance away from $x = 0$. Owing to the non–resonance property (30) we obtain

$$\Im\big(\bar{u}(x)\partial_x u(x)\big)_{x=x_2} \longrightarrow \frac{|g(i)|^2}{2\pi x_2} \quad \text{as } x_2 \to \infty\,. \tag{41}$$

Remember that $g(i)$ never vanishes. Substituting (41) into (40), we get finally the desired result

$$\Im\big(r_0\,\bar{u}(x)\partial_x u(x)\big)_{x=0} = r_0|u_0|^2\Im(b_\nu) = \frac{|g(i)|^2}{2\pi} > 0\,.$$

To prepare the following chapter, we summarize the properties of b_ν:

1. $b_\nu \xrightarrow{\nu\to\infty} -\nu/r_0$.
2. $|b_\nu/\nu| \leq K < \infty$ for all ν, which follows from (35) and (36).
3. $\Im(b_\nu) = |g(i)|^2/(2\pi r_0|u_0|^2) > 0$ for all $0 \leq \nu < \infty$.
4. $|b_\nu| \geq K > 0$ for all $\nu \geq 0$, which follows from (35) and (36) and item 3.

Similar results were shown by Harari and Hughes [7] and Ihlenburg [8, Chapter 3.2.2] dealing with properties of Hankel's functions.

7 Mapping properties

By construction, the boundary condition (23) supplies a DtN-map for the νth Fourier mode $u_\nu(r,\phi)$, where from now we label the νth mode by the subscript ν,

$$
\begin{aligned}
&\partial_r u_\nu(r,\phi)|_{r=r_0} = b_\nu u_\nu(r_0,\phi) \\
\text{with}\quad & u_\nu(r_0,\phi) = c_\nu \mathrm{e}^{i\nu\phi}, \quad c_\nu = \text{const.}
\end{aligned}
$$

The set of functions $\{u_\nu^{(0)} := \mathrm{e}^{i\nu\phi}/\sqrt{2\pi},\ \nu = -\infty,\dots,\infty\}$ supplies an orthonormal basis of $L^2(\partial\mathcal{B}_{r_0})$ and we construct the DtN-operator by superposition (see §3)

$$
\partial_r u(r,\phi)|_{r=r_0} = \sum_{\nu=-\infty}^{\infty} b_\nu \frac{\mathrm{e}^{i\nu\phi}}{2\pi} \left(\mathrm{e}^{i\nu\phi}, u(r_0,\phi)\right)_{L^2}. \tag{42}
$$

Here we have $b_{-\nu} = b_\nu$ for $\nu = 1,2,\dots$, as it follows from (23). Eq. (42) defines an operator M via

$$
\partial_r u(r_0,\phi))|_{r=r_0} = Mu(r_0,\phi)\,. \tag{43}
$$

For the solution of the interior problem (3), which we do not discuss here, it is necessary that we have $\partial_r u(r_0,\phi) \in H^{-1/2}(\partial\mathcal{B}_{r_0})$. In fact, in the context of the corresponding variational formulation, we show that

$$
M:\ H^{1/2}(\partial\mathcal{B}_{r_0}) \to H^{-1/2}(\partial\mathcal{B}_{r_0})
$$

supplies a continuous and bijective mapping between data $u_0 \in H^{1/2}$ and data $u_x \in H^{-1/2}$. That is, instead of (43), we consider the corresponding variational equation

$$
\int_{-\pi}^{\pi} \bar{v}\partial_r u\, d\phi = \int_{-\pi}^{\pi} \bar{v} M u\, d\phi \quad \text{for } u,v \in H^{1/2}(\partial\mathcal{B}_{r_0})\,,
$$

and define, corresponding to M, the map b

$$
\begin{aligned}
b:\ H^{1/2}(\partial\mathcal{B}_{r_0}) &\to \left(H^{1/2}(\partial\mathcal{B}_{r_0})\right)^* = H^{-1/2}(\partial\mathcal{B}_{r_0}) \\
\text{with } b(v)(u) &= \int_{-\pi}^{\pi} \bar{v} M u\, d\phi \quad \text{for all } u \in H^{1/2}(\partial\mathcal{B}_{r_0}) \\
&=: \langle b(v), u\rangle\,.
\end{aligned}
$$

We aim to show that the map b is continuous and bijective. To this end, we define the inner product in $H^{1/2}(\partial\mathcal{B}_{r_0})$ via the Fourier series expansion of two functions $u, v \in L^2(\partial\mathcal{B}_{r_0})$ by

$$
\begin{aligned}
(v,u)_{H^{1/2}} &= 2\pi \sum_{\nu=-\infty}^{\infty} (1+\nu^2)^{1/2} \bar{a}_\nu b_\nu \\
\text{with} \quad a_\nu &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-i\nu\phi} v(\phi)\, d\phi \\
\text{and} \quad b_\nu &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-i\nu\phi} u(\phi)\, d\phi\,.
\end{aligned}
$$

The inner product induces the norm

$$
\|u\|_{H^{1/2}} = (u,u)_{H^{1/2}}^{1/2} = \left(2\pi \sum_{\nu=-\infty}^{\infty} (1+\nu^2)^{1/2} |a_\nu|^2 \right)^{1/2} .
$$

The orthonormal basis of $L^2(\partial\mathcal{B}_{r_0})$ with elements $u_\nu^{(0)}$ transforms into an orthonormal basis $\{u_\nu^{(1/2)} := \mathrm{e}^{i\nu\phi}/(\sqrt{2\pi}(1+\nu^2)^{1/4}),\ \nu = -\infty, \dots, \infty\}$ of $H^{1/2}(\partial\mathcal{B}_{r_0})$, and every function

$$
u = \sum_{\nu=-\infty}^{\infty} c_\nu u_\nu^{(1/2)}, \quad c_\nu \in \mathbb{C}
$$

with $\sum |c_\nu|^2 < \infty$ belongs to $H^{1/2}(\partial\mathcal{B}_{r_0})$. We show first that each functional $\langle b(v_b), u\rangle$ corresponds uniquely to a functional $(v,u)_{H^{1/2}}$ such that $\langle b(v_b), u\rangle = (v,u)_{H^{1/2}}$ for all $u \in H^{1/2}$. More precisely, we define a mapping

$$
\begin{aligned}
T:\ H^{1/2}(\partial\mathcal{B}_{r_0}) &\to H^{1/2}(\partial\mathcal{B}_{r_0}) \\
\text{with } v = T(v_b), &\quad \text{such that} \\
(v,u)_{H^{1/2}} = \langle b(v_b), u\rangle &\quad \text{for all } u \in H^{1/2}.
\end{aligned}
$$

We show that T is bijective. The bijectivity of T then implies the bijectivity of the map b via the duality map. First, T is surjective. To each given $v \in H^{1/2}$ we compute explicitly a function $v_b \in H^{1/2}$. Let v given by

$$
v = \sum_{\nu=-\infty}^{\infty} c_\nu u_\nu^{(1/2)}, \quad \sum |c_\nu|^2 < \infty.
$$

Each $v_b \in H^{1/2}$ possesses a representation

$$
v_b = \sum_{\nu=-\infty}^{\infty} c_\nu^{(b)} u_\nu^{(1/2)}\,,
$$

and we must show that $\sum |c_\nu^{(b)}|^2 < \infty$. Due to the orthonormality of $u_\nu^{(1/2)}$ with respect to the inner product in $H^{1/2}$, it follows for all ν

$$(v, u_\nu^{(1/2)})_{H^{1/2}} = \bar{c}_\nu,$$

and further

$$\begin{aligned}
\bar{c}_\nu &= \langle b(v_b), u_\nu^{(1/2)} \rangle \\
&= \int_{-\pi}^{\pi} \bar{v}_b M u_\nu^{(1/2)} \, d\phi \\
&= \int_{-\pi}^{\pi} \bar{v}_b b_\nu u_\nu^{(1/2)} \, d\phi \\
&= \int_{-\pi}^{\pi} \overline{\left(\sum c_\nu^{(b)} u_\nu^{(1/2)}\right)} b_\nu u_\nu^{(1/2)} \, d\phi \\
&= \frac{b_\nu}{(1+\nu^2)^{1/2}} \, \bar{c}_\nu^{(b)} \, .
\end{aligned}$$

Because b_ν is bounded away from zero, and b_ν/ν is bounded away from infinity, the convergence of $\sum |c_\nu|^2$ implies the convergence of $\sum |c_\nu^{(b)}|^2$. Second, T is injective. Given two functions $v_{b,1}, v_{b,2} \in H^{1/2}$, it follows by the forgoing construction that there are two uniquely determined Riesz-elements $v_1, v_2 \in H^{1/2}$, such that

$$\begin{aligned}
&\langle b(v_{b,1}), u \rangle = (v_1, u)_{H^{1/2}} \\
\text{and} \quad &\langle b(v_{b,2}), u \rangle = (v_2, u)_{H^{1/2}} \quad \text{for all } u \in H^{1/2} \, .
\end{aligned}$$

Consequently,

$$\langle b(v_{b,1}), u \rangle = \langle b(v_{b,2}), u \rangle \quad \text{implies } (v_1, u)_{H^{1/2}} = (v_2, u)_{H^{1/2}}$$

for all $u \in H^{1/2}$. It follows $v_1 = v_2$ and $v_{b,1} = v_{b,2}$. Finally, because the duality map

$$\begin{aligned}
&J: \; H^{1/2}(\partial\mathcal{B}_{r_0}) \to H^{-1/2}(\partial\mathcal{B}_{r_0}) \\
\text{with} \quad &J(v) = (v, u)_{H^{1/2}} \quad \text{for all } v, u \in H^{1/2}(\partial\mathcal{B}_{r_0}) \\
&\quad =: \langle J(v), u \rangle
\end{aligned}$$

is continuous and bijective, see Zeidler [16, Chapter 2.11], so is the map b.

Summary

We presented a new view on the classic Sommerfeld radiation condition. Like Sommerfeld, we analyzed the far field of a scatterer in order to decide what are incoming and what are outgoing waves. This decision is taken by a comparison of the far field with local plane waves. Different to Sommerfeld, we formulated this insight not in a radiation condition containing an asymptotic term but in a structural property of the corresponding Laplace transformed field. We showed that outgoing fields are characterized by singularities in the upper half, and incoming fields by singularities in the lower half of the complex plane. Consequently, our radiation condition aims to remove the singularity in the lower half of the complex plane. We proved that, provided the radiation condition is satisfied, the field becomes even holomorphic there. Moreover, our formulation is constructive. It allows a direct computation of a radiation condition from the differential equation. We demonstrated this by a construction of a DtN-map on a circle, which is equivalent to known DtN-maps based on evaluating Hankel's functions. We were able to evaluate the coefficients without to use special functions, only by the computation of path integrals on the complex plane. By an analytic approximation of these integrals, the relevant mapping properties have been shown.

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