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ABSTRACT. Given an affine surjection of polytopes $\pi : P \to Q$, the Generalized Baues Problem asks whether the poset of all proper polyhedral subdivisions of Q which are induced by the map π has the homotopy type of a sphere. We extend earlier work of the last two authors on subdivisions of cyclic polytopes to give an affirmative answer to the problem for the natural surjections between cyclic polytopes $\pi : C(n, d') \to C(n, d)$ for all $1 \le d < d' < n$.

1. INTRODUCTION

The Generalized Baues Problem, posed by Billera, Kapranov and Sturmfels [4], is a question in combinatorial geometry and topology, motivated by the theory of fiber polytopes [5] [18, Lecture 9]. Given an affine surjection of polytopes $\pi : P \to Q$, the problem asks to determine whether the *Baues poset* $\omega(P \xrightarrow{\pi} Q)$ of all proper polyhedral subdivisions of Q which are induced in a certain way by the map π, endowed with a standard topology [6], has the homotopy type of a sphere of dimension $\dim(P) - \dim(Q) - 1$. We refer to [11] for a concise introduction and [15] for a recent survey.

Although the Generalized Baues Problem is known to have a negative answer in general [14], various special cases have remained of interest in the literature; see [15, Section 4]. One such relates to subdivisions of cyclic polytopes. Another is the case where P is a simplex, in which $\omega(P \xrightarrow{\pi} Q)$ is the poset of *all* proper polyhedral subdivisions of Q and is simply denoted $\omega(Q)$. In [9] an affirmative answer to the problem was given in the case of the poset of all subdivisions of cyclic polytopes of dimension at most 3. This was recently improved in [13] to all dimensions, as follows.

Theorem 1.1. ([13, Theorem 1.1]) *For all $1 \le d < n$, the Baues poset $\omega(C(n, d))$ of all proper polyhedral subdivisions of the cyclic polytope $C(n, d)$ is homotopy equivalent to an $(n - d - 2)$-sphere.*

For $1 \le d < d' < n$, one can consider the natural projections $\pi : C(n, d') \to C(n, d)$ between cyclic polytopes [1]. The Baues poset $\omega(C(n, d))$ in Theorem 1.1 is the Baues poset of the projection π for $d' = n - 1$. In this paper we use the "sliding" technique of [13] to give an affirmative answer to the Generalized Baues Problem for π for all d, d' and n.

Theorem 1.2. *For $1 \le d < d' < n$, the Baues poset $\omega(C(n, d') \xrightarrow{\pi} C(n, d))$ of all proper polyhedral subdivisions of the cyclic polytope $C(n, d)$ which are induced by π is homotopy equivalent to a $(d' - d - 1)$-sphere.*

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Theorem 1.2 was conjectured by Reiner [15] on the basis of the following special cases:

- $d = 2$, $d' = n - 2$ [1, Corollary 6.3],
- $d' = n - 1$ (Theorem 1.1),
- $d = 2$, $n < 2d' + 2$, $d' \geq 9$ [16, Corollary 15].

Other previously known special cases are those of $d = 1$ and $d' - d \leq 2$, which follow from more general results of [4] and [14], respectively: for any polytope projection $\pi : P \to Q$, the poset $\omega(P \xrightarrow{\pi} Q)$ of all proper π-induced subdivisions of Q is homotopy equivalent to a sphere whenever $\dim(Q) = 1$ or $\dim(P) - \dim(Q) \leq 2$.

Our argument is a modification of the one used in [13, Section 4] to prove Theorem 1.1 and therefore relies heavily on the constructions of [13]. In the next section we review some basic definitions and facts. In Section 3 we give a sketch of the proof of Theorem 1.2, thereby recalling those constructions from [13] that will be essential here. Section 4 contains the remaining details, which amount to proving that two certain posets of subdivisions are contractible.

2. Preliminaries

2.1. **Polyhedral subdivisions.** By a point configuration $\mathcal{A}$ in $\mathbb{R}^d$ we mean a finite labeled subset of $\mathbb{R}^d$. We allow $\mathcal{A}$ to have repeated points which are distinguished by their labels. The convex hull $\text{conv}(\mathcal{A})$ of $\mathcal{A}$ is a polytope.

A face of a subconfiguration $\sigma \subseteq \mathcal{A}$ is a subconfiguration $F^{\omega} \subseteq \sigma$ consisting of *all* points on which some linear functional $\omega \in (\mathbb{R}^d)^*$ takes its minimum over σ.

We say that two subconfigurations σ_1 and σ_2 of $\mathcal{A}$ *intersect properly* if the following two conditions are satisfied:

- $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2;
- $\text{conv}(\sigma_1) \cap \text{conv}(\sigma_2) = \text{conv}(\sigma_1 \cap \sigma_2)$.

A subconfiguration of $\mathcal{A}$ is said to be *full-dimensional*, or *spanning*, if it affinely spans $\mathbb{R}^d$. In that case we call it a *cell*. Following [3] and [10, Section 7.2] we say that a collection S of cells of $\mathcal{A}$ is a *(polyhedral) subdivision* of $\mathcal{A}$ if the elements of S intersect pairwise properly and cover $\text{conv}(\mathcal{A})$ in the sense that

$$\cup_{\sigma \in S} \text{conv}(\sigma) = \text{conv}(\mathcal{A}).$$

Cells that share a common facet are *adjacent*. The set of subdivisions of $\mathcal{A}$ is partially ordered by the *refinement* relation

$$S_1 \leq S_2 \qquad :\Longleftrightarrow \qquad \forall \sigma_1 \in S_1,\ \exists \sigma_2 \in S_2 :\ \sigma_1 \subset \sigma_2.$$

The poset of subdivisions of $\mathcal{A}$ has a unique maximal element which is the trivial subdivision $\{\mathcal{A}\}$. The minimal elements are the subdivisions all of whose cells are affinely independent, which are called *triangulations* of $\mathcal{A}$. We call subdivisions of a polytope Q the subdivisions of its vertex set.

2.2. **Induced subdivisions.** Now let $P \subset \mathbb{R}^p$ be a polytope, and let $\pi : \mathbb{R}^p \to \mathbb{R}^d$ be a linear projection map. We can consider the point configuration $\mathcal{A} = \pi(\text{vert}(P))$ arising from the projection of the vertex set of P. An element in $\mathcal{A}$ is labeled by the vertex of P of which it is considered to be the image. In other words, π induces a bijection between the vertex set of P and $\mathcal{A}$, even if different vertices of P have the same projection.

A subdivision S of $\mathcal{A}$ is said to be π-*induced* if every cell of S is the projection of the vertex set of a face of P. If P is a simplex then all subdivisions of $\mathcal{A}$ are π-induced. This concept of π-induced subdivisions was introduced in [5].

A π-induced subdivision S contains the same information as the collection of faces of P whose vertex sets are in S. In this sense one can say that a π-induced subdivision of $\mathcal{A}$ is a polyhedral subdivision whose cells are projections of faces of P (this statement is not accurate; see [11, 14, 18] for an accurate definition of π-induced subdivisions in terms of faces of P).

The poset of π-induced subdivisions excluding the trivial one is denoted by $\omega(P \xrightarrow{\pi} \pi(P))$. Its minimal elements are the subdivisions for which every cell comes from a $\dim(\mathcal{A})$-dimensional face of P. They are called *tight* π-induced subdivisions.

In [4] it was conjectured that the Baues poset $\omega(P \xrightarrow{\pi} \pi(P))$ is homotopy equivalent to a sphere of dimension $p-d-1$. Evidence for this were the cases $p-d=1$ (trivial) and $d=1$ (proved in [4]) together with the fact that $\omega(P \xrightarrow{\pi} \pi(P))$ always contains a subposet *homeomorphic* to a sphere of dimension $p-d-1$ (the poset of *coherent* π-induced subdivisions [5]). The conjecture was known as the *generalized Baues conjecture* since the case $d=1$ had been conjectured by J. Baues in a different form, until it was disproved in [14]. Still, several cases remain of interest. Theorem 1.1 is the case where π is the natural projection from a simplex to a cyclic polytope and our Theorem 1.2 is the case where π is the natural projection between two cyclic polytopes. Other cases where the statement is known to be true are when $p-d=2$ [14] and when P is a simplex and $d=2$ [8].

See [5, 15, 18] for more information on π-induced subdivisions and the Baues problem.

2.3. **Poset topology.** When refering to the topology of a finite poset we mean the topology of its *order complex*, i.e., the simplicial complex of chains in the poset [6]. For a poset P and $x \in P$ we denote by $P_{\le x}$ the set $\{y \in P : y \le x\}$. We will use the following tool from [2] to relate the homotopy type of two posets. A proof is given in [17, Section 3].

Lemma 2.1. (Babson) *Let* $f : \omega \to \omega'$ *be an order preserving map of posets. If*

(i) $f^{-1}(y)$ *is contractible for every* $y \in \omega'$ *and*
(ii) $\omega_{\le x} \cap f^{-1}(y)$ *is contractible for every* $x \in \omega$ *and* $y \in \omega'$ *with* $f(x) > y$

then f *induces a homotopy equivalence.*

2.4. **Cyclic polytopes.** The cyclic polytope $C(n,d)$ is the convex hull of any n points on the moment curve $\{(t, t^2, \dots, t^d) : t \in \mathbb{R}\}$ in $\mathbb{R}^d$. We consider it as the point configuration consisting of these n points, which are the vertices for $d \ge 2$. Hence, all the notions for induced subdivisions make sense for cyclic polytopes. Also, we extend the usual definition by the trivial case of $d=0$: the cyclic polytope $C(n,0)$ is just the set of n copies of the only point in $\mathbb{R}^0$. The cyclic polytope $C(n,1)$ consists of n distinct points in the real line $\mathbb{R}$.

As usual, we label the vertices of $C(n,d)$ with the numbers $1, \dots, n$, in the order they appear along the moment curve and refer to faces of $C(n,d)$ by the index sets of their vertices, i.e. as subsets of $[n] := \{1, 2, \dots, n\}$.

The face lattice of $C(n,d)$ is known to be independent of the choice of points on the curve and is characterized by Gale's evenness criterion, which is as follows (see also [18, p. 14] or [1, Theorem 5.2]). For a subset $F \subset [n]$ with complement $[n] \setminus F = \{a_1, a_2, \dots, a_k\}$, we divide F in its *initial interval* $\{1, \dots, a_1 - 1\}$, its *final interval* $\{a_k + 1, \dots, n\}$ and its *interior intervals* $\{a_i + 1, \dots, a_{i+1} - 1\}$, $i = 1, \dots, k-1$. The initial and final intervals may be empty. An interval is called odd if it has an odd number of elements and even otherwise. Then, F is a face of $C(n, d)$ if and only if the cardinality of F plus the number of odd interior intervals does not exceed d. Two obvious consequences of this description are that cyclic polytopes are simplicial and that faces of $C(n, d)$ are also faces of $C(n, d')$ for $d' > d$.

Moreover, if d is the smallest integer for which F is a face of $C(n, d)$, then F is an *upper* face of $C(n, d)$ (meaning that its normal cone contains only vectors with last coordinate positive) if the final interval in F is odd and F is a *lower* face (meaning that its normal cone contains only vectors with last coordinate negative) if the final interval in F is even (or empty).

2.5. **The canonical projections between cyclic polytopes.** For a fixed pair of dimensions $d' > d$ we will be interested in the surjection $\pi : C(n, d') \to C(n, d)$, induced by the map $\pi : \mathbb{R}^{d'} \to \mathbb{R}^d$ which forgets the last $d' - d$ coordinates. The fiber polytopes for this family of surjections were studied in [1]. The associated Baues posets were studied in the special case $d = 2$ in [16]. For the ease of notation, we will write $\omega_{d'}(C(n, d))$ for the Baues poset $\omega(C(n, d') \xrightarrow{\pi} C(n, d))$. This poset is also independent of the choice of points used to define $C(n, d')$. Note that the Baues poset $\omega_{d'}(C(n, 0)) = \omega(C(n, d') \xrightarrow{\pi} C(n, 0))$ is isomorphic to the poset of proper faces of $C(n, d')$ for all $d' > 0$, hence homeomorphic to a $(d' - 1)$-sphere.

3. Structure of the proof

The idea for proving Theorem 1.2 is as follows. Let us fix the dimensions $2 \leq d < d'$ and then use induction on the number n of vertices. The result is already known in the cases $d = 0, 1$. The base case $n = d' + 1$ for the induction is provided by Theorem 1.1. For the inductive step, we will use the same approach as in [13]: via the *deletion operation* of vertex n from a subdivision of $C(n, d)$, we will define a map between the posets $\omega_{d'}(C(n, d))$ and $\omega_{d'}(C(n - 1, d))$ and will prove it to be a homotopy equivalence. This deletion operation is a generalization of the deletion operation on triangulations of $C(n, d)$ from [12].

For two collections S and T of finite pointsets in $\mathbb{R}^d$ we define

$$\begin{aligned}
\operatorname{spanning}(S) &:= \{\, \sigma \in S \,:\, \sigma \text{ is spanning} \,\} \\
\operatorname{ast}_S(i) &:= \{\, \sigma \in S \,:\, i \notin \sigma \,\} \\
\operatorname{lk}_S(i) &:= \{\, \sigma - \{i\} \,:\, \sigma \in S,\ i \in \sigma \,\} \\
S * T &:= \{\, \sigma \cup \tau \,:\, \sigma \in S, \tau \in T \,\}.
\end{aligned}$$

As was discussed in [13, Section 4], if S is a subdivision of $C(n, d)$ then $\operatorname{lk}_S(n)$ is a subdivision of $C(n - 1, d - 1)$. Moreover, Gale's evenness criterion easily implies that if S is in $\omega_{d'}(C(n, d))$ then $\operatorname{lk}_S(n)$ is in $\omega_{d'-1}(C(n - 1, d - 1))$.

Definition 3.1. ([13]) *Given a subdivision S of $C(n, d)$, the deletion $S \backslash n$ is*

$$S \backslash n := \operatorname{spanning}\left(\{\sigma \backslash n : \sigma \in S\}\right),$$

where

$$\sigma\backslash n := \begin{cases} (\sigma - \{n\}) \cup \{n-1\}, & \textit{if } n \in \sigma, \\ \sigma, & \textit{otherwise.} \end{cases}$$

Equivalently,

$$S\backslash n := \mathrm{ast}_S(n) \cup \mathrm{spanning}\big(\mathrm{lk}_S(n) * \{n-1\}\big).$$

Using the idea of "sliding" vertex n to $n-1$, it is proved in [13, Theorem 3.2] that $S\backslash n$ is a subdivision of $C(n-1, d)$. The deletion of n defines a map between the posets $\omega_{d'}(C(n,d))$ and $\omega_{d'}(C(n-1,d))$:

Proposition 3.2. *Let $n \geq d'+2$. The deletion map $\Pi_{d'} : \omega_{d'}(C(n,d)) \to \omega_{d'}(C(n-1,d))$*

$$\Pi_{d'}(S) = S\backslash n$$

between the Baues posets of proper π-induced subdivisions is well-defined and order preserving.

Proof. In order to see that $\Pi_{d'}$ is well-defined we just need to check that if σ is a proper face of $C(n, d')$ then $\sigma\backslash n$, introduced in Definition 3.1, is a proper face of $C(n-1, d')$. It follows easily from Gale's evenness criterion that $\sigma\backslash n$ is a face of $C(n-1, d')$. Moreover, since σ is proper and $C(n, d')$ is simplicial, σ has at most $d' \leq n-2$ vertices. Thus $\sigma\backslash n$ has at most $n-2$ vertices and is a proper face of $C(n-1, d')$.

That $\Pi_{d'}$ is order preserving follows trivially from the fact that if $\sigma \subset \sigma'$ then $\sigma\backslash n \subset \sigma'\backslash n$. ☐

*

In order to apply Lemma 2.1 to the map $\Pi_{d'}$ we need to understand its fibers. The following concept of subdivisions of $C(n, d)$ *induced by a certain subdivision* $\overline{S}$ of $C(n, d+1)$ will be crucial for this.

Let $\overline{S}$ be a subdivision of the cyclic polytope $C(n, d+1)$ and S a subdivision of $C(n, d)$. Following [13], we say that S is induced by $\overline{S}$ if every cell $\sigma \in S$ is a face (not necessarily proper) of a cell $\sigma' \in \overline{S}$. We can think of S as a cellular section of the natural projection $C(n, d+1) \to C(n, d)$ which uses only cells in $\overline{S}$ or their faces. Observe that for every cell σ'' of $\overline{S}$ we can tell whether σ'' is above, on or below (the section corresponding to) a subdivision S induced by $\overline{S}$. We will denote by $\mathrm{above}(S, \overline{S})$ and $\mathrm{below}(S, \overline{S})$ the set of cells of $\overline{S}$ which lie above and below S, respectively.

We denote by $\omega(\overline{S})$ the poset of all subdivisions of $C(n, d)$ which are induced by $\overline{S}$, partially ordered by refinement, so that $\omega(\overline{S})$ is a subposet $\omega(C(n, d))$.

From the definition of the deletion $S\backslash n$ it follows trivially that $\mathrm{lk}_S(\{n, n-1\}) := \mathrm{lk}_{\mathrm{lk}_S(n)}(n-1) \subset \mathrm{lk}_{S\backslash n}(n-1)$ for any $S \in \omega(C(n, d))$. Let $T \in \omega_{d'}(C(n-1, d))$ and let $S \in \omega_{d'}(C(n, d))$ be such that $S\backslash n = T$, i.e., $S \in \Pi_{d'}^{-1}(T)$. Then the subdivision $\mathrm{lk}_S(\{n, n-1\})$ of $C(n-2, d-2)$ is induced by the subdivision $\mathrm{lk}_T(n-1) \in \omega(C(n-2, d-1))$. In other words, we have a map $\Pi_{d'}^{-1}(T) \to \omega(\mathrm{lk}_T(n-1))$ defined by $S \mapsto \mathrm{lk}_S(\{n, n-1\})$. The following much stronger statement follows from [13, Lemma 4.7].

Lemma 3.3. *Let $2 \le d < d' \le n-2$ and consider the deletion map $\Pi_{d'} : \omega_{d'}(C(n, d)) \to \omega_{d'}(C(n-1, d))$. Let $T \in \omega_{d'}(C(n-1, d))$ and $\overline{S} = \mathrm{lk}_T(n-1) \in \omega(C(n-2, d-1))$. Then:*

1. *The map $\omega(C(n, d)) \to \omega(C(n-2, d-2))$ given by $S \mapsto \mathrm{lk}_S(\{n-1, n\})$ restricts to a poset isomorphism between $\Pi_{d'}^{-1}(T)$ and a subposet $\omega_{d'}(\overline{S})$ of $\omega(\overline{S})$.*
2. *The inverse map $\tau : \omega_{d'}(\overline{S}) \to \Pi_{d'}^{-1}(T)$ is given by*

$$
\begin{aligned}
\tau(S) := &\{\, \sigma \in T \,:\, n-1 \notin \sigma \,\} \\
&\cup \left\{\, \sigma \cup \{n\} \,:\, \sigma \in \overline{S},\ \sigma \textit{ is below } S \,\right\} \\
&\cup \left\{\, \sigma \cup \{n-1\} \,:\, \sigma \in \overline{S},\ \sigma \textit{ is above } S \,\right\} \\
&\cup \{\, \sigma \cup \{n, n-1\} \,:\, \sigma \in S \,\}.
\end{aligned}
$$

   Moreover,

$$\omega_{d'}(\overline{S}) = \{S \in \omega(\overline{S}) : \tau(S) \in \omega_{d'}(C(n, d))\}.$$

3. *Let $T' \in \omega_{d'}(C(n, d))$ be such that $T' \backslash n$ is coarser than T and let $S_0 = \mathrm{lk}_{T'}(\{n, n-1\}) \in \omega(C(n-2, d-2))$. Then, the previous isomorphism restricts to an isomorphism between $\omega_{d'}(C(n, d))_{\le T'} \cap \Pi_{d'}^{-1}(T)$ and*

$$\omega_{d'}(\overline{S})_{\le S_0} := \{S \in \omega_{d'}(\overline{S}) : S \textit{ refines } S_0\} = \omega_{d'}(\overline{S}) \cap \omega(C(n-2, d-2))_{\le S_0}.$$

By Lemma 2.1 applied to the map $\Pi_{d'}$ introduced in Proposition 3.2, Lemma 3.3 implies that in order to prove Theorem 1.2 we just need to show that, under the assumptions of the lemma, both $\omega_{d'}(\overline{S})$ and $\omega_{d'}(\overline{S})_{\le S_0}$ are contractible. We will do this in the next section, following the ideas of [13].

4. The details

Throughout this section we assume that the hypotheses of Lemma 3.3 hold and we fix an element $T \in \omega_{d'}(C(n-1, d))$ and an element $T' \in \omega_{d'}(C(n, d))$ such that T refines $T' \backslash n$. We also let $\overline{S} = \mathrm{lk}_T(n-1)$ and $S_0 = \mathrm{lk}_{T'}(\{n, n-1\})$. Our task is to prove that both $\omega_{d'}(\overline{S})$ and $\omega_{d'}(\overline{S})_{\le S_0}$ are contractible. The proof for $\omega_{d'}(\overline{S})_{\le S_0}$ is easier and we do it in the following proposition. The proof for $\omega_{d'}(\overline{S})$ occupies the rest of this section.

Proposition 4.1. *Under the assumptions of part 3 of Lemma 3.3, let $\omega(\overline{S})_{\le S_0} := \omega(\overline{S}) \cap \omega(C(n-2, d-2))_{\le S_0}$. Then:*

1. *$\omega_{d'}(\overline{S})_{\le S_0} = \omega(\overline{S})_{\le S_0}$ and hence*
2. *$\omega_{d'}(\overline{S})_{\le S_0}$ is contractible.*

Proof. The second statement follows from [13, Corollary 4.6], where $\omega(\overline{S})_{\le S_0}$ is proved to be contractible.

For the first statement, let $T' \in \omega_{d'}(C(n, d))$ be such that $T' \backslash n$ is coarser than T and let $S_0 = \mathrm{lk}_{T'}(\{n-1, n\}) \in \omega(C(n-2, d-2))$. Observe that S_0 might not be in $\omega(\overline{S})$ but it is in $\omega(\overline{S'})$, where $\overline{S'} := \mathrm{lk}_{T' \backslash n}(n-1)$ is coarser than $\overline{S}$. By parts 1 and 2 of Lemma 3.3 we have that S_0 is in $\omega_{d'}(\overline{S'})$.

Let $S \in \omega(\overline{S})$ be a refinement of S_0. We will prove that $\tau(S)$ is in $\omega(C(n, d'))$, i.e. $S \in \omega_{d'}(\overline{S})$. Thus $S \in \omega_{d'}(\overline{S})_{\le S_0}$. For the proof we only use the fact that $S_0 \in \omega_{d'}(\overline{S'})$, that S refines S_0 and that $\overline{S}$ refines $\overline{S'}$.

Let $\sigma \in \text{above}(S, \overline{S})$ and choose $\sigma' \in \overline{S'}$ such that $\sigma \subset \sigma'$. Since S refines S_0, either $\sigma' \in \text{above}(S_0, \overline{S'})$ or $\sigma' \in S_0$. In both cases $\sigma' \cup \{n-1\}$, and hence $\sigma \cup \{n-1\}$, is a face of $C(n, d')$. In the same way, if $\sigma \in \text{below}(S, \overline{S})$ then $\sigma \cup \{n\}$ is a face of $C(n, d')$. Finally, if $\sigma \in S$, then there is $\sigma' \in S_0$ with $\sigma \subset \sigma'$ and since $\sigma' \cup \{n, n-1\}$ is a face of $C(n, d')$, $\sigma \cup \{n, n-1\}$ is a face too. □

We are now concerned with the poset $\omega(\overline{S}) \subset \omega(C(n-2, d-2))$ and its subposet

$$\omega_{d'}(\overline{S}) = \{S \in \omega(\overline{S}) : \tau(S) \in \omega_{d'}(C(n, d))\}.$$

Our goal is to prove that $\omega_{d'}(\overline{S})$ is contractible. Actually, we will never use the fact that $\overline{S}$ is a link of a subdivision $T \in C(n-1, d)$ but only that, since $T \in \omega_{d'}(C(n-1, d))$, its link $\overline{S}$ is in $\omega_{d'-1}(C(n-2, d-1))$. In other words, we will prove the following result.

Theorem 4.2. *Let $\overline{S}$ be a proper subdivision of $C(n-2, d-1)$, induced by $C(n-2, d'-1)$, and let $\omega_{d'}(\overline{S}) \subset \omega(\overline{S})$ be the poset of sections S of $\overline{S}$ which have the properties:*

1. *For any $\sigma \in \text{above}(S, \overline{S})$, $\sigma \cup \{n-1\}$ is a face of $C(n, d')$.*
2. *For any $\sigma \in \text{below}(S, \overline{S})$, $\sigma \cup \{n\}$ is a face of $C(n, d')$.*
3. *For any $\sigma \in S$, $\sigma \cup \{n, n-1\}$ is a face of $C(n, d')$.*

Then $\omega_{d'}(\overline{S})$ is contractible.

Let us recall the following technical property of subdivisions of cyclic polytopes, proved and called *stackability* in [13]. Let S be a subdivision of a cyclic polytope $C(n, d)$, for arbitrary $n > d$. For any two cells $\sigma_1, \sigma_2 \in S$ which share a facet, their common facet is an upper facet of one of σ_1, σ_2 and a lower facet of the other. If it is a lower facet of σ_2 and an upper facet of σ_1 we say that "σ_2 is above σ_1" and "σ_1 below σ_2".

Lemma 4.3. ([13, Lemma 2.16]) *The relation "σ_1 is below σ_2" just defined has no cycles. Hence, its transitive closure is a partial order on the collection of all cells of S.*

In the sequel we denote by $<_{st}$ this partial order on the cells of the subdivision $\overline{S} \in \omega_{d'-1}(C(n-2, d-1))$.

Lemma 4.4. *Let $\overline{S} \in \omega_{d'-1}(C(n-2, d-1))$. Let $\sigma \subset [n-2]$ be a face of $C(n-2, d'-1)$ (not necessarily a cell of $\overline{S}$) and let σ_+ and σ_- be cells in $\overline{S}$ such that $\sigma_- <_{st} \sigma_+$. Then:*

1. *At least one of $\sigma \cup \{n-1\}$ and $\sigma \cup \{n\}$ is a proper face of $C(n, d')$.*
2. *If $\sigma \cup \{n-1\}$ and $\sigma \cup \{n\}$ are both proper faces of $C(n, d')$, then so is $\sigma \cup \{n, n-1\}$.*
3. *If $\sigma_+ \cup \{n\}$ and $\sigma_- \cup \{n-1\}$ are both proper faces of $C(n, d')$ then so is either $\sigma_- \cup \{n\}$ or $\sigma_+ \cup \{n-1\}$.*

Proof. 1. If σ is a face of $C(n-2, d'-2)$ then $\sigma \cup \{n, n-1\}$ is a face of $C(n, d')$. Hence, both $\sigma \cup \{n-1\}$ and $\sigma \cup \{n\}$ are faces of $C(n, d')$ as well.

If σ is not a face of $C(n-2, d'-2)$ then σ is either an upper or a lower face of $C(n-2, d'-1)$. In the first case $\sigma \cup \{n-1\}$ is a face of $C(n, d')$ and in the second case $\sigma \cup \{n\}$ is a face of $C(n, d')$, as follows easily from Gale's evenness criterion.

2. We will show that $\sigma \cup \{n, n-1\}$ has at least one interior component of odd length less than either $\sigma \cup \{n\}$ or $\sigma \cup \{n-1\}$. Taking m to be the maximum element in $[n-2] \backslash \sigma$, we observe that this is the case for $\sigma \cup \{n-1\}$ if $n-m$ is even and for $\sigma \cup \{n\}$ if $n-m$ is odd.

3. Let m_+ (respectively m_-) be the maximum in $[n-2]\backslash\sigma_+$ (respectively $[n-2]\backslash\sigma_-$). We will prove that either $\{m_-+1,\dots,n-2\}$ has an even number of elements (and then $\sigma_-\cup\{n\}$ is a face of $C(n,d')$), or $\{m_++1,\dots,n-2\}$ has an odd number of elements (and then $\sigma_+\cup\{n-1\}$ is a face of $C(n,d')$).

For this let $\sigma_-=\sigma_0<_{st}\sigma_1<_{st}\cdots<_{st}\sigma_k=\sigma_+$ be a chain of cells of $\overline{S}$ such that every two consecutive ones share a facet. Let m be the maximum integer in $[n-2]\backslash\cap_{i=0}^k\sigma_i$. We will consider separately the following three cases: (i) $m\notin\cup_{i=0}^k\sigma_i$, (ii) $m\in\cup_{i=0}^k\sigma_i$ and $n-m$ is even and (iii) $m\in\cup_{i=0}^k\sigma_i$ and $n-m$ is odd.

(i) If $m\notin\cup_{i=0}^k\sigma_i$ then $m=m_+=m_-$. Obviously, $\{m+1,\dots,n-2\}$ has either an even or an odd number of elements.

(ii) If $n-m$ is even then the common interval $\{m+1,\dots,n-2\}$ to all the σ_i has an even number of elements. This implies that if $m\in\sigma_i$ for an $i\le k-1$ then $m\in\sigma_{i+1}$ too. Indeed, the common facet τ between σ_i and σ_{i+1} is an upper facet of σ_i and, hence, its last interval has odd length. So it is impossible to have $m\in\sigma_i\backslash\tau$ and $\{m+1,\dots,n-2\}\subset\tau$. In particular, m cannot be in $\sigma_-=\sigma_0$ because then it would be in $\cap_{i=0}^k\sigma_i$. Hence, $m=m_-$ and $\{m_-+1,\dots,n-2\}$ has an even number of elements.

(iii) This case is analogous: If $n-m$ is odd then the common interval $\{m+1,\dots,n-2\}$ to all the σ_i has an odd number of elements. This implies that if $m\in\sigma_i$ for an $i\ge 1$ then $m\in\sigma_{i-1}$ too. Indeed, the common facet τ between σ_i and σ_{i-1} is a lower facet of σ_i and, hence, its last interval has even length. So it is impossible to have $m\in\sigma_i\backslash\tau$ and $\{m+1,\dots,n-2\}\subset\tau$. In particular, m cannot be in $\sigma_+=\sigma_k$ because then it would be in $\cap_{i=0}^k\sigma_i$. Hence, $m=m_+$ and $\{m_++1,\dots,n-2\}$ has an odd number of elements.

□

Our next goal is to prove that $\omega_{d'}(\overline{S})$ is not empty, and hence that the map $\Pi_{d'}$ is surjective. Clearly, $\omega(\overline{S})$ is not empty so we are interested in which elements $S\in\omega(\overline{S})$ lie also in $\omega_{d'}(\overline{S})\subset\omega(\overline{S})$.

Lemma 4.5. *Let $S\in\omega(\overline{S})$. Then $S\in\omega_{d'}(\overline{S})$ if and only if for any cell σ of $\overline{S}$ we have:*

$$\text{if } \sigma\in S\cup\mathrm{above}(S,\overline{S}) \text{ then } \sigma\cup\{n-1\} \text{ is a face of } C(n,d'),$$
$$\text{if } \sigma\in S\cup\mathrm{below}(S,\overline{S}) \text{ then } \sigma\cup\{n\} \text{ is a face of } C(n,d').$$

Proof. Necessity of the conditions is obvious by the definition of $\omega_{d'}(\overline{S})$ in Lemma 3.3. Sufficiency is not obvious since a cell σ of S might not be a (spanning) cell of $\overline{S}$. We need to prove under the conditions in the statement that for such a cell σ, $\sigma\cup\{n,n-1\}$ is a face of $C(n,d')$.

Let σ be in $S\setminus\overline{S}$. Then σ is a simplex of dimension $d-2$ in $C(n-2,d-1)$ and there is a cell $\sigma_+\in\overline{S}$ (respectively σ_-) of which σ is a lower (respectively upper) facet unless σ is an upper (respectively lower) facet of $C(n-2,d-1)$. We will prove that $\sigma\cup\{n\}$ and $\sigma\cup\{n-1\}$ are faces of $C(n,d')$. Then by part 2 of Lemma 4.4 we conclude that $\sigma\cup\{n,n-1\}$ is a face of $C(n,d')$.

If σ is an upper (respectively lower) facet of $C(n-2,d-1)$ then $\sigma\cup\{n-1\}$ (respectively $\sigma\cup\{n\}$) is a lower (respectively upper) facet of $C(n,d)$, hence a face of $C(n,d')$. If σ is an upper (respectively lower) facet of $\sigma_-\in\overline{S}$, (respectively of σ_+) then σ_- is below S (respectively σ_+ is above S) and by hypothesis $\sigma_-\cup\{n\}$

(respectively $\sigma_+ \cup \{n-1\}$) is a face of $C(n, d')$. Thus $\sigma \cup \{n\}$ (respectively $\sigma \cup \{n-1\}$) is also a face.

□

For the sequel, for $\overline{S} \in \omega_{d'-1}(C(n-2, d-1))$, let us define the following collections, which depend on d':

$$\text{above}(\overline{S}) = \{\sigma \in \overline{S} : \forall \sigma' \in \overline{S} \text{ with } \sigma \leq_{st} \sigma',\ \sigma' \cup \{n-1\} \text{ is a face of } C(n, d')\},$$
$$\text{below}(\overline{S}) = \{\sigma \in \overline{S} : \forall \sigma' \in \overline{S} \text{ with } \sigma' \leq_{st} \sigma,\ \sigma' \cup \{n\} \text{ is a face of } C(n, d')\}.$$

By definition, below$(\overline{S})$ and above$(\overline{S})$ are a lower and an upper ideal respectively in $<_{st}$. This implies that the upper envelope S_{up} of below$(\overline{S})$ and the lower envelope S_{down} of above$(\overline{S})$ are valid sections in $\omega(\overline{S})$. We show that they are also in $\omega_{d'}(\overline{S})$. Observe that above$(\overline{S})$ = above$(S_{down}, \overline{S})$ and below$(\overline{S})$ = below$(S_{up}, \overline{S})$.

Lemma 4.6. *We have:*

1. below$(\overline{S}) \cup$ above$(\overline{S}) = \overline{S}$.
2. *Let $S \in \omega(\overline{S})$. Then $S \in \omega_{d'}(\overline{S})$ if and only if*

$$\text{above}(S, \overline{S}) \cup \text{below}(\overline{S}) = \overline{S},$$

$$\text{below}(S, \overline{S}) \cup \text{above}(\overline{S}) = \overline{S}.$$

3. *In particular, S_{up} and S_{down} are in $\omega_{d'}(\overline{S})$.*

Proof. 1. Let $\sigma \in \overline{S}$ and suppose $\sigma \notin$ below$(\overline{S})$. By definition of below$(\overline{S})$ this means that there is a $\sigma' \leq_{st} \sigma$ such that $\sigma' \cup \{n\}$ is not a face of $C(n, d')$. Since σ' is a face of $C(n-2, d'-1)$, parts 1 and 3 of Lemma 4.4 imply, respectively, that $\sigma' \cup \{n-1\}$ and any $\sigma'' \cup \{n-1\}$ with $\sigma'' \geq_{st} \sigma'$ are faces of $C(n, d')$. In particular, $\sigma \in$ above$(\overline{S})$.

2. We first prove the necessity of the conditions. If $\sigma \in \overline{S} \setminus (\text{above}(S, \overline{S}) \cup \text{below}(\overline{S}))$ then $\sigma \in S \cup$ below$(S, \overline{S})$ and there is a $\sigma' <_{st} \sigma$ such that $\sigma' \cup \{n\}$ is not a face of $C(n, d')$. This σ' will also be in $S \cup$ below$(S, \overline{S})$ and hence $S \notin \omega_{d'}(\overline{S})$ by Lemma 4.5. The case of a $\sigma \in \overline{S} \setminus (\text{below}(S, \overline{S}) \cup \text{above}(\overline{S}))$ is analogous.

For the sufficiency, let $S \in \omega(\overline{S})$ be such that above$(S, \overline{S}) \cup$ below$(\overline{S}) = \overline{S}$ and below$(S, \overline{S}) \cup$ above$(\overline{S}) = \overline{S}$. We will prove that $S \in \omega_{d'}(\overline{S})$ using Lemma 4.5. Let $\sigma \in \overline{S}$ and suppose that $\sigma \in S \cup$ above$(S, \overline{S})$. This is equivalent to $\sigma \notin$ below$(S, \overline{S})$ and hence $\sigma \in$ above$(\overline{S})$. Hence, $\sigma \cup \{n-1\}$ is a face of $C(n, d')$. In the same way, if $\sigma \in S \cup$ below$(S, \overline{S})$ we prove that $\sigma \cup \{n\}$ is a face of $C(n, d')$.

3. From the definition of S_{up}, it follows that S_{up} does not contain any cell of $\overline{S}$ (i.e. above$(S_{up}, \overline{S}) \cup$ below$(S_{up}, \overline{S}) = \overline{S}$) and also that below$(S_{up}, \overline{S})$ = below$(\overline{S})$. Putting these two facts together and using part 1, we conclude that S_{up} satisfies the conditions of part 2. The same holds for S_{down}. □

Remark 4.7. The last result can be interpreted using the following poset structure on the collection of subdivisions induced by $\overline{S}$.

Definition 4.8. *Let* St$(\overline{S})$ *be the set of subdivisions of* $C(n, d)$ *induced by* $\overline{S}$*, partially ordered by* $S_1 \leq S_2$ *if and only if* S_1 *lies below* S_2 *as a cellular section of the natural projection* $C(n, d+1) \to C(n, d)$ *or, equivalently, if* above$(S_2, \overline{S}) \subset$ above$(S_1, \overline{S})$ *and* below$(S_1, \overline{S}) \subset$ below$(S_2, \overline{S})$.

Let $\mathrm{St}_{d'}(\overline{S})$ *be the induced subposet of* $\mathrm{St}(\overline{S})$ *on the subset* $\omega_{d'}(\overline{S})$*. We call* $\mathrm{St}(\overline{S})$ *and* $\mathrm{St}_{d'}(\overline{S})$ *the* Stasheff orders *on* $\omega(\overline{S})$ *and* $\omega_{d'}(\overline{S})$*.*

The above definition reminds of the second higher Stasheff-Tamari order on the set of all triangulations of a cyclic polytope and its characterization as closed sets in dimensions 2 and 3 [7]. In this context the structure is well-behaved in all dimensions.

Using the Stasheff order, Lemma 4.6 can be rewritten as follows.

Lemma 4.9. *An element* S *of* $\omega(\overline{S})$ *is in* $\omega_{d'}(\overline{S})$ *if and only if* $S_{\mathrm{down}} \leq_{\mathrm{St}} S \leq_{\mathrm{St}} S_{\mathrm{up}}$*. Thus,* $\mathrm{St}_{d'}(\overline{S})$ *is a nonempty interval in* $\mathrm{St}(\overline{S})$*.*

It is also easy to see that $\omega_{d'}(\overline{S})$ is a lattice, where for every $S_1, S_2 \in \omega_{d'}(\overline{S})$ the join $S_1 \vee S_2$ and the meet $S_1 \wedge S_2$ are the elements satisfying

$$\begin{aligned}
\mathrm{above}(S_1 \vee S_2, \overline{S}) &:= \mathrm{above}(S_1, \overline{S}) \cap \mathrm{above}(S_2, \overline{S}),\\
\mathrm{below}(S_1 \vee S_2, \overline{S}) &:= \mathrm{below}(S_1, \overline{S}) \cup \mathrm{below}(S_2, \overline{S});\\
\mathrm{above}(S_1 \wedge S_2, \overline{S}) &:= \mathrm{above}(S_1, \overline{S}) \cup \mathrm{above}(S_2, \overline{S}),\\
\mathrm{below}(S_1 \wedge S_2, \overline{S}) &:= \mathrm{below}(S_1, \overline{S}) \cap \mathrm{below}(S_2, \overline{S}).
\end{aligned}$$

In a sense, $S_1 \vee S_2$ and $S_1 \wedge S_2$ are the common upper and lower envelopes of S_1 and S_2, except that if a cell σ is in $S_1 \cap S_2$ then σ (instead of its upper or lower envelope) is also in $S_1 \vee S_2$ and $S_1 \wedge S_2$.

In what follows we argue that the proof of [13, Theorem 4.5], showing that $\omega(\overline{S})$ is contractible, can be modified to prove that $\omega_{d'}(\overline{S})$ is contractible. The original proof is based on a total ordering of the cells of $\overline{S}$ compatible with the partial order $<_{\mathrm{st}}$. Here we also want our total order to behave nicely with respect to $\mathrm{above}(\overline{S})$ and $\mathrm{below}(\overline{S})$.

Lemma 4.10. *There is a total order, i.e. a numbering* $\overline{S} = \{\sigma_1, \ldots, \sigma_k\}$*, of the cells of* $\overline{S}$ *such that for every* $i, j \in \{1, \ldots, k\}$ *we have:*

1. *If* $\sigma_i <_{\mathrm{st}} \sigma_j$ *then* $i < j$*.*
2. *If* $\sigma_i \in \mathrm{below}(\overline{S})$ *and* $\sigma_j \notin \mathrm{below}(\overline{S})$ *then* $i < j$*.*
3. *If* $\sigma_i \in \mathrm{above}(\overline{S})$ *and* $\sigma_j \notin \mathrm{above}(\overline{S})$ *then* $i > j$*.*

Proof. We first order the cells not in $\mathrm{above}(\overline{S})$ with the numbers from 1 to k_1, in a way compatible with the partial order $<_{\mathrm{st}}$. Then we order the cells in $\mathrm{above}(\overline{S}) \cap \mathrm{below}(\overline{S})$ with numbers $k_1 + 1, \ldots, k_2$ and then those not in $\mathrm{below}(\overline{S})$ with $k_2 + 1, \ldots, k$, both times again in a way compatible with $<_{\mathrm{st}}$. This can be done since $<_{\mathrm{st}}$ is a partial order.

The order so obtained satisfies conditions 2 and 3 by construction and it also satisfies condition 1 since $\mathrm{below}(\overline{S})$ is a lower ideal in $<_{\mathrm{st}}$ (so that if $\sigma_i <_{\mathrm{st}} \sigma_j$, it is impossible that $\sigma_j \in \mathrm{below}(\overline{S})$ and $\sigma_i \notin \mathrm{below}(\overline{S})$) and $\mathrm{above}(\overline{S})$ is an upper ideal in $<_{\mathrm{st}}$ (so that if $\sigma_i <_{\mathrm{st}} \sigma_j$, it is impossible that $\sigma_i \in \mathrm{above}(\overline{S})$ and $\sigma_j \notin \mathrm{above}(\overline{S})$). ☐

The proof of the following proposition follows closely the one of [13, Theorem 4.5] but we include it for the sake of completeness. It establishes Theorem 4.2 and finishes the proof of Theorem 1.2.

Proposition 4.11. *The subposet* $\omega_{d'}(\overline{S})$ *of* $\omega(C(n-2, d-2))$ *is contractible.*

Proof. Let the cells of $\overline{S}$ be totally ordered as in Lemma 4.10, so that $\{\sigma_1, \dots, \sigma_{k_1}\} = \text{below}(\overline{S}) \setminus \text{above}(\overline{S})$, $\{\sigma_{k_1}+1, \dots, \sigma_{k_2}\} = \text{below}(\overline{S}) \cap \text{above}(\overline{S})$ and $\{\sigma_{k_2}+1, \dots, \sigma_k\} = \text{above}(\overline{S}) \setminus \text{below}(\overline{S})$.

For any $S \in \omega(\overline{S})$ we call *height* of S the maximum index i of a cell σ_i on or below S. For each $i = 0, \dots, k$ we denote by $\omega_{d'}(\overline{S}; i)$ the subposet of $\omega_{d'}(\overline{S})$ consisting of the subdivisions of height at most i.

By definition, S_{down} has height k_1 and S_{up} has height k_2. Moreover, by Lemma 4.6, $\omega_{d'}(\overline{S}) = \omega_{d'}(\overline{S}; k_2)$ and $\omega(\overline{S}; k_1)$ has only the element S_{down}. We will prove that $\omega_{d'}(\overline{S}; i)$ and $\omega_{d'}(\overline{S}; i-1)$ are homotopically equivalent for every $i = k_1 + 1, \dots, k_2$.

Consider first the following situation. Let $S \in \omega_{d'}(\overline{S})$ with $\sigma_i \in S$. We can get two new elements S_{σ_i+} and S_{σ_i-} of $\omega_{d'}(\overline{S})$ by substituting σ_i in S for its upper and lower envelope, respectively.

We now construct the homotopy equivalence $f_i : \omega_{d'}(\overline{S}; i) \to \omega_{d'}(\overline{S}; i-1)$. We define f_i to be the identity on those $S \in \omega_{d'}(\overline{S}; i)$ with height at most $i-1$. If S has height i then either S contains σ_i, in which case we take $f_i(S) = S_{\sigma_i-}$, or S contains the upper envelope of σ_i. In this case $S = T_{\sigma_i^+}$ for some $T \in \omega_{d'}(\overline{S})$. We then define $f_i(T_{\sigma_i^+}) = T_{\sigma_i^-}$. In this way, the inverse image of an element $S \in \omega_{d'}(\overline{S}; i-1)$ is given as follows:

(i) It is S itself, if S does not contain the lower envelope of σ_i.
(ii) If S contains the lower envelope of σ_i, then $S = T_{\sigma_i-}$ for some $T \in \omega_{d'}(\overline{S}; i)$ and $f^{-1}(S) = f^{-1}(T_{\sigma_i-}) = \{T, T_{\sigma_i-}, T_{\sigma_i+}\}$.

Define the following order-preserving map:

$$g_i : \begin{cases} \omega(\overline{S}; i-1) & \to \quad \omega(\overline{S}; i), \\ \qquad S & \mapsto \quad \begin{cases} S & \text{in case (i)}, \\ T & \text{in case (ii)}. \end{cases} \end{cases}$$

Then $f_i \circ g_i = \mathrm{id}_{\omega_{d'}(\overline{S}; i-1)}$ and $g_i \circ f_i \geq \mathrm{id}_{\omega_{d'}(\overline{S}; i)}$, which means that f_i and g_i are homotopy inverses to each other by Quillen's order homotopy theorem [6, 10.11]. Thus, $\omega_{d'}(\overline{S}; i)$ is homotopy equivalent to $\omega_{d'}(\overline{S}; i-1)$. □

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