Konrad-Zuse-Zentrum
für Informationstechnik Berlin

Takustraße 7
D-14195 Berlin-Dahlem
Germany

BIRKETT HUBER JÖRG RAMBAU FRANCISCO SANTOS

The Cayley Trick, lifting subdivisions and the Bohne-Dress theorem on zonotopal tilings

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THE CAYLEY TRICK, LIFTING SUBDIVISIONS AND THE BOHNE-DRESS THEOREM ON ZONOTOPAL TILINGS

BIRKETT HUBER, JÖRG RAMBAU, AND FRANCISCO SANTOS

ABSTRACT. In 1994, Sturmfels gave a polyhedral version of the Cayley Trick of elimination theory: he established an order-preserving bijection between the posets of *coherent* mixed subdivisions of a Minkowski sum $\mathcal{A}_1 + \cdots + \mathcal{A}_r$ of point configurations and of *coherent* polyhedral subdivisions of the associated Cayley embedding $\mathcal{C}(\mathcal{A}_1, \ldots, \mathcal{A}_r)$.

In this paper we extend this correspondence in a natural way to cover also *non-coherent* subdivisions. As an application, we show that the Cayley Trick combined with results of Santos on subdivisions of Lawrence polytopes provides a new independent proof of the Bohne-Dress Theorem on zonotopal tilings. This application uses a combinatorial characterization of lifting subdivisions, also originally proved by Santos.

1. INTRODUCTION

The investigations in this paper are motivated from several directions. Our point of departure is the polyhedral version of the *Cayley Trick* of elimination theory given by STURMFELS in [20, Section 5]. The Cayley Trick is originally a method to rewrite a certain resultant of a polynomial system as a discriminant of one single polynomial with additional variables [8, pp. 103ff. and Chapter 9, Proposition 1.7]. Its applications are in the area of sparse elimination theory and computation of mixed volumes [6, 9, 10, 12, 13, 22].

Mixed subdivisions of the Minkowski sum of a family $\mathcal{A}_1, \ldots, \mathcal{A}_r \subset \mathbb{R}^d$ of polytopes were introduced in [10, 13, 20]. The polyhedral Cayley Trick of Sturmfels says that *coherent* mixed polyhedral subdivisions of the Minkowski sum of $\mathcal{A}_1, \ldots, \mathcal{A}_r \subset \mathbb{R}^d$ are in one-to-one refinement-preserving correspondence to *coherent* polyhedral subdivisions of their Cayley embedding $\mathcal{C}(\mathcal{A}_1, \ldots, \mathcal{A}_r) \subset \mathbb{R}^{r-1} \times \mathbb{R}^d$. (For definitions of this and the following see Section 2.) More precisely, it establishes a strong isomorphism between certain fiber polytopes. In Theorem 3.1, we extend this isomorphism to an isomorphism between the refinement posets of *all* induced subdivisions, no matter whether coherent or not. This extension needs a more combinatorial approach than the one used in [20]. We carry it out in Section 3 after introducing the relevant concepts in Section 2.

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Our second motivation is that there are applications of the Cayley trick in specific cases which are of intrinsic interest. The most striking one is the *Bohne-Dress Theorem* [4] (see also [5, 17, 23]) about zonotopal tilings, to which we devote Section 5, after giving a preliminary result in Section 4. Other applications of the Cayley trick to triangulations of hypercubes and of products of simplices will appear in [19].

A *zonotope* is the affine projection of a hypercube, or equivalently, a Minkowski sum of segments. A *zonotopal tiling* is a subdivision induced by this projection (i.e., a subdivision into smaller zonotopes in certain conditions, see for example [23]). The Bohne-Dress Theorem states that there is a one-to-one correspondence between the zonotopal tilings of a zonotope Z and *the single-element lifts* of the oriented matroid $\mathcal{M}(Z)$ associated to Z. Our version of the Cayley trick, in turn, tells us that zonotopal tilings of Z are in one-to-one correspondence with polyhedral subdivisions of its Cayley embedding, which in this case is a *Lawrence polytope*. (Lawrence polytopes have been studied in connection to oriented matroid theory, see [5, 23], but their property of being Cayley embeddings of segments has never been pointed out before.) To close the loop, polyhedral subdivisions of a Lawrence polytope where shown to correspond to single-element lifts of the oriented matroid by SANTOS [18], via the concept of *lifting subdivisions* introduced in [5, Section 9.6]. We include a proof of this last equivalence in the realizable case (Proposition 5.2). It is based on a geometric characterization of lifting subdivisions (Theorem 4.2), also originally contained in [18], to which we devote Section 4. In this way, this paper contains a complete new proof of the Bohne-Dress Theorem (Theorem 5.1). It turns out that of the three equivalences in Theorem 5.1, the most transparent is the one given by the Cayley trick, which is exhibited in this paper for the first time.

Our final motivation concerns functorial properties of subdivision posets. Given an affine map between polytopes, can one draw conclusions about the induced map between the corresponding posets of polyhedral subdivisions? For example, the intersection of a subdivision with an affine subspace yields again a subdivision of the intersection polytope. In fact, it turns out that the isomorphism given by the Cayley Trick is exactly a map of this type. We think it would be of interest to investigate such maps in a more general framework (even if they do not produce isomorphisms), in relation to the so-called *generalized Baues problem* for polyhedral subdivisions (see [15, 16] for information on this problem).

2. PRELIMINARIES

2.1. **Subdivisions of point configurations.** By a point configuration $\mathcal{A}$ in $\mathbb{R}^d$ we mean a finite labeled subset of $\mathbb{R}^d$. We allow $\mathcal{A}$ to have repeated points which are distinguished by their labels. The convex hull $\operatorname{conv}(\mathcal{A})$ of $\mathcal{A}$ is a polytope.

A face of a subconfiguration $B \subseteq \mathcal{A}$ is a subconfiguration $F^\omega \subseteq B$ consisting of *all* the points on which some linear functional $\omega \in (\mathbb{R}^d)^*$ takes its minimum over $\mathcal{A}$. Given two subconfigurations B_1 and B_2 of $\mathcal{A}$ we say that they *intersect properly* if the following two conditions are satisfied:

- $B_1 \cap B_2$ is a face of both B_1 and B_2;

- $\mathrm{conv}(B_1) \cap \mathrm{conv}(B_2) = \mathrm{conv}(B_1 \cap B_2)$.

A subconfiguration of $\mathcal{A}$ is said to be full-dimensional if it affinely spans $\mathbb{R}^d$. In that case we call it a *cell*. It is *simplicial* if it is an affinely independent configuration. Following [2] and [8, Section 7.2] we say that a collection S of cells of $\mathcal{A}$ is a *(polyhedral) subdivision* of $\mathcal{A}$ if the elements of S intersect pairwise properly and cover $\mathrm{conv}(\mathcal{A})$ in the sense that

$$\cup_{B\in S}\mathrm{conv}(B) = \mathrm{conv}(\mathcal{A}).$$

Cells that share a common facet are *adjacent*. The set of subdivisions of $\mathcal{A}$ is partially ordered by the *refinement* relation

$$S_1 \leq S_2 \qquad :\Longleftrightarrow \qquad \forall B_1 \in S_1,\ \exists B_2 \in S_2 :\ B_1 \subset B_2.$$

The poset of subdivisions of $\mathcal{A}$ has a unique maximal element which is the trivial subdivision $\{\mathcal{A}\}$. The minimal elements are the subdivisions all of whose cells are simplicial, which are called *triangulations* of $\mathcal{A}$.

The following characterization has already been proved for triangulations by de Loera et al. in [7]. (It is a consequence of parts (i) and (ii) of their Theorem 1.1.) Here we include a proof for subdivisions, whose final part follows the proof of their Theorem 3.2.

Lemma 2.1. *Let $\mathcal{A}$ by a point configuration. Let S be a collection of cells of $\mathcal{A}$. Then, S is a subdivision if and only if the following conditions are satisfied:*

(i) There is a point in $\mathrm{conv}(\mathcal{A})$ *that is contained in the convex hull of exactly one cell of S.*
(ii) Any two adjacent cells in S lie in opposite halfspaces with respect to their common facet.
(ii) For every $B \in S$ and for every facet F of B, either F lies in a facet of $\mathrm{conv}(\mathcal{A})$ *or there is another $B' \in S$ adjacent to B in the facet F.*

Proof. If S is a subdivision, it is easy to verify that it satisfies (i), (ii), and (iii): First, no point in the relative interior of $\mathrm{conv}(B)$ for a cell $B \in S$ can lie in the convex hull of any other cell in S, or the two cells would intersect improperly. This proves (i). If two adjacent cells lie in the same side of the hyperplane supporting their common facet then they cannot intersect properly, which proves (ii). Finally, if a facet F of a cell $B \in S$ does not lie in a facet of $\mathrm{conv}(\mathcal{A})$, let p be a point *beyond* that facet (i.e., outside $\mathrm{conv}(B)$ but very close to a relative interior point of $\mathrm{conv}(F)$). Since the subdivision S covers $\mathcal{A}$, the point p has to lie in $\mathrm{conv}(B')$ for some cell $B' \in S$. The only way in which B and B' can intersect properly is being adjacent in the facet F. This proves (iii).

Let us now suppose that S satisfies (i), (ii), and (iii). We will prove that S is a subdivision. Consider the union H of all the hyperplanes spanned by subsets of $\mathcal{A}$. The connected components of $\mathrm{conv}(\mathcal{A}) \setminus H$ are called *chambers* of $\mathcal{A}$. They are open sets whose closures are convex polytopes, cover $\mathrm{conv}(\mathcal{A})$, and intersect properly. Two different points in the same chamber are contained in the same number (actually in the same collection) of convex hulls of cells of S. We call this number the *covering number* of a specific chamber.

Let C_1 and C_2 be two chambers which are adjacent (i.e., whose closures have a common facet D). Properties (ii) and (iii) imply that C_1 and C_2 have the same covering number, equal to the number of cells in S which cover both C_1 and C_2 plus the number of facets of cells of S whose convex hull contains D. (Such facets are facets of exactly one cell covering C_1 and one covering C_2.) Since any two chambers can be connected by a sequence of adjacent chambers (e.g., take generic points in the two chambers and consider the chambers which intersect the segment joining them) we conclude that all the chambers have the same covering number.

On the other hand, let p be a point satisfying the conditions in (i) and let B be the unique cell of S with $p \in \mathrm{conv}(S)$. Let C be a chamber contained in $\mathrm{conv}(B)$ and with p in its closure. Then C has covering number 1 and, thus, all the chambers have covering number 1. As a conclusion, the union $\mathrm{conv}(\mathcal{A}) \setminus H$ of all the chambers is an open dense subset of $\mathrm{conv}(\mathcal{A})$ each of whose points lies in the convex hull of exactly one cell of S. This implies in particular that S covers $\mathcal{A}$, since the subset $\cup_{B\in S}\mathrm{conv}(B)$ is closed.

Finally we prove that every pair of cells in S intersect properly. Let $B_1, B_2 \in S$. The inclusion $\mathrm{conv}(B_1 \cap B_2) \subset \mathrm{conv}(B_1) \cap \mathrm{conv}(B_2)$ always holds. For the reverse one, let F_i be the minimal face of B_i with $\mathrm{conv}(B_1) \cap \mathrm{conv}(B_2) \subset \mathrm{conv}(F_i)$, $i = 1,2$. Below we will prove F_1 is a face of B_2 too. By symmetry, F_2 is a face of B_1, which clearly implies $B_1 \cap B_2 = F_1 = F_2$. Thus, $B_1 \cap B_2$ is a common face of B_1 and B_2. From this we get $\mathrm{conv}(B_1) \cap \mathrm{conv}(B_2) \subset \mathrm{conv}(F_1) \subset \mathrm{conv}(B_1 \cap B_2)$. This finishes the proof.

Thus, we only need to prove that F_1 is a face of B_2 using the above hypotheses. For each cell $B \in S$ having F_1 as a face, consider the convex polyhedral cone

$$F_1 + pos(B - F_1) \quad = \quad \{\lambda q + (1-\lambda)p \,:\, p \in F_1, q \in B, \lambda \geq 0\}.$$

We claim that $conv(\mathcal{A})$ is contained in the union of such cones. Suppose a point b of $conv(\mathcal{A})$ lies outside their union. Then b "sees" a facet τ of some cone $F_1 + pos(B - F_1)$, where $B \in S$. Let F be the corresponding facet of B. It contains F_1. By the choice of τ, there is no $B' \in S$ having F as a facet and lying in the halfspace containing b. This violates either condition (ii) or (iii) for B.

Let a be any point in $\mathrm{conv}(B_1) \cap \mathrm{conv}(B_2)$ and in the relative interior of $\mathrm{conv}(F_1)$. (It exists since F_1 is the minimal face of B_1 covering $\mathrm{conv}(B_1) \cap \mathrm{conv}(B_2)$, which is convex.) The above implies that a neighborhood of a in $conv(\mathcal{A})$ is covered by cells in S which have F_1 as a face. Since there are generic points of $\mathrm{conv}(B_2)$ arbitrarily close to a and no generic point can be covered by two different cells in S, one of the cells having F_1 as a face is B_2. $\square$

2.2. **Induced subdivisions.** Now let $P \subset \mathbb{R}^p$ be a polytope, and let $\pi : \mathbb{R}^p \to \mathbb{R}^d$ be a linear projection map. We can consider the point configuration $\mathcal{A}$ arising from the projection of the vertex set of P. An element in $\mathcal{A}$ is labeled by the vertex of P of which it is considered to be the image. In other words, π induces a bijection from the vertex set of P into $\mathcal{A}$, even if different vertices of P have the same projection.

A subdivision S of $\mathcal{A}$ is said to be π-*induced* if every cell of S is the projection of the vertex set of a face of P. With these conditions, S contains the same information as the collection of faces of P whose vertex sets are in S. In this sense one can say that a π-induced subdivision of $\mathcal{A}$ is a polyhedral subdivision whose cells are projections of faces of P. (This statement is not very accurate; see [14, 15, 23] for an accurate definition of π-induced subdivisions in terms of faces of P.)

Every non-zero linear functional $\phi \in (\mathbb{R}^p)^*$ defines a π-induced subdivision S_ϕ as follows: ϕ gives a factorization of π into a map $(\pi,\phi) : \mathbb{R}^p \to \mathbb{R}^d \times \mathbb{R}$ and the map $\rho : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ which forgets the last coordinate. For any element $a \in \mathcal{A}$ let a_P denote the unique vertex of P of which it is considered to be the image by π. For any face F of the $(d+1)$-dimensional polytope $(\pi,\phi)(P)$ we denote by $\mathcal{A}_F$ the collection of points $\mathcal{A}_F := \{a \in \mathcal{A} : (\pi,\phi)(a_P) \in F\}$. A face F of $(\pi,\phi)(P)$ is called *lower* if its exterior normal cone contains a vector whose last coordinate is negative. With this notation, $S_\phi := \{\mathcal{A}_F \subset \mathcal{A} : F \text{ is a lower face of } (\pi,\phi)(P)\}$ is a π-induced subdivision of $\mathcal{A}$. The subdivision S_ϕ is called the π-*coherent* subdivision of $\mathcal{A}$ induced by ϕ, and a π-induced subdivision is called π-*coherent* if it equals S_ϕ for some ϕ.

Said in a more compact form, a subset $B \subset \mathcal{A}$ is a cell of S_ϕ if and only if there is a linear functional $\phi' : \mathbb{R}^d \to \mathbb{R}$ such that B is the subset of $\mathcal{A}$ where $\phi' \circ \pi + \phi$ takes its minimum value. (For example, S_ϕ is the trivial subdivision if and only if ϕ factors by π.)

The poset of π-induced subdivisions excluding the trivial one is denoted by $\omega(P,\pi)$. The minimal elements in it are the subdivisions for which every cell comes from a $dim(\mathcal{A})$-dimensional face of P. They are called *tight* π-induced subdivisions. The subposet of π-coherent subdivisions is denoted by $\omega_{\mathrm{coh}}(P,\pi)$. It is isomorphic to the face lattice of a certain polytope of dimension $dim(P) - dim(\mathcal{A})$, called the *fiber polytope* $\Sigma(P,\pi)$.

See [1, 23] for more information on π-induced subdivisions and fiber polytopes.

2.3. **Weighted Minkowski sums. Mixed subdivisions.** Let $\mathcal{A}_i := \{a_i^{(1)}, \dots, a_i^{(m_i)}\}$ be point configurations in $\mathbb{R}^d$.

Their *Minkowski sum* $\sum_{i=1}^r \mathcal{A}_i$ is defined to be the set of all points which can be expressed as a sum of a point from each $\mathcal{A}_i$, i.e.,

$$\sum_{i=1}^r \mathcal{A}_i := \{\, a_1 + \cdots + a_r \,:\, a_i \in \mathcal{A}_i \,\}.$$

A vector $\lambda = (\lambda_1, \dots, \lambda_r)$ in $\mathbb{R}^{r-1}$ with $\sum_{i=1}^r \lambda_i = 1$ and $0 < \lambda_1, \dots, \lambda_r < 1$ is a *weight vector*. For a weight vector λ the *weighted* Minkowski sum is defined by

$$\sum_{i=1}^r \lambda_i \mathcal{A}_i := \{\, \lambda_1 a_1 + \cdots + \lambda_r a_r \,:\, a_i \in \mathcal{A}_i \,\}.$$

The configuration $\sum_{i=1}^r \lambda_i \mathcal{A}_i$ has $\prod_{i=1}^r m_i$ points, some perhaps repeated.

A cell (i.e., full-dimensional subset) $B \subset \sum_{i=1}^r \lambda_i \mathcal{A}_i$ will be called a *Minkowski cell* if $B = \lambda_1 B_1 + \cdots + \lambda_r B_r$ for some non-empty subsets $B_i \subset \mathcal{A}_i$, $i = 1, \dots, r$. A *mixed subdivision* of the weighted Minkowski sum of $\mathcal{A}_1, \dots, \mathcal{A}_r$ is a subdivision of the configuration $\sum_{i=1}^r \lambda_i \mathcal{A}_i$ whose faces are all Minkowski cells. (There is not complete agreement in the literature concerning this definition. See Remark 2.4.) Minkowski cells are called *fine* if it does not properly contain any other Minkowski cell. A mixed subdivision is *fine* if all its faces are fine.

We can consider the *cartesian product* of point configurations as a Minkowski sum where all the point configurations lie in complementary affine subspaces. This leads to the following natural projection.

Definition 2.2 (Weighted Minkowski Projection)**.** *Let $\mathcal{A}_1,\dots,\mathcal{A}_r$ be point configurations in $\mathbb{R}^d$, and let $P_1,\dots,P_r$ be polytopes in $\mathbb{R}^{p_1},\dots,\mathbb{R}^{p_r}$, resp., the vertex sets of which affinely project to $\mathcal{A}_1,\dots,\mathcal{A}_r$ via*

$$\mathcal{P}_i := \operatorname{vert}(P_i) \xrightarrow{\pi_i} \mathcal{A}_i, \quad 1 \le i \le r.$$

Moreover, let $\lambda = (\lambda_1,\dots,\lambda_r)$ be a weight vector. We define

$$\lambda\Pi_M := \lambda_1\pi_1 + \cdots + \lambda_r\pi_r : \begin{cases} \mathcal{P}_1 \times \cdots \times \mathcal{P}_r & \to & \lambda_1\mathcal{A}_1 + \cdots + \lambda_r\mathcal{A}_r, \\ (p_1,\dots,p_r) & \mapsto & \lambda_1\pi_1(p_1) + \cdots + \lambda_r\pi_r(p_r); \end{cases}$$

The projection $\lambda\Pi_M$ is specially interesting if the polytopes P_i involved are simplices. The proof of the following fact is just a check of definitions.

Lemma 2.3. *Suppose that the polytopes P_i of Definition 2.2, are all simplices. Then, a subdivision of $\lambda_1\mathcal{A}_1 + \cdots + \lambda_r\mathcal{A}_r$ is (fine) mixed if and only if it is (tight) $\lambda\Pi_M$-induced.*

Remark 2.4. There is some confusion in the literature concerning the definition of mixed subdivisions of the Minkowski sum $\sum_{i=1}^r \mathcal{A}_i$ of the family of point configurations $\{\mathcal{A}_1,\dots,\mathcal{A}_r\}$. First of all, in most of the literature it is assumed that the number of configurations equals the dimension of the ambient space (i.e., $d = r$) because this is the case in the applications to zero-dimensional polynomial systems. However, the geometric proofs involved work the same without this assumption.

Pedersen and Sturmfels [13, page 380] defined mixed subdivisions to be the subdivisions Π_M-induced for the projection $\Pi_M : \mathcal{P}_1 \times \cdots \times \mathcal{P}_r \to \mathcal{A}_1 + \cdots + \mathcal{A}_r$ of our Lemma 2.3. Sturmfels [20, page 213] defined coherent mixed subdivisions as the ones which are Π_M-coherent. This is the same as we do. However, for the applications it is interesting to pose the following additional property: that in every cell $B = B_1 + \cdots + B_r$ of the subdivision the different B_i's lie in complementary subspaces. (This assumption allows to compute the *mixed volume* of $\mathcal{A}_1 + \cdots + \mathcal{A}_r$ by summing up the volumes of some cells of the subdivision.) It seems that Pedersen and Sturmfels [13] implicitly assume that all mixed subdivisions have this property, since they say (p. 380) “the mixed volume … is the sum of volumes of the parallelotopes in Δ”. In [20] the additional property is explicitly mentioned and said to hold for all *fine* mixed subdivisions (which are called *tight* there). In other literature the property is taken as part of the definition of mixed subdivision [10, 12]; Π_M-induced subdivisions without this property are just called *subdivisions* of the r-tuple $(\mathcal{A}_1,\dots,\mathcal{A}_r)$.

Finally, there seems to be agreement to call *tight* subdivisions the minimal elements in the poset of subdivisions induced by a projection in general [1, 15, 16, 23] and *fine mixed* those for the particular case of mixed subdivisions [10, 12], with the exception of [20] mentioned above. We have chosen to follow this convention.

2.4. **The Cayley embedding.** We call the *Cayley embedding* of $\mathcal{A}_1,\dots,\mathcal{A}_r$ the following point configuration in $\mathbb{R}^{r-1}\times\mathbb{R}^d$. Let $e_1,\dots,e_r$ be a fixed affine basis in $\mathbb{R}^{r-1}$ and $\mu_i:\mathbb{R}^d\to\mathbb{R}^{r-1}\times\mathbb{R}^d$ be the affine inclusion given by $\mu_i(x)=(e_i,x)$. Then we define

$$\mathcal{C}(\mathcal{A}_1,\dots,\mathcal{A}_r) := \cup_{i=1}^r \mu_i(\mathcal{A}_i)$$

The Cayley embedding of point configurations from complementary affine subspaces equals the *join product* of the point configurations. (For the purpose of this paper we can define the join product $\mathcal{P}_1*\dots*\mathcal{P}_r$ of several point configurations with $\mathcal{P}_i\subset\mathbb{R}^{p_i}$ to be their Cayley embedding $\mathcal{C}(\mathcal{P}_1,\dots,\mathcal{P}_r)\subset\mathbb{R}^{r-1}\times\mathbb{R}^{p_1}\times\dots\times\mathbb{R}^{p_r}$.) Hence, we have the following natural projection.

Definition 2.5 (Cayley Projection)**.** *Let $\mathcal{A}_1,\dots,\mathcal{A}_r$ be point configurations in $\mathbb{R}^d$, and let $P_1,\dots,P_r$ be polytopes in $\mathbb{R}^{p_1},\dots,\mathbb{R}^{p_r}$, resp., the vertex sets of which affinely project to $\mathcal{A}_1,\dots,\mathcal{A}_r$ via*

$$\mathcal{P}_i := \mathrm{vert}(P_i) \xrightarrow{\pi_i} \mathcal{A}_i, \quad 1\le i\le r.$$

Define

$$\Pi_C := \mathcal{C}(\pi_1,\dots,\pi_r) : \begin{cases} \mathcal{P}_1*\dots*\mathcal{P}_r & \to & \mathcal{C}(\mathcal{A}_1,\dots,\mathcal{A}_r), \\ (e_i,p_i) & \mapsto & (e_i,\pi_i(p_i)). \end{cases}$$

Again, the following lemma is obvious since a join of simplices is a simplex.

Lemma 2.6. *If $\mathcal{P}_i$ is a simplex for all $1\le i\le r$ then every subdivision of $\mathcal{C}(\mathcal{A}_1,\dots,\mathcal{A}_r)$ is Π_C induced.*

3. The Cayley Trick

In this section we state and prove the Cayley Trick for induced subdivisions.

Theorem 3.1 (The Cayley Trick for Induced Subdivisions)**.** *Let $\mathcal{A}_1,\dots,\mathcal{A}_r$ be point configurations in $\mathbb{R}^d$. Moreover, let $P_1,\dots,P_r$ be polytopes in $\mathbb{R}^{p_1},\dots,\mathbb{R}^{p_r}$, resp., the vertex sets of which affinely project to $\mathcal{A}_1,\dots,\mathcal{A}_r$ via*

$$\mathcal{P}_i := \mathrm{vert}(P_i) \xrightarrow{\pi_i} \mathcal{A}_i, \quad 1\le i\le r.$$

Then for all weight vectors $\lambda=\lambda_1,\dots,\lambda_r$ there are the following isomorphisms of posets:

$$\omega(\mathcal{P}_1\times\dots\times\mathcal{P}_r, \lambda_1\pi_1+\dots+\lambda_r\pi_r) \cong \omega(\mathcal{P}_1*\dots*\mathcal{P}_r, \mathcal{C}(\pi_1,\dots,\pi_r));$$

$$\omega_{\mathrm{coh}}(\mathcal{P}_1\times\dots\times\mathcal{P}_r, \lambda_1\pi_1+\dots+\lambda_r\pi_r) \cong \omega_{\mathrm{coh}}(\mathcal{P}_1*\dots*\mathcal{P}_r, \mathcal{C}(\pi_1,\dots,\pi_r)).$$

The second of the two equivalences above follows from [20, Theorem 5.1] and is stated only for completeness. The structure of the proof of the first one is as follows: first, we represent the Minkowski sum as a section of the Cayley embedding, then we define an explicit order-preserving map that carries the isomorphism. Finally, we show that the canonical inverse construction is well-defined and order-preserving. A "guide line" of the proof is indicated in Figure 1.

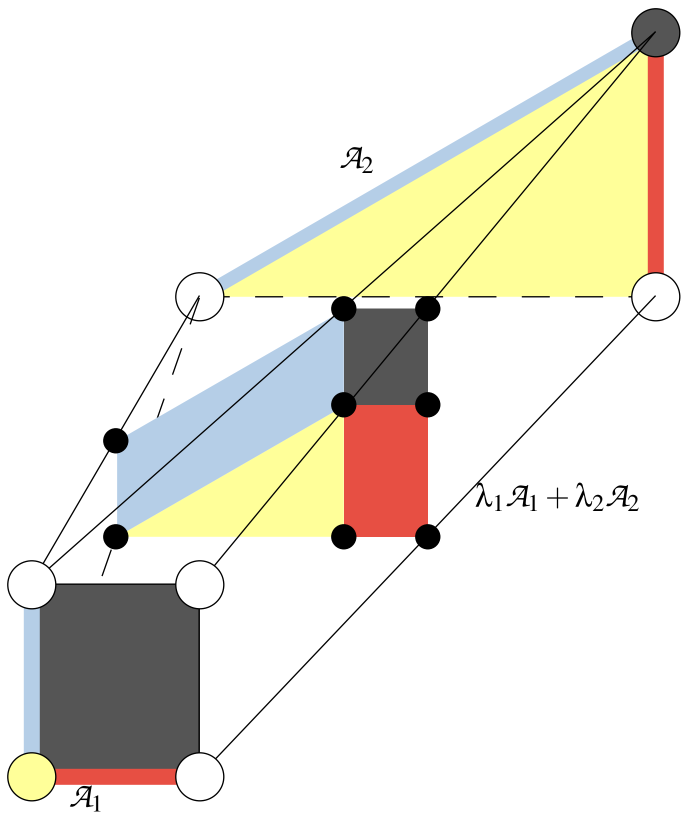

FIGURE 1. A "One-picture-proof" of the Cayley Trick.

Lemma 3.2. *Let $\mathcal{A}_1,\dots,\mathcal{A}_r\subset\mathbb{R}^d$ be point configurations. Moreover, let $\lambda=(\lambda_1,\dots,\lambda_r)$ be a weight vector. (Recall this implies that $\lambda_i>0\ \forall i$ and $\sum_{i=1}^r\lambda_i=1$.) Moreover, let $W(\lambda):=\{\lambda_1e_1+\cdots+\lambda_re_r\}\times\mathbb{R}^d\subset\mathbb{R}^{r-1}\times\mathbb{R}^d$.*

Then the scaled Minkowski sum $\lambda_1\mathcal{A}_1+\cdots+\lambda_r\mathcal{A}_r\subset\mathbb{R}^d$ has the following representation as a section of the Cayley embedding $\mathcal{C}(\mathcal{A}_1,\dots,\mathcal{A}_r)$ in $\mathbb{R}^{r-1}\times\mathbb{R}^d$:

$$\begin{aligned}&\lambda_1\mathcal{A}_1+\cdots+\lambda_r\mathcal{A}_r\cong\mathcal{C}(\mathcal{A}_1,\dots,\mathcal{A}_r)\wedge W(\lambda)\\&\qquad:=\Big\{\operatorname{conv}\big\{(e_1,a_1),\dots,(e_r,a_r)\big\}\cap W(\lambda):(e_1,a_1),\dots,(e_r,a_r)\in\mathcal{C}(\mathcal{A}_1,\dots,\mathcal{A}_r)\Big\},\end{aligned}$$

Moreover, F is a facet of $\lambda_1\mathcal{A}_1+\cdots+\lambda_r\mathcal{A}_r$ if and only if it is of the form $F=F'\wedge W(\lambda)$ for a facet F' of $\mathcal{C}(\mathcal{A}_1,\dots,\mathcal{A}_r)$ containing at least one point (e_i,a_i) for all $1\le i\le r$.

Remark 3.3. *On the level of convex hulls the above representation for the Minkowski sum polytope is nothing else but the ordinary intersection of the Cayley embedding polytope with the affine subspace $W(\lambda)$. We need the slightly more complicated version for point configurations stated above because in convex hulls—as subsets of a Euclidean space—we cannot keep track of multiple points.*

Proof of Lemma 3.2. Define $q_e(\lambda) := \lambda_1 e_1 + \cdots + \lambda_r e_r \in \mathbb{R}^{r-1}$, so that

$$W(\lambda) = \{q_e(\lambda)\} \times \mathbb{R}^d.$$

Analogously, for any sequence $a = (a_1, \ldots, a_r)$ of points with $a_i \in \mathcal{A}_i$ we set $q_a(\lambda) := \lambda_1 a_1 + \cdots + \lambda_r a_r \in \mathbb{R}^d$. Then the intersection point $\operatorname{conv}\big((e_1,a_1),\ldots,(e_r,a_r)\big) \cap W(\lambda)$ equals $(q_e(\lambda), q_a(\lambda)) \in \mathbb{R}^{r-1} \times \mathbb{R}^d$. But this is, by definition, a point in the scaled Minkowski sum—via the natural identification $W(\lambda) \cong \{q_e(\lambda)\} \times \mathbb{R}^d = W(\lambda)$—and every point in the Minkowski sum has this description.

The remark about the facets follows from the fact that a facet F' of $\mathcal{C}(\mathcal{A}_1,\ldots,\mathcal{A}_r)$ in $\mathbb{R}^{r-1} \times \mathbb{R}^d$ intersects $W(\lambda)$ if and only if it contains at least one point (e_i, a_i) for each $1 \le i \le r$ and that a linear functional is minimized on F' over $\mathcal{C}(\mathcal{A}_1,\ldots,\mathcal{A}_r)$ if and only if its projection to $W(\lambda)$ is minimized on $F \wedge W(\lambda)$. ☐

In order to keep the notation lean, we identify the embedding of the weighted Minkowski sum into $\mathbb{R}^{r-1} \times \mathbb{R}^d$ in the previous proof with the ordinary weighted Minkowski sum. The Cayley embedding $\mathcal{C}(\mathcal{A}_1,\ldots,\mathcal{A}_r)$ corresponding to the weighted Minkowski sum $\lambda_1\mathcal{A}_1 + \cdots + \lambda_r\mathcal{A}_r$ will be denoted by $(\lambda_1\mathcal{A}_1 + \cdots + \lambda_r\mathcal{A}_r) \vee W(\lambda)$. That is, we have

$$(\lambda_1\mathcal{A}_1 + \cdots + \lambda_r\mathcal{A}_r) \vee W(\lambda) = \mathcal{C}(\mathcal{A}_1,\ldots,\mathcal{A}_r),$$
$$\mathcal{C}(\mathcal{A}_1,\ldots,\mathcal{A}_r) \wedge W(\lambda) = \lambda_1\mathcal{A}_1 + \cdots + \lambda_r\mathcal{A}_r.$$

Of course, this notation extends to subconfigurations as well.

The following proposition states that the "intersection" with $W(\lambda)$ induces an order-preserving map from $\omega(\mathcal{P}_1 * \cdots * \mathcal{P}_r, \Pi_C)$ to $\omega(\mathcal{P}_1 \times \cdots \times \mathcal{P}_r, \lambda\Pi_M)$.

Proposition 3.4. *Let S be a Π_C-induced subdivision of $\mathcal{C}(\mathcal{A}_1,\ldots,\mathcal{A}_r)$ and*

$$S \wedge W(\lambda) := \{B \wedge W(\lambda) : B \in S\}.$$

Then

(i) $S \wedge W(\lambda)$ is a $\lambda\Pi_M$-induced subdivision of $\lambda_1\mathcal{A}_1 + \cdots + \lambda_r\mathcal{A}_r$;
(ii) $S < S'$ implies $(S \wedge W(\lambda)) < (S' \wedge W(\lambda))$;
(iii) $S \wedge W(\lambda)$ is tight if S is tight;
(iv) $S \wedge W(\lambda)$ is Π_C-coherent if S is $\lambda\Pi_M$-coherent.

Proof. Every cell B in a subdivision of a Cayley embedding is again a Cayley embedding. Therefore, by Lemma 3.2, $B \wedge W(\lambda)$ is a mixed subconfiguration in the Minkowski sum. Since for a cell in a Π_C-induced subdivision S of $\mathcal{C}(\mathcal{A}_1,\ldots,\mathcal{A}_r)$ to be full-dimensional it must contain a point (e_i, a_i) with $a_i \in \mathcal{A}_i$ for every $1 \le i \le r$, every cell in S intersects $W(\lambda)$ in a full-dimensional subconfiguration of $\lambda_1\mathcal{A}_1 + \cdots + \lambda_r\mathcal{A}_r$, thus defining a cell. This cell is clearly a projection of a face of the product $\mathcal{P}_1 \times \cdots \times \mathcal{P}_r$ under $\lambda\Pi_M$.

The incidence structure and proper intersections are not affected by intersection with $W(\lambda)$ by Lemma 3.2. Hence, by Lemma 2.1 we get (i).

Property (ii) is obvious, (iv) is part of [20, Theorem 5.1] and (iii) follows from (ii). ☐

The following proposition provides the inverse order-preserving map. Its proof is not difficult but nevertheless non-trivial; the extension of the polyhedral Cayley Trick from coherent to general induced subdivisions requires ingredients that are not necessary for the coherent case.

Proposition 3.5. *Let S be a $\lambda\Pi_M$-induced subdivision of $\lambda_1\mathcal{A}_1 + \cdots + \lambda_r\mathcal{A}_r$ and*

$$S \vee W(\lambda) := \{B \vee W(\lambda) : B \in S\}.$$

Then

(i) $S \vee W(\lambda)$ is a Π_C-induced subdivision;
(ii) $S < S'$ implies $(S \wedge W(\lambda)) < (S' \wedge W(\lambda))$;
(iii) $S \vee W(\lambda)$ is tight if S is tight;
(iv) $S \vee W(\lambda)$ is coherent if S is coherent.

Proof. Again, properties (ii) and (iii) are obvious, and (iv) follows from [20].

In order to prove (i), let S be a $\lambda\Pi_M$-induced subdivision of $\lambda_1\mathcal{A}_1 + \cdots + \lambda_r\mathcal{A}_r$. For every cell B in S there is a unique cell $B \vee W(\lambda)$ in $\mathcal{C}(\mathcal{A}_1, \ldots, \mathcal{A}_r)$ with $B \vee W(\lambda) \wedge W(\lambda) = B$. Let $W'(\lambda) = \{q_e(\lambda)\} \times \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_r}$ be the fiber of $W(\lambda)$ under $\Pi_C : \mathbb{R}^{r-1} \times \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_r} \to \mathbb{R}^{r-1} \times \mathbb{R}^d$. The cell B is a projection of a face F of $\mathcal{P}_1 \times \cdots \times \mathcal{P}_r$, and therefore the face $F \vee W'(\lambda)$ of $\mathcal{P}_1 * \cdots * \mathcal{P}_r$—recall that this equals $\mathcal{P}_1 \times \cdots \times \mathcal{P}_r \vee W'(\lambda)$—projects to $B \vee W(\lambda)$.

For the collection of cells $S \vee W(\lambda)$ we need to show—by Lemma 2.1—that

(i) there is a point in $\operatorname{conv} \mathcal{C}(\mathcal{A}_1, \ldots, \mathcal{A}_r)$ that is contained in exactly one cell of $S \vee W(\lambda)$
(ii) adjacent cells lie on different sides of the hyperplane that supports their common facet;
(iii) for every facet F of a cell $B \in S \vee W(\lambda)$ either F is contained in a facet of the configuration $\mathcal{C}(\mathcal{A}_1, \ldots, \mathcal{A}_r)$ or there is another cell $B' \in S$ containing F as a facet.

First, we prove (i). Since the Minkowski sum is contained in the Cayley embedding as a section and S is a subdivision of the Minkowski sum, i.e., S satisfies conditions (i), (ii), and (iii), we find a point $p \in \operatorname{conv}(\lambda_1\mathcal{A}_1 + \cdots + \lambda_r\mathcal{A}_r)$ that is contained in the convex hull $\operatorname{conv} B$ of exactly one cell B of S. Therefore, p is uniquely contained in $\operatorname{conv}(B \vee W(\lambda)) \supset \operatorname{conv} B$ where $B \vee W(\lambda) \in S \vee W(\lambda)$, which completes (i). Let $B_1 \vee W(\lambda)$ and $B_2 \vee W(\lambda)$ be two adjacent cells in $S \vee W(\lambda)$ with common facet F. Let H be the hyperplane supporting F. We show that $B_1 \vee W(\lambda)$ and $B_2 \vee W(\lambda)$ lie on different sides of H, which proves (ii). To this end, assume $B_1 \vee W(\lambda)$ and $B_2 \vee W(\lambda)$ lie on the same side of H. Then $B_1 = B_1 \vee W(\lambda) \wedge W(\lambda)$ and $B_2 = B_2 \vee W(\lambda) \wedge W(\lambda)$ lie on the same side of $H \wedge W(\lambda)$ while $F \wedge W(\lambda)$ is the common facet of B_1 and B_2, supported by $H \cap W(\lambda)$: contradiction to (ii) for S.

In order to prove (iii) we only need to observe that incidences are preserved by $\vee W(\lambda)$. ☐

See Figure 2 for an illustration of the situation.

Remark 3.6. *It is not true in general that a proper intersection of non-adjacent cells in the Minkowski sum implies a proper intersection of the corresponding cells in the Cayley embedding. See Figure 3 for an easy example.*

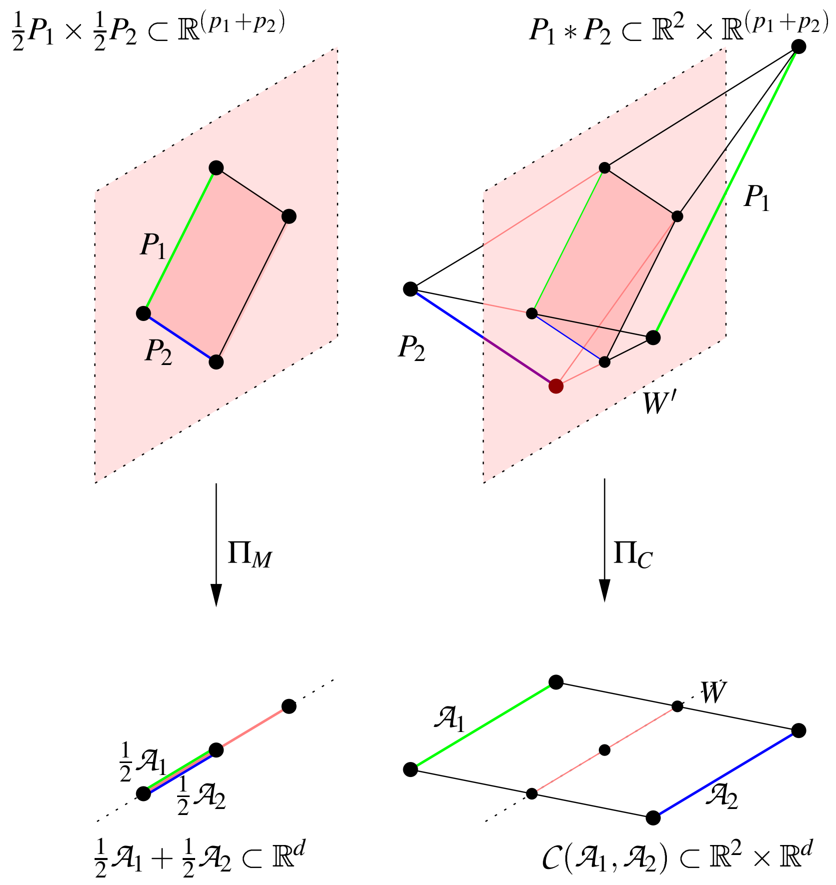

FIGURE 2. Affine picture for $r = 2$ and $P_1 = P_2 = [0,1]$: product and Minkowski sum are intersections of join resp. Cayley embedding with the affine subspace $W = \{x_1 = x_2, x_1 + x_2 = 1\}$.

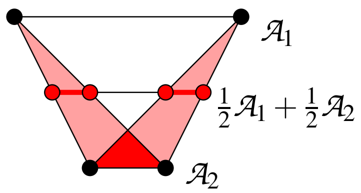

FIGURE 3. Two properly intersecting cells in the Minkowski sum whose counterparts in the Cayley embedding intersect improperly.

Propositions 3.4 and 3.5 imply Theorem 3.1. This one, in turn, has the following corollaries. The first one is straightforward.

Corollary 3.7. *Weighted Minkowski sums $\sum_{i=1}^{r} \lambda_i A_i$ of a point configuration $\mathcal{A}_1, \ldots, \mathcal{A}_r$ have isomorphic posets of subdivisions for all weights λ.*

In the following result we call *geometric (polyhedral) subdivision* of a convex polytope $\mathcal{P}$ a family of polytopes contained in $\mathcal{P}$ which cover $\mathcal{P}$ and intersect properly. If

$\mathcal{P} = \text{conv}(\mathcal{A})$ for a point configuration $\mathcal{A}$ then any subdivision S of $\mathcal{A}$ has an associated geometric subdivision $\{\text{conv}(B) : B \in S\}$ of $\mathcal{P}$. Reciprocally, a geometric subdivision K of $\mathcal{P}$ equals $\{\text{conv}(B) : B \in S\}$ for some subdivision S of $\mathcal{A}$ if and only if every element of K has vertex set contained in $\mathcal{A}$ (but the subdivision S of $\mathcal{A}$ is not unique, in general).

Given a family $\mathcal{A}_1, \dots, \mathcal{A}_r$ of point configurations and a geometric subdivision K of the polytope $\text{conv}(\sum_{i=1}^r \lambda_i \mathcal{A}_i)$ we say that K is *mixed* if there is a mixed subdivision S of $\sum_{i=1}^r \lambda_i \mathcal{A}_i$ with $K = \{\text{conv}(B) : B \in S\}$. A necessary condition for this to happen is that each polytope Q in K can be written as $Q = \text{conv}(\sum_{i=1}^r \lambda_i B_i\})$ for certain subsets $B_i \subset \mathcal{A}_i$, $i = 1, \dots, r$. But this condition is not sufficient, as the following example shows: Consider the Minkowski sum of two squares of side 1 divided into four squares of side 1. There are 96 ways of introducing two diagonals in the four squares, and all of them provide geometric subdivisions satisfying the extra condition. But not all are mixed. (In this example a necessary and sufficient condition is that the two diagonals be drawn in non-adjacent squares.)

Corollary 3.8. *Let $\mathcal{A}_1, \dots, \mathcal{A}_r$ be a family of point configurations, and let K, K' be geometric subdivisions of* $\text{conv}(\sum_{i=1}^r \lambda_i A_i)$. *Suppose that K is a refinement of K' (i.e., every cell of K' is a union of cells of K) and that K is mixed. Then K' is mixed too.*

Proof. An easy consequence of Theorem 3.1 is that a geometric subdivision of the geometric Minkowski sum $\text{conv}(\sum_{i=1}^r \lambda_i A_i)$ is mixed if and only if it is the intersection of the geometric subdivision of $\text{conv}(\mathcal{C}(\mathcal{A}_1, \dots, \mathcal{A}_r))$ associated to some subdivision of $\mathcal{C}(\mathcal{A}_1, \dots, \mathcal{A}_r)$ with the affine subspace $W(\lambda)$.

We suppose that K is the intersection with $W(\lambda)$ of a geometric subdivision $\overline{K}$ of $\text{conv}(\mathcal{C}(\mathcal{A}_1, \dots, \mathcal{A}_r))$ and that $\overline{K}$ equals $\{\text{conv}(B) : B \in \overline{S}\}$ for some subdivision $\overline{S}$ of $\mathcal{C}(\mathcal{A}_1, \dots, \mathcal{A}_r)$. Let $K = \{Q_1, \dots, Q_k\}$, $K' = \{Q'_1, \dots, Q'_l\}$ and $\overline{K} = \{\overline{Q}_1, \dots, \overline{Q}_k\}$ with $Q_i = \overline{Q_i} \cap W(\lambda)$ for each $i = 1, \dots, k$.

Since K refines K', for each $j = 1, \dots, l$ we can write Q'_j as a union of some of the Q_i's. We define $\overline{Q'_j}$ to be the union of the corresponding $\overline{Q}_i$'s, and let $\overline{K'} := \{\overline{Q'_1}, \dots, \overline{Q'_l}\}$. We claim that $\overline{K'}$ is a geometric subdivision of $\text{conv}(\mathcal{C}(\mathcal{A}_1, \dots, \mathcal{A}_r))$. If this is true then it is obvious that $\overline{K'}$ is the geometric subdivision associated to some subdivision $\overline{S'}$ of $\mathcal{C}(\mathcal{A}_1, \dots, \mathcal{A}_r)$ and that K' is the intersection of $\overline{K'}$ with $W(\lambda)$, which finishes the proof.

The only non-obvious parts in the claim are that the unions $\overline{Q'_j}$ are convex and that they intersect pairwise properly. We prove these two facts in the following lemma. $\square$

Lemma 3.9. *Let K be a geometric subdivision of the geometric Cayley embedding* $\text{conv}(\mathcal{C}(\mathcal{A}_1, \dots, \mathcal{A}_r))$. *Let Q and R denote unions of cells in K.*

1. *If there is a weight vector λ for which $Q \cap W(\lambda)$ is convex, then Q is convex.*
2. *Suppose Q and R are convex. If there is a weight vector λ_0 for which $Q \cap W(\lambda_0)$ and $R \cap W(\lambda_0)$ intersect properly then Q and R intersect properly.*

Proof. 1. Let $Q = \{Q_1, \dots, Q_l\}$ where the Q_i's are cells in the subdivision K. Since the Q_i's intersect properly, for every weight vector λ the intersections $Q_1 \cap W(\lambda), \dots, Q_l \cap W(\lambda)$ intersect properly. Also, the polytopes $Q_i \cap W(\lambda)$ for different values of λ are normally equivalent. Thus, if $Q_i \cap W(\lambda_0)$ and $Q_j \cap W(\lambda_0)$ share a face then $Q_i \cap W(\lambda)$

and $Q_j \cap W(\lambda)$ must share "the same" face for every λ (or otherwise Q_i and Q_j intersect improperly). This implies that $Q \cap W(\lambda_0)$ and $Q \cap W(\lambda)$ are combinatorially equivalent polyhedral complexes and their boundaries are combinatorially and normally equivalent convex polytopes. Even more, their faces are labeled in the same (unique) way as intersections of faces of Q with $W(\lambda_0)$ and $W(\lambda)$ respectively. In particular, $Q \cap W(\lambda)$ is a convex polytope for every λ.

Suppose now that Q is not convex. Let p and q be points in Q such that the segment $[p,q]$ is not contained in Q and sufficiently generic so that $[p,q]$ intersects the boundary of Q in the relative interior of a facet F of Q. Let F^+ be the exterior open halfspace to that facet. One of p and q is in F^+, suppose that it is p and let λ be the weight for which $p \in W(\lambda)$. Then, $F^+ \cap W(\lambda)$ is the halfspace exterior to the facet $F \cap W(\lambda)$ of $Q \cap W(\lambda)$ and $p \in F^+ \cap W(\lambda)$. This means $p \notin Q \cap W(\lambda)$, a contradiction.

2. Let $F_0 = Q \cap R \cap W(\lambda_0)$ be the common face in which $Q \cap W(\lambda_0)$ and $R \cap W(\lambda_0)$ intersect. F_0 can be expressed as a union $(F_1 \cup \cdots \cup F_k) \cap W(\lambda_0)$ where each F_i is a face of one of the Q_j's in K whose union equals Q. This expression is unique (up to reordering) if it is not redundant (i.e., if $F_i \cap W(\lambda_0)$ has the same dimension as F_0 for every i). In the same way, $F_0 = (G_1 \cup \cdots \cup G_{k'}) \cap W(\lambda_0)$, where the G_i's are now faces of the cells of K whose union is R. The fact that the F_i's and G_j's intersect properly (since they are all faces of cells of the subdivision K) together with $(F_1 \cup \cdots \cup F_k) \cap W(\lambda_0) = (G_1 \cup \cdots \cup G_{k'}) \cap W(\lambda_0)$ for the weight λ_0 implies that each F_i equals a G_j and vice versa. Thus, Q and R intersect properly, in the face $F_1 \cup \cdots \cup F_k = G_1 \cup \cdots \cup G_{k'}$. □

4. Lifting subdivisions

Throughout this section let $\mathcal{A} = \{a_1, \ldots, a_n\} \subset \mathbb{R}^d$ be a fixed point configuration of dimension d, and let $\mathcal{M}$ denote the oriented matroid of affine dependences of $\mathcal{A}$, which has rank $d+1$ and ground set $\{1, \ldots, n\}$. A *lift* of $\mathcal{M}$ is an oriented matroid $\widehat{\mathcal{M}}$ of rank $d+2$ with ground set $\{1, \ldots, n+1\}$ which satisfies $\widehat{\mathcal{M}}/(n+1) = \mathcal{M}$.

Every lift $\widehat{\mathcal{M}}$ of $\mathcal{M}$ induces a subdivision $S_{\widehat{\mathcal{M}}}$ of $\mathcal{A}$ as follows: a subset $\sigma \subset \{1, \ldots, n\}$ is (the set of indices of the elements of) a cell in S if and only if σ is a facet of $\widehat{\mathcal{M}}$ not containing $n+1$ (a *facet* in an oriented matroid is the complement of a positive cocircuit [5, Chapter 9]). The subdivisions of $\mathcal{A}$ which can be obtained in this way are called *lifting subdivisions*. They were formally introduced in [5, Section 9.6], with some of the ideas coming from [3]. The process is a combinatorial abstraction (as well as a generalization) of the definition of *regular subdivisions* of $\mathcal{A}$. In particular, every regular subdivision of $\mathcal{A}$ is a lifting subdivision. The converse is not true since a subdivision being regular or not does not depend only on the oriented matroid $\mathcal{M}$ of affine dependences of $\mathcal{A}$.

This section is devoted to providing a characterization of lifting subdivisions of $\mathcal{A}$ which does not explicitly involve the oriented matroid $\mathcal{M}$. The results of this section come from [18], where they are proved in a more general context: the oriented matroid $\mathcal{M}$ involved needs not be realizable. (A concept of subdivision of a non-realizable oriented matroid was also introduced in [5, Section 9.6].) We include here a proof in the realized case for completeness.

Definition 4.1. *Let S be a subdivision of the point configuration $\mathcal{A}$. For each subset $B \subset \mathcal{A}$, let S_B be a subdivision of $\mathcal{A}$. We say that the family of subdivisions $\mathcal{S} = \{S_B\}_{B \in \mathcal{A}}$ is* consistent *if for every subset $B \subset \mathcal{A}$ the following happens:*

(i) For every cell $\tau \in S_B$ and for every $B' \subset B$, $\tau \cap B'$ is a face of a cell of $S_{B'}$.

(ii) For every affine basis σ of $\mathbb{R}^d$ contained in B if σ is contained in a cell of $S_{\sigma \cup \{b\}}$ for every $b \in B \setminus \sigma$, then σ is contained in a cell of S_B as well.

We say that the family is consistent with S *if, moreover, $S = S_{\mathcal{A}}$.*

Condition (i) says that the subcomplex of S_B induced by the elements of any $B' \subset B$ is a subcomplex of $S_{B'}$. Condition (ii) is void unless B affinely spans $\mathcal{A}$ and has at least $d+3$ elements. The main result of this section is:

Theorem 4.2. *Let S be a subdivision of a point configuration $\mathcal{A}$. Then, the following conditions are equivalent:*

(i) S is a lifting subdivision.

(ii) There is a family $\mathcal{S}$ of subdivisions of the subsets of $\mathcal{M}$ which is consistent with S.

Lifts of an oriented matroid $\mathcal{M}$ are duals to the extensions of the dual oriented matroid $\mathcal{M}^*$ and vice versa. The following statements are the dualized version of results by Las Vergnas [11] on extensions of oriented matroids (see also [5, Section 7.1]): If $(\widehat{\mathcal{M}}, \{1,\ldots,n+1\})$ is a lift of $(\mathcal{M}, \{1,\ldots,n\})$, then for every circuit $C = (C^+, C^-)$ of $\mathcal{M}$ precisely one of $(C^+ \cup \{n+1\}, C^-)$, $(C^+, C^- \cup \{n+1\})$, and (C^+, C^-) is a circuit of $\widehat{\mathcal{M}}$. Thus, a lift is characterized by its *circuit signature*, which is a function $\lambda \colon \mathcal{C} \to \{+1,-1,0\}$ where $\mathcal{C}$ is the set of circuits of $\mathcal{M}$ and $s(C) = +$, $-$ or 0 in the three cases mentioned above, respectively. The function λ clearly satisfies $\lambda(-C) = -\lambda(C)$, but this property is not enough for a function $\lambda : \mathcal{C} \to \{+1,-1,0\}$ to represent a lift. The necessary and sufficient condition for this is that λ defines a lift on every corank 2 restriction of $\mathcal{M}$. Even more, in corank 2 there is a list of only three forbidden subconfigurations which can prevent λ from representing a lift [5, Theorem 7.1.8].

For proving Theorem 4.2 we will first show how a consistent family of subdivisions of $\mathcal{A}$ induces a circuit signature function λ. We recall that a point configuration B of corank 1 (i.e., with two more points than its affine dimension) has exactly one circuit $C = (C^+, C^-)$ (up to sign reversal) and three subdivisions, defined as follows:

$$S(B, C^+) := \{B \setminus \{a\} \mid a \in C^+\},$$
$$S(B, C^-) := \{B \setminus \{a\} \mid a \in C^-\}, \qquad S(B, C^0) := \{B\}.$$

We will say that the three subdivisions above give positive, negative and zero sign to the circuit C, respectively. We recall that $\overline{C} = C^+ \cup C^-$ denotes the support of C.

Lemma 4.3. *Let $\mathcal{M}$ denote the oriented matroid of affine dependences of $\mathcal{A}$ and $\mathcal{C}$ its set of circuits. Let $\mathcal{S} = \{S_B\}_{B \subset \mathcal{A}}$ be a family of subdivisions of the subconfigurations of $\mathcal{A}$ which is consistent with S.*

Define a circuit signature function $\lambda_{\mathcal{S}} : \mathcal{C} \to \{-1,0,+1\}$ as follows. For each circuit C of $\mathcal{M}$, let B be a corank 1 subset of $\mathcal{A}$ having C as a circuit. Let $\lambda_{\mathcal{S}}(C)$ be $+1$, -1 or 0 if S_B equals $S(B, C^+)$, $S(B, C^-)$, and $S(B, C^0)$, respectively. Then, *(i) The function $\lambda_{\mathcal{S}}$ is well-defined (it does not depend on the choice of the subset B) and satisfies $\lambda_{\mathcal{S}}(-C) = -\lambda_{\mathcal{S}}(C)$.*

(ii) If $\lambda_{\mathcal{S}}$ defines a lift $\widehat{\mathcal{M}}$ of $\mathcal{M}$, then $S_{\mathcal{A}}$ is the lifting subdivision induced by that lift.

Proof. (i) Let C be a circuit of $\mathcal{M}$ and $\overline{C}$ its support. Then, $\overline{C}$ is already a corank 1 subset of $\mathcal{A}$ having C as a circuit. Moreover, any other such subset B contains $\overline{C}$, so that the first condition of consistency easily implies that S_B gives the same sign to the circuit C as $S_{\overline{C}}$. That $\lambda(-C) = -\lambda(C)$ is trivial.

(ii) Suppose that $\lambda_{\mathcal{S}}$ defines a lift $\widehat{\mathcal{M}}$ of $\mathcal{M}$. We want to prove that $S_{\mathcal{A}}$ equals the lifting subdivision of $\mathcal{A}$ induced by $\widehat{\mathcal{M}}$ A subdivision of a point configuration can be specified by saying which simplices (i.e., affine bases) of $\mathcal{A}$ are contained in cells of the subdivision. Thus, it will suffice to show that for every basis σ of $\mathcal{A}$, σ is contained in a cell of $S_{\mathcal{A}}$ if and only if it is contained in a facet of $\widehat{\mathcal{M}}$ not containing the additional element $n+1$.

If σ lies in a facet of $\widehat{\mathcal{M}}$ not containing $n+1$, then σ lies in a facet of $\widehat{\mathcal{M}}(\sigma \cup \{b, n+1\})$ not containing $n+1$ for every $b \in \mathcal{A} \setminus \sigma$. Thus, σ lies in a cell of the restriction $S_{\sigma\cup\{b\}}$ for every such b and in a cell of $S_{\mathcal{A}}$ by condition (ii) of consistency.

Conversely, suppose that σ is contained in a cell of $S_{\mathcal{A}}$. Since σ is a basis in $\mathcal{M}$, $\sigma \cup \{n+1\}$ is a basis in $\widehat{\mathcal{M}}$. Lat C_σ denote the cocircuit of $\widehat{\mathcal{M}}$ which vanishes in σ, oriented so that it is positive at $n+1$. We will prove that C_σ is non-negative, which implies that σ lies in a facet not containing $n+1$ of $\widehat{\mathcal{M}}$. If C_σ is negative on some element $b \in A \setminus \sigma$, let $C = (C^+, C^-)$ be the unique circuit of $\mathcal{A}$ contained in $\sigma \cup \{b\}$, oriented so that $b \in C^+$. (Since σ is independent, b is in the support of C.) Since $S_{\sigma\cup\{b\}}$ is clearly the lifting subdivision induced by the lift of $\sigma \cup \{b\}$ given by $\lambda_{\mathcal{S}}(C)$ and since σ is in a cell of the subdivision $S_{\sigma\cup\{b\}}$ by condition (i) of consistency, we have that $\lambda_{\mathcal{S}}(C)$ is different from $+$ (the sign of C at b). But this implies that either (C^+, C^-) or $(C^+, C^- \cup \{n+1\})$ is a circuit in the lifted oriented matroid $\widehat{\mathcal{M}}$, which violates orthogonality with the cocircuit C_σ: contradiction. (Observe that the support of the circuit and the cocircuit intersect only in b in the first case and in b and $n+1$ in the second, but not orthogonally.) $\square$

Lemma 4.4. *In the same conditions of Lemma 4.3, suppose moreover that $\mathcal{A}$ has corank 2. Then, the circuit signature $\lambda_{\mathcal{S}}$ induced is the circuit signature of a lift of $\mathcal{M}$.*

Proof. Without loss of generality, we assume that $\mathcal{A}$ has no coloops. In other words, that for every element $a \in A$ its deletion $\mathcal{A} \setminus \{a\}$ has corank 1. Otherwise the statement follows easily by induction on the cardinality of $\mathcal{A}$, by simply removing that coloop.

In these conditions, for each element $a \in \mathcal{A}$ the deletion $\mathcal{A} \setminus a$ has a unique circuit C_a (up to a sign), which is given a certain sign by $\lambda_{\mathcal{S}}$. The Gale transform $\mathcal{A}^*$ of $\mathcal{A}$ is a vector configuration of rank 2, whose cocircuits are the complements of the lines generated by vectors of the configuration. We can picture $\lambda_{\mathcal{S}}(C_a)$ by putting a $+$ and a $-$ sign on the two sides of the vector a, in the way indicated by $\lambda_{\mathcal{S}}(C_a)$ if this is non-zero and putting zeroes if $\lambda_{\mathcal{S}}(C_a) = 0$.

One of the characterization by Las Vergnas of valid cocircuit signatures for extensions of the oriented matroid $\mathcal{M}^*$ is a list of three forbidden subconfigurations of rank three

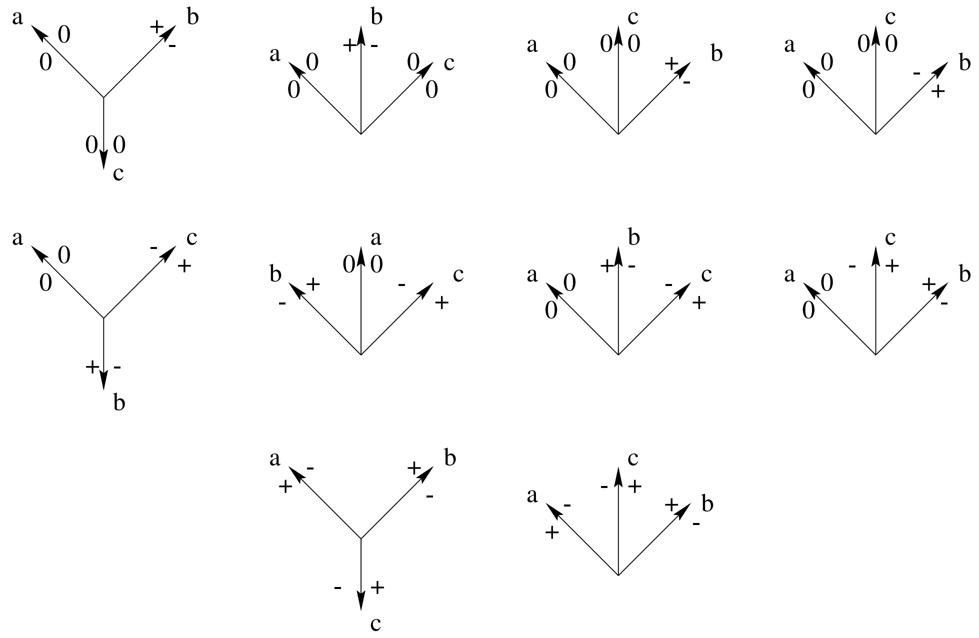

FIGURE 4. Forbidden subconfigurations for a cocircuit signature in rank 2.

with three elements a , b and c (see [5, Theorem 7.1.8, part (3)]). They correspond with the three rows of pictures in Figure 4, respectively. The pictures in a row are all the different reorientations of each forbidden subconfiguration. We only need to show that none of them can appear in the Gale diagram of $\mathcal{A}$, when we picture $\lambda_{\mathcal{S}}$ as indicated above. Observe that a zero in a vector v of the picture means that $S_{\mathcal{A}\setminus\{v\}}$ is a trivial subdivision, while a $+$ on one side of v means that, for every w on that side of v, $\mathcal{A}\setminus\{v,w\}$ is a cell in $S_{\mathcal{A}\setminus\{v\}}$. With this we can discard the different possibilities as follows:

(1) In the first row of pictures we have zero signs for C_a and C_c, but not for C_b. This implies that $S_{\mathcal{A}\setminus\{a\}}$ and $S_{\mathcal{A}\setminus\{c\}}$ are trivial subdivisions, so that the simplex $\sigma = \mathcal{A}\setminus\{a,c\}$ is contained in a cell of each. Taking $B = \mathcal{A}$ in condition (ii) of consistency we conclude that σ is contained in a cell τ of S. Taking $B = \mathcal{A}$ and $B' = \mathcal{A}\setminus\{c\}$ in the first condition of consistency, that $\tau\setminus\{c\}$ is a cell in $S_{\mathcal{A}\setminus\{c\}}$. Since $S_{\mathcal{A}\setminus\{c\}}$ is the trivial subdivision, $a\in\tau$. In the same way one proves $c\in\tau$. But then, τ contains $\mathcal{A}\setminus\{b\}$ and this would imply that $S_{\mathcal{A}\setminus\{b\}}$ is trivial as well, which is not the case.

(2) In the pictures of the second row we have a unique zero sign, in C_a. Again this implies that $S_{\mathcal{A}\setminus\{a\}}$ is the trivial subdivision. We have labeled all the cases so that the vector a of the Gale transform lies on the positive side of the vector b and the negative side of the vector c. In terms of the subdivisions, this implies that $\mathcal{A}\setminus\{a,b\}\in S_{\mathcal{A}\setminus\{b\}}$ but $\mathcal{A}\setminus\{a,c\}\notin S_{\mathcal{A}\setminus\{c\}}$.

Taking $\sigma = \mathcal{A}\setminus\{a,b\}$ and $B = \mathcal{A}$, the second condition of consistency tells us that σ lies in a cell τ of S. In the same way as before we can prove that $b\in\tau$, so that either $\tau = \mathcal{A}$ or $\tau = \mathcal{A}\setminus\{a\}$. But then, the first condition of consistency with $B' = \mathcal{A}\setminus\{c\}$

implies that either $S_{\mathcal{A}\setminus\{c\}}$ is trivial (which would imply a zero on c in the picture) or $\mathcal{A}\setminus\{a,c\}\in S_{\mathcal{A}\setminus\{c\}}$ (which we have said to be false).

(3) Here we consider the two reorientation cases separately:

(3.a) In the picture of the left, $\{a,b,c\}$ is the support of a spanning positive circuit of $\mathcal{A}^*$, so that its complement is a simplicial facet of $\mathcal{A}$. Thus, there is a cell τ in S containing $\mathcal{A}\setminus\{a,b,c\}$. Suppose that $a\in\tau$, i.e., $\mathcal{A}\setminus\{b,c\}\subset\tau$. Then, the first condition of consistency with $B'=\mathcal{A}\setminus\{b\}$ tells us that $\mathcal{A}\setminus\{b,c\}$ lies in a cell $\tau\setminus\{b\}$ of $S_{\mathcal{A}\setminus\{b\}}$. But this is impossible since the picture implies that $\mathcal{A}\setminus\{a,b\}$ is a cell in $S_{\mathcal{A}\setminus\{b\}}$ and the two simplices $\mathcal{A}\setminus\{b,c\}$ and $\mathcal{A}\setminus\{a,b\}$ intersect improperly. Similar contradictions are obtained by assuming $b\in\tau$ or $c\in\tau$.

(3.b) Now the hyperplane spanned by $\mathcal{A}\setminus\{a,b,c\}$ has b in one side and a and c in the other. The picture tells us that $\mathcal{A}\setminus\{a,b\}$ is a simplex in both $S_{\mathcal{A}\setminus\{b\}}$ and $S_{\mathcal{A}\setminus\{a\}}$, so that it is contained in a cell τ of S, by condition 2 of consistency with $B'=\mathcal{A}\setminus\{a,b\}$. Moreover, $a\in\tau$ or $b\in\tau$ would imply, respectively, that $S_{\mathcal{A}\setminus\{b\}}$ is trivial or $S_{\mathcal{A}\setminus\{a\}}$ is trivial, which is not true since we have no zeroes in $\lambda_{\mathcal{S}}$. Thus, $\tau=\mathcal{A}\setminus\{a,b\}$ is a cell in S.

But then, the first condition of consistency implies that $\mathcal{A}\setminus\{a,b,c\}$ is a face of a cell in $S_{\mathcal{A}\setminus\{c\}}$. Since $S_{\mathcal{A}\setminus\{c\}}$ is not trivial, either $\mathcal{A}\setminus\{a,c\}$ or $\mathcal{A}\setminus\{b,c\}$ is in $S_{\mathcal{A}\setminus\{c\}}$. This would imply that one of a or b is in the positive side of the vector c in the picture, which is not the case. □

Poof of Theorem 4.2. For the implication from (i) to (ii), let $\widehat{\mathcal{M}}$ be the lifting of $\mathcal{M}$ inducing the lifting subdivision S. Then the different restrictions of $\widehat{\mathcal{M}}$ provide liftings of the restrictions of $\mathcal{M}$ and, in particular, a family $\mathcal{S}$ of (lifting) subdivisions of the different subsets of $\mathcal{A}$. It can be checked easily (see [18]) that $\mathcal{S}$ is consistent with S.

The implication from (ii) to (i) follows from lemmas 4.3 and 4.4. Part 1 of Lemma 4.3 implies that $\mathcal{S}$ defines a cocircuit signature, which is the cocircuit signature of a lift by Lemma 4.4 and then part 2 of Lemma 4.3 implies that S is the associated lifting subdivision. □

5. ZONOTOPES, LAWRENCE POLYTOPES AND THE BOHNE-DRESS THEOREM

Let $\mathcal{A}=\{a_1,\dots,a_n\}$ be a point configuration spanning the affine space $\mathbb{R}^d$. Let us consider $\mathbb{R}^d$ embedded as the affine hyperplane of $\mathbb{R}^{d+1}$ where the last coordinate equals 1. A usual way of representing such a point configuration is by an $n\times(d+1)$ matrix whose i-th column has the coordinates of a_i in the first d rows and a 1 in the last one. This matrix, which we still denote $\mathcal{A}$, has rank $d+1$. In these conditions the Lawrence lifting of $\mathcal{A}$ is defined (see [21]) to be the point configuration corresponding to the matrix

$$\Lambda(\mathcal{A}) := \begin{pmatrix} \mathcal{A} & \mathbf{0} \\ I & I \end{pmatrix},$$

where I is the identity matrix of size $n\times n$ and $\mathbf{0}$ the zero matrix of size $n\times(d+1)$. (The $2n$ column vectors of the matrix $\Lambda(\mathcal{A})$ affinely span a non-linear affine hyperplane of $\mathbb{R}^{n+d+1}$, so it represents a point configuration with $2n$ points in dimension $n+d$ which we still denote $\Lambda(\mathcal{A})$.) The convex hull of this configuration is called the Lawrence polytope associated with $\mathcal{A}$. It turns out that all the points in $\Lambda(\mathcal{A})$ are vertices of this polytope.

By reordering the columns of $\Lambda(\mathcal{A})$ we see that the Lawrence polytope can be regarded as the Cayley embedding of the n segments $\overline{Oa_i} \subset \mathbb{R}^{d+1}$. I.e:

$$\Lambda(a_1,\dots,a_n) = \mathcal{C}(\overline{Oa_1},\dots,\overline{Oa_n}).$$

The Minkowski sum of a collection of segments is a *zonotope* and its mixed subdivisions are usually called *zonotopal tilings* [23, Section 7.5]. We will call *zonotope associated* with the point configuration $\mathcal{A}$ (and denote $\mathcal{Z}(\mathcal{A})$) the Minkowski sum $\sum_{i=1}^{n} \overline{Oa_i}$. Thus, the Cayley trick gives a correspondence between zonotopal tilings of the zonotope $\mathcal{Z}(\mathcal{A})$ and polyhedral subdivisions of the Lawrence polytope $\Lambda(\mathcal{A})$.

Finally, let $\mathcal{M}_{\mathcal{A}}$ be the oriented matroid of affine dependences between the points in $\mathcal{A}$. (It coincides with the oriented matroid realized by the columns of the $n\times(d+1)$ matrix defined at the beginning of this section.) The *lifts* of $\mathcal{M}_{\mathcal{A}}$ defined in the previous section are partially ordered by weak maps, a lift being lower in this poset if it is "more generic" or "more uniform" see [5, Chapter 7]. (More precisely, the circuit signature function of the lower lift is obtained from that of the higher by setting some zeroes to $+$ or $-$.)

This section is devoted to prove the following Theorem:

Theorem 5.1 (Bohne-Dress, Santos)**.** *Let $\mathcal{A}$ be a point configuration. The following posets are isomorphic:*

(i) The poset of zonotopal tilings of $\mathcal{Z}(\mathcal{A})$.
(ii) The poset of lifts of the oriented matroid $\mathcal{M}_{\mathcal{A}}$.
(iii) The poset of subdivisions of the Lawrence polytope $\Lambda(\mathcal{A})$.

The equivalence of the first two posets is the so-called Bohne-Dress theorem for polytopes (see [5, Theorem 2.2.13], [23, Theorem 7.32], [17]). We provide a new proof of the Bohne-Dress theorem as follows: Our Theorem 3.1 directly implies the isomorphism between the first and last posets. The equivalence of the last two was proved in [18, Section 4.2] in the general case of perhaps non-realizable oriented matroids; the proof is reproduced below for completeness.

Proposition 5.2. *Let $\mathcal{A}$ be a point configuration with oriented matroid $\mathcal{M}_{\mathcal{A}}$, and let $\Lambda(\mathcal{A})$ be the associated Lawrence polytope, with oriented matroid $\mathcal{M}_{\Lambda(\mathcal{A})}$. Then:*

(i) Two different lifts of $\mathcal{M}_{\Lambda(\mathcal{A})}$ produce different associated lifting subdivisions.
(ii) Every subdivision of $\Lambda(\mathcal{M})$ is a lifting subdivision.
(iii) The poset of lifts of $\mathcal{M}_{\Lambda(\mathcal{M})}$ and the poset of lifts of $\mathcal{M}_{\mathcal{A}}$ are isomorphic.

Thus, the poset of lifts of $\mathcal{M}_{\mathcal{A}}$ and the poset of subdivisions of $\Lambda(\mathcal{A})$ are isomorphic.

Proof. Throughout the proof we will denote by $b_1,\dots,b_n,e_1,\dots,e_n$ the vertices of the Lawrence polytope, that is to say the columns of the matrix

$$\Lambda(\mathcal{A}) := \begin{pmatrix} \mathcal{A} & \mathbf{0} \\ I & I \end{pmatrix}.$$

Observe that the complement of every pair $\{e_i, b_i\}$ is a facet of the Lawrence polytope. The following are some other very special properties of $\Lambda(\mathcal{A})$.

Let $C = (C^+, C^-)$ be a circuit of $\Lambda(\mathcal{A})$. The structure of the matrix clearly implies that whenever an element b_i or e_i is in C^+ the companion e_i or b_i is in C^- and vice versa. In other words, the support of every circuit has the form $\{\, b_i : i \in J\,\} \cup \{\, e_i : i \in J\,\}$, for some $J \subset \{1, \dots, n\}$. On the other hand, the structure of the matrix also shows that such a subset of vertices is always (the set of vertices of) a face of $\Lambda(\mathcal{A})$.

If B is now an arbitrary subset of the vertices of $\Lambda(\mathcal{A})$, let $B_0 = \{\, b_i \in B : e_i \in B\,\} \cup \{\, e_i \in B : b_i \in B\,\}$. Every element $p \in B \setminus B_0$ is a coloop in B. In other words, for every subset B of the vertices of $\Lambda(\mathcal{A})$, $\mathrm{conv}(B)$ is an iterated cone over the face $\mathrm{conv}(B_0)$ of $\Lambda(\mathcal{A})$. These facts will be crucial in the proof of the three statements:

(i) The circuit signature functions of two different lifts will necessarily give different sign to a certain circuit C of $\Lambda(\mathcal{A})$. But this implies that the associated lifting subdivisions are different, since they are different in the face of $\Lambda(\mathcal{A})$ spanned by the support of that circuit.

(ii) This is a sort of converse of the previous assertion. Since every subset B of the vertices of $\Lambda(\mathcal{A})$ is an iterated cone over a face $\mathrm{conv}(B_0)$, a subdivision S of $\Lambda(\mathcal{M})$ gives a unique way to subdivide B in a way consistent with S: cone the subdivision of the face $\mathrm{conv}(B_0)$ induced by S over the elements in $B \setminus B_0$. Let $\{S_B\}_{B \subset \Lambda(\mathcal{A})}$ denote the family of subdivisions so obtained. The first condition of consistency is trivially satisfied by this family. For proving the second one we will use induction on the dimension of the subset B involved.

Let σ be a basis contained in B such that for every $b \in B \setminus \sigma$ we have that σ is in a cell of the subdivision $S_{\sigma \cup \{b\}}$. Since σ is full-dimensional, it must contain at least one of each pair of vertices b_i and e_i of $\Lambda(\mathcal{A})$, for every $i \in \{1, \dots, n\}$. On the other hand, since the case $\sigma = \Lambda(\mathcal{A})$ is trivial, σ contains an element e_i or b_i whose companion e_i or b_i is not in σ. Let a be such an element, and let us denote its companion by $\overline{a}$.

Since $\{a, \overline{a}\}$ is the complement of the set of vertices of a facet of $\Lambda(\mathcal{A})$, by induction on the dimension we assume that $\sigma \setminus \{a\}$ lies in a cell of $S_{B \setminus \{a, \overline{a}\}}$. If $\overline{a} \notin B$ this implies that σ lies in a cell of S_B. If $\overline{a} \in B$ we still can conclude that either σ or $\sigma \setminus a \cup \{\overline{a}\}$ lie in a cell of S_B. So suppose that the second happens, and let τ be that cell. We will proof that $a \in \tau$ as well.

Consider the corank 1 subconfiguration $B' = \sigma \cup \{\overline{a}\}$ of B. By the first condition of consistency, $\tau \cap B'$ is a face of a cell in $S_{B'}$. On the other hand, since B' is of the form $\sigma \cup \{b\}$, σ lies in a cell of $S_{B'}$ by hypothesis. Thus, both $B' \setminus \{\overline{a}\} = \sigma$ and $B' \setminus \{a\} \subset \tau \cap B'$ lie in cells of $S_{B'}$. Since $B' \setminus \{a, \overline{a}\}$ is a face of B', this implies that $S_{B'}$ is the trivial subdivision. Finally, since $\tau \cap B'$ is full dimensional because it contains $\sigma \setminus \{a\} \cup \{\overline{a}\}$, $\tau \cap B'$ is a cell of S'_B and, thus, $a \in \tau$, as we wanted to prove.

(iii) Let A^* be a Gale transform of $\mathcal{A}$, represented as a matrix of size $n \times (n-d-1)$ whose row space $row(\mathcal{A}^*)$ is an orthogonal complement of $row(\mathcal{A})$. Then, the matrix $(\mathcal{A}^*, \mathcal{A}^*)$ of size $2n \times (n-d-1)$ represents a Gale transform of $\Lambda(\mathcal{A})$. In other words, the oriented matroid dual to $\mathcal{M}_{\Lambda(\mathcal{A})}$ is obtained from the dual of $\mathcal{M}_{\mathcal{A}}$ by adjoining an antiparallel element to every element. Then, it is trivial that the two duals have the same posets of extensions (for example, via the topological representation theorem of oriented matroids; also via Las Vergnas characterization of extensions by cocircuit signature functions). Since lifts of an oriented matroid are duals to extensions of its dual, the result is proved.

Once we have proved parts 1, 2, and 3 we have a bijection between the two posets we are interested in. That this bijection is a poset isomorphism is trivial. □

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BIRKETT HUBER, MATHEMATICAL SCIENCES RESEARCH INSTITUTE, 1000 CENTENNIAL DRIVE, BERKELEY, CA 94720-5070
E-mail address: birk@isc.tamu.edu

JÖRG RAMBAU, KONRAD-ZUSE-ZENTRUM FÜR INFORMATIONSTECHNIK BERLIN, TAKUSTR. 7, 14195 BERLIN
E-mail address: rambau@zib.de

FRANCISCO SANTOS, DEPTO. DE MATEMÁTICAS, ESTADÍSTICA Y COMPUTACIÓN, UNIVERSIDAD DE CANTABRIA, E-39005 SANTANDER, SPAIN
E-mail address: santos@matesco.unican.es