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# A symmetry test for quasilinear coupled systems

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Preprint SC 99-02 (January 1999)

# A symmetry test for quasilinear coupled systems

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January 8, 1999

## 1 Classification result.

It is well known that the following class of systems of evolution equations

$$\begin{cases} u_t = u_{xx} + F(u, v, u_x, v_x), \\ v_t = -v_{xx} + G(u, v, u_x, v_x), \end{cases} \tag{1}$$

is very rich in integrable cases. In the papers [1]-[5] by Mikhailov, Shabat and Yamilov, all systems (1), possessing higher conservation laws, were classified. Hence, these authors have found all systems (1) that can be integrated by the inverse scattering method (S-integrable equations in the terminology by F. Calogero [6]).

However, there are integrable cases that are not in their classification. As an example, consider the system

$$\begin{cases} u_t = u_{xx} - 2uu_x - 2vu_x - 2uv_x + 2u^2v + 2uv^2, \\ v_t = -v_{xx} + 2vu_x + 2uv_x + 2vv_x - 2u^2v - 2uv^2, \end{cases} \tag{2}$$

first discussed in [7] (see [8] for generalizations). It can be reduced to

$$U_t = U_{xx}, \qquad V_t = -V_{xx}$$

by the following substitution of the Cole-Hopf type

$$u = \frac{U_x}{(U+V)}, \qquad v = \frac{V_x}{(U+V)}.$$

The system (2) has no higher order conservation laws, but it has higher order symmetries. This is a typical feature of linearizable systems like the Burgers equation (C-integrable equations). Therefore, it would be interesting to classify the systems (1), which have higher order symmetries. As a result, all S-integrable and C-integrable systems would be found.

The complete classification problem is very difficult. Here we consider only the most interesting (from our opinion) subclass of systems (1). Namely, we consider equations linear in all derivatives of the form

$$\begin{cases} u_t = u_{xx} + A_1(u,v)u_x + A_2(u,v)v_x + A_0(u,v) \\ v_t = -v_{xx} + B_1(u,v)v_x + B_2(u,v)u_x + B_0(u,v). \end{cases} \tag{3}$$

without any restrictions on the functions $A_i(u, v), B_i(u, v)$. We apply to such systems the simplest version of the symmetry test (see [9]-[12]).

**Lemma.** If system (3) has a fourth order symmetry

$$\begin{cases} u_\tau = u_{xxxx} + f(u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx}), \\ v_\tau = -v_{xxxx} + g(u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx}) \end{cases} \tag{4}$$

then the system is of the following form

$$\begin{cases} u_t = u_{xx} + (a_{12}uv + a_1u + a_2v + a_0)u_x + (p_2v + p_{11}u^2 + p_1u + p_0)v_x + A_0(u, v), \\ v_t = -v_{xx} + (b_{12}uv + b_1v + b_2u + b_0)v_x + (q_2u + q_{11}v^2 + q_1v + q_0)u_x + B_0(u, v), \end{cases}$$

where $A_0$ and $B_0$ are polynomials of at most fifth degree.

The coefficients of the last system satisfy an overdetermined system of algebraic equations. The most essential equations are

$$\begin{array}{lll} p_2(b_{12} - q_{11}) = 0, & p_2(a_{12} - p_{11}) = 0, & p_2(a_{12} + 2b_{12}) = 0, \\ q_2(b_{12} - q_{11}) = 0, & q_2(a_{12} - p_{11}) = 0, & q_2(b_{12} + 2a_{12}) = 0. \end{array}$$

$$\begin{array}{ll} a_{12}(a_{12} - b_{12} + q_{11} - p_{11}) = 0, & b_{12}(a_{12} - b_{12} + q_{11} - p_{11}) = 0, \\ (a_{12} - p_{11})(p_{11} - q_{11}) = 0, & (b_{12} - q_{11})(p_{11} - q_{11}) = 0, \\ (a_{12} - p_{11})(a_{12} - b_{12}) = 0, & (b_{12} - q_{11})(a_{12} - b_{12}) = 0. \end{array}$$

As usual, such factorized equations lead to a tree of variants, which was investigated by the computer algebra program CRACK [19],[20].

Solving the overdetermined system, we don't consider so called triangular systems like the following

$$u_t = u_{xx} + 2uv_x, \qquad v_t = -v_{xx} - 2vv_x. \tag{5}$$

Here the second equation is separated and the first is linear with the variable coefficients defined by a given solution of the second equation.

**Theorem.** Any nonlinear nontriangular system (3), having a symmetry (4), up to scalings of $t$, $x$, $u$, $v$, shifts of $u$ and $v$, and the involution

$$u \leftrightarrow v, \qquad t \leftrightarrow -t \tag{6}$$

belongs to the following list:

$$\begin{cases} u_t = u_{xx} + (u + v)u_x + uv_x, \\ v_t = -v_{xx} + (u + v)v_x + vu_x, \end{cases} \tag{7}$$

$$\begin{cases} u_t = u_{xx} - 2(u + v)u_x - 2uv_x + 2u^2v + 2uv^2 + \alpha u + \beta v + \gamma, \\ v_t = -v_{xx} + 2(u + v)v_x + 2vu_x - 2u^2v - 2uv^2 - \alpha u - \beta v - \gamma, \end{cases} \tag{8}$$

$$\begin{cases} u_t = u_{xx} + vu_x + uv_x, \\ v_t = -v_{xx} + vv_x + u_x, \end{cases} \tag{9}$$

$$\begin{cases} u_t = u_{xx} + 2vu_x + 2uv_x + 2uv^2 + u^2 + \alpha u + \beta v + \gamma, \\ v_t = -v_{xx} - 2vv_x - u_x, \end{cases} \tag{10}$$

$$\begin{cases} u_t = u_{xx} + \alpha v_x + (u + v)^2 + \beta(u + v) + \gamma, \\ v_t = -v_{xx} + \alpha u_x - (u + v)^2 - \beta(u + v) - \gamma, \end{cases} \tag{11}$$

$$\begin{cases} u_t = u_{xx} + (u+v)u_x + 4\alpha v_x + \alpha(u+v)^2 + \beta(u+v) + \gamma, \\ v_t = -v_{xx} + (u+v)v_x + 4\alpha u_x - \alpha(u+v)^2 - \beta(u+v) - \gamma, \end{cases} \tag{12}$$

$$\begin{cases} u_t = u_{xx} + 2\alpha u^2 v_x + 2\beta uvu_x + \alpha(\beta - 2\alpha)u^3v^2 + \gamma u^2 v + \delta u, \\ v_t = -v_{xx} + 2\alpha v^2 u_x + 2\beta uvv_x - \alpha(\beta - 2\alpha)u^2v^3 - \gamma uv^2 - \delta v, \end{cases} \tag{13}$$

$$\begin{cases} u_t = u_{xx} + 2uvu_x + (\alpha + u^2)v_x, \\ v_t = -v_{xx} + 2uvv_x + (\beta + v^2)u_x, \end{cases} \tag{14}$$

$$\begin{cases} u_t = u_{xx} + 2\alpha uvu_x + 2\alpha u^2 v_x - \alpha\beta u^3v^2 + \gamma u, \\ v_t = -v_{xx} + 2\beta v^2 u_x + 2\beta uvv_x + \alpha\beta u^2v^3 - \gamma v, \end{cases} \tag{15}$$

$$\begin{cases} u_t = u_{xx} + 2uvu_x + 2(\alpha + u^2)v_x + u^3v^2 + \beta u^3 + \alpha uv^2 + \gamma u, \\ v_t = -v_{xx} - 2uvv_x - 2(\beta + v^2)u_x - u^2v^3 - \beta u^2 v - \alpha v^3 - \gamma v, \end{cases} \tag{16}$$

$$\begin{cases} u_t = u_{xx} + 4uvu_x + 4u^2v_x + 3vv_x + 2u^3v^2 + uv^3 + \alpha u, \\ v_t = -v_{xx} - 2v^2u_x - 2uvv_x - 2u^2v^3 - v^4 - \alpha v, \end{cases} \tag{17}$$

$$\begin{cases} u_t = u_{xx} + 4uu_x + 2vv_x, \\ v_t = -v_{xx} - 2vu_x - 2uv_x - 3u^2v - v^3 + \alpha v, \end{cases} \tag{18}$$

$$\begin{cases} u_t = u_{xx} + vv_x, \\ v_t = -v_{xx} + u_x, \end{cases} \tag{19}$$

$$\begin{cases} u_t = u_{xx} + 6(u+v)v_x - 6(u+v)^3 - \alpha(u+v)^2 - \beta(u+v) - \gamma, \\ v_t = -v_{xx} + 6(u+v)u_x + 6(u+v)^3 + \alpha(u+v)^2 + \beta(u+v) + \gamma, \end{cases} \tag{20}$$

$$\begin{cases} u_t = u_{xx} + vv_x, \\ v_t = -v_{xx} + uu_x. \end{cases} \tag{21}$$

We omitted the term $(cu_x, cv_x)^T$ on the right hand sides of all systems. It is a Lie symmetry, corresponding to the invariance of our classification problem with respect to the shift of $x$.

## 2 Discussion.

**1. Admissible transformations.** Some equations in the list contain arbitrary constants $\alpha, \beta, \gamma, \delta$. Not all of them are essential.

Let us consider, for instance, the equations (14). It is easy to see that those constants $\alpha$ and $\beta$ which are not equal to zero can be reduced to 1 via scalings of $t, x, u$ and $v$. In this way, (14) actually describes three different equations without parameters. These equations correspond to $\alpha = \beta = 1$, $\alpha = \beta = 0$ and $\alpha = 1, \beta = 0$.

The following parameters: $\alpha$ in (11) and (12), $\gamma$ in (13), both $\alpha$ and $\beta$ in (16) are not essential in the same sense. For (13) and (15) the essential parameter is the ratio of $\alpha$ and $\beta$. Note, that if $\alpha = \beta = 0$ then (13) coincides with the nonlinear Schrödinger equation, which is not a separate equation in our list.

For some equations from the list, there exist admissible transformations of the form

$$u \to p(x,t)u + q(x,t), \quad v \to r(x,t)v + s(x,t). \tag{22}$$

"Admissible" means that the resulting equation does not depend explicitly on $x$ and $t$ and has the same form (3). Using such admissible transformations, one can remove some of constants in the equations of the list.

In particular, with the help of the transformation $u \to \exp(ct)u,\ \ v \to \exp(-ct)v$ one can remove the terms $(cu, -cv)^T$ in (13), (15), (16).

Using the transformations $u \to u + \lambda t + \mu x,\ \ v \to v - \lambda t - \mu x$ it is possible to remove $\beta$ and $\gamma$ in (11), (12). The equation (20) can be reduced to the form

$$\begin{cases} u_t = u_{xx} + 6(u+v)v_x - 6(u+v)^3 + cv_x, \\ v_t = -v_{xx} + 6(u+v)u_x + 6(u+v)^3 + cu_x \end{cases} \tag{23}$$

by such a transformation and by shifts of $u$ and $v$. It seems to us that the essential constant $c$ was missed in the classification result of [3].

More general transformations are described in [4],[5] which reduce some of the equations (7)-(21) to others in this list. For simplicity in applying our results, we will not rely on these non-trivial transformations, and will instead operate with the complete list (7)-(21).

**2. Three groups of equations.** All equations of the list can be divided into three groups. The first group contains the so called NLS-type equations (7), (9),(12),(13),(14). Besides a higher symmetry (4) every such equation possesses a symmetry of the form

$$\begin{cases} u_\tau = u_{xxx} + \varphi(u, v, u_x, v_x, u_{xx}, v_{xx}), \\ v_\tau = v_{xxx} + \psi(u, v, u_x, v_x, u_{xx}, v_{xx}). \end{cases} \tag{24}$$

This is typical for equations having the Lax representations in $sl(2)$.

The equations of the Boussinesq type form the second group (11),(19),(20),(21). They have no symmetries of third order. This indicates the existence of a Lax representation in $sl(3)$. We have chosen the existence of the symmetry (4) as a criterion of integrability for (3) since the choice of the simplest ansatz (24) leads to the loss of all equations of the second group.

The last group (8),(10),(15),(16),(17), (18) consists of "linearizable" equations, which have no higher conservation laws. Some of them seem to be new.

*Equations (15), (16) :* In [18],[6] one can find a linearization procedure for (15). Namely, a non-local substitution

$$U = u \exp\left(\alpha \int uv\, dx\right), \qquad V = v \exp\left(-\beta \int uv\, dx\right) \tag{25}$$

reduces it to the linear equation $U_t = U_{xx} + \gamma U,\ V_t = -Vxx - \gamma V$.

Consider equation (16). It is easy to see that it has the following symmetry

$$\begin{cases} u_\tau = u_{xxx} + 3uvu_{xx} + 6uu_xv_x + 3vu_x^2 + 3u^2v^2u_x, \\ v_\tau = v_{xxx} + 3uvv_{xx} + 6vu_xv_x + 3uv_x^2 + 3u^2v^2v_x, \end{cases} \tag{26}$$

for any $\alpha, \beta, \gamma$. Under a reduction $u = v$, (26) coincides with the well known equation (see [13],[11], [6])

$$u_\tau = u_{xxx} + 3u^2u_{xx} + 9uu_x^2 + 3u^4u_x,$$

which can be linearized by the substitution $U = u \exp \int u^2 dx$. It is easy to verify that the following very similar substitution (cf. also with (25))

$$U = u \exp\left(\int uv\, dx\right), \qquad V = v \exp\left(\int uv\, dx\right) \tag{27}$$

reduces (26) to $U_\tau = U_{xxx},\ V_\tau = V_{xxx}$. The same substitution reduces (16) to a linear system

$$U_t = U_{xx} + 2\alpha V_x + \gamma U, \qquad V_t = -V_{xx} - 2\beta U_x - \gamma V.$$

Generalizing the formula (27) one can find the following vector generalizations of the systems (16) and (26):

$$\begin{cases} u_t = u_{xx} + 2 < u,v > u_x + 2 < u, v_x > u + < u,v >^2 u + \\ \qquad 2\alpha v_x + \beta < u,u > u + 2\alpha < u,v > v - \alpha < v,v > u + \gamma u, \\ \\ v_t = -v_{xx} - 2 < u,v > v_x - 2 < v, u_x > v - < u,v >^2 v - \\ \qquad 2\beta u_x - \alpha < v,v > v - 2\beta < u,v > u + \beta < u,u > v - \gamma v, \end{cases}$$

$$\begin{cases} u_\tau = u_{xxx} + 3 < u,v > u_{xx} + 3u < u_x, v_x > + 3 < u,v >_x u_x + 3 < u,v >^2 u_x, \\ v_\tau = v_{xxx} + 3 < u,v > v_{xx} + 3v < u_x, v_x > + 3 < u,v >_x v_x + 3 < u,v >^2 v_x, \end{cases}$$

where $u$ and $v$ are $N$-dimensional vectors and $< >$ is a scalar product. Both of them can be linearized just as in the scalar case:

$$U = u \exp \int < u,v > dx, \qquad V = v \exp \int < u,v > dx. \tag{28}$$

In contrast with (15), (16), equations (8) and (10) are related to linear systems

$$U_t = U_{xx} + c_1 U_x + c_2 V_x + c_3 U, \qquad V_t = -V_{xx} - k_1 V_x - k_2 U_x - k_3 V \tag{29}$$

via **local** differential substitutions.

*Equation (8):* For $\alpha = \beta = \gamma = 0$ the substitution is given by (1). In the general case such a substitution is defined by

$$u = \frac{U_x}{(U+V)} + \frac{(c_1 - k_2)U + c_2 V}{2(U+V)}, \qquad v = \frac{V_x}{(U+V)} + \frac{(c_1 - c_2)V + k_2 U}{2(U+V)},$$

where the constants in (29) and (8) satisfy the following conditions

$$k_1 = c_1, \qquad k_3 = c_3 = \frac{c_1(c_1 - c_2 - k_2)}{4},$$

$$2\alpha = -k_2(c_1 + c_2), \qquad 2\beta = -c_2(c_1 + k_2), \qquad 2\gamma = c_1 k_2 c_2.$$

*Equation (8):* The substitution is of the form

$$u = \frac{c_2 U_x}{V} + \frac{c_2^2}{2}, \qquad v = V_x \; V + \frac{c_1}{2}.$$

The relations between constants are the following

$$k_1 = c_1, \qquad k_2 = c_2, \qquad k_3 = 0,$$

$$2\alpha = -c_1^2 - 2c_2^2 + 2c_3, \qquad \beta = -c_1 c_2^2, \qquad 4\gamma = c_2^2(2c_1^2 + c_2^2 - 2c_3).$$

*Equation (17), (18):* These equations are related to triangular systems. Equations (18) have been obtained in [17]. The substitution

$$u = \frac{U_x}{2U}, \qquad v = \frac{V}{\sqrt{U}};$$

found first by Marihin [15], links equations (18) with

$$U_t = U_{xx} + 2V^2, \qquad V_t = -V_{xx} + \alpha V.$$

The last system is linear in the following sense. To find $V$ we need to solve a linear equation. For a given function $V$, the function $U$ satisfies a linear equation with variable coefficients.

Similarly the following substitution

$$u = \frac{1}{3}U^{-2/3}V^{-1}U_x, \qquad v = U^{-1/3}V,$$

reduces (17) to

$$U_t = U_{xx} - 2V^{-1}V_xU_x + 3\alpha U + 3V^3, \qquad V_t = -V_{xx}.$$

**3. Master-symmetries.** For all equations (7)-(21) we have found all symmetries of order less or equal than four using the computer program LiePde [21]. It turns out that many of the equations have symmetries depending on $t$ and $x$ explicitly. For example, computing for equation (15) the general symmetry of $n^{\text{th}}$ order of the form

$$u_\tau = P(t)u_{nx} + \ldots, \qquad v_\tau = -P(t)v_{nx} + \ldots$$

the polynomial $P(t)$ is an arbitrary polynomial of degree $n$.

It is well known that symmetries which are linearly depending on $t$ and $x$ are closely related to local master-symmetries [14]. The equations (7)-(10), (14)-(16) have such symmetries. To obtain the master-symmetries one has to simply put $t$ equal to zero in these time-dependent symmetries.

The resulting master-symmetry is of the form

$$\begin{cases} u_\tau = 2x\ (u_{xx} + F(u,v,u_x,v_x)) + f(u,v,u_x,v_x), \\ v_\tau = 2x\ (-v_{xx} + G(u,v,u_x,v_x)) + g(u,v,u_x,v_x), \end{cases} \tag{30}$$

where $F$ and $G$ are the right hand sides of the corresponding equation (1) and $f$ and $g$ are given by the following list

$$\begin{array}{rll} (7): & f = u^2 + 3uv + 4u_x, & g = v^2 + 3uv - 4v_x, \\ (8): & f = 2u^2 + 4uv + \beta + 3u_x, & g = -2v^2 - 4uv - \alpha - 3v_x, \\ (9): & f = 4uv + 5u_x, & g = v^2 + 4u - 5v_x, \\ (10): & f = 4uv + \beta + 3u_x, & g = -2v^2 - 2u - \alpha - 3v_x, \\ (14): & f = 2u^2v + 2\alpha v + 3u_x, & g = 2uv^2 + 2\beta u - 3v_x, \\ (15): & f = 2\alpha u^2v - 2\gamma xu + 2u_x, & g = 2\beta v^2u + 2\gamma xv - 2v_x, \\ (16): & f = 2u^2v + 2\alpha v + 2u_x, & g = -2uv^2 - 2\beta u - 2v_x. \end{array}$$


# Acknowledgement

The first author (V.S.) was supported, in part, by RFBR grant 98-, EPSRC grant GR/L99036 and LMS(fSU) grant 5324. He is grateful to the Queen Mary & Westfield College (Univ. of London) for its hospitality. The second author (T.W.) thanks the Symbolic Computation Group at the University of Waterloo for its hospitality during the authors sabbatical in the fall of 1998.


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