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# Conformational Dynamics: Modelling, Theory, Algorithm, and Application to Biomolecules

Preprint SC 99-18 (July 1999)

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## Abstract

The function of many important biomolecules comes from their dynamic properties and their ability to switch between different *conformations*. In a conformation, the large scale geometric structure of the molecule is understood to be conserved, whereas on smaller scales the system may well rotate, oscillate or fluctuate. In a recent article [J. Comp. Phys., 151,1 (1999)], the present author and his coworkers demonstrated that (a) conformations can be understood as almost invariant sets of some Markov chain being defined via the Hamiltonian system governing the molecular dynamics and that (b) these sets can efficiently be computed via eigenvectors of the corresponding Markov operator. The present manuscript reviews the mathematical modelling steps behind the novel approach, includes a rigorous analytical justification of this approach and the corresponding numerical realization, and illustrates the performance of the algorithm when applied to realistic molecular systems.

# Contents

# 1 Introduction

This manuscript presents a novel approach to the *direct* numerical approximation of the so-called "conformational dynamics", that is, the essential dynamical behavior of mechanical systems moving on multi-minima energy surfaces. It includes the derivation of the underlying mathematical model, its theoretical analysis, and a first proposal for its efficient numerical realization, all this tailored to the application to biomolecules. Hence, this introduction should start with a brief description of the importance of this application and of the origin of the phrase "conformational dynamics":

**Conformational Dynamics** The classical description of molecular processes deals with the molecule's *microscopic configuration* (positions $q$ and momenta $p$ of all atoms) and leads to a mathematical model in terms of some Hamiltonian differential equation (cf. Sec. 2.1). The solution of this equation, given by the associated Hamiltonian flow, is understood as the representation of the motion of the molecular system.

The chemically interesting *function* of many important biomolecules, like proteins or enzymes, results from their *dynamical* properties, particularly from their ability to undergo so-called *conformational transitions* (cf. [116]). In a *conformation*, the large scale geometric structure of the molecule is understood to be conserved, whereas on smaller scales, that is, in the details of each microscopic configuration visited, the system may well rotate, oscillate or fluctuate. Thus, the phrase "conformation" means a meta-stable quasi-equilibrium of the molecule. In comparison to the configurational fluctuations inside every conformation, transitions between different conformations are extremely rare events. As an implication, the computational characterization of such conformational changes via direct simulation of the associated Hamiltonian system often requires inaccessibly long time spans. Even worse, long-term simulation of a single trajectory comes out to be ill-conditioned (cf. Sec. 2.1.5). Nevertheless, most applications of molecular dynamics (MD) to the characterization of conformations deal with some kind of statistical analysis based on averages over long term trajectories or with remodelling steps for artificial acceleration of the process, compare [4, 47, 104]. Herein, a different line of method is advocated: it is suggested to attack the determination of conformations and the transition probabilities between them *directly*, i.e., without long term simulation or artificial remodelling.

**Dynamical System Approach** The key insight having finally led to the approach presented herein goes back to P. Deuflhard. He observed that the problem of algorithmic characterization of conformations is related to the problem of the identification of *almost invariant sets* of dynamical systems as studied by M. Dellnitz and O. Junge: If the conformations were *invariant sets* of the flow of the Hamiltonian system, then there could not be any transitions between different conformations. Since such transitions exist but are rare, we may understand every conformation as being an "almost" invariant subset of the phase space of the Hamiltonian system.

In [19], M. DELLNITZ and O. JUNGE suggest the direct computation of invariant and almost invariant sets of deterministic discrete dynamical systems via eigenmodes of the associated *Frobenius-Perron operator* without long-term simulations. In broad terms, their key idea is the following: since the natural invariant sets and measures of the dynamical system are given by the eigenvectors of the Frobenius-Perron operator for its (maximal) eigenvalue $\lambda_0 = 1$ (infinite relaxation time), the eigenvectors for eigenvalues $|\lambda_j| < 1$ near $\lambda_0 = 1$ should correspond to "almost" invariant structures ("large" finite relaxation times). The eigenmodes of the Frobenius-Perron operator are approximated by means of a discretization of this operator embedded in a *multilevel subdivision algorithm*. Dellnitz and Junge show that this technique is efficient whenever the part of the phase space, which is important for the long-term dynamics of the system, is some "low-dimensional" object [19, 18]. Otherwise, even the subdivision technique produces an exploding number of discretization boxes (a pitfall which we will call the "curse of dimension" in the following).

Deuflhard's suggestion to use similar techniques for the identification of conformation as almost invariant sets has been realized for small Hamiltonian systems. As reported in [21], the numerical results are intriguing and seem to catch the essential features of the dynamics. However, the main problem was obvious: the essential dynamics of highly-dimensional Hamiltonian systems is *not* supported on any low-dimensional object so that any kind of deterministic discretization —whether adaptive or not— must inevitably suffer from the curse of dimension. But the investigation also revealed some unexpected, deep-lying theoretical problems: In the space of measures with $L^p(\Gamma)$-densities, the Frobenius–Perron operator for Hamiltonian systems has infinitely many invariant densities and its entire spectrum lies *on* the unit circle (so that the identification of (almost) invariant sets via eigenvalues near $\lambda = 1$ but *inside* the unit circle makes no sense).[1] Despite these problems, this "dynamical system approach" to the identification of conformations has been an important intermediate step for the development of the "ensemble approach" advocated herein. This ensemble approach does also exploit the intriguing idea of computing conformations as almost invariant structures via an eigenvalue problem. However, the underlying statistical operator and the notion of "almost invariance" are fundamentally different (see below).

**Ensemble Approach** The starting point of this approach is the following insight: at least in the biomolecular context, molecular dynamics (MD) deals with

[1]The underlying reason is that the unavoidable discretization of the Hamiltonian flow in time destroys some of the conservation properties of the flow such that the essential dynamics of the discrete solution is supported on subsets of the full dimension of $\Gamma$. Hence, the spatial discretization of the Frobenius–Perron operator has to deal with measures supported on such sets, and thus, numerically, with $L^p(\Gamma)$-densities. The above statements concerning invariant densities and spectrum of the Frobenius–Perron operator are substantiated in Secs. 2.2 and 2.3 below. For more details see [21, 94].

*statistical ensembles* of molecules instead of single molecular systems, since only such ensembles can be an object of experimental investigation. Consequently, the rate of conformational transitions has to be characterized with respect to some experimentally given stationary ensemble, i.e., in terms of statistical mechanics and *not* for any single Hamiltonian system (cf. Sec. 2.2):

Suppose that the probability of systems in the ensemble to be in some state $x \in \Gamma$ at time $t = 0$ is given by the density $f_0 : \Gamma \to [0, 1]$. Then, the *transition probability* $w(B, C, \tau)$ from $B \subset \Gamma$ to $C \subset \Gamma$ during some *fixed* observation time $\tau$, is given by the fraction of systems in the ensemble, which are found in $B$ at $t = 0$ and in $C$ at $t = \tau$. Since all systems move due to the Hamiltonian flow $\Phi^\tau$, this transition probability can be expressed as

$$w(B, C, \tau) \;=\; \left(\int_B f_0(x)dx\right)^{-1} \int_B \chi_C(\Phi^\tau x)\, f_0(x)\, dx.$$

We are interested in almost invariant subsets, i.e., in sets $B \subset \Gamma$ with large probabilities to stay within, which, for the time being, can be expressed as $w(B, B, \tau) \approx 1$. In particular, conformations are given by sets of configurations with similar large scale geometric structure, that is, they are *spatial subsets* $A$ of positions $q \in A$ such that the associated phase space fiber

$$\Gamma(A) \;=\; \{(q, p) \in \Gamma, \quad q \in A\}$$

is almost invariant in the above sense. It should again be emphasized that there are **two fundamentally different notions of almost invariance**:

1. We may call some set $A$ almost invariant if the single dynamical system under consideration remains inside of $A$ for some long period of time before leaving it. The "dynamical system approach" to the identification of conformations due to [21] should be understood in this sense.[2]

2. In the ensemble approach, $A$ is called almost invariant, if the fraction of systems in the ensemble, which leave $A$ during some *fixed* observation time $\tau$, is small.

In [95], the differences between the two notions are discussed in detail by applying both concepts to the same kind of randomly perturbed dynamical system.

In terms of statistical mechanics, the Frobenius–Perron operator of the Hamiltonian system under investigation has to be interpreted as the *propagator* of the ensemble governing the evolution of the corresponding probability density. As illustrated in detail in Sec. 2.2 below, this observation implies that

[2] Not only the "dynamical system approach" to the identification of conformations but the entire approach of Dellnitz and Junge is often interpreted in this sense. However, in order to give a rigorous justification of their approach for general discrete dynamical systems, Dellnitz and Junge have to add small random perturbations to the discrete mapping, cf. [19]. For nonvanishing perturbations, their approach may also be interpreted as an ensemble approach in the above sense with exactly the same interpretation of almost invariance. What is herein called the "dynamical system approach" corresponds to the limit of vanishing random perturbations.

every invariant density of the Frobenius–Perron operator corresponds to the *initial experimental preparation* of some specific stationary ensemble. This insight led the present author to the definition of some *transition operator* $T$ as a certain "restriction" of the Frobenius–Perron operator to the unique invariant density induced by the ensemble under consideration (for more details see SCHÜTTE et al. [94] and Sec. 2.3 herein). This can be realized such that $T$ in fact describes the corresponding transition probabilities within the ensemble (cf. Sec. 3.1). As we will see in detail in Sec. 3, the transition operator $T$ is a *Markov operator* (in an appropriate $L^1$-space) and self-adjoint (in the associated $L^2$-space). This implies that its spectrum is real-valued and satisfies $\sigma(T) \subset [-1, 1]$. Similar as in the dynamical system approach, the *basic algorithmic idea* is the *identification of almost invariant sets of the ensemble via the eigenvectors of the transition operator* for eigenvalues near the (maximal) eigenvalue $\lambda_1 = 1$.

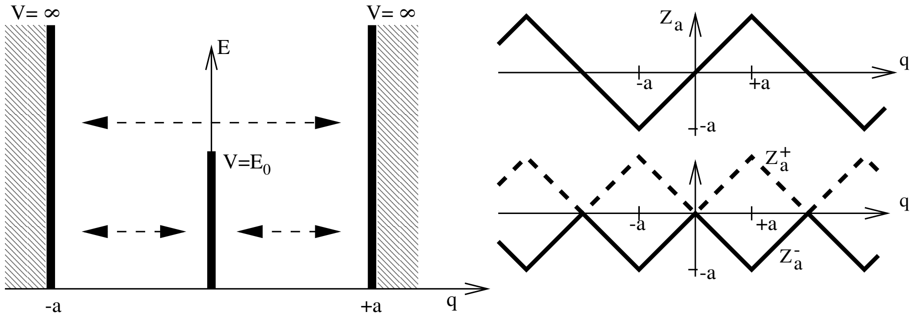



Figure 1: Illustration of the particles-in-a-box ensemble. A statistical ensemble of free particles is moving without interactions between three ideally reflecting walls. Every particle (position $q$ and momentum $p$) with energy $E = p^2/2 \le E_0$ is locked between $q = 0$ and $q = \pm a$ by a reflecting barrier of energetic height $E_0$; particles with $p^2/2 > E_0$ do not feel this barrier and move between $q = -a$ and $q = a$.

**Guiding Example** In order to illustrate the key idea "*Identification of conformational subsets via eigenvectors of the transition operator*", consider the simple particles-in-a-box ensemble explained in Fig. 1. In terms of the zigzag functions from Fig. 1, the position $z = z(q,p)$ of some particle with initial position $q$ and momentum $p$ after some time span $\tau$ can be denoted as

$$z(q,p) = \begin{cases} Z_a(q + \tau p), & \frac{1}{2}p^2 > E_0 \\ Z_a^{\pm}(q + \tau p), & \text{otherwise}, \pm q > 0 \end{cases}$$

Let the initial distribution of energy $E = p^2/2$ for the particles in the ensemble be given by the Boltzmann distribution $\mathcal{P}(p) = \exp(-\beta p^2/2)/\mathcal{Z}$ with $\beta$ being Boltzmann's inverse temperature and $\mathcal{Z}$ such that $\int_{\mathbb{R}} \mathcal{P}(p)dp = 1$. In addition, suppose that the initial positions are equidistributed in $(-a, a)$. Then, the transition probability in the ensemble from $B \subset (-a, a)$ to $C \subset (-a, a)$ is given by

$$w(\Gamma(B),\Gamma(C),\tau) \;=\; \left(\int_B dq\right)^{-1} \int_B\int_{\mathbb{R}} \chi_C(z(q,p))\,\mathcal{P}(p)\,dp\,dq.$$

Whenever $\beta$ and $E_0$ are chosen such that particles with energy $E > E_0$ are rare, that is, whenever $\epsilon = \int_{p^2/2>E_0} \mathcal{P}(p)dp$ is small, then the two sets $B = (-a,0)$ and $C = (0,a)$ are *almost invariant* in the sense that only a small fraction of the particles can move from $B$ to $C$. In this case, the fundamental difference between the two notions of almost invariance is particularly significant: the ensemble has two obvious almost invariant sets, while for none of the *single* particles the notion of "almost invariant sets" makes sense.

The associated transition operator $T$ acts on functions $u : (-a,a) \to \mathbb{R}$ and is defined via

$$Tu(q) = \int_{\mathbb{R}} u\,(z(q,p))\,\mathcal{P}(p)\,dp.$$

In fact, this transition operator allows to represent the transition probabilities: Using the usual scalar product $\langle\cdot,\cdot\rangle$ in the Hilbert space $L^2(-a,a)$, we find that

$$w(\Gamma(B),\Gamma(C),\tau) \;=\; \frac{1}{\langle\chi_B,\chi_B\rangle}\,\langle T\chi_C,\chi_B\rangle.$$

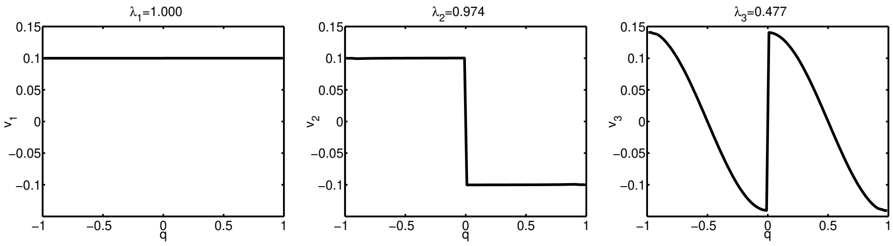



Figure 2: Eigenvectors for the largest eigenvalues $\lambda_1 = 1$, $\lambda_2 = 0.974$, and $\lambda_3 = 0.477$ (from the left to the right) of the transition operator $T$ for the particles-in-a-box ensemble from Fig. 1 above ($a = 1$, $\beta = 25$, $\tau = 2$, $E_0 = 1/10$, leading to $\epsilon \approx 0.025$). The eigenvectors for $\lambda_1 = 1$ and $\lambda_2 = 0.974$ are constant on the two almost invariant sets $(-1,0)$ and $(0,1)$ and their *signs* suffice to decompose $(-1,1)$ into these two almost invariant sets. For details see Sec. 3.2.

We will see in detail in Sec. 3.2 that, with respect to $L^2(-a,a)$, the spectrum of $T$ lies in the interval $(-1,1]$ and is discrete. For $\epsilon = \int_{p^2/2>E_0} \mathcal{P}(p)dp$ being small, it shows some significant gap between the two dominant eigenvalues $\lambda_1 = 1$ and $\lambda_2 \approx 1-\epsilon$ and the remaining eigenvalues. As can be seen in Fig. 2, the eigenfunctions $v_1$ and $v_2$ for these two dominant eigenvalues suffice to decompose $(-1,1)$ into these two almost invariant sets simply by the two different combination of signs ((positive,positive) for $(-1,0)$ and (positive,negative) for $(0,1)$).

**Properties of the Transition Operator** It is the fundamental strategy of our approach to compute conformational subsets from eigenstates of $T$ for eigenvalues near $\lambda = 1$. It is, thus, of main importance, whether such eigenvalues exist and the eigenvalue $\lambda = 1$ is simple. Since we are interested in a numerically stable approximation result, we have to demand for the existence of *isolated* eigenvalues near $\lambda = 1$. Hence, the second part of the manuscript (Sec. 4) is concerned with the construction of conditions which guarantee that the essential spectrum $\sigma_{\text{ess}}(T)$ of the transition operator is strictly bounded away from $\lambda = 1$ (Sec. 4.2). Under some additional mixing assumption (open set accessibility), the specific properties of the Markov operator $T$ also guarantee that the eigenvalue $\lambda = 1$ is simple and dominant (Sec. 4.4). So far the manuscript follows the "operator–oriented" approach mainly by using classical results from linear functional analysis.

The investigation of the fundamental properties of the transition operator $T$ reveals another crucial insight: $T$ is associated with some specific *stochastic dynamical system*, which can be simulated via the *corresponding Markov chain* (Sec. 3.6). We will see that some results of the operator–oriented approach (e.g., that $\lambda = 1$ is simple and dominant) can also be shown under weaker conditions by means of the well-established convergence theory for Markov chains (Sec. 4.5).

The abstract conditions for the above mentioned results are worth the effort only if we can give explicit evidence that they are in fact valid for some realistic "biomolecular" type of Hamiltonian systems. This is the case as we will see in the final subsection of Sec. 4. Thus, at the end of Sec. 4, it will be obvious that our novel approach is built on solid mathematical ground.

**Numerical Realization** Typical biomolecular systems contain hundreds or thousands of atoms such that any direct spatial discretization of the transition operator $T$ suffers from the curse of dimension. This problem can be (at least partly) circumvented by two decisive insights:

1. Chemical observations reveal that conformational transitions of biomolecules can be described via relatively few *conformational degrees of freedom* or *essential variables*. Hence, only the *essential configuration space* associated with these variables has to be discretized which leads to a tremendous reduction of dimension. Therefore, some *restricted transition operator* has to be introduced which now acts on the essential configuration space only (Sec. 3.5) but inherits all the crucial spectral properties of the full spatial transition operator discussed above (Sec. 4.6).

2. Since the underlying invariant density is given in advance by the experimental realization of the ensemble, one can use appropriate *Monte-Carlo* (MC) schemes to sample this distribution. Hence, the transition operator

can be discretized via some Galerkin ansatz and the entries of the resulting *transition matrix* can be evaluated simply by counting the transitions between discretization boxes during the MC sampling. The details of the MC scheme will result from the deep connection between our transition operator $T$ and the associated Markov chain, which will lead us to so-called Hybrid Monte-Carlo (HMC) schemes. The specific Markov chain induced by HMC can be seen as an approximation of the original chain associated with $T$ and inherits all its fundamental properties (see Sec. 5.4).

Last but not least, we need some algorithm for the final identification of almost invariant sets on the basis of the discrete eigenvectors of $T_n$. The molecular dynamics group at the Zuse Center developed some prototype which is based on the interpretation of almost invariant sets as perturbed invariant sets (see Deuflhard et al. [24]). This *identification algorithm* is discussed in detail in Sec. 5.3. It exploits that the transition matrix $T_n$ is a stochastic matrix with a cluster of eigenvalues near $\lambda = 1$ and associated eigenvectors that are approximately constant on the underlying almost invariant sets (compare Fig. 2). With the HMC-based evaluation of the transition matrix $T_n$ discretizing $T$ the application to realistic molecular systems comes into reach. The applicability of the ensemble approach to realistic molecular systems including Galerkin discretization of the transition operator and identification of conformational subsets is documented in Sec. 6.


## Acknowledgments

It is a pleasure to thank all those who made this work possible by providing support, ideas, and encouragement. I would like to mention a few to whom I am especially thankful:

Peter Deuflhard, for guiding me into this field, offering continuous support, and letting me share his impressive mathematical intuition which among others initiated the project this manuscript reports on.

Michael Dellnitz and Oliver Junge, for many stimulating discussions considering the same key problems again and again with (almost) invariant patience, and for their encouragement and criticism on the way towards a solution.

The molecular dynamics group at the Zuse Center, Frank Cordes, Peter Deuflhard, Alexander Fischer, Wilhelm Huisinga, Peter Nettesheim, and Sebastian Reich, for the stimulating interdisciplinary atmosphere, many important questions and answers, various hints, and simply for enduring me in the last six months.

Folkmar Bornemann, for asking exactly the right questions in some decisive moments when I was able to develop answers to them.

And finally, my family, from which this work has taken much time off that should have been spent with them.

# 2 Mathematical and Physical Modelling

In this section, we complement the introductory comments on the modelling issue. In the first part of this section, some aspects of molecular dynamics are collected which are of special importance for the understanding of the side-conditions of realistic applications to biomolecular systems: the structure of the energy function determining the Hamiltonian equations of motion, the necessity of periodic boundary conditions, the existence and origin of conformations and conformational degrees of freedom, and the numerical background of the fundamental difficulties of long-term simulation. The second part is concerned with some notions and concepts from statistical mechanics which are necessary for the final definition of transition probabilities and almost invariance. It also includes a short discussion of the Frobenius–Perron operator in the context of statistical mechanics and molecular dynamics.

## 2.1 Classical Molecular Dynamics

### 2.1.1 Hamiltonian and Flow

In classical MD (cf. textbook [1]) a molecule is modeled by a Hamiltonian function

$$H(q,p) \;=\; \tfrac{1}{2}\,p^T M^{-1} p \,+\, V(q), \tag{1}$$

where $q$ and $p$ are the corresponding positions and momenta of the atoms, $M$ the diagonal mass matrix, and $V$ a differentiable potential. The Hamiltonian $H$ is defined on the phase space $\Gamma \subset \mathbb{R}^{2d}$. Realistic MD-simulations typically include a large number $N$ of atoms resulting in $d = 3N$ spatial coordinates. Thus, the dimension of $\Gamma$ is $2d = 6N$. The corresponding canonical equations of motion

$$\dot q \;=\; M^{-1}p, \qquad \dot p \;=\; -\mathrm{grad}\, V(q) \tag{2}$$

describe the dynamics of the molecule. The formal solution of (2) with initial state $x_0 = (q(0), p(0))$ is given by $x(t) = (q(t), p(t)) = \Phi^t x_0$, where $\Phi^t$ denotes the flow.

It is well–known that Hamiltonian flows have several important conservation and invariance properties. For the considerations herein, three of these properties are of main importance [5]: First, the *conservation of energy* $H(\Phi^t x) = H(x)$; second, the *symplecticness* of the flow, which implies the well-known volume conservation property, the local expression of which is

$$\det D\Phi^t x \;=\; 1, \qquad \text{for all } x \in \Gamma, \tag{3}$$

where $D\Phi^t$ denotes the Jacobian matrix of the flow; and, third, the *reversibility* of $\Phi^t$:

LEMMA 2.1 *Let $R$ denote the momentum reversion, i.e., $R(q,p) = (q,-p)$. Then, the flow $\Phi^t$ is $R$-reversible, i.e.,*

$$R \circ \Phi^{-t} \circ R = \Phi^t.$$

*Proof:* The equations of motion (2) are invariant under the substitution $t \to -t$ and $p \to -p$, a fact which directly implies $\Phi^{-t} \circ R = R \circ \Phi^t$. ∎

An $x \in \Gamma$ is called a state of the system. For a given state $x = (q,p)$, we will make frequent use of the notations

$$q = \pi_1 x \quad \text{and} \quad p = \pi_2 x, \tag{4}$$

which allow us to extract the position and momentum information from $x$. The set $\Omega = \pi_1 \Gamma \subset \mathbb{R}^d$ is then called the *position space*. In most cases the phase space is simply given by $\Gamma = \Omega \times \mathbb{R}^d$, i.e., for every position $q \in \Omega$ arbitrary momenta $p \in \mathbb{R}^d$ are allowed.

### 2.1.2 Potentials and Atomic Interactions

In classical molecular dynamics, the interaction potential $V$ of a molecular system is modelled as a sum of contributions from different types of interaction. To explain this, let the $N$ atoms in the molecule be numbered by $1, \ldots, N$ and let the position of the $k$th atom in the molecule be denoted $q_k \in \mathbb{R}^3$ such that the molecule's entire position state is given by $q = (q_1^T, \ldots, q_N^T)^T \in \Omega$. The typical interactions and associated types of potentials can be divided into "local" interactions and "long-range" interactions. Typically, local interactions are induced by the bond structure of the molecule as, e.g.,

- stretching of a covalent bond between two atoms $k$ and $l$, modelled by some potential with radial symmetry, for example, by a harmonic potential $V_{\mathrm{b}}(q_k, q_l) = \alpha(r_{kl} - r_0)^2$ with $r_{kl} = |q_k - q_l|$;
- changes in the angles between some bonds, modelled via *bond angle potentials* $V_{\mathrm{ba}}$ (depending only on the angle between two bonds, i.e., depending on three atomic positions),[3] and via so-called *dihedral angle potentials* $V_{\mathrm{dih}}$ (depending on the "dihedral angle" between the two planes which are spanned by four neighboring atoms, compare Fig. 3),

while typical long-range interactions are independent of the bond structure, e.g.,

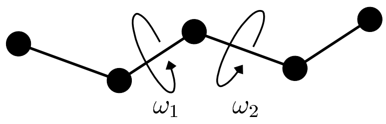
- electrostatic interactions induced by the (partial) charges of the atoms $k$ and $l$, modelled by the Coulomb potentials $V_{\mathrm{c}} \sim r_{kl}^{-1}$;
- van der Waals interactions, modelled by Lennard-Jones potentials $V_{\mathrm{LJ}}$ of the form $br^{-12} - ar^{-6}$.

[3] The bond angle contribution of the two bonds between atoms $k$ and $l$, and $l$ and $j$ depends only on the bond angle $\phi_{klj}$ given by $\cos\phi_{klj} = (q_k - q_l)^T(q_l - q_j)/r_{kl}r_{lj}$, i.e., is of the form $V_{\mathrm{ba}} = V_{\mathrm{ba}}(\phi_{klj})$.

The resulting total potential then reads:

$$\begin{aligned} V(q) &= \sum_{k,l} V_{\mathrm{b}}(q_k,q_l) + \sum_{k,l,j} V_{\mathrm{ba}}(q_k,q_l,q_j) + \sum_{k,l,j,m} V_{\mathrm{dih}}(q_k,q_l,q_j,q_m) \\ &\quad + \sum_{k,l} V_{\mathrm{LJ}}(q_k,q_l) + \sum_{k,l} V_{\mathrm{c}}(q_k,q_l), \end{aligned}$$

where the sums run over each pair, triple, or quadruple of atoms which contributes to the corresponding type of interaction. Generally speaking, all other many-body interactions are incorporated in an "average" way via these pair-, triple-, quadruple-additive terms. Errors in one term are compensated by parameter adjustments in other terms, so that the applicability of the resulting force field is always limited: it is never more than *semi-empirical.* There are some other types of interaction like, e.g., polarizability, which are included in essentially different ways. For more details, the interested reader is referred to [1].



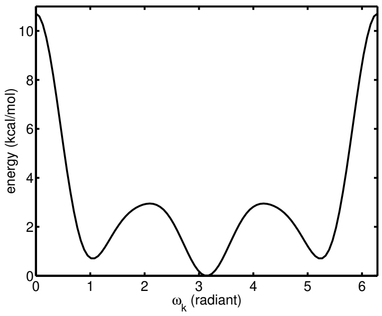




Figure 3: United atom model of n-pentane with the two dihedral angles $\omega_1$ and $\omega_2$. On the right: Dihedral angle potential $V_{\mathrm{dih}}$ as a function of the dihedral angle due to [89]. The main minimum corresponds to the so-called trans orientation of the angle, the two side minima to the so-called $\pm$gauche orientations.

Remark 2.2 Some of the contributing potentials are unbounded. This is an artifact contradicting the physical situation: when excited with enough energy, any bound will break (thus the harmonic form of $V_{\mathrm{b}}$ is inappropriate for large energies); for any pair of atoms the situation $|q_k - q_l| = 0$ is impossible, i.e., Coulomb and Lenard-Jones potential are inappropriate for $|q_k - q_l| < \delta$ for some $\delta$ representing, e.g., the size of the atoms.[4] One can deal with these problems by simply adapting the potentials: For example, we may introduce hard core collision conditions for two atoms $k$ and $l$ whenever $|q_k - q_l| = \delta$. As a consequence, we may assume that all types of potentials are *smooth* and *bounded* but we have to pay for this by accepting *collision boundary conditions.*

[4] Obviously, these problems can only occur when the kinetic energy of the system is huge which typically is not the case. Nevertheless, in typical models (as, for example, in the canonical ensemble, see Sec. 2.2.2) such a situation is a rare event but not impossible.

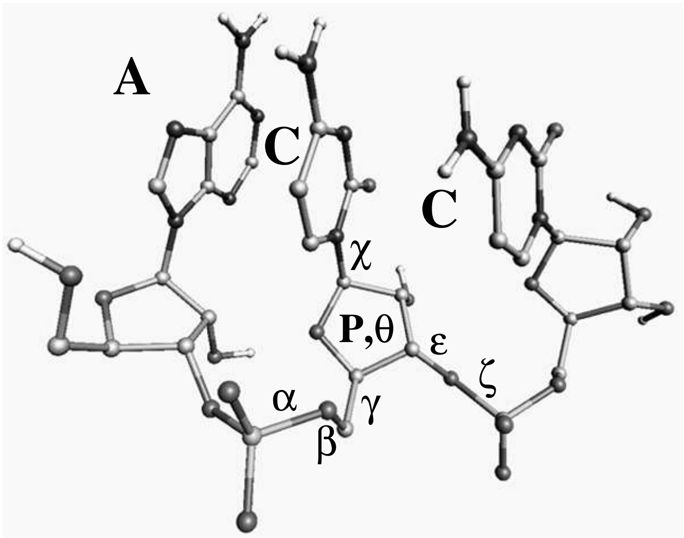



Figure 4: The triribonucleotide adenylyl*(3'-5')*cytidylyl*(3'-5')*cytidin [*r(ACC)*] in the extended atom representation of GROMOS96 [109]. A and C denote the bases adenine and cytosine. Small greek letters refer to the set of torsion angles, which is necessary for a rough reconstruction of the molecule's configuration. The torsion angles of the ribose can be approximated by the pseudorotation angle $P$ and the phase $\theta$ [3].

### 2.1.3 Essential Variables and Conformations

The typical molecular force fields are mixtures of bond-structure effects and strong long-range interactions. The potentials modelling bond interactions are functions of certain *internal degrees of freedom* like bond angles or dihedral angles. It suggests itself to rewrite the equations of motion in terms of these internal coordinates. Unfortunately, this provokes a whole bunch of nasty problems, e.g., with the efficiency of the evaluation of the forces, in particular of the long-range forces. However, the internal coordinates represent the (spatial) geometry of the molecule so that changing some internal coordinate affects the molecule's "form". But most of the energetically possible changes are of minor importance; they can be seen as small fluctuations around the actual (meta)stable "global" molecular geometry, called the *conformation*. Biomolecules typically appear in different conformations and the coordinate changes which transform one conformation into another one are object of main chemical interest. Normally, these *conformational transitions* can be described in terms of only a few internal coordinates, which are therefore called *conformational degrees of freedom* or *essential variables*. In many cases, essential variables simply are specific dihedral angles connecting some otherwise nearly rigid subgroups of the molecule (cf. Fig. 4), but they may also be combinations of different internal coordinates (see [4] or Sec. 3.3 of Berendsen's survey in [23]). Clearly, whether a certain internal variable may be an essential variable, depends on the structure of the whole molecule and can be made sure only by inspection of its dynamical behavior.

Nevertheless, chemical experience and intuition can often point out a collection of candidates and the statistical analysis of simulation data (for example via diagonalization of the covariance matrix [4, 30]) can supply other candidates.

The dynamics of every (bio)molecular system contains a large number of extremely different time scales: On the smallest time scales (around 1 femtosecond), the motion of the molecule consists of fast oscillations around equilibrium positions,[5] while all chemically significant molecular processes like conformational changes will show up, e.g., on a millisecond time scale. Thus, investigation of conformational transitions requires extremely long time spans so that it still is inaccessible to conventional simulation methods (compare Sec. 2.1.5 below).

Today, a varied collection of methods for describing conformational dynamics is available. These approaches are substantially different; they range from "simply" visualizing a hypothetical path for conformational transitions via interpolation between experimentally observed crystal structures [111] to methods artificially changing the atomistic description of molecular dynamics for allowing the acceleration of conformational transitions. The latter kind of approach includes such different concepts as the combination of molecular dynamics with reaction path methods [84], so-called "conformational flooding" via subsequent modifications of the original potential energy surface [47], or "steered molecular dynamics" by simulating atomistic force microscope experiments [59, 48]. Despite all differences, these methods share the same basic idea: to circumvent the inaccessibility of conformational transitions by means of changing the physical model. In contrast to this, our direct approach tries to leave the (reliable) atomistic model intact but replaces long-term simulation by an appropriately chosen ensemble of short subtrajectories.

### 2.1.4 Boundary Conditions

Typically, biomolecular experiments are concerned with large numbers of some certain type of biomolecule embedded in a crystal or in solution. For modelling a crystal, it suggests itself to use *periodic boundary conditions*,[6] because a crystal may be understood as an infinite repetition of some elementary cell, where each of these cells contains, e.g., one of the molecular systems under investigation. Similarly, periodic boundary conditions are also used to model biomolecular solute/solvent systems in the typical test-tube situation: a large number of biomolecules of the same type is irregularly, but nearly homogeneously distributed in a solute which itself consists of (small) molecules (e.g., water and ions). Each of the "large" biomolecules is surrounded by its "hydration shell" consisting of many solute molecules, so that the molecule together with its shell can be understood as a large "biomolecular unit". These large units are only loosely coupled to each other via the exchange of electrostatic energy and so-

[5] These are nothing but fluctuations (e.g., bond length or bond angle vibrations) inside the otherwise (meta-)stable conformation of the system.

[6] In molecular dynamics the phrase "periodic boundary conditions" means the reformulation of the associated Hamiltonian equation of motion on some *torus* $\Omega$.

lute molecules. Typically this situation is modelled by a system containing one of the biomolecular units with periodic boundary conditions allowing for some exchange with its neighbors (=copies), see Fig. 5.[7] In this sense, periodic boundaries are of main importance for modelling the intermolecular exchange.

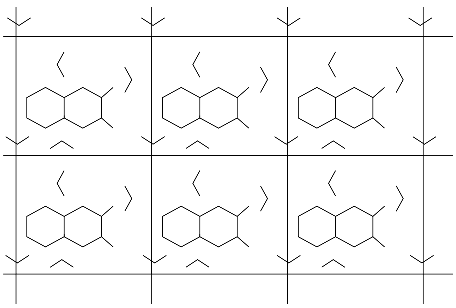


Figure 5: Illustration of periodic boundary conditions for modelling intermolecular exchange. In the lower left corner of each periodicity cell a water molecule is exchanged between neighboring cells.

While efficient algorithmic realization of periodic boundary conditions for the long-range interactions has achieved a lot of attention (cf. [67, 17]), there is no general strategy for dealing with bond-interactions in a periodic setting. Normally, one tries to fix the biomolecule inside the periodicity cell and implements periodic boundaries only for the motion of the small solute molecules and for the long-range interactions.

Summarizing, for crystals or pure liquids, it is realistic to assume that the potential used in MD simulations is periodic, while this is a crude but not totally misleading assumption for biomolecular systems in solution.

**Assumptions concerning Potentials and Position Space** As a consequence of the above considerations the subsequent investigation is restricted to the following cases: We always assume that the potential is smooth and that singularities in the interior of the position space $\Omega$ are avoided, for example by means of collision boundary conditions. Whenever $\Omega$ is unbounded (that is, in most cases, $\Omega = \mathbb{R}^d$) we always suppose that the potential $V$ is *binding*, i.e., satisfies $\lim_{|q|\to\infty} V(q) = \infty$. The case that the position space $\Omega$ is bounded is always considered in context with periodic boundary conditions: Then, $\Omega$ is some rectangular box in $\mathbb{R}^d$ (that is, in particular, $\Omega$ is compact) and the potential $V$ is assumed to be bounded and smooth at the boundary (that is, it can be extended smoothly as $\Omega$-periodical function).

[7] Without the periodic boundary, i.e., if the model would only include a single "free" biomolecular unit, the polarization of the water molecules in the hydration shell would be significantly different. But the hydration shell has important influence on the behavior of the biomolecule, which illustrates the importance of the periodic boundary.

### 2.1.5 Problems with Long-term Simulation

Despite all problems mentioned so far, let us assume, now and in the following, that we are concerned with some potential $V$ and corresponding boundary conditions which together appropriately model the interactions of a certain molecular system. In order to analyze the dynamical behavior of this system, we then have to approximate the associated flow $\Phi^t$, i.e., we have to apply some numerical discretization technique to the equation of motion (2). Typically, one uses symplectic one-step methods like the well-known "Verlet" scheme, named after its early inventor L. Verlet [110]. Generally, in the process of one-step numerical integration of (2) we replace $\Phi^t$ by a *discrete flow* $\Psi^{\Delta t}$, so that

$$x_{k+1} \;=\; \Psi^{\Delta t} x_k \quad \Rightarrow \quad x_k \;=\; \left(\Psi^{\Delta t}\right)^k x_0,$$

with stepsize $\Delta t$ (assumed to be constant, for the time being).

It is an important feature of molecular processes that *long term predictions* over periods tremendously longer than the time steps applied in the discretization are required. As already mentioned, the dynamics of every (bio)molecular system contains extremely different time scales, from fast vibrations on scales around 1 femtosecond to chemically significant molecular processes on, e.g., a millisecond time scale. Unfortunately, every numerical discretization schemes is forced to use time-steps of the order of magnitude of the fastest vibrations; already time steps of about 5 femtoseconds result in dramatic instabilities [98]. Consequently, inspection of most chemically relevant processes by direct long-term simulation requires such a huge number of time steps that it still is inaccessible to conventional MD methods.

But numerical long term predictions seem to be inappropriate also for another, perhaps more important reason: *Numerical analysis of present discretizations restricts the validity of the discrete solution to only short time spans and to comparatively small discretization steps.* Let us shortly illuminate this statement by summarizing the results of so-called "forward" and "backward" analysis:

In "forward" analysis, one is interested in the propagation of initial perturbations $\delta x_0$ along the flow $\Phi^t$ of (2), i.e., in the growth of the perturbations $\delta x(t; x_0) = \Phi^t(x_0 + \delta x_0) - \Phi^t x_0$. The *condition number* $\kappa(t)$ may be defined as the worst case error propagation factor (cf. textbook [20]), so that, in first order perturbation analysis and with a suitable norm $|\cdot|$:

$$|\delta x(t; x_0)| \;<\; \kappa(t)|\delta x_0| \quad \text{for all } x_0.$$

Note that this number $\kappa(t)$ is *independent of any discretization.* From this point of view, numerical integration is reasonable only over time intervals $[0, T]$ with $\kappa(T)$ sufficiently small compared to expected input errors. In real life MD problems, however, $\kappa$ seems to be exponentially increasing (see [1, 21] for examples).

The results of "backward" analysis [91, 49, 7] are more specific: For symplectic discretizations, the discrete solution of a certain Hamiltonian system with Hamiltonian $H$ is "exponentially close" to the exact solution of some *perturbed* Hamiltonian system, in which, for consistency order $p$ and stepsize $\Delta t$, the perturbed Hamiltonian has the form

$$\tilde{H} \;=\; H \;+\; \sum_{k=0}^{N} \;\Delta t^{p+k}\, H_k \tag{5}$$

This means that the discrete solution nearly conserves the Hamiltonian $\tilde{H}$ and, thus, conserves $H$ up to $\mathcal{O}(\Delta t^p)$.[8] In fact, numerical observations show that the average of the *total energy* is nearly constant over rather long time spans for large stepsizes, say $\Delta t \approx 1$ femtosecond. Whenever one is not interested in a single discrete trajectory but in approximating *time averages* of (macro-)observables over a time interval $[0,T]$ via associated mean values of $x_k$, $k = 1 \ldots T/\Delta t$, the results of backward analysis may lead to much better error estimates than the worst case estimates of forward analysis (but clearly, only as long as $T$ (or $\Delta t$) are small enough). Compare [87] for more details.

## 2.2 Statistical Mechanics

It is not the only problem that long term prediction of *single* solutions of the Hamiltonian system (2) is *numerically* ill-conditioned. There also are purely *physical reasons* which let it seem questionable to compute any single solution, even if this solution were arbitrarily accurate. This may come as a surprise, but it has a simple reason: We can never know the precise initial state —all the positions and momenta— of the whole molecule, simply because we *in principle* always have to accept measurement uncertainties when determining the initial state. When modelling the physical reality, this simple insight always forces us to propagate a *collection* of trajectories which "samples" the distribution of possible initial states. In this sense, we always have to simulate an *ensemble* of molecular systems which represents the *distribution of possible initial states determined via the initial measurement.* Then, every comparison of later measurements with simulation results will concern *mean* or *expectation values* and *not* any single system in the ensemble. Hence, we have to consider an ensemble of systems described by a *time dependent probability density* $f = f(x,t)$ in phase space, which obviously has to satisfy

$$f(x,t) \;=\; f_0(\Phi^{-t}x), \qquad \text{with } f_0 \;=\; f(\cdot, t=0), \tag{6}$$

i.e., the probability $f_0(x)$ of being in $x \in \Gamma$ at time $t = 0$ is simply transported along the trajectory $\Phi^t x$ of the system. Even if the initial density $f_0$ is concentrated near an initial position $x_0$, it may become disintegrated or "smeared out", so that the trajectory $\Phi^t x_0$ alone cannot describe the situation appropriately.

[8] In general, however, the above formal series diverges as $N \to \infty$ and the term "exponentially close" has to be specified carefully. See [50] for details.

**Initial Preparation of an Ensemble** The density $f_0$ describes the initial probability distribution in the statistical ensemble, i.e., $f_0(x)$ is interpreted as the relative frequency in the ensemble of systems in state $x$ at time $t = 0$. Therefore, the density $f_0$ has to be *defined in accordance with the initial experimental preparation* of the ensemble. The phrase "preparation" reflects that the ensemble should be imagined as a collection of copies of the same system, each initially in one of the possible states with the collector having to "prepare" the collection such that the correct relative frequencies are achieved. In this sense, the evolution of the density $f = f(x,t)$ should *not* be interpreted as describing the "possibility" of finding a certain *single* system in a certain state, but as a "relative frequency" of systems in the ensemble occupying this certain state: The latter can be measured, the first not at all.

### 2.2.1 Liouville Equation

Another formulation of the evolution (6) of the probability density uses the Liouville equation associated with the Hamiltonian $H$:

$$\partial_t f \;=\; i\mathcal{L}f \;=\; \{H, f\}, \qquad f(t=0) \;=\; f_0, \tag{7}$$

where $\{\cdot,\cdot\}$ denotes the well–known Poisson bracket.[9] $\mathcal{L} = -i\{H,\cdot\}$ is a self-adjoint operator on the Hilbert space $L^2(\Gamma)$, called the associated Liouville operator (cf. [65, 62]). The solution of (7) in fact satisfies (6). On the other hand, it can be denoted using the semi-group generated by $\mathcal{L}$:

$$f(\cdot,t) \;=\; \exp(it\mathcal{L})\, f_0 \;=\; f_0 \circ \Phi^{-t}, \tag{8}$$

for example, on the Hilbert space $L^2(\Gamma)$.

### 2.2.2 Stationary Ensembles and Invariant Densities

By far the most experiments are performed using *equilibrium ensembles*, i.e., ensembles which are described by *stationary densities* of the Liouville equation. In view of (6), these stationary densities $f$ are given by *invariant densities* of the flow, i.e., densities $f$ such that $f(x) = f(\Phi^{-t}x)$ for all instances $t$ and all $x \in \Gamma$. In particular, for *arbitrary* smooth functions $\mathcal{F} : \mathbb{R} \to \mathbb{R}_0^+$ with

$$\int_\Gamma \mathcal{F}(H(x))\, dx \;=\; 1,$$

the associated densities $f(x) = \mathcal{F}(H(x))$ are invariant. In our context the most important features of these "energy prepared" densities are the following two:

$$\boxed{\begin{aligned} f &= f \circ \Phi^{-\tau}, \quad \text{i.e., } f \text{ is invariant}, && (9)\\ f &= f \circ R, \quad \text{i.e., } f \text{ is } p\text{-symmetric}, && (10)\end{aligned}}$$

[9]That is, for smooth functions $f, g : \Gamma \to \mathbb{R}$: $\{f, g\} = D_q f \cdot D_p g - D_q g \cdot D_p f$, with $D_q$ and $D_p$ denoting the derivatives with respect to positions and momenta, respectively.

where $R$ denotes the momentum reversion $R(q,p) = (q, -p)$.

**Canonical Density** Most experiments on molecular systems are performed under the equilibrium conditions of constant temperature, particle number, and volume. The corresponding stationary density is the *canonical density* associated with the Hamiltonian $H$

$$f_{\text{can}}(x) = \frac{1}{\mathcal{Z}} \exp\left(-\beta\, H(x)\right), \quad \text{with} \quad \mathcal{Z} = \int_\Gamma \exp\left(-\beta\, H(x)\right) dx,$$

where $\beta = 1/k_B\mathcal{T}$, with $\mathcal{T}$ being the system's temperature and $k_B$ Boltzmann's constant.[10] Since $H$ was assumed to be separable, $f_{\text{can}}$ is a product

$$f_{\text{can}}(x) = \underbrace{\frac{1}{\mathcal{Z}_p} \exp\left(-\frac{\beta}{2}\, p^T M^{-1} p\right)}_{=\mathcal{P}(p)} \underbrace{\frac{1}{\mathcal{Z}_q} \exp\left(-\beta\, V(q)\right)}_{=\mathcal{Q}(q)}, \tag{11}$$

where we normalize $\mathcal{P}$ and $\mathcal{Q}$ such that

$$\int \mathcal{P}(p)dp = \int \mathcal{Q}(q)dq = 1. \tag{12}$$

For the case that the position space $\Omega$ is unbounded, we have to guarantee that the partition function $\mathcal{Z}$ is finite and the normalization (12) is possible. Thus, we restrict our consideration to the case of binding potentials, for which we always assume that the asymptotic growth is fast enough to guarantee that $\int_\Gamma \exp(-\beta H(x))dx < \infty$.

## 2.3 Frobenius–Perron and Koopman Operators

One can analyze the statistical properties of rather general deterministic dynamical systems independent of any connection to statistical mechanics and its interpretation. Typically, this is realized by means of the *Frobenius–Perron operator* of the dynamical system on the set $\mathcal{M}$ of probability measures. For the discrete Hamiltonian system $x_{k+1} = \Phi^\tau x_k$, the Frobenius–Perron operator can be defined by

$$(P\mu)(B) = \mu\left(\Phi^{-\tau}(B)\right), \quad \text{for all measurable } B \subset \Gamma \text{ and } \mu \in \mathcal{M}.$$

We are mainly interested in absolutely continuous measures $\mu$ and, thus, in the form of the operator acting on the associated densities. When restricted to densities, the Frobenius–Perron operator takes a particularly simple form for

[10] The canonical density is often called the *Boltzmann distribution* or *Gibb's canonical distribution.* It is known to be the maximizer of Boltzmann's entropy function $S(f) = -\int f \log f dx$ in the space of all densities under the condition of given energy expectation $\langle H\rangle$ of the ensemble. If $\beta$ is associated with the temperature $\mathcal{T}$, the maximal entropy $S(f_{\text{can}}) = \log\mathcal{Z} + \beta\langle H\rangle$ is just the thermodynamic entropy for systems with fixed temperature $\mathcal{T}$ and internal energy $\langle H\rangle$ and for given volume and particle number (cf.[117], Chap. 1.3).

measure-preserving maps like our discrete Hamiltonian system. In this case it is defined by

$$Pf = f \circ \Phi^{-\tau}, \tag{13}$$

a definition, which we may extend, e.g., to the usual function spaces $L^p(\Gamma) = \{f : \int_\Gamma |f(x)|^p \, dx < \infty\}$, $p = 1, 2$. Our short encounter with the Liouville equation, especially equality (8), allows us to rewrite our Frobenius–Perron operator $P$ in terms of the Liouville operator $\mathcal{L}$ as

$$P = \exp(i\tau\mathcal{L}) \qquad \text{in } L^p(\Gamma),\ p = 1, 2. \tag{14}$$

The associated adjoint operator $P^* = \exp(-i\tau\mathcal{L})$ is called Koopman operator due to B.O. Koopman [62]. Koopman's lemma (cf. Lemma A.16 in Appendix A) states that $P^*$, and herein also $P$, are *unitary* as operators on $L^2(\Gamma)$.[11] Consequently, for Hamiltonian systems, the $L^2$-spectrum of the Frobenius–Perron operator lies on the unit circle, i.e., it has *no eigenvalues inside the unit circle.* As already mentioned in the introduction of this manuscript, this is the central difficulty of the "dynamical system approach" to the identification of conformations.

**Ensembles versus Single Systems** There are at least two significantly different interpretations of the Frobenius–Perron operator $P$ for Hamiltonian systems:

- Due to (14), we can understand $P$ in the context of Statistical Mechanics as the *propagator of an ensemble.* Its norm-preserving properties guarantee the possibility of this statistical interpretation (no "loss" of probability).
- One can also interpret $P$ in a *probabilistic* sense for *single systems*, as typically done in the theory of dynamical systems, via its *invariant measures*, i.e., measures $\mu \in \mathcal{M}$ such that $P\mu = \mu$. If an invariant measure $\mu$ is ergodic,[12] Birkhoff's ergodic theorem states that we have

$$\overline{\mathcal{A}}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{A}(\Phi^{k\tau} x) = \int_\Gamma \mathcal{A}(y)\, \mu(dy), \tag{15}$$

  for $\mu$-almost every $x \in \Gamma$ and every integrable function $\mathcal{A}$. Hence, $\mu$ describes the relative frequency with which the single system visits a certain phase space region during its evolution in time.

The total energy of our Hamiltonian system (2) is preserved along its trajectory so that the system stays on a certain energy surface, i.e., on a $2d-1$-dimensional submanifold of the phase space $\Gamma$. This implies

[11] For the special case of Hamiltonian systems, $P$ is norm-preserving in $L^1(\Gamma)$, too. This results from [66], Prop. 3.1.2.

[12] That is, it satisfies $\mu(B) \in \{0, 1\}$ for every invariant set $B \subset \Gamma$.

Proposition 2.3 *Every ergodic invariant measure of our Hamiltonian system (2) is* singular *with respect to the volume measure on* $\Gamma$.

In the context of molecular dynamics, the two above interpretations are mostly expected to be equivalent, in the sense of the ***physical ergodicity hypothesis*** which states that the ensemble average (expectation value for the stationary ensemble) equals the running average $\overline{\mathcal{A}}(x)$ for some single system in the ensemble. Prop. 2.3 states that this can be true only for ensembles which are distributed according to singular measures on $\Gamma$, that is, only such ensembles can be simulated by following the long-term dynamics of some single system in the ensemble.[13]

In turn, the *ensembles with* $L^1(\Gamma)$*-densities* considered herein (like the canonical ensemble, for example), *cannot be generated by iterates of a single system.*

In general, the invariant density associated with the evolution of any single system is determined by the corresponding initial state, while every stationary ensemble density is determined by the initial preparation of the ensemble.

## 2.4 Conformations as Almost Invariant Sets

Assume that an (arbitrary) stationary density $f_0$ is given. How to define the *transition probability* from one region $B \subset \Gamma$ of the phase space to another one, $C \subset \Gamma$? We are herein only interested in considering transition probabilities which allow for an *experimental* determination. The typical measurement process for any kind of transition probabilities is the following two-step experiment:

1. *Pre-Selection:* Select from the ensemble $f_0$ at $t = 0$ all such systems with states $x \in B$. This selection prepares a new ensemble, which now has the density

$$f_B(x) = \left( \int_B f_0(x)\, dx \right)^{-1} \chi_B(x)\, f_0(x).$$

2. *Transition-Counting:* After a time span $\tau$, determine the relative frequency of systems in the ensemble $f_B$ with states in $C$. Since all systems evolve due to $\Phi^t$, this relative frequency is equal to

$$\int_B \chi_C\left(\Phi^\tau x\right)\, f_B(x)\, dx.$$

Hence, in order to get a measurable quantity, the *transition probabilities* have to be defined as

[13] The physical ergodicity hypothesis is mostly used in the context of the *micro-canonical ensemble* which is given by the equidistribution on a certain energy surface (with respect to the projected Lebesgue measure). Then, it has to be understood as the assumption that Birkhoff's ergodic theorem holds with $\mu$ being this micro-canonical measure. Systems satisfying this condition are called "physically ergodic".

$$w(B,C,\tau) \;=\; \frac{\int_\Gamma \chi_C\left(\Phi^\tau x\right)\; f_0(x)\,dx}{\int_B f_0(x)\,dx}. \tag{16}$$

Using this definition we can introduce our notion of "almost invariance": A subset $B \subset \Gamma$ is called *invariant* under the flow $\Phi^t$ iff, for all $t \in \mathbb{R}$,

$$\Phi^t(B) \;=\; B \quad \text{and, thus,} \quad w(B,B,t) \;=\; 1,$$

where the last equality is *independent* of the choice of the stationary density $f_0$. We are interested in subsets $B$ with $w(B,B,\tau)$ sufficiently close to 1, to be denoted as *almost invariant* subsets. This first rough "definition",

$$B \subset \Gamma \text{ almost invariant} \quad \Leftrightarrow \quad w(B,B,\tau) \approx 1, \tag{17}$$

clearly depends on the interpretation of "sufficiently close to 1". For the next steps we will ignore the question how to define $\approx 1$ precisely; it will later come out to be problem-dependent and related to the eigenvalue structure of the associated transition operator. However, our definition of almost invariance depends on the choice of the stationary density $f_0$ and on the time span $\tau$.

**Conformations** Before introducing a mathematical definition of the phrase "conformation", let us collect the main aspects of the chemical intuition behind this phrase. Every conformation contains a lot of configurations, that is, it is a *set of configurations* characterized by the following properties:

- *Geometric similarity:* Every configuration in the set induces nearly the same global geometry of the molecule which can be described in terms of a certain set of internal variables, the so-called essential variables of the molecule (cf. Sec. 2.1.3).
- *Meta-stability:* The trajectory of a single system including all its fast oscillations around equilibrium positions remains inside this set for a long period of time before leaving it eventually.
- *Hierarchy of conformations:* Every conformation corresponds to one of the "main wells" of the potential. The potential has a huge number of local minima. Thus, every such main well must contain many local minima and must be separated from the remaining parts of the potential energy surface by substantially large energy barriers such that the trajectory of a single system is trapped in this well for some long period of time. Thus, there is a hierarchy of potential wells (every main well will decompose into several wells with less significant meta-stability). In turn, we also have to deal with a hierarchy of conformations.

Consequently, the connections between these three aspects are as follows: We need some measure of meta-stability in order to define the hierarchy of conformations. Then, we have to decide which level of this hierarchy we are willing to resolve and this decision determines whether two different kinds of global geometry are distinguished as indicating two different conformations or not.

Unfortunately, the above characterization of meta-stability is related to the concept of a single system. In order to define the notion "conformation" *in terms of the ensemble* under consideration, we have to transfer these characterization to the statistical level of description. The statistical concept of meta-stability is given by the notion of almost invariance due to (17), which leads us to the following statistical definition: *every conformation is an almost invariant set of the ensemble* in the sense of (17).

As the above considerations indicate, the chemical usage of the phrase "conformation" never refers to any momentum information. Consequently, we are only interested in *spatial* subsets, i.e., subsets of the position space $\Omega$. The *transition probability* between such spatial subsets $B \subset \Omega$ and $C \subset \Omega$ is given by the transition probability between the associated phase space fibers $\Gamma(B)$ and $\Gamma(C)$:

$$w(B, C, \tau) = w(\Gamma(B), \Gamma(C), \tau) \quad \text{with} \quad \Gamma(B) = \{(q, p) \in \Gamma,\, q \in B\}, \tag{18}$$

where the notational ambiguity is accepted for the sake of simplicity; in every case, the meaning of $w(B, C, \tau)$ is clear from the context. Consequently, some spatial subset $B \subset \Omega$ is called *almost invariant* iff $w(B, B, \tau) \approx 1$.

The probability $w(B, B, \tau)$ to stay within some set $B \subset \Omega$ induces our kind of a *statistical hierarchy*: an almost invariant set $B$ may contain almost invariant subsets $B_j$ but, whenever $w(B_j, B_j, \tau) < w(B, B, \tau)$, the decomposition of $B$ is interesting at most on finer levels of resolution. We will see that this statistical hierarchy induces an associated hierarchy of potential wells. In this sense, a decomposition of the potential energy landscape into several "main wells" corresponds to a decomposition of the position space into almost invariant sets with superior probability to stay within.

If we —due to the usual belief in chemistry— suppose that conformational transitions can be characterized via some few essential variables only, then we may further restrict the form of the almost invariant sets of interest: We do no longer consider arbitrary spatial sets $B \subset \Omega$ or the associated fibers $\Gamma(B) \subset \Gamma$ but only such sets which can be characterized in terms of the essential variables alone. This final restriction to such *conformational* subsets will be discussed in Sec. 3.5 in detail.

Summarizing, in order to characterize the conformational dynamics of the molecular system, almost invariant (spatial or conformational) subsets with superior probability to stay within and the transitions between them are the objects of interest.

# 3 Problem-Adapted Transition Operators

We are now ready to define the transition operator $T$ for replacing the inappropriate Frobenius–Perron operator $P$. The needs explained above require that $T$ must have the following properties:

- $T$ must have a unique invariant density reflecting the distribution in the experimentally prepared ensemble.
- $T$ has to represent the correct transition probabilities between subsets of the position space.
- Considered in appropriate spaces, $T$ must have isolated eigenvalues which allow to identify the conformations via the associated eigenvectors.

To this end, we will first define a *spatial transition operator*, which acts on functions living on the entire position space. After studying its basic properties, we will generalize this definition for allowing to include the restriction to essential variables (Sec. 3.5).

## 3.1 Spatial Transition Operator

Let us now assume, that the statistical ensemble under consideration is described by a (nonnegative) invariant phase space density $f_0 \in L^1(\Gamma)$ which satisfies conditions (9) and (10) and leads to a positive *reduced density*

$$F(q) = \int_{\mathbb{R}^d} f_0(q,p)\,dp, \tag{19}$$

which is smooth and finite on $\Omega$. The transition operator is given by

$$\boxed{Tu(q) = \frac{1}{F(q)} \int\limits_{\mathbb{R}^d} u\left(\pi_1 \Phi^{-\tau}(q,p)\right)\, f_0(q,p)\,dp, \qquad (20)}$$

where $u = u(q)$ is a function $u : \Omega \to \mathbb{C}$. Thus, $T$ is defined by a suitable *weighted average* of the Frobenius–Perron operator over the momenta in each of the trivial fibers

$$\Gamma(q) = \{x \in \Gamma,\, \pi_1 x = q\} = \{q\} \times \mathbb{R}^d,$$

where the weights are given by the experimentally prescribed stationary density $f_0$. Hence, the transition operator describes the statistics of the redistribution of systems in the ensemble via the flow $\Phi^\tau$ with respect to the time scale $\tau$. Since $f_0$ is stationary the shape of the ensemble distribution does not change. It is thus more adequate to say that $T$ describes the spatial *fluctuations* inside the ensemble $f_0$ induced by the flow $\Phi^\tau$.

We consider $T$ as an operator on the weighted spaces

$$L_F^p(\Omega) \;=\; \{u : \Omega \to \mathbb{C}, \quad \int_\Omega |u(q)|^p F(q)\, dq < \infty\}, \qquad p = 1, 2.$$

Obviously, $L_F^2(\Omega)$ is a Hilbert space with scalar product

$$\langle u, v\rangle_F \;=\; \int_\Omega u^*(q)\, v(q)\, F(q)\, dq$$

and induced norm $\|u\|_F^2 = \langle u, u\rangle_F$. On $L_F^1(\Omega)$, we use the canonical norm $\|u\|_{1,F} = \int_\Omega |u(q)|F(q)dq$. In the subsequent paragraphs we will discuss the important properties of $T$ with respect to these spaces. But before going into details, let us consider the special case of the canonical ensemble:

Example 3.1 For $f_0 = f_{\text{can}}$, the definition (11) yields $F(q) = \mathcal{Q}(q)$, so that, together with (20),

$$Tu(q) \;=\; \int_{\mathbb{R}^d} u(\pi_1 \Phi^{-\tau}(q,p))\, \mathcal{P}(p)\, dp. \tag{21}$$

Hence, in this case, $T$ describes the momentum weighted fluctuations inside the canonical ensemble with respect to the time scale $\tau$. In the following, always if $f_0 = f_{\text{can}}$, we simplify our notation and denote the weighted spaces $L_{\mathcal{Q}}^p(\Omega)$ from above by $L^p(\Omega)$, $p = 1, 2$, and, consequently, the associated norms by $\|\cdot\|_{p,\mathcal{Q}} = \|\cdot\|_p$.

### 3.1.1 Transition Operator for Periodic Potentials

Let us now discuss the case of a periodic potential, where

$$V(q + m l_j e_j) \;=\; V(q), \quad \forall m \in \mathbb{Z},$$

with $e_j$ being the unit vector in the $j$th coordinate direction. The associated position space is the "periodicity cell" $\Omega = \prod_{j=1}^d [0, l_j)$ and all considered ensemble densities $f_0$ are supposed to be normalized with respect to the restricted phase space $\Omega \times \mathbb{R}^d$. Hence, the transition operator $T$ should also be restricted to spaces of periodic functions on $\Omega$. That is, we may consider $T$ as acting on $L_{\text{per}}^p(\Omega)$ instead of $L^p(\mathbb{R}^d)$. It will later turn out that it is convenient to do this in the following way: Let us first define the *periodicity map* $\xi : \mathbb{R}^d \to \Omega$ as follows: For every $y \in \mathbb{R}^d$ there is a unique $q \in \Omega$ such that there is a tuple $(m_1, \ldots, m_d) \in \mathbb{Z}^d$ yielding

$$y \;=\; q + \sum_{j=1}^d m_j\, l_j.$$

This function $y \mapsto q$ is the periodicity map $\xi$, i.e., $\xi(y) = q$. Via $\xi_\Gamma(q,p) = (\xi(q), p)$, the map $\xi$ induces a periodicity map $\xi_\Gamma : \Gamma \to \Omega \times \mathbb{R}^d$ for the original phase space $\Gamma = \mathbb{R}^d \times \mathbb{R}^d$. With its help, we restrict the flow $\Phi^t$ to the torus given by $\Omega$ and the periodic boundary conditions by considering the flow

$$\Phi^t_\xi \;=\; \xi_\Gamma \circ \Phi^t \circ \xi_\Gamma \tag{22}$$

instead of $\Phi^t$. The transition operator is then defined by

$$Tu(q) \;=\; \frac{1}{F(q)} \int_{\mathbb{R}^d} u\left(\pi_1 \Phi_\xi^{-\tau}(x)\right)\, f_0(q,p)\, dp, \tag{23}$$

now acting on the spaces $L^p_F(\Omega)$ of functions $u : \Omega \to \mathbb{C}$ weighted with the density $F|_\Omega$.

### 3.1.2 Exemplifying Spectral Properties of $T$

In the following three examples, we will always consider the case of canonical ensembles ($f_0 = f_{\text{can}}$) with reduced density $F = \mathcal{Q}$.

EXAMPLE 3.2 As a first example consider the one-dimensional harmonic oscillator ($H(q,p) = (q^2+p^2)/2$ and $\Omega = \mathbb{R}$). From (21) we get, with the abbreviations $c = \cos\tau$ and $s = \sin\tau$:

$$Tu(q) \;=\; \int_{\mathbb{R}} u(cq - sp)\, \mathcal{P}(p)\, dp \quad \text{ with } \quad \mathcal{P}(p) = \sqrt{\frac{\beta}{2\pi}} \exp\left(-\frac{\beta}{2}p^2\right).$$

Thus, we have to distinguish between two essentially different cases:

First, assume $c = 1$, i.e., $\tau = 2m\pi$, $m \in \mathbb{Z}$. Then, we immediately observe that $Tu = u$ for all $u \in L^2(\Omega)$. That is, $T$ is the identity with spectrum $\sigma(T) = \{1\}$ and any subset $B \subset \mathbb{R}$ is invariant. For $c = -1$, $T$ is the identity on the subspace of all symmetric functions ($u(q) = u(-q)$).

Second, for $|c| < 1$ we can generate a sequence of eigenvectors by the following construction: Assume that $u \in L^2(\Omega)$ is a smooth eigenvector for the eigenvalue $\lambda$, i.e., that $Tu = \lambda u$. Differentiation of this equation with respect to $q$ yields

$$D_q(Tu) \;=\; c\, T D_q u \;=\; \lambda D_q u.$$

Thus, $\lambda_* = \lambda/c$ is an eigenvalue with eigenvector $u_* = D_q u$. Since $\chi_{\mathbb{R}} \in L^2(\Omega)$ satisfies $T\chi_{\mathbb{R}} = \chi_{\mathbb{R}}$, we can find a sequence of eigenvectors given by polynomials $u_n \in L^2(\Omega)$, $n \in \mathbb{N}$, satisfying

$$D_q u_n \;=\; u_{n-1} \quad \text{ and } \quad D_q u_1 = \chi_{\mathbb{R}}.$$

If we additionally choose these $u_n$ to be pairwise orthogonal with respect to $\langle\cdot,\cdot\rangle_F$, we end up with

$$u_1(q) = q, \quad u_2(q) = q^2 - s^2, \quad u_3(q) = q^3 - 3s^2 q, \ldots,$$

with $s = \int p^2 \mathcal{P}(p) dp = 1/\beta$. The corresponding eigenvalues are $\lambda_n = \cos^n(\tau)$. These eigenvectors are illustrated in Figure 6. We will see in Section 4.2 that, for $|c| < 1$, $T$ indeed has purely discrete spectrum with a single accumulation point at zero.

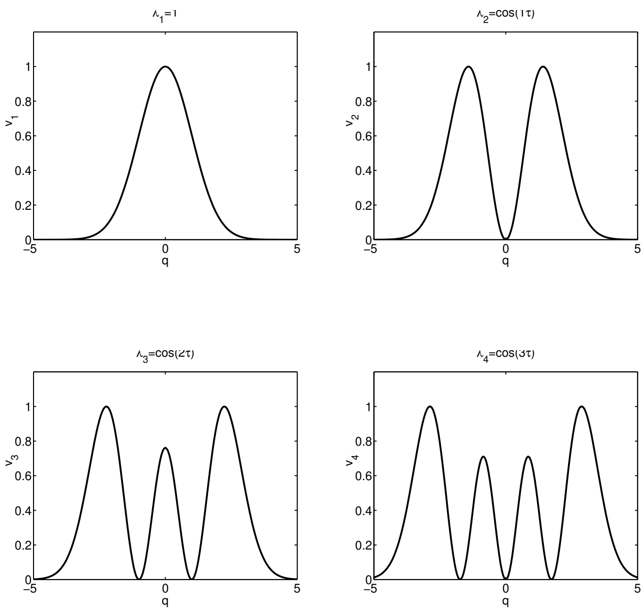



Figure 6: Probability density $v_n(q) = |u_n(q)|^2 \mathcal{Q}(q)$ of the eigenvectors $u_n$ for a harmonic oscillator ($H(q,p) = (q^2 + p^2)/2$ and $\beta = 1$).

EXAMPLE 3.3 As a worst case example let us consider a canonical ensemble of free particles in space dimension one (i.e., $V = 0$, $M = 1$).[14] For simplicity, set $\tau = 1$. From (21) we then get

$$Tu(q) = \int_{\mathbb{R}} u(q-p)\,\mathcal{P}(p)\,dp \quad \text{with} \quad \mathcal{P}(p) = \sqrt{\frac{\beta}{2\pi}}\exp\left(-\frac{\beta}{2}p^2\right).$$

Applying the Fourier transform $\hat{u}(k) = (1/2\pi)\int u(q)\exp(ikq)dq$, the convolution is reduced to a simple multiplication:

$$(\hat{Tu})(k) = \sqrt{2\pi}\,\hat{\mathcal{P}}(k)\,\hat{u}(k) = \hat{T}\hat{u}(k).$$

Since the Fourier transform is unitary on $L^2(\mathbb{R})$ and the transformed operator $\hat{T}$ is a multiplication operator, we simply have[15]

$$\sigma(T) = \sigma(\hat{T}) = (2\pi)^{1/2}\,\overline{\text{Range}(\hat{\mathcal{P}})} = \overline{\text{Range}(\exp(-k^2/2\beta))} = [0,1].$$

EXAMPLE 3.4 The simplest example for a bounded system is the free particle in a box with reflections at the walls: choose $\Omega = [-a, a]$ for an $a > 0$, with

[14] In this case our initial assumption $f_0 \in L^1(\Gamma)$ is hurt. For the scope of this example we ignore this.

[15] Compare Appendix B, Thm. B.41.

$V(q) = 0$ for $-a < q < a$ and reflecting walls ($V(q) = \infty$ for $|q| \geq a$). The flow map associated with this "irregular" potential is

$$\Phi^\tau(q,p) \;=\; (Z_a(q+\tau p),\, p\, S_a(q+\tau p))\,,$$

with the $4a$-periodic zigzag (cf. Fig. 1) and sign functions $Z_a$ and $S_a$ given on $[-2a, 2a]$ by

$$Z_a(q) \;=\; \begin{cases} -2a-q, & q \in [-2a,-a) \\ q, & q \in [-a,a] \\ 2a-q, & q \in (a,2a] \end{cases} \qquad \text{and} \qquad S_a(q) \;=\; \begin{cases} -1, & |q| \in (a,2a] \\ 1, & q \in [-a,a] \end{cases}.$$

Hence, the definition of the transition operator reads

$$Tu(q) \;=\; \int_{\mathbb{R}} u(Z_a(q-\tau p))\, \mathcal{P}(p)\, dp,$$

acting on $L^2(-a,a)$.[16] We now choose a somewhat unusual representation of $L^2(-a,a)$: The trigonometric functions

$$c_k(q) \;=\; \cos\left(k\frac{\pi}{a}q\right) \quad \text{and} \quad s_k(q) \;=\; \sin\left((k+\frac{1}{2})\frac{\pi}{a}q\right), \qquad k = 0,1,\ldots$$

indeed[17] span $L^2(-a,a)$. Fortunately, these basis functions satisfy $c_k \circ Z_a = c_k$ and $s_k \circ Z_a = s_k$, which directly leads us to

$$\begin{aligned} Tc_k \;&=\; \left(\int_{\mathbb{R}} \cos(k\frac{\pi}{a}\tau p)\mathcal{P}(p)\, dp\right)\, c_k \;=\; \exp\left(-k^2\frac{\pi^2\tau^2}{2a^2\beta}\right)\, c_k \\ Ts_k \;&=\; \left(\int_{\mathbb{R}} \cos((k+\frac{1}{2})\frac{\pi}{a}\tau p)\mathcal{P}(p)\, dp\right)\, s_k \;=\; \exp\left(-(k+\frac{1}{2})^2\frac{\pi^2\tau^2}{2a^2\beta}\right)\, s_k, \end{aligned}$$

showing that $c_k$ and $s_k$ are eigenvectors of $T$. Moreover, we may expand any $u \in L^2(-a,a)$ in this basis, yielding:

$$u \;=\; \sum_{k=0}^{\infty} a_{2k}\, c_k \;+\; \sum_{k=0}^{\infty} a_{2k+1}\, s_k.$$

Hence, we may rewrite $T$ in the form of a discrete multiplication operator:

$$T\,(a_k)_{k\in\mathbb{N}_0} \;=\; (\lambda_k\, a_k)_{k\in\mathbb{N}_0} \quad \text{with} \quad \lambda_k \;=\; \exp\left(-k^2\frac{\pi^2}{8a^2\beta}\tau^2\right), \quad k = 0,1,2,\ldots$$

This proves that its spectrum is discrete: $\sigma(T) = \{\lambda_k,\, k \in \mathbb{N}_0\} \cup \{0\}$.

[16] The reduced density is $F(q) = \mathcal{Q}(q) = 1/2a$.

[17] With $B_1 = \{\exp(ik\pi q/a), k \in \mathbb{Z}\}$ also $B_2 = \{\exp(i(k+1/2)\pi q/a), k \in \mathbb{Z}\}$ is a basis system in $L^2(-a,a)$, because with $u \in L^2(-a,a)$ also the function $u(q)\exp(-i\pi q/2a)$ lies in $L^2(-a,a)$. The basis chosen herein consists of the symmetric part $\{c_k\}$ of $B_1$, and the anti-symmetric part $\{s_k\}$ of $B_2$.

These examples taught us several important lessons about the properties of the transition operator $T$: If $T$ is considered as an operator in $L^2$, the spectrum of $T$ is real-valued, and may be discrete. The eigenvalues depend on the time length $\tau$, converging to one with $\tau \to 0$, while the eigenvectors need not depend on $\tau$. For $\tau > 0$, the largest eigenvalue $\lambda = 1$ is simple and the associated eigenvector is the constant function $\chi_\Omega$. But we must not forget that, in worst case situations (e.g., $T = \mathrm{Id}$ if $\cos(\tau) = 1$ in Example 3.2), the spectrum may degenerate (no eigenvalues of finite multiplicity).

But the above examples do not contain anything like an almost invariant set. Thus, for illustrating the connection between certain eigenvectors of our transition operator $T$ and almost invariant structures, we have to consider another example:

## 3.2 The Guiding Example

Let us now consider a simple system for which the distinction between different "conformations" or almost invariant subsets makes sense. For this purpose we return to the particles-in-a-box ensemble from the introduction. That is, we add a thin reflecting barrier of energetic height $E_0$ at $q = 0$ to the system of Example 3.4 (cf. Fig. 1) and consider the Hamiltonian $H(q,p) = p^2/2 + V(q)$ in $\Omega = [-a, a]$ with the potential

$$V(q) = \begin{cases} 0 & \text{if } 0 < |q| < a \\ \infty & \text{if } |q| > a \\ E_0 & \text{if } q = 0 \end{cases}$$

This has to be understood such that the flow $\Phi^t$ is given by the zigzag functions from Fig. 1. More precisely, the flow consists of: reflections between $-a$ and $a$ if the total energy is sufficient to cross the barrier, i.e.,

$$\Phi^\tau(q,p) = \Phi_0^\tau(q,p) = (Z_a(q+\tau p), pS_a(q+\tau p), \quad \text{if } H(q,p) > E_0,$$

with the zigzag and sign functions $Z_a$ and $S_a$ from Example 3.4; and reflections between $\pm a$ and 0 if the total energy is too small, i.e.,

$$\begin{aligned} \Phi^\tau(q,p) = \Phi_\mp^\tau(q,p) &= \left(Z_{a/2}(q \pm a/2 + \tau p) \mp a/2, pS_{a/2}(q \pm a/2 + \tau p)\right), \\ &\quad \text{if } H(q,p) \le E_0, \end{aligned}$$

with $q \in (-a, 0)$ for the $+$ sign and $q \in (0, a)$ for the $-$ sign.

With respect to the canonical ensemble the probability that the total energy of some system in the ensemble is sufficient to cross the barrier is given by

$$\mathrm{prob}(H(q,p) > E_0) = \int_{|p| > \sqrt{2E_0}} \mathcal{P}(p)\, dp,$$

which is temperature-dependent via $\mathcal{P}$. Suppose that the temperature and the barrier height $E_0$ are chosen such that this probability is very small, i.e.,

$\mathrm{prob}(H(q,p) > E_0) = \epsilon$. Then, the two sets $(-a,0)$ and $(0,a)$ are *almost invariant*. In order to see how this intuitively obvious fact is described by the eigenvectors of the transition operator $T$, consider the following decomposition:

$$\begin{aligned} Tu(q) &= \underbrace{\left(\int_{|p|\leq\sqrt{2E_0}} u(\pi_1\Phi_+^{-\tau}(q,p))\,\mathcal{P}(p)\,dp\right)}_{=T_-u(q) \text{ if } q\in(-a,0)} \chi_{(-a,0)}(q) \\ &\quad + \underbrace{\left(\int_{|p|\leq\sqrt{2E_0}} u(\pi_1\Phi_-^{-\tau}(q,p))\,\mathcal{P}(p)\,dp\right)}_{=T_+u(q) \text{ if } q\in(0,a)} \chi_{(0,a)}(q) \\ &\quad + \underbrace{\left(\int_{|p|>\sqrt{2E_0}} u(\pi_1\Phi_0^{-\tau}(q,p))\,\mathcal{P}(p)\,dp\right)}_{=T_0u(q)} \end{aligned}$$

We know that $\chi_{(-a,0)}$ and $\chi_{(0,a)}$ are eigenvectors to the largest eigenvalue $\lambda = 1-\epsilon$ of $T_-$ and $T_+$, respectively. If we now consider

$$v = \chi_{(-a,0)} - \chi_{(0,a)},$$

we therefore find that

$$Tv(q) = (1-\epsilon)\,v(q) + T_0v(q) \leq (1-\epsilon)\,v(q) + \epsilon.$$

Thus, we may interpret $v$ as a good approximation of an eigenvector of $T$ to an eigenvalue $\lambda \approx 1-\epsilon$. We already know that $\lambda = 1$ is the largest eigenvalue of $T$ with eigenvector $\chi_\Omega$. As we already observed in the introduction (compare Fig. 2), $v$ approximates the eigenvector to the second largest eigenvalue $\lambda_2 \approx 1-\epsilon$ of $T$ (they are identical in "picture norm"). Moreover, this eigenvector indicates the two almost invariant sets via its *sign*, that is, the eigenfunction takes positive values on the first almost invariant set and negative values on the other. For the parameter values of Fig. 2 ($a = 1$, $\beta = 25$, $\tau = 2$, $E_0 = 1/10$, leading to $\epsilon \approx 0.025$), the dominating eigenvalues $\lambda_k$ of $T$ are given in the following list

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\lambda_k$ | 1 | 0.974 | 0.478 | 0.453 | 0.256 | 0.254 |

We observe that the third largest eigenvalue is well-separated from $\lambda_1 = 1$ and $\lambda_2$. As we will see in the following, this is the "generic" situation if the system mainly has two almost invariant sets: There is a cluster of two eigenvalues, $\lambda_1 = 1$ and $\lambda_2 \approx 1$, clearly separated from the remaining part of the spectrum of $T$, and the two corresponding eigenvectors indicate the almost invariant sets via their signs: Denote the two eigenvectors by $v_1 = \chi_{[-1,1]}$ and $v_2$ and define their *sign structures* $s(q) = (\mathrm{sign}(v_1(q)), \mathrm{sign}(v_2(q)))$ via the signum of the corresponding entries of the eigenvalues for every position $q \in [-1,1]$. Then, the almost invariant sets are given by all $q$ with the same sign structure.

REMARK 3.5 When considering the limit $E_0 \to \infty$, i.e., an unbounded increase of barrier energy leading to $\epsilon \to 0$, we observe that $\lambda_2(\epsilon) \to \lambda_1(\epsilon) = 1$. Hence, in the limit, the eigenspace of $T$ for the eigenvalue $\lambda = 1$ is two-dimensional and spanned by the two eigenfunctions $\chi_{(-a,0)}$ and $\chi_{(0,a)}$, so that $(-a, 0)$ and $(0, a)$ are strictly invariant sets. That is, we may interpret the above almost invariant case as a specific *perturbation* (with small perturbation parameter $\epsilon$) of the unperturbed invariant case. As a consequence, the eigenvectors of $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ for the perturbed situation (cf. Fig. 2) correspond to the basis $\{\chi_{(-a,0)} + \chi_{(0,a)}; \chi_{(-a,0)} - \chi_{(0,a)}\}$ of the "unperturbed" eigenspace for $\lambda = 1$.

Under this perturbation, the eigenvalue cluster $(\lambda_1(\epsilon), \lambda_2(\epsilon))$ remains isolated from the remaining part of the spectrum: For example, the third and forth largest eigenvalues, $\lambda_3(\epsilon)$ and $\lambda_4(\epsilon)$, both converge to $\exp(-\pi^2\tau^2/2a^2\beta)$ (this is an implication of the results of Example 3.3).[18]

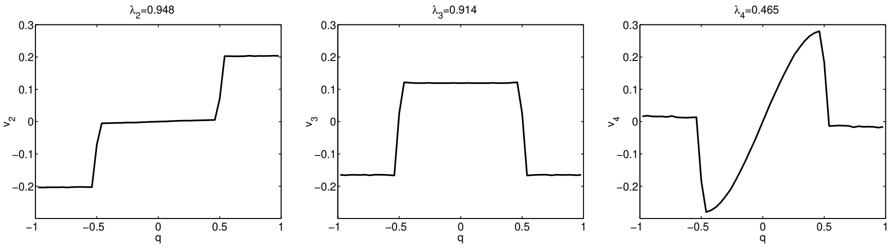



Figure 7: Eigenvectors to the second ($\lambda = 0.948$), third ($\lambda = 0.914$), and forth ($\lambda = 0.465$) largest eigenvalues of the transition operator $T$ for a free particle in a box with two reflecting barriers at $q = -0.5$ and $q = 0.5$ ($a = 1$, $\beta = 25$, $\tau = 2$, $E_0 = 1/10$, leading to $\epsilon \approx 0.025$). Results of a discretization of $T$ due to Sec. 5.

**Two Barriers** Figure 7 indicates that this concept can be generalized. It shows the interesting eigenvectors for a particle in a box with *two* reflecting barriers separating *three* almost invariant subsets. We observe that now there is a cluster of three eigenvalues near $\lambda = 1$ with a distinct gap to the remaining part of the spectrum and with eigenfunctions indicating the almost invariant subsets via their sign structure.[19]

Again, the step-like shape of the eigenfunctions indicates that they correspond to a certain "unperturbed" situation (given by the limit $E_0 \to \infty$, see Rmk. 3.5 above) where the eigenvalue $\lambda = 1$ is three-fold and the associated eigenspace is spanned by $\chi_{(-1,-1/2)}$, $\chi_{(-1/2,1/2)}$, and $\chi_{(1/2,1)}$.

[18] For the parameter values from above, $\exp(-\pi^2\tau^2/2a^2\beta) \approx 0.454$.

[19] The *sign structure* of the three eigenvectors $v_1 = \chi_{(-1,1)}$, $v_2$ (left in Fig. 7), and $v_3$ (middle in Fig. 7) for the three largest eigenvalues $\lambda_1 = 1$, $\lambda_2 = 0.948$, and $\lambda_3 = 0.914$, is as follows: $(+,-,-)$ for the almost invariant set $(-1,-1/2)$; $(+,\approx 0,+)$ for the almost invariant set $(-1/2,1/2)$; $(+,+,-)$ for the almost invariant set $(1/2,1)$.

Remark 3.6 Almost invariant sets can be interpreted as specific perturbations of invariant sets. The associated perturbation parameters are small crossing probabilities or large barrier energies, but *not* the fluctuation length $\tau$ or "external" parameters like the temperature. To see this, observe that in the limit $\tau \to 0$ or $\mathcal{T} \to \infty$, the specific isolation of the eigenvalue cluster near $\lambda = 1$ is destroyed, because most the eigenvalues (and/or the essential spectral radius) of $T$ converge to 1.

## 3.3 Consideration as Markov Operator

We now consider the probability space $(\Omega, \mathcal{B}, \mu)$, where $\mathcal{B}$ denotes the $\sigma$-algebra of Borel sets of $\Omega$ and the probability measure $\mu$ is given by

$$\mu(B) \;=\; \int_B F(q)\, dq, \quad \text{for } B \in \mathcal{B}.$$

The set of all densities (i.e., nonnegative functions) in the corresponding $L^1$-space $L^1_F(\Omega)$ is denoted $\mathcal{D}_F$ (cf. Def. A.2 in appendix A). We want to show that $T$ can be considered as an Markov operator on $L^1_F(\Omega)$ (cf. Def. A.3).

Lemma 3.7 *$T$ defines a bounded linear Markov operator $L^1_F(\Omega) \to L^1_F(\Omega)$.*

*Proof:* Consider an arbitrary $u \in L^1_F(\Omega)$. Via definition (20) we find

$$\begin{aligned}
\|Tu\|_{1,F} &= \int_\Omega \frac{1}{F(q)} \left| \int_{\mathbb{R}^d} u\left(\pi_1 \Phi^{-\tau}(q,p)\right)\, f_0(q,p)\, dp \right| F(q)\, dq \\
&\le \int_\Omega \int_{\mathbb{R}^d} \left| u\left(\pi_1 \Phi^{-\tau}(q,p)\right)\right|\, f_0(q,p)\, dp\, dq \\
&= \int_\Gamma \left| u\left(\pi_1 \Phi^{-\tau} x\right)\right|\, f_0(x)\, dx \\
&= \int_\Gamma \left| u\left(\pi_1 x\right)\right|\, f_0(x)\, dx \\
&= \int_\Omega |u\,(q)| \underbrace{\int_{\mathbb{R}^d} f_0(q,p)\, dp}_{=F(q)}\, dq \;=\; \|u\|_{1,F}
\end{aligned}$$

where the step from the third to the forth line uses the substitution $x \to \Phi^\tau x$ and exploits the invariance of $f_0$ with respect to this transformation and the volume conservation property of the flow. Thus, $T$ is well-defined on the entire space $L^1_F(\Omega)$. In the second line, equality holds iff $u \ge 0$, which shows that $\|Tu\|_{1,F} = \|u\|_{1,F}$ for $u \in \mathcal{D}_F$. Since $F, f_0 \ge 0$, we also have $Tu \ge 0$ for $u \in \mathcal{D}_F$. Hence, $T$ is a Markov operator. ∎

Remark 3.8 The property of being a Markov operator already implies the boundedness $\|T\|_{1,F} \le 1$ (cf. [66], Prop. 3.1.1).

### 3.4 Self-Adjointness and Transition Probabilities

Now, $T$ is considered as an operator on the Hilbert space $L^2_F(\Omega)$.

LEMMA 3.9 *The transition operator* $T : L^2_F(\Omega) \to L^2_F(\Omega)$ *is a bounded linear operator with* $\|Tu\|_F \leq \|u\|_F$.

The following proof is nothing but an application of the Cauchy-Schwarz inequality. Nevertheless, it is presented in detail because the same strategy will be used again in the subsequent.
*Proof:* For an arbitrary $u \in L^2_F(\Omega)$, definition (20) yields

$$\|Tu\|_F^2 \;=\; \int_\Omega \frac{1}{F(q)^2} \left| \int_{\mathbb{R}^d} u\left(\pi_1 \Phi^{-\tau}(q,p)\right)\; f_0(q,p)\, dp \right|^2 F(q)\, dq. \tag{24}$$

Consider the following family of Hilbert spaces $\mathcal{H}_q$: For $q \in \Omega$ the space $\mathcal{H}_q$ consists of all functions $w : \mathbb{R}^d \to \mathbb{C}$ with

$$\int_{\mathbb{R}^d} |w(p)|^2\, f_0(q,p)\, dp \;<\; \infty.$$

with the associated scalar product

$$(w_1, w_2)_q \;=\; \int_{\mathbb{R}^d} w_1(p)^*\, w_2(p)\, f_0(q,p)\, dp.$$

The induced norm is denoted $\|\cdot\|_q$. For all $q \in \Omega$, the constant function $\mathbf{1} = \chi_{\mathbb{R}^d}$ is an element of $\mathcal{H}_q$ because

$$\|\mathbf{1}\|_q^2 \;=\; \int_{\mathbb{R}^d} f_0(q,p)\, dp \;=\; F(q) < \infty.$$

Next, consider the family of functions $w_q : \mathbb{R}^d \to \mathbb{C}$ defined by $w_q(p) = u(\pi_1 \Phi^{-\tau}(q,p))$. A short calculation like in the proof of Lemma 3.7 reveals that

$$\int_\Omega \underbrace{\int_{\mathbb{R}^d} |w_q(p)|^2\, f_0(q,p)\, dp}_{=\|w_q\|_q^2 \text{ if } <\infty}\; dq \;=\; \|u\|_F^2 \;<\; \infty. \tag{25}$$

Thus, we have $w_q \in \mathcal{H}_q$ for almost every $q \in \Omega$. Then, the Cauchy-Schwarz inequality yields (again for almost every $q$):

$$\begin{aligned}
\left| \int_{\mathbb{R}^d} u(\pi_1 \Phi^{-\tau}(q,p)) f_0(q,p)\, dp \right|^2 \;&=\; \left|(\mathbf{1}, w_q)_q\right|^2 \\
&\leq\; \|w_q\|_q^2\; \|\mathbf{1}\|_q^2 \\
&=\; \|w_q\|_q^2\; F(q).
\end{aligned}$$

Inserting this into equation (24) yields

$$\|Tu\|_F^2 \;\le\; \int_\Omega \|w_q\|_q^2 \; dq \;=\; \|u\|_F^2,$$

where the last equality results from (25). ∎

Next, we are interested in the transition probabilities defined by $T$. Therefore, let us consider two arbitrary, measurable subsets $B, C \subset \Omega$. We again use the notation

$$\Gamma(B) \;=\; \{x \in \Gamma, \quad \pi_1(x) \in B\}.$$

The characteristic functions $\chi_B$ and $\chi_C$ are elements of $L^2_F(\Omega)$. Thus,

$$\begin{aligned}
\langle T\chi_B, \chi_C\rangle_F &= \int_\Omega \frac{1}{F(q)} \int_{\mathbb{R}^d} \chi_B\left(\pi_1 \Phi^{-\tau}(q,p)\right)\right) \, f_0(q,p)\, dp\, \chi_C(q)\, F(q)\, dq \\
&= \int_\Gamma \chi_B\left(\pi_1 \Phi^{-\tau} x\right) \, f_0(x)\, \chi_{\Gamma(C)}(x)\, dx \\
&= \int_\Gamma \chi_{\Gamma(B)}(x)\, \chi_{\Gamma(C)}\left(\Phi^\tau x\right) \, f_0(x)\, dx, \qquad (26)
\end{aligned}$$

where the last equality results from the transformation $x = \Phi^\tau y$ together with the invariance of $f_0$ and the volume conservation property of the flow. We also find that

$$\langle \chi_B, \chi_B\rangle_F \;=\; \int_B F(q)\, dq \;=\; \int_{\Gamma(B)} f_0(x)\, dx,$$

which together with equation (26) finally reveals that

$$\boxed{\frac{\langle T\chi_B, \chi_C\rangle_F}{\langle \chi_B, \chi_B\rangle_F} \;=\; \frac{\int_{\Gamma(B)} \chi_{\Gamma(C)}\left(\Phi^\tau x\right) \, f_0(x)\, dx}{\int_{\Gamma(B)} f_0(x)\, dx} \;=\; w(\Gamma(B), \Gamma(C), \tau), \qquad (27)}$$

showing that $T$ indeed represents the transition probabilities of our interest.

Using the other invariance of $f_0$, it is easy to prove another crucial property of $T$:

Lemma 3.10 *The transition operator $T : L^2_F(\Omega) \to L^2_F(\Omega)$ is self-adjoint. Hence, its spectrum satisfies $\sigma(T) \subset [-1, 1]$.*

*Proof:* First, consider two arbitrary, measurable subsets $B, C \subset \Omega$. Using (26) and the reversibility of the flow (Lemma 2.1) we get

$$\langle T\chi_B, \chi_C\rangle_F \;=\; \int_\Gamma \chi_{\Gamma(B)}(x)\, \chi_{\Gamma(C)}\left(R(\Phi^{-\tau} Rx)\right) \, f_0(x)\, dx.$$

Now, since $f_0$ is $p$-symmetric (eq.(10)) and the sets $\Gamma(B)$ and $\Gamma(C)$ include all possible momenta (i.e., $x \in \Gamma(B) \Rightarrow Rx \in \Gamma(B)$, for example), a transformation $y = Rx$ yields

$$\begin{aligned}\langle T\chi_B, \chi_C\rangle_F &= \int_\Gamma \chi_{\Gamma(B)}(y)\, \chi_{\Gamma(C)}\left(R(\Phi^{-\tau} y)\right)\, f_0(y)\, dy \\ &= \int_\Gamma \chi_{\Gamma(B)}(x)\, \chi_{\Gamma(C)}\left(\Phi^{-\tau} x\right)\, f_0(x)\, dx \;=\; \langle \chi_B, T\chi_C\rangle_F.\end{aligned}$$

Since the step-functions are dense in $L^2(\Omega)$, we get $\langle Tu, v\rangle_F = \langle u, Tv\rangle_F$ for all $u, v \in L^2_F(\Omega)$, i.e., the self-adjointness of $T$. Thus, its spectrum $\sigma(T)$ is real-valued, which together with the boundedness (Lemma 3.9) implies $\sigma(T) \subset [-1, 1]$. $\square$

## 3.5 Restriction to Essential Variables

In the following we study the consequences of the restriction from full spatial coordinates to other essential degrees of freedom. Let this set of essential variables be given in terms of the state of the system by a continuously differentiable function $\vartheta : \Gamma \to \mathbb{R}^\nu$,

$$\vartheta(x) \;=\; (\vartheta_1(x), \ldots, \vartheta_\nu(x)),$$

and denote the corresponding *essential configuration space* by $\Theta = \vartheta(\Gamma)$. We always assume that $\vartheta$ is *independent* of the momenta $p$, i.e., the function $\vartheta$ depends only on the positions $q$. For simplicity, we use the notation $\vartheta = \vartheta(q)$ as well as the more general form $\vartheta = \vartheta(x)$, where the meaning is always clear from the context. For any possible value $\theta \in \Theta$ we denote the corresponding level set by

$$\Gamma(\theta) \;=\; \{x \in \Gamma,\ \vartheta(x) = \theta\}.$$

We assume that these level sets are *smoothly embedded submanifolds* of dimension $2d - \nu$ in $\Gamma$.[20] Let $d\sigma_\theta(x)$ be the intrinsic volume element on $\Gamma(\theta)$.[21]

Let $\Omega \subset \mathbb{R}^d$ be the position space so that $\Gamma = \Omega \times \mathbb{R}^d$. Since $\vartheta$ does not depend on the momenta $p$, the volume elements have the special product form

$$d\sigma_\theta(x) \;=\; d\sigma_\theta(q) \wedge dp,$$

where $d\sigma_\theta(q)$ denotes the intrinsic volume form of

$$\Omega(\theta) \;=\; \{q \in \Omega,\ \vartheta(q) = \theta\},$$

[20] Thus, we assume that the associated Jacobian matrix $D\vartheta(q)$ has full rank for any $q \in \Omega$. Due to Sard's lemma, this is the generic situation.

[21] Now and in the following, we assume that $d\sigma_\theta(x)$ is appropriately defined on all connected submanifolds of $\Gamma(\theta)$. Moreover, it herein is of no importance whether $\Gamma(\theta)$ consists of more than one component or not.

which is a smoothly embedded, $(d-\nu)$-dimensional submanifold of $\Omega$.

For any invariant density $f_0$ we define the *reduced probability density* as

$$F_\vartheta(\theta) \;=\; \int_{\Gamma(\theta)} f_0(x)\, d\sigma_\theta(x),$$

which is an element of $L^1(\Theta)$ if $f_0 \in L^1(\Gamma)$.

The transition operator associated with this set of essential variables then is

$$\boxed{T_\vartheta u(\theta) \;=\; \frac{1}{F_\vartheta(\theta)} \int\limits_{\Gamma(\theta)} u\left(\vartheta(\Phi^{-\tau} x)\right)\, f_0(x)\, d\sigma_\theta(x), \qquad (28)}$$

where $u = u(\theta)$ is a function $u : \Theta \to \mathbb{C}$.

Thus, $T_\vartheta$ is again defined by a suitable $f_0$*-weighted average* of the Frobenius–Perron operator over every "essential fiber" $\Gamma(\theta)$, that is, the average includes all momenta $p$ and that part of the degrees of freedom which are "orthogonal" to the considered essential variables $\vartheta$.

We consider $T_\vartheta$ as an operator on the weighted spaces

$$L^p_{F_\vartheta}(\Theta) \;=\; \{u : \Theta \to \mathbb{C}, \quad \int_\Theta |u(\theta)|^p F_\vartheta(\theta)\, d\theta < \infty\}, \qquad p = 1, 2,$$

with the scalar product

$$\langle u, v\rangle_{F_\vartheta} \;=\; \int_\Omega u^*(\theta)\, v(\theta)\, F_\vartheta(\theta)\, d\theta,$$

for the Hilbert space $L^2_{F_\vartheta}(\Theta)$, and induced norm $\|u\|^2_{F_\vartheta} = \langle u, u\rangle_{F_\vartheta}$.

For subsets $B \subset \Theta$, the union of all fibers $\Gamma(\theta)$ with $\theta \in B$ is denoted by

$$\Gamma(B) \;=\; \bigcup_{\theta \in B} \Gamma(\theta) \;=\; \{x \in \Gamma :\; \vartheta(x) \in B\},$$

in analogy to the notation used above. By repeating the computations from Sec. 3.4, we observe that $T_\vartheta$ in fact describes the transition probabilities between subsets $B, C \subset \Theta$, that is,

$$\frac{\langle T_\vartheta \chi_B, \chi_C\rangle_{F_\vartheta}}{\langle \chi_B, \chi_B\rangle_{F_\vartheta}} \;=\; w(\Gamma(B), \Gamma(C), \tau).$$

Example 3.11 For the above considered spatial case, we have to choose $\vartheta = \pi_1$, that is, $\vartheta(q) = q$ with $\nu = d$. Then, $\Omega(q) = \{q\}$ and $\Gamma(q) = \{q\} \times \mathbb{R}^d$, implying $d\sigma_q(x) = dp$ and the reduced probability density is $F_\vartheta(q) = F(q)$ so that definitions (28) and (20) coincide.

REMARK 3.12 Suppose that we are dealing with the canonical ensemble $f_0 = f_{\text{can}}$. Due to [64, 104], the so-called *conformational free energy* $A = A(\theta)$ for a set of essential variables may be defined via

$$A(\theta) \;=\; -\beta^{-1}\ln\mathcal{Z}(\theta), \quad \text{with} \quad \mathcal{Z}(\theta) \;=\; \int_{\Gamma(\theta)} \exp(-\beta H(x))\, d\sigma_\theta(x),$$

that is, it is defined via the contribution $\mathcal{Z}(\theta)$ of the $\Gamma(\theta)$ to the classical partition sum $\mathcal{Z} = \int_\Gamma \exp(-\beta H) dx$. With this definition, we obviously have

$$\mathcal{Z}(\theta) \;=\; \exp(-\beta A(\theta)) \;=\; \mathcal{Z}\, F_\vartheta(\theta),$$

which allows to rewrite the transition operator as

$$T_\vartheta u(\theta) \;=\; \int\limits_{\Gamma(\theta)} u\left(\vartheta(\Phi^{-\tau}x)\right)\, \exp\left(-\beta\left(H(x) - A(\theta)\right)\right)\, d\sigma_\theta(x).$$

Consequently, the transition operator $T_\vartheta$ describes the *fluctuations* inside the ensemble $f_0$ induced by the flow $\Phi^\tau$ and weighted by the *difference between the potential energy surface and the conformational free energy surface.*

**Restriction Operator and Adjoint** We next show that $T_\vartheta$ may be written as a specific restriction of the full spatial transition operator $T$. The associated embedding is given by the following

DEFINITION 3.13 Assume $p = 1, 2$ and let $\vartheta : \Gamma \to \Theta \subset \mathbb{R}^k$ be a set of essential coordinates with the reduced probability density $F_\vartheta = F_\vartheta(\theta)$. The restriction operator $R_\vartheta : L^p_F(\Omega) \to L^p_{F_\vartheta}(\Theta)$ is defined by

$$R_\vartheta u(\theta) \;=\; \frac{1}{F_\vartheta(\theta)} \int_{\Gamma(\theta)} u(\pi_1 x)\, f_0(x)\, d\sigma_\theta(x),$$

while the prolongation operator $B_\vartheta : L^p_{F_\vartheta}(\Theta) \to L^p_F(\Omega)$ is simply given by $(B_\vartheta u)(q) = u(\vartheta(q))$.

We next show that these operators allow to express the transition operator for the given essential coordinates as $T_\vartheta = R_\vartheta T B_\vartheta$. To see this, first consider

$$TB_\vartheta u(q) \;=\; \frac{1}{F(q)} \int_{\mathbb{R}^d} u(\vartheta(\Phi^{-\tau}(q,p)))\, f_0(q,p)\, dp.$$

Since also

$$\begin{aligned} R_\vartheta u(\theta) \;&=\; \frac{1}{F_\vartheta(\theta)} \int_{\Omega(\theta)} u(q) \underbrace{\left(\int_{\mathbb{R}^d} f_0(x)\, dp\right)}_{=F(q)} d\sigma_\theta(q) \\ &=\; \frac{1}{F_\vartheta(\theta)} \int_{\Omega(\theta)} u(q)\, F(q)\, d\sigma_\theta(q), \end{aligned} \tag{29}$$

we indeed end up with

$$R_\vartheta T B_\vartheta u(\theta) \;=\; \frac{1}{F_\vartheta(\theta)} \int_{\mathbb{R}^d} \int_{\Omega(\theta)} u(\vartheta(\Phi^{-\tau}(q,p)))\, f_0(q,p)\, d\sigma_\theta(q)\, dp \;=\; T_\vartheta u(\theta).$$

Thus, for further purpose we note the following

PROPOSITION 3.14 *For any integrable function $u : \Theta \to \mathbb{C}$, the restriction and prolongation operators $R_\vartheta$ and $B_\vartheta$ allow to rewrite the transition operator $T_\vartheta$ as*

$$T_\vartheta u \;=\; R_\vartheta T B_\vartheta u.$$

*Moreover, $B_\vartheta$ is an isometry, i.e., for $p = 1, 2$ we have*

$$\|B_\vartheta u\|_{p,F} \;=\; \|u\|_{p,F_\vartheta},\ \forall u \in L^p_{F_\vartheta}(\Theta),$$

*and $R_\vartheta$ is a contraction with Markov property, i.e., for $p = 1, 2$*

$$\|R_\vartheta u\|_{p,F_\vartheta} \;\le\; \|u\|_{p,F},\ \forall u \in L^p_F(\Omega),\ \text{and}\ \|R_\vartheta u\|_{1,F_\vartheta} \;=\; \|u\|_{1,F},\ \forall u \in \mathcal{D}_F(\Omega).$$

*In addition, $R_\vartheta : L^2_F(\Omega) \to L^2_{F_\vartheta}(\Theta)$ and $B_\vartheta : L^2_{F_\vartheta}(\Theta) \to L^2_F(\Omega)$ are adjoint to each other, i.e., $R^*_\vartheta = B_\vartheta$ and $B^*_\vartheta = R_\vartheta$. Thus, in particular, if $T : L^2_F(\Omega) \to L^2_F(\Omega)$ is self-adjoint, then $T_\vartheta : L^2_{F_\vartheta}(\Theta) \to L^2_{F_\vartheta}(\Theta)$ is, too.*

*Proof:* The isometry of $B_\vartheta$ directly results from $u \circ \vartheta$ being constant on every submanifold $\Gamma(\theta)$:

$$\begin{aligned} \|B_\vartheta u\|^p_{p,F} &= \int_\Omega |u(\vartheta(q))|^p F(q)\, dq \;=\; \int_\Gamma |u(\vartheta(x))|^p\, f_0(x)\, dx \\ &= \int_\Theta |u(\theta)|^p \underbrace{\int_{\Gamma(\theta)} f_0(x)\, d\sigma_\theta(x)}_{=F_\vartheta(\theta)}\, d\theta \;=\; \|u\|^p_{p,F_\vartheta}. \end{aligned}$$

The contraction property for $R_\vartheta$ for $p = 1$ and the associated Markov property are obvious. The contraction property for $p = 2$ can be proved with the similar technique as the $L^2$-boundedness of $T$ in the proof of Lemma 3.9. Therefore, define a family of Hilbert spaces $\mathcal{H}_\theta$ with scalar product

$$\langle u, v\rangle_\theta \;=\; \int_{\Omega(\theta)} u^*(q) v(q) F(q)\, d\sigma_\theta(q)$$

and associated norm $\|\cdot\|_\theta$. Then, with similar arguments as in the proof of Lemma 3.9, we can show for arbitrary $u \in L^2_F(\Omega)$ that the Cauchy-Schwarz inequality yields for almost every $\theta \in \Theta$:

$$\left| \int_{\Omega(\theta)} u(q) F(q)\, d\sigma_\theta(q) \right|^2 \;=\; |\langle \chi_\Omega, u\rangle_\theta|^2 \;\le\; \|u\|^2_\theta\, F_\vartheta(\theta),$$

which permits us to estimate

$$\begin{aligned}
\|R_\vartheta u\|_{2,F_\vartheta} &= \int_\Theta |R_\vartheta u(\theta)|^2 \, F_\vartheta(\theta)\, d\theta \\
&= \int_\Theta \frac{1}{F_\vartheta(\theta)} \left| \int_{\Omega(\theta)} u(q) F(q)\, d\sigma_\theta(q) \right|^2 \, d\theta \\
&\le \int_\Theta \|u\|_\theta^2 \, d\theta \;=\; \int_\Omega |u(q)|^2 F(q)\, dq \;=\; \|u\|_{2,F}.
\end{aligned}$$

In order to show $B_\vartheta^* = R_\vartheta$ choose arbitrary $v \in L_F^2(\Omega)$ and $u \in L_{F_\vartheta}^2(\Theta)$. Then $B_\vartheta u \in L_F^2(\Omega)$ and

$$\begin{aligned}
\langle B_\vartheta u, v\rangle_F &= \int_\Omega u(\vartheta(q))^* v(q)\, F(q)\, dq \\
&= \int_\Theta \int_{\Omega(\theta)} u(\vartheta(q))^* v(q)\, F(q)\, d\sigma_\theta(q)\, d\theta \\
&= \int_\Theta u(\theta)^* \underbrace{\int_{\Omega(\theta)} v(q)\, F(q)\, d\sigma_\theta(q)}_{=F_\vartheta(\theta)(R_\vartheta v)(\theta)} \, d\theta \\
&= \langle u, R_\vartheta v\rangle_{F_\vartheta},
\end{aligned}$$

which demonstrates that $B_\vartheta^* = R_\vartheta$. Since we are working in Hilbert spaces, this also implies $R_\vartheta^* = B_\vartheta^{**} = B_\vartheta$. $\square$

### 3.6 Associated Stochastic Dynamical System

Let us return to the case $\vartheta = \pi_1$, that is, to the spatial transition operator and the position space $\Omega$. The following paragraph is crucial for the final interpretation of our approach and its results. For the sake of conceptional simplicity, we restrict the presentation to the case of the canonical ensemble, i.e., we only consider $f_0(q,p) = f_{\text{can}}(q,p) = \mathcal{Q}(q)\mathcal{P}(p)$.

Assume $\mathcal{B} = \mathcal{B}(\Omega)$ to be the $\sigma$-algebra of Borel subsets of $\Omega$. Moreover, let $\mathcal{M}_f$ and $\mathcal{M}_1 \subset \mathcal{M}_f$ be the spaces of all finite and probability measures $\mu : \mathcal{B} \to \mathbb{R}^+$, respectively.

We now define a specific stochastic dynamical system, which will later be identified as the dynamical system associated with the spatial transition operator $T$. For a given initial position $q_0 \in \Omega$ we define

$$q_{k+1} \;=\; \pi_1 \Phi^\tau(q_k, p_k), \quad k = 0, 1, \ldots \tag{30}$$

with every $p_k \in \mathbb{R}^d$ being randomly chosen from the probability distribution $\mathcal{P}$ on $\mathbb{R}^d$. According to [66], Chap. 12.4, this defines a regular stochastic dynamical system which is described by a sequence of probability measures $\mu_k \in \mathcal{M}_1$ given by the probability of finding $q_k$ in a subset $B \in \mathcal{B}$ of $\Omega$, i.e.,

$$\mu_k(B) \;=\; \text{prob}(q_k \in B).$$

The sequence $\{\mu_k\}$ is also given by the iterates of the so-called *Foias operator* $P : \mathcal{M}_f \to \mathcal{M}_f$ defined by

$$P\mu(B) \;=\; \int_\Omega \left\{ \int_{\mathbb{R}^d} \chi_B \left(\pi_1 \Phi^\tau(q,p)\right) \mathcal{P}(p)\, dp \right\} \mu(dq), \quad \forall B \in \mathcal{B}, \tag{31}$$

in the sense that $\mu_k = P^k \mu_0$ if $\mu_0 \in \mathcal{M}_1$ is the probability measure according to which the initial random position $q_0$ is distributed.

In the following we consider measures $\mu_u \in \mathcal{M}_f$ with densities $u \in L^1_{\mathcal{Q}}$, i.e.,

$$\mu_u(B) \;=\; \int_B u(q)\, \mathcal{Q}(q)\, dq, \quad \forall B \in \mathcal{B}.$$

According to (31), the Foias operator $P$ acts on such measures as follows:

$$\begin{aligned} P\mu_u(B) \;&=\; \int_\Gamma \chi_B \left(\pi_1 \Phi^\tau(x)\right)\, u(\pi_1 x)\, f_{\text{can}}(x)\, dx \\ &=\; \int_{\Gamma(B)} u(\pi_1 \Phi^{-\tau} x)\, f_{\text{can}}(x)\, dx \\ &=\; \int_B \left\{ \int_{\mathbb{R}^d} u(\pi_1 \Phi^{-\tau}(q,p))\, \mathcal{P}(p)\, dp \right\} \mathcal{Q}(q)\, dq. \end{aligned}$$

Thus, $P\mu_u$ has the density

$$Tu(q) \;=\; \int_{\mathbb{R}^d} u(\pi_1 \Phi^{-\tau}(q,p))\, \mathcal{P}(p)\, dp,$$

given by the transition operator $T$. That is, $T : L^1(\Omega) \to L^1(\Omega)$ is the density operator associated to the Foias operator $P$ and therefore also associated with the stochastic dynamical system (30). Consequently: If the initial position $q_0$ of (30) is distributed according to the probability density $u \in \mathcal{D}(\Omega)$, the probability density $u_k \in \mathcal{D}(\Omega)$ of finding $q_k = q$ is given by $u_k(q) = T^k u(q)$.

**Associated Markov Chain and Control Model** In addition, we know from Sec. A.1 of the appendix, that the Markov operator $T$ induces a *Markov chain*. Every iteration of (30) is a realization of this Markov chain. Meyn and Tweedie [79] call the stochastic dynamical system (30) a "nonlinear state space model" and also discuss its interpretation as a Markov chain (see Sec. 3.5.5 in [79]). And they stress another important point: (30) may also be interpreted as a "control model", which describes the control of the positions $q$ via the "control variables" $p$. If we use the recursively defined notation

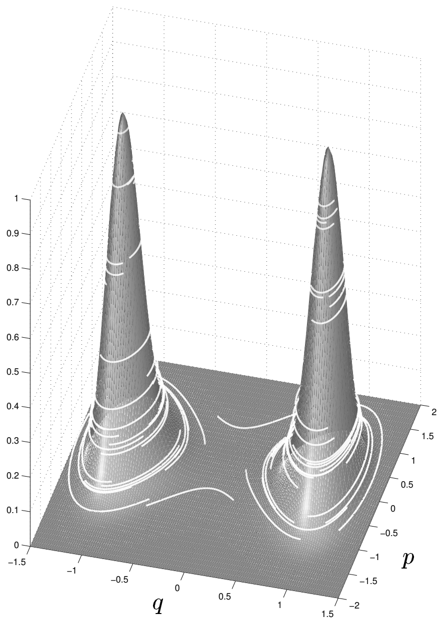
$$\Psi_{k+1}(q, p_0, \ldots, p_k) \;=\; \pi_1 \Phi^\tau \left(\Psi_k(q, p_0, \ldots, p_{k-1}), p_k\right), \quad k \in \mathbb{N} \tag{32}$$

with $\Psi_1(q, p_0) = \pi_1 \Phi^\tau(q, p_0)$, the iterates of (30) can be denoted as

$$q_k \;=\; \Psi_k(q_0, p_0, \ldots, p_{k-1}), \tag{33}$$

Hence, we see that $(p_0, \ldots, p_{k-1})$ may be interpreted as some *control sequence* which can be designed such that some desired final position $q_k$ is *accessible* from the initial position $q_0$.

rast to the mathematical analogy between
.e and the stochastic dynamical system—
een the *physical meaning* of the two levels
)f physical reality (the ensemble) and the
the associated Markov chain) (cf. Fig. 8).



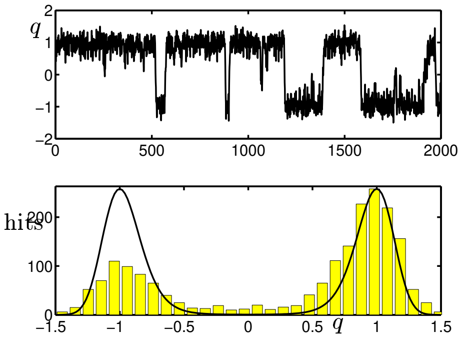




Figure 8: Illustration of transitions in an ensemble in contrast to the evolution of the associated stochastic dynamical system for the double well potential $V(q) = (q^2 - 1)^2$. Left: Canonical ensemble $f_{\text{can}}$ in the two-dimensional phase space. The white lines indicate trajectories of single systems during some time span $\tau$ inside the ensemble. Since $f_{\text{can}}$ is invariant under the flow, all trajectories are parts of isolines. Only some trajectories of systems with small probability cross the separation line $q = 0$ between the two almost invariant sets around $q = -1$ and $q = 1$. Right: First 2000 steps of the discrete trajectory of a single realization of the associated stochastic dynamical system. Again, jumps across the line $q = 0$ are rare. Below: Histogram of the distribution of positions after these 2000 steps compared with the appropriately scaled canonical position density $\mathcal{Q}$. Asymptotically, the distributions converges to $\mathcal{Q}$.

### 3.7.1 Ensemble versus Stochastic Dynamical System

The transition operator $T$ describes the *redistribution* or *fluctuation* in a *stationary ensemble*, i.e., it describes a statistical collection of single systems with different actual states and measures how many of these single systems may perform some kind of transition during a *single* "time step" $\tau$. Hence, some power $T^m$ of this operator can *not* necessarily be interpreted as describing *fluctuations in the ensemble* on time scales $m\tau$. As a consequence, the spatial transition operator has *no semigroup property*: Let $T^\tau$ denote the transition operator for fixed time span $\tau$ as in (20), then, in general,

$$T^\tau T^{\tau'} \neq T^{\tau+\tau'}.$$

This can easily be illustrated if we assume that $T^\tau$ is the transition operator of Example 3.2 for the harmonic oscillator $H(q,p) = (q^2+p^2)/2$ with $|\cos(\tau)| < 1$.

Then, $T^\tau u_1 = \cos(\tau)\, u_1$ with $u_1(q) = q$ for all such $\tau$. Hence,

$$T^\tau T^{\tau'} u_1 = \cos(\tau)\cos(\tau')\, u_1 \neq \cos(\tau + \tau')\, u_1 = T^{\tau+\tau'} u_1,$$

if only $\sin(\tau)\sin(\tau') \neq 0$. Consequently, we cannot simply "link" fluctuations on some short time scale to get fluctuations on longer scales. One can explain this observation by considering the underlying measurement processes: $\langle (T^\tau)^2 \chi_B, \chi_C \rangle / \langle \chi_B, \chi_B \rangle$ belongs to a two-fold repetition of the two-step experiment from page 19 —therefore including two pre-selection steps—, while $\langle T^{2\tau} \chi_B, \chi_C \rangle / \langle \chi_B, \chi_B \rangle$ corresponds to a single realization with double stepsize but with only one pre-selection step. But the physical observables associated with the pre-selection step and the transition counting procedure do *not* commute with respect to the Poisson bracket $\{\cdot, \cdot\}$. Hence, the additional pre-selection step changes the ensemble irreversibly so that, in general, the two transition probabilities are different.

Independent of the interpretation with respect to an ensemble, the operator $T$ is associated with the stochastic dynamical system (30) and the corresponding Markov chain $\{X_k\}$. On the one hand, the running time averages of the Markov chain approximate the ensemble averages (see Sec. 4.5 below). But on the other hand, the multiple-step transition probabilities $P(X_m \in B | X_0 \in A)$ of the Markov chain from $A \subset \Omega$ to $B \subset \Omega$ after time $m\tau$ can be expressed via the powers $T^m$ of the transition operator (see Sec. A.1 in Appendix A), although these powers have no direct interpretation for the ensemble.

Summarizing, multiple-step fluctuations ($m > 1$) of the Markov chain $\{X_k\}$ *cannot* be interpreted as fluctuations in the ensemble on time scales $m\tau$; only single-step fluctuations and the invariant distribution of the Markov chain represent properties of the ensemble.

In this sense, the stochastic dynamical system and the corresponding Markov chain are only *artificial* representations of the ensemble in an iterative way, in form of some stochastically linked chain of single systems from the ensemble. But in addition, the stochastic dynamical system (30) should *not* be taken as a model of a *single* physical system.

These considerations are typical for the discussion of the correspondence between statistical ensembles and stochastically embedded single systems. The contributions to this discussion are varied and range from modelling decisions,[22] over algorithmically oriented realizations[23] to systematic investigation in, e.g., "stochastic realisation theory".[24] For the context discussed herein, it is only of importance that the stochastic dynamical system (30) correctly represents the

[22] A typical example is the representation of a heat bath by means of adding some stochastic excitation —external "noise"— as in Langevin dynamics [1, 115].

[23] For example, so-called "constant–temperature" embeddings of the Hamiltonian system via Nosé-Hoover dynamics [81, 82] or its variants are often used in real-life applications.

[24] Stochastic realization theory or *dilation theory* stands for an overlap between systems theory and statistical mechanics: dilations are embeddings of "small" systems into "large" ones ("heat baths"), which have the property that the time-reversible, conservative motion of the large system reduces to a dissipative, irreversible evolution of the small system. Use [73] as a pointer to the literature.

fluctuations in our ensemble moderating the transitions between certain subsets of phase space on the given time scale $\tau$.

**Algorithmic Differences** In addition to this conceptual differences between ensemble and stochastic dynamical systems, we also have to distinguish between transition operator and Markov chain on the *algorithmic* level:

1. If the system really contains almost invariant sets, then direct *long term iteration of the associated chain is algorithmically inappropriate*: Whenever some set $B \subset \Omega$ is almost invariant with respect to the ensemble, it is almost invariant for the Markov chain in the dynamical sense ("long relaxation time"), that is, the chain is trapped in $B$ for many iterations before it undergoes some conformational transition which then allows the chain to sample other regions of the phase space.

2. As we will see in Sec. 5.4 below, it is difficult enough to construct an efficient algorithmic realization of the Markov chain associated with the spatial transition operator for the canonical ensemble $f_{\text{can}}$. In this case we will exploit the specific multiplicative form of $f_{\text{can}}$. However, for some arbitrary set of essential variables $\vartheta$, we in general will not even have any explicit expression for the associated reduced density $F_\vartheta$. Therefore, there is no way —at least not with the strategies considered herein— to construct any efficient realization of the Markov chain associated with the transition operator $T_\vartheta$.

Despite these problems, we will exploit the mathematical analogy between transition operator, Markov chain and stochastic dynamical system not only for proving convergence results (see Sec. 4.5) but also to construct an appropriate numerical algorithm for evaluating the transition probabilities (cf. Sec. 5.4).

### 3.7.2 Fluctuation Length $\tau$ and Almost Invariance

How important is the choice of the "fluctuation length" $\tau$ for the identification of almost invariant sets? In the context of the examples in Sec. 3.1, we already observed that the eigenvectors for eigenvalues of $T$ near $\lambda = 1$ do *not* show any dramatic dependence on the actual value of $\tau$, while these eigenvalues tend to one with $\tau \to 0$. We will now present two additional insights which may help to understand this observation and may support our hope that $\tau$ may not have decisive influence on the shape of almost invariant sets.

The first of these insights is illustrated in Fig. 9 and states roughly that mainly the (topological) properties of the potential energy surface determine the shape of the almost invariant sets of the associated system: Around the main minima of the energy landscape there are large flow-invariant regions. Such a region is the "core" $C$ of an almost invariant set, if the probability to be within this region is large enough. In addition, each of these regions is surrounded by some set $S_C$ of unlikely states (shaded in light grey in Fig. 9), which have enough energy to leave $S_C$ under the action of the flow. The almost

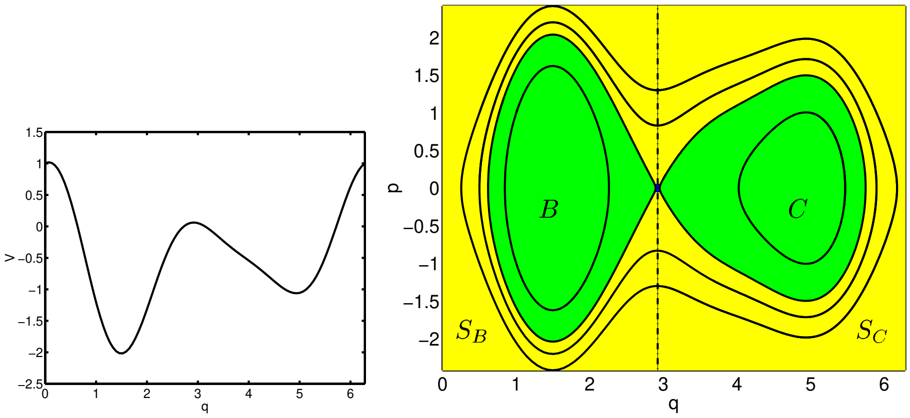



Figure 9: Phase portrait of the $2\pi$-periodic potential shown in the left figure. The saddle point at $(q_s, 0)$ between the minima separates two regions, shaded in dark grey and marked $B$ and $C$, which, both, are invariant under the flow. With the minima of the potential, the maxima of the canonical density are located inside $B$ and $C$. Thus, if the temperature is only small enough, the canonical density is (exponentially) small for any state $x \notin B \cup C$. But only trajectories starting in such unlikely states from the surroundings $S_B \cup S_C$ of $B \cup C$ can cross the line $q = q_s$, so that $A_1 = \{q : q < q_s\}$ and $A_2 = \{q : q > q_s\}$ are almost invariant sets. This implies that some variation of the fluctuation length $\tau$ may change the transition probability between $A_1$ and $A_2$, but *not* the almost invariant sets themselves.

invariant sets are given by the projection of these objects onto the essential coordinates chosen. Whenever the degree of almost invariance is large enough, the value of $\tau$ will have only minor influence, since it merely determines how many of the unlikely states from $S_C$ will have enough time to finish the possible transition between two almost invariant subsets (see Fig. 10). In this sense, $\tau$ controls a kind of "melting process" for the flow-induced mixing; this mixing generates conformational transitions but only in regions of the phase space with insignificant probability to be within.

To understand the second observation, we have to introduce some suitable notation: Let $\Phi_V^\tau$ denote the flow associated with the equations of motion (2) induced by the potential $V$. A simple rescaling of time in (2) reveals that

$$\Phi_V^{\alpha\tau}(q,p) \;=\; \Phi_{\alpha^2 V}^{\tau}(q, \alpha p), \quad \alpha > 0,$$

which shows that we can map an increase of the fluctuation length ($\alpha > 1$) onto an increase of potential and momenta. When considering the canonical density, and using a generalized notation for the transition probabilities, we find that

$$w\left(A, B, \Phi_V^{\alpha\tau}, e^{-\beta(T+V)}\right) \;=\; w\left(A, B, \Phi_{\alpha^2 V}^{\tau}, e^{-\frac{\beta}{\alpha^2}(T+\alpha^2 V)}\right),$$

where $T = p^T M^{-1} p/2$ denotes the kinetic energy and the exponential prefactor $\beta/\alpha^2$ belongs to some increased temperature $\mathcal{T}_\alpha = \alpha^2 \mathcal{T}$. Thus, this simple trick reveals that an increase in the fluctuation length $\tau$ may be understood as a certain rescaling of parameters of the ensemble. Again, $\tau$ appears as some kind of temperature-like "melting parameter". This identification of changes in $\tau$ with

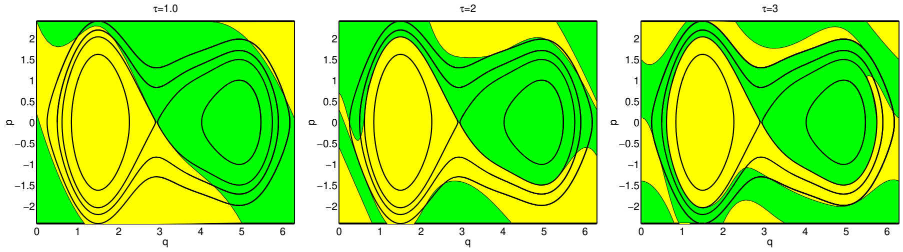



Figure 10: Illustration of the flow-induced mixing of the surroundings $S_C$ and $S_B$ of the system from Fig. 9. The three figures correspond to three different values (1.0/2.0/3.0) of the fluctuation length $\tau$. In all three cases the sets $A_1 = \{q : q < q_s\}$ and $A_2 = \{q : q > q_s\}$ are almost invariant (if the temperature is small enough). States $(q,p)$, which are transported to $A_2$ under the action of the flow during $\tau$, i.e., for which $\pi_1\Phi^\tau(q,p) > q_s$, are colored in dark grey. States ending up in $A_1$ are colored in light grey.

simply rescaling the ensemble is of particular interest, since biophysical intuition states that (mainly) the interactions (=potential) determine the conformations, while the temperature only redistributes the probabilities to be within.

While the conformational subsets may be relatively insensitive to changes in $\tau$, this is different for all quantities characterizing the actual conformational *dynamics*: The transition probabilities between the conformational subsets depend crucially on $\tau$ and converge to zero in the limit $\tau \to 0$. The same holds for the eigenvalues of $T$ near $\lambda = 1$. This should not come as a surprise: the fluctuation length $\tau$ is determined by the two-step experiment from page 19 as the time span during which the ensemble can fluctuate freely; that is, the transition probabilities can only be understood *relative* to the measurement procedure defining them.

# 4 Spectral and Asymptotic Properties

We now switch to the consideration of the spectrum $\sigma(T) \subset [-1,1]$ of our spatial transition operator $T : L^2_F(\Omega) \to L^2_F(\Omega)$. We will see in Sec. 4.6, that it is possible to get all crucial results for the spectrum of $T_\vartheta$ for some set of essential variables by transferring them from the spatial transition operator $T$ to $T_\vartheta$ via the restriction and prolongation operators introduced in Sec. 3.5.

We already observed that the simple constant function $\chi_\Omega$ satisfies

$$(T\chi_\Omega)(q) = \frac{1}{F(q)} \int_{\mathbb{R}^d} f_0(x)\, dp = 1, \quad \text{for all } q \in \Omega,$$

that is, $T\chi_\Omega = \chi_\Omega$. Thus, $\chi_\Omega$ is an invariant density of $T$. The reader might notice, that $\chi_\Omega$ is an element of $L^2_F(\Omega)$ because we initially assumed $f_0 \in L^1(\Gamma)$.

It is the fundamental strategy of this approach to compute conformational subsets from eigenstates of $T$ for eigenvalues near $\lambda = 1$, and to quantify the degree of invariance of the subsets. It is, thus, of main importance, under which conditions such eigenvalues exist and the eigenvalue $\lambda = 1$ is simple.

Since we are interested in a numerically stable approximation result, we have to demand for the existence of *isolated* eigenvalues near $\lambda = 1$, i.e., for eigenvalues of finite multiplicity which are separated from continuous parts of the spectrum by a finite gap. According to the well-known spectral theory of linear operators, the spectrum may be decomposed into two disjoint subsets: the set of all isolated eigenvalues of finite multiplicity and the essential spectrum (cf. Appendix B). Thus, our first crucial question is:

A. Under which conditions can $T$ have isolated eigenvalues of finite multiplicity near $\lambda = 1$, i.e., which conditions guarantee that the essential spectrum $\sigma_{\text{ess}}(T)$ is bounded away from $\lambda = 1$?

For an answer to this question, we will show that, in the Hilbert space $L^2_F(\Omega)$, our transition operator $T$ is *quasi-compact*. More precisely: We will show that $T$ —as an operator in $L^2_F(\Omega)$— can be decomposed into two linear operators, $T = T_1 + T_2$, where $T_1$ is a strict contraction ($\|T_1\|_{2,F} < 1$), while $T_2$ is compact. Thus, due to the general results of spectral theory (cf. Appendix B), the essential spectrum of $T$ is given by that of $T_1$ and, therefore, is bounded away from $\lambda = 1$ (cf. Sec. 4.2).

For the uniqueness of the invariant density, the following two questions are of comparable importance:

B. Under which conditions is $\lambda = 1$ a simple eigenvalue, i.e., when is $\chi_\Omega$, up to a factor, the unique eigenvector for $\lambda = 1$? And, when is $\lambda = 1$ the dominant eigenvalue, that is, $-1 \notin \sigma(T)$?

In order to find such conditions, we will exploit the stability theory for Markov operators. That is, we will analyze the asymptotic behavior of the iterates $T^n u$ for *densities* $u \in L^1_F(\Omega)$, in order to prove *asymptotic stability* of $T : L^1_F(\Omega) \to L^1_F(\Omega)$ (cf. Secs. 4.3 and 4.4). This will imply that, in $L^1_F(\Omega)$ and $L^2_F(\Omega)$,

$\lambda = 1$ is simple and dominant. It will come out to be relatively easy to realize this "operator–oriented" approach for bounded position space $\Omega$ (Secs. 4.3 and 4.4). However, we will also see that we can easily show the same results under weaker conditions when exploiting the well-established convergence theory for the associated Markov chains (Sec. 4.5). In particular, we will then be able to include the case of unbounded position space $\Omega$. Unfortunately, this "Markov chain theory" approach can only be applied to the *spatial* transition operator. But we will be able to include the case of unbounded $\Omega$ also in the operator oriented approach, when considering the transition operator $T_\vartheta$ associated with some set of essential variables.

In this section, we will always consider the case of canonical ensembles ($f_0 = f_{\text{can}}$). This simplifies a lot of arguments and avoids some nasty computations. But the reader might notice, that, if not explicitly stated otherwise, the following steps can also be realized for an arbitrary smooth stationary density $f_0 \in L^1(\Gamma)$.

## 4.1 Transition Kernels

We now want to examine whether the spatial transition operator $T$, or at least a part of it, has a representation as an integral operator with transition kernel. That is, we ask whether there is a nonnegative measurable function $k : \Omega \times \Omega \to \mathbb{R}$ such that

$$Tu(q) \;=\; \int_\Omega k(q,y)\, u(y)\, \mathcal{Q}(y)\, dy. \tag{34}$$

To this end, let us first proceed purely heuristically: Assume that for every $q \in \Omega$ the function $y_q(p) = \pi_1 \Phi^{-\tau}(q,p)$ is invertible for all $p \in \mathbb{R}^d$ and let $v_q = v_q(y)$ denote the inverse of $y_q$. Then, the transformation $p \mapsto y = y_q(p)$ applied to the integral

$$Tu(q) \;=\; \int_{\mathbb{R}^d} u(y_q(p))\, \mathcal{P}(p)\, dp$$

results in

$$Tu(q) \;=\; \int_\Omega \mathcal{P}\left(v_q(y)\right)\, \left|\det Dv_q(y)\right|\, u(y)\, dy,$$

so that we have to define our transition kernel as

$$k(q,y) \;=\; \frac{1}{\mathcal{Q}(y)}\, \mathcal{P}\left(v_q(y)\right)\, \left|\det Dv_q(y)\right|,$$

in order to achieve a representation like (34).

### 4.1.1 Momentum Invertibility

Certainly, our functions $y_q$ will in general *not* be invertible for all momenta. Therefore, we define the following weaker notion of invertibility:

DEFINITION 4.1 The Hamiltonian flow $\Phi^\tau$ is called *momentum-invertible* if both of the following two conditions hold:

1. For almost every $q \in \Omega$ there is an open set $M(q) \subset \mathbb{R}^d$ such that the function $y_q(p) = \pi_1 \Phi^{-\tau}(q,p)$ is locally invertible in $M(q)$, i.e., $\det Dy_q(p) \neq 0$ for $p \in M(q)$.

2. There is an $\eta > 0$ such that

$$\operatorname*{ess-inf}_{q\in\Omega} \int\limits_{M(q)} \mathcal{P}(p)\, dp \;=\; \eta. \tag{35}$$

In this case, the set $\mathcal{I} = \{(q,p) \in \Gamma \;:\; q \in \Omega, p \in M(q)\} \subset \Gamma$ is called the "accessible phase space" with respect to the sets $M(q)$.

REMARK 4.2 Every set $M(q)$ may consist of disjoint subsets. It follows from the Inverse Function Theorem that in every of these subsets $m(q) \subset M(q)$ there exists a smooth function $v_q : y_q(m(q)) \subset \Omega \to \mathbb{R}^d$ such that $v_q(y_q(p)) = p$ for all $p \in m(q)$. In most of the subsequent, we mostly refer to all of these functions as to "the inverse" $v_q$ defined on the whole of $y_q(M(q))$ without stating the different subsets explicitly.

For compact position spaces, momentum invertibility holds under considerably weak conditions:

LEMMA 4.3 *Let the position space $\Omega$ be compact and the potential $V$ be smooth. Assume that for every $q \in \Omega$ there is a momentum $p \in \mathbb{R}^d$ such that* $\det Dy_q(p) \neq 0$*. Then, the flow is momentum-invertible.*

*Proof:* Consider arbitrary $q \in \Omega$ and $p \in \mathbb{R}^d$ such that $\det Dy_q(p) \neq 0$. Since $y_q$ is smooth, there is an open neighborhood of $p$ where $y_q$ is invertible. Moreover, since the entire flow is smooth, there even is an open neighborhood $U(q) \subset \Omega$ of $q$ such that, for all $\tilde{q} \in U(q)$, $y_{\tilde{q}}$ is invertible in an open neighborhood $O$ of $p$. We may, without loss of generality, assume that the sets $M(\tilde{q})$ contain $O$. Then, there is an $\alpha = \int_O \mathcal{P}(p)dp$ such that

$$\eta(\tilde{q}) \;=\; \int_{M(\tilde{q})} \mathcal{P}(p)\, dp \;\geq\; \alpha \;>\; 0, \qquad \text{for all } \tilde{q} \in U(q). \tag{36}$$

Since $q$ has been arbitrary, such $\alpha > 0$ and neighborhood $U(q)$ exists for all $q \in \Omega$. Consequently, there is a covering of $\Omega$ consisting of such open neighborhoods $U(q)$. Since $\Omega$ is compact, this system of open sets covering $\Omega$ contains a *finite* subsystem $U(q_1), \ldots, U(q_m)$, $m \in \mathbb{N}$, also covering $\Omega$. Let $\alpha_l$ denote the $\alpha$-value for the neighborhoods $U(q_l)$. Hence, $\min_{l=1,\ldots,m} \alpha_l > 0$ which implies the assertion. $\square$

**Decomposition of $T$** If the system is momentum-invertible in the sense of Def. 4.1, we may define the following *partial transition operator*

$$T_2 u(q) = \int_{M(q)} u\left(\pi_1 \Phi^{-\tau}(q,p)\right) \mathcal{P}(p)\, dp. \tag{37}$$

relative to the sets $M(q)$ chosen. Because of our assumption, the so-defined operator has a transition kernel, i.e.,

$$T_2 u(q) = \int_{\Omega} k(q,y)\, u(y)\, \mathcal{Q}(y)\, dy.$$

The kernel can be written as:

$$k(q,y) = \frac{1}{\mathcal{Q}(y)} \sum_{\substack{p \in M(q) \\ y_q(p)=y}} \mathcal{P}\left(v_q(y)\right) \left|\det Dv_q(y)\right|, \tag{38}$$

where the sum runs over the disjoint subsets of $M(q)$, and we used the simplifications introduced in Remark 4.2, i.e., $v_q$ denotes all different branches on $y_q(M(q))$. In order to guarantee that the summation in (38) is well-defined, we introduce the following simplification:

DEFINITION 4.4 Let $n(q)$ denote the number of disjoint subsets in $M(q)$, where $n(q) = \infty$ whenever $M(q)$ contains infinitely many disjoint subsets. For every family of sets $M(q)$ define $\mathcal{N}_M = \sup_{q \in \Omega} n(q)$.

It is obvious that, if the flow is momentum-invertible, it is always possible to choose the sets $M(q)$ such that each $M(q)$ contains at most a given number of disjoint subsets[25] so that $\mathcal{N}_M < \infty$. This suggests to make the following

ASSUMPTION 4.5 *In the following, we mostly suppose that the sets $M(q)$ are chosen such that $\mathcal{N}_M < \infty$. This simplifies some arguments drastically but does not change anything crucially. Whenever the case $\mathcal{N}_M = \infty$ is considered, we suppose that every $M(q)$ contains at most countably many disjoint subsets.*

REMARK 4.6 The reader might notice that we may choose specific sets $M(q)$ *without loss of generality*, as long as we are only interested in *qualitative properties* of the full transition operator $T$, because, in this case, some freedom is left concerning the selection of the decomposition $T = T_1 + T_2$ induced by the $M(q)$. For example, in order to show that $T$ is quasi-compact, it suffices to prove the *existence* of some family $M(q)$ that leads to a compact $T_2$ with the property that $T_1 = T - T_2$ is a strict contraction (thus, we are free to show this under the side-condition $\mathcal{N}_M < \infty$, cf. Sec. 4.2). We are not necessarily interested in the "maximal" family of sets $M(q)$.

[25] The conditions 1 and 2 from Def. 4.1 remain intact, only the value of $\eta$ may change.

REMARK 4.7 The kernel $k$ is an $L^1$-function, i.e., it is $k \in L^1(\Omega \times \Omega)$, since

$$\int_\Omega \int_\Omega k(q,y)\mathcal{Q}(y)\mathcal{Q}(q)\,dy\,dq \;=\; \int_\Omega \int_{M(q)} \mathcal{P}(p)\,dp\,\mathcal{Q}(q)\,dq \;\leq\; 1.$$

As another direct consequence of the definition, we find that

$$\underbrace{\int_\Omega k(q,y)\;\mathcal{Q}(y)\,dy}_{=\eta(q)} \;=\; \int_{M(q)} \mathcal{P}(p)\,dp.$$

That is, $\eta(q)$ is the probability with which $q$ is *accessible* via $\Phi^\tau$ in the canonical ensemble (relative to the position density $\mathcal{Q}$). Thus, the constant from Def. 4.1,

$$\eta \;=\; \operatorname*{ess-inf}_{q\in\Omega} \eta(q) \;=\; \operatorname*{ess-inf}_{q\in\Omega} \int_{M(q)} \mathcal{P}(p)\,dp,$$

may be interpreted as the *minimal accessibility* via $\Phi^\tau$ in the canonical ensemble (with respect to the sets $M(q)$).

The two parts $T_1$ and $T_2$ of the decomposition $T = T_1 + T_2$ are linear bounded operators on the $L^p$-spaces:

LEMMA 4.8 *Let the flow $\Phi^\tau$ be momentum-invertible. Then, the transition operators $T_2$ and $T_1 = T - T_2$ are bounded operators $T_j : L^p(\Omega) \to L^p(\Omega)$ for $j = 1,2$ and $p = 1,2$. Moreover, in $L^2(\Omega)$, the operator norm of $T_1$ satisfies $\|T_1\|_2 < 1$, i.e., more precisely,*

$$\|T_1 u\|_2 \;\leq\; \kappa\,\|u\|_2, \quad \forall u \in L^2(\Omega), \qquad \textit{with } \kappa \;=\; \sqrt{1-\eta} \;<\; 1,$$

*where $\eta > 0$ is the constant from Def. 4.1.*

REMARK 4.9 In Lemma 4.8, the stated property of $T_1$ depends on the specific sets $M(q)$ which were selected from all the possible sets with respect to which the flow $\Phi^\tau$ is momentum-invertible. The reader should be aware that, now and in the following, statements like that of Lemma 4.8 should always be understood in this sense "relative to the specific sets $M(q)$".

*Proof of Lemma 4.8:* Clearly, the assertions for $T_2$ follow from those for $T_1$ and the properties of $T$ itself. The operator $T_1$ has the representation

$$T_1 u(q) \;=\; \int\limits_{\mathbb{R}^d \setminus M(q)} u\left(\pi_1 \Phi^{-\tau}(q,p)\right) \mathcal{P}(p)\,dp,$$

which immediately implies the assertion for $L^1(\Omega)$. In order to establish the asserted bound in $L^2(\Omega)$, consider the measurable function $g : \Gamma \to \{0,1\}$ defined by $g(q,p) = 1 - \chi_{M(q)}(p)$ and its integral

$$\gamma(q) \;=\; \int_{\mathbb{R}^d} g(q,p)\,\mathcal{P}(p)\,dp \;=\; \int_{\mathbb{R}^d \setminus M(q)} \mathcal{P}(p)\,dp,$$

for which the normalization $\int_{\mathbb{R}^d} \mathcal{P}(p)\,dp = 1$ and condition 2 of the momentum-invertibility implies that

$$\operatorname*{ess-sup}_{q\in\Omega} \gamma(q) \;=\; 1 \;-\; \operatorname*{ess-inf}_{q\in\Omega} \int\limits_{M(q)} \mathcal{P}(p)\,dp \;=\; 1 \;-\; \eta \;<\; 1. \tag{39}$$

In close analogy to the proof of Lemma 3.9, the following estimate results from using the Cauchy-Schwarz inequality in the Hilbert space $L^2_{\mathcal{P}} = \{w : \mathbb{R}^d \to \mathbb{C} : \int_{\mathbb{R}^d} |w(p)|^2 \mathcal{P}(p)\,dp < \infty\}$ for the inner integration:

$$\begin{aligned}
\|T_1 u\|_2^2 \;&=\; \int_\Omega \left[ \left| \int_{\mathbb{R}^d} g(q,p)\, u\left(\pi_1 \Phi^{-\tau}(q,p)\right)\, \mathcal{P}(p)\,dp \right|^2 \right] \mathcal{Q}(q)\,dq \\
&\le\; \int_\Omega \left[ \underbrace{\int_{\mathbb{R}^d} g(q,p)\mathcal{P}(p)\,dp}_{=\gamma(q)} \;\cdot\; \int_{\mathbb{R}^d} \left| u\left(\pi_1 \Phi^{-\tau}(q,p)\right)\right|^2\, \mathcal{P}(p)\,dp \right] \mathcal{Q}(q)\,dq \\
&\le\; \operatorname*{ess-sup}_{q\in\Omega} \gamma(q) \;\cdot\; \|u\|_2^2.
\end{aligned}$$

Together with (39), this yields the assertion. ∎

Let us now check what is needed to establish a similar bound for $T_1$ in the $L^1$-space. Obviously,

$$\begin{aligned}
\|T_1 u\|_1 \;&\le\; \int_\Omega \int_{\mathbb{R}^d \setminus M(q)} |u(\pi_1 \Phi^{-\tau}(q,p))|\, \mathcal{P}(p)\,dp\, \mathcal{Q}(q)\,dq \\
&=\; \|T|u|\|_1 \;-\; \|T_2|u|\|_1,
\end{aligned}$$

where we have equality in the first line if $u$ is nonnegative. Hence, for proving $\|T_1 u\|_1 < 1$, we have to show that there is a $\kappa > 0$ such that $\|T_2 u\|_1 \ge \kappa$ for all densities $u \in \mathcal{D}(\Omega)$. But for these $u \in \mathcal{D}(\Omega)$:

$$\begin{aligned}
\|T_2 u\|_1 \;&=\; \int_\Omega \int_{M(q)} u(\pi_1 \Phi^{-\tau}(q,p))\, \mathcal{P}(p)\, \mathcal{Q}(q)\,dp\,dq \\
&=\; \int_{\mathcal{I}} u(\pi_1 \Phi^{-\tau} x)\, f_0(x)\,dx \;=\; \int_{\Phi^\tau \mathcal{I}} u(\pi_1 x)\, f_0(x)\,dx,
\end{aligned}$$

where $\mathcal{I}$ is the set introduced in Def. 4.1. We thus define new momenta subsets $\tilde{M}(q)$ such that

$$\Phi^\tau \mathcal{I} \;=\; \{(q,p) \in \Gamma :\; q \in \Omega,\, p \in \tilde{M}(q)\}, \tag{40}$$

which, inserted into the equation above, yields the required bound:

$$\|T_2 u\|_1 \;\ge\; \operatorname*{ess-inf}_{q\in\Omega} \int_{\tilde{M}(q)} \mathcal{P}(p)\,dp.$$

Hence, one observes that the condition for $\|T_1\|_1 < 1$ requires an analogy to our condition (35) (which implies $\|T_1\|_2 < 1$ as we have already observed). Now, the condition concerns the *final* momenta of trajectories starting in the accessible phase space $\mathcal{I}$, instead of the *initial* momenta in (35).

PROPOSITION 4.10 $T_1 : L^1(\Omega) \to L^1(\Omega)$ *is bounded by* $\|T_1\|_1 \leq 1 - \tilde{\eta}$ *with*

$$\tilde{\eta} = \operatorname*{ess\text{-}inf}_{q\in\Omega} \int_{\tilde{M}(q)} \mathcal{P}(p)\, dp,$$

*where the sets* $\tilde{M}(q)$ *are given by (40).*

Merging the above construction with the strategy of the proof of Lemma 4.3, we end up with

PROPOSITION 4.11 *Let the flow be smooth and momentum-invertible with accessible phase space* $\mathcal{I}$ *and suppose that the position space* $\Omega$ *is compact. Assume that for every* $y \in \Omega$ *there is some* $(q,p) \in \mathcal{I}$ *such that* $y = y_q(p)$*. Then, there is some* $\rho > 0$ *such that* $\|T_1\|_1 \leq 1 - \rho$.

### 4.1.2 Symmetric Momentum Invertibility

Let $\tilde{\eta}$ again be the constant from Prop. 4.10. Unfortunately, $\tilde{\eta} > 0$ is *not* a consequence of our condition (35). Because of this, we introduce a stronger notion of invertibility:

DEFINITION 4.12 The Hamiltonian flow $\Phi^\tau$ is called *symmetrically momentum-invertible* if it is momentum-invertible and if the sets $M(q)$ in Def. 4.1 can be chosen such that, simultaneously to the two conditions in Def. 4.1, the following two conditions are satisfied, too. For all $q \in \Omega$:

1. $M(q)$ is almost everywhere momentum reversible, that is, $p \in M(q) \Leftrightarrow -p \in M(q)$.
2. If $p \in M(q)$ and $(y,v) = \Phi^{-\tau}(q,p)$ then $v \in M(y)$.

REMARK 4.13 Since $\Phi^\tau$ is (momentum) reversible, the two conditions imply that, for $p \in M(q)$, also $v \in M(y)$ when $(y,v) = \Phi^\tau(q,p)$. Hence, if $\Phi^\tau$ is symmetrically momentum-invertible:

$$\begin{aligned}\Phi^\tau \mathcal{I} &= \{\Phi^\tau(q,p) : \; q \in \Omega, \, p \in M(q)\} \\ &= \{(y,v) = \Phi^\tau(q,p) : \; y \in \Omega, \, v \in M(y)\} \; = \; \mathcal{I},\end{aligned}$$

which directly implies that $M(q) = \tilde{M}(q)$ for almost all $q \in \Omega$.

Thus, as a consequence of Prop. 4.10:

PROPOSITION 4.14 *If $\Phi^\tau$ is symmetrically momentum-invertible, then $T_1 : L^1(\Omega) \to L^1(\Omega)$ is bounded by $\|T_1\|_1 \le 1 - \eta$ with $\eta$ given by*

$$\eta \;=\; \operatorname*{ess-inf}_{q\in\Omega} \int\limits_{M(q)} \mathcal{P}(p)\,dp.$$

Moreover, this kind of momentum-invertibility has other advantages:

LEMMA 4.15 *If $\Phi^\tau$ is symmetrically momentum-invertible, then the transition operator $T_2 : L^2(\Omega) \to L^2(\Omega)$ is self-adjoint.*[26]

*Proof:* Since the step-functions are dense in $L^2(\Omega)$, it is sufficient to prove

$$\langle T_2\chi_B, \chi_C\rangle \;=\; \langle \chi_B, T_2\chi_C\rangle,$$

for the characteristic functions of two arbitrary measurable sets $B, C \subset \Omega$. For notational convenience, let us introduce for every subsets $A \subset \Omega$:

$$\mathcal{I}(A) \;=\; \{(q,p) \in \Gamma :\; q \in A,\, p \in M(q)\} \subset \mathcal{I}.$$

Following the same strategy as in the proof of Lemma 3.10, we find, by exploiting the reversibility of $\Phi^\tau$ and the invariances of $f_0$, that:

$$\begin{aligned}
\langle \chi_B, T_2\chi_C\rangle &= \int_{\mathcal{I}(B)} \chi_{\Gamma(C)}(\Phi^{-\tau}x)\, f_0(x)\,dx \;=\; \int_{\mathcal{I}(B)} \chi_{\Gamma(C)}(R\Phi^{\tau}Rx)\, f_0(x)\,dx \\
&= \int_{R\mathcal{I}(B)} \chi_{\Gamma(C)}(\Phi^{\tau}x)\, f_0(x)\,dx \;=\; \int_{\Gamma(C)} \chi_{R\mathcal{I}(B)}(\Phi^{-\tau}x)\, f_0(x)\,dx \\
&= \int_{\Gamma(C)\cap\Phi^\tau R\mathcal{I}(B)} f_0(x)\,dx.
\end{aligned}$$

But we also have

$$\langle T_2\chi_B, \chi_C\rangle \;=\; \int_{\mathcal{I}(C)} \chi_{\Gamma(B)}(\Phi^{-\tau}x)\, f_0(x)\,dx \;=\; \int_{\mathcal{I}(C)\cap\Phi^\tau\Gamma(B)} f_0(x)\,dx.$$

Hence, the assertion is proved if

$$\Gamma(C) \cap \Phi^\tau R\mathcal{I}(B) = \mathcal{I}(C) \cap \Phi^\tau\Gamma(B).$$

In order to finally see this, we exploit conditions 1 and 2 of Def. 4.12 and Rmk. 4.13:

$$\begin{aligned}
\mathcal{I}(C) \cap \Phi^\tau\Gamma(B) &= \{(q,p) :\; q \in C,\, \pi_1\Phi^{-\tau}(q,p) \in B,\, p \in M(q)\} \\
&= \{(q,p) :\; q \in C,\, (y,v) = \Phi^{-\tau}(q,p),\, y \in B,\, v \in M(y)\} \\
&= \Gamma(C) \cap \{\Phi^\tau(y,v) :\; y \in B,\, v \in M(y)\} \\
&= \Gamma(C) \cap \Phi^\tau\mathcal{I}(B) \;=\; \Gamma(C) \cap \Phi^\tau R\mathcal{I}(B).
\end{aligned}$$

∎

[26] That is, $T_2$ is self-adjoint if the sets $M(q)$, with respect to which $T_2$ is defined, satisfy the conditions for symmetric momentum-invertibility. Cf. Rmk. 4.9.

Remark 4.16 Self-adjointness of $T_2$ implies the symmetry of the kernel:[27]

$$T_2 \text{ self-adjoint} \quad \Rightarrow \quad k(q,y) \;=\; k(y,q) \text{ almost everywhere in } \Omega \times \Omega. \tag{41}$$

This observation allows a symmetric interpretation of the minimal accessibility of the system (cf. Rmk. 4.7). In other words: we no more have to distinguish between transitions to and from $q$. Let us introduce the set $\mathcal{A}(q)$ of all positions $y$ which are accessible from $q$ via $\Phi^{-\tau}$:

$$\mathcal{A}(q) \;=\; \{y \in \Omega,\ \text{there is } p \in M(q) \text{ s.t. } y = y_q(p)\} \;=\; y_q(M(q)), \tag{42}$$

which now is also the set of all positions $y$ which are accessible from $q$ via $\Phi^{\tau}$. In particular, the symmetry of $k$ yields

$$y \in \mathcal{A}(q) \quad \Leftrightarrow \quad q \in \mathcal{A}(y).$$

### 4.1.3 Illustrative Example

Consider the Hamiltonian system $H(q,p) = p^2/2 + V(q)$ with the smooth, periodic potential (cf. Fig. 11)

$$V(q) \;=\; \begin{cases} 1 \,-\, (q+2)^2/2, & \text{for } -2 \le q \le -1 \\ \quad q^2/2, & \text{for } -1 \le q \le 1 \\ 1 \,-\, (q-2)^2/2, & \text{for } 1 \le q \le 2 \end{cases} \qquad \text{with } V(q) = V(q+4). \tag{43}$$

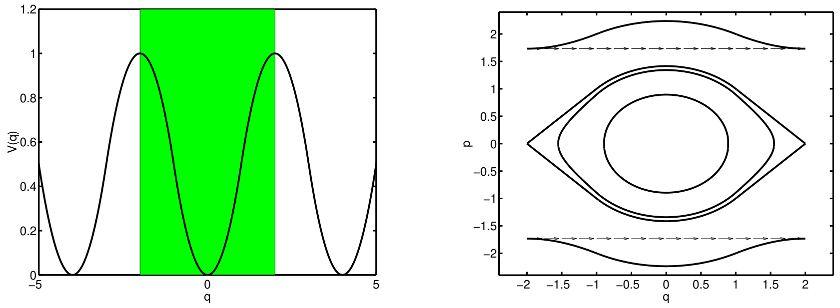



Figure 11: Left: Periodic potential $V$ of (43). The shaded domain is the periodicity cell $\Omega$. Right: Phase portrait of the flow $\Phi_\xi^{-\tau}$ for total energies $E = 0.4/0.9/1.0/2.5$. The thin vector lines indicate that the end points of the curves for $E = 2.5$ are connected via the periodicity map $\xi$. Along the flow $\Phi^{-\tau}$ the curves are circled clockwise.

[27] It is easy to see that the adjoint of $T_2$ may be represented as $T_2^* u(q) = \int k(y,q)u(y)\mathcal{Q}(y)dy$, cf. [113], Chap. VII. If we define $T_2 u(q) = \int_\Omega \tilde{k}(q,y)u(y)dy$, the kernel $\tilde{k}$ is not directly symmetric but satisfies the well-known *detailed balance condition*: $\mathcal{Q}(q)\tilde{k}(q,y) = \mathcal{Q}(y)\tilde{k}(y,q)$.

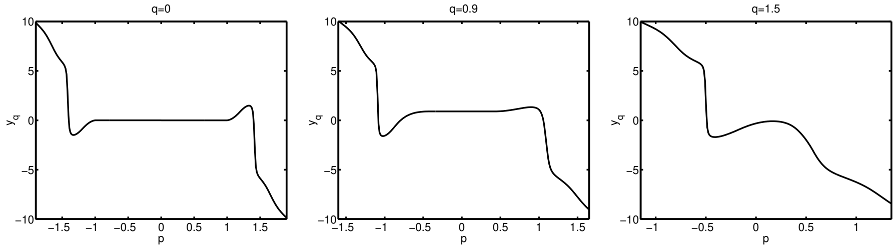


Figure 12: The function $y_q$ for different $q$ for the system with potential (43), i.e., $y_q(p) = \pi_1 \Phi^{-\tau}(q,p)$ for $q = 0/0.9/1.5$ versus momentum $p$.

Let $\Phi^t : \mathbb{R}^2 \to \mathbb{R}^2$ be the associated flow. We restrict our description to the periodicity cell $\Omega = [-2, 2]$, i.e., we define the flow $\Phi^t_\xi$ due to (22) via the periodicity map $\xi$ and the transition operator acting on functions $u : \Omega \to \mathbb{C}$ due to (23) via

$$Tu(q) = \int_{\mathbb{R}} u\left(\pi_1 \Phi_\xi^{-\tau}(q,p)\right) \mathcal{P}(p)\, dp, \quad \text{for } q \in \Omega. \tag{44}$$

Let us choose $\tau = 2\pi$. Then, we immediately observe that, for the harmonic part of the potential and low energy, $\Phi_\xi^{-\tau}$ is the identity. More precisely

$$q \in [-1,1],\, |p| < \sqrt{1-q^2} \quad \Rightarrow \quad \Phi_\xi^{-\tau}(q,p) = (q,p).$$

One also finds[28] that sufficient kinetic energy guarantees invertibility of $y_q(\cdot) = \pi_1 \Phi_\xi^{-\tau}(q,\cdot)$, i.e., for every $q \in \Omega$:

$$\frac{1}{2}p^2 > 1 \quad \Rightarrow \quad \det Dy_q(p) \neq 0.$$

Thus, $\Phi_\xi^\tau$ is symmetrically momentum-invertible with $M(q) = \{p : p^2/2 > 1\}$. But we also observe that, particularly for $|q| > 1$, the function $y_q$ is also invertible for some momenta with $p^2/2 \leq 1$ (cf. Fig. 12). This nicely illustrates, that there is a significant freedom in the choice of the sets $M(q)$, and that it is a remarkable problem to identify “maximal” sets $M(q)$ such that the flow is (symmetrically) momentum-invertible.

**Worst Case Scenario** Let us go to the extreme of this example and consider the Hamiltonian $H(q,p) = p^2/2 + V(q)$ with the potential $V(q) = q^2/2$ for $|q| \leq 1$ as a 2-periodic function, i.e., with $V(q+2) = V(q)$.[29] We again take $\tau = 2\pi$, define the transition operator by equation (44) via the periodic flow $\Phi_\xi^\tau$, and denote $y_q(\cdot) = \Phi_\xi^{-\tau}(q,\cdot)$ (cf. Fig. 13). The flow is again symmetrically momentum invertible, now, e.g., with

$$M(q) = \{p \in \mathbb{R} : \quad H(q,p) > 1/2\} = \{p \in \mathbb{R} : \quad |p| > \sqrt{1-q^2}\},$$

[28] Compare Sec. 4.7 for more details.

[29] Within the limited scope of this example, the discontinuity of the force $DV$ does not matter.

with associated minimal accessibility

$$\eta \,=\, 2\int_1^\infty \mathcal{P}(p)\,dp.$$

But this time, $p \notin M(q)$ implies $\Phi_\xi^{-\tau}(q,p) = (q,p)$, so that the sets $M(q)$ are "maximal". For the associated operator $T_1$ this yields

$$T_1 u(q) \,=\, \int\limits_{|p|\le\sqrt{1-q^2}} u(\pi_1\Phi_\xi^{-\tau}(q,p))\,\mathcal{P}(p)\,dp \,=\, \gamma(q)\,u(q),$$

with $\gamma(q) = \int_{|p|\le\sqrt{1-q^2}} \mathcal{P}(p)dp$. That is, $T_1$ is a multiplication operator. Due to Thm. B.41 and Thm. B.41, its spectrum is given by

$$\sigma(T_1) \,=\, \sigma_{\text{ess}}(T_1) \,=\, \overline{\mathrm{Ran}(\gamma)} \,=\, [0, 1-\eta],$$

i.e., the spectrum of $T_1$ contains no discrete part.

Explicit computations show that the Jacobian $Dv_q$ is uniformly bounded in $q$ and $y$ (cf. Fig. 13). The sums in the definition (38) of the kernel $k$ contain infinitely many terms but come out to be also uniformly bounded in $q$ and $y$.[30] Moreover, the weight factor $\mathcal{Q}(y)^{-1}$ is bounded since $\inf_{[-1,1]}\mathcal{Q} = \exp(-\beta/2)/\mathcal{Z}_q > 0$. Thus, $k$ is bounded. We will see in the subsequent sections that this implies that the associated operator $T_2 = T - T_1$ is compact,[31] so that, due to Thm. B.43, the essential spectrum of the full transition operator $T$ is identical to that of $T_1$. This illustrates that, in the "worst case", the (maximal) minimal accessibility $\eta$ describes the *spectral gap* between the essential spectrum of $T$ and the dominant eigenvalue $\lambda = 1$.

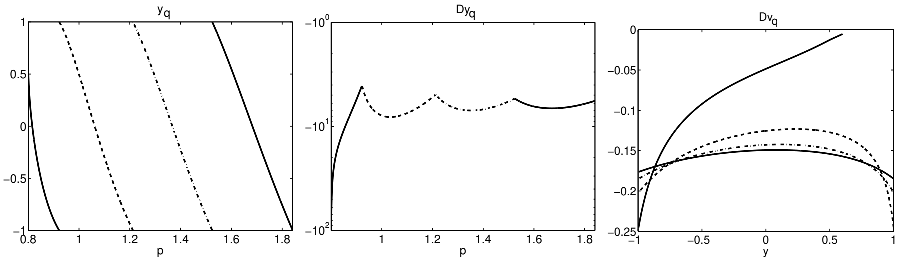



Figure 13: Worst case scenario: Functions $y_q = y_q(p)$, $Dy_q = Dy_q(p)$, and $Dv_q = Dv_q(y)$ for $q = 0.6$ and $p > \sqrt{1-q^2}$. Observe that $Dy_q$ is bounded and the branches of $Dy_q$ for increasing momenta in $y_q$ converge to $-2\pi = -\tau$ so that $Dv_q$ converges to $-1/\tau$.

[30] The Jacobian is uniformly bounded and the sequence of the possible values of $v_q(y)$ is asymptotically equidistributed so that the decay properties of $\mathcal{P}$ together with the majorant criterion yield a uniform bound.

[31] Compare Prop. 4.17.

## 4.2 Compactness in $L^2$

Let us assume all over this subsection that $\Phi^\tau$ is momentum-invertible so that the transition kernel of $T_2$ is given by (38). Now, we want to collect useful criteria guaranteeing compactness of $T_2$ in $L^2(\Omega)$.

To this end, we will exploit that $T_2$ is a Hilbert-Schmidt operator (and, thus, compact) in $L^2(\Omega)$ if and only if its kernel satisfies $k \in L^2(\Omega \times \Omega)$,[32], i.e., if and only if

$$I(k) \;=\; \int_\Omega \int_\Omega k(q,y)^2\, \mathcal{Q}(y)\, \mathcal{Q}(q)\, dy\, dq \;<\; \infty. \tag{45}$$

As a first observation, the boundedness of the kernel is enough to guarantee compactness of $T_2$, since $k \in L^\infty(\Omega \times \Omega)$ implies $k \in L^2(\Omega \times \Omega)$ (cf. Prop. A.1). Thus, we note that

PROPOSITION 4.17 *The transition operator $T_2 : L^2(\Omega) \to L^2(\Omega)$ is a Hilbert-Schmidt operator, and thus compact, if its kernel $k$ is bounded, i.e., there is a $C < \infty$ such that*

$$\operatorname*{ess\text{-}sup}_{q,y\in\Omega} k(q,y) \;=\; C.$$

This criterion suffices in many situations, particularly if the position space $\Omega$ is bounded (cf. the worst case scenario in Sec. 4.1.3). But we can easily find weaker criteria:

**Bounded Position Space** If $\Omega$ is bounded, we have $\inf_{q\in\Omega} \mathcal{Q}(q) > 0$. Due to Ass. 4.5 the sums in (38) defining $k$ contain at most $\mathcal{N}_M < \infty$ terms. Thus, $k$ is bounded, if only the Jacobian $\det Dv_q$ is uniformly bounded:

PROPOSITION 4.18 *Let $\Omega$ be bounded and assume that the sets $M(q)$ are chosen such that $\mathcal{N}_M < \infty$ and the Jacobian $\det Dv_q$ is uniformly bounded, i.e., there is a $C > 0$ such that for almost every $q \in \Omega$:*

$$|\det Dv_q(y_q(p))| \;<\; C, \qquad \textit{for all } p \in M(q). \tag{46}$$

*Then, $T_2$ is a Hilbert-Schmidt operator.*

REMARK 4.19 The uniform boundedness of the Jacobian $\det Dv_q$ will also be of importance for other considerations. Therefore, let us call the flow $\Phi^\tau$ *(symmetrically) momentum invertible with uniform bound*, if the sets $M(q)$ can be chosen such that the above condition (46) is satisfied together with the conditions for (symmetric) momentum invertibility.

[32] Compare Thm. B.47 and Prop. B.46 in Appendix B.

REMARK 4.20 Whenever $\mathcal{N}_M = \infty$ in the sense of Ass. 4.5, we need some additional condition on the convergence of the sum in (38) in order to prove that $k$ is bounded. The following condition is sufficient: Let the disjoint subsets of $M(q)$ be denoted by $M_l(q)$, $l \in \mathbb{N}$, and the different branches of the functions $y_q$ and $v_q$ on these sets by $y_q^l$ and $v_q^l$. Moreover, suppose that the flow is momentum-invertible with uniform bound. Then, there must be some $C > 0$ such that

$$\sum_{y \in y_q^l(M_l(q))} \mathcal{P}\left(v_q^l(y)\right) < C, \qquad \text{uniformly for all } q, y \in \Omega.$$

This condition is valid, for example, if $v_q^l(y)$ asymptotically increases like some power of $l$ (as it is the case for periodic potentials, see Rmk. 4.48).

**Unbounded Position Space** If $\Omega$ is unbounded, we cannot hope (to show) that $k$ is bounded or that a condition as simple as in Prop. 4.18 can be valid. Thus, we need an alternative criterion for $k \in L^2(\Omega \times \Omega)$. Since we will consider the case of unbounded $\Omega$ in Sec. 4.6 again, we may herein restrict the presentation to the following result which exploits particular properties of the canonical density. In order to derive this result we have to introduce some notation:

Let us denote the disjoint subsets of $M(q)$ by $M_l(q)$, $l \in \{1, \ldots, \mathcal{N}_M\}$, and the corresponding branches of the functions $y_q$ and $v_q$ by $y_q^l$ and $v_q^l$. Some $y \in \Omega$ may be accessible via different branches of $y_q$. Therefore, introduce the index set $I_l(q,p) = \{j : y_q^l(p) \in y_q^j(M_j(q))\}$. For $j \in I_l(q,p)$, the position $y = y_q^l(p)$ is accessible from $q$ via the branch $y_q^j$ with initial momentum $v_q^j(y)$. Taking this branch, we arrive at $y = y_q^l(p)$ with final momentum

$$v^{lj}(q,p) = \pi_2 \Phi^{-\tau}\left(q, v_q^j(y_q^l(p))\right). \tag{47}$$

The distribution of these final momenta influences whether $T_2$ is Hilbert-Schmidt or not:

LEMMA 4.21 *Suppose that the flow is momentum invertible with uniform bound. Then, the transition operator $T_2 : L^2(\Omega) \to L^2(\Omega)$ is a Hilbert-Schmidt operator if the following integrability condition is satisfied:*

$$\int_\Omega \left\{ \sum_{l=1}^{\mathcal{N}_M} \int_{M_l(q)} \mathcal{P}(p) \sum_{j \in I_l(q,p)} \mathcal{P}\left(v^{lj}(q,p)\right) dp \right\} dq < \infty, \tag{48}$$

*where the functions $v^{lj}$ are given by (47).*

*Proof:* Inserting the definition (38) of the kernel $k$ into the condition $I(k) < \infty$ from (45) yields

$$I(k) = \int_\Omega \int_\Omega \sum_{l,j=1}^{\mathcal{N}_M} \frac{\mathcal{P}(v_q^l(y))\mathcal{P}(v_q^j(y))}{\mathcal{Q}(y)} \left|\det Dv_q^l(y) \det Dv_q^j(y)\right| \mathcal{Q}(q)\, dy\, dq < \infty.$$

Applying the transformations $y \mapsto y_q^l(p)$ and using the boundedness of the Jacobians transforms results in the following stronger condition:

$$\int_\Omega \left\{ \sum_{l=1}^{\mathcal{N}_M} \int_{M_l(q)} \frac{\mathcal{P}(p)}{\mathcal{Q}(y_q^l(p))} \sum_{j \in I_l(q,p)} \mathcal{P}(v_q^j(y_q^l(p)))\, dp \right\} \mathcal{Q}(q)\, dq \;<\; \infty. \tag{49}$$

The flow-invariance of the canonical density $f_0(q,p) = \mathcal{Q}(q)\mathcal{P}(p)$ leads to

$$\mathcal{Q}(q)\mathcal{P}(v_q^j(y_q^l(p))) \;=\; \mathcal{Q}(y_q(p))\mathcal{P}(v^{lj}(q,p)).$$

Inserting this into (49) finally yields the assertion. $\square$

In order to realize that this condition is satisfied, it is, e.g., sufficient to show that $|v^{lj}(q,p)| \to \infty$ for $q \to \infty$ fast enough for every fixed momentum $p$ and every pair of indices for which it is defined. For an illustration of the usefulness of Lemma 4.21, consider the following

EXAMPLE 4.22 Let us consider the one-dimensional harmonic oscillator, i.e., $H(q,p) = (q^2+p^2)/2$ in $\Gamma = \mathbb{R}^2$. Choose $\tau$ such that $\sin(\tau) \neq 0$. Then, the associated flow is symmetrically momentum-invertible with $M(q) = \mathbb{R}$ (and, thus, $\mathcal{N}_M = 1$) and

$$y_q(p) \;=\; q\cos(\tau) \;-\; p\sin(\tau) \quad \text{implying} \quad |\det Dv_q| \;=\; |\frac{1}{\sin(\tau)}| \;<\; \infty.$$

Thus, condition (48) of Lemma 4.21 is satisfied, because $\pi_2\Phi^{-\tau}(q,p) = p\cos(\tau) + q\sin(\tau)$ increases linearly with $q$. Consequently, the transition operator $T = T_2$ is a Hilbert-Schmidt operator in this case.

**Essential Spectral Radius** Suppose that our partial transition operator $T_2$ is a Hilbert-Schmidt operator and, thus, compact. Moreover, let $T_1 = T - T_2$ be a strict contraction in $L^2(\Omega)$, i.e., let there be some $\kappa < 1$ such that $\|T_1\|_2 \leq \kappa$. Then, the essential spectrum of the full transition operator $T$ is bounded away from one.[33] We have just observed that $T_1$ satisfies this condition, if only the flow is momentum-invertible (Lemma 4.8), and that, in this case, the operator norm of $T_1$ can be estimated via the minimal accessibility $\eta > 0$ due to $\|T_1\|_2 \leq \sqrt{1-\eta}$. That is, the *essential spectral radius* $r_{\text{ess}}(T)$ is strictly smaller than one and can be estimated via the minimal accessibility:

$$r_{\text{ess}}(T) \;=\; \max_{\lambda \in \sigma_{\text{ess}}(T)} |\lambda| \;\leq\; \sqrt{1-\eta}.$$

We, thus, are interested in sets $M(q) \subset \mathbb{R}^d$ for which the associated operator $T_2$ is a Hilbert-Schmidt operator and the minimal accessibility

$$\eta \;=\; \operatorname*{ess-inf}_{q \in \Omega} \int_{M(q)} \mathcal{P}(p)\, dp$$

[33]Compare Thms. B.43 and B.35 in Appendix B. It should be noticed that, for the application of Thm. B.43, it is *not* necessary that $T_2$ is self-adjoint: The essential spectrum of $T$ is real-valued and bounded by $\|T_1\|_2$, whether $T_2$ is self-adjoint or not.

is *maximal.* These sets would gain the best estimate for the essential spectral radius. Thus, maximize the minimal accessibility in choosing the sets $M(q)$.

The worst case example of Sec. 4.1.3 illustrates that the essential spectrum of $T$ may indeed be as large as these estimates indicate so that the maximal minimal accessibility in fact gives the gap between the essential spectrum of $T$ and its dominant eigenvalue $\lambda = 1$.

In other words, our best estimate for the essential spectral radius is given by the *maximal statistical weight* of the momenta for which one of the functions $y_q = y_q(p)$ is *not* invertible. However, the present author wants to express his opinion in form of the following

CONJECTURE 4.23 *Generically, for potentials modelling realistic molecular systems, the functions $y_q$ are invertible except for very few momenta, i.e., the essential spectral radius of $T$ in $L^2(\Omega)$ will be very small.*

REMARK 4.24 Due to FOGUEL [39], there exists an "optimal" decomposition of $T$ such that the kernel $k$ is "maximal" and the remaining part $T_1$ is "singular" (for details, see Thm. A.6). However, this maximal kernel $k$ will not necessarily define a *compact* operator $T_2$ on $L^2(\Omega)$. Nevertheless, the worst case example in Sec. 4.1.3 is one example for a situation, in which we may adopt FOGUEL's maximal decomposition (which in this case yields a compact $T_2$ and a multiplication operator $T_1$). According to FOGUEL, the idea of studying a Markov process, which is a sum of a kernel and a singular part, goes back to DOEBLIN [26, 27] and YOSIDA AND KAKUTANI [114].

## 4.3 Constrictiveness in $L^1$

We herein are interested in proving that our Markov operator $T : L^1(\Omega) \to L^1(\Omega)$ has a unique invariant density. As a first step to this aim, we prove that $T$ is *constrictive. Constrictiveness rules out the possibility that the iterates $T^n u$ of some density $u \in \mathcal{D}$ eventually concentrate on a set of very small or vanishing measure* (cf. Def. A.7).

This is of importance, since for constrictive Markov operators a strong *spectral decomposition theorem* holds. Roughly speaking: Whenever $T$ is constrictive, there is a finite dimensional subspace $S$ of $L^1$, such that the iterates $T^n u$ converge towards $S$ for every $u \in L^1$. Moreover, there is a (disjoint) partition $\{B_j\}$ of $\Omega$ such that $S = \text{span}\{\chi_{B_j}\}$, and $T$ acts on $\{\chi_{B_j}\}$ as a weighted permutation.[34] In other words: The iterates $T^n u$ for constrictive $T$ are *asymptotically periodic.*

For this subsection, assume that our decomposition $T = T_1 + T_2$ guarantees that $T_1$ is a strict $L^1$-contraction, i.e., that $\|T_1\|_1 < 1$ (cf. Props. 4.10, 4.11, and Prop. 4.14).

In order to prove the (direct) constrictiveness of $T$, we have to show that there exists some $\delta > 0$, and $\gamma < 1$ and a measurable set $B \subset \Omega$, such that for

[34] For a rigorous formulation see Thm. A.12 in Appendix A.

every density $u \in \mathcal{D}$

$$\int_{(\Omega\setminus B)\cup E} Tu(q)\, \mathcal{Q}(q)\, dq \;\le\; \gamma,$$

for all measurable subsets $E \subset B$ with $\int_E \mathcal{Q}(q)\, dq \,<\, \delta$. We will do this by exploiting again the decomposition $T = T_1 + T_2$.

**Bounded Position Space** Let us first analyse the rather simple case that $\Omega$ is bounded and fix $B = \Omega$. Our assumption $\|T_1\|_1 \le \tilde{\eta} < 1$ implies that for arbitrary $u \in \mathcal{D}(\Omega)$ and measurable $E \subset \Omega$

$$\begin{aligned}
\int_E Tu(q)\, \mathcal{Q}(q)\, dq \;&\le\; \underbrace{\int_\Omega T_1 u(q)\, \mathcal{Q}(q)\, dq}_{=\|T_1 u\|_1} + \int_E T_2 u(q)\, \mathcal{Q}(q)\, dq \\
&\le\; \tilde{\eta} + \int_E T_2 u(q)\, \mathcal{Q}(q)\, dq. \qquad (50)
\end{aligned}$$

Consequently, direct constrictiveness were proved if there exist some $\delta > 0$ and $K > 0$, such that for every density $u \in \mathcal{D}$

$$\int_E T_2 u(q)\, \mathcal{Q}(q)\, dq \;\le\; K \int_E \mathcal{Q}(q)\, dq, \qquad (51)$$

for all measurable subsets $E \subset \Omega$ with $\int_E \mathcal{Q}(q) dq < \delta$. This leads to the following

Proposition 4.25 *Let $\Omega$ be bounded and assume that the flow is momentum-invertible with uniform bound*[35] *with $\|T_1\|_1 < 1$. Then, the transition operator $T : L^1(\Omega) \to L^1(\Omega)$ is directly constrictive.*

*Proof:* We will show that the assumed boundedness implies condition (51) from above. To this end, we first remember that we may (without loss of generality) adjust the sets $M(q)$ such that $\mathcal{N}_M < \infty$ (cf. Rmk. 4.6). This bound for the number of terms and the bound $|\det Dv_q| < C$ imply $k \in L^\infty(\Omega \times \Omega)$, i.e.,

$$\operatorname*{ess\text{-}sup}_{y\in\Omega} k(q,y) \;<\; C \min_{y\in\Omega} \frac{1}{\mathcal{Q}(y)}\, \mathcal{N}_M \max_{p\in\mathbb{R}^d} \mathcal{P}(p) \;=\; K \;<\; \infty.$$

Moreover, the particular form of $T_2$ yields:

$$\int_E T_2 u(q)\, \mathcal{Q}(q)\, dq \;=\; \int_\Omega \int_E k(q,y)\, \mathcal{Q}(q)\, u(y)\, \mathcal{Q}(y)\, dq\, dy.$$

Since $u \in \mathcal{D}(\Omega)$, we thus have

$$\int_E T_2 u(q)\, \mathcal{Q}(q)\, dq \;\le\; \Big(\operatorname*{ess\text{-}sup}_{q,y\in\Omega} k(q,y)\Big) \int_E \mathcal{Q}(q)\, dq,$$

which implies condition (51) and, therefore, direct constrictiveness of $T$. ∎

[35] Compare Rmk. 4.19.

**Unbounded Position Space** For unbounded $\Omega \subset \mathbb{R}^d$ we not only may have problems in finding a bound for $k$ but in addition need something which prevents the iterates $T^n u$ to be dispersed throughout the entire space. Therefore, we may combine our above strategy with the approach outlined in [66], Chap. 5.7, showing that the existence of a certain type of *Ljapunov function* suffices to make $T$ constrictive. This, in fact, is possible and one can easily show that, e.g., for the one-dimensional harmonic oscillator $H(q,p) = (q^2 + p^2)/2$ with $\Omega = \mathbb{R}$ and $\sin(\tau) \neq 0$ such a Ljapunov function exists. But unfortunately, one again needs the guaranty that $\|T_1\|_1 < 1$ and the construction of such a Ljapunov function for some realistic potential seems to be a significant problem. Since Sec. 4.5 will provide us with a more convenient way, we avoid to go into the details now.

However, as we will see in Sec. 4.6, constrictiveness may be established, independent of any boundedness of $k$ or $\Omega$, for the transition operator $T_\vartheta$ generated by some set of essential variables.

**Contrictiveness and Compactness** The reader might expect that there is a general connection between an appropriate compactness result for $T_2$ and constrictiveness of $T$. Therefore, it should be pointed out that, in fact, constrictiveness of $T$ results from $T_2$ being compact as an operator acting on $L^1(\Omega)$: Since the set $\mathcal{D}(\Omega)$ of all densities is bounded in $L^1(\Omega)$, the operator $T_2$ takes it into a precompact set in $L^1(\Omega)$.[36] Due to Lemma A.11 from Appendix A, this precompactness implies that for all $\epsilon > 0$ there is a $\delta > 0$ such that for all measurable subsets $F \subset \Omega$ with $\int_F \mathcal{Q}(q)dq < \delta$:

$$\int_F T_2 u(q) \mathcal{Q}(q)\, dq \, < \, \epsilon, \qquad \text{for all } u \in \mathcal{D}(\Omega). \tag{52}$$

Now, repeat estimate (50) to get

$$\int\limits_{\Omega\setminus B\cup E} Tu(q)\, \mathcal{Q}(q)\, dq \, \leq \, \|T_1\|_1 + \int_{\Omega\setminus B} T_2 u(q)\, \mathcal{Q}(q)\, dq + \int_E T_2 u(q)\, \mathcal{Q}(q)\, dq,$$

and choose a bounded $B \subset \Omega$ large enough such that $\int_{\Omega\setminus B} \mathcal{Q}(q)dq < \delta$ for the $\delta$ associated with $\epsilon = \|T_1\|_1/3$. Then, (52) directly yields the following

PROPOSITION 4.26 *Let the flow be momentum-invertible, $T_2$ be compact as an operator acting on $L^1(\Omega)$, and $\|T_1\|_1 < 1$. Then, the transition operator $T : L^1(\Omega) \to L^1(\Omega)$ is directly constrictive.*

## 4.4 Asymptotic Stability

As already explained, contrictiveness implies *asymptotic periodicity.* Hence, whenever $T$ is constrictive, we are in the same situation as for Markov chains in finite dimensional state space: in order to get a unique invariant density we need some additional *accessibility condition* which guarantees aperiodicity.

[36] Compare Def. B.36 in Appendix B.

### 4.4.1 Open Set Accessibility

Our accessibility assumption is a rather general *mixing assumption*:

ASSUMPTION 4.27 *The flow has the following mixing property: For every pair of open subsets $B, C \subset \Omega$ there is an $n_0 \in \mathbb{N}$ such that*

$$\int_B T^n \chi_C(q)\, \mathcal{Q}(q)\, dq \;=\; \langle \chi_B, T^n \chi_C \rangle_{\mathcal{Q}} \;>\; 0, \quad \forall n \geq n_0.$$

This kind of mixing should *not* be confused with the notions of "mixing" for the flow map $\Phi^\tau$ or for the Markov operator $T$ (see Appendix A.4). Assumption 4.27 alone does imply neither the one nor the other. But the next lemma will show that, whenever additionally $T$ is constrictive, assumption 4.27 implies that $T$ is mixing (which, for constrictive Markov operators, is equivalent to asymptotic stability, see Thm. A.15).

LEMMA 4.28 *Let the assumption 4.27 be valid and let the transition operator $T : L^1(\Omega) \to L^1(\Omega)$ be constrictive. Then, $T$ has the unique invariant density $\chi_\Omega \in \mathcal{D}(\Omega)$ and is asymptotically stable, that is: For every $u \in \mathcal{D}(\Omega)$ we have*

$$\|T^n u \,-\, \chi_\Omega\|_1 \;\to\; 0, \qquad \textit{for} \quad n \to \infty.$$

*Proof:* $T$ is constrictive and satisfies $T\chi_\Omega = \chi_\Omega$. For such Markov operators, the literature provides strong results concerning the asymptotic behaviour (cf. Appendix A). According to this, $T$ has the spectral decomposition (98) as explained in Thm. A.12. Let the integer $r$, the permutation $\alpha$, and the sets $B_j$ be as in Thm. A.12 and the weighted characteristic functions of Thm. A.12 be denoted by

$$\mathbf{1}_{B_i} \;=\; \left( \int_{B_i} \mathcal{Q}(q)\, dq \right)^{-1} \chi_{B_i}, \quad i = 1, \ldots, r.$$

In a first step we prove that each of these sets $B_j$ contains an open subset. To this end, choose an arbitrary $j \in \{1, \ldots, r\}$ and denote $l = \alpha^{-1}(j)$, such that $T\mathbf{1}_{B_l} = \mathbf{1}_{B_j}$ due to Thm. A.12. Moreover, consider an arbitrary $q \in B_l$. Then, $T\mathbf{1}_{B_l} = \mathbf{1}_{B_j}$ guarantees that

$$y_q(p) \;=\; \pi_1 \Phi^{-\tau}(q, p) \in B_j, \quad \forall p \in \mathbb{R}^d. \tag{53}$$

But since the system is assumed to be momentum invertible, there is a momentum $p_* \in \mathbb{R}^d$ such that $y_q$ is invertible in an open neighborhood $\mathcal{U}(p_*) \subset \mathbb{R}^d$ of $p_*$. Thus, $y_q\mathcal{U}(p_*) \subset \Omega$ is an open neighborhood of $y_q(p_*) \in B_j$, and, because of (53), $y_q\mathcal{U}(p_*)$ is a subset of $B_j$. Consequently, every $B_j$ contains an open subset $\tilde{B}_j \subset B_j$.

For proving asymptotic stability we have to show that $r = 1$. To this end, assume $r > 1$ and choose $j, l \in \{1, \ldots, r\}$, $j \neq l$. Let $\sigma \in \mathbb{N}$ be the period of the permutation $\alpha$. Due to Thm. A.12 we then have for all $n \in \mathbb{N}$:

$$T^{n\sigma}\mathbf{1}_{B_j} \;=\; \mathbf{1}_{B_j} \quad \text{and} \quad B_l \cap B_j = \emptyset,$$

and, therefore, for all $n \in \mathbb{N}$

$$\langle \mathbf{1}_{B_l}, T^{n\sigma}\mathbf{1}_{B_j}\rangle_{\mathcal{Q}} \;=\; \langle \mathbf{1}_{B_l}, \mathbf{1}_{B_j}\rangle_{\mathcal{Q}} \;=\; 0,$$

which is in contradiction to the mixing assumption 4.27, since

$$\langle \mathbf{1}_{B_l}, \mathbf{1}_{B_j}\rangle_{\mathcal{Q}} \;\geq\; \langle \mathbf{1}_{\tilde{B}_l}, \mathbf{1}_{\tilde{B}_j}\rangle_{\mathcal{Q}} \;>\; 0, \qquad \text{because of } \tilde{B}_i \subset B_i \;\forall i.$$

Thus, $r = 1$, and the representation (98) implies for all $u \in L^1(\Omega)$ that

$$\lim_{n\to\infty} T^n u \;=\; \lambda(u)\,\chi_\Omega.$$

If $u \in \mathcal{D}(\Omega)$, then $\lim_{n\to\infty} T^n u \in \mathcal{D}(\Omega)$ and thus $\lambda(u) = 1$. ∎

Remark 4.29 We will see in Sec. 4.5 below, that asymptotic stability can also be proved without the "detour" via constrictiveness.

### 4.4.2 The Dominant Eigenvalue is Simple

From the previous section we know that, under certain assumptions, the transition operator $T : L^1(\Omega) \to L^1(\Omega)$ may be asymptotically stable.

Lemma 4.30 *Assume that the transition operator $T$, considered as a Markov operator acting on $L^1(\Omega)$, is asymptotically stable. Then, for every normalized $u \in L^1(\Omega)$ (that is, $\|u\|_1 = 1$) the condition $Tu = u$ implies that $u = \chi_\Omega$ or $u = -\chi_\Omega$ and there is no normalized $u \in L^1(\Omega)$ such that $Tu = -u$. Moreover, if we consider $T$ as an operator on $L^2_{\mathcal{Q}}(\Omega)$ with $\pm 1 \notin \sigma_{\rm ess}(T)$, then, $\lambda = 1$ is a* simple *eigenvalue of $T$.*[37] *In addition, $-1$ is not an eigenvalue so that $\lambda = 1$ is dominant.*

*Proof:* The weight function $\mathcal{Q}$ in the definition of the spaces $L^p(\Omega)$, $p = 1, 2$, satisfies $\int_\Omega \mathcal{Q}(q)\,dq = 1$. Thus, every element of $L^2(\Omega)$ is contained in $L^1(\Omega)$, i.e., $L^2(\Omega) \subset L^1(\Omega)$ (cf. Prop. A.1). Hence, the assertions for $L^1(\Omega)$ imply the assertions for the eigenvalues in $L^2(\Omega)$.

In $L^1(\Omega)$, we may use a standard argument (cf. [66], proof of Prop. 5.6.1): Consider an arbitrary normalized $u \in L^1(\Omega)$. Then, split $u$ into its positive and negative part, i.e., $u = u_+ - u_-$ with $u_+(q) = \max(0, u(q))$ and $u_- = \max(0, -u(q))$. Next, introduce $v_\pm = u_\pm/\|u_\pm\|_1$ such that $v_\pm \in \mathcal{D}(\Omega)$. Since $T$ is asymptotically stable with $T\chi_\Omega = \chi_\Omega$, we have the convergence:

$$\lim_{n\to\infty} T^n v_\pm \;=\; \chi_\Omega \quad \text{ in } L^1(\Omega).$$

Now, $Tu = u$ implies that in $L^1(\Omega)$ for $n \to \infty$:

$$\begin{aligned} u \;=\; T^n u \;&=\; \|u_+\|_1\, T^n v_+ \;-\; \|u_-\|_1\, T^n v_- \\ &\to\; \|u_+\|_1 \chi_\Omega \;-\; \|u_-\|_1 \chi_\Omega \;=\; (\|u_+\|_1 \;-\; \|u_-\|_1)\, \chi_\Omega. \end{aligned}$$

[37] That is, $\{u \in L^2_{\mathcal{Q}},\, Tu = u\}$ is one-dimensional.

Hence, $Tu = u$ with $\|u\|_1 = 1$ implies $u = \chi_\Omega$ or $u = -\chi_\Omega$. In addition, $Tu = -u$ yields $T^n u = (-1)^n u$, so that the above convergence is only possible if $u = 0$, which obviously contradicts the normalization $\|u\|_1 = 1$. ∎

## 4.5 Asymptotic Stability via Markov Chain Theory

We already know from Sec. 3.6 that the spatial transition operator $T$ induces an associated stochastic dynamical system given by (30) or (33), that is,

$$q_{k+1} = \pi_1 \Phi^\tau(q_k, p_k) = \Psi_{k+1}(q_0, p_0, \dots, p_k),$$

with the momenta $p_j$ being randomly chosen from the canonical momentum density $\mathcal{P}$ on $\mathbb{R}^d$. We are now again interested in the interpretation of this system as a Markov chain $\{X_k\}$. As explained in Sec. 3.6 and Sec. A.1, the transition function of this chain is

$$T(q, A) = T^* \chi_A(q) = T \chi_A(q) = \int_{\mathbb{R}^d} \chi_A(\pi_1 \Phi^{-\tau}(q,p)\, \mathcal{P}(p)\, dp.$$

We also know that this chain has the probability measure $\mu_{\mathcal{Q}}$, given by $\mu_{\mathcal{Q}}(A) = \int_A \mathcal{Q}(q) dq$, as an invariant distribution. The well-developed theory of general state space Markov chains (cf. [83, 79]) yields strong results concerning the convergence of the chain $\{X_k\}$ to its invariant distribution $\mu_{\mathcal{Q}}$. We will now exploit some of these results for our purpose. Therefore, we will make use of the definitions and notations for Markov chains as given in the Appendix (Sec. A.1 and Sec. A.5).

LEMMA 4.31 *Let the flow be momentum-invertible and let the mixing assumption 4.27 be satisfied. Then, the Markov chain $\{X_k\}$ associated with the transition operator $T$ is irreducible, aperiodic, and positive Harris recurrent.*

*Proof:* This lemma is a simple corollary to the results of [79] for the "nonlinear state space model": According to [79] Prop. 7.1.5, momentum-invertibility implies that our Markov chain $\{X_k\}$ is a so-called "T-chain". Due to [79] Thm. 6.0.1, a T-chain is irreducible if $P(\tau_A < \infty | X_0 = q) > 0$ for all open sets $A \subset \Omega$ and all $q \in \Omega$. But momentum-invertibility and mixing assumption, together, in fact imply this last property so that $\{X_k\}$ is irreducible. In the same way, Prop. 7.3.4 and Thm. 7.3.5 of [79] together with our assumptions guarantee that $\{X_k\}$ is aperiodic. Finally, momentum-invertibility and Thm. A.23 of the appendix imply that our chain is positive Harris recurrent.

In order to illustrate the role of momentum-invertibility and mixing assumption in these proofs, we now add a *direct* proof for the *irreducibility* assertion which mimics the strategy of the proofs in [79]. To this end, consider the decomposition $T = T_1 + T_2$ induced by the sets $M(q)$ chosen in accordance with the momentum invertibility of the flow and denote the corresponding accessibility weight of the position $q \in \Omega$ by $\eta(q) = \int_\Omega k(q,y) \mathcal{Q}(y) dy$. Then, we choose some

$q_* \in \Omega$ such that $\eta(q_*) > 0$ and consider the associated probability distribution $\varphi$ defined by $\varphi(A) = T_2\chi_A(q_*)/\eta(q_*)$ for all measurable $A \subset \Omega$. After these preparations, consider another arbitrary position $q \in \Omega$ and a set $A \subset \Omega$ for which $\varphi(A) > 0$. Due to Def. A.21, it is sufficient to show that there is an $n \in \mathbb{N}$ such that $T^{n+2}(q, A) > 0$. We will do this in three steps:

Step 1: We already know that $\delta = \eta(q_*)\varphi(A) > 0$, or equivalently,

$$\delta \;=\; T_2\chi_A(q_*) \;=\; \int_A k(q_*, z)\mathcal{Q}(z)dz \;>\; 0.$$

Since the flow is smooth, we can always choose the sets $M(q)$ and with it the kernel $k$ such that there is an open neighborhood $O \subset \Omega$ such that[38]

$$T(y, A) \;\geq\; T_2\chi_A(y) \;=\; \int_A k(y, z)\mathcal{Q}(z)dz \;>\; \delta/2, \quad \text{for all } y \in O. \tag{54}$$

Step 2: Now, return to the $q \in \Omega$ chosen and consider some $p \in M(q)$. Set $\gamma = k(q, y_q(p))$. We can assume, without loss of generality, that there is an open, bounded neighborhood $m(p) \subset M(q)$ of $p$ such that the smoothness of $y_q$ guarantees that there is an open, bounded neighborhood $U = y_q(m(p)) \subset \Omega$ of $y_q(p)$ such that $k(q, z) > \gamma/2$ for all $z \in U$. As a consequence:

$$\begin{aligned} T^{n+2}(q, A) \;&\geq\; T_2 T^{n+1}\chi_A(q) \;\geq\; \textstyle\int_U T^{n+1}\chi_A(z)\, k(q, z)\, \mathcal{Q}(z)\, dz \\ &\geq\; \tfrac{\gamma}{2} \textstyle\int_U T^{n+1}(z, A)\, \mathcal{Q}(z)\, dz. \end{aligned} \tag{55}$$

Step 3: Finally, introduce the set of all sequences $(p_0, \ldots, p_{n-1})$ of momenta which guide the system from some $z \in U$ into the set $O$, that is, define

$$M_n(z, O) \;=\; \{(p_0, \ldots, p_{n-1}) \in (\mathbb{R}^d)^n : \Psi_n(z, p_0, \ldots, p_{n-1}) \in O\},$$

with the function $\Psi_n$ defined in (32). As a consequence:

$$T^n(z, O) \;=\; T^n\chi_O(z) \;=\; \int\limits_{M_n(z,O)} \prod_{j=0}^{n-1} \mathcal{P}(p_j)\; d^n p,$$

with $d^n p = dp_0 \ldots dp_{n-1}$, so that we gain the estimate

$$\begin{aligned} T^{n+1}(z, A) \;&\geq\; \int\limits_{M_n(z,O)} \underbrace{T\left(\Psi_n(z, p_0, \ldots, p_{n-1}), A\right)}_{>\delta/2,\ \text{because of (54)}} \prod_{j=0}^{n-1} \mathcal{P}(p_j)\; d^n p, \\ &>\; \frac{\delta}{2} T^n(z, O) \;=\; \frac{\delta}{2} T^n\chi_O(z). \end{aligned}$$

Inserted into (55), this yields

$$T^{n+2}(q, A) \;>\; \frac{\gamma\delta}{4}\langle \chi_U, T^n\chi_O\rangle_{\mathcal{Q}}.$$

[38]We just have to make appropriate choices for the sets $M(y)$, $y \in O$, such that $k(\cdot, z)$ is continuous in $O(q_*)$.

Since $U$ and $O$ are open sets, the mixing assumption finally guarantees that there is an $n \in \mathbb{N}$ such that $T^{n+2}(q, A) > 0$. ∎

Remark 4.32 It should be emphasized that the quoted results of [79] allow to gain the same result under weaker conditions. It is, for example, possible to replace the supposed momentum-invertibility by a so-called *rank condition* on the function $\Psi_k$ defining the iterates of the stochastic dynamical system (30):

(RC) For all initial positions $q \in \Omega$, there is an integer $k \in \mathbb{N}$ and a sequence of momenta $v = (p_0, \dots, p_{k-1}) \in (\mathbb{R}^d)^k$ such that the rank of the *generalized controllability matrix* $C^k(q, v)$ (cf. Sec. 7.1.4 in [79]) is maximal.

For $k = 1$, it is $C^1(q_0, p_0) = Dy_q(p_0)$, i.e., condition (RC) is equivalent to the invertibility of $y_q$ in $p_0$. For $k > 1$, we get generalized invertibility conditions for $\Psi_k$ at $(p_0, \dots, p_{k-1})$. The reader might notice, that we could have introduced a similar generalization in the operator approach presented above by considering decompositions of some power $T^n$ of the transition operator; this would just fit perfectly into our approach since, in general, quasi-compactness of $T$ means that there is a decomposition $T^n = K + R$ for some $n \in \mathbb{N}$ with $K$ being compact and $R$ being a strict contraction.

Lemma 4.31 guarantees the convergence in distributions of iterative realizations of our Markov chain (see, e.g., Thm. A.24, Thm. A.26, and Cor. A.27 in Appendix A). For our purpose, the following immediate consequence of Thm. A.26 is of main importance:

Corollary 4.33 *Let the flow be momentum-invertible and let the mixing assumption 4.27 be satisfied. Then, the spatial transition operator $T : L^1(\Omega) \to L^1(\Omega)$ is asymptotically stable.*

*Proof:* See Cor. A.27 in Appendix A. ∎

Hence, asymptotic stability can be proved *without* any boundedness condition on $\Omega$ and *without* any condition which guarantees $\|T_1\|_1 < 1$ like, e.g., the *symmetric* momentum invertibility of the flow. In this sense, the application of Markov chain theory leads to stronger results than those presented in Sec. 4.3 and Sec. 4.4.1 (cf. Rmk. 4.29).

## 4.6 Essential Variables

Let $\vartheta : \Gamma \to \Theta \subset \mathbb{R}^\nu$ denote a set of essential variables, exactly as in Sec. 3.5, where we established the representation of the associated transition operator $T_\vartheta$ as a specific restriction of the spatial transition operator $T$.[39] We now exploit this representation of $T_\vartheta$ to finally transfer the decomposition result for $T$ to all sets of essential degrees of freedom.

[39] Since we restricted our considerations to the canonical density, i.e., to $f_0 = f_{\text{can}}$, the reduced spatial density now is $F(q) = \mathcal{Q}(q)$.

LEMMA 4.34 *Assume that the spatial transition operator $T : L^2(\Omega) \to L^2(\Omega)$ decomposes as $T = T_1 + T_2$ with $T_2$ being a compact operator and $T_1$ a strict contraction satisfying*

$$\|T_1 u\|_2 \le \kappa \|u\|_2, \quad \forall u \in L^2(\Omega),$$

*with a constant $\kappa < 1$. Then, the transition operator $T_\vartheta$ generated by a set of essential coordinates $\vartheta : \Gamma \to \Theta \subset \mathbb{R}^\nu$ may also be decomposed as $T_\vartheta = T_{1,\vartheta} + T_{2,\vartheta}$ with $T_{2,\vartheta}$ being compact and $T_{1,\vartheta}$ a strict contraction satisfying*

$$\|T_{1,\vartheta} u\|_{2,F_\vartheta} \le \kappa \|u\|_{2,F_\vartheta}, \quad \forall u \in L^2_{F_\vartheta}(\Theta),$$

*with the same constant $\kappa < 1$ as for $T_1$. Moreover, if $T_2$ is even self-adjoint, then $T_{2,\vartheta}$ is self-adjoint, too.*
*Last but not least, $T_\vartheta$ is a directly constrictive Markov operator, if $T$ is a directly constrictive Markov operator and the position space $\Omega$ is bounded.*

*Proof:* Let $R_\vartheta$ and $B_\vartheta$ the restriction and prolongation operators from Sec. 3.5. Then we have $T_\vartheta = R_\vartheta T B_\vartheta$. Choose $T_{2,\vartheta} = R_\vartheta T_2 B_\vartheta$. Its compactness follows from the general composition theorem for compact operators (cf. [2], Lemma 8.2 or [56], Thm. 9.5). $T_{2,\vartheta}$ is also self-adjoint if $T_2$ is, since $B_\vartheta$ and $R_\vartheta$ are adjoint due to Prop. 3.14. The contraction property for the remaining part $T_{1,\vartheta} = R_\vartheta T_1 B_\vartheta$ then is a simple consequence of the isometry of $B_\vartheta$ and the contraction property of $R_\vartheta$ stated in Prop. 3.14.

With $T$, also $T_\vartheta$ is a Markov operator because of the isometry of $B_\vartheta$ and the contraction and Markov properties of $R_\vartheta$ stated in Prop. 3.14. When $T$ is assumed to be directly constrictive, there are constants $\delta > 0$ and $0 < \gamma < 1$, such that for all densities $u \in \mathcal{D}(\Omega)$ and every measurable subset $B \subset \Omega$ with $\int_B \mathcal{Q}(q)dq < \delta$, it is

$$\int_B Tu(q)\,\mathcal{Q}(q)\,dq \le \gamma. \tag{56}$$

We have to show the corresponding statement for $T_\vartheta$. To this end, consider an arbitrary $E \subset \Theta$ with $\int_E F_\vartheta(\theta)d\theta < \delta$. For an also arbitrary density $u \in \mathcal{D}_{F_\vartheta}(\Theta)$ we then observe by using the spatial form (29) of the restriction operator $R_\vartheta$:

$$\begin{aligned}
\int_E T_\vartheta u(\theta)\,F_\vartheta(\theta)\,d\theta &= \int_E R_\vartheta T B_\vartheta u(\theta)\,F_\vartheta(\theta)\,d\theta \\
&= \int_E \int_{\Omega(\theta)} (TB_\vartheta u)(q)\,\mathcal{Q}(q)\,d\sigma_\theta(q)\,d\theta \\
&= \int_{\Omega(E)} (TB_\vartheta u)(q)\,\mathcal{Q}(q)\,dq \le \gamma,
\end{aligned}$$

where the last inequality results from (56), since $B_\vartheta u \in \mathcal{D}(\Omega)$ and the choice of $E$ implies $\int_E F_\vartheta(\theta)d\theta = \int_{\Omega(E)} \mathcal{Q}(q)dq < \delta$. ∎

**Compactness and Constrictiveness for Unbounded Position Space** In Sec. 4.2, it has been demonstrated that it is easy to see that the transition operator $T_2$ is compact in $L^2(\Omega)$, more exactly, that $T_2$ is a Hilbert-Schmidt operator, whenever its kernel $k$ is bounded (cf. Prop. 4.17). Due to Sec. 4.3, the same condition suffices to guarantee direct constrictiveness of the full transition operator $T$. If the position space $\Omega$ is unbounded, establishing a bound for $k$ is difficult. In by far the most cases the essential variables will be given by internal *angles* of the molecule (cf. Sec. 2.1.3) so that $\Theta$ is of the form $\Theta = [0, 2\pi]^\nu$. Hence, the essential configuration space $\Theta$ may be bounded, even if $\Omega$ is unbounded. Thus, we may generalize our results by analyzing the kernel of $T_{2,\vartheta} = R_\vartheta T_2 B_\vartheta$. After some simple calculations starting with the definition of the kernel $k = k(q, y)$ of $T_2$, we end up with:

$$T_{2,\vartheta} u(\theta) \;=\; \int_\Theta k_\vartheta(\theta, \varphi)\, u(\varphi)\, F_\vartheta(\varphi)\, d\varphi,$$

with the *averaged kernel*

$$k_\vartheta(\theta, \varphi) \;=\; \frac{1}{F_\vartheta(\theta)} \frac{1}{F_\vartheta(\varphi)} \int_{\Omega(\theta)} \int_{\Omega(\varphi)} k(q, y)\, \mathcal{Q}(q)\, \mathcal{Q}(y)\, d\sigma_\varphi(y)\, d\sigma_\theta(q). \qquad (57)$$

By simply repeating the proofs of Prop. 4.17 and Prop. 4.25, one realizes that the following statement is valid ($\mathcal{N}_M < \infty$):

PROPOSITION 4.35 *Let the essential configuration space $\Theta$ and the kernel $k_\vartheta$ of $T_{2,\vartheta}$ be bounded. Then, $T_{2,\vartheta} : L^2_{F_\vartheta}(\Theta) \to L^2_{F_\vartheta}(\Theta)$ is a Hilbert-Schmidt operator and the the transition operator $T_\vartheta : L^1_{F_\vartheta}(\Theta) \to L^1_{F_\vartheta}(\Theta)$ is directly constrictive.*

REMARK 4.36 Assume that the sets $M(q)$, defining the original kernel $k$, may be chosen such that $k$ is continuous. Then, even if $\Omega$ is unbounded, the averaged kernel $k_\vartheta$ is continuous in $\Theta \times \Theta$, and, thus, bounded if $\Theta$ is bounded. If $k_\vartheta$ were continuous and $\Theta$ bounded, we may infer from Thm. B.39 that $T_{2,\vartheta}$ is also compact in $L^1_{F_\vartheta}(\Theta)$.[40]

**Asymptotic Stability** In order to answer the question concerning the simplicity of the eigenvalue $\lambda = 1$ for $T_\vartheta$ we again have to introduce a mixing assumption:

ASSUMPTION 4.37 *For every pair of open subsets $B, C \subset \Theta$ there is a $n_0 \in \mathbb{N}$ such that*

$$\int_B T^n_\vartheta \chi_C(\theta)\, F_\vartheta(\theta)\, d\theta \;=\; \langle \chi_B, T^n_\vartheta \chi_C \rangle_{F_\vartheta} \;>\; 0, \quad \forall n \geq n_0.$$

REMARK 4.38 As one might expect, this form of the mixing assumption is an implication of the previous form (assumption 4.27) and the momentum invertibility of the flow.

[40]Compare the paragraph "Constrictiveness and Compactness" on page 60.

With assumption 4.37 being valid and Lemma 4.34 in mind, we may prove the following lemma in exactly the same way as we did for Lemma 4.28 above.

LEMMA 4.39 *Assume the assumption 4.37 to be valid and the transition operator* $T_\vartheta : L^1_{F_\vartheta}(\Theta) \to L^1_{F_\vartheta}(\Theta)$ *to be constrictive. Then,* $T_\vartheta$ *has the unique invariant density* $\chi_\Theta \in \mathcal{D}_{F_\vartheta}(\Theta)$ *and is asymptotically stable, that is:*

1. *For every* $u \in \mathcal{D}_{F_\vartheta}(\Theta)$ *we have*
$$\|T_\vartheta^n u \,-\, \chi_\Theta\|_{1,F_\vartheta} \;\to\; 0, \qquad \textit{for } n \to \infty.$$
2. *For every normalized* $u \in L^1_{F_\vartheta}(\Theta)$ *(that is,* $\|u\|_{1,F_\vartheta} = 1$*) the condition* $Tu = u$ *implies that* $u = \chi_\Theta$ *or* $u = -\chi_\Theta$*. Moreover, there is no* $u \in L^1_{F_\vartheta}(\Theta)$ *such that* $T_\vartheta u = -u$*.*

## 4.7 Periodic Boundary Conditions

In this subsection, we will show that for *smooth periodic potentials*:

1. The flow is (symmetrically) momentum invertible with uniform bound. As we have seen, this is sufficient to guarantee that the essential spectrum of the transition operator $T$ is bounded away from $\lambda = 1$, i.e., that the essential spectral radius is strictly smaller than one.
2. The mixing assumption 4.27 is valid for smooth periodic potential. As we have seen, this moreover guarantees that the eigenvalue $\lambda = 1$ is simple and dominant.

In the following, we again use the notation introduced above: For all $q \in \Omega$ we define $y_q : \mathbb{R}^d \to \Omega$ by
$$y_q(p) \;=\; \pi_1\left(\Phi^{-\tau}(q,p)\right).$$
The statements 1 and 2 from above will both be proved by analyzing the functions $y_q = y_q(p)$ asymptotically, i.e., by studying their behavior for $|p| \to \infty$. The idea behind this is that, in a periodic —and, thus, bounded— potential, a particle with high kinetic energy will asymptotically move like a free particle. For a free particle, symmetric momentum invertibility and the mixing assumption are valid. Hence, for proving the statements 1. and 2., we first show that the flow in some periodic potential asymptotically approximates the free flow (Sec. 4.7.1). Then, the mixing assumption and symmetric momentum invertibility will be implications.

Without loss of generality, we may assume that the mass matrix $M$ associated with the system is given by the identity matrix (this requires only a rescaling of the potential).

In order to study the asymptotic behavior rigorously, choose $p_0 \in \{p \in \mathbb{R}^d : |p| = 1\}$ and consider the equations of motion
$$\dot{q}_\epsilon = -p_\epsilon, \quad \dot{p}_\epsilon = DV(q_\epsilon), \quad q_\epsilon(0) = q, \quad p_\epsilon(0) = p_0/\epsilon. \tag{58}$$

That is, we analyze the asymptotic behavior of $y_q$ in $p$ by studying the limit behavior of $q_\epsilon(\tau) = y_q(p_0/\epsilon)$ for $\epsilon \to 0$.

### 4.7.1 Asymptotic Accessibility for Periodic Potentials

In order to prove the desired convergence $|q_\epsilon(t) - (q - p_0 t/\epsilon)| \to 0$ for $\epsilon \to 0$, we will exploit the *averaging principle* of *perturbation theory of integrable Hamiltonian systems* as it can be found, e.g., in the survey [6], Chap. 5. We could also follow the line of argumentation of [12, 13, 93], using appropriate *compactness principles*, or exploit some clever two-time scale ansatz (cf., e.g., [61], Chap. 5). However, the *averaging principle* provides us with a nice tool, fitting perfectly to our problem:

THEOREM 4.40 *([6], Thm. 4 and Thm. 5 in Chap. 5) Assume $f, g : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ to be smooth functions $f = f(I, \varphi, \epsilon)$, $g = g(I, \varphi, \epsilon)$ which, both, are $2\pi$-periodic with respect to the second variable $\varphi$.[41] Let $(I, \varphi)$ be the solution of the following equation of motion:*

$$\begin{aligned} \dot{I} &= \epsilon f(I, \varphi, \epsilon), & I(0) &= I_0, \\ \dot{\varphi} &= \omega + \epsilon g(I, \varphi, \epsilon), & \varphi(0) &= \varphi_0, \end{aligned} \tag{59}$$

*with a vector $\omega \in \mathbb{R}^d$ of constant "frequencies". Moreover, denote by $J$ the solution of the "averaged" equation of motion*

$$\dot{J} = \epsilon F(J), \quad J(0) = I_0, \quad \text{with} \quad F(J) = \frac{1}{(2\pi)^d} \int\limits_{[0,2\pi]^d} f(J, \varphi, 0)\, d\varphi. \tag{60}$$

*Then, the following averaging results hold:*

1. *Let $\omega$ be strongly incommensurable. Then, there is a constant $C > 0$, such that $|I(t) - J(t)| < C\epsilon$ for all $t \in [0, 1/\epsilon]$.*
2. *Let $\omega$ be incommensurable. Then, for every $\eta > 0$ there is an $\epsilon_* > 0$ such that for all $\epsilon < \epsilon_*$, it is $|I(t) - J(t)| < \eta$ for all $t \in [0, 1/\epsilon]$.*

The used notions of incommensurability are the following

DEFINITION 4.41 A vector $\omega \in \mathbb{R}^d$ is called *incommensurable* if $j^T\omega \neq 0$ for all nonzero integer vectors $j \in \mathbb{Z}^d \setminus \{0\}$. It is called *strongly incommensurable* if there are constants $c, b > 0$ such that

$$|j^T\omega| > c^{-1}|j|^{-b}, \quad \text{for all } j \in \mathbb{Z}^d \setminus \{0\}.$$

[41] That is, the functions $f(I, \cdot, \epsilon)$ and $g(I, \cdot, \epsilon)$ are periodic with respect to $[0, 2\pi]^d$ for all $(I, \epsilon)$.

Remark 4.42 According to [80], the set of vectors $\omega \in \mathbb{R}^d$, which are *not* strongly incommensurable, is of measure zero. Thus, the set of strongly incommensurable vectors is everywhere dense in $\mathbb{R}^d$.

In order to apply the averaging result to our situation, we, first, have to introduce periodicity assumptions for the potential $V$, and, second, have to transform our equation of motion (58) into an appropriate form.

We summarize our *periodicity assumptions* in the following

Assumption 4.43 *Let the potential $V$ be smooth on the entire space $\mathbb{R}^d$, and assume it to be* periodic *with periodicity domain $\Omega = \prod_{j=1}^{d}[0, l_k]$ with periods $l_k > 0$, i.e., for all $q \in \Omega$ and every $j \in \{1, \dots, d\}$*

$$V(q + m\, l_j\, e_j) = V(q), \quad \forall m \in \mathbb{Z},$$

*where $e_j \in \mathbb{R}^d$ is the $j$th unit vector. Without loss of generality we may assume that $l_k = 2\pi$ for all $k = 1, \dots, d$, so that $V$ is periodic with respect to $[0, 2\pi]^d$.*

To transform our equation of motion (58) into an appropriate form, we introduce

$$z_\epsilon(t) = q_\epsilon(\epsilon\tau t) \quad \text{and} \quad v_\epsilon(t) = p_\epsilon(\epsilon\tau t) - p_0/\epsilon, \tag{61}$$

which, inserted into (58), yields new equations of motion

$$\begin{aligned} \dot{z}_\epsilon &= -\tau p_0 + \epsilon\,\tau\, v_\epsilon, & z_\epsilon(0) = q, \\ \dot{v}_\epsilon &= -\epsilon\tau\, DV(z_\epsilon), & v_\epsilon(0) = 0, \end{aligned} \tag{62}$$

We observe, that, via the identifications $z_\epsilon \to \varphi$ and $v_\epsilon \to I$, (61) gets the form of (59) with $\omega = \tau p_0$ and $f = -\tau DV$. Consequently, whenever $p_0$ is (strongly) incommensurable, the conditions of Thm. 4.40 are satisfied. The periodicity of $V$ yields $F = 0$, so that the theorem states that $v_\epsilon$ remains small everywhere in $[0, 1/\epsilon]$. Retransformation due to (61) gives

$$\begin{aligned} q_\epsilon(t) &= q - p_0 t/\epsilon + \epsilon\tau \int_0^{t/\epsilon\tau} v_\epsilon(s)\, ds \\ p_\epsilon(t) &= p_0/\epsilon - v_\epsilon(t/\epsilon\tau), \end{aligned} \tag{63}$$

which results in

Corollary 4.44 *Let the periodicity assumption 4.43 be valid. Then, the averaging theorem 4.40 states the desired convergence $|q_\epsilon(t) - (q - p_0 t/\epsilon)| \to 0$ for $\epsilon \to 0$ if the initial momentum $p_0$ satisfies certain incommensurability conditions:*

1. *If $p_0$ is strongly incommensurable, then there is a constant $C > 0$ such that for all $t \in [0, \tau]$:*

$$|p_\epsilon(t) - p_0/\epsilon| < C\epsilon \quad \textit{and} \quad |q_\epsilon(t) - (q - p_0 t/\epsilon)| < Ct\epsilon. \tag{64}$$

2. *If $p_0$ is only incommensurable, then, for every constant $\eta > 0$ there is a $\epsilon_* > 0$ such that for all $\epsilon < \epsilon_*$ and $t \in [0, \tau]$:*

$$|p_\epsilon(t) - p_0/\epsilon| < \eta \quad \text{and} \quad |q_\epsilon(t) - (q - p_0 t/\epsilon)| < \eta t.$$

This insight finally allows us to show that the mixing assumption 4.27 is valid for periodic potentials, see Cor. 4.46 below. This corollary will be a simple consequence of the following well-known theorem stating that harmonic oscillations with incommensurable (=nonresonant) frequencies fill out the corresponding torus densely:

THEOREM 4.45 *([6], Thm. 4 in Chap. 4.1) Let $C \subset \Omega$ be an arbitrary open subset in $\Omega$ and assume that the initial momentum $p_0$ is incommensurable. Then, for every $q \in \Omega$:*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \chi_C(\xi(q - p_0 s))\, ds > 0,$$

*where $\xi : \mathbb{R}^d \to \Omega$ is the periodicity map defined in Sec. 3.1.*

Together with Cor. 4.44 this implies the mixing assumption for the transition operator $T$ for periodic potentials (cf. Def. (23)):

COROLLARY 4.46 *Let the above periodicity assumptions 4.43 be valid. Then, for every pair of open subsets $B, C \subset \Omega$ there is a $n_0 \in \mathbb{N}$ such that*

$$\int_B T^n \chi_C(q)\, \mathcal{Q}(q)\, dq = \langle \chi_B, T^n \chi_C \rangle_{\mathcal{Q}} > 0, \quad \forall n \geq n_0.$$

*Proof:* Consider arbitrary open sets $B, C \subset \Omega$. We have to show that

$$\int_B T\chi_C(q)\, \mathcal{Q}(q)\, dq = \langle \chi_B, T\chi_C \rangle_{\mathcal{Q}} > 0. \tag{65}$$

Instead of showing (65) directly, we will prove that for every $q \in B$ and strongly incommensurable $p_0$ there is some $\epsilon > 0$ such that $\xi(y_q(p_0/\epsilon)) \in C$. This is sufficient, since the maps $y_q$ and $q \mapsto \pi_1 \Phi^{-\tau}(q, p_0/\epsilon)$ are smooth so that there must be open neighborhoods $O(p_0/\epsilon) \subset \mathbb{R}^d$ and $U(q) \subset B$ such that $\xi(y_{\tilde{q}}(p)) \in C$ for all $p \in O(p_0/\epsilon)$ and $\tilde{q} \in U(q)$, which implies (65).

In order to demonstrate that there is some $\epsilon > 0$ such that $\xi(y_q(p_0/\epsilon)) \in C$, consider an arbitrary $q \in B$. Since $p_0$ is strongly incommensurable, Cor. 4.44 holds for the solution $q_\epsilon$ of (58). Consequently, since $C$ is open, we find another open set $E \subset C$ together with an $\epsilon_* > 0$ such that for all $\epsilon < \epsilon_*$:

$$\xi(q - p_0\tau/\epsilon) \in E \quad \Rightarrow \quad \xi(q_\epsilon(\tau)) = \xi(y_q(p_0/\epsilon)) \in C.$$

With the replacement $\alpha = 1/\epsilon$ this implies

$$\chi_E\left(\xi(q - \alpha p_0 \tau)\right) \leq \chi_C\left(\xi(y_q(\alpha p_0))\right), \quad \text{for all } \alpha > \alpha_* = 1/\epsilon_*.$$

Hence, a simple transformation ($s = \alpha\tau$) yields:

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{t} \int_{\alpha_*}^{t} \chi_C(\xi(y_q(\alpha p_0)))\, d\alpha \;\; &\geq \;\; \lim_{t\to\infty} \frac{1}{t} \int_{\alpha_*}^{t} \chi_E\left(\xi(q - \alpha p_0 \tau)\right)\, d\alpha \\
&= \;\; \lim_{t\to\infty} \frac{1}{\tau t} \int_{\tau\alpha_*}^{\tau t} \chi_E\left(\xi(q - p_0 s)\right)\, ds \; > \; 0,
\end{aligned}$$

where the last strict inequality results from Thm. 4.45. Thus, there must be momenta $p = \alpha p_0$ such that $\xi(y_q(p)) \in C$. ∎

### 4.7.2 Asymptotic Invertibility for Periodic Potentials

LEMMA 4.47 *Let the above periodicity assumptions 4.43 be valid. Then, the associated flow is momentum invertible with uniform bound.*[42]

REMARK 4.48 We again want to consider the periodic flow $\Phi_\xi^\tau$, introduced via the periodicity map $\xi$ due to (22). The replacement $\Phi^\tau \to \Phi_\xi^\tau$ does not touch the validity of the statement of Lemma 4.47. That is, the flow $\Phi_\xi^\tau$ also is momentum invertible with uniform bound with the same minimal accessibility. The only difference is that, for $\Phi_\xi^\tau$, the sets $M(q)$ contain infinitely many subsets (from which we perhaps have to choose countably many) each inducing a different branch of the functions $y_q$ and $v_q$. This results from the fact that in the process $\epsilon \to 0$ the curve $y_q(p_0/\epsilon)$ with incommensurable "direction" $p_0$ crosses the periodicity cell $\Omega$ again and again (infinitely many times).

*Proof of Lemma 4.47:* The strategy of the proof is as follows: We will use the asymptotic properties of $q_\epsilon(\tau) = y_q(p_0/\epsilon)$ to show that there are constants $0 < c < \tau$, $C > \tau$ such that for all $q \in \Omega$ and all strongly incommensurable momenta $p_0$ the following holds: There is an $\epsilon_* > 0$ such that

$$c \;\leq\; |\det Dy_q(p_0/\epsilon)| \;\leq\; C, \qquad \text{for all } |\epsilon| < \epsilon_*. \tag{66}$$

If we suppose that (66) holds, we have that $y_q$ is invertible for the momenta $p_0/\epsilon$ with $\epsilon < \epsilon_*$. Thus, its inverse $v_q$ satisfies the estimate $|\det Dv_q| < 1/c$, since $\det Dy_q \cdot \det Dv_q = 1$. In general, the values $\epsilon_*$ in (66) depend on the specific momentum $p_0$ and the initial position $q$ chosen, a fact which may be reflected by writing $\epsilon = \epsilon(q, p_0)$. But the smoothness of $y_q$ guarantees that we find some open neighborhood

$$M(q, p_0) \;\supset\; \{p_0/\epsilon, \quad |\epsilon| < \epsilon_*(q, p_0)\}$$

for each strongly incommensurable $p_0$, in which $y_q$ is invertible with an uniform bound for the Jacobian of the inverse. Then, the set $M(q)$ may be defined as the union of these sets $M(q, p_0)$. Since $\Omega$ is compact, the system is momentum invertible with uniform bound (see Lemma 4.3).

[42] Compare Rmk. 4.19.

Hence, we have to prove (66) for arbitrary $q \in \Omega$ and strongly incommensurable $p_0$. Therefore, we consider in the dependence of $q_\epsilon(\tau) = y_q(p_0/\epsilon)$ on its initial momentum $p_0/\epsilon$. Consequently, we are interested in the Jacobian matrix $D_2 q_\epsilon(\tau) = Dy_q(p_0/\epsilon)$, where $D_2$ denotes the derivative of $q_\epsilon = q_\epsilon(t; p_0/\epsilon)$ with respect to the second variable (the initial momentum). Instead of $D_2 q_\epsilon$, we analyze the extended Jacobian

$$J_\epsilon(t) \;=\; D_2 q_\epsilon(t) \;+\; t\,\mathrm{Id}.$$

Differentiation of the equation of motion (58) governing $q_\epsilon$ then results in:

$$\ddot{J}_\epsilon \;=\; D^2V(q_\epsilon)\,\cdot\,J_\epsilon, \quad J_\epsilon(0) = 0, \quad \dot{J}_\epsilon(0) \;=\; 0,$$

which after two-fold integration in time yields

$$J_\epsilon(t) \;=\; \int_0^t \int_0^s D^2V(q_\epsilon(\sigma))\;(J_\epsilon(\sigma) \;-\; \sigma\,\mathrm{Id})\;d\sigma\,ds.$$

Using the matrix norm $\|\cdot\|$ induced by the norm $|\cdot|$ used on $\mathbb{R}^d$, the regularity of $V$ gives us the bound $\|D^2V\| \le C$ which then implies

$$\begin{aligned} \|J_\epsilon(t)\| \quad &\le \quad \| \textstyle\int_0^t \int_0^s \sigma\; D^2V(q_\epsilon(\sigma))\,d\sigma\,ds\| \\ &\qquad + C\; \textstyle\int_0^t \int_0^s \|J_\epsilon(\sigma)\|\,d\sigma\,ds. \end{aligned} \tag{67}$$

Clearly, there are numbers $\rho_\epsilon > 0$ such that

$$\| \int_0^t \int_0^s \sigma\; D^2V(q_\epsilon(\sigma))\,d\sigma\,ds\| \;\le\; \rho_\epsilon, \qquad \text{for all } t \in [0,\tau]. \tag{68}$$

In fact, the bound $\|D^2V\| \le C$ ensures that (68) is satisfied with $\rho_\epsilon = C\tau^3/3$. But we can find significantly sharper estimates when using the asymptotic properties of $q_\epsilon$: We will see below that —if the initial momentum $p_0$ is strongly incommensurable— these asymptotic properties and the periodicity of $V$ indeed imply $\rho_\epsilon = \mathcal{O}(\epsilon)$. However, estimates (67) and (68) together with a generalization of Gronwall's lemma (see Lemma 4.49 below) imply that

$$\|J_\epsilon(\tau)\| \;\le\; \rho_\epsilon\;\exp\left(\frac{C}{2}\tau^2\right). \tag{69}$$

If we suppose $\rho_\epsilon = \mathcal{O}(\epsilon)$ for a moment, (69) shows $J_\epsilon(\tau) = \mathcal{O}(\epsilon)$, which means that for $\epsilon \to 0$:

$$Dy_q(p_0/\epsilon) \;=\; J_\epsilon(\tau) - \tau\,\mathrm{Id} \;\to\; -\tau\,\mathrm{Id} \quad \text{implying} \quad \det Dy_q(p_0/\epsilon) \;\to\; -\tau,$$

which obviously implies the assertion (66).

Hence, it only remains to show that we were allowed to assume $\rho_\epsilon = \mathcal{O}(\epsilon)$ for some fixed strongly incommensurable initial momentum $p_0$. To this end, we first introduce

$$I_\epsilon(t) \;=\; \int_0^t \int_0^s \sigma\; D^2V(q \;-\; \frac{p_0}{\epsilon}\sigma)\,d\sigma\,ds.$$

Cor. 4.44 and the Lipschitz continuity of $D^2V$ then yield that there is some $C > 0$ such that for sufficiently small $\epsilon$:

$$\| \int_0^t \int_0^s \sigma \; D^2V(q_\epsilon(\sigma))\, d\sigma\, ds \;-\; I_\epsilon(t)\| \;\leq\; C\,\epsilon, \qquad \text{for all } t \in [0,\tau],$$

so that it suffices to show that $\max_{[0,\tau]} \|I_\epsilon(t)\| = \mathcal{O}(\epsilon)$. To this end, we use the Fourier series expansion of $V$:

$$V(q) \;=\; \sum_{j\in\mathbb{Z}^d} \alpha_j \, \exp\left(-i\, j^T q\right),$$

with coefficients $\alpha_j \in \mathbb{C}$. Thus, by differentiation

$$D^2V(q) \;=\; -\sum_{j\in\mathbb{Z}^d, j\neq 0} \alpha_j j\, j^T \, \exp\left(-i\, j^T q\right).$$

The smoothness of $V$ guarantees that the expansion coefficients $\alpha_j$ decay exponentially with $|j|$. If we insert the Fourier expansion into the definition of $I_\epsilon$, the single terms behave like:

$$\left| \int_0^t \int_0^s \sigma \, \exp\left(-i\, j^T \left(q \,-\, \frac{p_0}{\epsilon}\sigma\right)\right) \, d\sigma\, ds \right| \;\leq\; tC \left(\frac{\epsilon}{j^T p_0}\right)^2$$

for every $j \in \mathbb{Z}^d \backslash \{0\}$ with a constant $C$ independent of $j$. The strong incommensurability of $p_0$ guarantees $1/(j^T p_0) < c|j|^b$ and, together with the exponential decay of the Fourier coefficients, the uniform convergence of the series. This implies $I_\epsilon = \mathcal{O}(\epsilon^2)$, which completes the proof. ∎

In order to fill the last gap in the proof of Lemma 4.47 we now prove the following generalization of Gronwall's Lemma for "second order" differential inequalities:

Lemma 4.49 *Assume that the functions $\varphi, \psi : [0,\tau] \to \mathbb{R}$ are nonnegative and satisfy the differential inequality*

$$\psi(t) \;\leq\; \rho \,+\, \int_0^t \int_0^s \varphi(\sigma)\psi(\sigma)\, d\sigma\, ds,$$

*for all $t \in [0,\tau]$ with $\rho > 0$. Then, the following estimate holds*

$$\psi(t) \;\leq\; \rho\, \exp\left(\int_0^t \int_0^s \varphi(\sigma)\, d\sigma\, ds\right), \qquad \textit{for all } t \in [0,\tau].$$

*Proof along the lines of the proof of Lemma 3.9 in [20]:* Consider

$$\Psi(t) \;=\; \rho \,+\, \int_0^t \int_0^s \varphi(\sigma)\psi(\sigma)\, d\sigma\, ds,$$

such that $\Psi \geq \rho > 0$ and $\Psi \geq \psi$ due to the assumptions. Differentiation of $\Psi$ with respect to $t$ yields $\Psi'' = \varphi\psi \leq \varphi\Psi$. This implies the estimate

$$(\log \Psi)'' \;=\; \Psi''/\Psi \;-\; (\Psi'/\Psi)^2 \;\leq\; \Psi''/\Psi \;\leq\; \varphi,$$

which yields by integration that

$$\log \Psi(t) \;\leq\; \log \Psi(0) \;+ t\frac{\Psi'(0)}{\Psi(0)} \;+\; \int_0^t \int_0^s \varphi(\sigma)\, d\sigma\, ds.$$

Since $\Psi(0) = \rho$ and $\Psi'(0) = 0$, this implies the assertion. ∎

REMARK 4.50 The entire asymptotic construction can be generalized in the following way: We can allow for initial momenta $p_\epsilon(0) = p_\epsilon^0 = p_0/\epsilon + p_1$ (with $p_1$ being $\epsilon$-independent), without changing the asymptotic results. That is, if only $p_0$ is strongly incommensurable, we have $|q_\epsilon(t) - (q - p_0 t/\epsilon - p_1 t)| < C_1 t\epsilon$, $|p_\epsilon - (p_0/\epsilon + p_1)| < C_2\epsilon$, and

$$c \;<\; |\det D_2 q_\epsilon(t)| \;<\; C,$$

for all $\epsilon$ being small enough. Consequently, the sets $M(q)$, with respect to which the system is momentum-invertible with uniform bound, can be chosen such that

$$\bigcup_{|\epsilon|<\epsilon_*(q,p_0)} B_1\left(p_0/\epsilon\right) \;=\; \{p_0/\epsilon + p_1 \,:\, |\epsilon| < \epsilon_*(q,p_0),\, |p_1| \leq 1\} \subset M(q), \tag{70}$$

for all strongly incommensurable $p_0$ and $q \in \Omega$, where $\epsilon_*(q, p_0)$ is used in the same sense as in the proof of Lemma 4.47. In general, $\epsilon_*(q,p_0)$ will depend on $q$. However, the smoothness of the map $q \mapsto \Phi^{-\tau}(q,p)$ guarantees that, for every $q \in \Omega$ there is an open neighborhood $O(q) \subset \Omega$ such that (70) is valid for all $\tilde{q} \in O(q)$ with the same $\epsilon_*(q,p_0)$. Since $\Omega$ is compact this finally implies,[43] that there even is some $\epsilon_* = \epsilon_*(p_0)$ such that for all $q \in \Omega$:

$$\bigcup_{|\epsilon|<\epsilon_*(p_0)} B_1\left(p_0/\epsilon\right) \;=\; \{p_0/\epsilon + p_1 \,:\, |\epsilon| < \epsilon_*(p_0),\, |p_1| \leq 1\} \subset M(q). \tag{71}$$

This enables us to show the following extension of Lemma 4.47:

LEMMA 4.51 *Let the above periodicity assumptions 4.43 be valid. Then, the sets $M(q)$ can be chosen such that the flow is momentum invertible with uniform bound and, simultaneously, $\|T_1\|_1 < 1$. Thus, due to Prop. 4.25, the spatial transition operator $T$ is directly constrictive.*

[43] Compare the proof of Lemma 4.3.

*Proof:* Due to Prop. 4.11, we have to show that the sets $M(q)$ can be chosen such that every $y \in \Omega$ is accessible, that is, for every $y \in \Omega$ there are $q \in \Omega$ and $p \in M(q)$ such that $y = y_q(p)$. Therefore, consider an arbitrary $y \in \Omega$, a strongly incommensurable momentum $p_0$, and define

$$v(\epsilon) = \pi_2 \Phi^{\tau}(y, p_0/\epsilon) = -\pi_2 \Phi^{-\tau}(y, -p_0/\epsilon).$$

Cor. 4.44 guarantees that there are some constants $C > 0$ and $\epsilon_0 > 0$ such that $|v(\epsilon) - p_0/\epsilon| < C\epsilon$ for all $\epsilon < \epsilon_0$. Suppose that the sets $M(q)$ are chosen according to the construction of Rmk. 4.50 and let us use the notation $\epsilon_*(p_0)$ introduced therein. Now, choose some

$$\epsilon < \frac{1}{2} \min\{\epsilon_*(p_0), 1/C, \epsilon_0\},$$

and realize that then there is some $q \in \Omega$ such that

$$\Phi^{\tau}(y, p_0/\epsilon) = (q, v(\epsilon)), \quad \text{that is,} \quad (y, p_0/\epsilon) = \Phi^{-\tau}(q, v(\epsilon)),$$

with $v(\epsilon) \in B_1(p_0/\epsilon) \subset M(q)$. Thus, $y$ is accessible and the assertion is proved. $\square$

As a final remark, it should be stated that, along a similar line of argument, we can also construct the sets $M(q)$ such that the flow is symmetrically momentum invertible with uniform bound with respect to these $M(q)$.

## 4.8 Commentary

The results presented so far are connected to different branches of the literature. Some have already been indicated; some further comments should be added: We proved quasi-compactness of the spatial transition operator $T$ in $L^2(\Omega)$ in order to show that the essential spectrum of $T$ is strictly bounded away from $\lambda = 1$. Then, we exploited Markov operator theory in $L^1$ to show that $\lambda = 1$ is a simple and dominant eigenvalue of $T$. So far, our considerations have been based on *operator theory/functional analysis*. But we also observed, that, via the stochastic dynamical system associated with $T$, *Markov chain theory* permitted us to get similar and even more general results. As far as the author can see, these are the two main categories for approaches in this field: via operator theory and via Markov chain theory, with the latter mainly dealing with the asymptotic behavior of the chain (convergence of distribution, rate of convergence), while the first is also interested in non-asymptotic properties of the operator.

In the literature on operator theory, quasi-compactness[44] of an operator has extensively been discussed in its connection to *uniform ergodicity* of the operator.[45] Following an early contribution of Kyrlov and Bogoliubov [63], it was shown that any quasi-compact Markov operator on a Banach space is

[44] In general, an operator $T$ is called quasi-compact, if $\|T^n - K\| < 1$ for some $n \in \mathbb{N}$ with $K$ being compact.

[45] A positive operator $T$ on a Banach lattice is uniform ergodic iff the sequence of averages $T_n = \sum_{k=1}^{n} T^k/n$ converges in the uniform operator topology, cf. [72].

uniformly ergodic and that, reversely, an operator is quasi-compact, if it is uniformly ergodic and the eigenspace $E = \{u : Tu = u\}$ is finite dimensional (for a survey see [72]). The connection to Markov chain theory was initialized by Yosida and Kakutani [114], who proved that Markov chains satisfying Doeblin's condition [25] are associated with quasi-compact operators, for example, in $L^\infty$.

Hence, Doeblin's condition —abbreviated (D) in the following— can be used as a link between quasi-compact operators and special Markov chains. It is, again, the book of Meyn and Tweedie [79] which provides us with criteria under which Markov chains satisfy (D): For irreducible and aperiodic Markov chains, (D) is equivalent to the chain being *uniformly ergodic*[46] (cf. Thm. 16.2.3 in [79]). This result is important for the context considered herein, because any irreducible and aperiodic T-chain —like the stochastic dynamical system associated with the spatial transition operator (cf. Sec. 4.5)— is known to be uniformly ergodic, if the state space is compact (Thm. 16.2.5 in [79]) or if the chain satisfies certain stability conditions (Thm. 16.2.6 in [79]). Hence, this line of argumentation may serve to establish more general results concerning quasi-compactness of transition operators, at least for such operators which are directly associated with some Markov chain like our spatial transition operator.

In addition, uniform and "geometric" ergodicity of Markov chains imply *central limit theorems* (see Thm. A.30), describing the rate of convergence to the invariant distribution of the chain. The reader will find some connections between central limit theorems and compactness of the associated transition operator in the literature on Metropolis-Hastings and Gibbs Sampler Markov chains, e.g., in [92, 70] and in the appendix of [100]. However, in these articles compactness of the transition operator is used as an assumption implying uniform or geometric ergodicity and different central limit theorems; compare the contribution of Kung Sik Chang and Geyer to the discussion in [106]. Some aspects of this discussion will reappear in the construction process of some appropriate discretization for the spatial transition operator (see Sec. 5.4).

[46] The definition of "uniform ergodicity" for a Markov chain can be found in Appendix A (see Def. A.28); as the above results indicate, this definition is much stronger than that used in operator theory.

# 5 Discretization

For simplicity of notation, from now on, the position space or, respectively, essential configuration space under consideration is denoted by $\Omega$, which always is assumed to be a Lipschitz-bounded domain. Moreover, we write $L^2_F$ for the associated weighted Hilbert spaces $L^2_F(\Omega)$ or $L^2_{F_\vartheta}(\Theta)$, respectively, and denote the corresponding scalar products and norms by $\langle \cdot, \cdot \rangle_F$ and $\| \cdot \|_F$. In accordance to this, the weight function $F = F(q)$ or $F = F(\theta)$, respectively, are simply denoted by $F = F(q)$.

In the following, we follow the line of arguments of SCHÜTTE et al. [94] (in Sec. 5.1 and Sec. 5.4) and DEUFLHARD et al. [24] (in Sec. 5.3). However, a significant part the material is new (in particular the approximation results in Sec. 5.4).

## 5.1 Spatial Discretization

In the weighted Hilbert space $L^2_F$, we (as in [19, 21]) derive a *Galerkin procedure* to discretize the eigenvalue problem $Tu = \lambda u$. Let $B_1, \dots, B_n \subset \Omega$ be a covering of $\Omega$ so that $B_k \cap B_l$ is of (Lebesgue) measure zero for $k \neq l$ and $\cup_{k=1}^n B_k = \Omega$. Then, the sets $\Gamma(B_k)$, $k = 1, \dots, n$, are a covering of $\Gamma$. Our finite dimensional ansatz space $\mathcal{V}_n = \operatorname{span}\{\chi_1, \dots, \chi_n\}$ is spanned by the associated characteristic functions $\chi_k = \chi_{B_k}$. The Galerkin projection $\Pi_n : L^2_F \to \mathcal{V}_n$ of $u \in L^2_F$ is defined by

$$\Pi_n u = \sum_{k=1}^n \frac{1}{\rho(B_k)} \langle \chi_k, u \rangle_F \, \chi_k, \qquad \text{with} \qquad \rho(B_k) \;=\; \langle \chi_k, \chi_k \rangle_F \;=\; \int_{B_k} F(q)\, dq.$$

The resulting discretized transition operator $\Pi_n T \Pi_n$ induces the approximate eigenvalue problem $\Pi_n T \Pi_n u = \lambda u$ in $\mathcal{V}_n$. Let $\lambda$ be one of the corresponding eigenvalues and let the related eigenvector be $u = \sum_{k=1}^n \alpha_k \chi_k$. Then, the discretized eigenvalue problem has the form

$$\sum_{l=1}^n \langle T\chi_k, \chi_l \rangle_F \; \alpha_l \;=\; \lambda\, \rho(B_k)\, \alpha_k, \qquad \forall k = 1, \dots, n.$$

After division by $\rho(B_k)$ (known to be positive), we end up with the convenient form

$$T_n \alpha \;=\; \lambda \alpha \quad \text{with} \quad \alpha = (\alpha_1, \dots, \alpha_n)^T,$$

where in fact the entries of the $n \times n$ matrix $T_n$ are given by the transition probabilities from $B_k$ to $B_l$:

$$\boxed{T_{n,kl} \;=\; \tfrac{\langle T\chi_k, \chi_l \rangle_F}{\rho(B_k)} \;=\; w(B_k, B_l, \tau).} \tag{72}$$

Hence, $T_n$ is the matrix representation of $T$ in $\mathcal{V}_n$ with respect to the orthonormal basis $\{\phi_j = \chi_j / \sqrt{\rho(B_j)}\}$. For simplicity, we often identify $T_n$ with

$\Pi_n T \Pi_n$ (and $\mathbb{R}^n$ with $\mathcal{V}_n$) in the following and call $T_n$ as well as $\Pi_n T \Pi_n$ the Galerkin discretization of $T$ or, alternatively, the *transition matrix*.

REMARK 5.1 The result (72) is the main reason for the choice of the simple box function ansatz spaces $\mathcal{V}_n$ suggested herein. In fact, one could also apply finite element ansatz spaces, for example, which would result in better approximation properties of the eigenelements (cf. Rmk. 5.5 in Sec. 5.2). But only our box function ansatz spaces allow for a direct interpretation of the entries of the discretization matrix as transition probabilities — a property, which will be important for the identification algorithm to be presented in Sec. 5.3 and for the efficient evaluation of the entries of $T_n$ via Hybrid Monte Carlo (cf. Sec. 5.4).[47]

**Properties of the Transition Matrix** Since $T$ is a Markov operator, its Galerkin discretization $T_n$ is a (row) stochastic matrix, i.e., $T_{n,kl} \geq 0$ and $\sum_{l=1}^{n} T_{n,kl} = 1$ for all $k = 1, \ldots, n$ (for details about stochastic matrices see [8]). Hence, all its eigenvalues $\lambda$ satisfy $|\lambda| \leq 1$.

Indeed, for arbitrary coverings $B_1, \ldots, B_n \subset \Omega$, the discretization matrices $T_n$ are also inheriting other important properties of the operator $T$:

PROPOSITION 5.2 *The Galerkin discretization $T_n$ of the transition operator $T$ has the following properties:*

1. *The discretized invariant density, i.e., the row vector $\rho = (\rho_1, \ldots, \rho_n)$, $\rho_k = \rho(B_k)$, is a left eigenvector to the eigenvalue $\lambda = 1$ of $T_n$, i.e., $\rho T_n = \rho$.*
2. *For every integer $m$, $T_n^m$ is the Galerkin discretization of the $m$th power $T^m$ of the transition operator $T$ — as usual.*
3. *If $T$ is asymptotically stable, then $T_n$ is* primitive,[48] *i.e., the eigenvalue $\lambda = 1$ is simple and dominant. Hence, the discretized invariant density $\rho$ is the* unique *stationary distribution of $T_n$.*
4. *Since the discretized invariant density $\rho$ has positive entries, it induces a weighted scalar product $\langle \cdot, \cdot \rangle_\rho$ on $\mathcal{V}_n$.*[49] *$T_n$ is symmetric with respect to $\langle \cdot, \cdot \rangle_\rho$. As a consequence, $T_n$ is* reversible, *i.e., $T_n$ satisfies the condition of* detailed balance:[50]

$$\rho_k T_{n,kl} = \rho_l T_{n,lk}, \qquad \forall k, l \in \{1, \ldots, n\}. \tag{73}$$

[47] In fact, if finite element ansatzes are used, the entries of the transition matrix would only be the *coefficients* of the polynomial approximation on the discretization cells. Thus, the reconstruction of the transition probabilities would always require some addition computations.

[48] A nonnegative square matrix $P$ is called primitive, if there is a positive integer $m$ such that $P^m > 0$ (entrywise). Any primitive matrix is *irreducible* and *aperiodic* and conversely (cf. [96], Thm. 1.4.).

[49] That is, for column vectors $y, z \in \mathcal{V}_n$, we define $\langle y, z \rangle_\rho = y^T \mathrm{diag}(\rho_j)\, z$.

[50] Stochastic matrices satisfying a detailed balance condition have been extensively analyzed in the context of Monte-Carlo techniques. For details and implications like such in the proof below, compare, e.g., [38].

*Moreover, obviously, all eigenvalues of $T_n$ are real-valued, $\sigma(T_n) \subset [-1,1]$.*

*Proof:* Let us check the assertions as they are numbered:

Ad 1) Exploiting $T\chi_\Omega = \chi_\Omega$, simple calculus reveals that

$$\begin{aligned}(\rho T_n)_k &= \sum_{j=1}^{n} T_{n,jk}\ \rho_j = \sum_{j=1}^{n} \frac{1}{\rho_j} \langle T\chi_j, \chi_k\rangle_F\ \rho_j \\ &= \langle T\chi_\Omega, \chi_k\rangle_F = \rho_k.\end{aligned}$$

Ad 2) Using the orthonormal basis $\{\phi_j = \chi_j/\sqrt{\rho_j}\}$ of $\mathcal{V}_n$, (72) and the self-adjointness of $T$ yield

$$\left(T_n^2\right)_{kl} = \sum_{j=1}^{n} \frac{1}{\rho_k} \langle T\chi_k, \phi_j\rangle_F \cdot \langle \phi_j, T\chi_l\rangle_F = \frac{1}{\rho_k} \langle T^2\chi_k, \chi_l\rangle_F.$$

A simple induction argument then proves the assertion for all $m \in \mathbb{N}$.

Ad 3) Every row vector $y = (y_1, \dots, y_n)$ with nonnegative entries $y_j \geq 0$, satisfying $\sum_j y_j = 1$, is called a ***probability vector***. Since $\rho$ is positive, we can prove that $T_n$ is primitive by showing that $yT_n^m \to \rho$ for $m \to \infty$ for every probability vector $y$. To this end, choose an arbitrary probability vector $y$. Consequently, the function $u = \sum_j y_j\chi_j/\rho_j$ is a density, i.e., $u \in \mathcal{D}_F$, and the asymptotic stability of $T$ together with assertion 2) yield for every $k \in \{1, \dots, n\}$:

$$(yT_n^m)_k = \langle T^m u, \chi_k\rangle_F \to \langle \chi_\Omega, \chi_k\rangle_F = \rho_k, \quad \text{for } m \to \infty.$$

Thus, $T_n$ is primitive and the assertion concerning the eigenvalue $\lambda = 1$ then follows from the well-known Perron-Frobenius Theorem (cf. [96], Thm. 1.1).

Ad 4) First, introduce the diagonal matrix $\mathbf{D}_\rho = \mathrm{diag}(\sqrt{\rho_j})$ and define $S = \mathbf{D}_\rho T_n \mathbf{D}_\rho^{-1}$. Simple calculus, using the self-adjointness of $T$, then shows that $S$ is symmetric with respect to the Euclidean scalar product, i.e., that $S^T = S$. In turn, this reveals that, for $T_n$ itself,

$$S^T = S \quad \Leftrightarrow \quad \mathbf{D}_\rho T_n \mathbf{D}_\rho^{-1} = \mathbf{D}_\rho^{-1} T_n^T \mathbf{D}_\rho \quad \Leftrightarrow \quad \mathbf{D}_\rho^2 T_n = T_n^T \mathbf{D}_\rho^2,$$

which is nothing else than the matrix notation of the detailed balance condition (73). Rewritten in terms of the $\rho$-weighted scalar product, this finally means for every pair of (column) vectors $y, z \in \mathcal{V}_n$:

$$\langle y, T_n z\rangle_\rho = y^T \mathrm{diag}(\rho_j) T_n z = y^T \mathbf{D}_\rho^2 T_n z = y^T T_n^T \mathbf{D}_\rho^2 z = \langle T_n y, z\rangle_\rho,$$

i.e., the asserted symmetry of $T_n$. $\square$

Let the stochastic matrix $T_n$ have a *right* eigenvector $r \in \mathbb{R}^d$, i.e., assume that there is a $\lambda \in [-1,1]$ such that $T_n r = \lambda r$. Then,

$$u = \sum_{j=1}^{n} r_j\, \chi_j$$

is an eigenvector of $\Pi_n T \Pi_n$ with associated eigenvalue $\lambda$. Thus, the *right* eigenvectors of $T_n$ are to be interpreted as approximate eigenvectors of $T$. The *left* eigenvector of $T_n$ associated with $r$ and $\lambda$ then is given by componentwise multiplication of $r$ with $\rho$, i.e., by the row vector

$$l = r^T \mathbf{D}_\rho^2 \quad \Rightarrow \quad l\, T_n = \lambda\, l.$$

Consequently, if $r$ is the right eigenvector approximating the eigenfunction $u$ of $T$, the left eigenvector $l = r^T \mathbf{D}_\rho^2$ approximates the *weighted* eigenfunction $u \cdot F$. For $T_n$, as for every primitive stochastic matrix, the unique right eigenvector to the eigenvalue $\lambda = 1$ is $e = (1, \ldots, 1)^T$, which results in $e^T \mathbf{D}_\rho^2 = \rho$ for the associated left eigenvector.

Finally, let $R = [r_1, \ldots, r_n]$ denote the matrix whose columns are the $\rho$-normalized[51] right eigenvectors of $T_n$ for the $n$ eigenvalues[52] $(\lambda_1, \ldots, \lambda_n)$. Let $L^T = [l_1, \ldots, l_n]$ denote the matrix whose rows are the left eigenvectors $l_k = r_k^T \mathbf{D}_\rho^2$. Then, $T_n$ has the following *spectral decomposition*:

$$T_n \;=\; R \;\mathrm{diag}(\lambda_j)\, L^T \quad \text{with} \quad L^T R = \mathrm{Id}, \;\; \text{since} \;\; \langle r_k, r_l \rangle_\rho = \delta_{kj}.$$

Remark 5.3 This technique of discretizing the transition operator is similar to the approach to discretizing the Frobenius-Perron operator of a discrete dynamical system via subdivision techniques due to Dellnitz and Junge [19] and to the concept of "cell-to-cell-mapping" due to Hsu [57]. However, the main difference is obvious: we have to deal with a stochastic dynamical system whose essential dynamics covers significantly large parts of some highly-dimensional state space; both of the other approaches have been designed for deterministic dynamical systems in state spaces with relatively small dimension. For more details see [21, 94].

## 5.2 Convergence of Eigenvalues and Eigenvectors

From our results concerning the spectral properties of the transition operator $T$ we may infer that the following situation is valid:

There are some integers $l$ and $k$ such that the discrete spectrum of $T$ can be written as

$$\sigma_{\text{discr}}(T) = \{\lambda_1, \ldots, \lambda_l\} \cup \{\lambda_{-1}, \ldots, \lambda_{-k}\},$$

where the second set may be empty $(k = 0)$.[53] The eigenvalues may be ordered such that

$$1 = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_l > 0 \quad \text{and} \quad -1 < \lambda_{-1} \leq \lambda_{-2} \leq \ldots \leq \lambda_{-k} < 0,$$

[51] That is, $\langle r_k, r_k \rangle_\rho = 1$.

[52] Note, that each eigenvalue has to be repeated with respect to its multiplicity

[53] In fact, $l$ and $k$ may also be infinite, e.g., if $T$ is compact. In this case, consider only the $l$ largest and $k$ smallest eigenvalues of $T$.

where some of the $\lambda_j$'s may occur repeatedly according to their multiplicity. The associated eigenvalues are denoted $u_j$, $j \in \{1, \ldots, l\} \cup \{-1, \ldots, -k\}$; for every repeated eigenvalue they are assumed to be normalized and to be chosen orthogonal with respect to $\langle \cdot, \cdot \rangle_F$.

For the discretized representation $\Pi_n T \Pi_n$ of the transition operator, we denote the eigenvalues by $\lambda_j^n$. For $n \geq \max(l, k)$ the largest and smallest ones are assumed to be ordered accordingly:

$$1 = \lambda_1^n > \lambda_2^n \geq \ldots \geq \lambda_l^n \quad \text{and} \quad -1 < \lambda_{-1}^n \leq \lambda_{-2}^n \leq \ldots \leq \lambda_{-k}^n,$$

again repeated due to their multiplicity. The associated eigenvectors are denoted by $u_j^n$, again being assumed to be orthogonal and normalized with respect to $\langle \cdot, \cdot \rangle_F$.

Finally, let us to assume that we are considering a sequence $\mathcal{V}_n \subset \mathcal{V}_{n+1} \subset \ldots$ of our simple box function ansatz spaces $\mathcal{V}_n = \text{span}\{\chi_{B_j^n}\}$, which is dense in $L_F^2$.[54] Hence, the associated coverings $B_j^n$, $j = 1, \ldots, n$, of $\Omega$ are supposed to result from an appropriate iterative subdivision process, such that the maximal "diameter" $h_n = \max_j \text{diam}(B_j^n)$ vanishes asymptotically, i.e., $h_n \to 0$ for $n \to \infty$.

Under these assumptions we can prove the following simple convergence result for the eigenelements associated with the discrete part of the spectrum.

Corollary 5.4 *For every $j \in \{1, \ldots, l\}$ and $i \in \{1, \ldots, k\}$ there is an integer $n_0 \in \mathbb{N}$ such that for all $n > n_0$*

$$\alpha_j^n + \lambda_j \leq \lambda_j^n \leq \lambda_j \quad \textit{and} \quad \lambda_{-i} \leq \lambda_{-i}^n \leq \alpha_{-i}^n + \lambda_{-i}, \tag{74}$$

*with $\alpha_j^n < 0$ and $\alpha_{-i}^n > 0$, but $\alpha_j^n \to 0$ and $\alpha_{-i}^n \to 0$ for $n \to \infty$. Moreover, for all $j \in \{1, \ldots, l\} \cup \{-1, \ldots, -k\}$, the convergence of the associated eigenvectors is guaranteed, i.e., after an appropriate adjustment of eigenvectors for repeated eigenvalues:*

$$\lim_{n \to \infty} \|u_j - u_j^n\|_2 = 0.$$

*Proof:* The asserted convergence is a simple corollary of the Lemmas B.48 and B.49 from Sec. B.3 of Appendix B. In order to meet the assumptions of these lemmas for the $l$ largest eigenvalues $\lambda_j$, $j \in \{1, \ldots, l\}$, of our transition operator $T$, consider the self-adjoint operator $A = (1 + \gamma)\text{Id} - T$ with an arbitrary $\gamma > 0$.[55] ∎

Remark 5.5 The explanations in Appendix B concerning Lemma B.49 illustrate that the rate of the convergence $u_j^n \to u_j$ crucially depends on the distribution of the eigenvalues in the spectrum of $T$. Moreover, as explained in

[54] That is, for every $u \in L_F^2$ and every $\epsilon > 0$, there is an $M \in \mathbb{N}$ and a $v \in \mathcal{V}_M$ such that $\|u - v\|_2 < \epsilon$.

[55] For the eigenvalues $\lambda_{-i}$, $i \in \{1, \ldots, k\}$ we may follow exactly the same idea using $A = (1 + \gamma)\text{Id} + T$.

more detail in Rmk. B.50, we can establish explicit *convergence rates* for the convergence in Cor. 5.4, if we use finite element ansatz spaces instead of our crude box function ansatz spaces.

## 5.3 Identification of Almost Invariant Aggregates

In Sec. 3.2, we already observed that almost invariant sets of the system under investigation may be identified via the eigenvectors $u_j$ of the largest eigenvalues of the transition operator $T$. Let us now switch to the situation *after* proper discretization of $T$, assuming that the discrete eigenvectors of the resulting transition matrix $T_n$ are appropriate approximations of these $u_j$.

The problem remains how to approximate the almost invariant sets on the basis of these discrete eigenvectors. Obviously, we have to identify the almost invariant sets in the discrete state space $\{1,\dots,n\}$ of the transition matrix $T_n$, which then can be interpreted as approximate almost invariant sets in the essential configuration space $\Omega$ by identifying index sets $A \subset \{1,\dots,n\}$ with subsets $\cup_{j\in A} B_j \subset \Omega$. In order to avoid confusion, we adopt the phrase "aggregate" for any (nonvoid) index sets $A \subset \{1,\dots,n\}$.[56] Thus, our strategy is to approximate *almost invariant sets* of $T$ via *almost invariant aggregates* of $T_n$.

In the following, the connection between almost invariant aggregates and eigenvectors of the dominant eigenvalues of $T_n$ is worked out. We will see that such almost invariant aggregates exist if the Markov chain associated to $T_n$ is *nearly uncoupled*, which means that it can be interpreted as a *perturbation* of an *uncoupled* Markov chain (UMC). Then, it will be demonstrated that the decoupling of an UMC can be managed by inspection of the *sign structure* of the dominant eigenvectors of the associated transition matrix. This technique will be transferred to the nearly uncoupled case via an appropriate perturbation result, which finally yields the required identification algorithm.

The idea of analyzing nearly uncoupled Markov chains as perturbed UMCs is far from being new (compare, e.g., the approaches via coupling matrices [78, 102, 103], via the "conductance" of a Markov chain [97], or via the "uncoupling measure" [52].). However, a special *algorithmic* concept for the *identification* of almost invariant aggregates has (to the author's knowledge) first been presented in our article [24] and we will closely follow this presentation herein. In [19], an essentially different treatment of the identification problem has been demonstrated, mainly aiming at identifying *almost cyclic* aggregates (whereas the case of almost invariant aggregates has only been vaguely indicated).

We first have to give a suitable definition of "almost invariance" of aggregates. To this end, we start with the following

Definition 5.6 Given a Markov chain and its transition matrix $P$ (not necessarily primitive) and a stationary distribution $\rho > 0$.[57] Let $A$ and $B$ be two

[56] This phrase has its origin in the literature concerning Markov chains with discrete state space. Hence, it may also be used for the associated stochastic transition matrices.

[57] That is, the row vector $\rho$ is a probability vector with positive entries which satisfies $\rho P = \rho$.

arbitrary aggregates. Then the transition probability between $A$ and $B$ with respect to $\rho$ is given by

$$w_\rho(A,B) \;=\; \left(\sum_{a\in A}\rho_a\right)^{-1} \sum_{a\in A, b\in B} \rho_a\, P_{ab}.$$

Let the aggregates $A_1,\dots,A_k$ be a disjoint decomposition of the state space. Then, the stochastic matrix $W_\rho$, the entries of which are the transition probabilities between the $A_j$, i.e.,

$$W_{\rho,jl} \;=\; w_\rho(A_j, A_l),$$

is called the *coupling matrix* of the decomposition.

The statistical characterization of uncoupled Markov chains (UMC) is based on transition probabilities between aggregates. An aggregate $A$ is said to be *invariant*, if $w_\rho(A,A) = 1$.[58] A Markov chain is then called *uncoupled*, if it allows to decompose the state space into disjoint *invariant* aggregates $A_1,\dots,A_k$, i.e., $w_\rho(A_i,A_j) = \delta_{ij}$, or equivalently, $W_\rho = \mathrm{Id}$.

Before discussing more details about UMCs, let us switch to the case of some Markov chain with *primitive* transition matrix. Then, its stationary distribution is unique and we may simply write $w(A,B)$ instead of $w_\rho(A,B)$. An aggregate $A$ is said to be *almost invariant*, if $w(A,A) \approx 1$. The Markov chain is called *nearly uncoupled* (NUMC), if its state space can be decomposed into disjoint *almost invariant* aggregates $A_1,\dots,A_k$ such that

$$w(A_i,A_j) \;\approx\; \delta_{ij}, \qquad \text{that is,} \qquad W \;\approx\; \mathrm{Id}. \tag{75}$$

Thus, the almost invariant character of the decomposition $A_1,\dots,A_k$ means that the conditional probability of leaving $A_j$, i.e., $\sum_{l\neq j} w(A_j,A_l)$, is small for every $j = 1,\dots,k$. That is, we demand that

$$\|W \,-\, \mathrm{diag}(W)\|_\infty \;=\; \epsilon$$

is sufficiently small.[59]

Before being able to present the algorithm for identifying almost invariant aggregates, we have to collect some consequences of these definitions.

### 5.3.1 Uncoupled and Nearly Uncoupled Markov Chains

As a consequence of (75), the states of a NUMC with $k$ almost invariant aggregates *can be ordered* such that the transition matrix $P$ is of *block–diagonally*

[58] Note that in the case of an UMC, the stationary distribution is not unique, because the corresponding transition matrix is not irreducible. However, in this special case the probabilities are independent of the chosen stationary distribution.

[59] Herein, $\mathrm{diag}(W)$ denotes the diagonal matrix whose diagonal entries are identical to that of $W$, i.e., $\mathrm{diag}(W) = \mathrm{diag}(w(A_j,A_j))$.

*dominant* form

$$P = D + E = \begin{pmatrix} D_{11} & E_{12} & \cdots & E_{1k} \\ E_{21} & D_{22} & \cdots & E_{2k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ E_{k1} & E_{k2} & \cdots & D_{kk} \end{pmatrix}, \tag{76}$$

where the matrix $E$ is small. Here, smallness of $E$ means

$$\|\mathbf{D}_\rho^2 E\|_\infty \leq \|W - \mathrm{diag}(W)\|_\infty = \epsilon,$$

where $\mathbf{D}_\rho = \mathrm{diag}(\sqrt{\rho_i})$ is the weighting matrix from above. We are interested in the limit $\epsilon \to 0$.

**Uncoupled Markov Chains** For $\epsilon = 0$, i.e., for $E = 0$, we end up with a block-diagonal transition matrix associated with an *uncoupled* Markov chain. In fact, as a consequence of the definition of an UMC, any associated transition matrix takes this form after suitably renumbering the state space. Thus, each of the block matrices $D_{ii}$ is a stochastic matrix, and it is primitive if $k$ is the maximal number of uncoupled aggregates. Then, each block $D_{ii}$ possesses a unique eigenvector $e_i = (1, \ldots, 1)^T$ of length $\dim(D_{ii})$ corresponding to its Perron root $\lambda_i = 1$. Therefore, $\lambda = 1$ is a $k$–fold eigenvalue of the transition matrix $P$ and the corresponding eigenspace is spanned by the vectors

$$\chi_{A_i} = (0, \ldots, 0, e_i^T, 0, \ldots, 0)^T, \qquad i = 1, \ldots, k.$$

Here our notation deliberately emphasizes that the eigenvectors can be interpreted as *characteristic functions* of the uncoupled aggregates.

In general, any basis $\{r_i\}$, $i = 1, \ldots, k$, of the eigenspace corresponding to $\lambda = 1$ can be written as linear combinations of the characteristic functions $\chi_{A_i}$, i.e., there are coefficients $\alpha_{ij} \in \mathbb{R}$ such that

$$r_i = \sum_{j=1}^{k} \alpha_{ij}\, \chi_{A_j}, \qquad i = 1, \ldots k. \tag{77}$$

As a consequence, eigenvectors corresponding to $\lambda = 1$ are *constant on each aggregate*. This structure implies the following lemma, which is a simple consequence of the $\rho$-orthogonality of the eigenvectors (for a proof see [24]).

LEMMA 5.7 *Given a block–diagonal transition matrix $P$ consisting of reversible, primitive blocks, a stationary distribution $\rho > 0$ and a $\rho$-orthogonal basis $\{r_i\}_{i=1,\ldots,k}$ of its eigenspace corresponding to $\lambda = 1$. Associate with every state $s_i$ its sign structure*

$$s_i \longmapsto (\mathrm{sign}((r_1)_i), \ldots, \mathrm{sign}((r_k)_i)). \tag{78}$$

*Then*

*1. invariant aggregates are collections of states with common sign structure,*

*2. different aggregates exhibit different sign structures.*

Summarizing, Lemma 5.7 states that —for uncoupled Markov chains— the set of all $k$ eigenvectors can be used to *identify* all uncoupled aggregates via sign structures. In principle this can also be done by using both *left* eigenvectors and *right* eigenvectors, since their sign structures are the same: Just recall that for every left eigenvector $l$ there exists an associated right eigenvector $r$ with $l^T = \mathbf{D}_\rho r$, hence $\text{sign}(l^T)=\text{sign}(r)$.

**Perturbation of Uncoupled Markov Chains** In what follows we want to analyze nearly uncoupled Markov chains as weak perturbations of uncoupled Markov chains. In analogy to the nearly block-diagonal form (76) of the transition matrix of a NUMC, we assume that the perturbation can be embedded in a family of block–diagonally dominant matrices $P = P(\epsilon)$ which is analytic in a domain of the complex plane containing the origin:

$$P(\epsilon) \quad = \quad P(0) + \epsilon P^{(1)} + \epsilon^2 P^{(2)} + \dots\,,$$

where $P(0)$ represents the uncoupled situation, i.e., $P(0)$ is block-diagonal with $k$ primitive blocks $D_{ii}$, $i = 1, \dots, k$. As normalization, let $\|\mathbf{D}_\rho^2 P^{(1)}\|_\infty = 1$ to fix the scale of $\epsilon$. Since the discretization process of the transition operator results in a reversible and primitive transition matrix $T_n$, we, moreover, may assume that $P(\epsilon)$ also is symmetric with respect to $\langle\cdot,\cdot\rangle_\rho$ for $\epsilon \in \mathbb{R}$ and primitive for all $\epsilon \neq 0$. Consequently, we may exploit the usual theory for *symmetric* matrices for the details of the perturbation analysis (cf. [60]). Finally, we have to assume that the unique stationary distributions[60] $\rho = \rho(\epsilon)$ of the $P = P(\epsilon)$ are "uniformly bounded away from zero" (for all $\epsilon \geq 0$).

These *regularity conditions* imply that, for sufficiently small $\epsilon \in \mathbb{R}$, the eigenvalues are continuous in $\epsilon$ and the spectrum of $P(\epsilon)$ can be divided into three parts [60, 78, 102]: the Perron root $\lambda_1(\epsilon) \equiv 1$, a cluster of $k-1$ eigenvalues $\lambda_2(\epsilon), \dots, \lambda_k(\epsilon)$ that approach 1 for $\epsilon \to 0$, and the remaining part of the spectrum, which is bounded away from 1 for $\epsilon \to 0$.

In particular, for small real $\epsilon$ there is a spectral gap between the "Perron cluster" $\lambda_1(\epsilon), \dots, \lambda_k(\epsilon)$ and the remaining part of the spectrum. The following perturbation theorem gives a characterization of the eigenvectors $r_1(\epsilon), \dots, r_k(\epsilon)$ corresponding to the Perron cluster (for a proof see [24]).

THEOREM 5.8 *Let $P(\epsilon)$ be a family of matrices satisfying the above introduced regularity conditions. Then, for real $\epsilon$, there exist $\rho$–orthonormal eigenvectors $r_1(\epsilon), \dots, r_k(\epsilon)$ of the following form: The eigenvector $r_1(\epsilon) \equiv (1, \dots, 1)^T$ corresponds to the Perron root $\lambda = 1$. The other $k-1$ eigenvectors correspond to*

[60] For $\epsilon > 0$ the $P(\epsilon)$ are primitive. For $\epsilon = 0$, define $\rho(0) = \lim_{\epsilon\to 0}\rho(\epsilon)$.

*the eigenvalue cluster* $\lambda_2(\epsilon), \ldots, \lambda_k(\epsilon)$ *near* $\lambda = 1$ *and have the form*

$$r_i(\epsilon) = \underbrace{\sum_{j=1}^{k} (\alpha_{ij} + \epsilon\,\beta_{ij})\,\chi_{A_j}}_{=(I)} + \underbrace{\epsilon \sum_{j=k+1}^{n} \frac{1}{1-\lambda_j} \langle r_j, P^{(1)} r_i \rangle_\rho\, r_j + \mathcal{O}(\epsilon^2)}_{=(II)} \qquad (79)$$

*with appropriate coefficients* $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$, $\{r_i = r_i(0)\}$ *being eigenvectors of* $P(0)$, *and aggregates* $A_1, \ldots, A_k$ *corresponding to the block–diagonal form of* $P(0)$.

This result permits an intriguing observation: the first terms $(I)$ are just shifts (up or down) of the locally constant levels to be associated with the almost invariant aggregates, whereas the second terms $(II)$, which depend on the spectral gap between the Perron root and the remaining part of the spectrum, will spoil the constant level pattern to some extent. It may, however, affect the sign structure associated with any almost invariant aggregate only to a smaller extent. Indeed, it is one of the key ideas of this subsection to identify the almost invariant aggregates via their sign structure. Clearly, caution must be taken with respect to the perturbation of any "almost zero" levels (cf. Fig. 14).

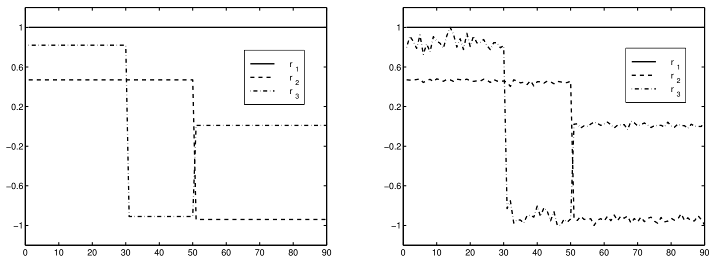


Figure 14: Illustration of Thm. 5.8 for a situation with $k = 3$ (almost) invariant aggregates. In this example, the $P(\epsilon)$ are $90 \times 90$ matrices acting on a state space $\{1, \ldots, 90\}$. Left: Eigenvectors spanning the eigenspace of $\lambda = 1$ for the uncoupled case $P(0)$. The state space is decomposed into three invariant aggregates by different sign structures. Right: Eigenvectors for the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 0.75$, and $\lambda_3 = 0.52$ of the Perron cluster for the nearly uncoupled case $P(\epsilon)$ versus the (discrete axis of the) 90 states. The eigenvectors are almost constant on the three almost invariant aggregates. The sign structures are the same as for the uncoupled case except for $r_3$ on the third aggregate, where perturbations introduce "erratic" sign structures. See [24] for the construction of the underlying transition matrices.

Remark 5.9 This approach to almost invariant aggregates as perturbation of invariant aggregates of uncoupled Markov chains should be compared to the understanding of almost invariant sets as perturbed invariant sets as illustrated in the guiding example, compare Sec. 3.2, Rmk. 3.5. An interpretation of the perturbation parameter $\epsilon$ is given in Rmk. 3.6.

**Error Indicator** The identification process exploiting the sign structure is justified only via the perturbation result. Therefore, it is of main importance to estimate the influence of the "weak modes" $r_i$, $i = k+1, \ldots, n$, on the coupling of the "dominant modes" $r_i$, $i = 1, \ldots, k$.

To this end, let us assume that we already know the almost invariant aggregates $A_1, \ldots, A_k$ for $P = P(\epsilon)$. In terms of the $n \times k$ matrices $\chi = [\chi_{A_1} \cdots \chi_{A_k}]$ and $R = [r_1 \cdots r_k] = R(\epsilon)$ the perturbation result (79) may also be expressed in the form $R = \chi \mathcal{A}^{-1} + \epsilon B$, with a $k \times k$ coefficient matrix $\mathcal{A} = \mathcal{A}(\epsilon)$ and a $n \times k$ matrix $B = B(\epsilon)$ representing the "weak-dominant" coupling terms $(II)$. This permits the computation of the associated coupling matrix in terms of the eigenvectors $R$ and the corresponding eigenvalues $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_k) = \Lambda(\epsilon)$:

$$W \;=\; (\chi^T \mathbf{D}_\rho^2 \chi)^{-1} \; (\chi^T \mathbf{D}_\rho^2 P \, \chi) \;=\; \mathcal{A}^{-1} \Lambda \, \mathcal{A} \,+\, \epsilon \, \Delta.$$

Herein, the matrix $\Delta$ represents the contribution of the "weak-dominant" interaction to the coupling matrix, while the first term, $\mathcal{A}^{-1}\Lambda\,\mathcal{A}$, describes the interaction between the dominant modes. The reader might remember, that, in the case of an *uncoupled* Markov chain, we had $B = 0$ and $\Lambda = \mathrm{Id}$, implying $\Delta = 0$ and $W = \mathcal{A}^{-1}\Lambda\,\mathcal{A} = \mathrm{Id}$. For the nearly decoupled Markov chains under consideration, we may expect $\mathcal{A}^{-1}\Lambda\,\mathcal{A} \approx \mathrm{Id}$ and interpret the $k \times k$ matrix

$$\mathrm{err}(A_1, \ldots, A_k) \;=\; W \;-\; \mathcal{A}^{-1}\Lambda\,\mathcal{A} \tag{80}$$

as *error indicator* measuring the influence of the weak modes on the coupling between the aggregates $(A_1, \ldots, A_k)$. Hence, if an entry of $\mathrm{err}(A_1, \ldots, A_k)$ is large, this may essentially have *two* different reasons:

1. Our assumptions concerning the perturbation were not valid (e.g., $\epsilon$ was too large, or the regularity conditions were hurt).
2. The identification process exploiting the sign structure resulted in "wrong" almost invariant aggregates. This may happen if the perturbations had crucially spoiled the sign structure resulting in an assignment of some states to the "wrong" aggregate (see below).

### 5.3.2 Identification Algorithm

In this section we present the basic concept of an algorithm for the identification of almost invariant aggregates. As derived above, this algorithm explores the sign structure of eigenvectors corresponding to an eigenvalue cluster near $\lambda = 1$.

In a first step we have to determine the number $k$ of almost invariant aggregates. This is done by computing the cluster of eigenvalues near $\lambda = 1$ which is well-separated from the remaining part of the spectrum by a gap (Thm. 5.8). Iterative eigenvalue solvers with simultaneous subspace iteration (see, e.g., [90, 68] or [22, Section 4]) are a natural way to perform this task.

We now assume the $k$ right eigenvectors $r_i$, associated with the Perron cluster, to be computed and explain the suggested identification algorithm in three steps:

**Step 1:** We are mainly interested in sign structures which are particularly *stable* with respect to the perturbations of the eigenvectors. Heuristically, the *sign* of an entry of one of the eigenvectors is the more likely to remain unperturbed, the larger this entry is. Hence, we are particularly interested in all those states $s \in \{1, \dots, n\}$, for which at least one of the eigenvectors $r_i$ has some "significantly large" entry $r_i(s)$. Since "significantly large" will vary for each eigenvector, we introduce an appropriate scaling: we normalize the positive and negative parts of each eigenvector by their maximum norm, yielding scaled eigenvectors $\tilde{r}_i$.[61] Now, we are interested in such states $s$ for which at least one of the scaled eigenvectors has an entry larger than a given threshold value $0 \ll \delta < 1$. Hence, the first step of the identification algorithm is

(S1) Determine $\Sigma = \{s : \max_{i=1,\dots,k} |\tilde{r}_i(s)| > \delta\}$.

**Step 2:** Based on the sign structures of the states in $\Sigma$, we now identify $k$ specific classes of "nonequivalent" sign structures. The sign structure $\sigma(s, \epsilon) = (\sigma_1, \dots, \sigma_k)$ of some state $s$ with respect to the *threshold value* $\epsilon$ is defined as follows: if $|r_i(s)| > \epsilon$, then $\sigma_i = \text{sign}(r_i(s))$, otherwise $\sigma_i = 0$. Then, the entry "0" is used as a kind of joker, being interpretable as "+1" as well as "-1". Consequently, two sign structures $\sigma_1$ and $\sigma_2$ are equivalent, denoted by $\sigma_1 \equiv \sigma_2$, iff their pointwise multiplication yields only nonnegative entries. Despite the fact that $\equiv$ defines no equivalence relation, it serves to decompose $\Sigma$ into "nonequivalent" classes of sign structures yielding a surjective map $a : \Sigma \to \{1, \dots, k\}$. For ease of presentation, we refer the reader for the details of these procedures to our article [24]. Summarizing, the second step of the identification algorithm contains:

(S2) Determine $k$ classes of (incommensurable) sign structures and the associated map $a$. Due to $a$, $\Sigma$ decomposes into $k$ disjoint subsets $\Sigma_1, \dots, \Sigma_k$, each of which represents the "core" of the almost invariant aggregates.

**Step 3:** We finally have to assign the remaining states $s \in \{1, \dots, n\} \setminus \Sigma$ to one of the sign structure classes, aiming at a complete decomposition of the state space into $k$ aggregates. Instead of using the possibly heavily perturbed sign structures of these states, we exploit the fact that the $k$ eigenvectors $r_i$ allow to approximate the $k$ characteristic functions of the aggregates. Since a subset $\Sigma_j$ of each aggregate is already available, we may again apply a least squares fit, now based only on the states from $\Sigma$, in order to determine approximate characteristic functions $\tilde{\chi}_j$. That is, evaluate coefficients $\alpha_{ij}$ such that

$$\|\chi_{\Sigma_i} - \sum_{j=1}^{k} \alpha_{ji} r_j \chi_\Sigma\|_\pi = \min!, \quad \text{for } i = 1, \dots, k, \tag{81}$$

[61] For every vector $X$, we define entrywise $X^+(s) = \max(0, X(s))$ and $X^-(s) = \min(0, X(s))$ such that $X = X^+ + X^-$. In this notation, we set $\tilde{r}_i = r_i^+/\|r_i^+\|_\infty + r_i^-/\|r_i^-\|_\infty$. In particular, the eigenvector for $\lambda = 1$ remains unchanged: $\tilde{r}_1 = (1, \dots, 1)^T$.

where $r_j\chi_\Sigma$ means pointwise multiplication. Then, define $\tilde{\chi}_j = \sum_{j=1}^{k} \alpha_{ji} r_j$. Having computed the $\tilde{\chi}_j$, a state $s$ is assigned to aggregate $j$ if $\tilde{\chi}_j(s)$ is maximal in the set $\{\tilde{\chi}_i(s), i = 1, \ldots, k\}$.[62] Thus, the third step of the algorithm is:

(S3) Evaluate coefficients $\alpha_{ij}$ due to (81), set $\tilde{\chi}_j = \sum_{j=1}^{k} \alpha_{ji} r_j$, and determine the aggregates via

$$A_j = \{s \in \{1, \ldots, n\} : \quad \tilde{\chi}_j(s) > \tilde{\chi}_i(s), \text{ for all } i \neq j\}.$$

The resulting aggregates can finally be validated with respect to different criteria, for example, via the entries of the error indicator matrix, via the deviation of the approximations $\tilde{\chi}_j$ from being true characteristic functions, or via a comparison between the Perron cluster and the diagonal entries of the coupling matrix.[63]

The performance of the resulting algorithm applied to a realistic problem will be illustrated in detail in Sec. 6 below.

## 5.4 Evaluation of the Transition Matrix via Hybrid Monte Carlo

Our next question is how to compute the matrix $T_n$ for given boxes $B_k \subset \Omega$. According to (72) we have to determine the transition probabilities between these boxes. For the scope of this section we restrict our consideration to the case of the canonical ensemble $f_0 = f_{\text{can}}$. That is, we have to compute the transition probabilities

$$\begin{aligned} w(B_k, B_l, \tau) &= \frac{\int_{\Gamma(B_k)} \chi_{\Gamma(B_l)}(\Phi^\tau x)\; f_{\text{can}}(x)\, dx}{\int_{\Gamma(B_k)} f_{\text{can}}(x)\, dx} \\ &= \frac{\int_{B_k} \left\{ \int_{\mathbb{R}^d} \chi_{B_l}(\pi_1 \Phi^\tau(q,p))\, \mathcal{P}(p)\, dp \right\}\; \mathcal{Q}(q)\, dq}{\int_{B_k} \mathcal{Q}(q)\, dq}, \end{aligned} \tag{82}$$

where the notation assumes for a moment, that $\Omega$ indeed denotes the *position space* of the system under consideration.

The two alternative formulae show that we have at least two options for evaluating $w(B_j, B_l, \tau)$ explicitly. We can realize the first formula by working in the phase space $\Gamma$ (**option x**), performing the following two steps

(X1) "Sampling of the canonical density": That is, we have to generate a sequence of states $\mathcal{S} = \{x_k, \quad k = 1, \ldots, M\} \subset \Gamma$ that is approximately distributed according to the canonical density $f_{\text{can}}$.

[62] In case of ambiguity, the state is assigned to an arbitrary aggregate with maximal $\tilde{\chi}_j$. However, this case was never observed in any of the numerical experiments performed.

[63] The deviation $\epsilon_P = 1 - \lambda_k$ of the Perron cluster $\lambda_1 \geq \ldots \geq \lambda_k$ from $\lambda_1 = 1$ should be comparable to the deviation $\epsilon_W = 1 - \min_j w(A_j, A_j)$ of the diagonal entries from 1, especially if the gap $g = \lambda_k - \lambda_{k+1}$ between the Perron cluster and the remaining part of the spectrum is large enough ($g \gg \epsilon_P$).

(X2) Approximation of the transition probabilities: Having computed $\mathcal{S}$, one has to count all such $x_j \in \mathcal{S}$ for which $x_j \in \Gamma(B_k)$ and $\Phi^\tau x_j \in \Gamma(B_l)$. For checking the last condition, sufficient approximations $\tilde{x}_j \approx \Phi^\tau x_j$ of all $M$ subtrajectories starting from $\mathcal{S}$ are needed.

The second formula of (82) allows to restrict the sampling problem to the position space $\Omega$ (**option q**) but requires an additional momenta sampling in the approximation step of the transition probabilities. We will see that the options can both be realized by suitable Monte-Carlo strategies.

### 5.4.1 Evaluation via Metropolis Monte-Carlo (MC)

The main problem with sampling the canonical density is that, in most realistic cases, we cannot explicitly evaluate its value $f_{\text{can}}(x) = \exp(-\beta H(x))/\mathcal{Z}$ for a given state $x \in \Gamma$, because the *partition function* $\mathcal{Z}$ is not known explicitly.[64] The typical approach to sampling the canonical density $f_{\text{can}}$ in the entire phase space is via Monte Carlo (MC) techniques. The literature on this topic is extremely rich and varied [11, 44] and we surely will not give particular merits to any particular MC variant. Let us shortly recall the basic steps of a Metropolis MC approach (for additional details see, e.g., [101]):

Metropolis MC realizes a Markov chain $\{x_j\}$ which is asymptotically distributed according to $f_{\text{can}}$ without evaluating $f_{\text{can}}$ itself. Each "update step" $x_j \to x_{j+1}$ of the Metropolis construction consists of two parts:

1. The *proposal step* $x_j \to \tilde{x}_j$: The numerical realization of the proposal step should exclude any evaluation of $f_{\text{can}}$ and must yield a final update step which satisfies the detailed balance condition.

2. The *acceptance step*: evaluate $\Delta E = H(\tilde{x}_j) - H(x_j)$ and choose $r$ randomly equidistributed from $[0,1]$. The state $\tilde{x}_j$ is accepted as $x_{j+1}$ if $r \leq \min\{1, \exp(-\beta\Delta E)\}$, otherwise we set $x_{j+1} = x_j$.

Thus, we may apply any such Metropolis MC variant to realize option x from above by producing a sequence $\mathcal{S} = \{x_k, \quad k = 1, \ldots, M\} \subset \Gamma$ sampling $f_{\text{can}}$. In an additional, second step we then have to realize problem (X2) via appropriate approximations of the flow. That is:

$$\frac{\text{rel.freq.}\,(x_j \in \Gamma(B_k) \wedge \Phi^\tau x_j \in \Gamma(B_l))}{\text{rel.freq.}\,(x_j \in \Gamma(B_k))} \to w(B_k, B_l, \tau). \tag{83}$$

It is obvious, that this idea of approximation is *not* restricted to the case where $\Omega$ is the position space and the $B_j$ are spatial boxes. On the contrary, we may use *every* box covering induced by a set $\vartheta$ of essential variables, only that, then, we have to replace (83) by

$$\frac{\text{rel.freq.}\,(\vartheta(x_j) \in B_k \wedge \vartheta(\Phi^\tau x_j) \in B_l)}{\text{rel.freq.}\,(\vartheta(x_j) \in B_k)} \to w(B_k, B_l, \tau). \tag{84}$$

[64] $\mathcal{Z} = \int_\Gamma f_{\text{can}}(x)dx$ is an integral in a highly dimensional space. Its computation of which is a tremendous task comparable to the evaluation of the transition probabilities itself.

**Evaluation via Associated Markov Chain** Let us now discuss how a realization of option q may look like, i.e., how we may restrict the sampling to the position space of the system under consideration. We know from Sec. 3.6, that our transition operator is associated with the Markov chain

$$q_{k+1} \;=\; \pi_1 \Phi^\tau(q_k, p_k), \tag{85}$$

if the $p_k$ are independently chosen randomly with respect to the canonical momentum distribution $\mathcal{P}$, and the initial position $q_0$ from an arbitrary initial distribution in position space. Thus, iterations of (85) realize sequences $\{q_k\}$ which are (asymptotically) distributed due to $\mathcal{Q}$. Simultaneously, such iterations allow us to determine the relative frequency of transitions $q_k \in B_j \to q_{k+1} \in B_l$ for arbitrary box numbers $j$ and $l$. The convergence guarantees that —as in step (X2) above— the relative frequencies approximate the desired transition probabilities in the sense similar to (83), i.e.,

$$\frac{\text{rel.freq.}\,(q_k \in B_j \wedge q_{k+1} \in B_l)}{\text{rel.freq.}\,(q_k \in B_j)} \;\to\; w(B_j, B_l, \tau),$$

or, for essential variables, in a sense similar to (84).

Thus, another idea for evaluating $T_n$ could be to realize the iteration (85) by replacing the exact flow $\Phi^\tau$ by an appropriate approximation. Therefore, let us apply a *symplectic and reversible* discretization $\Psi^t$ to the Hamiltonian system[65] and let us use, for example,

$$g \;=\; \left(\Psi^{\tau/m}\right)^m, \quad m \in \mathbb{N},$$

instead of $\Phi^\tau$ with $m$ being large enough such that the stepsize $\tau/m$ is adequate. Consequently, we are inheriting nearly all necessary properties of the flow, with only one disadvantage: The underlying stationary density $f_0$ is *not* invariant under the action of $g$, since $g$ does not preserve the energy of the system.[66] As a consequence, the iteration (85) with $\Phi^\tau$ replaced by $g$ will *not* sample the distribution $\mathcal{Q}$, thus destroying the basis of our approximation idea. Hence, we have to look for a Markov chain, iteration of which allows to sample $\mathcal{Q}$ while containing only $g$ and not the flow itself.

### 5.4.2 Hybrid Monte-Carlo (HMC)

So-called "hybrid" MC variants have to the author's knowledge first been introduced in the late 80's (cf. [28]) and have in MD mostly been used for condensed matter and polymer-like systems (cf. [76, 53, 40, 14], for example). The technique imitates the general Metropolis MC strategy of proposal and acceptance via a specification of the proposal step for separable Hamiltonians of form (1)

[65] For example, $\Psi^t$ may be the well-known Verlet scheme [110, 1].

[66] There is no discretization which is symplectic and reversible and simultaneously preserves energy exactly [42]. We may reduce the energy error, produced by $g$, to an arbitrary small value by increasing $m$, but this would lead to a totally inefficient computation scheme.

by means of a reversible and symplectic discretization (cf. [69]). We merely suggest the application of HMC herein, because it seems to be particularly appropriate for linking the subproblems of sampling the canonical density and approximating the transition probabilities by approximate trajectories.

For explaining HMC, let again the function $g$ from above denote the reversible and symplectic discretization. In difference to general MC techniques, HMC generates a sequence $\mathcal{S} = \{q_j\} \subset \Omega$ in position space. Starting with $q_j$, the first part of the proposal step is to choose momenta $p_j$ randomly from $\mathcal{P}$, gaining the state $x_j = (q_j, p_j)$. As the second part, compute the proposal state $\tilde{x}_j$ via a short approximate subtrajectory of the underlying Hamiltonian system, i.e., choose $\tilde{x}_j = g(x_j)$. Then, repeat the MC acceptance procedure with $x_j$ and $\tilde{x}_j$, let the accepted state be $x_{j+1}$, and finally set $q_{j+1} = \pi_1 x_{j+1}$. In other words, HMC realizes an iteration of the Markov chain

$$q_{j+1} \;=\; \pi_1 a(q_j, p_j, r_j) \;\text{ with }\; a(x,r) = \begin{cases} g(x), & \text{if } r \le \alpha(x), \\ x & \text{otherwise,} \end{cases} \tag{86}$$

$$\begin{aligned} &\text{setting} \quad \alpha(x) = \min\{1, \exp(-\beta \Delta E(x))\}, \\ &\text{with} \quad \Delta E(x) \;=\; H(g(x)) - H(x), \end{aligned}$$

with $p_j$ independently chosen randomly from $\mathcal{P}$ and $r_j$ randomly from the equidistribution in $[0,1]$. In this form, HMC has to be understood as a *pure position sampling* of the spatial canonical distribution $\mathcal{Q}$ such that the resulting Markov chain $\{q_j\}$ allows to approximate the expectation values of appropriate *spatial* observables $\mathcal{A} : \Omega \to \mathbb{R}$ in the sense that[67]

$$\lim_{M\to\infty} \frac{1}{M} \sum_{j=1}^{M} \mathcal{A}(q_j) \;=\; E_{\mathcal{Q}}(\mathcal{A}) \;=\; \int_{\Omega} \mathcal{A}(q)\mathcal{Q}(q)\,dq. \tag{87}$$

Heuristically, we thus are able to construct a scheme like (83) for the evaluation of the desired transition probabilities via HMC: First, rewrite the transition probabilities as spatial expectation values, yielding

$$w(B_k, B_l, \tau) \;=\; \frac{1}{E_{\mathcal{Q}}(\chi_{B_k})} \int_{\mathbb{R}^d} \left\{ E_{\mathcal{Q}}\left(\chi_{B_k} \circ \chi_{B_l}(\pi_1 \Phi^{-\tau}(\cdot, p))\right) \right\} \mathcal{P}(p)\,dp.$$

Then, use (87) to gain the approximation

$$w(B_k, B_l, \tau) \approx \frac{1}{\sum\limits_{j=1}^{M} \chi_{B_k}(q_j)} \sum_{j=1}^{M} \chi_{B_k}(q_j) \int_{\mathbb{R}^d} \chi_{B_l}(\pi_1 \Phi^{-\tau}(q_j, p))\,\mathcal{P}(p)\,dp. \tag{88}$$

Hence, in addition, we need to approximate the integral in (88) for any of the position $q_j$ from the HMC chain $\{q_j\}$. Consequently, this will again be done via

[67] For details see paragraph "Approximation Properties" below.

a momenta sampling $\{p_i\}$ of $\mathcal{P}$ yielding

$$\int_{\mathbb{R}^d} \chi_{B_l}(\pi_1 \Phi^{-\tau}(q_j,p))\,\mathcal{P}(p)\,dp \;\approx\; \frac{1}{L}\sum_{r=1}^{L} \chi_{B_l}(\pi_1 \Phi^{-\tau}(q_j,p_r)). \tag{89}$$

Hence, one obvious advantage of HMC in this context is that it is already based on approximations of $\Phi^\tau(q_j,p_r)$ which we need for the evaluation of the transition probabilities. We may indeed use the approximations $g(x_j)$ already evaluated in the proposal step (with the exception of such proposals $g(x_j)$, which could not be accepted).

Consequently, the transition matrix is allocated by executing (88) for any pair $(l,k)$ of box numbers after or during an HMC iteration of "sufficient" length.[68]

Theoretically, the transition matrix $T_n$ is reversible. In order to reproduce this property for its approximation, we may simply count each transition from $B_k$ to $B_l$ as a simultaneous transition $B_l \to B_k$ (thus exploiting the reversibility of the discretization $g$).

Remark 5.10 Whenever we generate the sampling data via some Metropolis MC method yielding a sequence $\{x_1,\dots,x_M\} \subset \Gamma$, we may replace equations (88) and (89) by the direct mean value, i.e., we may approximate the transition probabilities by

$$w(B_k,B_l,\tau) \;\approx\; \left(\sum_{j=1}^{M} \chi_{B_k}(\pi_1 x_j)\right)^{-1} \sum_{j=1}^{M} \chi_{B_k}(\pi_1 x_j)\,\chi_{B_l}(\pi_1 \Phi^{-\tau}(x_j)).$$

In comparison to the combination of (88) and (89) this option may lead to superior convergence properties. This possibility has not been analyzed yet (mainly because HMC has other conceptual advantages as we will see in Sec. 5.4.3 below).

### 5.4.3 Approximation Properties

In order to understand the necessity of the acceptance step for the convergence of an HMC chain to $\mathcal{Q}$, we may consider the Foias operator $P : \mathcal{M}_f \to \mathcal{M}_f$ associated with the HMC stochastic dynamical system (86). Due to the Sec. 3.6, it is given by

$$P\mu(B) \;=\; \int_\Omega \left\{ \int_{\mathbb{R}^d} \int_0^1 \chi_B(a(q,p,r))\mathcal{P}(p)\,dr\,dp \right\} \mu(dq),$$

for all Borel subsets $B \subset \Omega$. We first have to show that the canonical measure $\mu_{\mathcal{Q}}$, given by the density $\mathcal{Q}$, is a stationary measure of $P$, i.e., that $P\mu_{\mathcal{Q}} = \mu_{\mathcal{Q}}$.

[68] It should again be emphasized that there is no restriction to spatial boxes. (88) allows to compute $T_n$ for every box covering of the essential configuration space of an arbitrary set of essential variables $\vartheta$.

We now will see that this is the case only because the acceptance step allows to replace the perturbed invariance $f_{\text{can}}(g^{-1}(x)) \neq f_{\text{can}}(x)$ by a new one, the *HMC invariance*: We find that the reversibility of $g$ implies

$$\alpha(g^{-1}R(x))\, f_{\text{can}}(g^{-1}R(x)) \;=\; \alpha(x)\, f_{\text{can}}(x), \qquad \text{for every } x \in \Gamma. \tag{90}$$

This can be seen as follows: Per assumption on $g$ we have $Rg = g^{-1}R$, with the momentum reversion $R(q,p) = (q,-p)$. Thus, with $z(x) = \exp(-\beta\Delta E(x))$, one easily evaluates $f_{\text{can}}(g^{-1}R(x)) = f_{\text{can}}(Rg(x)) = f_{\text{can}}(g(x)) = f_{\text{can}}(x)z(x)$. Moreover, the symmetry $H(Rx) = H(x)$ yields after some computation that $\alpha(g^{-1}Rx) = \alpha(x)/z(x)$. Put together, these identities give the invariance (90).

In order to show that (90) in fact induces $P\mu_Q = \mu_Q$, rewrite $P$ as follows:

$$\begin{aligned} P\mu_Q(B) &= \int_\Gamma \left\{ \int_0^{\alpha(x)} \chi_{\Gamma(B)}(g(x))\, dr \;+ \int_{\alpha(x)}^1 \chi_{\Gamma(B)}(x)\, dr \right\} f_{\text{can}}(x)\, dx \\ &= \int_\Gamma \left\{ \alpha(x)\chi_{\Gamma(B)}(g(x)) \;+\; (1-\alpha(x))\chi_{\Gamma(B)}(x) \right\}\, f_{\text{can}}(x)\, dx \\ &= \mu_Q(B) \;+\; \underbrace{\int_\Gamma \alpha(x) \left\{ \chi_{\Gamma(B)}(g(x)) \;-\; \chi_{\Gamma(B)}(x) \right\} f_{\text{can}}(x)\, dx}_{=(I)} . \end{aligned}$$

Finally, using the transformation $y = Rg(x)$ for the first term in $(I)$ and the symplecticness of $g$, we end up with

$$(I) \;=\; \int_{\Gamma(B)} \big( \alpha(g^{-1}Rx)\, f_{\text{can}}(g^{-1}Rx) \;-\; \alpha(x)\, f_{\text{can}}(x) \big)\; dx,$$

which, as a consequence of the HMC invariance (90), indeed implies $(I) = 0$ for all subsets $B$. Hence, $\mu_Q$ is a stationary measure of the HMC Foias operator $P$.[69]

Let $T_{\text{HMC}}$ denote the density operator corresponding to the Foias operator $P$. Then, its adjoint operator is given by

$$\begin{aligned} T^*_{\text{HMC}}u(q) &= \int_{\mathbb{R}^d} \int_0^1 u\left(a(q,p,r)\right)\; \mathcal{P}(p)\, dr\, dp \\ &= r(q)u(q) \;+\; \int_{\mathbb{R}^d} \alpha(q,p)\, u(\pi_1 g(q,p))\, \mathcal{P}(p)\, dp, \end{aligned} \tag{91}$$

where $r(q) = \int (1-\alpha(q,p))\mathcal{P}(p)dp$. Together with the HMC invariance (90) and

[69]The whole procedure works in close analogy to the better-known strategy of Langevin dynamics simulations for sampling the canonical density: In Langevin simulations, the discretization similarly destroys the convergence to the original stationary density, a problem which is also solved by applying appropriate acceptance procedures after each step (cf. [45], Sec. 6.5). In [75], HMC is in fact interpreted as a simulation methods for a specific class of Langevin dynamics.

the reversibility and symplecticness of $g$, this implies for $u, v \in L^2$:

$$
\begin{aligned}
\langle u, T^*_{\text{HMC}} v\rangle_{\mathcal{Q}} &= \langle u, rv\rangle_{\mathcal{Q}} + \int_\Gamma \alpha(x)\, v(\pi_1 g(x))\, u(\pi_1 x)^*\, f_{\text{can}}(x)\, dx \\
&= \langle ru, v\rangle_{\mathcal{Q}} + \int_\Gamma \alpha(g^{-1}Rx)\, v(\pi_1 x)\, u(\pi_1 g^{-1}Rx)^*\, f_{\text{can}}(g^{-1}Rx)\, dx \\
&= \langle T_{\text{HMC}} u, v\rangle_{\mathcal{Q}},
\end{aligned}
$$

so that $T_{\text{HMC}}$ is self-adjoint[70] if restricted to $L^2(\Omega)$, and the Markov operator $T_{\text{HMC}} = T^*_{\text{HMC}}|_{L^1(\Omega)}$ is given by (91). Following the same strategy as in the proof of Lemma 3.9, we again see that $T_{\text{HMC}}$ is a bounded operator in $L^2(\Omega)$ with $\|T_{\text{HMC}}\|_1 \leq 1$.

Under certain conditions, $T_{\text{HMC}}$ inherits not only self-adjointness on $L^2(\Omega)$ but also all other crucial properties of the spatial transition operator $T$. To see this, let us introduce:

1. *Momentum invertibility of $g$:* We call $g$ momentum-invertible if it satisfies the same conditions as a momentum-invertible flow, only that this time the sets $M(q)$ for which $\gamma_q(p) = \pi_1 g(q,p)$ is invertible, have to satisfy

$$\operatorname*{ess-inf}_{q\in\Omega} \int_{M(q)} \alpha(q,p)\, \mathcal{P}(p)\, dp \;=\; \eta \;>\; 0.$$

   Accordingly, $g$ is called momentum-invertible with uniform bound (with respect to the sets $M(q)$) if the Jacobian of the inverse of $\gamma_q$ is uniformly bounded on the $M(q)$.

2. *Mixing assumption for $g$:* For every $q_0 \in \Omega$ and every open set $O \subset \Omega$ there is an $n \in \mathbb{N}$ and a sequence of momenta $p_0, \ldots, p_{n-1} \in \mathbb{R}^d$ such that

$$q_n \in O, \quad \text{if} \quad q_k \;=\; \pi_1 g(q_{k-1}, p_{k-1}), \quad k = 1, 2, \ldots, n.$$

For smooth potentials we have the same situation as for the spatial transition operator $T$: If $g$ is momentum-invertible and satisfies the mixing assumption, the HMC transition operator $T_{\text{HMC}}$ has the following properties:

(1) The associated HMC Markov chain is irreducible, aperiodic and Harris recurrent. Thus, the Markov operator $T_{\text{HMC}} : L^1(\Omega) \to L^1(\Omega)$ is asymptotically stable (cf. Sec. 4.5) and the convergence (87) is guaranteed almost surely (cf. Thm. A.24).

[70] Again, this self-adjointness is associated to a detailed balance condition for the corresponding Markov chain [76].

(2) If $g$ additionally is momentum-invertible with uniform bound and $\Omega$ is bounded, then $T_{\text{HMC}} : L^2(\Omega) \to L^2(\Omega)$ has a decomposition $T_{\text{HMC}} = T^1_{\text{HMC}} + T^2_{\text{HMC}}$ into a strict contraction ($\|T^1_{\text{HMC}}\|_2 < 1$) and a compact operator $T^2_{\text{HMC}}$. Thus, $\lambda = 1$ is simple and dominant and isolated from the remaining part of the spectrum. Consequently, due to Thm A.31, a *central limit theorem* is valid, that is, the rate of convergence in (87) is at least of order $M^{-1/2}$ (not explicitly depending on the dimension of $\Omega$).

Obviously, the properties of the discrete flow $g$ result from the properties of the discretization $\Psi^t$ defining it. Consider $\Psi^t$ to be the Verlet discretization, and assume that the potential satisfies the periodicity assumption 4.43 of Sec. 4.7. Then, it is easy to check that $g$ is momentum-invertible with uniform bound and satisfies the mixing assumption, if only the internal stepsize $\tau/m$ of the discretization is small enough.

Remark 5.11 In order to see that $g$ is momentum-invertible, let $q_1$ be the spatial component of a single Verlet step $(q_1, p_1) = \Psi^{\Delta t}(q, p)$ with stepsize $\Delta t = \tau/m$, that is,

$$q_1 = q_0 + \frac{\Delta t}{2}(p_0 + p_1) \quad \text{with} \quad p_1 = p_0 - \Delta t\, DV\left(q_0 + \frac{\Delta t}{2} p_0\right).$$

This implies

$$D_p q_1 = \Delta t - \frac{\Delta t^3}{4} D^2 V\left(q + \frac{\Delta t}{2} p\right).$$

Let $d$ denote the maximal Jacobian of $DV$, i.e., set $d = \max_\Omega |\det(D^2 V)|$. Since $d < \infty$ for periodic potentials, we may use $\Delta t < 1/d$ and assume (without loss of generality) that $\Delta t < 1$. This implies $|\det D_p q_1 - \Delta t| < \Delta t^2/4 < 1/4$. Hence, a single Verlet step with $\Delta t < 1/d$ is invertible with respect to $p$ for *every* momentum. This implies momentum invertibility of $g$.

Remark 5.12 In order to understand that $g$ satisfies the mixing assumption, choose arbitrary $q, q_1 \in \Omega$. It is sufficient to show that, for every single Verlet step, we find some momentum $p$ yielding $q_1 = \pi_1 \Psi^{\Delta t}(q, p)$. To see that this is true, denote $p = p_0 + \delta p$ with $p_0 = (q_1 - q)/\Delta t$, and observe that $\delta p$ has to satisfy

$$\delta p = \frac{\Delta t}{2} DV\left(q + \frac{\Delta t}{2}(p_0 + \delta p)\right) = h(\delta p).$$

Then, denote $C_k = \max_\Omega |D^k V|$, $k = 1, 2$, and $B = \{\delta p : |\delta p| \le \Delta t C_1/2\}$ so that the function $h$ maps the ball $B$ onto itself. It is easy to see that, if

$\Delta t^2 < 4/C_2$, we have $|D_{\delta p} h| < 1$ so that $h$ is contractive on $B$. Thus, the fixed point theorem guarantees that there indeed is some $\delta p_* \in B$ such that $\delta p_* = h(\delta p_*)$.

As indicated above, it is also possible to analyze HMC in close analogy to the general Metropolis Monte-Carlo algorithm. In the last years the problem of convergence results for Metropolis MC in continuous state space has attracted much attention, in particular concerning criteria for the geometric ergodicity of the chain and corresponding central limit theorems.[71]

We will not longer focus on analyzing HMC, because, for applications to realistic molecular systems, we cannot use HMC itself but have to introduce a *generalized* HMC variant in order to avoid some "trapping problems" (cf. next paragraph).

### 5.4.4 HMC with Adaptive Temperature

As is widely known, MC simulations for ensemble averages may suffer from possible "critical slowing down" [69]. This phenomenon occurs when the iteration $x_k \to x_{k+1}$ gets trapped near a local potential minimum due to high energy barriers so that a proper sampling of the phase space within reasonable computing times is prevented. Typically, this also happens to HMC applications [99]. Therefore, a novel approach combining HMC with the reweighting technique [31, 9] has been developed (see Fischer, Cordes, and Schütte [35]). This HMC variant generates the distribution of a mixed-canonical ensemble composed of two canonical ensembles at low and high temperature. Its analysis shows an efficient sampling of the canonical distribution at the low temperature, whereas the high temperature component facilitates crossing of the crucial energy barriers. We will therefore call this variant "adaptive temperature HMC" (ATHMC) in the following. The update steps using high temperature momenta have to be *reweighted* in order to guarantee overall convergence to the canonical distribution to the low temperature. Apart from this reweighting, the approximation of the transition probabilities due to (88) and (89) remains unchanged. In other words:

*The necessity of introducing generalizations of HMC is caused by the existence of almost invariant sets*: If there are almost invariant sets, $B$ and $C$, with very small transition probability $w(B, C, \tau)$, then both the Markov chain (85) associated with the transition operator and the original HMC Markov chains need a huge number of iterations in order to produce sufficiently many transitions between $B$ and $C$.[72] This problem is circumvented by introducing Markov

[71] For the analysis of the rate of convergence of MC Markov chains compare, e.g., [107, 77] or, for recent results, check the MCMC preprint server under ULR http://www.stats.bris.ac.uk/MCMC/.

[72] Compare Thm. A.31: If the variance $\sigma^2(\mathcal{A})$ therein gets too large, the desired convergence of the mean values is "slowed down" by a large constant. Due to the formula (106) in Thm. A.31, $\sigma^2(\mathcal{A})$ can be large especially if $T_{\text{HMC}}$ has an eigenvalue very close to one, i.e., if the HMC chain has an almost invariant set (which, hopefully, approximates an almost invariant set of the original flow).

chains that facilitate such transitions but which then have to be reweighted in order to yield samplings of the original canonical distribution.

For details of the ATHMC construction, the reader is referred to our article [35]. In this article a convergence result for ATHMC is presented in close analogy to (and on the same non-rigorous level as) the already mentioned HMC convergence result from [76]. Moreover, the convergence has successfully been checked in cases where analytical expressions for some expectation values are available (cf. [33]). Sec. 6 presents the performance of ATHMC in comparison to the original HMC variant.

ATHMC is not the only possibility to enforce barrier crossing. For example, A. Fischer recently developed another technique based on reweighted HMC, the so-called *Scaled Potential Hybrid Monte-Carlo* (SPHMC) variant [34], which seems to be more flexible and efficient. The reader might also notice, that other recent developments based on the original Monte-Carlo Markov chain construction also produced certain "barrier-crossing" variants.[73] It should again be emphasized that we may also apply every of these MC variants in order to evaluate $T_n$.

## 5.5 Algorithmic Considerations

We should not close this section without some final comments:

**Consequences of HMC Errors** In Secs. 5.1 to 5.3 we assumed that the exact transition matrix $T_n$ for some given box covering is available. But with the techniques introduced in Sec. 5.4 we are only able to approximate its entries, for example in the sense that denominator and numerator of $T_{n,jl} = t_{jl}/\rho_j$, both, are computed with some accuracy $\delta$, that is,

$$T^{\mathrm{HMC}}_{n,jl} = \tilde{t}_{jl}/\tilde{\rho}_j, \quad \text{with} \quad |\tilde{t}_{jl} - t_{jl}| < \delta \text{ and } |\tilde{\rho}_j - \rho_j| < \delta.$$

As a consequence, we may rewrite $T^{\mathrm{HMC}}_n$ as

$$T^{\mathrm{HMC}}_n = T_n + E \quad \text{with} \quad |E_{jl}| \leq \delta\,(1 + t_{jl})/\rho_j.$$

Hence, the evaluation of $T_n$ via HMC introduces an error matrix. Fortunately, this perturbation can be included in the perturbation analysis of Sec. 5.3 as an additional contribution with

$$\|\mathbf{D}^2_\rho E\|_\infty \leq \delta\,(1 + \|T_n\|_\infty) \leq 2\delta. \tag{92}$$

Hence, under the assumptions of Sec. 5.3, the identification of the conformational subsets will not be dramatically perturbed by the HMC errors $E$: In

[73] There are many examples which support barrier crossing like, e.g., the multicanonical algorithm [51], $1/k$ sampling [55], simulated tempering [74], J-walking [41], the fluctuating potential methods [71] or other potential smoothing techniques [85, 9]. Since we are herein interested in methods sampling the canonical ensemble, one needs to check carefully whether these techniques in fact produce Markov chains which, after reweighting, *efficiently* sample the canonical distribution.

Sec. 5.3, we discussed the "coupling-induced" perturbation of the eigenvectors used to identify the conformational subsets. Whenever (92) is satisfied, we may see the perturbations via HMC as additional coupling-induced perturbations.

**Sensitivity to Perturbations** In real-life applications, it is a tremendous task to control the accuracy of the HMC sampling. Whenever the invariant density is very small in some of the discretization boxes, the HMC perturbations of the corresponding entries of $T_n$ are in danger of being relatively large. We thus have to ask whether the identification of almost invariant sets will depend sensitively on such perturbations?

In the process of the derivation of the identification algorithm we already distinguished between the "cores" of the conformational subsets, which are particularly stable with respect to the perturbation of the eigenvectors, and all other states, which have to be assigned to these cores without explicit consideration of the (perhaps heavily perturbed) entries of the eigenvectors at these states. As long as (92) is satisfied, the identification of these "conformational cores" is relatively insensitive to the perturbations, while the assignment of the remaining states and thus the decomposition problem (that is, the full decomposition of the configurational state space into conformational subsets) obviously is sensitive.

The situation is comparable with the problem of separating domains of positive and negative entries of some function by finding its zeroes: if the evaluation of the function is significantly perturbed, the exact decomposition of the domain of interest may depend extremely sensitive on such perturbations (particularly if the function itself is small in the neighborhood of its zeroes) while we are still able to identify the "cores" of positive/negative domains.

Fortunately, the invariant density will in general be large only in the conformational cores which implies that the remaining states are statistically insignificant and we are content with the identification of the conformational cores. Nevertheless, the question remains whether the transition probabilities might depend sensitively on the details of the assignment of the remaining states. This may happen whenever there is a conformational subset with significantly small probability to be within. However, in all realistic cases analyzed up to now, the probability $w(B, B, \tau)$ to stay within some conformational subset $B$ has been significantly insensitive to perturbations.

**Cluster Analysis and Graph Partitioning** We have to distinguish between our dynamical definition of conformations via almost invariant sets and the static definition via clusters of geometrically similar configurations. Most chemical approaches to the identification of conformations use this static definition and determine the conformational subsets by means of *cluster analysis* of some time series produced by long-term MD simulations (e.g. via the associated covariance matrix as in [4]) or by means of *spectral graph theory* via some geometric similarity measure and decomposition of the associated Laplacian matrix (cf. [32, 54, 58]). Within such approaches, the dynamical information is neglected. However, the determined clusters include crucial information about significant geometrical differences between certain subsets of configurations and can thus be used as indicators for the choice of the essential variables and for optional additional refinements of the box discretization.

We can in fact apply cluster analysis in the context of our identification algorithm via certain eigenvectors $(v^1, \ldots, v^k)$ of the transition matrix: consider again the vector $b_j = (v_j^1, \ldots, v_j^k)$ of the entries of the eigenvectors for discretization box $B_j$; define some similarity measure for these vectors; and determine the clusters of boxes $B_j$ with similar vectors $b_j$ via standard techniques. The resulting conformational subsets are nearly identical with those evaluated via the approach presented in Sec. 5.3, cf. [58].

# 6 Numerical Experiments

## 6.1 Conformations of n-Pentane

In this section, the performance of the above derived algorithm in application to the n-pentane molecule $CH_3(CH_2)_3CH_3$ is presented. It is illustrated that the algorithmically identified almost invariant sets are in perfect agreement with the chemically observed conformations (cf. Fig. 15).

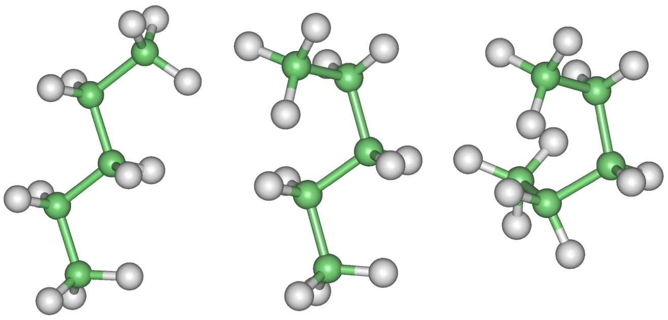


Figure 15: Different conformations of n-pentane: From the left to the right: trans-trans, trans-gauche, gauche-gauche orientations.

For modelling the n-pentane molecule, we use the united atom model (cf. Fig. 3 in Sec. 2.1.2 above) with the typical bound length and bond angle potentials, and a Lennard-Jones potential modelling the interaction between the first and the last of the united "atoms". The dihedral angle potentials are chosen according to [89], cf. Fig. 3 in Sec. 2.1.2 above. The form of the dihedral angle potential shows three different minima corresponding to the trans and gauche orientations of the angles. The vibrational frequencies induced by these potentials are considerably smaller than those induced by the bond interactions. Consequently, the conformations of the n-pentane molecule are described in terms of these dihedral angles, i.e., they are the *essential variables* of n-pentane.

Figures 16 to 20 below illustrate the execution of the algorithm for the temperature $\mathcal{T} = 300$K. The discretization boxes are constructed via uniform decomposition of the possible values $[0, 2\pi] \times [0, 2\pi]$ of the two dihedral angles $\omega_1$ and $\omega_2$ in $n = 20 \times 20 = 400$ boxes. The HMC sampling has been realized using the Verlet time discretization with a subtrajectory length of $\tau = 160$fs. Fig. 16 shows the resulting sequences of HMC steps in terms of the dihedral angles.

We observe frequent transitions between the different "trans" and "gauche" orientations of both angles. This observation nicely illustrates that it is not sufficient to know the probability to *be within* a particular orientation of the angles but that the essential dynamical information is given by the probability to *stay within* it until a transition into another orientation occurs.

From such an HMC sampling with $M = 200.000$ steps we computed the *transition matrix* $T_n$ by the procedure explained in Sec. 5.4. Within this sam-

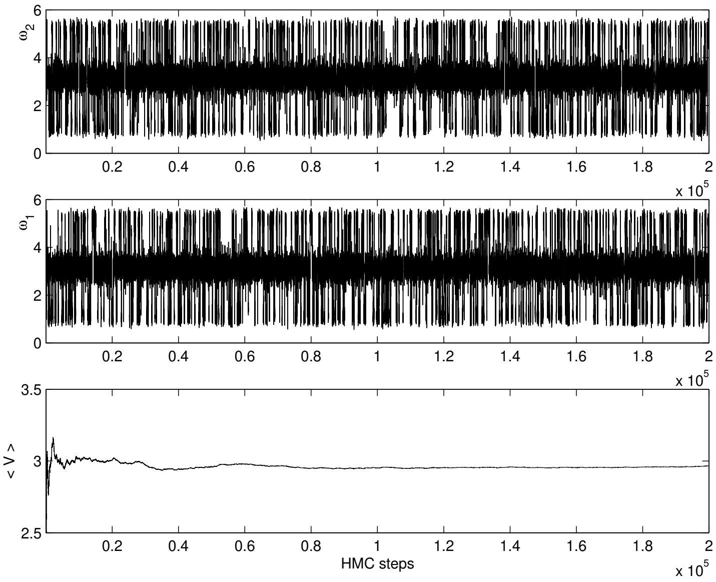



Figure 16: HMC simulation of n-pentane for $\mathcal{T} = 300$K. From top to bottom: The two dihedral angles (in radiant) versus the step number and the convergence of the potential energy expectation $\langle V \rangle$.

pling length, the HMC method produced a more than sufficient sampling of the canonical density (see the equilibration diagram on bottom of Fig. 16) — the question of whether $M$ could be smaller for HMC will be discussed below.

When repeating such a simulation for some smaller temperature (e.g., $\mathcal{T} < 250$K), we observe the trapping problem mentioned in Sec. 5.4.4: transitions between the distinct minima of the dihedral angle potential become rare events and the HMC simulation length $M$ has to be increased further and further. Then, ATHMC indeed helps to avoid drastic increases of $M$: it yields comparably reliable result with much smaller values of $M$ than HMC (cf. [33]).

From Sec. 5.1, Prop. 5.2, we know that the discrete invariant density $(\rho_k)$, $k = 1, \ldots, n$ is given by the left eigenvector of $T_n$ for the largest eigenvalue $\lambda_1 = 1$. The result produced herein is given in Fig. 17. As expected, the invariant density shows distinct local maxima at the minima of the dihedral angle potentials.

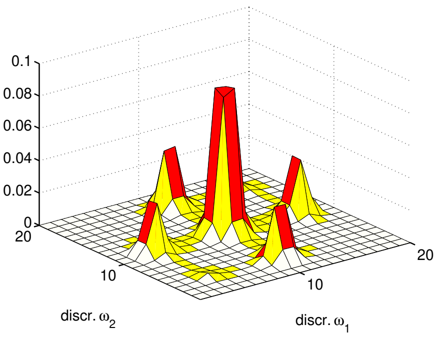



Figure 17: Discrete canonical distribution $\rho$ for n-pentane versus the indices of the discretization boxes of the two dihedral angles $\omega_1$ and $\omega_2$. $\mathcal{T} = 300$K.

### 6.1.1 Conformations

Following Sec. 5.3.2, the chemical conformations are analyzed via the right eigenvectors corresponding to an eigenvalue cluster near $\lambda = 1$. For this purpose, the 10 eigenvalues of $T_n$ with largest absolute value are arranged:

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_k$ | 1 | 0.986 | 0.984 | 0.982 | 0.975 | 0.941 | 0.938 | 0.599 | 0.590 | $-0.562$ |

The first nine ones are positive. From the 10th one on negative eigenvalues appear frequently. As can be seen, a first spectral gap arises between $\lambda_5$ and $\lambda_6$, and an even more significant one between $\lambda_7$ and $\lambda_8$. We decide to base the identification process on the first 7 eigenvectors (the results for the case $k = 5$ are comparable and can be found in [24]).

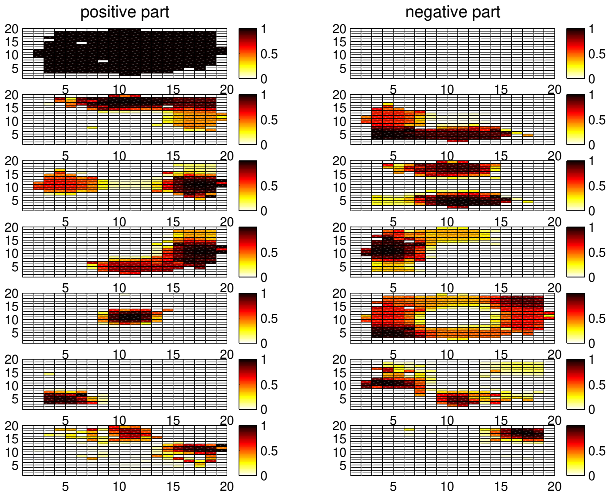



Figure 18: Positive and negative parts of the first seven *right* eigenvectors (right) scaled with respect to the maximum norm as described in Sec. 5.3.2 above. The zero entries in the first eigenvector (associated with the eigenvalue $\lambda = 1$) correspond to discretization boxes which were not visited by the hybrid Monte Carlo process, indicating that the probabilities to be within these boxes may be neglected. $\mathcal{T} = 300$K.

These first seven eigenvectors are illustrated in Fig. 18. Of course, the first eigenvector, associated with $\lambda_1 = 1$, is just a flat plateau (ignoring zeroes for cut-off states; that is, the state space of the associated Markov chain is represented only by the discretization boxes belonging to the plateau). The other six right eigenvectors contain more information. We can distinguish between different plateau … ates. The main pr… lassifying the entr… ero. The resulting…

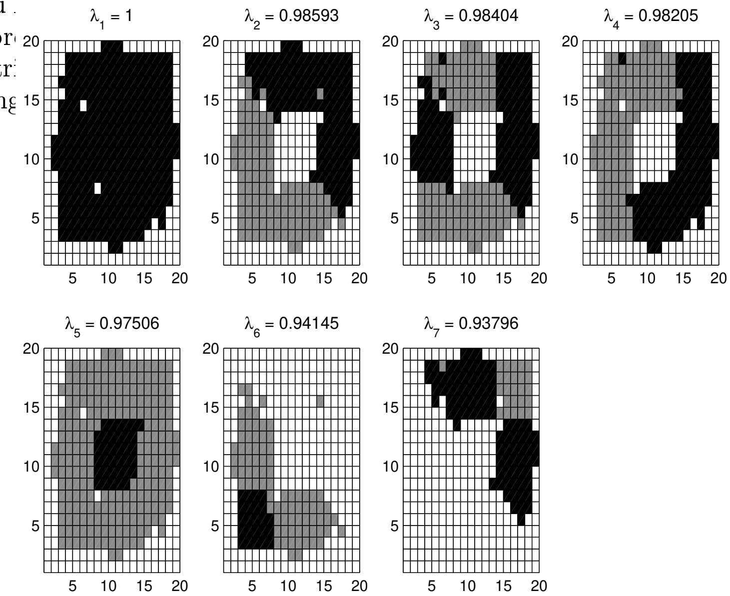



Figure 19: Schematic plot of the right eigenvectors corresponding to the seven largest eigenvalues $\lambda_1, \ldots, \lambda_7$ of $T_n$ versus the indices $(1, \ldots, 20) \times (1, \ldots, 20)$ of the discretization boxes of the two dihedral angles $\omega_1$ and $\omega_2$. Positive entries of the eigenvectors are indicated by black boxes, negative entries by gray boxes and white boxes indicate almost zero entries. $\mathcal{T} = 300$K.

From this picture one already can guess that the identification of conformational subsets via sign structures leads to the subsets shown in Fig. 20. As can be observed the automatic procedure in fact supplies the chemically expected information. After identifying the conformations, the corresponding probabilities to stay within each conformational subset $C_j$ composed of the discretization boxes $B_k$ with indices $k \in I_j$ can be computed due to

$$w(C_j, C_j, \tau) \;=\; \frac{1}{\sum\limits_{l \in I_j} \rho_l} \sum_{k,l \in I_j} \rho_k \, T_{n,kl}. \tag{93}$$

The resulting values $p_j = w(C_j, C_j, \tau)$ are also given in Fig. 20. We observe that the trans/trans conformation is slightly more stable than the different trans/gauche and gauche/trans conformations. As expected, the two gauche/gauche conformations are clearly less stable.

As already emphasized above, the probabilities to *stay within* should *not* be confused with the probability to *be within* a conformation, which is already given

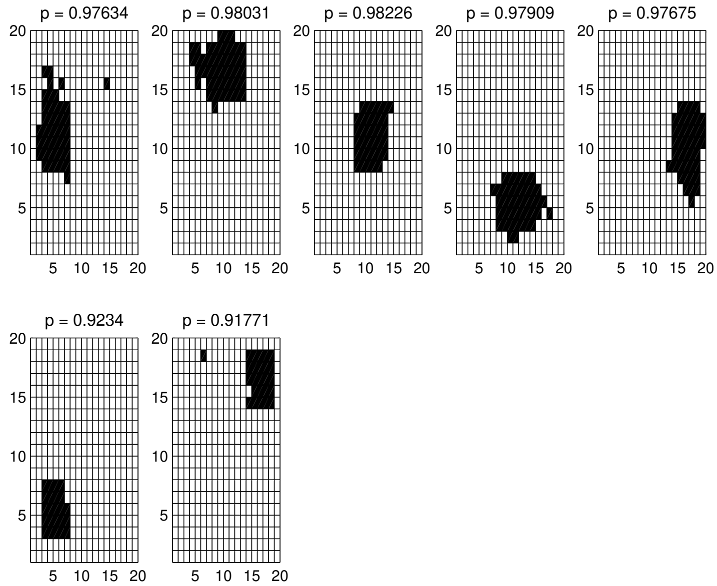



Figure 20: Almost invariant sets for $\mathcal{T} = 300$K. The numbers $p$ on top of each figure are the probabilities to stay within the corresponding subsets during the time span $\tau$. From the left hand side on top to the right hand side below we see the -gauche/trans, trans/+gauche, -gauche/-gauche, trans/trans, trans/-gauche, +gauche/trans, and +gauche/+gauche conformations (cf. Fig. 15).

by the invariant density (cf. Fig. 17). In the table below, these two different probabilities are enlisted for each of the conformational subsets shown in Fig. 20 (±g and t denote the ±gauche and trans orientations):

| conformation | -g/t | t/+g | -g/-g | t/t | t/-g | +g/t | +g/+g |
|---|---|---|---|---|---|---|---|
| prob. to be within | 0.120 | 0.132 | 0.012 | 0.473 | 0.117 | 0.132 | 0.013 |
| prob. to stay within | 0.976 | 0.980 | 0.910 | 0.982 | 0.979 | 0.970 | 0.865 |

The probability to be within the +gauche/-gauche or -gauche/+gauche orientations is less than 0.0005, showing that they are irrelevant in this context.

Indeed, the probabilities $w(C_j, C_j, \tau)$ to stay within each conformational subset $C_j$ are nothing but the diagonal entries of the coupling matrix $W \approx \mathrm{Id}$ obtained in this procedure. Correspondingly, the associated error indicator matrix from Sec. 5.3.2 contains only small entries (all entries smaller than $10^{-2}$; for details see [24]) indicating that the identification process is reliable.

### 6.1.2 Parameter Sensitivity

The presented results surely depend on a number of crucial parameters, some of them being of physical nature (e.g., the temperature $\mathcal{T}$), others being introduced by the algorithm (e.g., the number $n$ of discretization boxes or the length $M$ of the HMC sampling). We want to emphasize that the algorithm as it stands now is far from being perfectly tuned. We thus can only present some experiences from numerical experiments for the n-pentane molecule and some other comparably small systems.

At first, let us consider the dependence of the conformations on the temperature $\mathcal{T}$. Varying the temperature between $\mathcal{T} = 200$K and $\mathcal{T} = 600$K we do not observe an influence on the identified conformations. But, as to be expected, the probabilities to stay within these conformations are decreasing with increasing $\mathcal{T}$: Fig. 21 shows the corresponding decrease of the nine largest eigenvalues of the transition matrices $T_n = T_n(\mathcal{T})$. It also illustrates that in all cases tested so far there exists some distinct *spectral gap* between the seven largest eigenvalues used to iden ... uining part of the spectrum.

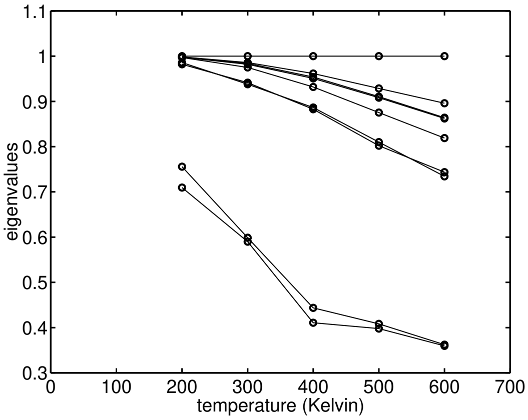



Figure 21: Temperature dependence of the nine largest eigenvalues of the transition matrix $T_n$.

The present version of the algorithmic realization does not include any automatic mechanism for controlling the length $M$ of the HMC sampling. If, for fixed temperature and spatial discretization, the number of steps is decreased from $M = 200.000$ down to $M = 50.000$, we observe that the approximation quality of the inv ... ponds to a slowly increasing distor ... sets.

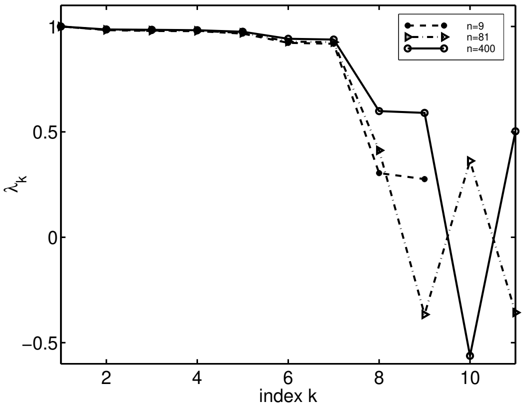



Figure 22: Sensitivity of the eigenvalues of $T_n$ with largest modulus for different uniform discretizations of $[0, 2\pi]^2$ with $n = 3 \times 3 = 9$ boxes (dashed line), $n = 9 \times 9 = 81$ (dashed-dotted), and $n = 20 \times 20 = 400$ boxes (dense line). Note that the seven largest eigenvalues – only these are used for the identification of the conformations – remain almost unperturbed if the grid gets coarser.

### 6.1.3 Dependence on Discretization

Finally, let us illustrate an extremely important property of the presented algorithm, the stability of the results even when significantly coarser discretizations are used. For the n-pentane molecule we indeed can reduce the decomposition of the discretization domain from $n = 20\times20$ boxes to $n = 3\times3$ boxes but the algorithm still identifies approximately the same conformations and nearly the same probabilities (both to stay and to be within). The reason for this is illustrated in Fig. 22: since the HMC procedure samples the phase space independent of the spatial discretization, the seven largest eigenvalues of the transition matrix $T_n$ are only insignificantly perturbed when the number of discretization boxes is reduced.

## 6.2 Conformations of a Trinucleotide

The proposed approach to identify conformations was applied to the triribonucleotide adenylyl(3'-5')cytidylyl(3'-5')cytidin (r(ACC)) model system in vacuum (see Fig. 4 on page 11 and Fig. 23 below). The physical representation of the molecule ($N = 70$ atoms) has been based on the GROMOS96 extended atom force field [109].

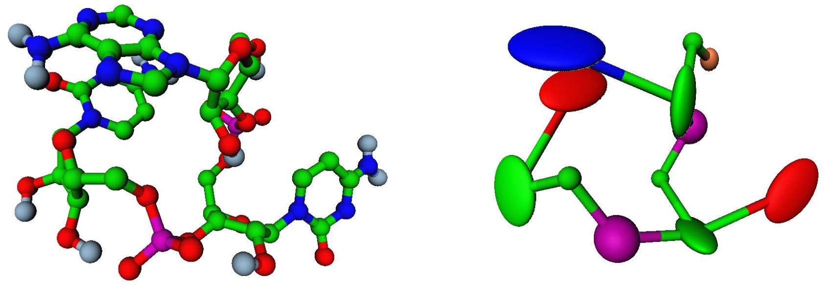


Figure 23: Configuration of the trinucleotide from Fig. 4. Left: Full ball-and-stick representation. Right: Simplified representation using ellipsoids for representing nearly rigid subgroup of the molecule.

It is beyond the scope of this section to discuss this application in detail or to judge the chemical relevance of the results. For such additional aspects the reader is referred to the article [58]. On the contrary, the following considerations should be understood as demonstration of, first, the superiority of ATHMC over HMC, second, the process of choosing essential variables in some realistic example, and, third, the illustration of typical results of the algorithmic procedure.

**Sampling of the Canonical Density** The simulation data were generated by means of an ATHMC sampling of the canonical density at $T = 300$K. The subtrajectories of length 80 femtoseconds were computed by means of the Verlet discretization with a stepsize of 2fs. For these parameters, HMC simulations typically require thousands of iterations only to leave the neighborhood of the initial configuration. Application of ATHMC (with adaptive temperatures between 300K and 400K) circumvents the problem: one observes frequent transitions in the crucial dihedral angles of the molecule (for details see [35]). The simulation was divided into 4 Markov chains, each starting with a different state chosen from a high temperature run at 500K, which allowed the molecule to move into different conformations. The sampling took about 12h on a workstation with MIPS R10.000 processor. It was terminated by a convergence indicator [43] associated with the potential energy and all 37 dihedral angles after 320.000 steps, resulting in the sampling sequence $q^{(1)}, \ldots, q^{(S)}$, $S = 32.000$ (considering only every 10th step). Since the temperature can change during the ATHMC run, each configuration is connected with a reweighting factor with respect to the canonical ensemble at 300K.

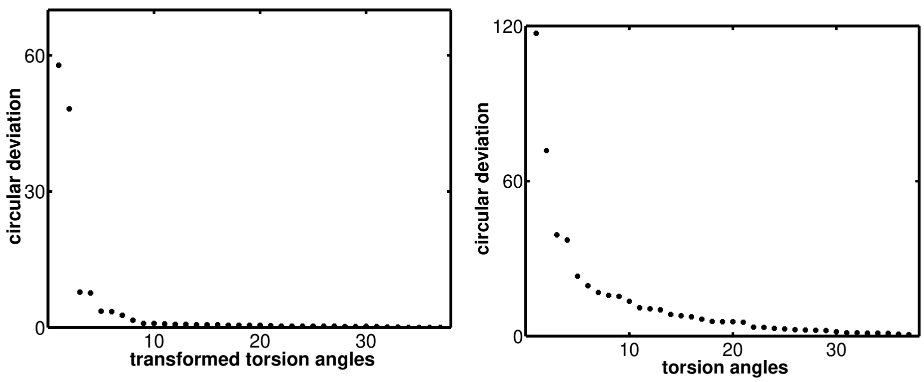



Figure 24: Top: circular deviation of the *transformed* dihedral angles ordered by magnitude (left) and circular deviation of the original dihedral angles (right).

**Essential Dihedral Angles** Since essential degrees of freedom should only reflect internal fluctuations of the molecule, we only consider the 37 dihedral angles of the r(ACC) molecule. The simulation data contains strong statistical correlations between the dihedral angles caused by the highly correlated motion of the molecule. In order to identify some minimal set of "independent" essential variables, we first have to remove the correlations, i.e., we have to introduce a set of *uncorrelated* "transformed dihedral angles". This is done via the following procedure going back to Amadei et al. [4]:

The correlations between atomic motions within the simulation data are expressed by the covariance matrix $C$.[74] Since $C$ is symmetric, it can always be

[74]To analyze the simulation data in terms of the dihedral angles we have to apply statistical methods for circular data [36, 37]; see [58] for resulting definition of the covariance matrix.

diagonalized, i.e., there is an orthonormal matrix $U$ such that $C = U^T \Lambda U$ with $\Lambda$ being the diagonal matrix whose entries are the eigenvalues of $C$. The matrix $U$ defines the transformation of the original dihedral angles into the uncorrelated transformed dihedral angles. The matrix $\Lambda$ is connected to the systems constraints in the following way [4]: Transformed dihedral angles corresponding to zero or nearly zero eigenvalues behave effectively as constraints; they have narrow Gaussian distributions with zero mean and do not contribute significantly to the fluctuations. In contrast to that, transformed dihedral angles corresponding to large eigenvalues have large deviations from their mean position, i.e., they belong to important fluctuations. Often, only a few coordinates see such important fluctuations; these are called essential degrees of freedom. In practice, one has to specify a set of largest eigenvalues of $C$, which, often, can only be done heuristically.

The transformation process for the dihedral angles based on the simulation data for r(ACC) is exemplified in Fig. 24 and Fig. 25. Figure 24 shows the circular deviations of the original and transformed dihedral angles in decreasing order of magnitude. Only the first four transformed dihedral angles have relevant circular deviation *and* are far from being Gaussian shaped (see Fig. 25), while the remaining transformed dihedral angles are Gaussian like.

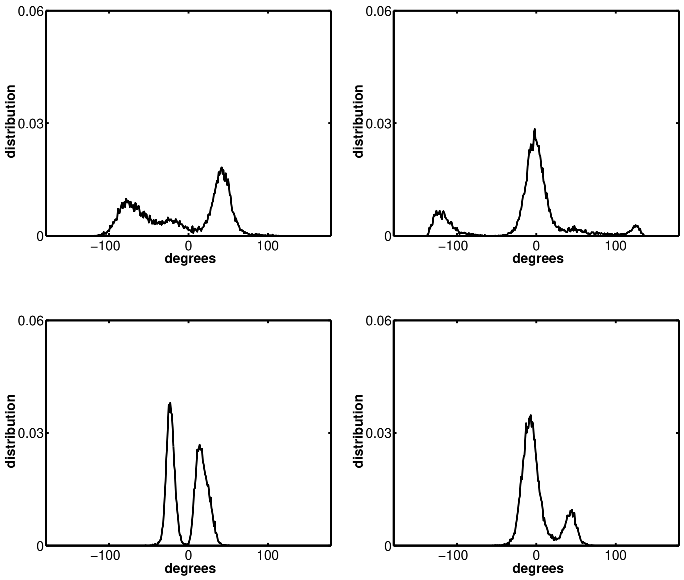



Figure 25: Distribution of the four essential dihedral angles. The distributions at the top allow to identify three maxima each, while there are two maxima for each distribution at the bottom.

**Single Configurations Representing Conformations** In order to identify representative configurations, we determine the maxima for each distribution of the essential dihedral angles. These maxima have been grouped to $3 \times 3 \times 2 \times 2 = 36$ combinations, from which we have selected four representative configurations to visualize characteristic differences (see Fig. 26).

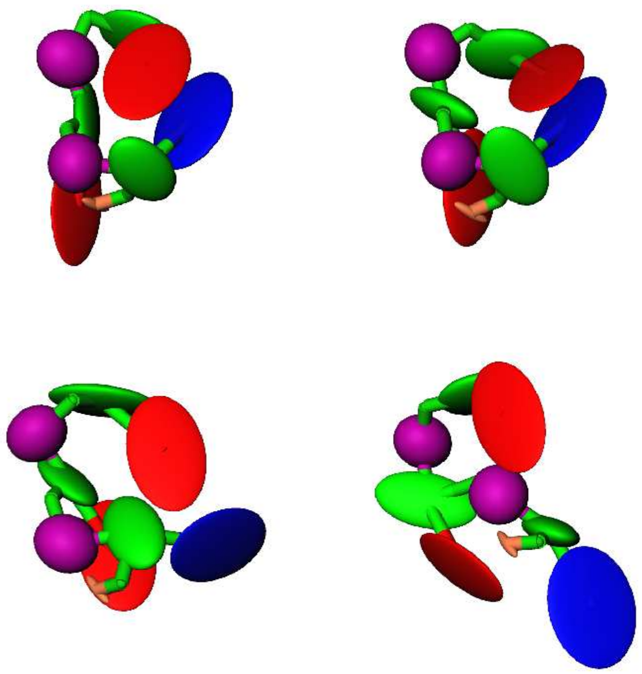


Figure 26: Different configurations representing the four dominant conformations of the trinucleotide in the simplified representation from Fig. 23.

**Transition Matrix** The dynamical fluctuations within the canonical ensemble were approximated by integrating four short trajectories of length $\tau = 80$fs starting from each sampling point $q^{(1)}, \ldots, q^{(S)}$. To facilitate transitions, analogous to the ATHMC sampling, the momenta were chosen according to the momenta distribution $\mathcal{P}(p)$ for 4 different temperatures between $300K - 400$K and reweighted afterwards. This resulted in a total of $4 \times 32.000 = 128.000$ transitions. This calculation took less than 25 % of the total computing time.

The configurational space was discretized into boxes $B_1, \ldots, B_d$, by means of all four essential degrees of freedom (see Fig. 25) resulting in $d = 36$ discretization boxes. Then the $36 \times 36$ transition matrix $P$ was computed based on the 128.000 transitions taking the different weighting factors into account. Since every box had been hit by sufficiently many transitions, the statistical sampling was accepted to be reliable. The computation of the eigenvalues of $T$ near 1 yielded a cluster of eight eigenvalues with a significant gap to the remaining part of the spectrum:

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ... |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_k$ | 1.000 | 0.999 | 0.989 | 0.974 | 0.963 | 0.946 | 0.933 | 0.904 | 0.805 | ... |

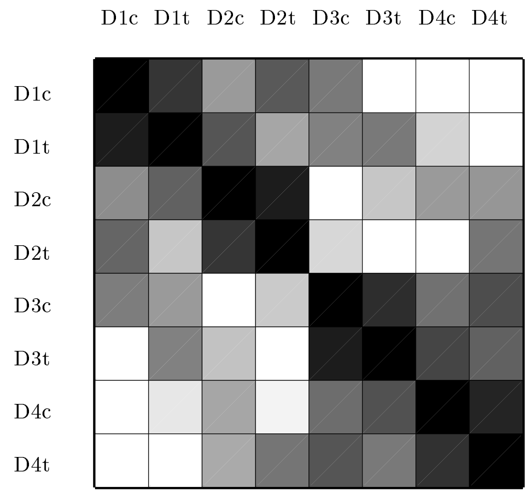



Figure 27: Schematical visualization of the transition probabilities $w_{kl}(\tau)$ between the conformation $D_{\text{from}}$ (row) and $D_{\text{to}}$ (column). The colors are chosen according to the logarithm of the corresponding entries; black: $w_{kl}(\tau) \approx 1$, white: $w_{kl}(\tau) \approx 0$.

**Identification of Conformations** Finally, the conformational subsets were computed based on the corresponding eight eigenvectors of $T$ via the identification algorithm presented above yielding eight conformations.

The conformational subsets identified turned out to be rather insensitive to further refinements of the discretization. The weighting factors within the canonical ensemble and the meta–stability of the eight identified conformations are given in the following table:

| conformations | D1c | D1t | D2c | D2t | D3c | D3t | D4c | D4t |
|---|---|---|---|---|---|---|---|---|
| weighting factor | 0.107 | 0.011 | 0.116 | 0.028 | 0.320 | 0.038 | 0.285 | 0.095 |
| meta–stability | 0.986 | 0.938 | 0.961 | 0.888 | 0.991 | 0.949 | 0.981 | 0.962 |

The transition probabilities between the different conformations are visualized schematically in Fig. 27 above (page 112). The matrix allows to define a hierarchy between the conformations, which is inherent to the algorithm. On the top level, there are two conformations, D1&D2 and D3&D4 corresponding to the two $4\times4$ blocks on the diagonal of $T$. On the next level, each of these conformations split up into two subconformations yielding D1, ... ,D4. On the bottom level, each conformation is further divided into a core (c) and a transition (t) part. The evaluation of the transition matrix together with the execution of the identification algorithm took less than 2% of the computing time required for evaluation of the simulation data.

We use a multidimensional scaling plot (Fig. 28 below) to visualize the conformations [10]. The 2d–plot shows a 2d least squares approximation of the $3N = 210$ dimensional position space in the sense that neighboring points correspond in general to structurally similar configurations, while distant points reflect in general structural differences.

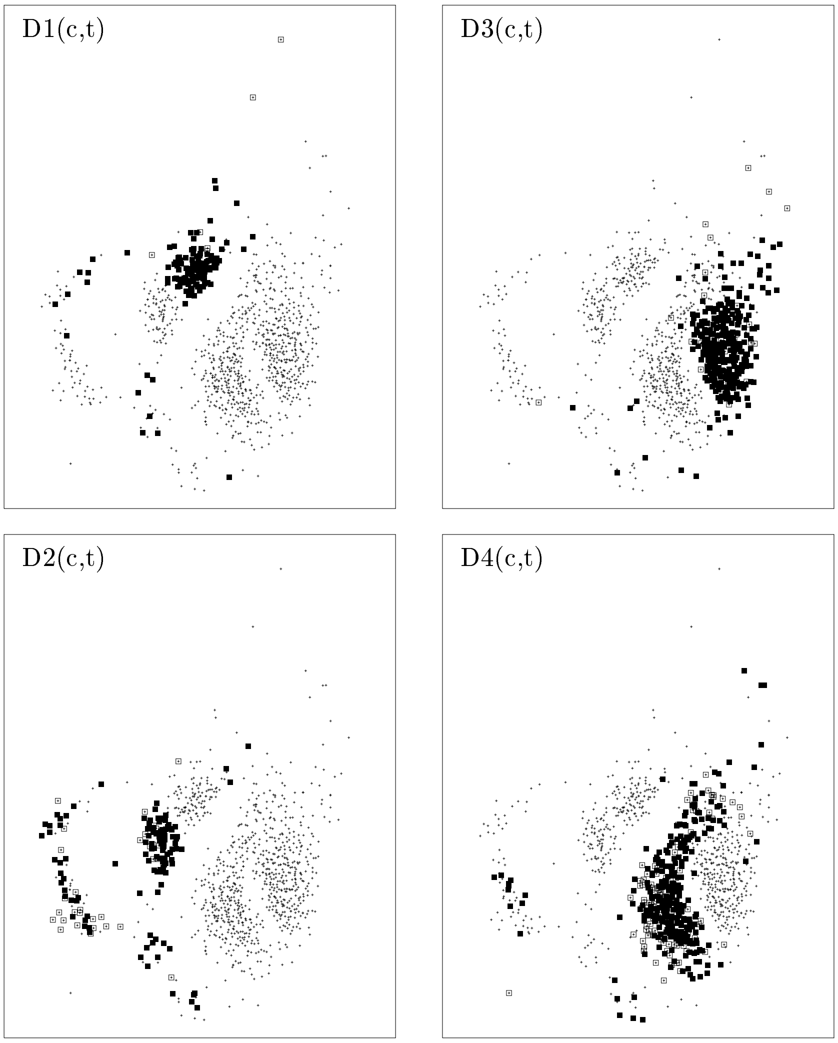



Figure 28: 2d plot of the four conformations D1, ... ,D4 (squares). The distinction between open and filled squares indicates a further splitting into eight conformations resulting from a partition into a core(c) and a transition(t) conformation.

# List of Symbols

Phase space operations:

| | |
|---|---|
| $\Phi^t$ | flow of Hamiltonian system (2) |
| $\pi_1$ | projection $\pi_1(q,p) = q$ |
| $\pi_2$ | projection $\pi_2(q,p) = p$ |
| $R$ | momentum inversion $R(q,p) = (q, -p)$ |
| $\vartheta$ | essential variables $\vartheta : \Gamma \to \Theta$, cf. Sec. 3.5 |
| $\xi$ | periodicity map, cf. page 24 |
| $V = V(q)$ | potential function, cf. (1) |

Sets:

| | |
|---|---|
| $\Gamma$ | phase space $\Gamma \subset \mathbb{R}^{2d}$ |
| $\Omega$ | position space $\Omega \subset \mathbb{R}^d$, cf. Sec. 2.1.1 |
| $\Gamma(\theta)$ | embedded submanifold induce by essential variables, cf. Sec. 3.5 |
| $\Gamma(A)$ | phase space fiber $\{x = (q,p) \in \Gamma,\ q \in A\}$ |
| $\Theta$ | essential phase space, cf. Sec. 3.5 |
| $\sigma(T)$ | spectrum of linear operator $T$, for definition see Appendix B |
| $\chi_B$ | characteristic function of a set $B$ |
| $M(q)$ | sets for momentum invertibility, cf. Def. 4.1 |

Operators:

| | |
|---|---|
| $T$ | spatial transition operator, cf. (20) |
| $T_\vartheta$ | transition operator w.r.t. essential variables, cf. (28) |
| $R_\vartheta$,$B_\vartheta$ | restriction and prolongation operators, s. Def. 3.13 |
| $T_1$,$T_2$ | parts of the decomposition of the spatial transition operator, cf. (37) and Lemma 4.8. |
| $k(q,y)$ | transition kernel of partial transition operator $T_2$, cf. (38) |
| $T_{1,\vartheta}$,$T_{2,\vartheta}$ | parts of the decomposition of the transition operator $T_\vartheta$, cf. Lemma 4.34. |
| $k_\vartheta(\theta,\phi)$ | transition kernel of partial transition operator $T_{2,\vartheta}$, cf. (57) |

Distributions:

| | |
|---|---|
| $f_{\mathrm{can}}$ | canonical density in phase space, cf. (11) |
| $\mathcal{Q}$ | normalized canonical density in position space, cf. (11) and (12) |
| $\mathcal{P}$ | normalized canonical density in momentum space, cf. (11) and (12) |
| $F$ | reduced density in position space, cf. (19) |
| $F_\vartheta$ | reduced density in essential phase space, cf. Sec. 3.5 |

Spaces:

| | |
|---|---|
| $L^p_F(\Omega)$ | $F$-weighted function space over position space $\Omega$, $p = 1,2$, see Sec. 3.1 |
| $L^p(\Omega)$ | space $L^p_F(\Omega)$ with weight $F = \mathcal{Q}$ for canonical density |
| $L^p_{F_\vartheta}(\Theta)$ | $F_\vartheta$-weighted function space over essential phase space $\Theta$, $p = 1,2$, see Sec. 3.5 |
| $\langle\cdot,\cdot\rangle_F$ | $F$-weighted scalar product in $L^2_F(\Omega)$, see Sec. 3.1 |
| $\langle\cdot,\cdot\rangle_{F_\vartheta}$ | $F_\vartheta$-weighted scalar product in $L^2_{F_\vartheta}(\Theta)$, see Sec. 3.5 |
| $\langle\cdot,\cdot\rangle_{\mathcal{Q}}$ | scalar product $\langle\cdot,\cdot\rangle_F$ with weight $F = \mathcal{Q}$ for canonical density |
| $\|\cdot\|_{p,F}$ | $F$-weighted norm in $L^p_F(\Omega)$, see Sec. 3.1 |
| $\|\cdot\|_{p,F_\vartheta}$ | $F_\vartheta$-weighted norm in $L^p_{F_\vartheta}(\Theta)$, see Sec. 3.5 |
| $\|\cdot\|_p$ | $\mathcal{Q}$-weighted norms $\|\cdot\|_{p,\mathcal{Q}}$ in $L^2(\Omega)$ for canonical density |
| $\mathcal{D}(\Omega)$ | set of all densities in $L^1(\Omega)$: $\mathcal{D}(\Omega) = \{u \in L^1(\Omega) : u \geq 0,\ \|u\|_1 = 1\}$ |

Other:

| | |
|---|---|
| $w(B,C,\tau)$ | transition probability between sets $B$ and $C$ wrt. time span $\tau$ |
| $d\sigma_\theta(x)$ | intrinsic volume form of submanifold $\Gamma(\theta)$, see Sec. 3.5 |
| $\rho_j = \rho(B_j)$ | probability to be within box $B_j$ of a disjoint decomposition $B_1, \ldots, B_n$ |

| | |
|---|---|
| $\mathbf{D}_\rho$ | diagonal weighting matrix $\mathbf{D}_\rho = \mathrm{diag}(\sqrt{\rho_j})$, cf. proof of Prop. 5.2 |
| $\mathcal{T}$ | temperature of canonical ensemble |
| $\beta$ | inverse temperature $\beta = 1/k_B\mathcal{T}$ |
| $\mathcal{N}_M$ | bound for number of disjoint subsets of the sets $M(q)$, see Def. 4.4 |

# Appendix A: Markov Operators and Mixing

In the subsequent we always consider a subset $\Omega \subset \mathbb{R}^d$ such that the set $\mathcal{B}$ of Borel sets is a $\sigma$-algebra on $\Omega$. Moreover, we consider an absolutely continuous measure $\mu$ on $\Omega$ and the associated $L^p$-spaces

$$L^p \;=\; \{u : \Omega \to \mathbb{R}, \quad \int_\Omega |u(q)|^p \,\mu(dq) \;<\; \infty\}, \quad 1 \le p < \infty,$$

with corresponding norm $\|u\|_p = (\int_\Omega |u(q)|^p \mu(dq))^{1/p}$. The space $L^\infty$ consists of all bounded, almost everywhere measurable functions $u : \Omega \to \mathbb{R}$. Since $L^\infty$ is the dual space of $L^1$, we use the duality to write

$$\langle u, v\rangle = \int_\Omega u^*(q) v(q)\,\mu(dq), \qquad \text{for all } u \in L^1, v \in L^\infty.$$

For $u, v \in L^2$ this defines the usual scalar product in $L^2(\Omega)$.

We always assume that $\mu$ is a probability measure, so that $\mu(\Omega) = 1$ and $\chi_\Omega \in L^p$, $1 \le p \le \infty$ for the characteristic function of $\Omega$. We then have (cf. Prop. 2.3.1 of [66])

Proposition A.1 *Let* $1 \le p_1 < p_2 \le \infty$. *Then*

$$\|u\|_{p_1} \;\le\; \|u\|_{p_2}, \qquad \textit{for all } u \in L^{p_2}.$$

*Thus, every element of* $L^{p_2}$ *belongs to* $L^{p_1}$.

Definition A.2 A nonnegative function[75] $u \in L^1$ with $\|u\|_1 = 1$ is called a *density function* in $L^1$; the set of all density functions is denoted $\mathcal{D}$.

Definition A.3 Every linear operator $P : L^1 \to L^1$ satisfying

$$Pu \;\ge\; 0, \qquad \text{for } u \in \mathcal{D}, \text{ and} \tag{94}$$

$$\|Pu\|_1 \;=\; \|u\|_1, \quad \text{for } u \in \mathcal{D}, \tag{95}$$

is called a *Markov operator*.

## A.1 Adjoint Operator and Induced Markov Chain

Let $P : L^1 \to L^1$ be a Markov operator satisfying $P\chi_\Omega = \chi_\Omega$ and let $P^* : L^\infty \to L^\infty$ denote the adjoint operator with respect to the product $\langle\cdot,\cdot\rangle$. Thus, $\langle Pu, g\rangle = \langle u, P^* g\rangle$ for all $u \in L^1$ and $g \in L^\infty$. $P^*$ allows the definition of a *Markov chain* associated with $P$. To see this, we have to introduce

Definition A.4 The function $t : \Omega \times \mathcal{B}(\Omega)$ is called a *transition function*, if it satisfies the following two conditions:

[75] The phrase "nonnegative" and the notation $u \ge 0$ are always used in the "almost everywhere" sense.

1. For every $B \in \mathcal{B}$, the function $t(\cdot, B)$ is nonnegative and measurable.
2. For every $q \in \Omega$, the map $t(q, \cdot)$ is a probability measure on $\mathcal{B}(\Omega)$.

$t(\cdot,\cdot)$ is called a *substochastic transition function* if it satisfies condition 1, but condition 2 only insofar that $t(q,\cdot)$ is a measure but not a probability measure (that is, $t(q,\Omega) < 1$ for some $q$).

LEMMA A.5 *([39], Chap. 1) Let $P^*$ be the adjoint of a Markov operator $P$ satisfying $P^*\chi_\Omega = \chi_\Omega$. Then, $P(q,B) = P^*\chi_B(q)$ defines a transition function.*

The transition function defines a time-homogeneous *Markov chain* $\{X_k,\, k = 0,1,2,\dots\}$ on the state space $\Omega$, insofar that the conditional probability of finding $X_{k+1} \in B$ after having been in $X_k = q$ is given by

$$P(X_{k+1} \in B | X_k = q) \;=\; P(q,B) \;=\; P^*\chi_B(q).$$

Hence, the conditional probability of finding $X_k \in B$ after having started in $X_0 = q$ is

$$P(X_k \in B | X_0 = q) \;=\; (P^*)^k\, \chi_B(q) \;=\; P^k(q,B).$$

Consequently, the *conditional transition probability* to move from subsets $A \in \mathcal{B}$ to $B \in \mathcal{B}$ in $k$ steps is

$$\begin{aligned} P(X_k \in B | X_0 \in A) &= \int_A P^k(q,B)\,\mu(dq) \\ &= \langle \chi_A, (P^*)^k\, \chi_B\rangle \;=\; \langle P^k\chi_A, \chi_B\rangle. \end{aligned} \tag{96}$$

One says that a probability measure $\nu$ is an *invariant probability distribution* for some Markov chain with transition function $P(\cdot,\cdot)$, if

$$\nu(A) \;=\; \int_\Omega P(q,A)\,\nu(dq), \qquad \text{for all } A \in \mathcal{B}.$$

Thus, in our case, the chain has the invariant probability distribution $\mu$, since we supposed $P\chi_\Omega = \chi_\Omega$.

Due to [39], Chap. V, transition functions and associated Markov operators exhibit the following nice decomposition into absolutely continuous and singular parts.

THEOREM A.6 *Let $P$ be a Markov operator satisfying $P^*\chi_\Omega(q) = 1$ for all $q \in \Omega$ so that the associated transition function $P(\cdot,\cdot)$ is defined everywhere. Then, there exists a decomposition $P^* = K^* + R^*$ into two linear operators $K^*$ and $R^*$ on $L^\infty$ such that*

1. *$K^* \le P^*$, that is, $K^*u \le P^*u$ almost everywhere in $\Omega$ for all positive $u \in L^\infty$.*

2. *$K^*$ has a transition kernel $k(\cdot,\cdot)$,*[76] *such that $K^*u(q) = \int k(q,y)u(y)\mu(dy)$ for all $u \in L^\infty$.*

3. *$K^*$ is maximal in the sense that $\mathcal{K}^* \le K^*$ for every other operator $\mathcal{K}^*$ having a transition kernel and satisfying $\mathcal{K}^* \le P^*$.*

4. *$R^* = P^* - K^*$ defines a (substochastic) transition function $R(q,A) = R^*\chi_A(q)$ such that $R(q,\cdot)$ is a singular measure with respect to $\mu$ for almost every $q \in \Omega$.*

## A.2 Asymptotic Properties of Markov Operators

DEFINITION A.7 A Markov operator $P : L^1 \to L^1$ is called *constrictive* if there exists $\delta > 0$, $\gamma < 1$ and a measurable set $B \subset \Omega$, such that for every density $u \in \mathcal{D}$ there is an $n_0(u) \in \mathbb{N}$ so that

$$\int_{(\Omega\setminus B)\cup E} P^n u(q)\,\mu(dq) \;\le\; \gamma, \qquad \text{for } n \ge n_0(u), \tag{97}$$

for all measurable subsets $E \subset \Omega$ with $\mu(E) < \delta$. If we can choose $n_0(u) = 1$ for all densities $u \in \mathcal{D}$, the operator $P$ is called *directly constrictive.*

REMARK A.8 In order to prove direct constrictiveness, it is sufficient to show (97) for $n = 1$ only, since with $u \in \mathcal{D}$ it is $v_n = P^n u \in \mathcal{D}$ for all $n \in \mathbb{N}$.

REMARK A.9 In checking the condition (97), it is not necessary to verify it for all densities $u \in \mathcal{D}$. If a set $\mathcal{D}_0 \subset \mathcal{D}$ is dense in $\mathcal{D}$, it is sufficient to check (97) for all $u \in \mathcal{D}_0$ (cf. [66], Chap. 5.3).

Constrictiveness is deeply connected with weak precompactness. To see this, let us first introduce

DEFINITION A.10 Let $X$ be a reflexive Banach space. A set $\mathcal{F} \subset X$ is called (strongly) precompact, if every sequence $\{u_n\} \subset \mathcal{F}$ contains a subsequence that converges strongly to an $u_* \in X$. The set $\mathcal{F} \subset X$ is called weakly precompact, if every sequence contains a subsequence that converges weakly to an $u_* \in X$.

In our Banach space $L^1$ we have the following criterion for weak precompactness (cf. [66] Chap. 5.1 or [29]):

LEMMA A.11 *A set of functions $\mathcal{F} \subset L^1$ is weakly precompact if and only if both of the following conditions are satisfied:*

1. *There is a $C > 0$ such that $\|u\|_1 \le C$ for all $u \in \mathcal{F}$.*

[76] That is, a nonnegative function $k : \Omega \times \Omega \to \mathbb{R}_0^+$, which is jointly measurable with respect to its two variables.

2. *For every $\epsilon > 0$ there is a $\delta > 0$ such that for all measurable subsets $E \subset \Omega$ with $\mu(E) < \delta$:*

$$\int_E |u(q)|\,\mu(dq) \;<\; \epsilon \quad \forall u \in \mathcal{F}.$$

**Spectral Decomposition** We are interested in the constrictiveness property, because the following strong *spectral decomposition theorem* holds (cf. Thm. 5.3.1 and Prop. 5.4.2 in [66]):

THEOREM A.12 *Let $P : L^1 \to L^1$ be constrictive with $P\chi_\Omega = \chi_\Omega$. Then there exists an $r \in \mathbb{N}$, a partition $\{B_k,\, k = 1, \dots, r\}$ of $\Omega$, i.e.,*

$$\bigcup_{k=1}^{r} B_k \;=\; \Omega \qquad \text{and} \qquad B_k \cap B_l = \emptyset \text{ for } k \neq l,$$

*a sequence of functions $K_j \in L^\infty(\Omega)$, $j = 1, \dots, r$, and an operator $Q : L^1 \to L^1$ such that for every $u \in L^1$ we have*

$$Pu(q) \;=\; \sum_{j=1}^{r} \lambda_j(u)\, \mathbf{1}_{B_j}(q) \;+\; Qu(q), \tag{98}$$

*with*

$$\lambda_j(u) \;=\; \int_\Omega u(q)\, K_j(q)\, \mu(dq) \qquad \text{and} \qquad \mathbf{1}_{B_j} \;=\; \frac{1}{\mu(B_j)}\chi_{B_j}.$$

*The operator $Q$ and the sets $B_j$ have the following properties:*

1. *For every $u \in L^1$ we have $\|P^n Q u\|_1 \to 0$ for $n \to \infty$.*
2. *For every $j \in \{1, \dots, r\}$ there exists a unique $\alpha(j)$ such that $P\mathbf{1}_{B_j} = \mathbf{1}_{B_{\alpha(j)}}$. Furthermore, $\alpha(j) \neq \alpha(l)$ for $j \neq l$ and thus the operator $P$ just permutes the functions $\mathbf{1}_{B_j}$.*

DEFINITION A.13 A Markov operator $P : L^1 \to L^1$ is called *asymptotically stable* if there exists a unique density $u_* \in \mathcal{D}$ such that $Pu_* = u_*$ and

$$\lim_{n\to\infty} \|P^n u \,-\, u_*\|_1 \;=\; 0, \quad \forall u \in \mathcal{D}.$$

DEFINITION A.14 A Markov operator $P : L^1 \to L^1$ with stationary density $\chi_\Omega$, i.e., $P\chi_\Omega = \chi_\Omega$, is called *mixing* if for all densities $u \in \mathcal{D}$ the sequence $\{P^n u,\, n \in \mathbb{N}\}$ is weakly convergent[77] to $\chi_\Omega$.

The connection to the asymptotic properties of $P$ are given by the following

[77] A sequence $\{u\} \subset L^1$ is called weakly convergent to $u \in L^1$, if $\int_\Omega u_k(q) g(q) \mu(dq) \to \int_\Omega u(q) g(q) \mu(dq)$ with $n \to \infty$ for all $g \in L^\infty$.

THEOREM A.15 *([66], Thm. 5.5.2 and 5.5.3) If the Markov operator $P : L^1 \to L^1$ with stationary density $\chi_\Omega$ is constrictive, the following two statements are equivalent:*

1. *$P$ is mixing.*
2. *$P$ is asymptotically stable, which is equivalent to $r = 1$ in the spectral decomposition (98) in Thm. A.12.*

Thus, for constrictive Markov operators $P$ with $P\chi_\Omega = \chi_\Omega$ the weak and strong convergence in Def. A.14 are equivalent.

## A.3 Frobenius–Perron and Koopman Operators

In the following $g : \Omega \to \Omega$ is always assumed to be an invertible nonsingular transformation which, in particular, is measure-preserving with respect to $\mu$, i.e., $\mu(B) = \mu(g^{-1}(B))$ for all $B \in \mathcal{B}$. Then, the operator $P : L^1 \to L^1$ given by

$$Pf \;=\; f \circ g^{-1}, \qquad \forall f \in L^1, \tag{99}$$

is called *Frobenius–Perron operator* to $g$ and is a Markov operator. The associated operator $P^* : L^\infty \to L^\infty$ defined via

$$P^* f \;=\; f \circ g, \qquad \forall f \in L^\infty, \tag{100}$$

is called *Koopman operator*. Koopman's lemma states, that $P^*$ can also be considered as an operator on $L^2$ (cf. [86] Chap. II.5 or [112]) and that then the measure preserving property of $g$ implies that $P^* : L^2 \to L^2$ is unitary:

LEMMA A.16 *(Koopman) $P^*$ is a* unitary *map of $L^2$ onto $L^2$.*

## A.4 Mixing Systems

Let $g : \Omega \to \Omega$ be as above.

DEFINITION A.17 The dynamical system $g$ is called *mixing* if for all $B, C \in \mathcal{B}$:

$$\lim_{n\to\infty} \mu\left(B \cap g^{-n}(C)\right) \;=\; \mu(B)\mu(C). \tag{101}$$

It is called *weakly mixing* if for all $B, C \in \mathcal{B}$:

$$\lim_{n\to\infty} \left[\frac{1}{n}\sum_{k=0}^{n-1} \mu\left(B \cap g^{-n}(C)\right) \;-\; \mu(B)\mu(C)\right] \;=\; 0. \tag{102}$$

The question of the connections between these concepts and the properties of the above defined operators is answered by the following theorem which summarizes the statements of Thm. VII.14 from [86] and Thm. 4.4.1 from [66].

THEOREM A.18 *1. If $g$ is weakly mixing, the associated Koopman operator $P^* : L^2 \to L^2$ has no eigenvalue other than $\lambda = 1$ and $\lambda = 1$ is a simple eigenvalue.*

*2. If $g$ is mixing, it is also weakly mixing.*

*3. $g$ is mixing if and only if the Frobenius–Perron operator $P$ associated to $g$ considered as a Markov operator is mixing.*

The same notion can be defined for the Markov chain $\{X_k\}$ associated with some Markov operator $P$.

DEFINITION A.19 The Markov chain $\{X_k\}$ is called *mixing* if for all $B, C \in \mathcal{B}$:

$$\lim_{k \to \infty} P(X_k \in C | X_0 \in B) \;=\; \mu(B)\mu(C),$$

where $P(X_k \in C | X_0 \in B)$ denotes the conditional transition probability given by (96).

Definition A.14 and equation (96) immediately show that

PROPOSITION A.20 *The Markov operator $P : L^1 \to L^1$ is mixing if and only if the associated Markov chain is mixing.*

## A.5 Some Aspects of Markov Chain Theory

This paragraph summarizes some aspects of general state space Markov chain theory as described in NUMMELIN [83] and MEYN AND TWEEDIE [79]. In order to remain concentrated on the necessary basics, the presentation follows the line of TIERNEY [107, 106].

In the following, let $\{X_k\}$ be some Markov chain with transition function $P(\cdot, \cdot)$. Whenever we assume that $\{X_k\}$ is associated with some Markov operator $P$ satisfying $P\chi_\Omega = \chi_\Omega$, we are in the nice situation that we already know that $\mu$ is an invariant distribution of the chain.

We now use the same notation as in Sec. A.1 and start with some definitions: First of all, the *first return time* of the chain to a set $A \subset \Omega$ is denoted

$$\tau_A = \inf\{k \in \mathbb{N} : \; X_k \in A\},$$

with the convention that $\tau_A = \infty$ if the chain never returns to $A$.

DEFINITION A.21 A Markov chain is $\varphi$-irreducible for a probability distribution $\varphi$ on $\Omega$, if $\varphi(A) > 0$ for a set $A \in \mathcal{B}$ implies

$$P\left(\tau_A < \infty | X_0 = q\right) \;>\; 0, \qquad \text{for all } q \in \Omega.$$

In this case, $\varphi$ is called an *irreducibility distribution* for the chain. The chain is called *irreducible* if there is some irreducibility distribution.

If the chain is irreducible it may have many different irreducibility distributions. But one can show that every irreducible chain has a *maximal irreducibility distribution* $\psi$ such that all other irreducibility distributions are absolutely continuous with respect to $\psi$. Different maximal irreducibility distributions are equivalent in the sense that they have the same null sets.

The number of hits of a set $A \in \mathcal{B}$ is denoted by

$$n_A = |\{k \in \mathbb{N}: \quad X_k \in A\}|,$$

where $|M|$ denotes the cardinal number of some set $M$ with $|M| = \infty$ for an $M$ containing infinitely many elements.

Definition A.22 An irreducible Markov chain with maximal irreducibility distribution $\psi$ is called *recurrent*, if for every set $A \in \mathcal{B}$ with $\psi(A) > 0$ the following two conditions are satisfied:

(R1) $P(n_A = \infty | X_0 = q) > 0, \quad$ for all $q \in \Omega$.

(R2) $P(n_A = \infty | X_0 = q) = 1, \quad$ for $\psi$-almost all $q \in \Omega$.

If the "almost everywhere" in the condition (R2) can be replaced by an "everywhere", i.e., if the following stronger condition

(HR) $P(n_A = \infty | X_0 = q) = 1, \quad$ for all $q \in \Omega$

is satisfied, then the chain is called *Harris recurrent.* An irreducible and recurrent chain is called *positive recurrent* if it has an invariant probability distribution.

Recurrence is guaranteed for irreducible chains with an invariant probability distribution:

Theorem A.23 *([107], Thm. 4.1 and Sec. 4.4) Assume that $\{X_k\}$ is irreducible and has an invariant distribution $\pi$. Then, $\pi$ is a maximal irreducibility distribution and the unique invariant distribution. In addition, the chain is positive recurrent. It, moreover, is Harris recurrent, if there is an $n \in \mathbb{N}$ such that the transition function $P^n(\cdot,\cdot)$ has a component that has a density with respect to $\pi$, that is, there is a decomposition $P^n = P_1^n + P_2^n$ such that $P_1^n$ has a representation as*

$$P_1^n(q, B) = \int_B k(q, y)\, \pi(dq), \qquad \textit{for all } q \in \Omega, B \in \mathcal{B}.$$

Irreducibility and the existence of an invariant density suffices to guarantee convergence of expectation values: A function $\mathcal{A} : \Omega \to \mathbb{R}$ is called an $\pi$-observable if its expectation value $E_\pi(\mathcal{A}) = \int_\Omega \mathcal{A}(q)\pi(dq)$ with respect to the the probability density $\pi$ exists.

Theorem A.24 *([107], Thm. 4.3) Assume that $\{X_k\}$ is irreducible and has an invariant distribution $\pi$. Let $\mathcal{A}$ be a $\pi$-observable satisfying $E_\pi(|\mathcal{A}|) < \infty$. Then, its mean value converges to its expectation value $E_\pi(\mathcal{A})$ in the sense that, for $n \to \infty$,*

$$P\left( \frac{1}{n+1} \sum_{k=0}^{n} \mathcal{A}(X_k) \to E_\pi(\mathcal{A}) \,\middle|\, X_0 = q \right) = 1, \quad \text{for } \pi\text{-almost all } q \in \Omega.$$

*If the chain additionally is Harris-recurrent, the result holds for all $q \in \Omega$.*

For even stronger results, we need that the chain is *aperiodic*:

Definition A.25 For an irreducible Markov chain, an $m$-cycle is a sequence of disjoint sets $E_1, \dots, E_m$ such that $P(q, E_l) = 1$ for all $q \in E_j$ with $l = j+1$ mod $m$. The period $M$ is the largest $m$ for which an $m$-cycle exists. The chain is aperiodic if $M = 1$.

For the next convergence result, we have to introduce a kind of distance of probability distributions. The *total variation distance* of two probability distributions $\nu_1$ and $\nu_2$ is defined by

$$\|\nu_1 - \nu_2\|_D = 2 \sup_{B \in \mathcal{B}} |\nu_1(B) - \nu_2(B)|. \tag{103}$$

Theorem A.26 *([107], Thm. 4.3) Assume that $\{X_k\}$ is irreducible and aperiodic and has an invariant distribution $\pi$. Then,*

$$\|P^n(q, \cdot) - \pi\|_D \to 0, \quad \text{for } \pi\text{-almost all } q \in \Omega. \tag{104}$$

*If the chain additionally is Harris-recurrent, the result holds for all $q \in \Omega$.*

A simple consequence is

Corollary A.27 *Whenever the Markov chain, associated with a Markov operator $P$ satisfying $P\chi_\Omega = \chi_\Omega$, is irreducible and aperiodic, the operator $P$ is asymptotically stable.*

*Proof:* Consider arbitrary $u \in \mathcal{D}$ and $\epsilon > 0$. We have to show that there is an $N_0 \in \mathbb{N}$ such that $\|P^n u - \chi_\Omega\|_1 < \epsilon$ for all $n > n_0$. To this end, choose a step-function $g = \sum_{k=1}^m \alpha_k \chi_{A_k} \in \mathcal{D}$ with sets $A_k \in \mathcal{B}$ and positive $\alpha_k$ such that $\|u - g\|_1 < \epsilon/2$. Consequently,

$$\|P^n u - \chi_\Omega\|_1 \leq \|P^n(u-g)\|_1 + \|P^n g - \chi_\Omega\|_1 \leq \epsilon/2 + \|P^n g - \chi_\Omega\|_1,$$

so that it is sufficient to show that there is an $N_0 \in \mathbb{N}$ such that $\|P^n g - \chi_\Omega\|_1 < \epsilon/2$ for all $n > n_0$. But this is an immediate consequence of (104) and

$\|g\|_1 = 1 = \sum_k \alpha_k \mu(A_k)$:

$$\begin{aligned}\|P^n g - \chi_\Omega\|_1 &\leq \int_\Omega \sum_{k=1}^m \alpha_k \, |P^n \chi_{A_k}(q) - \mu(A_k)| \, \mu(dq) \\ &\leq \sum_{k=1}^m \alpha_k \int_\Omega \|P^n(q,\cdot) - \mu(\cdot)\|_D \, \mu(dq),\end{aligned}$$

and the last term converges to zero by dominated convergence. ∎

One is often interested in the rate of convergence in (104):

DEFINITION A.28 An irreducible, aperiodic, and positive Harris recurrent Markov chain is called *ergodic*. An ergodic Markov chain with invariant distribution $\pi$ is called *geometrically ergodic*, if there is an function $M : \Omega \to \mathbb{R}_0^+$ with $E_\pi(M) < \infty$ and an $r \in (0,1)$ such that

$$\|P^n(q,\cdot) - \pi\|_D \leq M(q) r^n, \quad \text{for all } q \in \Omega \text{ and } n \in \mathbb{N}. \tag{105}$$

If the function $M$ may be chosen constant, the chain is called *uniform ergodic*.

REMARK A.29 The problem of identifying the constant $r$ in (105) or even of getting reasonable bounds on $r$ has received much attention in recent years. It has been mainly discussed in terms of regeneration and return times, see [88].

These versions of ergodicity are of particular interest, because they imply a *central limit theorem* for the convergence of the mean value to the expectation value in Thm. A.24. The following result can be found in [107]:

THEOREM A.30 *Let the chain $\{X_k\}$ be geometrically ergodic, and suppose that the $\pi$-observable $\mathcal{A}$ satisfies $\mathcal{A} \in L^{2+\epsilon}$ for some $\epsilon > 0$. Moreover, let $\overline{\mathcal{A}_n}$ denote the mean value $\sum_{k=0}^{n-1} \mathcal{A}(X_k)/n$. Then, for every initial distribution,*

$$\sqrt{n}\,\left(\overline{\mathcal{A}_n} - E_\pi(\mathcal{A})\right)$$

*converges weakly to a random variable which is normally distributed with mean 0 and finite variance $\sigma(\mathcal{A})^2$. If the chain is uniformly ergodic, the same convergence is valid for all $\mathcal{A} \in L^2$.*

For the discussion of in Sec. 4, another version is useful, which can be found in the contribution of CHANG and GEYER in the discussion part of [106], therein based on the results of [46]:

THEOREM A.31 *Let the chain $\{X_k\}$, associated with the Markov operator $P$ with $P\chi_\Omega = \chi_\Omega$, be ergodic. Denote by $L_0^2$ the subspace of $L^2$ orthogonal to the constants, i.e., set*

$$L_0^2 = \{u \in L^2 : \langle u, \chi_\Omega \rangle = 0\},$$

*and suppose that $Id - P^*$ is invertible on $L^2_0$. Moreover, consider a $\pi$-observable $\mathcal{A} \in L^2_0$. Then, for every initial distribution,*

$$\sqrt{n}\ \left(\overline{\mathcal{A}_n}\ -\ E_\pi(\mathcal{A})\right)$$

*converges weakly to a random variable which is normally distributed with mean* 0 *and variance*

$$\sigma(\mathcal{A})^2\ =\ \langle \mathcal{A}, \mathcal{A} \rangle\ +\ 2\sum_{k=1}^{\infty} \langle \mathcal{A}, P^k \mathcal{A} \rangle. \tag{106}$$

In the last formula, the terms

$$\langle \mathcal{A}, P^k \mathcal{A} \rangle\ =\ \int_\Omega \mathcal{A}(q)\mathcal{A}(y)P^k(q, dy)\mu(dq)$$

are the $k$th *auto-covariances* of the observable $\mathcal{A}$. The theorem may be applied to every $\mathcal{A} \in L^2$ by using $\mathcal{A}' = \mathcal{A} - E_\pi(\mathcal{A}) \in L^2_0$ instead of $\mathcal{A}$.

# Appendix B: The Spectrum of Linear Operators

In this part of the appendix we shortly summarize some aspects of the spectral theory of linear operators which are important for the arguments in the body of the manuscript. Many of the results stated herein can be generalized (compare, e.g., [60, 86, 56]).

Let $X$ be a Banach space with norm $\|\cdot\|$ and $A$ a closed linear operator on $X$ with domain $D(A) \subset X$. The kernel $\ker(A)$ and the range $\mathrm{Ran}(A)$ of this operator are defined by

$$\begin{aligned} \ker(A) &= \{x \in D(A),\, Ax = 0\}, \\ \mathrm{Ran}(A) &= \{y \in X,\, Ax = y \text{ for some } x \in D(A)\}. \end{aligned}$$

$A$ is called invertible if there is a *bounded* operator, which is called $A^{-1}$, such that $A^{-1} : X \to D(A)$ with $AA^{-1}$ being the identity on $X$ and $A^{-1}A$ the identity on $D(A)$.

Definition B.32 The spectrum of $A$, denoted $\sigma(A)$, is the set of all points $\lambda \in \mathbb{C}$ for which $A - \lambda$ is not invertible.

Remark B.33 There are basically three reasons why $A - \lambda$ fails to be invertible:

1. $\lambda \in \sigma(A)$ such that $\ker(A - \lambda) \neq \{0\}$, i.e., there is a $u \in D(A)$, $u \neq 0$ such that $Au = \lambda u$.

2. $\ker(A - \lambda) = \{0\}$, and $\mathrm{Ran}(A - \lambda)$ is dense in $X$. Then, $\lambda \in \sigma(A)$ states that $A - \lambda$ has a densely defined inverse which is unbounded.

3. $\ker(A - \lambda) = \{0\}$, and $\mathrm{Ran}(A - \lambda)$ is *not* dense in $X$. Then, $\lambda \in \sigma(A)$ states that $A - \lambda$ has an inverse which may be bounded on $\mathrm{Ran}(A - \lambda)$ but is not densely defined.

As a first step towards a characterization of the spectrum one defines the following notions:

Definition B.34 Consider $\lambda \in \sigma(A)$. Then:

1. If $\ker(A - \lambda) \neq \{0\}$, $\lambda$ is called an *eigenvalue* of $A$ and every $0 \neq u \in \ker(A - \lambda)$ is an associated *eigenvector*. Moreover, $\dim \ker(A - \lambda)$ is called the (geometric) *multiplicity* of $\lambda$ and $\ker(A - \lambda)$ itself the corresponding eigenspace. The set of all eigenvalues of $A$ is called the *point spectrum* of $A$ and denoted by $\sigma_p(A)$.

2. The set of all $\lambda \in \sigma(A)$ such that $\lambda$ is not an eigenvalue and $\mathrm{Ran}(A - \lambda)$ is *not* dense in $X$ is called the residual spectrum of $A$.

3. The set of all eigenvalues $\lambda \in \sigma(A)$, which are isolated[78] and have finite (algebraic)[79] multiplicity, is called the discrete spectrum of $A$, and is denoted as $\sigma_{\text{discr}}(A)$.
4. The *essential spectrum* $\sigma_{\text{ess}}(A)$ of $A$ is given by the complement of $\sigma_{\text{discr}}(A)$ in $\sigma(A)$.

For bounded operators, the spectrum is contained in a circle around 0 with a radius given by the operator norm[80] of the operator (cf. [60], Chap. III.6):

THEOREM B.35 *Let $A : X \to X$ be bounded. Then:*

$$\sup_{\lambda\in\sigma(A)} |\lambda| = \lim_{n\to\infty} \|A^n\|^{1/n}, \qquad \textit{that is,} \qquad \sup_{\lambda\in\sigma(A)} |\lambda| \le \|A\|.$$

A class of operators with "discrete" spectrum is the class of compact operators:

DEFINITION B.36 A linear operator $A : X \to X$ is called compact, if $A$ takes bounded sets in $X$ into (strongly) precompact[81] sets in $X$. We denote the set of all compact operators on $\mathcal{H}$ by $\mathcal{B}_c(X)$.

REMARK B.37 Another characterization of compact operators is the following: $A : X \to X$ is compact if and only if the image $\{Au_n\}$ of any bounded sequence $\{u_n\} \subset X$ contains a Cauchy subsequence.

REMARK B.38 For an operator $A \in \mathcal{B}_c(X)$, any nonzero $\lambda \in \sigma(A)$ is an isolated eigenvalue of finite multiplicity (due to the well-known Riesz-Schauder theory, cf. [60], Chap. III.6). Thus, we have $\sigma(A) \subset \sigma_{\text{discr}}(A) \cup \{0\}$.

As an example for compact operators on $L^1$-spaces, one may consider certain integral operators (cf. [60], Example 2.4 and 4.1 in Chap. III):

THEOREM B.39 *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space with $\Omega$ being a compact set and let $L^1_\mu(\Omega)$ be the associated $L^1$-space. Moreover, assume $k : \Omega \times \Omega \to \mathbb{R}$ to be a continuous function. Then*

$$Au(q) = \int_\Omega k(q,y)\, u(y)\, \mu(dy)$$

*defines a compact operator $A : L^1_\mu(\Omega) \to L^1_\mu(\Omega)$ which is defined on the whole of $L^1_\mu(\Omega)$.*

Before going into more details of the structure of the spectrum we restrict ourselves to the cases we are interested in:

[78]That is, for some $\epsilon > 0$, there is no $\kappa \in \sigma(A)$, $\kappa \neq \lambda$, such that $|\kappa - \lambda| < \epsilon$

[79]See [60], Chap. III.5 for a definition. In general, the algebraic multiplicity is larger or equal to the geometric multiplicity.

[80]The usual operator norm $\|A\|$ denotes the smallest constant $C > 0$ such that $\|Au\| < C\|u\|$ for all $u \in X$, $u \neq 0$.

[81]Compare Def. A.10 in Appendix A.

## B.1 Operators in Hilbert Space

For the remaining part of Appendix A.5, $\mathcal{H}$ denotes a separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|_2$.

For self-adjoint operators, one has the following well-known properties of the spectrum:

THEOREM B.40 *([86], Thms. VI.6 and VI.8) Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint linear operator. Then:*

1. *$A$ has no residual spectrum, i.e., case 3 of Rmk. B.33 does not appear.*
2. *$\sigma(A)$ is a subset of $\mathbb{R}$.*
3. *Eigenvectors corresponding to distinct eigenvalues are orthogonal.*
4. *If $A$ is bounded with operator norm $\|A\|_2$, its spectrum is bounded due to $\sup\limits_{\lambda \in \sigma(A)} |\lambda| = \|A\|_2$.*

We are mainly interested in a characterization of the spectrum which singles out its "discrete" part in contrast to a remaining "continuous" part of the spectrum. One possible decomposition in "discrete" and "continuous" parts of the spectrum of our self-adjoint operator $A$ is the following (cf. [60], Chap. X): Since $A$ has no residual spectrum, we may define the *continuous spectrum* $\sigma_c(A)$ as consisting of all $\lambda \in \sigma(A)$ for which case 2 of Rmk. B.33 is valid. Thus, $\sigma(A)$ can be decomposed in $\sigma_c(A)$ and the point spectrum $\sigma_p(A)$ (case 1 of Rmk. B.33). But $\sigma_p(A)$ may contain eigenvalues with infinite multiplicity and may even be dense in some interval $I \subset \mathbb{R}$.

Thus, the *disjoint* decomposition of the spectrum into its discrete and essential part (cf. Def. B.34) is of more interest herein. Obviously, the essential spectrum may contain eigenvalues of $A$, e.g., such with infinite multiplicity. More precisely, for self-adjoint $A$ (cf. Thm.VII.11 in [86]): $\lambda \in \sigma_{\text{ess}}(A)$ if and only if one or more of the following holds:

1. $\lambda \in \sigma_c(A)$.
2. $\lambda$ is a limit point of $\sigma_p(A)$.
3. $\lambda$ is an eigenvalue of infinite multiplicity.

As was already stated above, for compact operators we simply have $\sigma_{\text{ess}}(A) \subset \{0\}$, i.e., compact operators have (nearly) purely discrete spectrum. On the contrary, an example for a class of operators without discrete spectrum are certain multiplication operators (cf. [86], Chap VII):

THEOREM B.41 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with a smooth measure $\mu$ and $\mathcal{H} = L^2_\mu(\Omega)$.[82] Moreover, let $F : \Omega \to \mathbb{R}$ be a smooth bounded function on $(\Omega, \mathcal{A}, \mu)$ and $A_F$ be the multiplication operator on $\mathcal{H}$ defined by*

$$A_F u(x) = F(x)\, u(x).$$

[82] Compare Appendix A, page 117.

*Then, the spectrum $\sigma(A_F)$ is given by the closure of the range of $F$, that is,*

$$\sigma(A_F) = \sigma_{\text{ess}}(A_F) = \overline{Ran(F)} = \overline{\{F(x),\, x \in \Omega\}}.$$

For self-adjoint operators, the essential spectrum can be characterized by the well-known *Weyl-criterion* ([56], Thm. 7.2):

THEOREM B.42 *Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be self-adjoint. Then, $\lambda \in \sigma(A)$ if and only if there is a sequence $\{u_n\} \subset D(A)$ with $\|u_n\|_2 = 1$ for all $n \in \mathbb{N}$, such that $u_n$ converges weakly to $0$ but $(A - \lambda)u_n \to 0$ strongly.*

In our Hilbert space, compact operators map *any* weakly convergent sequence into a strongly convergent sequence (cf. [86], Thm. VI.11). Thus, we expect the Weyl-criterion to imply that a compact perturbation of a self-adjoint operator $A$ will not have any influence on the essential spectrum. This is true, even for non-self-adjoint operators, as the following theorem states:

THEOREM B.43 *Let $A : \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{H} \to \mathcal{H}$ be bounded operators and let $B$ moreover be compact. Then, $\sigma_{\text{ess}}(A + B) = \sigma_{\text{ess}}(A)$.*

REMARK B.44 This theorem is a corollary to the general theory in Chap. IV of [60] (cf. Thm. 5.35) for simplicity restricted to bounded operators. Therein we find, that the statement is valid even on a Banach space $X$, if only $A$ is bounded (and defined on the entire space $X$) and $B$ is compact.

## B.2 Hilbert–Schmidt Operators

Consider a bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ together with an arbitrary complete orthonormal family $\{u_n\}$ in $\mathcal{H}$, and define

$$\|A\|_S = \left( \sum_{n=1}^{\infty} \|Au_n\|_2^2 \right)^{1/2} .$$

If the series converge, i.e., $\|A\|_S < \infty$, we call $\|A\|_S$ the Schmidt norm of $A$.[83]

DEFINITION B.45 The set of all bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ with $\|A\|_S < \infty$ is called the class of Hilbert–Schmidt operators and is denoted by $\mathcal{B}_S(\mathcal{H})$.

Thus, by the definition, for every orthonormal sequence $\{u_n\} \subset \mathcal{H}$ the sequence $\{\|Au_n\|_2\}$ converges to zero. Together with Rmk. B.37, this yields:

PROPOSITION B.46 *Every $A \in \mathcal{B}_S(\mathcal{H})$ is compact, i.e., $\mathcal{B}_S(\mathcal{H}) \subset \mathcal{B}_c(\mathcal{H})$.*[84]

[83] It is easy to check, that the Schmidt norm then is independent of the choice of the orthonormal family $\{u_n\}$ employed in the definition.

[84] $\mathcal{B}_S(\mathcal{H})$ is a complete vector space. Another notation is quite usual: The class of compact operators is denoted $\mathcal{B}_0(\mathcal{H})$, the Hilbert-Schmidt operators $\mathcal{B}_2(\mathcal{H})$, and the trace class operators $\mathcal{B}_1(\mathcal{H})$, yielding a sequence of subspaces $\mathcal{B}_2(\mathcal{H}) \subset \mathcal{B}_1(\mathcal{H}) \subset \mathcal{B}_0(\mathcal{H})$. Then, the Schmidt norm of $A \in \mathcal{B}_2(\mathcal{H})$ usually is denoted $\|A\|_2$.

Interestingly, on $L^2$-spaces, any Hilbert–Schmidt operator can be expressed as an integral operator:

THEOREM B.47 *([86], Thm. VI.23) Let* $(\Omega, \mathcal{A}, \mu)$ *be a measure space and* $\mathcal{H} = L^2_\mu(\Omega)$ *the associated* $L^2$*-space. Then,* $A \in \mathcal{B}_S(\mathcal{H})$*, i.e.,* $A$ *is a Hilbert-Schmidt operator, if and only if there is a measurable function* $K : \Omega \times \Omega \to \mathbb{C}$ *with*

$$\|K\|^2 \;=\; \int_\Omega \int_\Omega |K(x,y)|^2 \, \mu(dx) \, \mu(dy) \;<\; \infty,$$

*such that for every* $u \in \mathcal{H}$*:*

$$Au(x) \;=\; \int_\Omega K(x,y) \, u(y) \, \mu(dy).$$

*Moreover,* $\|A\|_S = \|K\|$.

## B.3 Approximation of Isolated Eigenvalues

In this section, it is the question whether isolated eigenvalues $\lambda \in \sigma_{\mathrm{discr}}(A)$ of a bounded self-adjoint operator in a Hilbert space can be approximated by projection of the eigenproblem into appropriate finite-dimensional subspaces. We herein present some aspects of the application of the well-known Rayleigh-Ritz min-max principle [16] to this question. The reader should be aware that these aspect are chosen particularly for the needs of the analysis presented in Sec. 5.2; for more details about the usefulness and long history of the min-max principle the reader should consult the associated literature, e.g., [105, 15], or for the non-self-adjoint case [108].

For our purpose, we consider some separable Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle$ and norm associated $\| \cdot \|$, and a self-adjoint, bounded linear operator $A : \mathcal{H} \to \mathcal{H}$. Assume that there are constants $\gamma, C > 0$ such that

$$\gamma \|u\|^2 \;\le\; \langle u, Au \rangle \;\le\; C\|u\|^2, \quad \forall u \in \mathcal{H}. \tag{107}$$

Let the smallest $l$ eigenvalues of $A$ be

$$0 \;<\; \lambda_1 \;\le\; \lambda_2 \;\le\; \ldots \;\le\; \lambda_l$$

(including repetitions in case of multiplicities) and let the $\lambda_j$ belong to the discrete spectrum of $A$. Let $u_j \in \mathcal{H}$, $j \in \{1, \ldots, l\}$, be associated normalized eigenvectors, chosen pairwise orthogonal in case of multiplicities.

Property (107) of $A$ implies that $A$ defines the scalar product $\langle \cdot, A \cdot \rangle$ and the associated norm $\| \cdot \|_A$ which, because of (107) is norm-equivalent to the original norm $\| \cdot \|$. The min-max principle states that, for every $j \in \{1, \ldots, l\}$, the $j$th eigenvalue of $A$ is given via

$$\lambda \;=\; \min_{S_j \subset \mathcal{H}} \; \max_{0 \neq v \in S_j} R(v),$$

with the Rayleigh quotient $R(v) = \langle v, Av\rangle / \langle v, v\rangle$ for $v \neq 0$ and the minimum being taken over all linear subspaces $S_j \subset \mathcal{H}$ of dimension $j$.

We denote our sequence of ansatz spaces by $\mathcal{V}_1 \subset \mathcal{V}_2 \subset \ldots \subset \mathcal{V}_n \subset \mathcal{V}_{n+1} \subset \mathcal{H}$, and assume this sequence to be dense in $\mathcal{H}$.[85] Moreover, assume $A_n = \Pi_n A \Pi_n$ to denote the Galerkin projection of $A$ onto the ansatz spaces $\mathcal{V}_n$, and

$$\lambda_1^n \leq \lambda_2^n \leq \ldots \leq \lambda_l^n$$

to be the smallest eigenvalues of $A_n$ with associated eigenvectors $u_j^n$, $j \in \{1, \ldots, l\}$.

With respect to the scalar product $\langle \cdot, A\cdot\rangle$, we define an orthogonal projection $P_n : \mathcal{H} \to \mathcal{V}_n$ via

$$\langle u - P_n u, Av\rangle = 0, \quad \forall v \in \mathcal{V}_n.$$

It is easy to show, that for every $u \in \mathcal{H}$ we have $\|u - P_n u\|_A \to 0$ for $n \to \infty$.[86] Thus, the norm-equivalence (107) suffices to show that

$$\|u - P_n u\| \to 0, \qquad \text{for } n \to \infty, \qquad \forall u \in \mathcal{H}. \tag{108}$$

Under these assumptions, we have the following convergence result for the approximated eigenvalues:

LEMMA B.48 *Let $E_j$ denote the subspace spanned by the eigenvectors $u_1, \ldots, u_j$ and define*

$$\beta_n^j = \max_{u \in E_j \setminus \{0\}} \left( \frac{1}{\|u\|^2} \left| 2\langle u, u - P_n u\rangle - \langle u - P_n u, u - P_n u\rangle \right| \right). \tag{109}$$

*Then, for every $j \in \{1, \ldots, l\}$ and $n$ large enough, the following estimate holds:*

$$\lambda_j \leq \lambda_j^n \leq \frac{1}{1 - \beta_n^j} \lambda_j, \tag{110}$$

*with $\beta_n^j \downarrow 0$ for $n \to \infty$.*

*Proof:* A justification of the estimate (110) can be found in [105], Lemma 6.1, in form of a direct consequence of the min-max principle. The asserted convergence $\beta_n^j \downarrow 0$ for $n \to \infty$ then follows from (108). ∎

[85] That is, for every $u \in \mathcal{H}$ and every $\epsilon > 0$, there is an $M \in \mathbb{N}$ and a $v \in \mathcal{V}_M$ such that $\|u - v\| < \epsilon$.

[86] With a $v \in \mathcal{V}_M$ we find for all $n > M$ that

$$\|u - P_n u\|_A^2 = \langle u - P_n u, Au\rangle = \langle u - P_n u, A(u - v)\rangle \leq \|u - P_n u\|_A \cdot \|u - v\|_A.$$

Thus, $\|u - P_n u\|_A \leq \|u - v\|_A$. But since the $\mathcal{V}_n$ are dense in $\mathcal{H}$, Eq. (107) shows that, for $M$ large enough, for every $\epsilon > 0$ there is a $v \in \mathcal{V}_M$ such that $\|u - v\|_A < \epsilon$. This implies the stated convergence.

In order to establish a corresponding convergence result for the eigenvectors, we have to introduce a *separation constant* for the considered cluster of isolated eigenvalues. The convergence $\lambda_j^n \to \lambda_j$ guarantees that, for every $j \in \{1, \dots, l\}$, for $n$ large enough, there is a constant $s_j$ such that

$$\frac{\lambda_j}{|\lambda_j - \lambda_k^n|} \leq s_j, \qquad \text{for all } k \in \{1, \dots, l\} \text{ with } \lambda_k \neq \lambda_j.$$

With these separation constants, we have

LEMMA B.49 *For $n$ large enough and after an appropriate choice of eigenvectors for repeated eigenvalues, the following is true for all $j \in \{1, \dots, l\}$:*

$$\|u_j - u_j^n\| \leq 2\,(1 + s_j)\,\|u_j - P_n u_j\|, \tag{111}$$

*which, because of (108), implies the strong convergence $u_j^n \to u_j$ for $n \to \infty$.*

*Proof:* The asserted estimate follows in perfect analogy to equation (51) and the second equation in the proof of Thm. 6.2 in [105]. ∎

REMARK B.50 The convergence estimates (110) and (111) illustrate that the *rate of convergence* with $n$ crucially depends on the rate of the convergence (108) for the projections $P_n$. Let the discretization domain $\Omega$ be bounded with smooth boundary and consider $\mathcal{H} = L^2(\Omega)$. If we use finite element ansatz spaces in the discretization process associated with grids with maximal "grid-width" $\mathcal{O}(1/n)$, one can prove results of the form

$$\|u - P_n u\| \leq C(n)\,\|u\|, \qquad \text{with } C(n) \sim n^{-p},$$

and $p$ depending on the order of the elements used.

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