Konrad-Zuse-Zentrum
für Informationstechnik Berlin

Takustraße 7
D-14195 Berlin-Dahlem
Germany

Karin Gatermann[1]

Computer Algebra Methods for Equivariant Dynamical Systems

[1] FU Berlin

Preprint SC 99-26 (May 1999)

Preface

The topic of this work is a special sort of algorithms summarized under the name of Computer Algebra. Secondly, these algorithms are applied in the theory of dynamical systems. This rare combination of research interests is the result of being a member of the department Symbolik at ZIB and working together with people in the dynamical systems community. This rare combination gives the chance to attack problems in a challenging way.

What is Computer Algebra? Basically, Computer Algebra means computation with algebraic structures and answering questions from algebra and algebraic geometry in an algorithmic way. A good algorithm reflects and exploits the underlying mathematical stuctures as best as possible. While for combinatorial algorithms these are graphs, polytopes etc., for numerical algorithms these are analytical structures and structures from functional analysis, the underlying structures of Computer Algebra are from commutative and non-commutative algebra, algebraic geometry etc.

While in numerics the computed result may be a huge amount of real numbers encoding an approximate solution of a partial differential equation, in Computer Algebra the result is an algebraic object such as a Hilbert series of a module of splines encoding the dimensions of vector spaces of splines up to a certain degree [18, 42] or generators of Lie algebras encoding the symmetry group of self-similar solutions of partial differential equations [94, 95, 170, 193].

While numerical algorithms *approximate* symbolic computations are *exact*. Even with numbers this is obvious. The floating point numbers reflect the real numbers as a *Banach space*, but in exact computation the numbers are considered as elements of a *field*.

In both numerics and Computer Algebra Newton's method plays a prominent role. In numerics the family of inexact Newton methods and Quasi-Newton methods produces sequences in Banach spaces and Hilbert spaces [49], [105]. In symbolic computation the Newton method is used in order to find factorizations of polynomials in $\mathbf{Z}[x]$ or $\mathbf{Z}[x_1, \ldots, x_n]$ in a completion space exploiting Hensel's lemma [80], [196].

Although there are a lot of differences between symbolic computations and other areas of algorithmic mathematics they share some main principles.

- Restrict the computation to the essential information only.
- Reuse information which is available anyway.
- Use the underlying mathematical structure as best as possible.
- Exploit special structure.

The combination of algorithms of different types appears to be rather difficult. An appropriate combination of symbolic algorithms with numerics requires that both algebraic structures and analytic structures are simultaneously present without interfering each other. First symbolic computations are performed followed by the numerical algorithm. In general the data of a run of a numerical algorithm should not be the input of a symbolic algorithm. If the analytic and algebraic structures interfere each other the combination of symbolic and numerical methods appears to be inappropriate.

Successful examples of combination include the symbolic exploitation of equivariance with respect to a linear representation of a finite group and numerical pathfollowing and computation of bifurcation points as in *Symcon* [75, 69] and secondly the computation of a mixed subdivision of a tuple of Newton polytopes in order to determine all solutions of a sparse polynomial system numerically [187], [188].

But the most common use of Computer Algebra seems to be the automation of hand calculations. Especially for theoretical investigations it can save a lot of time and make tedious calculations more reliable.

Far beyond this Computer Algebra methods are alternative tools bringing a different point of view to a problem. The algebraic algorithms are able to exploit much more structure of a problem than numerical algorithms which basically approximate. The purpose of Computer Algebra is the computation of structural information. The demonstration of this in the context of dynamical systems is the aim of this work.

What is equivariant dynamics? In the theory of equivariant dynamical systems long time phenomena are investigated which are structured and classified by the symmetry. In engineering and science the symmetry enters in a natural way because geometric configurations may be symmetric. The formal description of symmetry is done using group theory, e.g. the linear representation of a compact Lie group. The main point is that the problem remains unchanged under the group action. First of all the differential equations remain unchanged, i.e. are equivariant. Secondly, the domain has the symmetry of this group. Besides the symmetry the problem may depend on some parameters which have a physical meaning such as temperature, aspect ratio, Rayleigh number etc. Of course the solution and long time behavior change with values of the parameters. Bifurcation theory deals with the study of dramatic changes in the solution quality depending on the parameters. It is well-known that the genericity of bifurcation phenomena is essentially dominated by the symmetry. Moreover, the dynamics is structured by the symmetry.

The theory of equivariant dynamical systems is motivated by several examples: In the Taylor-Couette problem [35, 87] (see the description in Section 4.3) the flow of some liquid between two rotating cylinders depends on the velocities of the cylinders. At different velocities different patterns of the flow appear. In the Bénard problem [87] a liquid in a thin plate is heated from below. The induced flow show hexagonal pattern and various other regular configurations. Another source of motivation is the magnetic field of the earth which has been going through a lot of pol reversals in the history of the earth [34]. This may be explained by heteroclinic cycles. For an introduction to heteroclinic cycles and symmetry see [57]. In the Faraday experiment light is reflected by the surface of a liquid which is vibrating and oscillating irregularly [141, pp. 255]. Especially in this experiment the phenomena of ordered chaos has been studied since the symmetry puts some structure on the chaotic attractors.

The aim of the theory of dynamical systems is the study of long time behavior and invariant sets. Secondly, the influence of a parameter which may cause dramatic changes is the goal of understanding. Analytical tools such as Liapunov-Schmidt reduction [86], and center manifold reduction [92, 122] reduce a given dynamical system to a smaller one which reflects the main phenomena. By now these became standard methods. For the study of attracting sets ergodic theory became more and more important in the recent years. Symmetric attractors using ergodic theory have been studied e.g. in [7, 9, 58], see also [70, 72].

Working in analysis one does not expect the occurrence of Computer Algebra within this context. But the symmetry brings in questions different from analysis. Symmetry goes along with algebraic structures such as groups, invariant rings, algebras, and varieties which are the objects of symbolic computation. Certainly, this modern tool will be used much more in the future when more people are familiar with it.

SUMMARY Although some attempts have been made to use Computer Algebra in dynamical systems the full power of constructive algebraic methods still needs to be discovered. This book is a start and will hopefully lead in the right direction.

In this work three topics within *equivariant* dynamical systems theory are treated. Once a symmetry of a problem class is described by a group action a general equivariant vector field is created in order to study generically occurring bifurcation phenomena. The derivation of the general equivariant vector field is the first point. Secondly, the application of Computer Algebra to local bifurcation theory is demonstrated. The most advanced topic is a special method known as *orbit space reduction*. The basic idea is to exploit the symmetry by dividing out the group action. Our investigation concerns the choice of symmetry adapted bases. In all three topics I use Gröbner bases, especially variants of their efficient computation. In order to prepare their application an introduction to Gröbner bases is presented in the first chapter.

Using Gröbner bases means algorithmic commutative algebra. Given a set of polynomials $f_1 \ldots, f_m \in \mathbf{C}[x_1 \ldots, x_n]$ one asks questions about their common zeros $V \subset \mathbf{C}^n$ such as finiteness, number, dimension or structure. These questions are attacked *indirectly* by investigating the ideal $I = \langle f_1, \ldots, f_m \rangle$ and the quotient ring $\mathbf{C}[x]/I$. The desired answers are given by the properties of the ring $\mathbf{C}[x]/I$ such as being a finite-dimensional vector space, its dimension as vector space or the structure over a subring. With the help of Gröbner bases these questions are reduced to examining *monomials*.

In the beginning of Chapter 1 the Gröbner basics are recalled and the efficient implementation details are discussed. Then three more advanced topics follow. The Hilbert series driven Buchberger algorithm exploits a priori information. Sometimes a set of given polynomials which needs to be investigated form already a Gröbner basis with respect to some term order, but which is not what one wants. On the other hand this enables the computation of the Hilbert series and thus the usage of the efficient Hilbert series driven variant. In a lot of cases the term order can be found by combinatorial methods (Structural Gröbner Basis detection). The exploitation of sparsity is treated here for the first time. The third topic is the change of the term order during the Buchberger algorithm (Dynamic Buchberger algorithm). In the last section of this chapter the standard application of Gröbner bases to the solution of polynomial equation systems is recalled for sake of completeness. Throughout the text and especially in this section I try to keep the description as simple as possible in order to address applied mathematicians as well.

In the third chapter the results from algorithmic invariant theory are presented which are needed in order to construct a generic equivariant vector field and guarantee the *unique* representation as required in the following chapters. The algorithms are classified by the structures they are exploiting. Examples illustrate the usefulness for equivariant dynamical systems theory. First the computations of invariants and equivariants using the Hilbert series are described in detail. The famous algorithm by Derksen using the algebraic group structure is recalled. Its generalization to the equivariant case is presented here for the first time. The algorithms exploiting the Cohen-Macaulay structure are given together with time comparisons. Then the algorithms follow which give generators such that a *unique* representation is guaranteed. The computation of a Hironaka decomposition by algorithmic Noether normalization and the computation of a Stanley decomposition of the module of equivariants are presented.

In Chapter 3 the typical argumentation of local symmetric bifurcation theory is outlined and illustrated by an example. The result on secondary Hopf bifurcation with cyclic symmetry shows the typical usage of Computer Algebra. The generic equivariant vector field is achieved by symbolic computation and also some other more simpler usages of symbolic computations are demonstrated. The invention of a different style of investigation within dynamical systems theory is the aim of this section.

In the last chapter the full power of the algorithms developed in the third chapter are exploited in the *orbit space reduction*, a very special method in equivariant dynamical systems theory. Treating the equivariant system modulo the group action yields a related system on a part of a real variety. In fact the domain of definition is stratified by semi-algebraic sets. These are submanifolds corresponding to different orbit types. Coordinates are introduced by choosing a Hilbert basis of the invariant ring. The group actions typically investigated in equivariant dynamical systems theory have a special property such the invariant ring is Cohen-Macaulay which implies a certain structure of the complex variety, the solutions of the relations of the Hilbert basis. The exploitation of Cohen-Macaulayness has not been done before. It yields new insight into the properties of the method of orbit space reduction. In appropriate coordinates the orbit space shows its structure in a clear form. Even more the differential equations on the orbit space reflect the fixed point spaces which are as flow invariant sets the main structure in the theory of equivariant dynamical systems. Also the Jacobians have a special structure simplifying the determination of eigenvalues and thus simplifying the bifurcation analysis. Examples such as the Taylor-Couette problem illustrate the advantages of special coordinates. The methods of Chapter 1, the computation of Gröbner bases, are exploited in multiples ways. They are used for the computation of the generic equivariant vector field, the computations of Chapter 2, rewriting an invariant in terms of a Hilbert basis, checking radicals and many more aspects.

Altogether this work links very different areas of mathematics. Hopefully, it helps to change the style in dynamical systems theory in the direction of making more usage of computers instead of doing work by pencil and paper.

Berlin, May 1999 *Karin Gatermann*

for my parents

Acknowledgments: This work would have never been written without the support of many people. They are too numerous to mention all, but I would like to express especial thanks to some of them. First of all I would like to thank my parents who gave me the chance to have a good school education and always help me as much as they can. Secondly, thanks are due to my 'doctor father' Bodo Werner who taught me to do research in mathematics and to write precisely in mathematical terms. The Frauenförderplan of the department of mathematics of the university of Hamburg motivated me to write a dissertation. I am grateful to Michael Möller for introducing me to Gröbner bases which became so important to me. I would like to thank Peter Deuflhard for his advice and for supporting me on my positions at ZIB and the Freie Universität Berlin as well. Claudia Wulff helped me with many discussions. Part of the revision has been done while I was visiting the Mathematical Science Research Institute Berkeley participating in the program *Symbolic Computation in Geometry and Analysis*. I like to thank all my coauthors of my publications, Andreas Hohmann, Reiner Lauterbach, Jan Verschelde, and Frederic Guyard. Especially Reiner Lauterbach helped me with many advices. I also like to thank Bernold Fiedler for motivating me. I benefited a lot by preprints and hints which Bernd Sturmfels sent to me. With Gregor Kemper and Mathias Rumberger I had very helpful conversations. Igor Hoveijn motivated me to learn about torus invariants and Frederic Guyard explained to me the Weyl integral formula and the Cartan decomposition. Finally, Patrick Worfolk helped me to get started with the Maple implementations. I am grateful to all of them.

Contents

Die eine Kerze zur Rechten. Wie immer. Kaum wahrnehmbar das Atmen, mit dem sie sich verzehrt. Ihr Licht taugt zu nichts. Die Helligkeit, die meinen Augen nötig ist, kommt von der Lampe links. Und doch habe ich sie angezündet. Wieder und wieder. Ein Jahr. Zwei Jahre. Ehe ich mich versah, waren es zwanzig und mehr. Solange ich an diesem Schreibtisch sitze. Wie auch die Lichtverhältnisse waren - ich wollte sie unbedingt, diese eine Kerze. Natürlich handelte es sich nicht immer um denselben Tisch, dieselbe Kerze. Für einen Moment will es mir sogar scheinen, als wäre ich niemals aufgestanden. Als wäre dieses Dasein, vornübergebeugt am Schreibtisch, mathematische Formeln auf ein Blatt Papier kritzelnd, das eigentliche Leben gewesen. O diese Lust! Diese Klarheit! Diese hochmütigen Konstruktionen! Aber dann die Zusammenbrüche. Der scharfe Schmerz in der Scheitelgegend, gegen den kein Haareraufen half. Der Zweifel an der eigenen Existenzberechtigung. Plötzlich, wenn schon alles verloren aussah: ein neuer Einfall. Also wieder von vorn. Verglichen mit diesem höllischen Pendeln zwischen Fegefeuer und Hosianna war das übrige Dasein fast eine Plattheit.

Helga Königsdorf
Respektloser Umgang
Luchterhand 736 S. 7

Chapter 1

Gröbner bases

> [Kronecker] believed that one could, and that one must, in these parts of mathematics, frame each definition in such a way that one can test in a finite number of steps whether it applies to any given quantity. In the same way, a proof of the existence of a quantity can only be regarded as fully rigorous when it contains a method by which the quantity whose existence is to be proved can actually be found.
>
> K. Hensel (cited in Mishra [145])

The algorithmic treatment of varieties is the topic of this chapter. This deals as preparation for the investigation of varieties in equivariant dynamical systems in Chapter 4. Secondly, the computation of invariants and equivariants in Chapter 2 necessitates algebraic computations based on Gröbner bases.

Since Gröbner bases are the most important tool of this work their basic theory and algorithmic determination is presented. The standard theory of Gröbner bases may be found in the books [3], [14], [41], [52], [61], [80], [145], [186], [192] and more advanced theory in [177]. In the first section some elementary notions of Gröbner bases are recalled emphazising on easy understandable presentation and illustration of ideas by pictures. Then the recent progress on this topic follows which is new or which is not included in text books. The main points are our implementation of the Hilbert series driven Buchberger algorithm, the exploitation of the sparsity of polynomials in the structural Gröbner basis detection, and the dynamic version of the Buchberger algorithm.

The exposition of the text is made for non-experts by including some elementary explanations where it seems appropriate. Especially Section 1.5 contains easy readable material and explains the purpose of Gröbner bases.

1.1 Buchberger's algorithm

For given polynomials $f_1, \ldots, f_m \in \mathbf{C}[x_1, \ldots, x_n]$ we would like to study the solutions $x \in \mathbf{C}^n$ of the system of algebraic equations given by $f_1(x) = 0$, $f_2(x) = 0, \cdots, f_m(x) = 0$. Since the solution set does not change by addition and multiplication one equivalently studies the *variety*

$$V(I) = \{x \in \mathbf{C}^n \mid f(x) = 0, \forall f \in I\}$$

of the *ideal* $I = \langle f_1, \ldots, f_m \rangle$ which is generated by the given polynomials. Even more it is possible to study the properties of the variety indirectly. The ideal I and the quotient ring $\mathbf{C}[x]/I$ give insight into the variety, e.g. on the number of isolated solutions or the dimension. So we are dealing algorithmically with ideals in a polynomial ring $K[x_1, \ldots, x_n]$ where the field[1] K is in most practical computations $\mathbf{Q}$. But we will as well discuss details of computation for extensions of $\mathbf{Q}$ in Section 1.2.4. A Gröbner basis is a special ideal basis depending on an order of the monomials $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ carrying essential information on the properties of the variety.

Definition 1.1.1 *([14] p. 189) The relation $\leq$ is called a* term order*, if for all monomials $x^\alpha, x^\beta, x^\gamma \in K[x_1, \ldots, x_n]$*

$$
\begin{array}{llll}
x^\alpha \leq x^\alpha & & & \textit{(reflexive)} \\
x^\alpha \leq x^\beta \textit{ and } x^\beta \leq x^\gamma & \Rightarrow & x^\alpha \leq x^\gamma & \textit{(transitive)} \\
x^\alpha \leq x^\beta \textit{ and } x^\beta \leq x^\alpha & \Rightarrow & x^\alpha = x^\beta & \textit{(antisymmetric)} \\
x^\alpha \leq x^\beta \textit{ or } x^\beta \leq x^\alpha & & & \textit{(connex)} \\
1 \leq x^\alpha & & & \textit{(Noetherian)} \\
x^\alpha \leq x^\beta \quad \Rightarrow \quad x^\alpha x^\gamma \leq x^\beta x^\gamma & & &
\end{array}
$$

The first conditions have the meaning of total ordering of the monomials while the last two conditions assure that the term order is admissible with the polynomial structure.

Example 1.1.2 *Examples of term orders on $K[x_1, \ldots, x_n]$ with variable order $x_1 > x_2 > \cdots > x_n$ include*

a.) the lexicographical ordering *($>_{lex}$) defined by*

$$x^\alpha >_{lex} x^\beta \quad \Leftrightarrow \quad \exists i \textit{ with } \alpha_j = \beta_j,\ 1 \leq j < i - 1 \textit{ and } \alpha_i > \beta_i,$$

b.) the graded lexicographical ordering *($>_{grlex}$) defined by*

$$x^\alpha >_{grlex} x^\beta \quad \Leftrightarrow \quad \sum_{k=1}^{n} \alpha_k > \sum_{k=1}^{n} \beta_k \textit{ or } \sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} \beta_k \textit{ and } x^\alpha >_{lex} x^\beta,$$

c.) the graded reverse lexicographical ordering *($>_{grevlex}$) defined by*

$$
\begin{array}{lll}
x^\alpha >_{grevlex} x^\beta & \Leftrightarrow & \sum_{k=1}^{n} \alpha_k > \sum_{k=1}^{n} \beta_k \quad \textit{or} \\
& & \\
& & \sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} \beta_k \textit{ and } \exists\, i \textit{ with} \\
& & \alpha_j = \beta_j,\ i - 1 \leq j < n \textit{ and } \alpha_i < \beta_i.
\end{array}
$$

In Maple this order is called tdeg.

The term orders b.) and c.) use the notion of the degree of a polynomial. There is a concept of a generalized degree which we will use later. Since it makes the notion of a term order more transparent we recall it here.

[1] Fields in general purpose Computer Algebra systems are $\mathbf{Q}, \mathbf{Z}_p$ or extensions. The fields of real numbers and complex numbers are not computable. Within symbolic computations the use of floating point numbers does not make sense. Also the conversion of floating point numbers to rationals and then computing exactly is of doubtful value.

degree				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
6	x_1^3	$x_1^2x_2^2$	$x_1x_2^4$	x_2^6
5	$x_1^2x_2$	$x_1x_2^3$	x_2^5	
4	x_1^2	$x_1x_2^2$	x_2^4	
3	x_1x_2	x_2^3		
2	x_1	x_2^2		
1	x_2			
0	1			

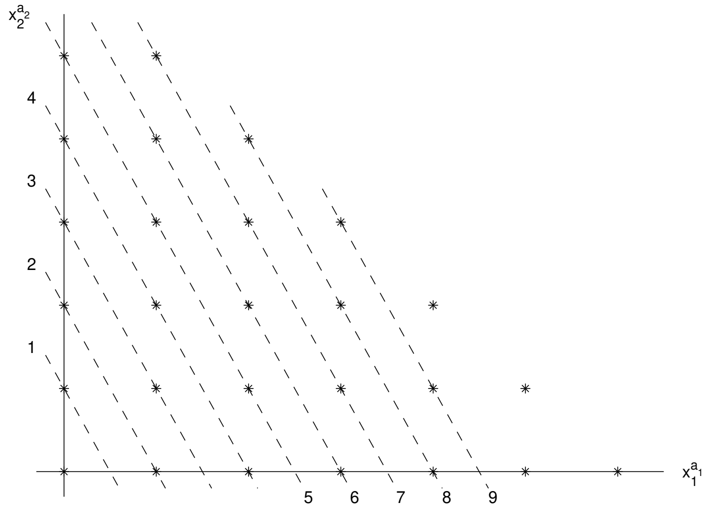

Figure 1.1: The ring $K[x_1, x_2]$ graded by $W(x_1) = 2, W(x_2) = 1$

Definition 1.1.3 *([52] p. 29) A ring is called* graded*, if a direct sum decomposition $R = \bigoplus_{i=-\infty}^{\infty} R_i$ exists such that*

$$R_i \cdot R_j \subseteq R_{i+j},$$

holds for all $i, j \in \mathbf{Z}$, where $R_i \cdot R_j$ is defined as $R_i \cdot R_j := \{r_i \cdot r_j | \; r_i \in R_i \text{ and } r_j \in R_j\}$.

Example 1.1.4 *: The polynomial ring $K[x]$ with the usual degree is a graded ring. Besides this natural grading there are other gradings: Let $w_1, \ldots, w_n \in \mathbf{Z}$ be weights on the variables $x_1, \ldots, x_n$. ($W : \{x_1, \ldots, x_n\} \to \mathbf{Z}, W(x_i) = w_i$). The* weighted degree *is defined by*

$$\deg_W(x^\alpha) = \sum_{i=1}^{n} w_i \alpha_i = w^t \alpha \,.$$

*Polynomials $f = \sum_{\alpha \in A} a_\alpha x^\alpha$ with $\deg_W(x^\alpha)$ equal for all $\alpha \in A \subset \mathbf{N}^n$ are called W-*homogeneous*. All homogeneous polynomials of degree i generate a vector space $H_i^W(K[x])$ yielding the graded structure*

$$K[x] = \bigoplus_{i=-\infty}^{\infty} H_i^W(K[x]) \,.$$

The natural grading $K[x] = \oplus_{i=0}^{\infty} H_i^N(K[x])$ is included by the weights $1, \ldots, 1$. All gradings of $K[x]$ are given by weights in this way ([14] p. 467).
In Figure 1.1 the example of the grading $W(x_1) = 2, W(x_2) = 1$ is illustrated. It is nice to observe that the word grading has its origin in the German word Grad.

Of course a ring may be graded several times. For examples with two gradings W_1, W_2 the vector space of all W_1-homogeneous and W_2-homogeneous polynomials of degree $\deg_{W_1,W_2}(p(x)) = (i, j)$ is given by

$$H_{i,j}^{W_1,W_2}(K[x]) := H_i^{W_1}(K[x]) \cap H_j^{W_2}(K[x]).$$

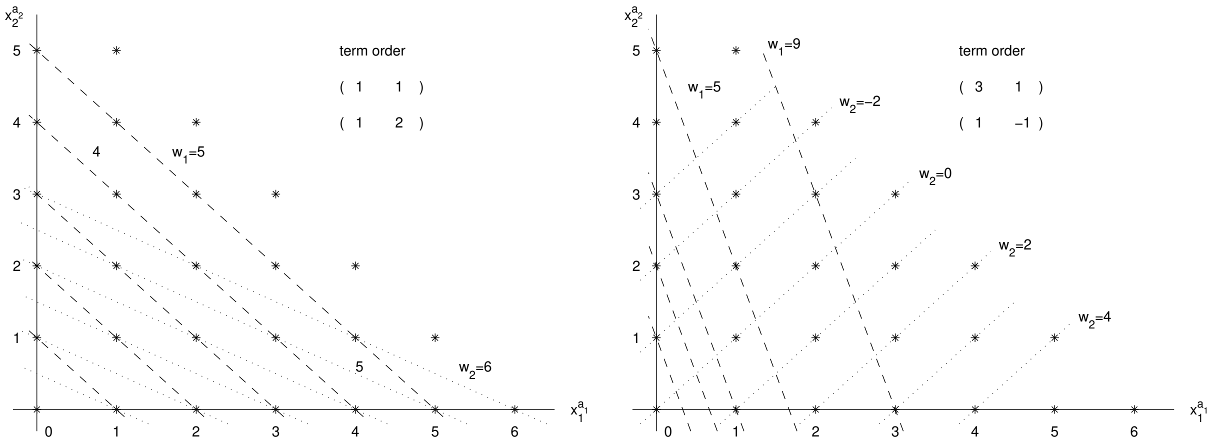

Figure 1.2: Two term orders defined by linear mappings and their interpretation as collection of gradings

Writing $W = (W_1, W_2)$ the bigrading is

$$K[x] = \bigoplus_{\alpha \in \mathbf{Z}^2} H_\alpha^W(K[x]) = \bigoplus_{i=-\infty}^{\infty} \bigoplus_{j=-\infty}^{\infty} H_{i,j}^{W_1,W_2}(K[x])\,.$$

Almost all term orders are defined by certain matrices whose rows define various gradings.

Definition 1.1.5 *: Let the matrix $M \in \mathbf{Z}^{n,n}$ have the following properties:*

i.) for each column j the first nonzero entry is positive.

$$\forall j\ \exists k \text{ with } m_{ij} = 0 \quad \forall i < k \text{ and } m_{kj} > 0.$$

ii.) M has full rank.

By $M\alpha < M\beta$ (which means there exists a k such that $(M\alpha)_i = (M\beta)_i$ for all $i < k$ and $(M\alpha)_k < (M\beta)_k$) a term order $<_M$ is defined.

The full rank of M assures that the relation $<_M$ is connex. The condition i.) on M assures that the constant is the smallest monomial, in other words the term order is Noetherian. In [161, 190] all term orders are classified by $(r \times n)$-matrices with real entries.

The interpretation is that each row of M defines a grading. First the grading of the first row sorts the monomials. The second grading sorts the monomials which have the same degree in the first grading and so on. Term orders are given by distinction with respect to different gradings. This principle is reflected in the examples in Figure 1.2.

For a polynomial $f \in K[x]$ one denotes by $ht(f)$ its *head term* or *leading term*, i.e. the monomial with highest order and non-vanishing coefficient $hc(f)$ in f:

$$f = hc(f) \cdot ht(f) + \text{lower order terms.}$$

Definition 1.1.6 *An ideal basis $\{f_1, \ldots, f_m\}$ of $I = \langle f_1, \ldots, f_m \rangle$ is called a* Gröbner basis, *if the ideal of the leading terms equals the* initial ideal *generated by all leading terms of elements of I:*

$$\langle ht(f_1), \ldots, ht(f_m) \rangle = \langle \{ht(f) | f \in I\} \rangle.$$

The initial ideal carries structural informations about the variety. Knowing the Gröbner basis enables the computations which involve the initial ideal such as computation of dimension, parameterization or the number of solutions for zero-dimensional ideals.

Before we discuss the details of the algorithmic computation of Gröbner basis the geometric interpretation of the leading term is presented.

Definition 1.1.7 *Let the polynomial $f \in K[x]$ have a representation $f(x) = \sum_{a\in\mathbf{N}^n} c_a x^a$, with coefficients $c_a \in K$ where only a finite number of coefficients is nonzero. Then*

$$supp(f) = \{a \in \mathbf{N}^n | \quad c_a \neq 0\},$$

is called the support *of f and the convex hull of $supp(f)$ in $\mathbf{R}^n$ is called the* Newton Polytope.

In general a polytope is a bounded convex set which is the intersection of a finite number of halfspaces given by hyperplanes.
Each vector $\omega \in \mathbf{R}^n$ defines a linear functional $\omega^* : \mathbf{R}^n \to \mathbf{R}, a \mapsto \omega^t a.$

Definition 1.1.8 *([195]) A subset of a polytope P in $\mathbf{R}^n$ is called* face *if a linear functional ω^* attains its maximum over the polytope at this set. The vector ω is called an* outer normal *of F and we write $F_\omega \subset P$. The* dimension *of a convex set is the dimension of the affine space generated by its points. A face F of dimension* $\dim F = \dim P - 1$ *is called* facet.

In most cases polytopes of dimension n in $\mathbf{R}^n$ are considered. Then the outer normal of a facet is unique up to normalization.

In the context of Newton Polytopes $NewP(f)$ we are interested in cases where the linear functional is given by a grading $W \in \mathbf{Z}^n$ or a row of a matrix defining a term order. Then $F := \mathbf{Z}^n \cap F_W$ picks some points of the lattice where F_W is the face of $NewP(f)$ corresponding to the grading W. The polynomial $in_W(f) = \sum_{a\in\mathbf{Z}^n\cap F_W} c_a x^a$ is called the *initial form.*

Lemma 1.1.9 *Let $M \in \mathbf{Z}^n$ fulfill the conditions in Definition 1.1.5 denoting by $<_M$ the associated term order and let the rows be $W_1, \ldots, W_n$. For each polynomial $f \in K[x]$*

$$hc_{<_M}(f) \cdot ht_{<_M}(f) = in_{W_n}(\cdots in_{W_2}(in_{W_1}(f))\cdots).$$

The example in Figure 1.3 illustrates that the first row (grading) of a matrix term order is often sufficient to determine the leading term.

We return to the definition of the Gröbner basis and its importance for practical computations. If $F = \{f_1, \ldots, f_m\} \subset K[x]$ is a Gröbner basis with respect to a term order then the question whether $f \in I := \langle f_1, \ldots, f_m\rangle$ can be decided in finitely many steps. If $j \in \{1, \ldots, m\}$ and a monomial x^α with $x^\alpha \cdot ht(f_j) = ht(f)$ and $c := hc(f)/hc(f_j)$ exists, the new polynomial $r := f - cx^\alpha f_j \in K[x]$ is easier than f in the sense that $ht(r) < ht(f)$. Observe that r is an element of the ideal I if and only if $f \in I$. Repeating this procedure which is called *top reduction* we end with a polynomial g which has lowest possible leading term, i.e. $ht(g) < \min_{j=1,\ldots,m}(ht(f_j))$. If F is a Gröbner basis then $f \in I$ is equivalent to $g = 0$. The repeated procedure is called *division algorithm* while the result g is called

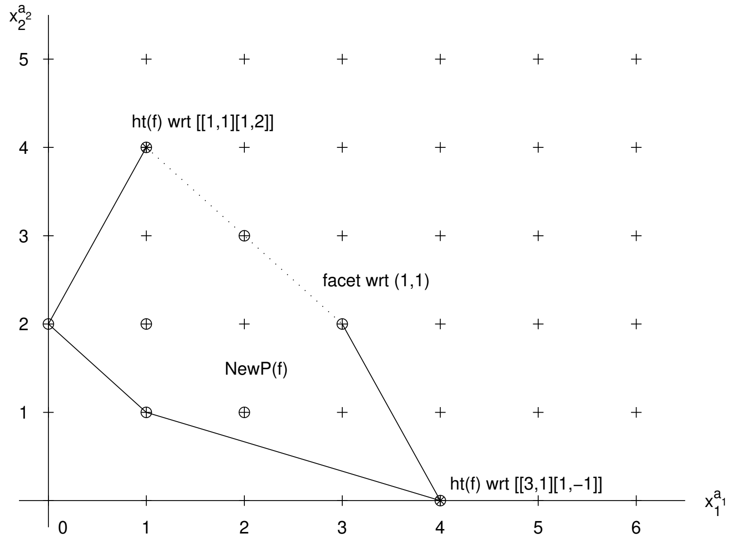

Figure 1.3: Support, Newton Polytope, and leading terms with respect to 2 term orders of $f(x) = x_1x_2^4 + 2x_1^2x_2^3 - x_1^3x_2^2 + 4x_1^4 + 6x_1^2x_2 + 3x_1x_2 + 2x_1x_2^2 + 9x_2^2$. The initial form $in_\omega(f) = x_1x_2^4 + 2x_1^2x_2^3 - x_1^3x_2^2$ with respect to the natural grading $\omega = (1,1)$ is visualized as corresponding facet of the Newton Polytope

normal form. It is denoted by $g = \text{normalf}(f)$ and more precisely $g = \text{normalf}_<(f(x), F)$, if we want to express the dependence on the set F and the term order $<$. Besides the top reductions of course also elimination of other monomials of f is possible. Observe that the reductions and thus the normal form depend on the term order. If F is a Gröbner basis the normal form does not depend on how the division algorithm is performed. For more details see [14] p. 195–204.

Lemma 1.1.10 *([177] Prop. 1.1, [14] p. 206) Let $F = \{f_1, \dots, f_m\} \subset K[x]$ be a Gröbner basis with respect to a term order $<$ and denote by $I = \langle f_1, \dots, f_m \rangle$ the ideal generated by them.*

i.) The polynomials g which are in normal form with respect to F and $<$ form unique representatives of classes $g + I$ of the quotient ring $K[x]/I$.

ii.) The classes of the monomials x^α with $x^\alpha \notin \langle ht(f_1), \dots, ht(f_m) \rangle$ form a vector space basis of the quotient ring $K[x]/I$.

The importance of i.) is that one can decide algorithmically whether f is a member of the ideal or not. One just applies the division algorithm.
The monomials in ii.) are called *standard monomials.*

Definition 1.1.11 *([41] p. 90, [177] Prop. 1.1) A Gröbner basis $\{f_1, \dots, f_m\}$ such that $\{ht(f_1), \dots, ht(f_m)\}$ form a minimal basis of the initial ideal and $hc(f_1) = 1 = \dots = hc(f_m)$ is called* minimal Gröbner basis. *If the f_i are additionally inter-reduced F is called a* reduced Gröbner basis.

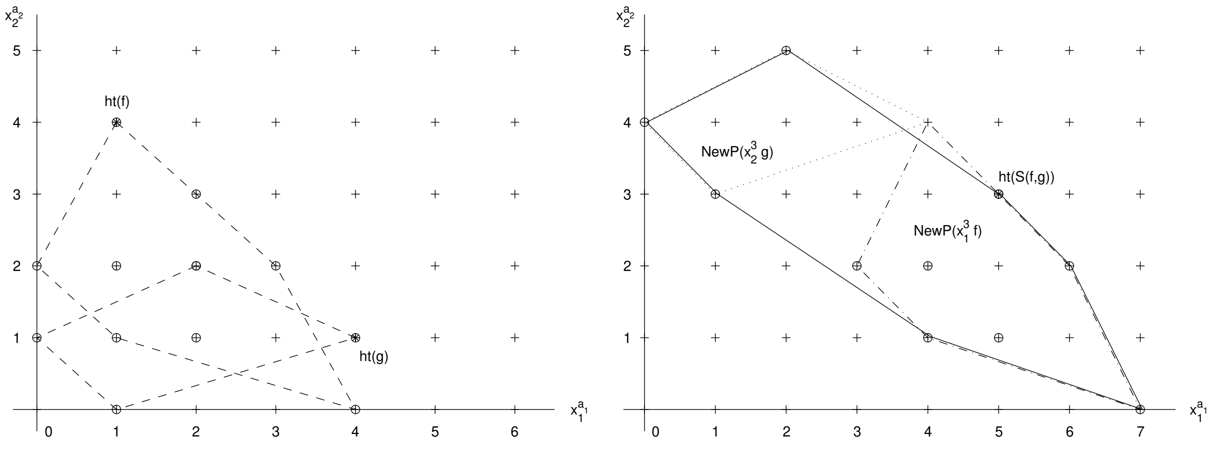

Figure 1.4: The Newton Polytopes of $g(x) = x_1^4x_2 + 5x_1^2x_2^2 - 3x_1 + 2x_2$ as well as of f (Figure 1.3) are illustrated in the picture on top. The picture at the bottom gives the support, the Newton Polytope, and leading term of the S-polynomial $S(f,g) = x_1^3f(x) - x_2^3g(x)$ along with the Newton Polytopes of $x_1^3f(x)$ and $x_2^3g(x)$

For each term order and each ideal in $K[x]$ there exists a Gröbner basis which generates this ideal. Of course there may be many Gröbner bases with this property. But the reduced Gröbner basis is *unique*. Although there exist infinitely many term orders for each ideal only a finite number of reduced Gröbner bases exists ([11, 149, 177]). This is a consequence of the polynomial ring being Noetherian, see [177] Thm. 1.2.

The following classification of Gröbner bases is the foundation of its algorithmic determination. The next example clearly shows the point.

Example 1.1.12 *Do the polynomials* $g(x) = x_1^4x_2 + 5x_1^2x_2^2 - 3x_1 + 2x_2$ *and* $f(x) = x_1x_2^4 + 2x_1^2x_2^3 - x_1^3x_2^2 + 4x_1^4 + 6x_1^2x_2 + 3x_1x_2 + 2x_1x_2^2 + 9x_2^2$ *with leading terms* $ht(g) = x_1^4x_2$ *and* $ht(f) = x_1x_2^4$ *with respect to the matrix term order* $[[1,1],[1,2]]$ *form a Gröbner basis? If they do form a Gröbner basis, then each element of* $I = \langle f, g\rangle$ *can be reduced to zero by the division algorithm. For example* $h(x) := x_1 \cdot x_2 \cdot g(x) + 5 \cdot x_2 \cdot f(x) \in I$. *Since* $ht(h) = x_2 \cdot ht(f) = x_1x_2^5$ *the polynomial* $r := h - 5x_2f$ *gives a top reduction* $h = 5x_2f + r$ *with* $ht(r) < ht(h)$. *The next reduction is with* $r = x_1 \cdot x_2 \cdot g$ *obvious and gives normalf*$(h) = 0$.

But for $k(x) := x_1^3f(x) - x_2^3g(x)$ *the normal form computation is less obvious since* $ht(k) < ht(x_1^3f) = ht(x_2^3g)$. *The cancellation of leading terms in* $k(x)$ *is the problem. For illustration see Figure 1.4.*

Theorem 1.1.13 *([41] p. 106) Let* $F = \{f_1, \ldots, f_m\} \subset K[x]$ *generate the ideal* $I = \langle f_1, \ldots, f_m\rangle$. *Fix a term order* $<$. *Then the following statements are equivalent*

i.) F *forms a Gröbner basis of* I *with respect to* $<$.

ii.) For all $f \in I$ *the division algorithm gives normalf*$_<(f, F) = 0$.

iii.) For all $(g_1, \ldots, g_m) \in K[x]^m$ *the division algorithm computes the normal form of* $\sum_{j=1}^m g_jf_j$ *to be zero, i.e. normalf*$_<(\sum_{j=1}^m g_j(x)f_j(x), F) = 0$.

iv.) For all $(s_1, \ldots, s_m) \in K[x]^m$ *with* $\sum_{j=1}^m s_j(x) \cdot hc(f_j) \cdot ht(f_j)(x) = 0$ *the division algorithm gives* $normalf_<(\sum_{j=1}^m s_j(x) f_j(x), F) = 0$.

Remark 1.1.14 *The Condition ii.) enables to explain the link between Gröbner bases and H-bases as they have been introduced in [134]. The set* $F = \{f_1, \ldots, f_m\}$ *of degrees* $d_1, \ldots, d_m$ *is called a H-*basis *(nowadays called* Macaulay basis*) if for each* $f \in I = \langle f_1, \ldots, f_m \rangle$ *of degree d there exists a representation* $f(x) = g_1(x) \cdot f_1(x) + \cdots + g_m(x) \cdot f_m(x)$ *with* $\deg(g_i) \leq d - d_i, i = 1, \ldots, m$. *Writing* $N = (1, \ldots, 1)$ *for the natural grading this is equivalent to* $\langle in_N(f_1) \ldots, in_N(f_m) \rangle = \langle \{ in_N(f) | f \in I \} \rangle$. *Instead of the natural grading the definition can be extended to* ω*-H-bases with respect to gradings* ω *given by the first row of a matrix term order. Condition ii.) means that each* f *has a representation* $f = g_1 f_1 + \cdots + g_m f_m$ *with* $ht(g_1 f_1) \leq ht(f), \ldots, ht(g_m f_m) \leq ht(f)$. *If* ω *is the first row of the matrix term order this includes* $\deg_\omega(g_1 f_1) \leq \deg_\omega(f), \ldots, \deg_\omega(g_m f_m) \leq \deg_\omega(f)$. *Thus condition ii.) shows that Gröbner bases are just a generalization of* ω*-H-bases. For more on the relation between Gröbner bases and Macaulay bases see [147] and [182] Section 2.3.*

Because of their importance the elements in iv.) have a special name.

Definition 1.1.15 *Given a set* $F = \{f_1, \ldots, f_m\}$ *with leading terms* $ht(f_i)$ *a* syzygy *is a tuple* $(s_1, \ldots, s_m) \in K[x]^m$ *such that*

$$\sum_{i=1}^{m} s_i \cdot hc(f_i) \cdot ht(f_i) = 0.$$

The set of all syzygies forms a $K[x]$-module denoted by $S(F)$. Each syzygy s corresponds to a polynomial in the ideal generated by F by defining $s \cdot F = \sum_{i=1}^m s_i f_i$.

The Buchberger algorithm is based on the fact that special sparse syzygies

$$S^{ij} \in S(F), \quad 1 \leq i < j \leq m, \tag{1.1}$$

form a module basis of $S(F)$, where $S_k^{ij} = 0, \forall\, k \neq i, k \neq j$, and

$$S_i^{ij} = \frac{\mathrm{lcm}(ht(f_i), ht(f_j))}{ht(f_i)} hc(f_j), \quad S_j^{ij} = -\frac{\mathrm{lcm}(ht(f_i), ht(f_j))}{ht(f_j)} hc(f_i).$$

$S(f_i, f_j) := S^{ij} \cdot F$ is called *S-polynomial.* Although S^{ij} is defined for the index order $i < j$ only, the notation is often used in a sloppy way by defining $S^{ji} := S^{ij}, S(f_j, f_i) := S(f_i, f_j)$.

Theorem 1.1.16 *([41] p. 106) Let* $F = \{f_1, \ldots, f_m\} \subset K[x]$ *generate the ideal* $I = \langle f_1, \ldots, f_m \rangle$ *and assume a term order* $<$. *The following statements are equivalent*

i.) F forms a Gröbner basis of I with respect to $<$.

ii.) All syzygies $s \in S(F)$ *have the property that* $s \cdot F$ *reduces to zero with respect to* $<$ *and F.*

iii.) All elements s in a module basis of $S(F)$ *have the property that* $normalf(s \cdot F, F) = 0$.

iv.) All S-polynomials $S(f_i, f_j), 1 \leq i < j \leq m$ *reduce to zero.*

The basic version of the classical Buchberger algorithm is based on criterion *iv.)*.

Algorithm 1.1.17 *(Buchberger [23, 24])*
Input: $F = \{f_1, \ldots, f_m\}$, *term order* $<$
Output: *Gröbner basis* $\mathcal{GB}$ *of* $I = \langle f_1, \ldots, f_m \rangle$

$\mathcal{GB} := F$
$m := |F|$
$S := \{(i,j) | 1 \le i < j \le m\}$
while $S \ne \{\}$ *do*
  choose $(i,j) \in S$,
  $S := S \setminus \{(i,j)\}$
  $g :=$*normalf*$(S(f_i, f_j))$ *wrt* $<$ *and* $\mathcal{GB}$
  if $g \ne 0$ *then* $\mathcal{GB} := \mathcal{GB} \cup \{g\}$
    $m := m + 1$
    $S := S \cup \{\, (i,m) \mid \quad i = 1, \ldots, m-1\}$
minimal Gröbner basis
for each $g \in \mathcal{GB}$ *do*
  if $ht(g) \in \langle \{ht(f) | f \in \mathcal{GB}, f \ne g\} >$ *then* $\mathcal{GB} := \mathcal{GB} \setminus \{g\}$
reduced Gröbner basis
for each $g \in \mathcal{GB}$ *do*
  $\tilde{g} :=$*normalf*(g) *wrt* $\mathcal{GB} \setminus \{g\}$ *and* $<$
  $\mathcal{GB} := \mathcal{GB} \setminus \{g\} \cup \{\tilde{g}\}$

Although the set of pairs S is enlarged the terminates because the polynomial ring is Noetherian.

Remark 1.1.18 *For each* $g \in \langle \mathcal{GB} \rangle$ *one easily finds the coefficient polynomials in the representation* $g(x) = \sum_{f \in \mathcal{GB}} g^f(x) \cdot f(x)$ *by the division algorithm. The coefficients* g_i *in the representation* $g = \sum_{i=1}^{m} g_i(x) \cdot f_i(x)$ *can be computed as well. But a bookkeeping of reduction steps during the Buchberger algorithm is required. For details see [3]. The complexity of Algorithm 1.1.17 is discussed in [139, 123].*

For the efficient computation various aspects are important.

- *software aspects*: good data structures for storing multivariate polynomials and determination of the leading term are important.
- *growth of coefficients*: the computations in $K = \mathbf{Q}$ or $\mathbf{Z}$ respectively may lead to enormous integer arithmetic with very long numbers. An attempt to get around this is the computation modulo a prime number as is done in Macaulay [88]. Secondly, the content of polynomials is extracted. A heuristic is necessary in order to decide when contents (after each normal form computation or after each reduction step) are computed and extracted.
- *parallel* implementation [160].
- *ambiguity of division algorithm*: Often several polynomials would fit for a reduction step. A strategy proven to be efficient is to choose the oldest polynomial (the first on the list which fits).

- *order of S-polynomials*: The description in Algorithm 1.1.17 does not clarify which of the S-polynomials to choose first for the normal form computation. A good strategy is the so-called *sugar selection strategy*, introduced in [83] where a ghost degree is associated to each polynomial and thus to each S-polynomial. For the input polynomials this is just their degree. For all after polynomials the sugar is determined recursively, see also [66].

- *superfluous S-polynomials*: most important for efficiency is to avoid the application of the division algorithm to S-polynomials which will reduce to zero anyway. The first point are the so-called Buchberger criteria which reflects the fact that the S-polynomials correspond to a module basis of the module of syzygies which do not form a minimal basis. Secondly, the Hilbert series may be exploited, see Section 1.2.

The main point for efficiency of the Buchberger algorithm is to avoid the treatment of superfluous S-polynomials. Proposition 4 in [41] shows that if $\mathrm{lcm}(ht(f_i), ht(f_j)) = ht(f_i) \cdot ht(f_j)$ the syzygy $S^{ij} \in S(F)$ leads to a S-polynomial which automatically reduces to zero ($\mathrm{normalf}_<(S(f_i, f_j)) = 0$ with respect to $\{f_i, f_j\}$). The occurrence of this case is easily checked since only a property of leading terms needs to be tested:

1. Buchberger criterion: If $ht(f_i), ht(f_j)$ are relatively prime then $S(f_i, f_j)$ is superfluous.

In [143], an improvement of the 1. criterion is formulated. But this criterion is more expensive since it requires the factorization of polynomials.

Criterion D: If $g(x) \in K[x]$ is a common divisor of f_i and f_j, i.e. $f_i(x) = g(x) \cdot f_i^*(x)$ and $f_j(x) = g(x) \cdot f_j^*(x)$ and $ht(f_i^*)$ and $ht(f_j^*)$ are coprime then $\mathrm{normalf}(S(f_i, f_j)) = 0$ with respect to $<$ and f_i, f_j.

A second group of superfluous S-polynomials is given by the fact that in general S^{ij} do not form a minimal module basis. If f_k divides $\mathrm{lcm}(f_i, f_j)$ then $S^{ij} = c_i x_i^\alpha \cdot S^{ik} + c_j x_j^\alpha S^{jk}$ for appropriate $c_i x_i^\alpha, c_j x_j^\alpha$, see Proposition 10 in [41] or Proposition 5.70 in [14].

2. Buchberger criterion: If $ht(f_k) | \mathrm{lcm}(ht(f_i), ht(f_j))$ and $\mathrm{normalf}(S(f_i, f_k)) = 0$ as well as $\mathrm{normalf}(S(f_j, f_k)) = 0$ are satisfied then $\mathrm{normalf}(S(f_i, f_j)) = 0$.

The criterion characterizes a case where among the generators S^{ik}, S^{jk}, S^{ij} the generator S^{ij} is superfluous in a generating set of the module of syzygies. Observe that

$$\begin{aligned} & ht(f_k) \,|\, \mathrm{lcm}(ht(f_i), ht(f_j)) \\ \Leftrightarrow \quad & \mathrm{lcm}(ht(f_i), ht(f_k)) \,|\, \mathrm{lcm}(ht(f_i), ht(f_j)) \\ \Leftrightarrow \quad & \mathrm{lcm}(ht(f_j), ht(f_k)) \,|\, \mathrm{lcm}(ht(f_i), ht(f_j)). \end{aligned}$$

The use of the 2. Buchberger criterion becomes complicated, if one tries to use it **before** $S(f_i, f_k), S(f_j, f_k)$ have been treated. In case $\mathrm{lcm}(ht(f_i), ht(f_k)) = \mathrm{lcm}(ht(f_i), ht(f_j))$ there is the danger to delete both pairs $(i, k), (i, j)$ without doing any normal form computation which is wrong in general. Either S^{ik} or S^{ij} is superfluous, but not both. That's why Gebauer and Möller [79] split it into three criteria. The index order $i < j$ is assumed.

Criterion M(i, j)**:** If k exists such that $k < j$ and $\mathrm{lcm}(ht(f_k), ht(f_j))$ is a divisor of $\mathrm{lcm}(ht(f_i), ht(f_j))$, but $\mathrm{lcm}(ht(f_k), ht(f_j)) \neq \mathrm{lcm}(ht(f_i), ht(f_j))$, then $S(f_i, f_j)$ is superfluous.

Criterion F(i, j)**:** If k exists such that $k < i$ and $\mathrm{lcm}(ht(f_k), ht(f_j)) = \mathrm{lcm}(ht(f_i), ht(f_j))$, then $S(f_i, f_j)$ is superfluous.

Criterion $\mathbf{B}_k(i,j)$: If k exists such that $k > j$ and $\mathrm{lcm}(ht(f_k), ht(f_j))$ is a divisor of $\mathrm{lcm}(ht(f_i), ht(f_j))$, and $\mathrm{lcm}(ht(f_k), ht(f_j)) \neq \mathrm{lcm}(ht(f_i), ht(f_j))$, $\mathrm{lcm}(ht(f_k), ht(f_i)) \neq \mathrm{lcm}(ht(f_i), ht(f_j))$, then the S-polynomial $S(f_i, f_j)$ is superfluous.

Experience shows that the correct exploitation of the second Buchberger criterion is a tricky task. It requires the distinction between new and old critical pairs and a sophisticated ordering of pairs and use of parts of the criterion, see [83], [66] and [14] p. 230.

It is well-known that the standard implementations in the general purpose systems Mathematica and Maple are rather pour. Better are special implementations such as GB [55], Macaulay [12], Macaulay 2 [88], the Groebner package in REDUCE [142], CoCoa [31], Singular [90], Magma [30], the package Mgfun [37] in Maple V.5, Bergman, Felix, MAS, and the package moregroebner [66] in Maple.

Besides the Gröbner bases in polynomial rings there are as well Gröbner bases for ideals where the coefficient field is a polynomial ring itself and secondly for ideals in non-commutative algebras. For the computation over rings the theoretical results on these so-called comprehensive Gröbner bases are given in [191] and an implementation is available in REDUCE. The non-commutative Gröbner bases are needed for the solution of differential equations and for the computation of recurrence formulas. In both cases it is not the general non-commutative case. For theoretical results we exemplary cite [103]. Non-commutative Gröbner bases are implemented in REDUCE, Bergman, and Mgfun.

The third type of generalization of Gröbner bases is given by a modification of the property of term orders. The standard bases are defined with respect to a term order which do not insist to be Noetherian (property (v) in Definition 1.1.1) thus being intermediate between the tangent cone algorithm by Mora and the traditional Gröbner bases as described in this chapter. Exemplary we cite [89].

Besides the generalization of Gröbner bases also the restriction to special cases such as the binomial ideals occurring in the context of integer programming ([42], Chapter 8) are important. As an example of several articles in the literature I refer to [117].

1.2 The consequence of grading

For special ideals the definition of a truncated Gröbner basis makes sense. The same idea enables the extension of the Gröbner basis concept to modules and gives rise to an efficient version of the Buchberger algorithm. The explanation of these results is the purpose of this section.

1.2.1 Definitions and the relation to Gröbner bases

Definition 1.2.1 *([52] p. 42) Consider a module M over the ring R which is assumed to be graded by $R = \oplus_{i=-\infty}^{\infty} R_i$. The module M is called* graded*, if it is a direct sum $M = \bigoplus_{j=-\infty}^{\infty} M_j$ such that $R_i \cdot M_j \subset M_{i+j}$ $\forall\, i, j \in \mathbf{Z}$.*

Example 1.2.2 *: Let $W : \{x_1, \ldots, x_n\} \to \mathbf{Z}, W(x_i) = w_i$ be a grading of $K[x_1, \ldots, x_n]$.*

i.) Each ideal of $K[x]$ which is generated by W-homogeneous polynomials is an example for a graded module:

$$I = \bigoplus_{i=-\infty}^{\infty} H_i^W(I)\,.$$

For $W(x_1) = 1, W(x_2) = 2$ the ideal $\langle x_2x_1+5x_1^3, x_2^2-2x_2x_1^2-x_1^4\rangle$ is W-homogeneous. On the other hand for each W-graded ideal there exists a set of generators which are W-homogeneous each.

ii.) If I is a W-homogeneous ideal the quotient ring $K[x]/I$ is a W-graded module over $K[x]$ as well.

Definition 1.2.3 *Let $K[x]$ be graded by W. A W-*homogeneous ideal *I of $K[x]$ is an ideal which respects the grading, i.e. is a W-graded module.*

It is well-known that homogeneous ideals are very valuable for the study of the solutions of a system of polynomial equations by interpreting the affine zeros as curves in the *weighted projective space*: Let $I = \langle f_1, \ldots, f_m\rangle \subset K[x_1, \ldots, x_n]$, and

$$V(I) = \{x \in \mathbf{C}^n |\ f(x) = 0, \forall\, f \in I\},$$

its affine variety and $w = (w_0, w_1, \ldots, w_n) \in N^{n+1}$ a grading W on $K[x_0, x_1, \ldots, x_n]$. Then

$$\tilde{f}_i(x_0, x_1, \ldots, x_n) := f_i(x_0^{-w_1}x_1^{w_0}, \ldots, x_0^{-w_n}x_n^{w_0}) \cdot x_0^{\deg_W(f_i)}, \quad i = 1, \ldots, m,$$

defines W-homogeneous polynomials of degrees $\deg(\tilde{f}_i) = w_o \deg_W(f_i)$ and thus a W-homogeneous ideal $\tilde{I} = \langle \tilde{f}_1, \ldots, \tilde{f}_m\rangle$. Since homogeneous ideals are generated by homogeneous polynomials the varieties have special properties:

$$x \in V(\tilde{I}) \quad \Rightarrow \quad (a^{w_0}x_0, a^{w_1}x_1, \ldots, a^{w_n}x_n) \in V(\tilde{I}), \quad \forall\, a \in \mathbf{C}.$$

This one-dimensional torus action implies an equivalence relation in the natural way. The classes $[(1, x_1, \ldots, x_n)] \in V(\tilde{I})/\!\sim$ correspond to the affine zeros $(x_1, \ldots, x_n) \in V(I)$. Secondly, it is sufficient to deal with Gröbner bases of $\tilde{I}$. Choosing a matrix term order

$$\begin{pmatrix} w_0 & w_1 \cdots w_n \\ 0 & \\ \vdots & M \\ 0 & \end{pmatrix} \tag{1.2}$$

a Gröbner basis of $\tilde{I}$ corresponds by substitution of $x_0 = 1$ to a Gröbner basis of I with respect to the term order defined by M. This is obvious by the elimination technique explained in Section 1.5. That's why one restricts in Macaulay to ideals which are homogeneous with respect to a grading.

Of course also multiple grading may occur in several situations. Its algorithmic exploitation is the purpose of this section.

Example 1.2.4 *Let $W = \{W_1, \ldots, W_r\}$ be a set of gradings of $K[x_1, \ldots, x_n]$ such that $K[x] = \oplus_{j=-\infty}^{\infty} H_j^{W_i}(K[x]), i = 1, \ldots, r$. Then*

$$K[x] = \bigoplus_{j \in \mathbf{Z}^r} H_j^W(K[x]) \text{ with } H_j^W(K[x]) = H_{j_1}^{W_1}(K[x]) \cap \cdots \cap H_{j_r}^{W_r}(K[x]),$$

is a multiple grading of the ring $K[x]$. Consider $K[x,z]$ with the Kronecker grading $\Gamma(x_k) = 0, k = 1,\dots,n, \Gamma(z_l) = 1, l = 1,\dots,s$. *Then the module $H_1^\Gamma(K[x,z])$ is multigraded by $W = \{W_1,\dots,W_r\}$ by extension to $K[x,z]$ by adding weights 0 on the z_l.*

$$H_1^\Gamma(K[x,z]) = \bigoplus_{j\in\mathbf{Z}^r} H_{1,j}^{\Gamma,W}(K[x,z]).$$

Grading enables the definition of truncated Gröbner bases and module Gröbner bases.

Definition 1.2.5 *Let $W = \{W_1,\dots,W_r\}$ be some gradings of $K[x_1,\dots,x_n]$ with $W_i \in \mathbf{N}^n, i = 1,\dots,r$ and I a W-homogeneous ideal. Let $<$ be a term order and $d \in \mathbf{N}^r$ be a fixed degree. A finite set of W-homogeneous polynomials $F \subset I$ is called a d-*truncated Gröbner basis *of I with respect to W and term order $<$, if*

$$\{ht(f)| \quad f \in F \text{ and } deg_{W_i}(f) \le d_i, i = 1,\dots,r\},$$

generates

$$\bigcap_{i=1}^{r} \bigoplus_{j_i=0}^{d_i} H_{j_i}^{W_i}(LT(I)) = \bigoplus_{j\le d} H_j^W(LT(I)),$$

where $LT(I) = \langle\{ht(f)|f \in I\}\rangle$ denotes the initial ideal. The truncated Gröbner basis is denoted by $\mathcal{GB}\left(\bigoplus_{j\le d} H_j^W(I)\right)$.

Remark 1.2.6 *i.) Observe that the grading is restricted such that the weights are non-negative. Then $H_0^W(K[x])$ is the smallest part and there are no components with negative index. ii.) In [26] the truncated Gröbner basis is defined in a similar way, but using W-compatible term orders, i.e. putting rows $W_1,\dots,W_r$ as first rows of the matrix representing the term order. But the compatibility is not necessary. In [26] is is only needed for the formulation. Even if the Buchberger algorithm is started with non-homogeneous generators of a W-homogeneous ideal and using a non-compatible term order the computed reduced Gröbner basis will consist of W-homogeneous polynomials. So the computation of a truncated Gröbner basis with respect to a non-compatible ordering is no problem as long as we start with homogeneous polynomials.*

This definition is useful in at least two ways.

Definition 1.2.7 *([14]) Consider $K[x,z]$ together with the* Kronecker grading $\Gamma(x_i) = 0, i = 1,\dots,n, \Gamma(z_j) = 1, j = 1,\dots,s$. *A* module Gröbner basis *of a submodule of the module $H_1^\Gamma(K[x,z])$ is a truncated Gröbner basis of Γ-degree 1.*

Since every finitely generated, free $K[x]$-module is isomorphic to a module $H_1^\Gamma(K[x,z])$ the truncated Gröbner bases enables to deal algorithmically with finitely generated modules and their submodules.

Lemma 1.2.8 *Let $W = \{W_1,\dots,W_r\}$ be a set of gradings of $K[x]$ and $d \in \mathbf{N}^r$ a fixed degree. Assume $\mathcal{GB} \subset K[x]$ is a d-truncated Gröbner basis of a W-homogeneous ideal I. Let $f \in K[x]$ be a polynomial with $deg_{W_i}(f) \le d_i$, $i = 1,\dots,r$. Then*

$$f \in I \Leftrightarrow normalf(f, \mathcal{GB}) = 0.$$

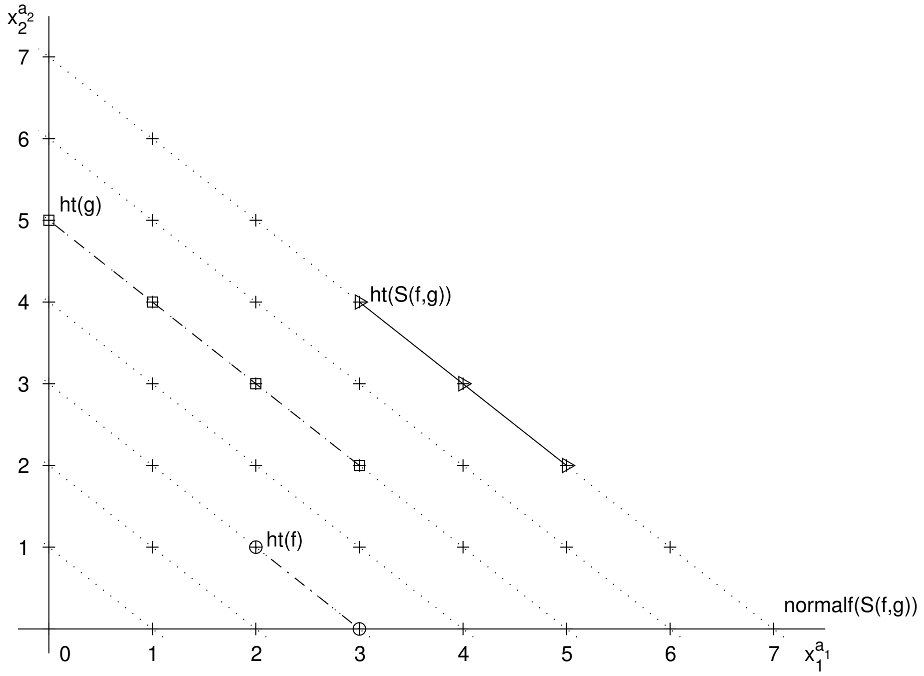

Figure 1.5: The supports of $g(x) = x_2^5 + 2x_2^4x_1 - x_2^3x_1^2 - 3x_2^2x_1^3$ and $f(x_1, x_2) = x_2x_1^2 + x_1^3$, and $S(f,g) = x_2^4 f - x_1^2 g$ as well as normalf$(S(f,g), \{f,g\}) = x_1^7$. The matrix term order is $[[1,1],[1,2]]$. Since f and g are homogeneous with respect to the natural grading so is $S(f,g)$ and each other intermediate polynomial of the Buchberger algorithm

This is the generalization of Thm. 10.39, p. 471 in [14] from one grading to multiple grading. As shown in [74] the restriction in degree can make difficult problems computable in short time.

In the following I consider the efficient computation of truncated Gröbner bases. The gradings $W = \{W_1, \ldots, W_r\}$ give rise to gradings on the module of syzygies $S(F)$ for a set $F = \{f_1, \ldots, f_m\}$ of homogeneous polynomials in the following way:

$$S \in H^{W_1 \ldots W_r}_{j_1 \ldots j_r}(S(F)) \quad :\Leftrightarrow S_k \cdot ht(f_k) \in H^{W_1 \ldots W_r}_{j_1 \ldots j_r}(K[x]), \quad k = 1, \ldots, m.$$

Especially the syzygies S^{ij} corresponding to the S-polynomials (defined in (1.1)) are homogeneous of degree

$$\deg_W(lcm(ht(f_i), ht(f_j))).$$

A W-homogeneous syzygy S of degree $(j_1, \ldots, j_r)$ gives a W-homogeneous polynomial $S \cdot F = S_1 \cdot f_1 + \cdots + S_m \cdot f_m$. Even more the degree of $S \cdot F$ is known before it is computed since it is the degree of the syzygy. Additionally the division algorithm preserves the homogeneity and the degree. Sloppy speaking the Buchberger algorithm is performing in the slices $H_j^W(K[x])$. Figure 1.5 gives an illustration of this abstract concept.

Consequently, the following heuristic is suggested as selection strategy of S-polynomials: sort the S-polynomials by their degree and treat the lowest first. On average this will give the best exploitation of Buchberger criteria.

Some structural information of an ideal is given by the grading which can be exploited by a special variant of the Buchberger algorithm. The following concept goes back to Hilbert.

Definition 1.2.9 *([8] p.116) Let M be a finitely generated module M graded by $M = \bigoplus_{i=0}^{\infty} M_i$ over a Noetherian graded ring $R = \bigoplus_{j=0}^{\infty} R_j$ such that $R_0 = K$ is a field. Then*

$$\mathcal{HP}(\lambda) = \sum_{i=0}^{\infty} \dim(M_i) \cdot \lambda^i,$$

is called the Hilbert-Poincaré series *of M and $h : \mathbf{N} \to \mathbf{N}, h(i) = \dim(M_i)$ is the* generating function *or* characteristic function. *Here $\dim(M_i)$ denotes the dimension of the K-vector spaces M_i.*

If M is multi-graded the multiple Hilbert series is defined in a similar way:

$$\mathcal{HP}_M^W(\lambda_1, \ldots, \lambda_r) = \sum_{i \in \mathbf{N}^r}^{\infty} \dim\left(H_{i_1,\ldots,i_r}^{W_1,\ldots,W_r}(M)\right) \cdot \lambda^{i_1} \cdots \lambda^{i_r}.$$

The series is always represented as a rational function in the variables λ. This enables the algorithmic use. Especially one compares the series of a module and some submodules. Once this rational representation is known its Taylor expansion gives the dimension of the vector spaces yielding some valuable structural information, see Section 1.2.3 and Section 2.1.

Under some restriction the gradings on the polynomial ring $K[x]$ fulfill the requirements in Definition 1.2.9 since $H_0^W(K[x])$ is a subring of $K[x]$ and the spaces $H_j^W(K[x])$ are $H_0^W(K[x])$-modules.

Definition 1.2.10 *([26]) A tuple of gradings $(W_1, \ldots, W_r)$ of $K[x]$ is a* weight system *if*

a.) $W_j(x_i) \geq 0$, for all $j = 1, \ldots, r, i = 1, \ldots, n$,

b.) for all $i = 1, \ldots, n$ exists $j \in \{1, \ldots, r\}$ with $W_j(x_i) > 0$,

c.) $W_1, \ldots, W_r$ are linear independent.

The weight system guarantees $H_0^W(K[x]) = K$. For each W-homogeneous ideal I the quotient ring $K[x]/I$ is a graded module and its series $\mathcal{HP}_{K[x]/I}^W(\lambda)$ is well-defined. Then $\dim(H_i^W(K[x]/I)$ equals the codimension of $H_i^W(I)$ in $H_i^W(K[x])$.

Example 1.2.11 *The Kronecker grading on $K[x, z]$ in Example 1.2.4 is not a weight system. Because $H_0^\Gamma(K[x, z]) = K[x]$ the condition for the definition of the Hilbert-Poincaré series is not fulfilled. But on $K[x_1, x_2, x_3]$ the grading $W_1(x_1) = 2, W_1(x_2) = 1, W_1(x_3) = 0$ together with the second grading $W_2(x_1) = 0, W_2(x_2) = 3, W_2(x_3) = 5$ forms a weight system. Since $H_{0,0}^{W_1,W_2}(K[x]) = K$ the homogeneous components are K-vector spaces of finite dimension. The series of $K[x]$ is*

$$\mathcal{HP}_{K[x]}^{W_1,W_2}(\lambda_1, \lambda_2) = \frac{1}{(1 - \lambda_1^2)(1 - \lambda_1\lambda_2^3)(1 - \lambda_2^5)}.$$

For $I = (x_2^2)$ the quotient ring is $K[x]/I = K[x_1, x_3] \oplus x_2 K[x_1, x_3]$ and its series is

$$\mathcal{HP}_{K[x]/I}^{W_1,W_2}(\lambda_1, \lambda_2) = \frac{1}{(1 - \lambda_1^2)(1 - \lambda_2^5)} + \frac{\lambda_1^1\lambda_2^3}{(1 - \lambda_1^2)(1 - \lambda_2^5)}.$$

Hilbert series are not only an abstract concept but have a meaning for varieties. Assume $J \subset K[x_1, \ldots, x_n]$ is an ideal and $\tilde{J} \subset K[x_0, \ldots, x_n]$ its homogenization with respect to the natural grading. Then the codimension of $J \cap \mathbf{P}_i$ in $\mathbf{P}_i = \oplus_{j=0}^{i} H_i^N(K[x])$ equals $\dim\left(H_i^N(K[x]/\tilde{J})\right)$ and thus $\mathcal{HP}^N_{K[x]/\tilde{J}}(\lambda)$ equals the affine Hilbert series of J, $\sum_{j=0}^{\infty} \operatorname{codim}(J \cap \mathbf{P}_j \text{ in } \mathbf{P}_j)) \cdot \lambda^j$, see [41] p. 434. The Hilbert polynomial satisfies $p(j) = \operatorname{codim}(\tilde{J} \cap \mathbf{P}_j \text{ in } \mathbf{P}_j))$ for all j sufficiently big. It may be easily computed from the Hilbert series, see [13]. An example is given in 1.2.17. The degree of p is the *dimension* of the variety $V(J)$. If the Hilbert polynomial is a constant then the ideal is called zero-dimensional and has only finitely many zeros. This constant (codimension of $J \cap \mathbf{P}_j$ in $\mathbf{P}_j$ for some big j) is the number of affine zeros of J in $\mathbf{C}^n$ (counted with multiplicity), see [14] Thm. 8.32. Consequently many Computer Algebra systems offer implementations of algorithms for computation of Hilbert series or Hilbert polynomials. For example REDUCE has an implementation of the affine Hilbert polynomial, based on [62].

1.2.2 Computation of a Hilbert series

Once a Gröbner basis is known the computation of the Hilbert series pulls down to a purely combinatorial problem.

Lemma 1.2.12 *(Macaulay [135]) Let $(W_1, \ldots, W_r)$ be a weight system for $K[x]$ and I a W-homogeneous ideal. Let $LT(I)$ be the initial ideal of I with respect to a term order of $K[x]$. Then the Hilbert series of I and $LT(I)$ are equal.*

Since the leading terms $\{ht(f), f \in \mathcal{GB}\}$ of a Gröbner basis $\mathcal{GB}$ generate the monomial ideal $LT(I)$ the series $\mathcal{HP}_{K[x]/I}$ is easily computed. We only need an algorithm for monomial ideals.

Suppose we have a weight system $\{W_1, \ldots, W_r\}$ for $K[x_1, \ldots, x_n]$ with $W_i(x_j) = w_{ij}$. Then the Hilbert-Poincaré series of the full ring is given by

$$\mathcal{HP}^W_{K[x]}(\lambda_1, \ldots, \lambda_r) = \frac{1}{(1 - \lambda_1^{w_{11}} \cdots \lambda_r^{w_{r1}}) \cdot \cdots \cdot (1 - \lambda_1^{w_{1n}} \cdots \lambda_r^{w_{rn}})}.$$

The Hilbert series of an ideal has always a representation

$$\mathcal{HP}^W_{K[x]/I}(\lambda_1, \ldots, \lambda_r) = \frac{g^I(\lambda)}{(1 - \lambda_1^{w_{11}} \cdots \lambda_r^{w_{r1}}) \cdot \cdots \cdot (1 - \lambda_1^{w_{1n}} \cdots \lambda_r^{w_{rn}})},$$

where the numerator $\operatorname{num}(I) := g^I(\lambda)$ is a polynomial in λ.

Lemma 1.2.13 *Let $x^{\beta_1}, \ldots, x^{\beta_m} \in K[x]$ be a minimal generating set of J and $x^\alpha \notin J$ such that $x^\alpha, x^{\beta_1}, \ldots, x^{\beta_m} \in K[x]$ is a minimal generating set of $I = \langle J \cup \{x^\alpha\}\rangle$. Let $\{W_1, \ldots, W_r\}$ be a weight system given by $W_i(x_j) = w_{ij}, i = 1, \ldots, r, j = 1, \ldots, n$. By $W_i(\alpha) = \deg_{W_i}(x^\alpha)$ an abbreviation for the degree of a monomial is used. Then*

a.) $num(\langle x^\alpha \rangle) = 1 - \lambda_1^{W_1(\alpha)} \cdots \lambda_r^{W_r(\alpha)}$,

b.) $num(J \cap \langle x^\alpha \rangle) = 1 - \lambda_1^{W_1(\alpha)} \cdots \lambda_r^{W_r(\alpha)} + \lambda_1^{W_1(\alpha)} \cdots \lambda_r^{W_r(\alpha)} \cdot num(J : x^\alpha)$, where $J : x^\alpha$ denotes the ideal quotient $\{f \in K[x] \mid f(x) \cdot x^\alpha \in J\}$.

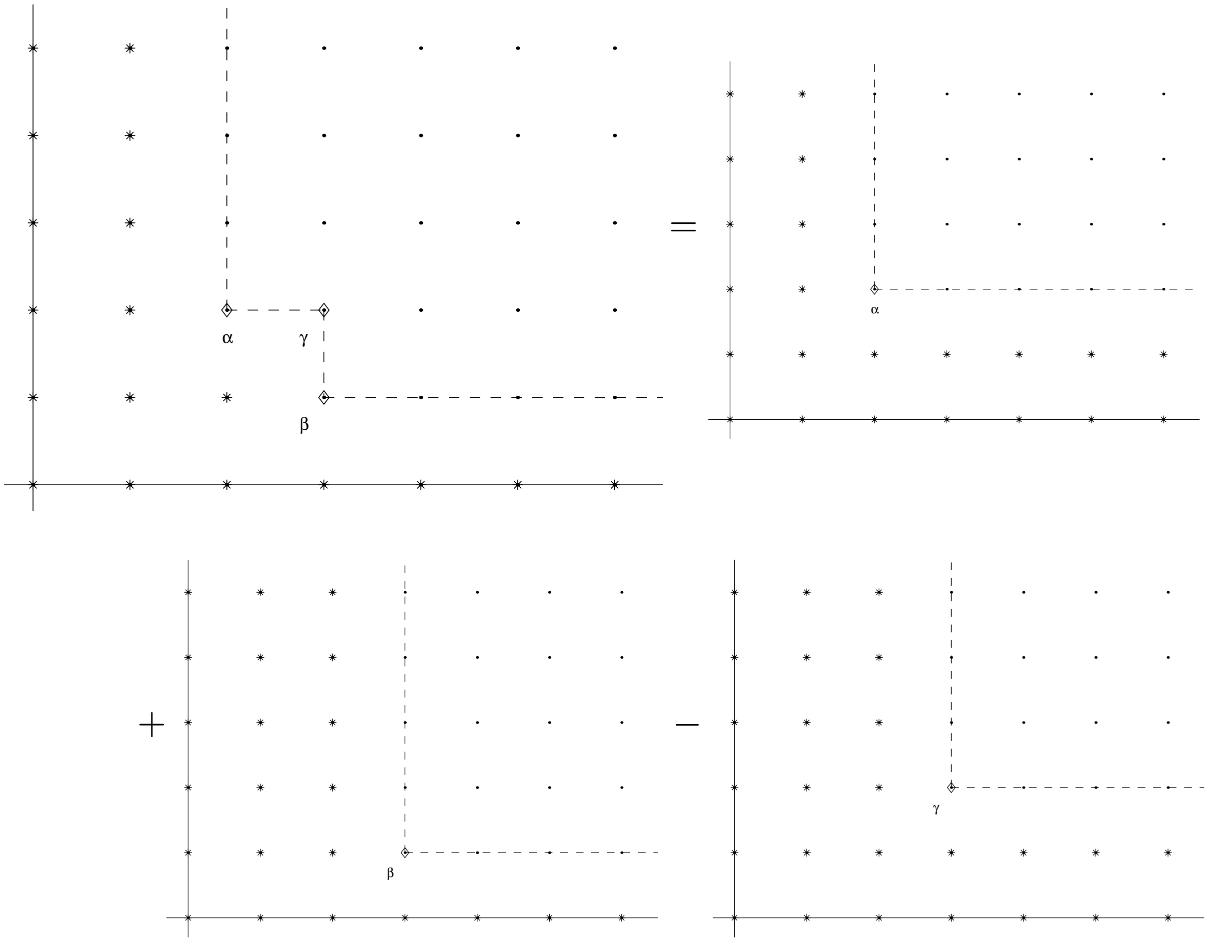

Figure 1.6: Illustration of Lemma 1.2.13: The monomials outside of $\langle J \cap \langle x^\alpha \rangle \rangle$ with $J = \langle x^\beta \rangle$ equal the monomials outside of $\langle x^\alpha \rangle$ plus those outside of J minus those outside of $J \cap \langle x^\gamma \rangle$ with $x^\gamma = \mathrm{lcm}(x^\alpha, x^\beta)$

c.) $num(I) = num(J) - \lambda_1^{W_1(\alpha)} \cdots \lambda_r^{W_r(\alpha)} \cdot num(J : x^\alpha)$.

Proof: a.) In order to determine $\dim \left(H_i^W(K[x]/\langle x^\alpha \rangle) \right)$ we need to count the monomials x^γ with $x^\gamma \notin \langle x^\alpha \rangle$. These are the monomials in $K[x]$ minus the monomials of type $x^\delta \cdot x^\alpha$ where x^δ runs through all monomials in $K[x]$. Thus $\dim \left(H_i^W(\langle x^\alpha \rangle) \right) = \dim \left(H^W_{i-\deg_W(x^\alpha)}(K[x]) \right)$ and

$$\begin{aligned}
\mathcal{HP}^W_{\langle x^\alpha \rangle}(\lambda) &= \textstyle\sum_{i \in \mathbf{N}^r} \dim \left(H_i^W(\langle x^\alpha \rangle) \right) \cdot \lambda_1^{i_1} \cdots \lambda_r^{i_r} \\
&= \textstyle\sum_{i \in \mathbf{N}^r} \dim \left(H^W_{i-deg_W(x^\alpha)}(K[x]) \right) \cdot \lambda_1^{i_1} \cdots \lambda_r^{i_r} \\
&= \textstyle\sum_{i \in \mathbf{N}^r} \dim \left(H_i^W(K[x]) \right) \cdot \lambda_1^{i_1 + W_1(\alpha)} \cdots \lambda_r^{i_r + W_r(\alpha)} \\
&= \lambda_1^{W_1(\alpha)} \cdots \lambda_r^{W_r(\alpha)} \cdot \mathcal{HP}^W_{K[x]}(\lambda)\,.
\end{aligned}$$

This yields $\mathrm{num}(\langle x^\alpha \rangle) = 1 - \lambda_1^{W_1(\alpha)} \cdots \lambda_r^{W_r(\alpha)}$.

b.) $K[x]/(J \cap \langle x^\alpha \rangle) \simeq K[x]/\langle x^\alpha \rangle \oplus \langle x^\alpha \rangle/(J \cap \langle x^\alpha \rangle)$ yields

$$\mathcal{HP}^W_{K[x]/(J\cap\langle x^\alpha\rangle)}(\lambda) = \mathcal{HP}^W_{K[x]/\langle x^\alpha\rangle}(\lambda) + \mathcal{HP}^W_{\langle x^\alpha\rangle/(J\cap\langle x^\alpha\rangle)}(\lambda)\,.$$

The first series is known from a.). For the second series $J \cap \langle x^\alpha \rangle \simeq x^\alpha(J : x^\alpha)$ gives $\langle x^\alpha \rangle/(J \cap \langle x^\alpha \rangle) \simeq K[x]/(J : x^\alpha)$. But a shift in the indexing has occurred.

$$\begin{aligned} \mathrm{num}(J \cap \langle x^\alpha \rangle) &= \mathrm{num}(\langle x^\alpha \rangle) + \lambda_1^{W_1(\alpha)} \cdots \lambda_r^{W_r(\alpha)} \cdot \mathrm{num}(J : x^\alpha) \\ &= 1 - \lambda_1^{W_1(\alpha)} \cdots \lambda_r^{W_r(\alpha)} \; + \; \lambda_1^{W_1(\alpha)} \cdots \lambda_r^{W_r(\alpha)} \cdot \mathrm{num}(J : x^\alpha)\,. \end{aligned}$$

c.) The monomials outside of $I = \langle J \cup \{x^\alpha\} \rangle$ are counted by those outside of J, those outside of $\langle x^\alpha \rangle$ minus those outside of $J \cap \langle x^\alpha \rangle$.

$$\begin{aligned} \mathrm{num}(I) = \;& \mathrm{num}(J) + \mathrm{num}(\langle x^\alpha \rangle) - \mathrm{num}(J \cap \langle x^\alpha \rangle) \\ =\;& \mathrm{num}(J) + 1 - \lambda^{W(\alpha)} - \Big(1 - \lambda^{W(\alpha)} + \lambda^{W(\alpha)} \cdot \mathrm{num}(J : x^\alpha)\Big) \\ =\;& \mathrm{num}(J) - \lambda_1^{W_1(\alpha)} \cdots \lambda_r^{W_r(\alpha)} \cdot \mathrm{num}(J : x^\alpha) \end{aligned}$$

□Multiple use of c.) gives a recursive formula exploited in Algorithm 1.2.15.

Corollary 1.2.14 *Let $I = \langle x^{\alpha_1}, \ldots, x^{\alpha_m} \rangle \subset K[x]$. Then*

$$num(I) = num(\langle x^{\alpha_1} \rangle) - \sum_{j=2}^{m} \lambda^{W(\alpha_j)} \cdot num(\langle x^{\alpha_1}, \ldots, x^{\alpha_{j-1}} \rangle : x^{\alpha_j})\,.$$

Lemma 1.2.15 *([13]) Let I be generated by monomials. Assume that the monomials split the set of variables into disjoint sets $X_i, i = 1, \ldots, j$. Furthermore assume that the set of generating monomials split into disjoint sets $M_1, \ldots, M_j$ such that the monomials in the set M_i depend on the set of variables X_i only. Define $I_i := I \cap K[X_i]$. Then*

$$num(I) = num(I_1) \cdot \cdots \cdot num(I_j).$$

Algorithm 1.2.16 *(Hilbert-Poincaré series)*
Input: *monomial ideal I minimally generated by $Ims := \{x^{\alpha_1}, \ldots, x^{\alpha_m}\}$,*
a weight system $W_i(x_j) = w_{ij}, i = 1, \ldots, r, j = 1, \ldots, n$,
variables $x_1, \ldots, x_n$,
variables $\lambda_1, \ldots, \lambda_r$
Output: $\mathcal{HP}^W_{K[x]/I}(\lambda) \in K(\lambda)$

```
g := hilbinumerator(Ims, x, W, λ)
HP := g
for i from 1 to n do
        HP = HP/(1 − λ_1^{w_{1i}} ··· λ_r^{w_{ri}})
return(HP)
 Subroutine hilbinumerator(Ims, x, W, λ)                 # I = ⟨Ims⟩
if Ims = {}                                    # Ims = {x^{α_1}, ..., x^{α_m}}
        then numI = 1
        else x^α = Ims[1]                                  # x^α = x^{α_1}
             numI = 1 − λ_1^{W_1(α)} ··· λ_r^{W_r(α)}           # num(⟨x^{α_1}⟩)
```

for j from 2 to m do # $J_j = \langle x^{\alpha_1}, \ldots, x^{\alpha_{j-1}} \rangle : x^{\alpha_j}$
  $Jms := \{lcm(x^{\alpha_1}, x^{\alpha_j})/x^{\alpha_j}, \ldots, lcm(x^{\alpha_{j-1}}, x^{\alpha_j})/x^{\alpha_j}\}$
  $Jms :=$*minimal set of generators of* J_j # $J_j = \langle Jms \rangle$
  $\{x_{j_1}, \ldots, x_{j_k}\} :=$ *linear monomials in* Jms
  $J_1ms := Jms \setminus \{x_{j_1}, \ldots, x_{j_k}\}$
  $numJ_1 := hilbinumerator(J_1ms, x, W, \lambda)$ # $num(J_j)$
  $numJ := (1 - \lambda_1^{w_{1i_1}} \cdots \lambda_r^{w_{ri_1}}) \cdots (1 - \lambda_1^{w_{1i_k}} \cdots \lambda_r^{w_{ri_k}}) \cdot numJ_1$
  $numI := numI - \lambda_1^{W_1(\alpha_j)} \cdots \lambda_r^{W_r(\alpha_j)} \cdot numJ$ # *Corollary 1.2.14*
return(numI) # $num(I)$

An implementation is available in [66] and Macaulay. The algorithm is essential Algorithm 2.6 variant A in [13] modified from the natural grading to multiple grading. Also Lemma 1.2.13 and Corollary 1.2.14 are the improved versions of results in [13]. The algorithmic computation of the affine Hilbert polynomial has first been attacked in [146] and improved in [62]. The implementation in REDUCE is based on the latter. More recently Bigatti [16] described an improved algorithm for the Hilbert series which makes use of the splitting of the set of monomials depending on disjoint sets of variables according to Lemma 1.2.15. Also in Maple V.5 Hilbert series and Hilbert polynomial are available.

Example 1.2.17 *The cyclo hexane problem ([143, 71, 78, 136]) is defined with the matrices*

$$A := \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 8/3 & x1 & 8/3 \\ 1 & 1 & 0 & 1 & 8/3 & x2 \\ 1 & 8/3 & 1 & 0 & 1 & 8/3 \\ 1 & x1 & 8/3 & 1 & 0 & 1 \\ 1 & 8/3 & x2 & 8/3 & 1 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 8/3 & x1 & 8/3 & 1 \\ 1 & 1 & 0 & 1 & 8/3 & x2 & 8/3 \\ 1 & 8/3 & 1 & 0 & 1 & 8/3 & x3 \\ 1 & x1 & 8/3 & 1 & 0 & 1 & 8/3 \\ 1 & 8/3 & x2 & 8/3 & 1 & 0 & 1 \\ 1 & 1 & 8/3 & x3 & 8/3 & 1 & 0 \end{bmatrix},$$

and the polynomials

$$f_4(x) = \det(B),$$
$$f_3(x) = \det(A), \ f_1(x_1, x_2, x_3) = f_3(x_2, x_3, x_1), \ f_2(x_1, x_2, x_3) = f_3(x_3, x_1, x_2).$$

The Gröbner basis with respect to $<_{grevlex}$ *has leading terms*

$$x_1{}^2 x_3{}^2, \ x_2{}^2 x_3{}^2, \ x_1 x_3{}^3, \ x_2 x_1{}^2, \ x_2{}^2 x_1, \ x_2 x_1 x_3 \,.$$

Homogenizing the polynomials of the Gröbner basis with respect to the natural grading gives a Gröbner basis of the homogeneous ideal. The set of leading terms remains the same. The Hilbert series computed by Algorithm 1.2.16 is

$$-\frac{\lambda^4 - \lambda^3 - 3\,\lambda^2 - 2\,\lambda - 1}{(-1+\lambda)^2} = \frac{-7}{1-\lambda} + \frac{6}{(1-\lambda)^2} - \lambda^2 - \lambda + 2$$

Since the degree of the Hilbert polynomial $p(\lambda) = 6(1+\lambda) - 7$ *is one there is a one-dimensional variety of solutions.*

This example clearly illustrates the advantages of Gröbner bases. Structural information such as the dimension can be computed.

1.2.3 The Hilbert series driven Buchberger algorithm

A variant of the Buchberger algorithm uses the structural information on the ideal given by the Hilbert series. With this the number of remaining elements in a Gröbner basis of a certain degree are known. Consequently, S-polynomials can be dropped which means a speed-up of the algorithm.

Lemma 1.2.18 *Let W be a single grading on $K[x]$ which forms a weight system. Let $F = \{f_1, \ldots, f_m\} \subset K[x]$ be a set of W-homogeneous generators of the homogeneous ideal $I = \langle f_1, \ldots, f_m \rangle$. Assume a term order $<$ and denote by $J = \langle ht(f_1), \ldots, ht(f_m) \rangle$ a subideal of the initial ideal $LT(I)$. Denote by $\mathcal{HP}^W_{K[x]/LT(I)}(\lambda) = \sum_{i=0}^{\infty} a_i \lambda^i$ and $\mathcal{HP}^W_{K[x]/J}(\lambda) = \sum_{i=0}^{\infty} b_i \lambda^i$ the Hilbert series of I and J. If $a_i = b_i, i = 0, \ldots, d$ then F forms a d-truncated Gröbner basis of I with respect to $<$. Furthermore if $a_{d+1} < b_{d+1}$ then d is the maximal degree with this property. A minimal Gröbner basis of I includes $b_{d+1} - a_{d+1}$ polynomials of degree $d+1$ which form (in $H^W_{d+1}(I)$) a direct complement of*

$$H^W_{d-\deg f_1+1}(\mathbf{C}[x]) \cdot f_1 + \cdots + H^W_{d-\deg f_m+1}(\mathbf{C}[x]) \cdot f_m \subset H^W_{d+1}(I)\,.$$

Proof: The module of syzygies is graded as well. For all syzygies S of degree $\leq d$ the associated polynomial $S \cdot F$ has degree d or less. Since $a_i = b_i$ for $i = 0, \ldots, d$ we have $\oplus_{i=0}^{d} H_i^W(J) = \oplus_{i=0}^{d} H_i^W(LT(I))$ and the division algorithm reduces $S \cdot F$ to zero. Considering the degree $d+1$ the $b_{d+1} - a_{d+1}$ additional polynomials have leading terms which are not members of J and are all different since we construct a minimal Gröbner basis. Amending these $b_{d+1} - a_{d+1}$ polynomials gives a truncated Gröbner basis of degree $d+1$. □

Lemma 1.2.18 reflects the fact that W-homogeneous Gröbner bases are W-H-bases. It means that at exactly $b_{d+1} - a_{d+1}$ linear independent polynomials of degree $d+1$ are missing in a Gröbner basis. This is the key for the Hilbert series driven Buchberger algorithm. The idea is to climb up the degree and add polynomials at degrees where some are missing. Once $b_{d+1} - a_{d+1}$ S-polynomials which are not reducing to zero are found all remaining S-polynomials of degree $d+1$ are dropped. The idea of the following algorithm has been presented in [182].

Algorithm 1.2.19 *(HP driven Buchberger algorithm)*
Input: *one grading W, forming a weight system*
set of W-homogeneous polynomials $F = \{f_1, \ldots, f_m\}$,
term order $<$
Hilbert series $\mathcal{HP}^W_{K[x]/I}(\lambda) = \sum_{i=0}^{\infty} a_i \lambda^i$

Output: *Gröbner basis $\mathcal{GB}$ of $I = \langle f_1, \ldots, f_m \rangle$*

$\mathcal{GB} := F$
$m := |F|$
$HT := \{ht(f_1), \ldots, ht(f_m)\}$ # *leading terms*
$HP := \mathcal{HP}^W_{K[x]/\langle HT \rangle}(\lambda) = \sum_{i=0}^{\infty} b_i \lambda^i$ # *tentative Hilbert series*
$d := \min\{i \mid b_i \neq a_i\}$ # *minimal deg. of missing pols.*
$c_d := b_d - a_d$ # *nr of missing pols. of deg. d*
$S := \{(i,j) \mid 1 \leq i < j \leq m,\ \deg_W(lcm(ht(f_i), ht(f_j))) \geq d\}$
while $S \neq \{\}$ *do*

```
choose (i,j) ∈ H_d^W(S),                          # a pair of degree d
S := S \ {(i,j)}
g :=normalf(S(f_i, f_j)) wrt < and GB
if g ≠ 0 then
        GB := GB ∪ {g}
        HT := HT ∪ {ht(g)}
        m := m + 1
        c_d := c_d − 1
        if c_d = 0 then                    # all pols. of degree d found
          HP := HP^W_{K[x]/⟨HT⟩}(λ)              # Algorithm 1.2.16
          if HP = HP^W_{K[x]/I}(λ) then break# GB found (Lem. 1.2.18)
          d := min{i | b_i ≠ a_i}               # new degree d
          c_d := b_d − a_d
          S := S \ {(i,j) | deg_W(S(f_i, f_j)) < d}    # skipping
        S := S ∪ { (i,m) |   i = 1,...,m − 1, deg_W(S(f_i, f_m)) ≥ d}
# reduce to minimal and reduced Gröbner basis
```

This is still a basic version since the exploitation of the Bucherger criteria for deletion of superfluous S-polynomials are missing. Also the possibilitity of truncation is not considered.

In the sequel I will generalize Algorithm 1.2.19 for multiple grading. Then the Hilbert series is used in a more sophisticated way.

Lemma 1.2.20 *(Lemma 3.5 in [26]) Let $W = (W_1, \dots, W_r)$ be a weight system of $K[x]$ such that the subsystem $(W_1, \dots, W_s)$ forms a weight system of minimal length. Let I be a W-homogeneous ideal and*

$$\mathcal{HP}^W_{K[x]/I}(\lambda) = \sum_{i \in \mathbf{N}^r} a_i \lambda^i, \quad \mathcal{HP}^{W_1,\dots,W_s}_{K[x]/I}(\lambda) = \sum_{j \in \mathbf{N}^s} c_j \lambda^j,$$

the corresponding Hilbert-Poincaré series. Then $c_j = \sum_{k \in \mathbf{N}^{r-s}} a_{(j,k)}$ and

$$\mathcal{HP}^{W_1,\dots,W_s}_{K[x]/I}(\lambda_1, \dots, \lambda_s) = \mathcal{HP}^W_{K[x]/I}(\lambda_1, \dots, \lambda_s, 1, \dots, 1).$$

Proof: The vector spaces are decomposed as

$$H^{W_1,\dots,W_s}_j(K[x]/I) = \sum_{k_{s+1}=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} H^{W_1,\dots,W_r}_{j_1,\dots,j_s,k_{s+1},\dots,k_r}(K[x]/I), \tag{1.3}$$

for all $j \in \mathbf{N}^s$. Since $W_1, \dots, W_s$ is a weight system the vector space at the left has finite dimension and consequently the sum on the right hand side only runs over a finite number of spaces. Denoting the dimensions as

$$c_j = \dim\left(H^{W_1,\dots,W_s}_j(K[x]/I)\right), \qquad a_{j,k} = \dim\left(H^W_{j,k}(K[x]/I)\right),$$

the formula $c_j = \sum_{k \in \mathbf{N}^{r-s}} a_{j,k}$ follows immediately. From this the formula for the Hilbert series is obvious. □

Using the decomposition (1.3) Lemma 1.2.18 is refined to multiple grading.

Lemma 1.2.21 *Let $W = (W_1, \ldots, W_r)$ be a grading of $K[x]$ and the subgrading $(W_1, \ldots, W_s)$ be a weight system. Let $I \subset K[x]$ be a W-homogeneous ideal and $F = \{f_1, \ldots, f_m\}$ a set of W-homogeneous generators. Assume a term order $<$ and denote by $J = \langle ht(f_1), \ldots, ht(f_m)\rangle$ a monomial ideal $J \subset LT(I)$. Denote by*

$$\begin{aligned}
\mathcal{HP}^{W}_{K[x]/LT(I)}(\lambda_1, \ldots, \lambda_r) &= \textstyle\sum_{i \in \mathbf{N}^r} c_i \lambda^i \quad \textit{and} \\
\mathcal{HP}^{W}_{K[x]/J}(\lambda_1, \ldots, \lambda_r) &= \textstyle\sum_{i \in \mathbf{N}^r} d_i \lambda^i \quad \textit{and} \\
\mathcal{HP}^{W_1,\ldots,W_s}_{K[x]/LT(I)}(\lambda_1, \ldots, \lambda_s) &= \textstyle\sum_{i \in \mathbf{N}^s} a_i \lambda^i \quad \textit{and} \\
\mathcal{HP}^{W_1,\ldots,W_s}_{K[x]/J}(\lambda_1, \ldots, \lambda_s) &= \textstyle\sum_{i \in \mathbf{N}^s} b_i \lambda^i
\end{aligned}$$

the Hilbert series of I and J. Let $d \in \mathbf{N}^s$ be a multi-degree. If $a_i = b_i, i \leq d$ then F forms a d-truncated Gröbner basis of I with respect to the order $<$ and with respect to $(W_1, \ldots, W_s)$. Let $\delta_s = d_s + 1, \delta_j = d_j, j = 1, \ldots, s-1$ be a greater multi-degree. If $a_i = b_i, i \leq d$ and $a_\delta < b_\delta$ then there exists a finite number of degrees $E = \{e \in \mathbf{N}^{r-s}\}$ with the following properties:

i.) There are $b_\delta - a_\delta$ linear independent polynomials of $(W_1, \ldots, W_s)$-degree δ missing in F in order to form a δ-truncated Gröbner basis of I.

ii.) For each degree $e \in E$ there are $d_{\delta,e} - c_{\delta,e}$ linear independent polynomials of W-degree (δ, e) missing in F in order to form a δ-truncated Gröbner basis of I.

iii.) Assume the Gröbner basis consists of $(W_1, \ldots, W_r)$-homogeneous polynomials. Each missing polynomial of degree δ has a W-degree (δ, e) with $e \in E$ and

$$b_\delta - a_\delta = \sum_{e \in E} d_{\delta,e} - c_{\delta,e}.$$

Lemma 1.2.21 is the basis for the multi-graded Hilbert series driven Buchberger algorithm in [26]. There the authors have restricted to the special case $s = 1$ which is reasonable since a sufficient linear combination of gradings $W_1, \ldots, W_r$ will be a weight system consisting of one single grading. Often we would like to compute truncated Gröbner bases. In most cases the truncation will be taken with respect to the weight system. But sometimes it is convenient to consider as well the truncation with respect to a different set of gradings.

Lemma 1.2.22 *For the ring $K[x]$ let $U = (U_1, \ldots, U_\nu)$ be a set of gradings and $W = (W_1, \ldots, W_r)$ be a weight system. Let the notations $F, I, J, <, a_i, b_i$ and the Hilbert series be as in Lemma 1.2.21. By $S(F)$ we denote the module of syzygies and by S^{kl} its sparse elements corresponding to the S-polynomials. Let $d \in \mathbf{N}^\nu$ be a given degree and*

$$j_{max} = \max\left\{ j \mid \text{ exists } i \leq d \text{ and } H^{U,W}_{i,j}\left(S(F)\right) \neq \{0\} \right\}.$$

The following conditions are sufficient for F being a d-truncated Gröbner basis of I with respect to U.

i.) $\forall\ S \in H^U_i(S(F)), i \leq d \quad normalf_<(F \cdot S, F) = 0,$

ii.) $\forall\ S^{kl} \in H^U_i(S(F)), i \leq d \quad normalf_<(F \cdot S^{kl}, F) = 0,$

iii.) $\forall\ S^{kl} \in H^{U,W}_{i,j}(S(F)), i \leq d, j \in \mathbf{N} \quad normalf_<(F \cdot S^{kl}, F) = 0,$

iv.) $\forall\ S^{kl} \in H^{W}_{j}(S(F)), j \leq j_{max} \quad normalf_<(F \cdot S^{kl}, F) = 0,$

v.) $\forall j \leq j_{max} \quad a_j = b_j.$

If for a degree $j \in \mathbf{N}^r$ *the coefficients* a_j *and* b_j *are equal, then for all syzygies* $S \in H^W_j(S(F))$ *we have* $normalf_<(F \cdot S, F) = 0$. *Especially all* $S^{kl} \in H^{U,W}_{i,j}(S(F))$ *reduce to zero.*

Proof: The induced grading on $S(F)$ enables to restrict to the vector spaces $H^U_i(S(F))$. The definition of j_{max} was chosen such that a j_{max}-truncated Gröbner basis with respect to W is a d-truncated Gröbner basis with respect to U. □

This suggests successive increasing of the degree $j \in \mathbf{N}^r$. For small degrees we expect $a_j = b_j$ and thus we have treated a lot of $H^{U,W}_{i,j}(S(F))$. This is the case right from the input polynomials or after some reduction of S-polynomials. If this criterion cannot be used one needs to reduce all S-polynomials in the vector space slices $H^{U,W}_{i,j}(S(F)), i \leq d$.

In [26] a multi-graded Hilbert series driven Buchberger algorithm is presented. But there no truncation to a different grading is considered and it is used that W_1 is a weight system. The following algorithm generalizes this concept for the case that one needs several gradings in order to form a weight system. In several situations (see Chapter 2) it is convenient to have an algorithm in this general form.

Algorithm 1.2.23 *(multi-truncated, multi-graded HP driven algorithm)*
Input: *weight system* $W = (W_1, \ldots, W_r)$,
s minimal such that the subsystem $(W_1, \ldots, W_s)$ *is a weight system,*
set of gradings $U = (U_1, \ldots, U_\nu)$,
set of (U, W)*-homogeneous polynomials* $F = \{f_1, \ldots, f_m\}$,
term order $<$
Hilbert series $\mathcal{HP}^W_{K[x]/I}(\lambda) = \sum_{i \in \mathbf{N}^r} c_i \lambda^i$
degree $d \in \mathbf{N}^\nu$

Output: *d-truncated Gröbner basis* $\mathcal{GB}$ *wrt* U *of* $I = \langle f_1, \ldots, f_m \rangle$

$\mathcal{GB} := F$
$m := |F|$
$\mathcal{HP}^{W_1,\ldots,W_s}_{K[x]/I}(\lambda) = \mathcal{HP}^{W_1,\ldots,W_r}_{K[x]/I}(\lambda_1, \ldots, \lambda_s, 1, \ldots, 1) = \sum_{i \in \mathbf{N}^r} a_i \lambda^i$
$HT := \{ht(f_1), \ldots, ht(f_m)\}$
$HP_{1s} := \mathcal{HP}^{W_1,\ldots,W_s}_{K[x]/\langle HT \rangle}(\lambda) = \sum_{i=0}^{\infty} b_i \lambda^i$ # *tentative Hilbert series*
if $\mathcal{HP}^{W_1,\ldots,W_s}_{K[x]/I}(\lambda) = HP_{1s}$ *then break*
$\delta := \min\{i \mid b_i \neq a_i\}$ # *minimal deg. of missing pols.*
if $d < \delta$ *then break*
$\mathcal{C}_\delta := b_\delta - a_\delta$ # *nr of missing pols. of deg.* δ
$HP := \mathcal{HP}^{W_1,\ldots,W_r}_{K[x]/\langle HT \rangle}(\lambda) = \sum_{i=0}^{\infty} d_i \lambda^i$
$E := \{e \in \mathbf{N}^{r-s} | c_{\delta,e} \neq d_{\delta,e}\}$ # *missing degrees* (δ, e)
$\epsilon := \min(E)$ # *minimal missing deg.* (δ, e)

$\mathcal{D}_{\delta,\epsilon} = d_{\delta,\epsilon} - c_{\delta,\epsilon}$ # *nr of missing pols of deg* (δ, e)
$S := \{(i,j) \mid 1 \leq i < j \leq m,\ \deg_U(lcm(ht(f_i), ht(f_j))) \leq d,$
$\qquad \deg_W(lcm(ht(f_i), ht(f_j))) \geq (\delta,\epsilon)\}$
if $S = \{\}$ *then break*
if $H^W_{\delta,\epsilon}(S) = \{\}$ *then* $update(\delta, \epsilon, E, \mathcal{C}_\delta, \mathcal{D}_{\delta,\epsilon})$
$\qquad S := S \setminus \{(i,j) \mid \deg_W(S^{i,j}) < (\delta,\epsilon)\}$
while $S \neq \{\}$ *do*
  choose $(i,j) \in \oplus_{k=0}^{d} H^U_k\left(H^W_{\delta,\epsilon}(S)\right)$, # *a pair of degree* (δ, ϵ)
  $S := S \setminus \{(i,j)\}$
  $g :=$*normalf*$(S(f_i, f_j))$ *wrt* $<$ *and* $\mathcal{GB}$
  if $g \neq 0$ *then*
    $\mathcal{GB} := \mathcal{GB} \cup \{g\}$
    $HT := HT \cup \{ht(g)\}$
    $m := m + 1$
    $\mathcal{C}_\delta := \mathcal{C}_\delta - 1$
    $\mathcal{D}_{\delta,\epsilon} := \mathcal{D}_{\delta,\epsilon} - 1$
    $S := S \cup \{ (i,m) \mid i = 1, \ldots, m-1,$
    $\qquad \deg_W(S^{im}) \geq (\delta,\epsilon), \deg_U(S^{im}) \leq d \}$
    if $\mathcal{D}_{\delta,\epsilon} = 0$ *then*
      $E := E \setminus \{\epsilon\}$
      if $E = \{\}$ *or* $\bigoplus_{e \in E} H^W_{\delta,e}(S) = \{\}$ *then*
        $update(HP_{1s}, HP)$
        $update(\delta, \epsilon, E, \mathcal{C}_\delta, \mathcal{D}_{\delta,\epsilon})$
        $S := S \setminus \{(i,j) \mid \deg_W(S^{i,j}) < (\delta,\epsilon)\}$
      else
        $update(\epsilon, E, \mathcal{D}_{\delta,\epsilon})$
        $S := S \setminus \{(i,j) \mid \deg_W(S^{i,j}) < (\delta,\epsilon)\}$
    if $\mathcal{C}_\delta = 0$ *then*
      $update(HP_{1s}, HP)$
      $update(\delta, \epsilon, E, \mathcal{C}_\delta, \mathcal{D}_{\delta,\epsilon})$
      $S := S \setminus \{(i,j) \mid \deg_W(S^{i,j}) < (\delta,\epsilon)\}$
reduce to minimal and reduced Gröbner basis

Algorithm 1.2.23 shows the full power of using gradings. The restriction with respect to grading and the exploitation of the Hilbert series with respect to a weight system and its refinement is done as best as possible. But there are still points which are not carried out in Algorithm 1.2.23. The exploitation of Buchberger criteria needs to be taken into account. Secondly, the computation of the tentative Hilbert series

$$\mathcal{HP}^{W_1,\ldots,W_s}_{K[x]/\langle ht(f_1),\ldots,ht(f_m)\rangle}(\lambda) \text{ and } \mathcal{HP}^{W}_{K[x]/\langle ht(f_1),\ldots,ht(f_m)\rangle}(\lambda)$$

can be simplified in the beginning (for small degrees δ). The computation of the series can be restricted to the set $\{ht(f_i) \mid \deg_{W_1,\ldots,W_s}(f_i) \leq \delta\}$ since only the dimensions $b_\delta - a_\delta$ and $d_{\delta,\epsilon} - c_{\delta,\epsilon}$ are essential. The third point is the consideration of the field. Since standard implementations are dealing with $\mathbf{Q}$ (or $\mathbf{Z}$ by according multiplication of input polynomials) for algebraic numbers such as $\sqrt{2}$ additional tricks are necessary, see Section 1.2.4.

Table 1.1: Timings in seconds on a Dec Alpha work station for the new Maple implementation *moregroebner* [66], the standard Maple package *grobner*, and the REDUCE 3.6 implementation [142]. The symbol − means that computation was not possible because the implementation of the term order is missing

		Maple 5.3		REDUCE 3.6
example	term order	moregroebner	grobner	groebner
cyclo hexane	lex	2.9	3.3	0.2
[143]	tdeg=revgradlex	3.3	3.4	0.1
	tdeg in matrix form	7.7	-	0.2
	gradlex	3.2	-	0.2
	gradlex in matrix form	7.7	-	0.1
Complex $n = 4$	tdeg=revgradlex	6.4	6.6	8
Complex $n = 5$	tdeg=revgradlex	33.5	37.3	13
Complex $n = 6$	tdeg=revgradlex	45.1	60.1	26
Complex $n = 7$	tdeg=revgradlex	148.8	236.4	50
Lotka Volterra	lex	46.1	422.8	871.9
[67], [71]	inv.+ lex.	20.1	913.0	0.9
	inv. + weighted lex	56.8	-	0.6

This algorithm including the mentioned details has been implemented in Maple in [66]. It has been tested successfully. Timings for typical test examples are shown in Table 1.1 in comparison with the standard Maple package and the REDUCE package.

The experience has shown that the timings are very sensitive against *good data structures* (REDUCE handles multivariate polynomials as it is required in Buchberger's algorithm much better than Maple), *implementation of basic routines* (the determination of leading terms depend at lot on the realization of the term order as condensed form or in general form as a matrix term order), and *use of tricks* (adding polynomials involving invariants of the symmetry of the system, see [71] and Section 4.1).

1.2.4 The computation with algebraic extensions

Since in some Computer Algebra Systems e.g. Maple the implementation of the Buchberger Algorithm is done for the field $\mathbf{Q}$ only one needs to use a special trick if the coefficients of the polynomials are elements of an algebraic extension of $\mathbf{Q}$.

Let the field K be given by $\mathbf{Q}$ and some additional elements $w_1, \ldots, w_r \in \mathbf{C}$ which satisfy the algebraic relations $h_1(y), \ldots, h_r(y) \in \mathbf{Q}[y]$. Thus

$$h_i(w_i) = 0, \quad i = 1, \ldots, r.$$

Let the polynomials $f_1, \ldots, f_s$ depend on the variables $x_1, \ldots, x_n$ and the coefficients be elements of $\mathbf{Q}(w_1, \ldots, w_r)$. By appropriate multiplication one can always assure that the coefficients are not fractions, but polynomials in $w_1, \ldots, w_r$. It is well-known that one may use $w_1, \ldots, w_r$ as additional variables, add $h_1(w_1), \ldots, h_r(w_r) \in \mathbf{Q}[w]$ to $f_1, \ldots, f_s$ and compute the Gröbner basis of the ideal in $\mathbf{Q}[x_1, \ldots, x_n, w_1, \ldots, w_r]$ generated by $f_1, \ldots, f_s, h_1, \ldots, h_r$ with respect to an appropriate term order.

Lemma 1.2.24 *Let $f_1, \ldots, f_s \in \mathbf{Q}[w][x]$ be given. Let $h_1, \ldots, h_r \in \mathbf{Q}[w]$ describe the algebraic extension of $\mathbf{Q}$. Let $I \subset \mathbf{Q}[x, w]$ denote the ideal generated by $f_1, \ldots, f_s, h_1, \ldots, h_r$ and $J \subset \mathbf{Q}(w)[x]$ the ideal generated by $f_1, \ldots, f_s \in \mathbf{Q}[w][x]$. Assume $\mathcal{GB}$ is a Gröbner basis of I with respect to a term order $<$ which eliminates the variables $x_1, \ldots, x_n$. Then the polynomials in $\mathcal{GB}$ which depend on x and w minus the polynomials in $\mathcal{GB}$ which depend on w only form a Gröbner basis of $J \subset \mathbf{Q}(w)[x]$ with respect to the term order $<_{|Q[x]}$ (order $<$ restricted to $\mathbf{Q}[x]$). If there are no polynomials in $\mathcal{GB}$ really depending on x then the Gröbner basis of J is* 1.

In the case of the Hilbert series driven Buchberger algorithm this trick can be modified. We additional assume that the polynomials $f_1, \ldots, f_s \in \mathbf{Q}[w][x]$ are homogeneous with respect to a weight system W of $K[x]$ with $K = \mathbf{Q}(w)$ and that the Hilbert series $\mathcal{HP}^W_{K[x]/J}$ of $J \subset K[x]$ generated by $f_1, \ldots, f_s$ is known. There are infinitely many ways of extending the weight system W to a set of gradings of $\mathbf{Q}[x, w]$. But it does not extend to a weight system of $\mathbf{Q}[x, w]$ such that $h_1(w_1), \ldots, h_r(w_r)$ are homogeneous since these polynomials define the algebraic extension of $\mathbf{Q}$. Thus there is no Hilbert series of the ideal $I \subset \mathbf{Q}[x, w]$ generated by $f_1, \ldots, h_r$.

Nevertheless one can combine the Hilbert series driven Buchberger algorithm with the trick described in Lemma 1.2.24. Let the ideal I and the term order $<$ be as in Lemma 1.2.24.

(1) Compute the Gröbner basis $\mathcal{GB}(h_1, \ldots, h_r)$ of $h_1, \ldots, h_r$ with respect to $<_{|Q[w]}$.

(2) Compute the Gröbner basis $\mathcal{GB}$ of I by climbing up the degree and using the modified tentative Hilbert series. For any set of intermediate polynomials g_i in the Buchberger algorithm choose the set of modified leading terms

$$LT := \{m(x) \in \mathbf{Q}[x] \mid \exists i \text{ and } n_i(w) \text{ with } m(x) \cdot n_i(w) = ht(g_i) \in \mathbf{Q}[x, w]\}$$

The Hilbert series $\mathcal{HP}^W_{\mathbf{Q}[x]/(LT)}$ measures how close the set is to forming a Gröbner basis of I.

(3) While counting those S-polynomials which do not reduce to zero one has to modify this count. If the S-polynomial is formed by one intermediate polynomial and one h_i the reduced S-polynomial is just a multiple of the previous polynomials in $\mathbf{Q}[w][x]$. Thus it does not contribute. This case is easily recognized since the intermediate polynomial and the new S-polynomial has the same degree with respect to the grading of $\mathbf{Q}[x]$ extended with weights zero on the variables w_i.

The first step is superfluous if the term order is chosen such that $h_1, \ldots, h_r$ already form a Gröbner basis.

Observe that the ideal membership problem in $\mathbf{Q}(w)[x]$ can not be transported immediately to the ideal membership problem in $\mathbf{Q}[w, x]$ since the leading coefficient may depend on w considering $\mathbf{Q}(w)[x]$.

Example 1.2.25 *Is $f(x) = x_1^2 + \frac{1}{2}\sqrt{2}x_2$ an element of $\langle \sqrt{2}x_1^2 + x_1, x_2 - x_1 \rangle$? The division algorithm in $\mathbf{Q}[w, x_1, x_2]$ with respect to the lexicographical term order and $x_2 > x_1 > w$ yields normalf(f) $= x_1^2 + \frac{1}{2}\sqrt{2}x_1$ which is not the correct answer. Only the division algorithm in $\mathbf{Q}[w][x]$ gives the desired answer.*

Either one uses a pseudo division algorithm allowing for multiplication with polynomials in w or one normalizes the system of polynomials by multiplication of the inverse of the leading coefficient in $\mathbf{Q}[w_i]/\langle h_i \rangle$ using the extended Euclidean algorithm.

1.3 Detection of Gröbner bases

The algorithms 1.2.19 and 1.2.23 in Section 1.2 necessitates the knowledge of the Hilbert-Poincaré series. There is only one chance in order to derive this series. One needs the knowledge of a Gröbner basis with respect to a term order and then compute the series with Algorithm 1.2.16. Observe that this term order might be different from the order in Algorithm 1.2.19. So the Gröbner basis computation still makes sense.

There are several situations where one knows the Gröbner basis with respect to the 'wrong' term order.

- Computations with respect to graded orders are much cheaper than with respect to purely lexicographical orders. So one might want to make a computation with respect to gradlex or even better graded reverse lex (which has been proven to be cheapest –remark in [41] p. 57) and then use the Hilbert series to speed up the interesting lex computations.

- Sometimes the term order is obvious for which the given polynomials form a Gröbner basis by the 1. Buchberger criterion. This will used in Section 2.1.

- In [91] a combinatorial algorithm is presented for the detection of the term order such that the input polynomials are a Gröbner basis with respect to this term order. Unfortunately, the complexity of this algorithm is too bad for practical computations. Considering a special case of a Gröbner basis as in [180] is much easier. The complexity is less bad than expected since the complexity analysis is done for dense polynomials which are rarely treated in Computer Algebra, see the description of the algorithm and the discussion below.

The Hilbert series driven Buchberger algorithm is an example of several algorithms which are exploiting the knowledge of one Gröbner basis for the computation of another. A straightforward way is the use of linear algebra technique as in the FGLM-class algorithms, see [56] and the improvements. Implementations are available in [55, 31, 37]. Secondly, for homogeneous ideals the structure of the *state polytope* (introduced in [11]) is exploited in the *Gröbner walk* [39]. Since the polynomial ring is Noetherian each ideal has only finitely many reduced Gröbner bases, although there exists infinitely many term orders. Each vertex of the state polytope corresponds to a reduced Gröbner basis and thus to a monomial ideal. Along an edge two monomial ideals are connected by a non-monomial ideal generated by all initial terms with respect to a vector not representing a term order. The Gröbner walk uses this intermediate ideal in the conversion from one Gröbner basis at one vertex to the Gröbner basis at the neighboring vertex. The corresponding fan to the state polytope was first introduced in Mora and Robbiano [149]. Implementations are given for example in Magma [30] and [4]. Since this theory is described nicely by Sturmfels in [177] there is no need to repeat it here. A recent improvement has been given in the Fractal walk by Amrhein and Gloor [5] where a step through a low-dimensional face is perturbed into several steps through facets.

In the final of this section I will discuss the Gröbner basis detection of a special case in [180]. The ingredients are Buchberger's 1. criterion, maximal weighted matching of a bipartite graph and linear programming.

Lemma 1.3.1 *Let $f_1, \ldots, f_m \in K[x]$ and $<$ be a term order. If the set of leading terms $ht(f_1), \ldots, ht(f_m)$ are relatively coprime then $\{f_1, \ldots, f_m\}$ forms a Gröbner basis with respect to $<$.*

Proof: By the 1. criterion of Buchberger (see Section 1.1) all S-polynomials reduce to zero. This implies the Gröbner basis property. $\square$

The lemma reduces the problem of Gröbner basis detection to inspection of the monomials in f_i and the appearance of variables in it. The second source of simplification is a lemma by Ostrowski. Since all computations are done with finitely many polynomials which are represented by finitely many monomials it suffices to order all monomials up to a certain degree d. This relaxes the requirements for a term order.

Lemma 1.3.2 *(Ostrowski, see [91] Lemma 1.3.1): Let $<$ be a term order and $d \in \mathbf{N}$. There exists a positive vector $\omega \in \mathbf{R}^n_+$ with the following property.*

For all monomials $x^\alpha, x^\beta \in \oplus_{i=0}^d H_i^N(K[x])$ it is true that $x^\alpha < x^\beta$ is equivalent to $\omega^t\alpha < \omega^t\beta$.

With this lemma the search of a term order is restricted to the search of a vector $\omega \in \mathbf{R}^n_+$. Consequently, Sturmfels and Wiegelmann [180] formulate the following *structural Gröbner basis detection problem* which is an important special case of the general detection problem.

(SGBD) Given $\{f_1, \ldots, f_m\} \subset K[x_1, \ldots, x_n]$. Does there exist a term order $\omega \in \mathbf{R}^n_+$ (and if so compute ω) such that the leading terms $ht(f_1), \ldots, ht(f_m)$ (with respect to ω) form a set of pairwise coprime monomials?

The solution of (SGBD) is based on results of Gröbner basis detection for square systems ($m = n$). In [180] a necessary condition is presented which leads to a bipartite maximum matching problem and linear programming problem. But the general case is treated in a way such that combinatorial bad behavior is expected. In contrast to [180] I exploit the sparsity of polynomials

which is reasonable since most of the time special polynomials are treated in Computer Algebra systems.

First the results for square systems are recalled. Here the problem (SGBD) is more special.

(SGBD)$_{n=m}$ Given $F = \{f_1, \ldots, f_n\} \subset K[x_1, \ldots, x_n]$. Does there exist a term order $\omega \in \mathbf{R}^n_+$ and a permutation σ of indices $\{1, \ldots, n\}$ and exponents $a_1, \ldots, a_n \in \mathbf{N} \setminus \{0\}$ such that $in_\omega(f_{\sigma(1)}) = c_1 x_1^{a_1}, \cdots, in_\omega(f_{\sigma(n)}) = c_n x_n^{a_n}$ with some non-zero constants c_i?

Obviously, it is important to see whether f_i is *monic* in x_j. Thus one rewrites each polynomial f_i as a polynomial in the variable x_j with coefficients which are polynomials in the rest of the variables. Using the notation $\tilde{X}_j := \{x_1, \ldots, x_n\} \setminus \{x_j\}$ we write

$$
\begin{aligned}
f_1 &= \underline{x_1^2} + (x_2 + x_3)x_1 + (x_2x_3^2 + x_2^3 - x_3^5 + 7) \\
&= \underline{x_2^3} + (x_1 + x_3^2)x_2 + (x_1^2 - x_3^5 + x_1x_3 + 7) \\
&= -\underline{x_3^5} + x_2x_3^2 + x_1x_3 + (x_2^3 + x_1^2 + x_1x_2 + 7) \\
f_2 &= \underline{x_1} + (x_3^2 + 2x_2x_3 + 1) \qquad \in K[x_2, x_3][x_1] \\
&= (2x_3)\underline{x_2} + (x_3^2 + x_1 + 1) \quad \in K[x_1, x_3][x_2] \\
&= \underline{x_3^2} + (2x_2)x_3 + (x_1 + 1) \quad \in K[x_1, x_2][x_3] \\
f_3 &= (x_2 + x_3)\underline{x_1} + x_2^3 + x_3^2 \\
&= \underline{x_2^3} + x_1x_2 + (x_1x_3 + x_3^2) \\
&= \underline{x_3^2} + x_1x_3 + (x_2x_1 + x_2^3)
\end{aligned}
$$

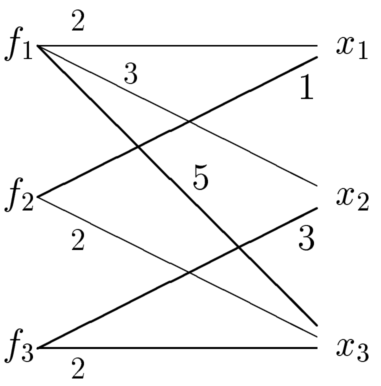

Figure 1.7: A simple example of a weighted bipartite graph associated to three polynomials in three variables. The maximal matching $(f_1, x_3), (f_2, x_1), (f_3, x_2)$ corresponds to a Gröbner basis with respect to the term order $\omega = (2.4, 1.3, 1)$. The number of complex solutions equals the matching value $5 \cdot 1 \cdot 3 = 15$

$f_i \in K[\tilde{X}_j][x_j]$ giving the representations for $i = 1, \ldots, n, \; j = 1, \ldots, n$

$$
f_i(x) = \begin{cases}
c_{ij} \cdot x_j^{a_{ij}} + \sum_{k=0}^{a_{ij}-1} p_k(x) \cdot x_j^k & \text{where} \quad a_{ij} \geq 0, c_{ij} \in K \setminus \{0\}, \\
 & \phantom{\text{where}} \quad \text{and } p_k \in K[\tilde{X}_j] \\
 & \text{or} \\
q_{ij}(x) \cdot x_j^{b_{ij}} + \sum_{k=0}^{b_{ij}-1} p_k(x) \cdot x_j^k & \text{where} \quad b_{ij} \geq 0, q_{ij} \in K[\tilde{X}_j] \setminus K, \\
 & \phantom{\text{where}} \quad \text{and } p_k \in K[\tilde{X}_j].
\end{cases}
$$

This gives rise to a *weighted bipartite graph* with one set of *vertices* $f_1, \ldots, f_m$ and a second set $x_1, \ldots, x_n$ and *edges* (f_i, x_j) between these two sets (see Figure 1.7 for an example). The tuple (f_i, x_j) is an edge if $a_{ij} > 0$ exists in the above representation. Then a_{ij} is the *weight* of this edge. If f_i does not depend on x_j or the coefficient of $x_j^{b_{ij}}$ depends on variables in $\tilde{X}_j$ (the second case above) then the tuple (f_i, x_j) is not an edge and we set $a_{ij} = 0$.

Lemma 1.3.3 *([180] Lemma 5) Let $F = \{f_1, \ldots, f_n\} \subset K[x_1, \ldots, x_n]$ be a system of polynomials and the integers a_{ij} be given by the representations above. Assume there exists a solution of $(SGBD)_{n=m}$, i.e. there exists a vector $\omega \in \mathbf{R}_+^n$ and a permutation σ of indices such that $in_\omega(f_i) = c_i x_{\sigma(i)}^{a_{i\sigma(i)}}$, $i = 1, \ldots, n$ with $c_i \in K \setminus \{0\}$. Then for each permutation ρ the following inequality holds*

$$\prod_{i=1}^{n} a_{i\sigma(i)} \geq \prod_{i=1}^{n} a_{i\rho(i)} \, .$$

Proof: Since ω represents a term order we have

$$\omega_{\sigma(i)} a_{i\sigma(i)} \geq \omega_{\rho(i)} a_{i\rho(i)}, \quad i = 1, \ldots, n,$$

where strong inequality holds for $\sigma(i) \neq \rho(i)$. Multiplication of the inequalities yields

$$\prod_{i=1}^{n} \omega_{\sigma(i)} a_{i\sigma(i)} \geq \prod_{i=1}^{n} \omega_{\rho(i)} a_{i\rho(i)},$$

and thus the statement is proved by dividing through the common factor $\prod_{i=1}^{n} \omega_i > 0$. □

This lemma is the key for the structural Gröbner basis detection problem since it reduces the problem to finding the permutation σ with maximal value of $\prod_{i=1}^{n} a_{i\sigma(i)}$ (*bipartite maximum matching problem*). Then one tests whether a term order ω exists by solving a linear programming problem as follows.

$$in_\omega(f_i) = c_i x_{\sigma(i)}^{a_{i\sigma(i)}}, \quad i = 1, \ldots, n \qquad \text{for } \omega \in \mathbf{R}_+^n,$$

is equivalent to

$$\begin{array}{ll} \omega_{\sigma(i)} a_{i\sigma(i)} > \omega^t b_i, & i = 1, \ldots, n, \ \forall\, b_i \in \mathcal{A}_i \setminus \{a_{i\sigma(i)} e_{\sigma(i)}\} \\ \omega_i > 0 & i = 1, \ldots, n, \end{array}$$

where x^{b_i} runs through all monomials of $f_i(x) = \sum_{b \in \mathcal{A}_i} c_b x^b$ except of $x_{\sigma(i)}^{a_{i\sigma(i)}}$.

That's why Algorithm 7 in [180] searches in the weighted bipartite graph the maximal matching and tests whether the associated linear programming problem has a solution. The largest matchings of a bipartite graph may be found by a reformulation as transshipment problem and solution of the associated linear programming problem [36].Comparing the values $\prod a_{i\sigma(i)}$ of all matchings gives the maximal one. More efficient algorithms may be found in the literature.

The key for the general problem (SGBD) is the reduction to the problem $(SGBD)_{n=m}$ for square systems. Solutions of $(SGBD)_{n=m}$ imply solutions of (SGBD).

Lemma 1.3.4 *Let $m \leq n$ and $F = \{f_1, \ldots, f_m\} \subset K[x_1, \ldots, x_n]$ be given. Let $S_1, \ldots, S_m$ be disjoint subsets of the variable set $\{x_1, \ldots, x_n\}$ ($S_i \cap S_j = \{\}$ for $i \neq j$). The remaining variable set is denoted by $R = \{x_1, \ldots, x_n\} \setminus \cup_{j=1}^{m} S_j$. The substitutions $\pi(x_i) = y_j$ if $x_i \in S_j$ and $\pi(x_i) = y_{m+1}$ for $x_i \in R$ results into a mapping on the polynomial ring $\pi : K[x_1, \ldots, x_n] \to K[y_1, \ldots, y_{m+1}]$. Assume there exists a term order $<_y$ on $K[y]$ such that (with respect to $<_y$) the leading terms $ht(\pi(f_i))$ are powers of $y_{\sigma(i)}$ for $i = 1, \ldots, m$, where σ denotes a permutation of indices. Then there exists a term order $<_x$ on $K[x]$ such that the leading terms $ht(f_i) = x^{\alpha_i}$ with respect to $<_x$ have a variable set $T_i = \{x_k \mid (\alpha_i)_k > 0\}$ which is a subset of $S_{\sigma(i)}$ for all $i = 1, \ldots, m$.*

Proof: We define $<_x$ by any refinement of $<_y$. □

Observe that the other direction is wrong in general.

Example 1.3.5 *The polynomials $f_1(x) = x_1 x_2^2 + x_1^4 x_3$, $f_2(x) = x_3^{11} - x_1$ have leading terms $x_1 x_2^2$ and x_3^{11} with respect to $\omega = (10, 20, 1)$. This is a solution of (SGDB). The variable sets are $S_1 = \{x_1, x_2\}$, $S_2 = \{x_3\}$. The projected polynomials are $\pi(f_1) = y_1^3 + y_1^4 y_2$, $\pi(f_2) = y_2^{11} - y_1$. There is no $\omega \in \mathbf{R}_+^2$ such that y_1^3 is the leading term of $\pi(f_1)$. But a different choice of π gives the desired solution of $(SGBD)_{n=m}$. $\pi(x_1) = y_1^{10}$, $\pi(x_2) = y_1^{20}$, $\pi(x_3) = y_2$ yields $\pi(f_1) = y_1^{50} + y_1^{40} y_2$, $\pi(f_2) = y_2^{11} - y_1^{10}$. With respect to $\omega = (1, 1)$ they have leading terms y_1^{50} and y_2^{11} and thus $\pi(f_1), \pi(f_2)$ form a Gröbner basis.*

Lemma 1.3.6 *Let $m \le n$ and $F = \{f_1, \dots, f_m\} \subset K[x_1, \dots, x_n]$ be given. Assume the vector $\omega \in \mathbf{N}_+^n$ represents a term order such that $ht(f_1), \dots, ht(f_m)$ have pairwise disjoint variable sets $S_i \subset \{x_1, \dots, x_n\}, i = 1, \dots, m$. There exists a mapping $\pi : K[x_1, \dots, x_n] \to K[y_1, \dots, y_{m+1}]$ such that $\pi(f_1), \dots, \pi(f_m)$ form a Gröbner basis because of coprime leading terms, i.e. exists a vector $\omega_y \in \mathbf{R}_+^{m+1}$ representing a term order on $K[y]$ such that $in_{\omega_y}(\pi(f_1)) = c_1 y_1^{\alpha_1}, \dots, in_{\omega_y}(\pi(f_m)) = c_m y_m^{\alpha_m}$ with $c_i \in K \setminus \{0\}, \alpha_i \in \mathbf{N}$.*

Proof: : The mapping π is given by the substitution $\pi(x_i) = y_j^{\omega_i}, x_i \in S_j$ and $\pi(x_i) = y_{m+1}$ for $x_i \in \{x_1, \dots, x_n\} \setminus (S_1 \cup \dots \cup S_m)$. □

The example and the lemmas show that Lemma 9 in [180] is wrong. Nevertheless the statements based on the lemma remain valid, but needs an alternative proof as has been communicated by Markus Wiegelmann. In [180] the idea of the algorithmic treatment is the following. In order to solve (SGBD) for a given set $\{f_1, \dots, f_m\} \subset K[x_1, \dots, x_n]$ it is suggested to try for all partitions $S_1 \cup \dots \cup S_m$ of $\{x_1, \dots, x_n\}$ to solve $(SGBD)_{n=m}$ for $\pi(f_1), \dots, \pi(f_m) \subset K[y_1, \dots, y_m]$.

For each partition this means the solution of a bipartite maximum weighting problem and a linear programming problem. Unfortunately, there is no guarantee that one will find an existing solution of $(SGBD)$ by this approach. Nevertheless the developed tools in [180] are very valuable and are able to find a lot of structural Gröbner bases. The only drawback is the expected complexity which is rather pessimistic although polynomial.

On the other hand this complexity analysis was done for dense polynomials. But in symbolic computations usually sparse polynomials are treated. We generalize the approach in [180] in two ways. First I exploit the sparsity of polynomials. Secondly, I deal with partitions $S_1 \cup \dots \cup S_m \cup R$ consisting of $m+1$ sets instead of m. Exploiting the sparsity obviously the treatment of all partitions $S_1 \cup \dots \cup S_m \cup R$ is reduced to a small number of partitions $S_1 \cup \dots \cup S_m \cup R$.

For $i = 1, \dots, m$ denote the set of all variable sets of monomials in f_i by

$$T^i = \{S \subset \{x_1, \dots, x_n\} \,|\, \text{exists a monomial } x^\alpha \text{ in } f_i \text{ with variables}(x^\alpha) = S\}.$$

Then $T = T^1 \cup \dots \cup T^m$ consists of variable sets $S \subset \{x_1, \dots, x_n\}$ encoding the appearance of variables in all monomials of $f_1, \dots, f_m$. Instead of trying all partitions it suffices to check all combinations $S_1 \cup \dots \cup S_m$ with

i.) $S_i \cap S_k = \{\}$ for all $i \neq k$.

ii.) There exists a permutation σ with $S_j \in T^{\sigma(j)}, j = 1 \dots, m$.

If $S_1 \cup \dots \cup S_m$ is not a partition there are $k = n - |\cup_{i=1}^m S_i|$ additional variables $x_{j_1}, \dots, x_{j_k}$. Thus $S_1 \cup \dots \cup S_m \cup \{x_{j_1}\} \cup \dots \cup \{x_{j_k}\}$ is a partition of $\{x_1, \dots, x_n\}$. A mapping π as in Lemma 1.3.4 is defined.

Then one needs to solve the maximal bipartite matching problem induced from the polynomials $\pi(f_1), \dots, \pi(f_m), x_{j_1}, \dots, x_{j_k}$ and the variables $y_1, \dots, y_m, x_{j_1}, \dots, x_{j_k}$.

Algorithm 1.3.7 *(Structural Gröbner basis detection)*
Input: *$n \ge m$, polynomials $F = \{f_1, \dots, f_m\} \subset K[x_1, \dots, x_n]$ in encoding*

$$f_i(x) = \textstyle\sum_{\alpha \in \mathcal{A}_i} c_\alpha \cdot x^\alpha, \;\; c_\alpha \in K \setminus \{0\} \;\; \forall\, \alpha \in \mathcal{A}_i \subset \mathbf{N}^n,\; i = 1, \dots, m$$

Output: Yes *or* No *(if* Yes *then term order $\omega \in \mathbf{R}_+^n$)*

1. *determine set of all variable sets in* F
$T = \{S \subset \{x_1, \ldots, x_n\} |\ \exists i$ *and* $\alpha \in \mathcal{A}_i$ *such that* x^α *has variable set* $S\}$
for each $T_j \in T$
associate the list $I_j = \{i_1, \ldots, i_{k_j}\}$ *and degrees* $A_j = \{a_{i_1 j}, \ldots, a_{i_{k_j} j}\}$
$I_j := \{\}$
for $i = 1, \ldots, m$ *do*
$a_{ij} := \max\Big\{\deg(\prod_{x_k \in T_j} x_k^{\alpha_k}) = \sum_{x_k \in T_j} \alpha_k \mid \ \alpha \in \mathcal{A}_i,$ *variables*$(x^\alpha) \subseteq T_j\}$
for all $\alpha \in \mathcal{A}_i$ *do*
if variables$(x^\alpha) \cap T_j \neq \{\}$ *and* $\prod_{x_k \in T_j} x_k^{\alpha_k} > a_{ij}$ *then* $a_{ij} = 0$
if $a_{ij} \neq 0$ *then* $I_j := I_j \cup \{i\}$, $A_j := A_j \cup \{a_{ij}\}$

2. *search for all* m*-tuples* $(j_1, \ldots, j_m)$ *of indices*
I. *such that*
$j_k \neq j_l$ *for* $k \neq l$
$T_{j_1}, \ldots, T_{j_m} \in T$
$T_{j_k} \cap T_{j_l} = \{\}$ *for* $k \neq l$
exists $\gamma : \{j_1, \ldots, j_m\} \to \{1, \ldots, m\}$ *such that*
a.) $\gamma(j_k) \in I_{j_k}, k = 1, \ldots, m$
b.) γ *is injective*
c.) $a_{\gamma(j_k), j_k} \neq 0, k = 1, \ldots, m$ # *matching in bipartite graph*
II. *for each tuple* $(j_1, \ldots, j_m)$ *with existing matching* γ
search in the bipartite graph of $\pi(f_1), \ldots, \pi(f_m)$ *defined*
by $\{j_1, \ldots, j_m\}$ *and degrees* a_{ij} *the maximal matching,*
i.e. the mapping $\gamma : \{j_1, \ldots, j_m\} \to \{1, \ldots, m\}$ *satisfying*
conditions a.), b.) and c.) such that $\prod_{k=1}^m a_{\gamma(j_k), j_k}$ *is maximal*
III. *if a maximal matching* γ *was found then solve the linear*
programming problem according to $\{T_{j_1}, \ldots, T_{j_m}\}$ *and*
the polynomials $\pi(f_{\gamma(j_1)}), \ldots, \pi(f_{\gamma(j_m)})$.
if solution ω *exists then STOP*

For dense polynomials the set T of variable sets is a huge set and combinatorial bad behavior can be expected. On the other hand if only a few monomials appear in each polynomial and even more only part of the variables appear this algorithm has a much better behavior than the one given in [180]. The following example is a typical example in algorithmic invariant theory as discussed in Chapter 2. In order to compute the algebraic dependencies of the three polynomials $x_1^2 + x_2^2$, $x_1^3 - 3x_1x_2^2$, $x_1^2x_2 - \frac{1}{3}x_2^3$ one considers polynomials with slack variables and computes a Gröbner basis with respect to a special term order, see Section 1.5.

Example 1.3.8 *In order to compute algebraic relations one considers*

$$\begin{array}{rcl} f_1 & = & x_3 - (x_1^2 + x_2^2) \\ f_2 & = & x_4 - (x_1^3 - 3x_1x_2^2) \\ f_3 & = & x_5 - (x_1^2x_2 - \frac{1}{3}x_2^3) \end{array} \quad \textit{with } \mathcal{A}_1 = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

Table 1.2: Illustration of the structural Gröbner basis detection as in Algorithm 1.3.7. All possible combinations in Example 1.3.8 are presented

T_j	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_4\}$	$\{x_5\}$	$\{x_1, x_2\}$	$\prod a_{ij}$	term order ω
I_j	$\{1,2\}$	$\{1,3\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2,3\}$		
degs a_{ij}	$\{2,3\}$	$\{2,3\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{2,3,3\}$		
T_j	x	x	x					
γ_1	2	3	1					
degree	3	3	1				9	–
T_j	x	x		x				
γ_1	1	3		2				
degree	2	3		1			6	–
T_j	x	x			x			
γ_1	2	1			3			
degree	3	2			1		6	–
T_j	x		x		x			
γ_1	2		1		3			
degree	3		1	1			3	$(1, 0.1, 10, 0.1, 10)$
T_j	x			x	x			
γ_1	1			2	3			
degree	2			1	1		2	$(1, 0.1, 0.1, 10, 10)$
T_j		x	x	x				
γ_1		3	1	2				
degree		3	1	1			3	$(0.1, 1, 10, 10, 0.1)$
T_j		x		x	x			
γ_1		1		2	3			
degree		2		1	1		2	$(0.1, 1, 0.1, 10, 10)$
T_j			x	x	x			
γ_1			1	2	3			
degree			1	1	1		1	$(0.1, 0.1, 1, 1, 1)$
T_j			x	x		x		
γ_1			1	2		3		
degree			1	1		3	3	$(2, 1, 10, 7, 0.1)$
T_j			x		x	x		
γ_1			1		3	2		
degree			1		1	3	3	$(1, 2, 10, 0.1, 7)$

$$\text{and } \mathcal{A}_2 = \left\{ \begin{pmatrix} 3\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\2\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1\\0 \end{pmatrix} \right\}, \quad \mathcal{A}_3 = \left\{ \begin{pmatrix} 2\\1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\3\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\0\\1 \end{pmatrix} \right\}.$$

The list of all variable sets is

$$T = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_1, x_2\}.\}$$

The associated index sets are

$$I_1 = \{1, 2\},\ I_2 = \{1, 3\},\ I_3 = \{1\},\ I_4 = \{2\},\ I_5 = \{3\},\ J_6 = \{1, 2, 3\},$$

and their degrees are

$$(a_{ij})_{i=1,\dots,3,j=1\dots,6} = \begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 2\\ 3 & 0 & 0 & 1 & 0 & 3\\ 0 & 3 & 0 & 0 & 1 & 3 \end{pmatrix}.$$

In Table 1.2 we list all possible combinations of variable sets and their weight of matching. There are only 10 *possibilities which we need to consider instead of the* 25 *partitions by neglecting the sparsity of the polynomials. There are* 7 *possibilities of classes of term orders such that the input forms a Gröbner basis. The maximal bipartite matching problems are all trivial since only one matching exists for each problem. Note that the algorithm would stop at the first Gröbner basis found. The three input polynomials are homogeneous with respect to the grading* $(1, 1, 2, 3, 3)$. *Thus the state polytope of the generated ideal* $\langle f_1, f_2, f_3\rangle$ *is a polytope in four-dimensional space. It has at least* 7 *vertices.*

1.4 Dynamic Buchberger algorithm

In several situations the input polynomials do not form a structural Gröbner basis as defined in Section 1.3 or the complexity of the detection problem prevents one from starting these algorithms. Nevertheless one might want to determine the Gröbner basis with respect to any term order in the cheapest way. There may be several reasons for this. Possibly, one is only interested in some structural information on the variety such as the dimension. Then it suffices to know the initial ideal with respect to any term order. Or one wants to convert to the Gröbner basis with respect to the desired term order. Then the knowledge of a Gröbner basis with respect to the 'wrong' term order can be exploited in the Hilbert series driven Buchberger algorithm (see Section 1.2) or in the FGLM-class algorithms or in the Gröbner walk.

Gritzmann and Sturmfels presented in [91] the idea of a dynamic Buchberger algorithm. It exploits the observation of Lemma 1.3.2 that term orders are represented by a vector and the fact that only finitely many term orders exists. First one uses the Hilbert series in order to measure the closeness to a Gröbner bases. Secondly, one changes the term order within the Buchberger algorithm whenever it seems to be appropriate.

Definition 1.4.1 *([91] p. 268) Let a weight system W on $K[x_1, \ldots, x_n]$ consist of one grading. Let the W-homogeneous ideal I be generated by W-homogeneous polynomials $f_1, \ldots, f_m$. A term order $\omega_1 \in \mathbf{R}_+^n$ is* preferable *to the term order $\omega_2 \in \mathbf{R}_+^n$ if in the tentative Hilbert-Poincaré series*

$$\mathcal{HP}^W_{K[x]/HT_1}(\lambda) = \sum_{i=0}^{\infty} a_i \lambda^i, \quad \mathcal{HP}^W_{K[x]/HT_2}(\lambda) = \sum_{i=0}^{\infty} b_i \lambda^i,$$

where $HT_1 := \langle in_{\omega_1}(f_1), \ldots, in_{\omega_1}(f_m)\rangle$ and $HT_2 := \langle in_{\omega_2}(f_1), \ldots, in_{\omega_2}(f_m)\rangle$ there exists an index j such that $a_i = b_i, i = 0, \ldots, j-1$ and $a_j < b_j$.

Remark 1.4.2 *It suffices to compare indices from 0 to $2R-1$ where R is the maximal degree $\deg_W(f_i)$ of the given polynomials. This follows from the fact that all S-polynomials which might reduce to something unequal zero have degree $\leq 2R-1$.*

The definition reflects the fact that the true Hilbert series of I bounds all tentative Hilbert series. Compare as well with the results in Section 1.2.

Of course Definition 1.4.1 has a generalization to multiple grading.

Algorithm 1.4.3 *(Dynamic Buchberger algorithm)*
Input: *one grading W on $K[x_1, \ldots, x_n]$ which forms a weight system*
W-homogeneous polynomials $F = \{f_1, \ldots, f_m\}$

Output: *Gröbner basis $\mathcal{GB}$ and term order ω*

1.) *Find an initial term order:*
determine maximal degree R of F with respect to W
$HP := \sum_{i=0}^{2R-1} a_i \lambda^i$ *with* $a_i = \dim\left(H_i^W(K[x])\right), i = 0, \ldots, 2R-1$
choose randomly several vectors $\omega_j \in \mathbf{R}_+^n, j = 1, \ldots, N$ and
– *compute by Algorithm 1.2.16 the Hilbert series*
$HP_j := \mathcal{HP}^W_{K[x]/\langle HT_j\rangle}(\lambda)$ *of the monomial ideal*
$\langle HT_j\rangle = \langle ht(f_1), \ldots, ht(f_m)\rangle$
– *expand into a Taylor series* $HP_j := \sum_{i=1}^{2R-1} b_i \lambda^i + \cdots$
– *for $i := 0$ to $2R-1$ do* # *choose the preferable term order*
if $a_i < b_i$ then stop;
if $a_i > b_i$ then stop; $HP := HP_j$; $\omega := \omega_j$ $HT := HT_j$

2.) *Buchberger algorithm with adaptive change of term order*
$\mathcal{GB} := F;\ t = 1;\ \omega_1 := \omega$ # *initialization*
$J := \{1, \ldots, m\}$ # *indices of elements in Gröbner basis*
$S := \{(i,j) \mid 1 \leq i < j \leq m\}$ # *generators for module of syzygies wrt ω_t*
while $S \neq \{\}$ do
choose $(i,j) \in S$; $S := S \setminus \{(i,j)\}$
$\deg_W(lcm(ht_{\omega_t}(f_i), ht_{\omega_t}(f_j)))$ *minimal*
if criteria1((i,j)) then # *use 1. Buchberger criterion*
$g := 0$ *else*
$g :=$*normalf*$(S(f_i, f_j))$ *wrt term order ω_t and $\mathcal{GB}$*
if lcm$(ht(f_i), ht(f_j)) = ht(f_j)$ *then*
$J := J \setminus \{j\}$ # *delete superfluous f_j*

if $lcm\,(ht(f_i), ht(f_j)) = ht(f_i)$ *then*
$J := J \setminus \{i\}$ # *delete superfluous* f_i
if $g \neq 0$ *then*
$f_{m+1} := g;\ J := J \cup \{m+1\};\ R := \max(R, \deg_W(f_{m+1}))$
$\mathcal{GB} := \mathcal{GB} \cup \{f_{m+1}\};\ HT := HT \cup \{ht_{\omega_t}(f_{m+1})\}$
$S := S \cup \{\,(j, m+1) \mid j \in J\}$
$m := m+1$
$HP := \mathcal{HP}^W_{K[x]/\langle HT\rangle}(\lambda)$
choose randomly several term orders $\omega \in \mathbf{R}^n_+$ *and*
compute the Hilbert series of $\langle ht_\omega(f_j), j \in J\rangle$
if the series is preferable in comparison to HP *then*
$\omega_{t+1} := \omega$ # *dynamic adaptation*
$t := t+1$
for $j \in J$ *do* # *update set* S
if $ht_{\omega_{t-1}}(f_j) \neq ht_{\omega_t}(f_j)$ *then*
$S := S \cup \{\,(j,i) \mid i \in J, i > j\} \cup \{\,(i,j) \mid i \in J, i < j\}$
$HT := \big(HT \setminus \{ht_{\omega_{t-1}}(f_j)\}\big) \cup \{ht_{\omega_t}(f_j)\}$
fi
fi
3.) *reduce* $\{f_j \mid j \in J\}$ *to minimal and reduced Gröbner basis wrt* ω_t

Remark 1.4.4 *While an inter-reduction of input polynomials is not appropriate in the classical Buchberger algorithm, the dynamic version might profit from that.*

The key idea of this algorithm appeared in [91] as Algorithm 3.1.3. Also Caboara suggests in [25] a dynamic version of the Buchberger algorithm. Choosing the first term order by comparing Hilbert series information as in [91] the term order is only refined after each computation of a new polynomial. Previous computations stay valid since only a refinement of the cone is done. In [25] the computational experience is reported that during the final phase the choice of new term orders should be dropped due to overhead computations.

Theorem 1.4.5 *Algorithm 1.4.3 computes a Gröbner basis and terminates.*

Proof: *Correctness:* The algorithm computes a Gröbner basis since all S-polynomials with respect to the last term order reduce to zero which is sufficient by Theorem 1.1.16. The S-polynomials are either reduced by the division algorithm and amended onto the list or the 1. Buchberger criterion implies that they reduce to zero. If the term order changes then the critical pairs are put on the list of S-polynomials again. The algorithm also includes the elimination of superfluous elements. If $\mathrm{lcm}\,(ht(f_i), ht(f_j)) = ht(f_j)$ then $S(f_i, f_j) = f_j - c_i x^\alpha f_i$ for an appropriate constant c_i and a monomial x^α. This shows $ht(S(f_i, f_j)) = ht(f_j)$ and that f_j is not necessary nor its critical pairs (i, j).

Termination: For each homogeneous ideal there exist only a finite number of inequivalent term orders $w_1, \ldots, w_r$. Thus the set $\{f_j \mid j \in J\}$ gives rise to a tuple of monomial ideals $\mathcal{HT}^m := \langle ht_{w_1}(f_j), j \in J\rangle \times \cdots \times \langle ht_{w_r}(f_j), j \in J\rangle$. Each new polynomial f_{m+1} gives a new tuple. These tuples are ordered by inclusion. Each ω_t is equivalent to one w_i.

Thus the actual HT in the algorithm is a generating set of a component $\mathcal{HT}_i^m$ of the tuple of monomial ideals. Since the polynomial ring is Noetherian each ascending chain of monomial ideals becomes stationary. This is true as well for tuples of monomial ideals. From this it follows that the algorithm terminates. □

An important point which needs to be discussed is the exploitation of Buchberger criteria.

The new term order is chosen by comparing the associated tentative Hilbert series. Alternatives might be able to exploit the faces of the Newton polytope of the new polynomial f_{m+1}. Especially, for fewnomial polynomial ideals as for example the binomial ideals in the context of integer programming the dynamic Buchberger algorithm might be an interesting alternative. Once a Gröbner basis is found the Hilbert series driven version will give the Gröbner basis with respect to the true term order.

1.5 Elimination

In the engineering literature e.g. [10] one finds the statement that a computation of a Gröbner basis is the method which converts a set of equations into upper triangular form

$$\begin{array}{rcl} g_1(x_1, x_2, \ldots, x_n) & = & 0 \\ g_2(x_2, \ldots, x_n) & = & 0 \\ \ddots & \vdots & \vdots \\ g_n(x_n) & = & 0\,. \end{array} \tag{1.4}$$

Indeed, given a system of polynomial equations $f_1(x) = 0, \ldots, f_m(x) = 0$ one manipulates this system into a different set $g_1(x) = 0, \ldots, g_l(x) = 0$ having the same set of solutions (even with the same multiplicity). This means that the two sets generate the same ideal. Obviously, the existence of one g_i depending only on one variable x_j is preferable. After the solution of this single equation the substitution of the solutions $x_j = \tilde{a}_k \in \mathbf{C}, k = 1 \ldots, \deg(g_i)$ yields a couple of smaller and easier systems. Below we will discuss the question which term orders have the property that the Gröbner basis of the ideal $\langle f_1, \ldots, f_m \rangle$ contains such a polynomial depending on just one variable. The best situation would be the case of an *upper triangular form* as in (1.4). Even if the original system $f_1(x) = 0, \ldots, f_m(x) = 0$ has finitely many solutions such a triangular form does not necessarily exist. A counterexample is given in Figure 1.8. On the other hand each system with finitely many solutions can be decomposed into smaller subsystems which can be written in upper triangular form, see Lazard [128].

There are other approaches in elimination theory such as the *characteristic sets of Ritt-Wu* ([14] p. 520, [145] Chapter 5, [41] Section 6.5) and the *resultants*. Even in numerics the concept of resultants is used (detection of
Hopf bifurcation, see [112] p. 289). Of course the sparsity of the polynomial system is used in the computation of the resultant. All resultant methods for multivariate homogeneous polynomials (Dixon, Macaulay, sparse) have in common that a matrix is built and some determinant is computed, for an introduction see [178], [42] Chapter 3. The sparse mixed resultant has been applied for problems in robotics.

For the case of infinitely many solutions one likes to identify a subgroup of variables (e.g. $x_1, \ldots, x_d$) such that the system of equations has for each value of $x = a \in \mathbf{C}^d$ only finitely many solutions $(a, b) \in \mathbf{C}^n$. Gröbner basis computation can test whether

$$
\begin{aligned}
g_1^2(x_1,x_2) &= {x_1}^2 - 3\,x_1 + 2,\\
g_1^1(x_1,x_2) &= x_1x_2 - 2\,x_1 - 2\,x_2 + 4,\\
&= (x_2-2)(x_1-1)\\
g_2(x_2) &= {x_2}^2 - 3\,x_2 + 2\\
&= (x_2-2)(x_2-1)
\end{aligned}
$$

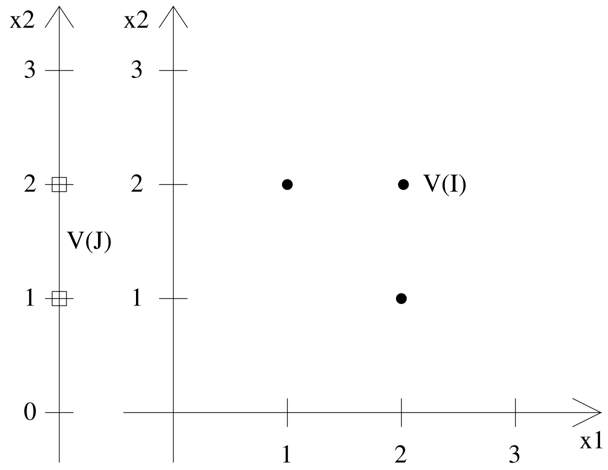

Figure 1.8: A variety consisting of three isolated points and a Gröbner basis of the associated ideal $I = \langle g_1^2, g_1^1, g_2\rangle$ with respect to the lexicographical order which is not in triangular form. The primary decomposition of I consists of three prime ideals having Gröbner bases in upper triangular form. The variety of the elimination ideal $J = I \cap \mathbf{C}[x_2] = \langle g_2\rangle$ consists of 2 points

this situation occurs. Obviously this is a *parameterization* of the variety. The number d is called *dimension* and equals the degree of the Hilbert polynomial. The variables $x_1, \ldots, x_d$ are called *parameters*. By linear or nonlinear change of coordinates one can always convert the variety into such a form, the so-called *normal form*. This process is called *Noether normalization* and can be performed algorithmically, see [132].

The advantage of Gröbner bases is that characteristics of the variety are computed without knowing the variety itself. The dimension is computed from the degree of the Hilbert polynomial and for zero-dimensional ideals the number of solutions is computed from the codimension of the ideal in the ring.

Let $I \subset K[x_1, \ldots, x_n, y_1, \ldots, y_l]$ be an ideal which is given for example by $f_1, \ldots, f_m$. We are searching for polynomials which depend on the variables $y_1, \ldots, y_l$ only and have the same solutions than I. This means one computes the generators of the *elimination ideal.*

Definition 1.5.1 *The ideal $I \cap K[y]$ is called* elimination ideal *of I in $K[y]$.*

A set of generators of the elimination ideal can be computed with the Buchberger algorithm, if one chooses a term order with a special property.

Definition 1.5.2 *A term order $\leq$ of $K[x,y]$ is called an* elimination order*, if for all cases with $x^\alpha \neq 1$ we have*

$$x^\alpha y^\gamma \geq y^\delta\,.$$

By [14] p. 257 Lemma 6.14 it is sufficient to demand $x^\alpha \geq y^\delta \quad \forall\, x^\alpha \neq 1, y^\delta$. One says the variables $x_1, \ldots, x_n$ are *eliminated.*

Example 1.5.3 *Elimination orders include*

a.) the lexicographical order,

b.) *all matrix orders with matrix $M = (M_{ij})$ with entries in the first column $M_{1i} = 1, i = 1, \ldots, n$, $M_{1i} = 0, i = n+1, \ldots, n+l$. This includes the elimination order by Bayer and Stillman.*

c.) *block orders $\geq$ consisting of orders $\geq_x$ and $\geq_y$ on $K[x]$ and $K[y]$, respectively. ($x^\alpha y^\gamma > x^\beta y^\delta \Leftrightarrow x^\alpha >_x x^\beta$ or $x^\alpha = x^\beta$ and $y^\gamma >_y y^\delta$).*

If $\mathcal{GB}$ forms a Gröbner basis of I with respect to an elimination order then $\mathcal{GB} \cap K[y]$ is a Gröbner basis of the elimination ideal with respect to $<_{|K[y]}$.

Remark 1.5.4 *In subsection 1.2.4 an elimination order was used for the handling of algebraic extensions in case the implementations are restricted to the field Q. It is also useful for the relation between Gröbner bases of the affine ideal and the associated projective ideal.*

If we want to compute an upper triangular form as above we need a term order which successively eliminates x_1 than x_1, x_2, next x_1, x_2, x_3 and so on. It is well-known that a lexicographical order fulfills these requirements of nested multiple elimination.

Remark 1.5.5 *In Section 1.4 the idea of the dynamic Buchberger algorithm is presented. The term order represented by one vector is updated during the Buchberger algorithm such that the polynomials are as close to a Gröbner basis as possible. A variant of this is possible in elimination although the elimination orders are restricted. One only has to assure that the weights on x_i are much larger than on y_j. Just start with one bound. If during computation the degrees of the intermediate polynomials become too high one enlarges the bound.*

Algorithm 1.5.6 *(Test for Noether normal form)*
Input: *one grading W forming a weight system*
W-homogeneous polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$
assume the parameters are $x_1, \ldots, x_d$

Output: Yes *or* No

1.) *Choose a term order*

$$M = \begin{pmatrix} w_1 & \ldots & w_{d-1} & w_d & w_{d+1} & \ldots & w_{n-1} & w_n \\ -w_1 & \ldots & -w_{d-1} & -w_d & 0 & \ldots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & & 1 & 0 \\ 1 & \cdots & 0 & 0 & 0 & \ldots & 0 & 0 \\ & \ddots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and compute the Gröbner basis of $\langle f_1, \ldots, f_m \rangle$ with respect to this term order.
2.) *If the leading terms of $\mathcal{GB}$ do not contain any monomial which depends on the parameter variables $x_1, \ldots, x_d$ only and for each of the remaining variables there exists a monic leading term in this variable then indeed the variables $x_1, \ldots, x_d$ form a homogeneous system of parameters and the output is YES.*

In the algorithm it is important to use an order which eliminates the non-parameter variables and then successively eliminates all variables one after each other.

Algorithm 1.5.6 is inspired by subroutine 3.9 in [179] where a gradrevlex order has been used. We come back to the topic of Noether normalization in Chapter 2, Section 4.

The algorithmic Noether normalization shows some typical features of Gröbner bases. The use of Gröbner bases pulls difficult questions of commutative algebra down to the question whether a variable appears in a monomial or not. In turn these algebraic properties give the structural information on varieties.

Chapter 2

Algorithms for the computation of invariants and equivariants

Haben Sie auch ein Beispiel gerechnet?

Did you compute an example?

Lothar Collatz[1]

In Chapter 3 and Chapter 4 the investigations are started with a symmetric vector field having arbitrary coefficients. This dynamical system might be thought of as the result of a center manifold reduction or of a Liapunov Schmidt reduction of a bigger system of ordinary differential equations or of a partial differential equation. Each equivariant vector field can be written as a combination of fundamental invariants and fundamental equivariants. These fundamental invariants and fundamental equivariants can be computed by algorithms in a systematic way. In this chapter I give several algorithms where I emphasize on those aspects which are essential for efficient computations. Some theoretical background is recalled where it is necessary. But the development of theory is not the aim since there are several textbooks on invariant theory, see e.g. [50], [174]. In [176] Sturmfels presents invariant theory together with its algorithmic treatment. Chapter 4 of [176] also includes Hilbert's classical algorithm which we do not follow here since it is not appropriate to generalize it to equivariants. Two alternatives of the classical way are both based on the canonical cone and algorithmic normalization, see Chapter 4.7 of [176] and [46] p. 85 using algebraic groups.

2.1 Using the Hilbert series

For the rest of this chapter G will denote a real compact Lie group and $\vartheta : G \to GL(\mathbf{R}^n)$ a linear representation on the vector space $\mathbf{R}^n$. Of course only faithful representations are considered, that means G and $\vartheta(G)$ are isomorphic as groups. Often ϑ is extended to a representation on $\mathbf{C}^n$.

[1]Prof. Dr. L. Collatz (1910–1990), a famous applied mathematicsm, used to ask this question. He liked to tell a story about a special publication. The topic of the paper was an algorithm for a highly specialized problem class with certain properties, but no example was included. After this paper had been published, people realized that the class of problems with these specific properties is the empty set.

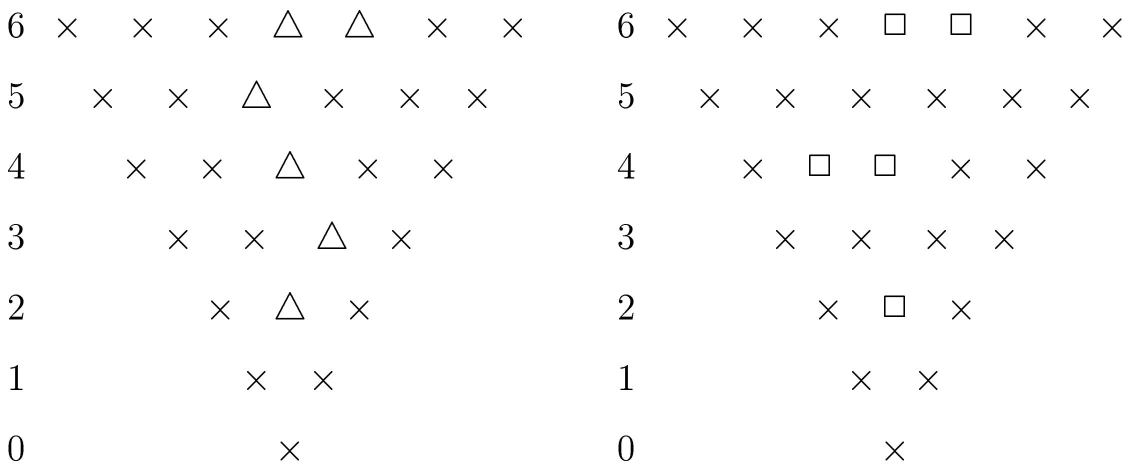

Figure 2.1: The polynomial ring $\mathbf{C}[x_1, x_2]$ and the dimensions of its homogeneous parts are symbolized by $\times, \triangle, \square$. The invariant rings $\mathbf{C}[x_1, x_2]^{D_3}$ and $\mathbf{C}[x_1, x_2]^{D_4}$ are graded $\mathbf{C}$-algebras. The dimensions of vector spaces of homogeneous D_3 or D_4- invariants are visualized by $\triangle, \square$, respectively

In equivariant dynamical systems one focuses on dihedral groups D_n, S_1, $H \times S_1$, $O(2) \times S_1$, SO(3), $O(3)$. That means that the group G is of one of the following types

a.) finite group

b.) a torus or a semi-simple Lie group (that means that the group is connected as manifold and has an associated Lie algebra which as complexified Lie algebra is semi-simple, i.e. has no nontrivial solvable ideals)

c.) a direct product $H \times G_0$ of a finite group H and a Lie group G_0 of type b.)

d.) a semi-direct product of a finite group H and a Lie group G_0 from b.) being normal in G.

The theoretical statements are as general as possible. But for practical computations we restrict to one of these types. Especially, the equivariant Reynolds projection exploits the Cartan decomposition valid for semi-simple Lie algebras.

In this section we start with definitions and give the first basic algorithms.

2.1.1 Invariants

Definition 2.1.1 *A polynomial $p(x) \in K[x_1, \ldots, x_n]$ is called* invariant *(with respect to a representation ϑ of a compact Lie group G), if*

$$p(\vartheta(s)x) = p(x), \ \forall \ s \in G.$$

The set of all invariant polynomials is called invariant ring *and is denoted by $K[x]^G$ or more precisely by $K[x]^{\vartheta(G)}$.*

In fact $K[x]^G$ is a *graded K-algebra* where the grading is induced by the natural grading on $K[x]$. That means that elements of $H_i(K[x]^G)$ are homogeneous polynomials consisting of monomials of degree i. In Figure 2.1 this is visualized for $\mathbf{R}[x]^{D_3}$ and $\mathbf{R}[x]^{D_4}$.

Theorem 2.1.2 *(Hilbert [97, 98]) The invariant ring $\mathbf{C}[x]^G$ of a compact Lie group G is generated by finitely many homogeneous invariant polynomials.*

A finite set of invariants which generates the invariant ring is called *Hilbert basis* and the elements are called *fundamental invariants*. If the fundamental invariants of $\mathbf{C}[x]^G$ have real coefficients they generate the real invariant ring $\mathbf{R}[x]^G$. That means that for theoretical purposes it is sufficient to deal with the field of complex numbers while practical computations often will be performed for a subfield of $\mathbf{R}$, especially $\mathbf{Q}$.

Definition 2.1.3 *A polynomial mapping $f : \mathbf{C}^n \to \mathbf{C}^n$ is called* equivariant*, if*

$$f(\vartheta(s)x) = \vartheta(s) \cdot f(x), \ \forall \ s \in G.$$

The set of all equivariants forms a module over the invariant ring and is denoted by $\mathbf{C}[x]^G_G$ or more precisely by $\mathbf{C}[x]^{\vartheta}_{\vartheta}$.

Also the module of equivariants is finitely generated over $\mathbf{C}[x]^G$ ([87] p. 51). The generators are called *fundamental equivariants*. Finite generation means the following. Given a Hilbert basis $\pi_1, \ldots, \pi_r \in \mathbf{R}[x]^G$ and a set of fundamental equivariants $b_1, \ldots, b_s \in \mathbf{R}[x]^G$ each equivariant polynomial mapping $f : \mathbf{R}^n \to \mathbf{R}^n$ can be written as

$$f(x) = p_1(\pi(x)) \cdot b_1(x) + \cdots + p_s(\pi(x)) \cdot b_s(x), \tag{2.1}$$

where the p_i are polynomials in r variables with real coefficients. The question of a unique representation will be discussed in Section 2.4, see also Remark 2.1.19 and Algorithm 2.3.20.

Sometimes one generalizes the concept of equivariants to mappings f which fulfill

$$f(\vartheta(s)x) = \rho(s) \cdot f(x), \ \forall \ s \in G,$$

where ρ is another representation of G. Of course the module of these ϑ-ρ-equivariants is finitely generated over $\mathbf{C}[x]^{\vartheta(G)}$ as well.

In the following we first present a naive algorithm for the computation of a Hilbert basis of a given group action followed by a similar algorithm for the equivariants. Climbing up the degree candidates for fundamental invariants are produced and the next degree is determined at which fundamental invariants are missing. While the production of candidates may be done with a certain projection the dimension of the vector space of homogeneous invariants of degree d is given by the Molien series.

Definition 2.1.4 *The* Molien series *of the invariant ring $\mathbf{C}[x]^G$ is defined by*

$$\mathcal{HP}_{\mathbf{C}[x]^G}(\lambda) = \sum_{i=0}^{\infty} \dim \left(H_i^N(\mathbf{C}[x]^G)\right) \cdot \lambda^i.$$

Remark 2.1.5 *i.) The Molien series is a special case of the Hilbert-Poincaré series in Definition 1.2.9. ii.) In case the representation of G is reducible and the coordinate system is such that the representation matrices have all the same block diagonal structure a modified series may be defined. Decomposing the variables $x = [X_1, \dots, X_l]$ into l groups according to the blocks some Kronecker gradings are defined by $W_i(x) = 1, x \in X_i$, $W_i(x) = 0$, $x \in X_j$, $j \neq i$, $j = 1, \dots, l$, $i = 1, \dots, l$. Since $W = (W_1, \dots, W_l)$ is a weight system the series $\mathcal{HP}^W_{\mathbf{C}[x]^G}(\lambda_1, \dots, \lambda_l)$ is well-defined.*

It is well-known that the Molien series can be computed, for finite groups see [176] p. 29 and for compact Lie groups see [174].

Lemma 2.1.6 *For finite groups G we have*

$$\mathcal{HP}_{\mathbf{C}[x]^G}(\lambda) = \frac{1}{|G|} \sum_{s \in G} \frac{1}{\det(id - \lambda\, \vartheta(s))}.$$

For compact Lie groups similarly it holds

$$\mathcal{HP}_{\mathbf{C}[x]^G}(\lambda) = \int_G \frac{1}{\det(id - \lambda\, \vartheta(s))} d\mu_G,$$

where μ_G is the unique normalized Haar measure of G.

In order to explain the evaluation of the integral I need to exploit the group structure. Let G_0 denote the connected component of the identity in G. The representation $\vartheta : G \to GL(\mathbf{R}^n)$ is restricted to the connected Lie group G_0. Each connected compact Lie group G_0 contains a maximal *torus group* T (maximal Abelian subgroup) which is non-unique in general. Then the *Weyl group* W is defined as the normalizer of T in G_o modulo the torus T ($N_{G_0}(T)/T \simeq W$). This is a finite group and its order is denoted by $|W|$. Moreover we denote by g the Lie algebra associated to G_0. Since we restrict to semi-simple Lie groups G_0 the theorem of Cartan-Chevalley (see e.g. [63]) is valid and the Cartan subalgebra is denoted by h. As a vector space the Lie algebra decomposes as $g = h \oplus g/h$ and the *adjoint mapping* $Ad : G_0 \to Aut(g)$, $Ad(g_0)(X) = \frac{d}{ds}\left(g_0 \exp(sX) g_0^{-1}\right)_{s=0}$ may be restricted to the complement g/h. With these notations the *Weyl integral formula* in [19] yields

$$\mathcal{HP}_{\mathbf{C}[x]^{G_0}}(\lambda) = \int_{G_0} f(g_0; \lambda) d\mu_{G_0} = \frac{1}{|W|} \int_T \det(id_{g/h} - Ad_{t^{-1}|g/h}) \cdot f(t; \lambda) d\mu_T,$$

$$\text{with} \quad f(g_0; \lambda) = \frac{1}{\det(id - \lambda\, \vartheta(g_0))},$$

where $|W|$ denotes the order of the Weyl group and μ_{G_0}, μ_T denote the unique normalized Haar measures. Moreover, t is an element of the torus group T and $g_0 \in G_0$. Observe that one uses the property of f to be equal on conjugacy classes of G_0. Analogously, the Molien series of G (direct or semi-direct product of finite H and G_0) is derived as

$$\mathcal{HP}_{\mathbf{C}[x]^G}(\lambda) = \frac{1}{|H|} \sum_{s \in H} \frac{1}{|W|} \int_T \det(id_{g/h} - Ad_{t^{-1}|g/h}) \frac{1}{\det(id - \lambda\, \vartheta(s\,t))} d\mu_T \, .$$

The integral over the torus can be written as a curve integral which is solved with the theorem of residues. Since Maple includes the computation of residues the Molien series is computed in [73] along this way. Experience shows that most of the time is spent in the simplification of expressions obtained from the theorem of residues.

Example 2.1.7 *The group $SO(3)$ is generated by three rotations around one coordinate axis. Thus the associated Lie algebra $so(3)$ is generated by*

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Considering $so(3)$ as a Lie algebra over $\mathbf{C}$ a different set of generators is $J_0 = -iL_3$, $J_\pm = iL_1 \pm L_2$. It has the advantage that the Cartan subalgebra h is generated by J_0 and J_-, J_+ are elements of the so-called root spaces. (For more on the Cartan decomposition of semi-simple Lie algebras see [63] and Thm. 2.1.15.) A maximal torus group T of $SO(3)$ is given by $T = \{\exp(sL_3) \mid s \in [0, 2\pi[\}$. The Weyl group $W = N_{SO(3)}(T)/T$ has order 2. The images of the adjoint action $Ad : SO(3) \to Aut(so(3))$ are represented by matrices with respect to a vector space basis of $SO(3)$. In oder to deal with the restriction $Ad_{|T} : T \to Aut(so(3))$ we use the splitting $so(3) = h \oplus so(3)/h$ and choose L_1, L_2 as a vector space basis of $so(3)/h$. This gives

$$Ad_{\exp(sL_3)|so(3)/h} = \begin{pmatrix} cos(s) & -sin(s) \\ sin(s) & cos(s) \end{pmatrix},$$

which is used in the Weyl integral formula *yielding the Molien series*

$$\begin{aligned} \mathcal{HP}_{\mathbf{C}[x]^{SO(3)}}(\lambda) &= \int_{SO(3)} \frac{1}{\det(id - \lambda\,\vartheta(g_0))} d\mu_{SO(3)} \\ &= \frac{1}{2} \int_T \frac{\det(id_{so(3)/h} - Ad_{t^{-1}|so(3)/h})}{\det(id - \lambda\,\vartheta(t))} d\mu_T . \end{aligned}$$

Representations ϑ of $SO(3)$ are usually presented by the actions ζ of the Lie algebra $so(3)$ for the generators $J_0, J_\pm$, see [166]: for each $l = 1, 2, \ldots$ there is a representation of degree $2l+1$. Denoting a vector space basis of $\mathbf{C}^{2l+1}$by $\{f_{-l}, f_{-l+1}, \ldots, f_0, f_1, \ldots, f_l\}$ the representation in this coordinate system is given by

$$\zeta(J_0) f_m = m f_m, \quad \zeta(J_\pm) f_m = \pm\gamma_m f_{m\pm1}, \quad m = -l, \ldots, l,$$
$$\gamma_m = \sqrt{(l-m)(l+m+1)}\,.$$

where we use the convention $f_{-l-1} = 0 = f_{l+1}$. Additionally, $\bar{f}_m = (-1)^m f_{-m}, m = 0, 1, \ldots, l$ can be chosen, see e.g. [166] p. 140. By $\vartheta(L_3) = \exp(-i\,\zeta(J_0)), \vartheta(L_1) = \exp(-\frac{i}{2}\zeta(J_+ + J_-)), \vartheta(L_2) = \exp(\frac{1}{2}\zeta(J_+ - J_-))$ a real representation of $SO(3)$ on the real vector space

$$\{(f_{-l}, \ldots, f_l) \in \mathbf{C}^{2l+1} \mid \bar{f}_m = (-1)^m f_{-m},\, m = 0, \ldots, l\}$$

is given. Using a parameterization of the torus group T the rules for integration yield for $R = \mathbf{C}[x]^{SO(3)}$

$$\begin{aligned} \mathcal{HP}_R(\lambda) &= \frac{1}{2} \int_{s=0}^{2\pi} \frac{\det(id_{so(3)/h} - Ad_{(\exp(sL_3))^{-1}|so(3)/h})}{\det(id - \lambda\,\vartheta(exp(sL_3)))} ds \\ &= \frac{1}{2} \int_{s=0}^{2\pi} \frac{(1 - (\cos(-s) + i\sin(-s)))(1 - (\cos(-s) - i\sin(-s)))}{\prod_{m=-l}^{l}(1 - \lambda\exp(i\,s\,m))} ds \\ &= \frac{1}{2\pi i} \frac{1}{2} \int_{|z|=1} \frac{(1-z)(1-z^{-1})}{\prod_{m=-l}^{l}(1 - \lambda z^m)} \frac{dz}{z}\,. \end{aligned}$$

The curve integral is evaluated by the theorem of residues. For $l = 2$ and $|\lambda| < 1$ the poles inside the curve are $\lambda, \pm\sqrt{\lambda}$. With

$$
\begin{aligned}
h(\lambda, z) &= \frac{(1-z)(1-z^{-1})}{z(1-\lambda z^{-2})(1-\lambda z^{-1})(1-\lambda)(1-\lambda z)(1-\lambda z^2)}, \\
\mathcal{HP}_{\mathbf{C}[x]^{SO(3)}}(\lambda) &= \frac{1}{2}\left(Res_\lambda h(\lambda, z) + Res_{\sqrt{\lambda}} h(\lambda, z) + Res_{-\sqrt{\lambda}} h(\lambda, z)\right) \\
&= \frac{1}{(1-\lambda^2)(1-\lambda^3)},
\end{aligned}
$$

where the residues are computed with the help of Maple. The package Symmetry [73] includes an operator for the computation of the Molien series along this lines.

Given some homogeneous invariants $\pi_1(x), \ldots, \pi_k(x)$ one would like to know how the invariant ring $\mathbf{C}[x]^G$ compares with the ring $\mathbf{C}[\pi_1(x), \ldots, \pi_k(x)]$ generated by the given invariants. The key to this question is given by the *relations* or *syzygies* $r \in \mathbf{C}[y_1, \ldots, y_k]$ with $r(\pi_1(x), \ldots, \pi_k(x)) \equiv 0$. Denoting by I the ideal of relations the ring $\mathbf{C}[\pi(x)]$ is isomorphic to $\mathbf{C}[y]/I$. Moreover they are isomorphic as graded $\mathbf{C}$-algebras where $\mathbf{C}[\pi(x)]$ is graded by the natural degree and $\mathbf{C}[y]$ by $W(y_i) = \deg_N(\pi_i(x)), i = 1, \ldots, k$, the induced grading. Since I is a W-homogeneous ideal the grading W on $\mathbf{C}[y]$ induces to the quotient ring $\mathbf{C}[y]/I$. This is illustrated in Figure 2.2. It is well-known that a basis of the ideal of relations is computed by means of Gröbner bases. Moreover a Gröbner basis of I is determined and thus the Hilbert series of $\mathbf{C}[y]/I$ (equal to the series of $\mathbf{C}[x]/init(I)$) is known by Algorithm 1.2.16. Since this is the same series as of $\mathbf{C}[\pi(x)]$ this gives the method in [176] Algorithm 2.2.5 p. 32 for the determination of complete generation of the invariant ring. In [74] it has been improved by two details, the truncation at degree d and the Hilbert series driven Buchberger algorithm for the algorithmic determination of algebraic relations.

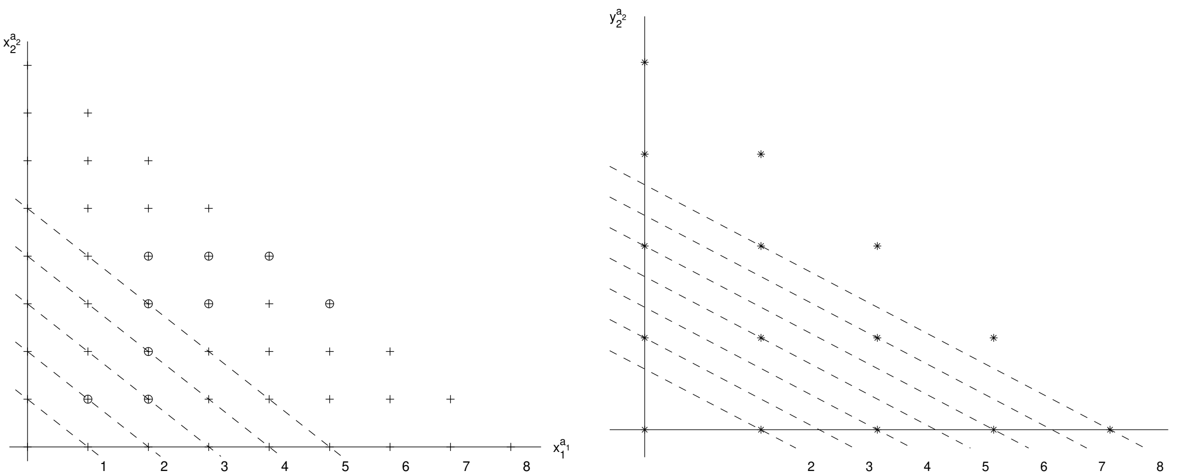

Figure 2.2: The invariant ring of D_3 acting on a plane is generated by two algebraic independent polynomials $\pi_1(x), \pi_2(x)$ of degrees 2 and 3, respectively. The dimensions of the homogeneous components of $K[\pi_1(x), \pi(x)]$ are given on the left. On the right $K[y_1, y_2]$ with respect to the induced grading $W(y_1) = 2, W(y_2) = 3$ is pictured

Algorithm 2.1.8 *(Completeness of invariant ring up to degree d, [176])*
Input: *Homogeneous invariant polynomials* $\pi_1(x), \ldots, \pi_k(x) \in K[x]^G$
Molien series $\mathcal{HP}_{K[x]^G}(\lambda)$
degree d

Output: *TRUE or minimal degree of missing invariant*

$Js := \{y_1 - \pi_1(x), \ldots, y_k - \pi_k(x)\}$ # *generators of ideal* J
define grading $W(x_i) = 1, i = 1 \ldots, n, W(y_i) = \deg_N(\pi_i), i = 1, \ldots, k$
$HP := 1/(1-\lambda)^n$ # *series of* $K[x,y]/J$ *wrt* W
compute a Gröbner basis $\mathcal{GB}(\sum_{i=0}^d H_i^W(J))$
with respect to an order which eliminates x
using the Hilbert series driven Buchberger algorithm with series HP
and truncation at degree d # *using Algorithm 1.2.23*
$Is := \mathcal{GB} \cap K[y]$
$HTs := \{ht(f),\ f \in Is\}$
$HP_2 := \mathcal{HP}^W_{K[y]/\langle HTs\rangle}(\lambda)$ # *using Algorithm 1.2.16*
if $HP_2 = \mathcal{HP}_{K[x]^G}(\lambda)$ *then* 'ring is completely generated'
else compare Taylor expansions $HP_2 = b_0 + b_1\lambda + b_2\lambda^2 + \cdots$
$\mathcal{HP}_{K[x]^G}(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots$
$d_{min} := \min\{i \mid a_i > b_i\}$
if $d < d_{min}$ *then TRUE else OUTPUT*(d_{min})
fi

The polynomials $y_1 - \pi_1(x), \ldots, y_k - \pi_k(x)$ with slack variables y_i generate an ideal $J \subset \mathbf{C}[x,y]$. Since an elimination order is chosen a Gröbner basis of the ideal of relations $I = J \cap \mathbf{C}[y]$ is computed. So far this is standard. The natural grading on $\mathbf{C}[x]$ gives rise to the weighted grading W on $\mathbf{C}[x,y]$. Then the polynomials $y_i - \pi_i(x)$ are W-homogeneous which enables truncation. With respect to a suitable term order the monomials y_i are the leading terms of the polynomials $y_i - \pi_i(x)$. Since the y_i are coprime the set $\{y_i - \pi_i(x)\}$ forms a Gröbner basis. Consequently, the Hilbert series of $\mathbf{C}[x,y]/J$ equals the series of the ring $\mathbf{C}[x,y]/\langle y_1, \ldots, y_k\rangle$. Obviously, the series is $1/(1-\lambda)^n$. This enables the use of the Hilbert series driven Buchberger algorithm as it has been pointed out in [182] and has been used independently in [74]. These two details gain efficiency.

Example 2.1.9 *In order to investigate a Takens-Bogdanov point with D_3-symmetry in [138] a generic equivariant vector field is investigated for the action of D_3 decomposing as two times the two-dimensional natural representation. The real representation $\vartheta : D_3 \to GL(\mathbf{R}^4)$ is written in a different way by changing coordinates from $(v_r, v_i, w_r, w_i) \in \mathbf{R}^4$ to $(v, \bar{v}, w, \bar{w}) \in \mathbf{C}^4$ by $v = v_r + i \cdot v_i, w = w_r + i \cdot w_i$ and identifying $\{(x_1, x_2, x_3, x_4) \in \mathbf{C}^4 \mid x_1 = \bar{x}_2, x_3 = \bar{x}_4\}$ as real vector space. In these complex coordinates the group action is nicely written as*

$$flip(v,w) = (\bar{v}, \bar{w}), \ \ and\ rotation(v,w) = (e^{i\frac{2\pi}{3}}v, e^{i\frac{2\pi}{3}}w)\,.$$

Since the author of [138] did not know about Algorithm 2.3.17 he suggested the invariants

$$\begin{array}{ll} s_1 = v\bar{v} = {v_r}^2 + {v_i}^2, & s_2 = w\bar{w} = {w_r}^2 + {w_i}^2, \\ t_0 = w^3 + \bar{w}^3 = 2\,{w_r}^3 - 6\,w_r\,{w_i}^2, & t_3 = v^3 + \bar{v}^3 = 2\,{v_r}^3 - 6\,v_r\,{v_i}^2, \end{array}$$

$$\begin{aligned} s_3 &= v\bar{w} + \bar{v}\,w = 2\,v_r\,w_r + 2\,v_i\,w_i, \\ t_1 &= vw^2 + \bar{v}\,\bar{w}^2 = 2\,v_r\,{w_r}^2 - 2\,v_r\,{w_i}^2 - 4\,v_i\,w_r\,w_i, \\ t_2 &= v^2w + \bar{v}^2\bar{w} = 2\,{v_r}^2w_r - 4\,v_r\,v_i\,w_i - 2\,{v_i}^2w_r, \end{aligned}$$

and showed that they generate the invariant ring $\mathbf{C}[v, \bar{v}, w, \bar{w}]^{D_3}$ *completely. Since they are real polynomials they generate* $\mathbf{R}[v_r, v_i, w_r, w_i]^{D_3}$ *as well. In [74] this was alternatively shown by Algorithm 2.1.8. By Lemma 2.1.6 the Molien series is*

$$\frac{\lambda^6 + \lambda^4 + 2\,\lambda^3 + \lambda^2 + 1}{(1-\lambda^3)^2\,(1-\lambda^2)^2}\,.$$

The relations with respect to the revgradlex term order are

$$\begin{array}{l} 4\,{s_2}^3t_2 - 4\,{s_2}^2t_1s_3 + \underline{t_0s_2{s_3}^2} - t_2{t_0}^2 + t_0{t_1}^2, \\ \underline{t_3{s_3}^2s_1} - t_1{t_3}^2 + t_3{t_2}^2 + 4\,{s_1}^3t_1 - 4\,{s_1}^2t_2s_3, \\ 4\,s_3{s_2}^2t_2 - 4\,s_2{s_3}^2t_1 + \underline{t_0{s_3}^3} - t_3{t_0}^2 + t_1t_2t_0, \\ 4\,s_3{s_1}^2t_1 - 4\,s_1{s_3}^2t_2 + \underline{t_3{s_3}^3} - {t_3}^2t_0 + t_2t_1t_3, \\ 4\,\underline{s_2{s_1}^2} - {s_3}^2s_1 + t_1t_3 - {t_2}^2,\ 4\,\underline{{s_2}^2s_1} - {s_3}^2s_2 + t_2t_0 - {t_1}^2, \\ 4\,\underline{s_3s_2s_1} - {s_3}^3 + t_3t_0 - t_2t_1,\ t_2s_2 + \underline{t_0s_1} - t_1s_3,\ \underline{t_3s_2} + t_1s_1 - t_2s_3, \end{array} \tag{2.2}$$

with $t_0s_2{s_3}^2$, $t_3{s_3}^2s_1$, $t_0{s_3}^3$, $t_3{s_3}^3$, $s_2{s_1}^2$, ${s_2}^2s_1$, $s_3s_2s_1$, t_0s_1, t_3s_2 *as leading terms. The Hilbert series of the quotient ring with respect to the ideal generated by these leading terms with respect to the induced grading*

$$W(s_1) = W(s_2) = 2,\ W(t_0) = W(t_3) = 3,\ W(s_3) = 2,\ W(t_1) = W(t_2) = 3,$$

equals the Molien series. Thus the invariant ring is completely generated.

Now we turn our attention to the *Reynolds projection*

$$\mathcal{R} : \mathbf{C}[x] \to \mathbf{C}[x]^G, \tag{2.3}$$

which obviously respects the degree: $\mathcal{R}_d : H_d(\mathbf{C}[x]) \to H_d(\mathbf{C}[x]^G)$.

For finite groups the projection is easily realized by (see e.g. [176] p. 25)

$$\mathcal{R}(p(x)) = \frac{1}{|G|}\sum_{s\in G} p(\vartheta(s)x).$$

For compact Lie groups the description of the projection is more involved.

Recall that G_0 denotes the connected component of the identity of G and $\mathrm{Res}(\vartheta, G_0)$ the restriction of $\vartheta : G \to GL(\mathbf{R}^n)$. The representation $\mathrm{Res}(\vartheta, G_0)$ corresponds to a representation $\zeta : g \to M(\mathbf{R}^n)$ of the associated *Lie algebra* g of G_0 where $\zeta(Y)$ are real $(n{\times}n)$-matrices. Each $Y \in g$ is given by a path $\gamma : [-1, 1] \to G_0$, $\gamma(0) = e$, $\frac{d}{ds}\gamma(s)_{|s=0} = Y$.

The group action $\mathrm{Res}(\vartheta, G_0)$ gives rise via the representation

$$\Theta : G_0 \to Aut(\mathbf{C}[x]), \quad \Theta(t)(p(x)) = p(\vartheta(t^{-1})x), \quad t \in G,$$

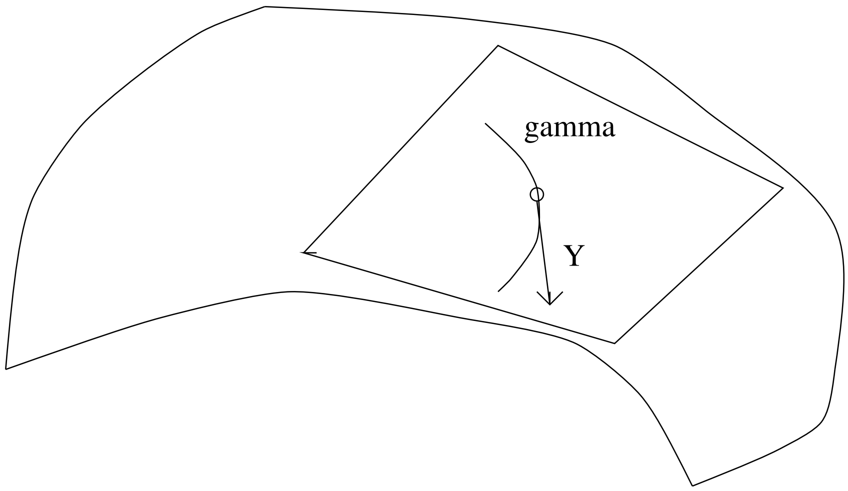

Figure 2.3: A connected compact Lie group G viewed as a differential manifold. The tangent space at the identity is the associated Lie algebra g. A path $\gamma(\mu)$ through the identity in G defines by differentiation an element $Y \in g$. The conjugation $G \to G, s \mapsto tst^{-1}$ of group elements gives rise to the Lie bracket $[X, Y] = ad_X(Y)$ in the Lie algebra

Table 2.1: Data structure of a compact Lie group which is semi-simple or a direct or a semi-direct product of a semi-simple compact Lie group with a finite group H

structure of the abstract group G:	**representation** ϑ of G:
– connected component G_0,	– action $\zeta(Y)$ of generators Y of the Lie algebra g,
– finite subgroup H,	– action $Res(\vartheta, H)(s)$ for all $s \in H$
– generators t of torus group T	– indices for maximal weight spaces
– order of the Weyl group W,	– matrix for change of coordinates
– generators of the Lie algebra g associated to G_0, distinguished by elements of the Cartan subalgebra and of the root spaces,	
– Adjoint action $Ad_{t^{-1}\|g/h}$	

to an action of the Lie algebra on the polynomials. Elements of the tangent space of $T_{\vartheta(e)}\vartheta(G_0)$ are given by paths

$$\gamma : [-1, 1] \to G_0, \quad \gamma(0) = e, \quad \gamma'(0) = Y \in \ g,$$

$$\zeta : \ g \to M(\mathbf{C}^n), \quad \zeta(Y) = \frac{d}{ds}\vartheta(\gamma(s))_{|s=0}.$$

Analogously, we have $T_{\Theta(e)}\Theta(G_0)$ by $\theta : \ g \to Aut(\mathbf{C}[x])$

$$\begin{aligned}
\theta(Y)(f(x)) &= \tfrac{d}{ds}[\Theta(\gamma(s))(f(x))]_{|s=0} \\
&= \tfrac{d}{ds}[f\,(\vartheta(\gamma(s)^{-1})x)]_{|s=0} \\
&= \nabla f(x) \cdot \tfrac{d}{ds}\vartheta(\gamma(s)^{-1})_{|s=0} \cdot x = \nabla f(x) \cdot \zeta(-Y) \cdot x \\
&= -\nabla f(x) \cdot \zeta(Y) \cdot x.
\end{aligned}$$

From this construction it is obvious that a polynomial f is invariant with respect to $\mathrm{Res}(\vartheta, G_0)$ iff

$$\theta(Y)(f(x)) = -\nabla f(x)\zeta(Y)x = 0, \tag{2.4}$$

for a set of generators of g. This is well-known, see e.g. [186] p. 206. It is the key for Algorithm 2.1.10.

Provided a vector space basis $\tilde{V} = \{v_1(x), \ldots, v_m(x)\}$ of a subvector space V of $\mathbf{C}[x]$ (e.g. the monomials in $H_d(\mathbf{C}[x])$ is given the projection $\mathcal{R}_{|V} : V \to V \cap \mathbf{C}[x]^{G_0}$ is realized in the following way: We make an ansatz $f(x) = \sum_{v \in \tilde{V}} a_v v(x) \in \mathbf{C}[a_{v_1}, \ldots, a_{v_m}][x]$. The conditions (2.4) result in a linear system of equations in the unknowns a_v by comparing coefficients. By solving this system and substitution of the result back into f one easily determines a vector space basis of $V \cap \mathbf{C}[x]^{G_0}$.

If G is a direct product $G = H \times G_0$ with a finite group H the application of the Reynolds projection of H on G_0-invariants yields G-invariants. Since H and G_0 commute the Reynolds projections commute and thus we have $\mathcal{R}^G = \mathcal{R}^H \mathcal{R}^{G_0} = \mathcal{R}^{G_0} \mathcal{R}^H$.

In case of a semi-direct product the system of linear equations above is amended by the equations resulting from comparing coefficients for all generators s of the finite group H in $f(x) - f(\vartheta(s)x)$.

Algorithm 2.1.10 *(Computation of fundamental invariants up to degree d)*
Input: *representation ϑ of G:*

- *action $\zeta(Y)$ of the generators Y of the Lie algebra g*
- *action $Res(\vartheta, H)$*

Molien series $\mathcal{HP}_{K[x]^G}(\lambda) = \sum_{i=0}^{\infty} a_i \lambda^i$
maximal degree d

Output: *invariants*

$\Pi := \{\}, m := 0$ # *set of invariants*
choose term order $<$ on $K[x]$
$GB := \{\}$
$k := \min(\{i \mid a_i \neq 0,\ i \geq 1\}$ # *minimal missing degree*
$HP := 1$ # *tentative Hilbert series $\mathcal{HP}_{K[\pi(x)]}(\lambda) = \sum_i b_i \lambda^i$*
while $k \leq d$ do
  $s := a_k - b_k$ # *number of missing invariants*
  $\tilde{V} :=$ { *monomials in $H_k^N(K[x])$*}
  $M :=$ { *monomials in $H_k^W(K[y_1, \ldots, y_m])$*}
   # *$W \simeq$ induced weighted grading*
  $\mathcal{M} := \{q(ht(p_1), \ldots, ht(p_m)) \in K[x] \mid \quad q \in M\}$
  $\tilde{V} := \tilde{V} \setminus \mathcal{M}$
  $Q := \mathcal{R}(\tilde{V})$ # *vector space basis $Q = \{q_1, \ldots, q_l\}$ of $H_k^N(K[x]^G) \cap V$*
  $P := \{\}$ # *new invariants of degree k*
  for i from 1 to l do
   $h := normalf_<(q_i, GB)$
   if $h(x,y) \notin K[y_1, \ldots, y_m]$ then
    $p := q_i - h(0, \pi(x))$
     # *$p \in \mathcal{P}$ with $H_k^N(K[x]^G) = H_k^N(K[\pi(x)]) \oplus \mathcal{P}$*
    $\tilde{p} := normalf_<(p, P)$

$\qquad\qquad$ *# linear combination yields* $ht(\tilde{p}) \neq ht(p), p \in P$
$\qquad$ *if* $\tilde{p} \neq 0$ *then* $P := P \cup \{\tilde{p}\}$ $\quad$ *#* $ht(\tilde{p}) \notin \langle ht(\pi_1), \ldots, ht(\pi_m)\rangle$
$\Pi := \Pi \cup \{p_1, \ldots, p_s\} = \Pi \cup P$ $\qquad$ *# invariants found*

$\tilde{k} := \min(\{i \mid a_i \neq b_i,\, i > k\}$
if $\tilde{k} > d$ *then* $OUTPUT(\Pi)$
else $\qquad$ *# compute relations*
$\quad$ *extend elimination order* $<$ *to* $K[x, y_1, \ldots, y_{m+s}]$ *eliminating* x
$\quad$ $GB := \mathcal{GB}(GB \cup \{y_{m+1} - p_1, \ldots, y_{m+s} - p_s\})$ $\quad$ *# Gröbner basis*
$\quad$ *- using the Hilbert series driven version*
$\quad$ *- using truncation at degree d*
$\quad$ $m := m + s$
$\quad$ $HT := \{ht(f) \mid f \in GB \cap K[y]\}$
$\quad$ $HP := \mathcal{HP}^W_{K[y]/\langle HT\rangle} = \mathcal{HP}_{K[\pi(x)]}(\lambda) = \sum_i b_i \lambda^i$ $\quad$ *# Algorithm 1.2.16*
$\quad$ $k := \min(\{i \mid a_i \neq b_i\}$ $\qquad$ *# minimal missing degree*
$\quad$ *if* $k > d$ *then* $OUTPUT(\Pi)$

Remark 2.1.11 *i.) The step* $h = normalf_<(q_i, GB)$ *rewrites the polynomial* $q_i(x) = h(\pi_1(x), \ldots, \pi_m(x))$ *in terms of invariants if the result* $h(x, y)$ *depends on* y *only. (Computation in a ring, see [14] p. 270.) Once a term order has been fixed the representation in fundamental invariants is unique, but* h *depends on the term order in general. ii.) The linear combination from* p_i *to* $\tilde{p}_i$ *is necessary in order to skip some monomials in the set* $\tilde{V}$*. For this it is preferable that all* $p \in P$ *have different leading terms. Nevertheless in general it is not possible to derive a direct complement* $\mathcal{P}$ *of* $H_k(\mathbf{C}[\pi(x)])$ *in* $H_k(\mathbf{C}[x]^G)$ *since* $\mathbf{C}[\pi_1(x), \ldots, \pi_r(x)]$ *and* $\mathbf{C}[ht(\pi_1(x)), \ldots, ht(\pi_r(x))]$ *are not isomorphic as graded algebras in the general case. (If they were* $\{\pi_i\}$ *would be called a* SAGBI basis *[162, 104], Chapter 11 in [177], p. 199 in [186]) iii.) A part of the input polynomials of the Buchberger algorithm forms a Gröbner basis. An implementation of the Buchberger algorithm taking advantage of this fact would be desirable. Secondly, as far as the computation of the Hilbert series is concerned the computation of a minimal Gröbner basis suffices. The final intermediate reduction could be skipped. iv.) The advantage of this algorithm is that it also works if only a Taylor series expansion of the Molien series* $\mathcal{HP}_{K[x]^G}(\lambda) = 1 + a_1\lambda + \cdots + a_d\lambda^d + \cdots$ *up to degree* d *is known. v.) The second advantage is that it is very flexible. In case a set of fundamental invariants of a subgroup* $\tilde{G}$ *of* G *is known one only needs to replace the set of monomials* $\tilde{V}$ *by a vector space basis of* $\tilde{G}$*-invariants of degree* k*. Secondly, in case the user knows some candidates of invariants these can easily be incorporated.*

Example 2.1.12 *In [129] Leis has investigated a bifurcation problem with* $O(2) \times S_1$*-symmetry and computed the fundamental invariants and equivariants. With Algorithm 2.1.10 the systematic computation of the Hilbert basis is possible. The real representation* $\vartheta : O(2) \times S_1 \to GL(\mathbf{R}^6)$ *is written as a representation on* $\{y \in \mathbf{C}^6 \mid$ *exists* $z_{-2}, z_0, z_2 \in \mathbf{C}$ *with* $y_1 = z_{-2}, y_2 = \bar{z}_{-2}, y_3 = z_0, y_4 = \bar{z}_0, y_5 = z_2, y_6 = \bar{z}_2\}$ *which is a real 6-dimensional vector space. By* $z_{-2} = \frac{1}{\sqrt{2}}(x_1 + ix_2), z_0 = \frac{1}{\sqrt{2}}(x_3 + ix_4), z_2 = \frac{1}{\sqrt{2}}(x_5 + ix_6)$ *a change of coordinates* $y = Ax$ *is defined resulting in*

$$\begin{aligned} \tilde{\vartheta} : O(2) \times S_1 \to GL(\mathbf{C}^6), \quad \tilde{\vartheta}(s) &= A^{-1}\vartheta(s)A, \quad s \in O(2) \times S_1, \\ \tilde{\vartheta}(\kappa)(z_{-2}, z_0, z_2) &= (z_2, z_0, z_{-2}), \end{aligned}$$

$$\begin{aligned}\tilde{\vartheta}(r_\theta)(z_{-2}, z_0, z_2) &= (e^{-i\theta}z_{-2}, z_0, e^{i\theta}z_2),\\ \tilde{\vartheta}(\phi)(z_{-2}, z_0, z_2) &= (e^{i\phi}z_{-2}, e^{i\phi}z_0, e^{i\phi}z_2).\end{aligned}$$

A torus group is generated by r_θ and ϕ. The action ζ of the associated Lie algebra on $\mathbf{C}^6$ is given by

$$\zeta(Y_1) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \zeta(Y_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

Additionally, there is the finite group $\mathbf{Z}_2 = \{id, \kappa\}$. In the Maple package symmetry [73] the representation matrices $\zeta(Y_1), \zeta(Y_2)$ with respect to the complex coordinate system and $\vartheta(\kappa)$ with respect to the original real coordinate system are stored as well as the transformation matrix A. The invariant generators of $\mathbf{C}[z]^{O(2)\times S_1}$ are easily computed to be

$$\bar{z}_2 z_2 + \bar{z}_{-2} z_{-2},\ \bar{z}_0 z_0,\ \bar{z}_2 z_2 \bar{z}_{-2} z_{-2},\ z_2 \bar{z}_0^2 z_{-2},\ \bar{z}_2 {z_0}^2 \bar{z}_{-2},$$

once the Molien series

$$\mathcal{HP}_{SO(2)\times S_1}(\lambda) = \frac{1+\lambda^4}{(1-\lambda^2)^2(1-\lambda^4)^2},$$

is known. The invariants of $\mathbf{R}[y]^{O(2)\times S_1}$ with real coefficients are easily obtained by linear combination. The experience shows that the possibility to restriction in degree is very valuable for efficiency.

2.1.2 Equivariants

The treatment of equivariants is analogously. The most important fact on ϑ-ϑ-equivariants is the correspondence to invariants of degree one.
The representation $\vartheta : G \to GL(\mathbf{C}^n)$ induces a representation

$$\tilde{\vartheta} : G \to GL(\mathbf{C}^{2n}), \quad \tilde{\vartheta}(s)(x, z) = (\vartheta(s)x, \vartheta(s)z), \quad \forall\, s \in G, x, z \in \mathbf{C}^n. \tag{2.5}$$

Naturally the ring $\mathbf{C}[x,z]^{\tilde{\vartheta}}$ is bigraded by the weight system (U, W) with $U_{|K[x]} = 0, U_{|K[z]} = N, W_{|K[x]} = N, W_{|K[z]} = 0$. The module of equivariants $\mathbf{C}[x]^{\vartheta}_{\vartheta}$ is isomorphic to $H_1^U(\mathbf{C}[x,z]^{\tilde{\vartheta}})$ by identification of an equivariant $f(x)$ with a $\tilde{\vartheta}$-invariant $(z, f(x))$, where $(\cdot,\cdot)$ is a ϑ-invariant inner product on $\mathbf{C}^n$. By this isomorphy a lot of results for invariants immediately generalize to the module of equivariants. Moreover, this relation is important for computations. In practice $\vartheta(s)$ are orthogonal (unitary) matrices. (For compact Lie groups a unique normalized Haar measure exists such that one has a G-invariant inner product. For groups with G-invariant inner product the coordinate system may be chosen such that the representation matrices are unitary or orthogonal, respectively.) Instead of computing with tuples $f(x)$ one computes with polynomials $z^t f(x)$.

$\mathbf{C}[x]^{\vartheta}_{\vartheta}$ is graded by the natural degree and thus the Hilbert-Poincaré series is well-defined. Analogous to the case of invariants we obtain

$$\mathcal{HP}^N_{\mathbf{C}[x]^{\vartheta}_{\vartheta}}(\lambda) = \frac{1}{|G|}\sum_{s\in G}\frac{\operatorname{trace}(\vartheta(s^{-1}))}{\det(id-\lambda\,\vartheta(s))}, \tag{2.6}$$

for finite groups G (see e.g. [166], [194] or [71]) and

$$\mathcal{HP}_{\mathbf{C}[x]^{\vartheta}_{\vartheta}}(\lambda) = \int_G \frac{\operatorname{trace}(\vartheta(s^{-1}))}{\det(id-\lambda\,\vartheta(s))}d\mu_G,$$

for compact Lie groups [166]. For groups of cases b.), c.) and d.) this is evaluated by

$$\mathcal{HP}_{\mathbf{C}[x]^{\vartheta}_{\vartheta}}(\lambda) = \frac{1}{|H|}\sum_{s\in H}\frac{1}{|W|}\int_T \det(id_{g/h} - Ad_{t^{-1}|g/h})\frac{\operatorname{trace}(\vartheta(g_0^{-1}s^{-1}))}{\det(id-\lambda\,\vartheta(sg_0))}d\mu_T\,, \tag{2.7}$$

where t is an element of the torus T and $|W|$ denotes the order of the Weyl group $N_{G_0}(T)/T$.

Given some homogeneous invariant polynomials $\pi_1(x),\ldots,\pi_k(x)$ and some homogeneous equivariants $f_1(x),\ldots,f_l(x)$ one likes to know whether they generate the module of equivariants. Comparing the Hilbert series $\mathcal{HP}^N_{\mathbf{C}[x]^{\vartheta}_{\vartheta}}(\lambda)$ of the $\mathbf{C}[x]^G$-module of equivariants with the $\mathbf{C}[\pi(x)]$-module generated by $f_1(x),\ldots,f_l(x)$ gives the answer. Analogous to the syzygies of invariants the computation of the latter series is based on the relations

$$R = \{r\in\mathbf{C}[y,u]\,|\,r(y,u) = r_1(y)u_1+\cdots+r_l(y)u_l,\ r(\pi(x),f(x))\equiv 0\}\,.$$

Lemma 2.1.13 *Given some homogeneous polynomials $\pi_1(x),\ldots,\pi_k(x)\in\mathbf{C}[x]$ and some homogeneous polynomial tuples $f_1(x),\ldots,f_l(x)\in\mathbf{C}[x]^n$ we denote by $\mathcal{M}$ the module generated by $f_1,\ldots,f_l$ over $\mathbf{C}[\pi_1(x),\ldots,\pi_k(x)]$. We define a Kronecker grading $U_{|\mathbf{C}[x,y]} = 0$, $U_{|\mathbf{C}[z,u]} = N$ and denote by $M\subset\oplus_{\nu=0}^1 H^U_\nu(\mathbf{C}[x,y,z,u])$ the $\mathbf{C}[x,y]$-module generated by*

$$y_1-\pi_1(x),\ldots,y_k-\pi_k(x),u_1-\sum_{j=1}^n(f_1(x))_jz_j,\ldots,u_l-\sum_{j=1}^n(f_l(x))_jz_j\,.$$

Denote by $\mathcal{GB}$ a Gröbner basis of M which is truncated at degree 1 with respect to U with respect to a term order which eliminates x and z. Then the Hilbert series $\mathcal{HP}^N_{\mathcal{M}}(\lambda)$ of the $\mathbf{C}[\pi(x)]$-module generated by $f_1,\ldots,f_l$ is given by

$$\mathcal{HP}^N_{\mathcal{M}}(\lambda) = \lambda^{\deg(f_1)}\cdot\mathcal{HP}^W_{\mathbf{C}[y]/\langle HT_1\rangle}(\lambda)+\cdots+\lambda^{\deg(f_l)}\cdot\mathcal{HP}^W_{\mathbf{C}[y]/\langle HT_l\rangle}(\lambda),$$

where W is the induced weighted grading with $W(y_i) = \deg_N(\pi_i), i = 1,\ldots,k$ and HT_j denote the sets of all monomials y^α where $y^\alpha u_j$ is a leading term of an element of $\mathcal{GB}$.

Proof: Recall the ideal $J = \langle y_1-\pi_1(x),\ldots,y_k-\pi_k(x)\rangle$ and the elimination ideal $I = J\cap\mathbf{C}[y]$. The rings $\mathbf{C}[\pi(x)]$ and $\mathbf{C}[y]/I$ are isomorphic as rings and even more as graded algebras where the natural grading and the induced weighted grading W are taken. The Gröbner basis of the theorem includes a Gröbner basis of J and of I.

The module of relations R as defined above is equal to $M\cap H^U_1(\mathbf{C}[y,u])$. It is a $\mathbf{C}[y]/I$-module. The quotient ring $\mathbf{C}[y]/I$ is graded by W and so R is a W-graded module where we define $W(u_i) = \deg_N(f_i)$. As W-graded module R is isomorphic to $\mathcal{M}$ graded by the natural degree. Thus R can be used for the computation of the Hilbert series. By Lemma 1.2.12 the leading terms of the Gröbner basis give the Hilbert series. $\square$

Remark 2.1.14 *i.) The Hilbert series driven version of the Buchberger algorithm 1.2.23 can be used with the bigrading $(U+W, W)$ and the series*

$$\mathcal{HP}^{U+W,W}_{\mathbf{C}[x,y,z,u]/\langle M\rangle}(\lambda_1,\lambda_2) = \frac{1}{(1-\lambda_1)^n\,(1-\lambda_1\lambda_2)^n},$$

since $y_1,\ldots,y_k,u_1,\ldots,u_l$ are the leading terms of a Gröbner basis with respect to an appropriate term order and $\mathbf{C}[x,y,z,u]/\langle y,u\rangle = \mathbf{C}[x,z]$.
ii.) For efficiency it is important to observe that truncation of the Gröbner basis of J with respect to W is possible. Then only a truncation of the Hilbert series is guaranteed to be correct.

In [74] this Lemma has been used in order to show that the module of $O(2)\times S_1$-equivariants is generated completely by the equivariants suggested by Leis in [129].

The natural grading of $\mathbf{C}[x]^{\vartheta}_{\vartheta}$ is respected by the *equivariant Reynolds projection*

$$\mathcal{R} : (\mathbf{C}[x])^n \to \mathbf{C}[x]^{\vartheta}_{\vartheta}.$$

For finite groups G the projection is realized by

$$\mathcal{R}(f(x)) = \frac{1}{|G|}\sum_{s\in G}\vartheta(s^{-1})f(\vartheta(s)x), \tag{2.8}$$

see [71] for more theoretical background.

For connected compact Lie groups G_0 the computation of the projection is done with the help of the associated Lie algebra g. Analogous to equation (2.4) for the case of invariants we have: A polynomial mapping $f(x)\in(\mathbf{C}[x])^n$ is equivariant, iff

$$\theta(-Y)(f(x)) = \zeta(Y)f(x), \tag{2.9}$$

for a set of generators Y of g. Observe that this easily generalizes to the ϑ-ρ-equivariants by using the representation of the Lie algebra g associated to ρ.

The realization of the projection is similar to the case of invariants. Given a vector space basis of $H_d((\mathbf{C}[x])^n)$ a projection to $H_d((\mathbf{C}[x])^n\cap\mathbf{C}[x]^{\vartheta}_{\vartheta})$ is realized by a formal ansatz with unknown coefficients and condition (2.9) for a set of generators of g which gives by comparing coefficients of monomials x^α in the tuple a system of linear equations in the unknown coefficients. Additionally, the vector space basis can be chosen much smaller, if one exploits that the group G_0 is semi-simple and thus the associated Lie algebra has special properties. This was first described in [166] for $SO(3)$ and generalized for arbitrary semi-simple Lie groups by Guyard in [74].

Assume the Lie algebra g is semi-simple. Then there exists a decomposition of g as a $\mathbf{C}$-vector space

$$g = h\oplus g_1\oplus\cdots\oplus g_{2s}\,, \tag{2.10}$$

having the additional property that for each g_i a linear from $\alpha_i : h\to\mathbf{C}$ exists such that

$$[X,Y] = ad(X)(Y) = \alpha_i(X)\cdot Y,\quad \forall\, X\in h, Y\in g_i.$$

That means that $ad(X)$ acts diagonally on g. Moreover, for each α_i there exists one α_j with $\alpha_i = -\alpha_j$. Collecting the forms α_i in a set R, one usually uses α as index and writes the well-known *Cartan decomposition* (2.10) as $g = h\oplus\bigoplus_{\alpha\in R} g_\alpha$. The g_α are called *root spaces.*

Theorem 2.1.15 *(Cartan-Weyl, see [63]) Assume the Lie algebra g is semi-simple. Then there exists a vector space basis $\{H_1, \dots, H_r\}$ of the Cartan subalgebra h and for each $\alpha \in R$ some Y_α generates g_α such that*

$$\begin{aligned}
&ad(H_i)Y_\alpha = [H_i, Y_\alpha] = \alpha(H_i)Y_\alpha,\\
&ad(Y_\alpha)Y_{-\alpha} = [Y_\alpha, Y_{-\alpha}] \in h,\\
&ad(Y_\alpha)Y_\beta = [Y_\alpha, Y_\beta] = N_{\alpha,\beta}Y_{\alpha+\beta} \text{ with } N_{\alpha,\beta} = 0 \text{ unless } \alpha+\beta \in R,
\end{aligned}$$

with $N_{\alpha,\beta} = -N_{\beta,\alpha} = N_{-\beta,-\alpha} = -N_{-\alpha,-\beta}$.

Example 2.1.16 *The Lie algebra $so(3)$ is generated by L_1, L_2, L_3 or over $\mathbf{C}$ by J_0, J_+, J_- (see Example 2.1.7). There are two root spaces $g_{\alpha_1} = span(J_+), g_{\alpha_2} = span(J_-)$. The linear forms $\alpha_i : h \to \mathbf{C}, i = 1, 2$ are defined by $\alpha_1(J_0) = 1, \alpha_2(J_0) = -1$ since $[J_0, J_\pm] = \pm J_\pm$. Additionally we have $[J_+, J_-] = 2J_0$.*

Moreover, one might choose a special coordinate system on $\mathbf{C}^n$ such that the representation matrices of $\zeta : g \to M(\mathbf{C}^n)$ are sparse. The vector space is decomposed into *weight spaces*

$$\mathbf{C}^n = \oplus_{\beta \in \mathcal{W}} V_\beta,$$

where $\mathcal{W}$ is a set of linear forms on h, such that

$$\forall v \in V_\beta, \forall X \in h, \quad \zeta(X)(v) = \alpha(X) \cdot v.$$

The rest of the Lie algebra g acts in the following way

$$\begin{aligned}
\zeta(g_\alpha)(V_\beta) \subset V_{\alpha+\beta} &\quad \text{if } \alpha+\beta \in \mathcal{W},\\
\zeta(g_\alpha)(V_\beta) = 0 &\quad \text{if } \alpha+\beta \notin \mathcal{W}.
\end{aligned}$$

Furthermore there exists a decomposition into *positive* and *negative* root spaces $R = R^+ \cup R^-$ and there exists a *maximal weight space* V_{β_0} such that

$$\zeta(Y_\alpha)V_{\beta_0} = 0 \quad \forall\ \alpha \in R^+.$$

It also holds that for each $v \in V_{\beta_0}$ the vectors $v, \zeta(Y_\alpha)v, \alpha \in R^-$ generate a vector space such that the restriction of ζ to this space is an irreducible representation. In an irreducible representation the maximal weight space is one-dimensional and unique.

In order to work with an efficient Reynolds projection we assume that the matrices $\zeta(Y)$ are given in a coordinate system such that the spaces V_β are generated by unit vectors. For an equivariant F this means that each component F_i corresponds to an element of one of the V_β. One makes an ansatz $F_i(x) = \sum_{m \in M} a^i_m m(x)$ (M basis of $H_k(\mathbf{C}[x])$) for the components $F_i, i \in I_{max}$ corresponding to the maximal weight space. Conditions for the coefficients a^i_m are given by the Lie algebra elements of the Cartan subalgebra and the positive roots. The conditions $\theta(-H)(F(x)) = \zeta(H)F(x)$ for generators H of h and $\theta(-Y_\alpha)(F(x)) = \zeta(Y_\alpha)F(x)$ for $\alpha \in R^+$ result into

$$\begin{aligned}
(\theta(-H)(F(x))_i = \zeta(H)_{ii}F_i(x), &\qquad i \in I_{max},\ H \in h,\\
(\theta(-Y_\alpha)(F(x))_i = 0, &\qquad i \in I_{max},\ \alpha \in R^+.
\end{aligned}$$

The other components $F_i, i \notin I_{max}$ of the equivariant are derived successively by shifts. The condition $(\theta(-Y_\alpha)F(x))_i = \zeta(Y_\alpha)_{ij}F_j(x)$, $\alpha \in R^-$ for some $i \in I_{max}$ and some $j \notin I_{max}$ implies

$$F_j(x) = \frac{1}{\zeta(Y_\alpha)_{ij}} \left(\theta(-Y_\alpha)F(x)\right)_i .$$

Once F_j is known other components are derived analogously. Of course this only works if the matrices $\zeta(Y)$ are taken in a coordinate system according to the weight spaces V_β.

Example 2.1.17 *Consider the representation $l = 1$ of the Lie algebra $so(3)$. With respect to a basis $\{f_{-1}, f_0, f_1\}$ of $\mathbf{C}^3$ the coordinates are denoted by z_{-1}, z_0, z_1 and the representation matrices are*

$$\zeta(J_+) = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}, \qquad \zeta(J_-) = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}.$$

The third coordinate corresponds to the maximal weight space. An equivariant of degree 3 *is*

$$F(z) = [2\, z_1 {z_{-1}}^2 - {z_0}^2 z_{-1},\, 2\, z_1 z_0 z_{-1} - {z_0}^3,\, -z_1 {z_0}^2 + 2\, {z_1}^2 z_{-1}]^t.$$

Once $F_3(z)$ is determined $F_2(z), F_1(z)$ are derived by J_- by

$$F_2(z) = \frac{1}{\sqrt{2}} \nabla F_3(z) \cdot \zeta(J_-)\, z, \quad F_1(z) = \frac{1}{\sqrt{2}} \nabla F_2(z) \cdot \zeta(J_-)\, z.$$

The Lie algebra $so(3)$ has one shift operator only. For other Lie algebras there might be more. Then a component F_j may be derived in several ways. The package symmetry [73] contains a routine which automatically finds out which root $\alpha \in R^-$ and which previous index to use provided the matrices $\zeta(Y)$ and the index set I_{max} of the maximal weight space are known.

If G is a direct or semi-direct product of a semi-simple Lie group G_0 with a finite group H the Reynolds projection of G consists of the Reynolds projection of G_0 and the projection for the finite group H. In the direct case one may just apply the Reynolds projection of H to an G_0-equivariant. In the semi-direct case one considers G_0-equivariants $f_1, \ldots, f_r \in \mathbf{R}[x]^G_G$. Then comparing coefficients in $f(\vartheta(s)x) = \vartheta(s)f(x)$ for a set of generators s of H and for $f = \sum a_i f_i$ yields a system of linear equations in the unknowns a_i. While the usage of the Cartan decomposition produces G_0-equivariants with complex coefficients it is important here to start with G_0-equivariants with *real* coefficients.

Algorithm 2.1.18 *(Computation of fundamental equivariants up to degree d)*
Input: *representation ϑ of G:*

- *action $\zeta(Y)$ of generators Y of the Cartan subalgebra h of g and of generators Y of the root spaces*
- *action $Res(\vartheta, H)$*

homogeneous invariants $\pi_1(x), \ldots, \pi_r(x)$
Hilbert-Poincaré series $\mathcal{HP}_{K[x]^G_G}(\lambda) = \sum_{i=0}^\infty a_i \lambda^i$
maximal degree d

Output: *homogeneous equivariants $f_1, \ldots, f_l$ which generate the module of equivariants over $\mathbf{C}[\pi(x)]$ up to degree d*

$M := \{\}, m := 0$ # *set of equivariants generating module* $\mathcal{M}$ *over* $K[\pi(x)]$
choose term order $<$ *on* $K[x_1, \ldots, x_n, y_1, \ldots, y_r]$ *eliminating* x
$GBinv := \mathcal{GB}(y_1 - \pi_1(x), \ldots, y_r - \pi_r(x))$ # *compute Gröbner basis wrt* $<$
    - using Hilbert series driven version
    - using truncation at degree d
$GB := GBinv$ # *Gröbner basis of module in* $\oplus_{i=0}^{1} H_i^W(K[x,y,z,u])$
$k := \min(\{i \mid a_i \neq 0\})$ # *minimal missing degree*
$HP := 0$ # *Hilbert series* $\mathcal{HP}_{\mathcal{M}}^N(\lambda) = \sum_i b_i \lambda^i$ *of module* $\mathcal{M}$
while $k \leq d$ *do*
  $s := a_k - b_k$ # *number of missing equivariants*
  $\tilde{V} := \{$ *monomials in* $H_k^N(K[x])\}$
  for $j \in \{j_1, \cdots, j_{dim}\}$ *do* # *dim - dimension of weight space*
    $\hat{V}_j = \{w(x) \in K[x]^n \mid w_i(x) = 0, i = 1, \ldots, n,\ i \neq j,\ w_j(x) \in \tilde{V}\}$
  $\hat{V} := \hat{V}_1 \cup \cdots \cup \hat{V}_{dim}$
  $Q := \mathcal{R}(\hat{V})$ # *vector space basis* $Q = \{q_1, \ldots, q_l\}$ *of* $H_k^N(K[x]_G^G) \cap \hat{V}$
  $F := \{\}$ # *new equivariants of degree* k
  for i *from 1 to* l *do*
    $p := normalf_<(\sum_{j=1}^n (q_i)_j z_j, GB)$
    if $p(x,y,z,u) \notin K[y_1, \ldots, y_r, u_1, \ldots, u_m]$ *then*
      $p := \sum_{j=1}^n (q_i)_j z_j$
        $-p(0, \pi(x), 0, \sum_{j=1}^n (M_1(x))_j z_j, \ldots, \sum_{j=1}^n (M_m(x))_j z_j)$
      $p := normalf_<(p, F)$ # $p(x,z) = f_1(x) z_1 + \cdots + f_n(x) z_n$
      if $p \neq 0$ *then* $F := F \cup \{f\}$
  $m := m + s$
  $M := M \cup \{f_1, \ldots, f_s\} = M \cup F$
    # *equivariants* $M = \{M_1, \ldots, M_m\}$ *found*
  $\tilde{k} := \min(\{i \mid a_i \neq b_i,\ i > k\}$
  if $\tilde{k} > d$ *then* $OUTPUT(M)$
  else # *compute relations*
    extend $<$ *to* $K[x, y_1, \ldots, y_r, z_1, \ldots, z_n, u_1, \ldots, u_m]$
      eliminating x *and* z
    $\mathcal{J} := GB \cup \left\{u_{m-s+1} - \sum_{j=1}^n (f_1)_j z_j, \ldots, u_m - \sum_{j=1}^n (f_s)_j z_j\right\}$
    $GB := \mathcal{GB}(\mathcal{J})$ # *compute Gröbner basis by Algorithm 1.2.23*
    $-$ *using grading* W *and Kronecker grading* U
      $W(x_i) = 1, W(y_i) = \deg_N(\pi_i(x)),$
      $W(z_i) = 0, W(u_i) = \deg_N(f_i(x))$
      $U(x_i) = 0, U(y_i) = 0, U(z_i) = 1, U(u_i) = 1$
    $-$ *using the Hilbert series*
      $\mathcal{HP}_{K[x,y,z,u]/\langle\mathcal{J}\rangle}^{U+W,W}(\lambda_1, \lambda_2) = \frac{1}{(1-\lambda_1)^n (1-\lambda_1\lambda_2)^n}$
    $-$ *using truncation at degree* d *(wrt* W*) and* 1 *(wrt* U*)*
    $HT := \{ht(p) \mid p \in GB \cap K[y,u]\}$ # *leading terms of relations*
    $HP := 0$
    for $j = 1, \ldots, m$ *do* # *Lemma 2.1.13*
      $HT_j := \{p \mid u_j \cdot p \in HT\}$

$HP := HP + \lambda^{\deg(f_j)} \mathcal{HP}^W_{K[y]/\langle HT_j \rangle}(\lambda)$ # *Algorithm 1.2.16*
$HP = \mathcal{HP}_{\mathcal{M}}(\lambda) = \sum_i b_i \lambda^i$ # *Hilbert series of* $\mathcal{M}$
$k := \min(\{i \mid a_i \neq b_i\}$ # *minimal missing degree*
if $k > d$ *then* $OUTPUT(M)$

Remark 2.1.19 *i.) Experience shows that it is more efficient to compute the relations for the invariants first. ii.) If the degree* d *is* ∞ *the algorithm terminates provided the module of equivariants is finitely generated over* $K[\pi(x)]$ *where* $\pi_1(x), \ldots, \pi_r(x)$ *are the given invariants. This is fulfilled for a homogeneous system of parameters. iii.) The division algorithm (step* $p := normalf_<(\sum_{j=1}^n (q_i)_j z_j, GB)$*) computes a representation of an equivariant* q_i *of degree* $\leq d$ *in terms of invariants and equivariants since an elimination order has been chosen and by Lemma 1.2.8. Once the term order has been fixed the representation in (2.1) is unique. iv.) The algorithm is very flexible. If the user already knows some fundamental equivariants this knowledge can easily be incorporated. v.) If the group is a connected semi-simple group then less slack variables are needed. In the right coordinates system the components of the equivariants* f_i, M_j *correspond to the weight spaces. The other components of the tuple are given by the shift operators in the root spaces of the Lie algebra. For efficiency this is an important point since the complexity of the Buchberger algorithm increases dramatically with the number of variables.*

Example 2.1.20 *In [34] P. Chossat, F. Guyard and R. Lauterbach investigate mode interaction, a dynamical system with symmetry of* $O(3)$ *acting on* $\mathbf{R}^8$*. The representation decomposes into two irreducible representations. The restrictions to* $SO(3)$ *act as the irreducible representations denoted by* $l = 1$ *and* $l = 2$ *(see Example 2.1.17). Since* $O(3)$ *is the product of* $SO(3)$ *with* $\mathbf{Z}_2$ *it remains to explain the action of the reflection. For both representations the natural action is assumed. That means that for* $l = 1$ *the action is faithful and for* $l = 2$ *it is non-faithful. The first step in [34] is the determination of the general equivariant vector field. The Molien series of the invariant ring is*

$$\mathcal{HP}_{\mathbf{C}[x]^{O(3)}}(\lambda) = \frac{1}{(1-\lambda^2)^2 (1-\lambda^3)^2 (1-\lambda^4)},$$

while the equivariant Molien series is

$$\mathcal{HP}_{\mathbf{C}[x]^{O(3)}_{O(3)}}(\lambda) = \frac{2\lambda + 3\lambda^2 + 2\lambda^3 + \lambda^4}{(1-\lambda^2)^2 (1-\lambda^3)^2 (1-\lambda^4)}.$$

By Algorithm 2.1.10 we computed the invariants

$$\begin{aligned}
[\quad & x_1 x_{-1} - \tfrac{1}{2} {x_0}^2, \quad y_2 y_{-2} - y_1 y_{-1} + \tfrac{1}{2} {y_0}^2, \\
& y_2 {x_{-1}}^2 + y_{-2} {x_1}^2 + \tfrac{1}{3}\sqrt{6}\, y_0 x_1 x_{-1} + \tfrac{1}{3}\sqrt{6}\, y_0 {x_0}^2 - y_{-1} x_1 x_0 \sqrt{2} - x_0 x_{-1} y_1 \sqrt{2}, \\
& -\tfrac{1}{4} y_2 {y_{-1}}^2 \sqrt{6} - \tfrac{1}{4} {y_1}^2 y_{-2} \sqrt{6} + \tfrac{1}{2} y_1 y_0 y_{-1} + y_2 y_0 y_{-2} - \tfrac{1}{6} {y_0}^3, \\
& \tfrac{13}{6}\sqrt{6}\, y_1 y_{-1} x_1 x_{-1} - \tfrac{5}{6}\sqrt{6}\, y_1 y_{-1} {x_0}^2 + \tfrac{4}{3}\sqrt{6}\, y_2 y_{-2} {x_0}^2 - \tfrac{1}{4}\sqrt{6}\, {y_{-1}}^2 {x_1}^2 \\
& \quad + y_{-2} {x_1}^2 y_0 - \tfrac{1}{2}\sqrt{6}\, y_2 y_{-1} x_0 x_{-1} \sqrt{2} - \tfrac{1}{2}\sqrt{6}\, y_1 y_{-2} x_1 x_0 \sqrt{2} \\
& \quad + \tfrac{1}{2} y_0 y_{-1} x_1 x_0 \sqrt{2} + y_2 y_0 {x_{-1}}^2 + \tfrac{1}{2} y_1 y_0 x_0 x_{-1} \sqrt{2} - \tfrac{5}{3}\sqrt{6}\, y_2 y_{-2} x_1 x_{-1} \\
& \quad - \tfrac{1}{4}\sqrt{6}\, {y_1}^2 {x_{-1}}^2 + \tfrac{1}{3}\sqrt{6}\, {x_0}^2 {y_0}^2 - \tfrac{7}{6}\sqrt{6}\, {y_0}^2 x_1 x_{-1} \;].
\end{aligned}$$

The following equivariants have been computed with truncation at degree 3 within two hours.

$$
\begin{aligned}
[[\ & x_{-1}, x_0, x_1, 0,0,0,0,0], \ [0,0,0,y_{-2},y_{-1},y_0,y_1,y_2], \\
[\ & y_0x_{-1} - \sqrt{3}y_{-1}x_0 + \sqrt{2}\sqrt{3}y_{-2}x_1, \sqrt{3}y_1x_{-1} - 2\,y_0x_0 + \sqrt{3}y_{-1}x_1, \\
& y_0x_1 - y_1x_0\sqrt{3} + y_2x_{-1}\sqrt{2}\sqrt{3}, 0,0,0,0,0\,], \\
[\ & 0,0,0,3\,x_{-1}{}^2, 3\,\sqrt{2}x_{-1}x_0, \sqrt{6}x_{-1}x_1 + \sqrt{6}x_0{}^2, 3\,\sqrt{2}x_0x_1, 3\,x_1{}^2\,], \\
[\ & 0,0,0,4\,y_0y_{-2} - y_{-1}{}^2\sqrt{6}, -2\,y_0y_{-1} + 2\,y_1y_{-2}\sqrt{6}, 4\,y_2y_{-2} + 2\,y_1y_{-1} - 2\,y_0{}^2, \\
& 2\,y_{-1}y_2\sqrt{6} - 2\,y_1y_0, -y_1{}^2\sqrt{6} + 4\,y_2y_0\,], \\
[\ & x_0\sqrt{3}y_{-1}y_0 - 3\,\sqrt{2}x_0y_1y_{-2} + 2\,y_{-2}\sqrt{2}\sqrt{3}y_0x_1 - 3\,y_{-1}{}^2x_1 - 5\,y_1y_{-1}x_{-1} + 2\,x_{-1}y_0{}^2 \\
& +8\,y_2y_{-2}x_{-1}, \ -x_{-1}y_1y_0\sqrt{3} + 3\,\sqrt{2}y_1y_{-2}x_1 - \sqrt{3}y_{-1}y_0x_1 + 3\,\sqrt{2}x_{-1}y_2y_{-1} + 5\,x_0y_0{}^2 \\
& -8\,x_0y_1y_{-1} + 2\,y_2y_{-2}x_0, -3\,y_2y_{-1}x_0\sqrt{2} + 2\,y_2y_0x_{-1}\sqrt{2}\sqrt{3} + 8\,y_2y_{-2}x_1 \\
& +2\,x_1y_0{}^2 - 3\,y_1{}^2x_{-1} + y_1y_0x_0\sqrt{3} - 5\,y_1y_{-1}x_1, 0,0,0,0,0\,], \\
[\ & 0,0,0,6\,y_0x_{-1}{}^2 - 6\,x_{-1}x_0y_{-1}\sqrt{3} + 6\,x_{-1}y_{-2}x_1\sqrt{2}\sqrt{3}, \\
& 3\,x_{-1}{}^2y_1\sqrt{2}\sqrt{3} + 3\,x_{-1}y_{-1}x_1\sqrt{2}\sqrt{3} - 3\,\sqrt{2}\sqrt{3}y_{-1}x_0{}^2 + 6\,x_0y_{-2}x_1\sqrt{3} - 3\,x_{-1}x_0y_0\sqrt{2}, \\
& 3\,y_1x_0x_{-1}\sqrt{2} + 6\,y_{-2}x_1{}^2 + 2\,\sqrt{2}\sqrt{3}y_0x_1x_{-1} - 4\,\sqrt{2}\sqrt{3}y_0x_0{}^2 + 3\,y_{-1}x_1x_0\sqrt{2} + 6\,y_2x_{-1}{}^2, \\
& 6\,x_0y_2x_{-1}\sqrt{3} + 3\,y_{-1}x_1{}^2\sqrt{2}\sqrt{3} + 3\,x_{-1}y_1x_1\sqrt{2}\sqrt{3} - 3\,x_0{}^2y_1\sqrt{2}\sqrt{3} - 3\,x_0y_0x_1\sqrt{2}, \\
& 6\,y_2x_1x_{-1}\sqrt{2}\sqrt{3} + 6\,y_0x_1{}^2 - 6\,y_1x_1x_0\sqrt{3}\,]\,].
\end{aligned}
$$

2.2 Using the nullcone

In this section we concentrate on algebraic groups. While Lie groups are additionally to the group structure endowed with the structure of a manifold, algebraic groups have the additional structure of a variety. An algorithm by Derksen [48] (see also [186] p. 205) for the computation of invariants is recalled and generalized to equivariants. Hilbert's first proof has been criticized to be non-constructive last century. Derksen replaced the non-constructive argument by an algorithm exploiting Gröbner bases. We start with the definition of the *nullcone* which is the most important object in invariant theory.

Definition 2.2.1 *Let $\vartheta : G \to GL(\mathbf{C}^n)$ be a faithful representation of a group G and $\mathbf{C}[x]^\vartheta$ the ring of invariants. The ideal generated by all homogeneous invariants over $\mathbf{C}[x]$ is denoted by $I_\mathcal{N} \subset \mathbf{C}[x]$. The variety $V(I_\mathcal{N}) \subset \mathbf{C}^n$ is called* nullcone.

Of course all homogeneous invariants generate an ideal in $\mathbf{C}[x]^\vartheta$ as well. Projecting a set of generators of $I_\mathcal{N}$ to a set of generators of $I_\mathcal{N} \cap \mathbf{C}[x]^\vartheta$ gives the Hilbert basis. (This is recalled in Lemma 2.2 in [48].) The non-constructive argument of existence of generators of $I_\mathcal{N}$ could be replaced by a computational way, see the algorithm below.

A variety G with the structure of a group is called *algebraic group* if $\mu : G \times G \to G$, $\mu(x,y) = xy$ and $\iota : G \to G$, $\iota(x) = x^{-1}$ are mappings of varieties. Finite groups and most of the compact Lie groups are algebraic groups.

Algorithm 2.2.2 *(Computation of invariants of an algebraic group)*
(Derksen [48])
Input: *algebraic group, represented by $h_1(z), \ldots, h_r(z) \in K[z_1, \ldots, z_s]$ and*
$(a_{ij}(z))_{i,j=1,\ldots,n}$ where $a_{ij}(z) \in K[z]/J_G$ with $J_G = \langle h_1, \ldots, h_r \rangle$

and its representation as a compact Lie group
degree d

Output: *invariants*

1.) *Compute a Gröbner basis $\mathcal{GB}$ of*

$$I_\Gamma := \left\langle h_1(z), \ldots, h_r(z), y_1 - \sum_{j=1}^{n} a_{1j}(z)x_j, \ \ldots\ , \ y_n - \sum_{j=1}^{n} a_{nj}(z)x_j \right\rangle$$

in $K[x,y,z]$ with respect to a term order which eliminates z,
using truncation at degree d with respect to
$W(x_i) = 1 = W(y_i), i = 1, \ldots, n, W(z_j) = 0, j = 1, \ldots, s$.

2.) *Substitute zeros: $I_N = \{f(x,0) \mid f(x,y) \in \mathcal{GB} \cap K[x,y]\}$.*
This gives generators of $I_{\mathcal{N}}$.

3.) *Compute a Gröbner basis $GB = \{f_1, \ldots, f_m\}$ of the ideal $\langle I_N \rangle$*
(truncate at degree d).

4.) *Apply the Reynolds projection: $\mathcal{R}(f)$, $f \in GB$ in the following way:*
Denote by $i_1, \ldots, i_k$ the sorted degrees of elements in GB
Invs $:= \{f \in GB \mid deg(f) = i_1\}$
for $j = 2, \ldots, k$ do
$i := i_j$
Apply the projection to the vector space spanned by
$\{m(x)f_l(x) \mid f_l \in GB, \deg(f_l) \leq i, \deg(m) = i - \deg(f_l),$
$\quad m(x) \text{ monomial}, ht(f_s) \not| m(x)ht(f_l) \text{ for all } s < l\}$
Update(Invs)

Remark 2.2.3 *i.) The examples presented in [48] show that in general the set of generators is not minimal. ii.) The ideal I_Γ is homogeneous with respect to the grading W given by $W(x_i) = 1 = W(y_i), i = 1, \ldots, n, W(z_i) = 0, i = 1, \ldots, s$. So the computation of the Gröbner basis may be restricted in degree. Consequently, only the low-degree part of $I_\Gamma \cap K[x,y]$ and the low-degree part of $I_{\mathcal{N}}$ is computed. iii.) For finite groups the Reynolds projection is easily realized while for compact Lie groups the Reynolds projection may be realized as in Section 2.1 with the Lie algebra associated to the connected compact Lie group.*

The proof in [48] is based on the ideal $b = \{h \in \mathbf{C}[x,y] \mid h(x, \vartheta(g)x) = 0\}$. Since in Section 4.1 analogous arguments are given the proof is not recalled here.

Algebraic groups include especially torus groups. A torus action is defined as a matrix $B = (b_{ij})_{i=1,\ldots,n, j=1,\ldots,d}$ with $b_{ij} \in \mathbf{Z}$. It is most convenient to split it uniquely into $B = \tilde{B} - \hat{B}$ with $\tilde{B}, \hat{B} \in \mathbf{N}^{n \times d}$ with $\tilde{B}, \hat{B}$ as sparse as possible. The matrix $A = (a_{ij})$ is defined by a diagonal matrix

$$a_{ii}(z_1, \ldots, z_{2d}) = \prod_{j=1}^{d} z_j^{\tilde{b}_{ij}} z_{d+j}^{\hat{b}_{ij}}, \quad a_{ij} = 0, \ i \neq j.$$

The restrictions in z are given by $h_1(z) = z_1 z_{d+1} - 1, \ldots, h_d(z) = z_d z_{2d} - 1$. This shows that Algorithm 2.2.2 is a generalization of Algorithm 1.4.5 in [176] which computes invariants of torus group actions.

Example 2.2.4 *In [100] Igor Hoveijn investigates Hamiltonian systems where the Jacobian has eigenvalues $\pm i, \pm i, \pm 2i$. This case is called 1:1:2 resonance. In the Birkhoff normal form of the dynamical system this leads to the study of the torus group action on a 6-dimensional space. The group acts with $t = e^{2\pi i\phi}$ on the real vector space*

$$\mathbf{R}^6 \simeq \{x \in \mathbf{C}^6 \mid x_4 = \bar{x}_1,\ x_5 = \bar{x}_2,\ x_6 = \bar{x}_3\},$$

as $(x_1, x_2, x_3, x_4, x_5, x_6) \to (t\,x_1, t\,x_2, t^2\,x_3, t^{-1}\,x_4, t^{-1}\,x_5, t^{-2}\,x_6)$.

This may be considered as a one-dimensional torus action with $t \in \mathbf{C}$ on $\mathbf{C}^6$. It is written as algebraic group by introducing $t = z_1, t^{-1} = z_2$ and $h_1(z) = z_1\,z_2 - 1$. The group action is a diagonal matrix

$$A(z) = \begin{pmatrix} z_1 & 0 & \cdots & & & 0 \\ 0 & z_1 & 0 & \cdots & & \vdots \\ \vdots & 0 & {z_1}^2 & 0 & \cdots & \\ \vdots & & 0 & z_2 & 0 & \\ \vdots & \cdots & & 0 & z_2 & 0 \\ 0 & \cdots & & & 0 & {z_2}^2 \end{pmatrix} \quad mod\ J_G = \langle h_1 \rangle.$$

There are 11 *invariants:*

$$x_1x_4,\ x_6x_3,\ x_5x_2,\ x_2x_4,\ x_1x_2x_6,\ {x_1}^2x_6,\ x_4x_5x_3,\ x_3{x_4}^2,\ x_5x_1,\ {x_5}^2x_3,\ {x_2}^2x_6.$$

The algorithmic computation of torus invariants is also interesting for other group actions. In case the torus T is normal in the group the group G/T is acting on the set of T-invariants and their conjugates. Then G-invariants are derived from G/T-invariants. This is described in [110].

Equivariants

Since equivariants correspond to certain invariants of degree one, Algorithm 2.2.2 immediately generalizes to the computation of equivariants. The invariants of degree one are taken with respect to the representation in (2.5). We restrict to orthogonal (unitary) group representations.

Let the matrix $A(z) \in \mathbf{C}[z]^{n,n}$ and the polynomials $h_1(z), \ldots, h_r(z)$ define an algebraic group. Assume $\tilde{A}(z)$ is a matrix with

$$A(z) \cdot \tilde{A}(z) \equiv Id \mod J_G = \langle h_1, \ldots, h_r \rangle.$$

Decompose the variables $x_1, \ldots, x_{2n}$ and $y_1, \ldots, y_{2n}$ into two groups $X_1 = \{x_1, \ldots, x_n\}, X_2 = \{x_{n+1}, \ldots, x_{2n}\}$ and $Y_1 = \{y_1, \ldots, y_n\}, Y_2 = \{y_{n+1}, \ldots, y_{2n}\}$. Then

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} A(z) & 0 \\ 0 & A(z) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

is the algebraic group action corresponding to the representation in (2.5) for the case of equivariants. Defining the grading W with $W(x_i) = W(y_i) = 0, i = 1, \ldots, n, W(x_i) = W(y_i) = 1, i = n+1, \ldots, 2n, W(z_i) = 0, i = 1, \ldots$ the ideal I_Γ defined in Algorithm 2.2.2 is W-homogeneous. Thus a truncation to degree one with respect to W is possible. Since the projection $I_\Gamma \cap K[x, y] \to I_\mathcal{N}, f(x, y) \to f(x, 0)$ and the Reynolds projection respects the degree a computation of a generating set of equivariants is possible.

Algorithm 2.2.5 *(Computation of invariants and equivariants of an algebraic group)*

Input: *algebraic group, represented by*
$h_1(z), \dots, h_r(z) \in K[z_1, \dots, z_s]$ *generating* $J_G = \langle h_1, \dots, h_r \rangle$
and matrix $A = (a(z)_{ij})_{i,j=1,\dots,n}$
and its representation as compact Lie group
degree d

Output: *invariants and equivariants generating the ring and the module up to degree* d

1.) a.) *Compute a Gröbner basis* GB_1 *of* $I_\Gamma^1 \subset K[X_1, Y_1, z]$ *generated by*

$$h_1(z), \dots, h_r(z)\ ,\ y_1 - \textstyle\sum_{j=1}^n a_{1j}(z)x_j\ ,\ \dots\ ,\ y_n - \sum_{j=1}^n a_{nj}(z)x_j,$$

with respect to a term order which eliminates z.
One may truncate at degree d *with respect to*
$W(x_i) = W(y_i) = 1, i = 1, \dots, n, W(z_j) = 0, j = 1, \dots, s.$

b.) *Substitute the second group of variables*

$$GB_2 := \{f(X_2, Y_2, z) \mid f(X_1, Y_1, z) \in GB_1,\ deg_U(f) \leq 1\}$$

where the degree is taken with respect to the Kronecker grading
$U(x_i) = 1 = U(y_i), i = 1, \dots, n,\ U(z_j) = 0, j = 1, \dots, s.$

c.) *Compute a truncated Gröbner basis* $\mathcal{GB}$ *of the* $K[z]$*-module* $H_{0,1}^W(I_\Gamma)$ *generated by* GB_1 *and* GB_2 *with respect to a term order which eliminates* z*. Here* $I_\Gamma \subset K[x, y, z]$ *denotes the ideal which is generated by*

$$\begin{aligned} &h_1(z), \dots, h_r(z),\\ &y_1 - \textstyle\sum_{j=1}^n a_{1j}(z)x_j,\ \dots\ ,\ y_n - \sum_{j=1}^n a_{nj}(z)x_j,\\ &y_{n+1} - \textstyle\sum_{j=1}^n a_{1,j}(z)x_{n+j},\ \dots\ ,\ y_{2n} - \sum_{j=1}^n a_{n,j}(z)x_{n+j}, \end{aligned}$$

The Gröbner basis is truncated at degree 1 *with respect to the grading* U *given by* $U(x_i) = U(y_i) = 0, i = 1, \dots, n,$
$U(x_i) = U(y_i) = 1, i = n+1, \dots, 2n$
and truncated at degree d *with respect to the grading*
$W(x_i) = W(y_i) = 1,\ i = 1, \dots, 2n, W(z_j) = 0, j = 1, \dots, s.$

2.) *Substitute zeros:* $I_N = \{f(x, 0) \mid f(x, y) \in \mathcal{GB} \cap K[x, y]\}$.

3.) *Compute a truncated Gröbner basis* GB *of* $\bigoplus_{i=0}^1 \bigoplus_{j=0}^d H_{i,j}^{U,W}(\langle I_N \rangle)$.

4.) *Apply the Reynolds projection:* $\mathcal{R}(f)\ ,\ f \in GB$.

5.) *Polynomials of degree zero with respect to* U *are invariant polynomials. Polynomials of degree one correspond to equivariants.*

Remark 2.2.6 *The precomputation of* GB_1 *and* GB_2 *are just done for efficiency reasons. One may generalize the algorithm to the case of general* ϑ*-*ρ*-equivariants, but then Step 1.) b.) is not valid any longer.*

Example 2.2.7 *In [87] p. 328 the group action of* $O(2) \times S_1$ *on the four-dimensional real vector space* $\{(x_1, \mathfrak{x}_1, x_2, \mathfrak{x}_2) \in \mathbf{C}^4 \mid \bar{x}_1 = \mathfrak{x}_1,\ \bar{x}_2 = \mathfrak{x}_2\}$ *is considered. In these complex* *coordinates this group action is nicely written as*

$$\begin{array}{ll} \vartheta(\theta)(x_1,x_2) = (e^{i\theta}x_1, e^{i\theta}x_2), & \theta \in S_1, \\ \vartheta(\phi)(x_1,x_2) = (e^{-i\phi}x_1, e^{i\phi}x_2), & \phi \in O(2), \\ \vartheta(\kappa)(x_1,x_2) = (x_2,x_1), & \kappa \textit{ flip in } O(2). \end{array}$$

As algebraic group this group representation is written with 5 *variables* Z_1, Z_2, Z_3, Z_4, Z_5 *where* $h_1(Z) = Z_1Z_3 - 1$, $h_2(Z) = Z_2Z_4 - 1$, $h_3(Z) = Z_5^2 - 1$ *and*

$$A(Z) = \begin{bmatrix} \frac{Z4\ Z1\ Z5\ (Z5+1)}{2} & 0 & \frac{Z4\ Z1\ Z5\ (Z5-1)}{2} & 0 \\ 0 & \frac{Z2\ Z3\ Z5\ (Z5+1)}{2} & 0 & \frac{Z2\ Z3\ Z5\ (Z5-1)}{2} \\ \frac{Z2\ Z1\ Z5\ (Z5-1)}{2} & 0 & \frac{Z2\ Z1\ Z5\ (Z5+1)}{2} & 0 \\ 0 & \frac{Z4\ Z3\ Z5\ (Z5-1)}{2} & 0 & \frac{Z4\ Z3\ Z5\ (Z5+1)}{2} \end{bmatrix}.$$

Denoting by $z_1, \bar{z}_1, z_2, \bar{z}_2$ *the second group of variables for the correspondence between equivariants and linear invariants the* 3 *Gröbner basis computations yield the generators of* $I_{\mathcal{N}}$*:*

$$\begin{array}{lllll} [& {x_2}^2{\bar{x}_2}^2, & x_2{\bar{x}_2}^2 z_2, & {x_2}^2\bar{x}_2\bar{z}_2, & z_1x_2\bar{x}_2 - x_1z_2\bar{x}_2, \\ & \bar{z}_1x_2\bar{x}_2 - \bar{x}_1\bar{z}_2x_2, & x_2\bar{x}_2 + x_1\bar{x}_1, & \bar{x}_2z_2 + z_1\bar{x}_1, & x_2\bar{z}_2 + \bar{z}_1x_1 &] \end{array}$$

such that the Reynolds projection gives the well-known Hilbert basis

$$x_1\bar{x}_1 + x_2\bar{x}_2, \quad (x_2\bar{x}_2 - x_1\bar{x}_1)^2,$$

and the set of fundamental equivariants

$$\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right), \quad (x_2\bar{x}_2 - x_1\bar{x}_1)\cdot\left(\begin{array}{c} x_1 \\ -x_2 \end{array}\right).$$

A truncation to degree 4 *has been used. During the big Gröbner basis computation* 14024 *S-polynomials have been treated.* 22022 *pairs were neglected because of the restriction in degree and* 11747 *pairs were superfluous because of the Buchberger criteria.*

Discussion of Table 2.2: We tested the algorithms for computation of invariants and equivariants for a couple of group actions. For a final judgment the algorithms have to be tested for more examples and with different and more efficient implementations of the Buchberger algorithm. Especially, one reason for different timings is that heuristics are involved in the computation of Gröbner bases which may be well tuned for one class of problems and for others not. Since Derksen's idea (computation of the basis of the zero-fiber ideal) is valid for algebraic groups, but the other algorithms work for compact Lie groups it is kind of unfair to compare the algorithms. Algorithm 2.2.2 implementing Derksen's idea and its generalization to equivariants (Algorithm 2.2.5) have *much wider* application than the other algorithms. But this is also the reason for their efficiency. By a rule of thumb algorithms exploiting more special structure are faster. Viewing a group as a differential manifold (compact Lie group) gives much more structure than the algebraic structure as a variety. The compact Lie group structure enables the computation of the Molien series and the implementation of the Reynolds projection which gives a lot of

Table 2.2: Time comparisons of the computation of invariants and equivariants for various group actions. The degree d indicates the restriction of the computation with respect to this degree. The sign $>$ means that the computation has been canceled or collapsed after the given timing

Group	Dim	Literature	Invariants				
			HP	d	Alg. 2.1.10	d	Alg. 2.2.2
$SO(3), l = 2$	5		38 sec		14 sec	2	54117 sec
$O(2) \times S_1$	6	[87, 144]	217 sec	∞	6309 sec	∞	56758 sec
		[163]		12	1497 sec	6	34617 sec
$O(2) \times S_1$	4	[87] p. 331	42 sec	∞	3 sec	∞	904 sec
$O(2) \times S_1$	6	[129]	263 sec	∞	59 sec	∞	70836 sec

Group	Literature	Equivariants				
		HP	d	Alg. 2.1.18	Alg. 2.3.21	Alg. 2.2.5
$SO(3), l = 2$		37 sec	∞	131 sec	5 sec	
$O(2) \times S_1$	[87, 144]	340 sec	∞	$>$ 4 days	189 sec	
	[163]		6	3839 sec		$>$ 2 weeks
			3	90 sec	42 sec	
$O(2) \times S_1$	[87] p. 331	30 sec	∞	61 sec	15 sec	13202 sec
$O(2) \times S_1$	[129]	278 sec	∞	37526 sec	202 sec	$>$ 5 days

structural insight of the invariant ring. Secondly, Algorithms 2.1.10 and 2.1.18 are able to use the Hilbert series driven Buchberger algorithm, are more flexible, and are able to use additional input (e.g. candidates of fundamental invariants) given by the user. For the equivariants Algorithm 2.3.21 behaves best because it additionally exploits the Cohen-Macaulay property of the module of equivariants for those cases where this property is known to be true. Derksen's idea is important because it replaces a non-constructive argumentation in Hilbert's first proof by an algorithmic step. But mathematical deepness does not necessarily equal algorithmic efficiency.

2.3 Using a homogeneous system of parameters

In this section I explain the algorithm by Sturmfels for computation of invariants for finite groups along with its implementation details and the generalization to equivariants. In order to explain the algorithm some concepts from commutative algebra are recalled and explained first. Along the way we derive Algorithm 2.3.11 testing for parameters and integral elements.

Definition 2.3.1 *([137] p. 71) Let R be a ring and $I_0 \supset I_1 \supset \cdots \supset I_h$ a chain of prime ideals. The maximal length of a prime chain is called* height *of I_0. The maximal height of a chain of prime ideals in R is called* Krull dimension.

The dimension of a quotient ring $\mathbf{C}[x]/J$ is computed by the degree of the Hilbert polynomial as mentioned in Example 1.2.17 or by inspecting the leading terms of a Gröbner basis of J with respect to a lexicographical order as suggested in [132] Proposition 2.1.

Definition 2.3.2 *([137] p. 73) Let R be a local ring and $\mathcal{M}$ its maximal ideal. An ideal I with $\mathcal{M}^\nu \subseteq I \subseteq \mathcal{M}$ for some $\nu > 0$ is called an* ideal of definition.

The ideal of the nullcone ($I_\mathcal{N}$ in Definition 2.2.1) is maximal and homogeneous, even more it is the unique maximal homogeneous ideal. Rings with unique maximal ideal are called *local rings.*

One can show that there are ideals of definition generated by d elements where d denotes the Krull dimension.

Definition 2.3.3 *([137] p. 78) Let R be a local ring of Krull dimension d and I an ideal of definition generated by $x_1, \ldots, x_d$. Then the set $x_1, \ldots, x_d$ is called a* system of parameters. *If R is graded, the maximal ideal is homogeneous, and $x_1, \ldots, x_d$ are chosen homogeneous then $x_1, \ldots, x_d$ are called a* homogeneous system of parameters.

Example 2.3.4 *Consider the ideal J generated by ${y_1}^3 - y_1{x_1}^2, y_1y_2, {y_2}^2 - y_2x_2, y_3 - x_1 - x_2$ in $\mathbf{C}[y_1, y_2, y_3, x_1, x_2]$. Since the polynomials are homogeneous the ring $\mathbf{C}[y, x]/J$ is a local ring. The maximal ideal $\mathcal{M}$ is generated by all homogeneous rest classes. The polynomials form a Gröbner basis with respect to the matrix term order with $y_1 > y_2 > y_3 > x_1 > x_2$ and with matrix*

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

which is eliminating y_1, y_2, y_3. They have leading terms ${y_1}^3, y_1y_2, {y_2}^2, y_3$. Since there are no monic terms in x_1 or x_2 an ideal of definition is given by $I = \langle [x_1], [x_2] \rangle$. For other rest classes $[y_1], [y_2], [y_3] \in \mathcal{M}$ the relations show that $\mathcal{M}^3 \subset I$, e.g. $[y_1]^3 = [y_1][x_1^2] \in I$. The element $[y_1] \in \mathbf{C}[y, x]/J$ solves the monic relation $y_1^3 - [x_1]^2 y_1 = 0$ with coefficient $x_1^2 \in \mathbf{C}[x_1, x_2]$. The parameters are x_1, x_2 and the dimension is two. Indeed the variety $V(J) = \{(y, x) \in \mathbf{C}^5 \mid y = (\pm x_1, 0, x_1 + x_2)$ or $y = (0, 0, x_1 + x_2)$ or $y = (0, x_2, x_1 + x_2), x_1, x_2 \in \mathbf{C}\}$ has a parameterization over x_1, x_2. The number of leaves equals four which is as well the number of standard monomials of J in $\mathbf{C}[y]$.

Definition 2.3.5 *([8]) Let $R \supset S$ be two rings. An element $r \in R$ is called* integral*, if it is the solution of a monic polynomial $y^n + \sum_{i=0}^{n-1} a_i y^i$ with coefficients a_i in S.*

Imagine we have a polynomial in two variables which is not monic (e.g. $f(u, y) = u^2y^2 - uy + u - 1$) by a change of coordinates one can always achieve that we have a monic polynomial. For example $x = u - y$ leads to $\tilde{f}(x, y) = y^4 + 2\,xy^3 + (x^2 - 1)y^2 - (x - 1)y + x - 1$ such that $[y] \in \mathbf{C}[y, x]/\langle \tilde{f} \rangle$ is integral over $\mathbf{C}[x]$. The advantage is that in these coordinates the variety $V(\tilde{f})$ is much clearer. For each $x \in \mathbf{C}$ there are four points $(x, y) \in V(\tilde{f})$. The ring $\mathbf{C}[x, y]/\langle \tilde{f} \rangle$ is a $\mathbf{C}[x]$-module generated by $1, [y], [y^2], [y^3]$ which are four generators. This principle is valid more generally. We refer for example to [22] Appendix A p. 370.

Theorem 2.3.6 *(Noether's normalization) Let R be an affine algebra over a field k, and let I be a proper ideal of R then there exist $x_1, \ldots, x_d \in R$ such that*

a.) $x_1, \ldots, x_d$ are algebraically independent over k;

b.) R is an integral extension of $k[x_1, \ldots, x_d]$ (and thus a finite $k[x_1, \ldots, x_d]$-module);

c.)

$$I \cap k[x_1, \ldots, x_d] = \sum_{i=r+1}^{d} x_i k[x_1, \ldots, x_d] = \langle x_{r+1}, \ldots, x_d \rangle\,,$$

for some $r, 0 \le r \le d$.

Moreover, if $x_1, \ldots, x_d$ satisfy a.) and b.), then $d = \dim R$.

Also a variant with respect to grading exists which is much older. It dates back to Hilbert's work from 1893 [98], see [22] p. 37 or [46] Thm. 1.7 p. 68.

Definition 2.3.7 *([22] p. 3) Let R be a ring and M a R-module. An element $x \in R$ is called M-*regular *if it is not a zero-divisor of M ($x \cdot z = 0$ with $z \in M$ implies $z = 0$). A sequence $x_1, \ldots, x_d \in R$ is called an M-*sequence *if*
(i) x_i is an $M/(x_1, \ldots, x_{i-1})M$-regular element for $i = 1, \ldots, d$ and
(ii) $M/(x_1, \ldots, x_d)M \neq 0$.
An R-sequence is simply called a regular sequence.

Definition 2.3.8 *([22] p. 10) If R is a local ring and $\mathcal{M}$ the maximal ideal then the maximal length of a $\mathcal{M}$-regular sequence $x_1, \ldots, x_d \in \mathcal{M}$ is called* depth.

The following example shows that in general the Krull dimension is unequal the depth.

Example 2.3.9 *(Example 2.3.4 modified) Additional to the generators of J in Example 2.3.4 we amend $(2x_2 - x_1)\,({y_1}^2 - {x_1}^2)\,x_2$. This modifies the variety. The variety has still the property that for each value of x_1, x_2 there are finitely many zeros, but their number varies.*

$$\begin{aligned} V \;=\; & \{(y,x) \in \mathbf{C}^5 \mid \; y = (\pm x_1, 0, x_1 + x_2),\;\; x_1 \neq 2x_2,\; x_2 \neq 0\,\} \\ & \cup \{(y,x) \in \mathbf{C}^5 \mid \; y = (\pm 2x_2, 0, 3x_2) \text{ or } y = (0,0,3x_2) \text{ or } y = (0, x_2, 3x_2), \\ & \qquad\qquad\qquad\quad\; x_1 = 2x_2,\; x_2 \neq 0\,\} \\ & \cup \{(y,x) \in \mathbf{C}^5 \mid \; y = (\pm x_1, 0, x_1) \text{ or } y = (0,0,x_1),\, x_1 \neq 2x_2,\, x_2 = 0\,\} \\ & \cup \{(0,0) \in \mathbf{C}^5 \quad \}\,. \end{aligned}$$

The leading terms of a Gröbner basis with respect to the term order used in Example 2.3.4 are

$$y_2 x_2 {x_1}^3,\; x_2 x_1 {y_1}^2,\; {y_1}^3,\; y_1 y_2,\; {y_2}^2,\; y_3.$$

Since there are no monic terms in x_1 or x_2 an ideal of definition is still given by $I = \langle [x_1], [x_2] \rangle$. For other rest classes $[y_1], [y_2], [y_3] \in \mathcal{M}$ the relations show that $\mathcal{M}^3 \subset I$, e.g. $[y_1]^3 = [y_1][x_1^2] \in I$. But the difference is that now some leading terms involve x_1 or x_2. Substituting special values of x_1 or x_2 the polynomials may or may not form a Gröbner basis. First consider the case that they still form a Gröbner basis. Then the variety for the specialization of x_1, x_2 is zero-dimensional and the number of isolated solutions is given by Thm. 8.32 in [14] by the codimension of the ideal in the ring. This is easily read off from the leading terms of a Gröbner basis and is in this case two. If the specialization destroys the Gröbner basis property there are more isolated solutions. But y_1^3, y_2^2, y_3 are leading monomials and thus there are at most four standard monomials in $\mathbf{C}[y]$. This shows that the number of leaves varies between two and four.

The depth of the homogeneous maximal ideal is one, since in the sequence x_1, x_2 the second element x_2 is not regular. Denoting by $R = \mathbf{C}[y,x]/J$ the ring we have to show that x_2 is not R/x_1R-regular. Although in R/x_1R we have $(2x_2 - x_1)(y_1^2 - x_1^2) \neq 0$ it holds $(2x_2 - x_1)(y_1^2 - x_1^2)x_2 = 0$ in R since this is an element of J.

Observe that R has a regular sequence is equivalent to the fact that R is a free module over the ring in the parameters. In [176] (Thm. 2.3.1 p. 38) it is proven: if a ring is a finitely generated free module over a system of homogeneous parameters then it is a finitely generated free module for every system of homogeneous parameters.

Definition 2.3.10 *([137] p. 103) A local ring is called* Cohen-Macaulay *if the dimension equals the depth.*

For example the quotient ring in Example 2.3.4 is Cohen-Macaulay. Algorithm 1.5.6 in Section 1.5 gives a procedure for testing whether some variables are the parameters of the quotient ring. It has been inspired by Subroutine 3.9 in [179] where a special property of the graded reverse lexicographical order is exploited.

Algorithm 2.3.11 *(Test for parameters, module basis and free module basis)*
Input: *W-homogeneous polynomials $p_1, \ldots, p_m \in K[x_1, \ldots, x_d, y_{d+1}, \ldots, y_n]$*
suggestion for parameters $x_1, \ldots, x_d$

1.) *Compute a* reduced *Gröbner basis of $J = \langle p_1, \dots, p_m \rangle$ with respect to a term order which*
 – *eliminates $y_{d+1}, \dots, y_n$ as a block ($y^\alpha x^\beta > x^\gamma \; \forall \, \alpha \neq 0, \beta, \gamma$)*
 – *and $y^\alpha > x^\beta y^\gamma \; \forall \, \alpha \neq 0, \beta \neq 0, \gamma$ with $\deg_W(y^\alpha) = \deg_W(x^\beta y^\gamma)$ (e.g. eliminates each variable $y_{d+1}, \dots, y_n$ successively for W-homogeneous ideals).*

2.) *Inspect leading monomials:*
If there are no monic leading terms $x_1^{a_1}, \dots, x_d^{a_d}$ then $K[x,y]/J$ is a $K[x_1, \dots, x_d]$-module.
If additionally there are monic leading terms $y_{d+1}^{a_{d+1}}, \dots, y_n^{a_n}$ then $K[x,y]/J$ is a finitely generated $K[x]$-module.
If additionally there are no mixed leading terms $x^\alpha y^\beta$ with $\alpha_i > 0, \beta_j > 0$ for at least one $i \in \{1, \dots, d\}$ and one $j \in \{d+1, \dots, n\}$ then $K[x,y]/J$ is Cohen-Macaulay.

Observe that the term order in [176] Subroutine 2.5.6 p. 52 for the representation in the Hironaka decomposition does not have the desired property.

This test is the key for an iterative Noether normalization in Algorithm 2.4.1.

Lemma 2.3.12 *Let $J = \langle p_1, \dots, p_m \rangle \subset K[x_1, \dots, x_d, y_{d+1}, \dots, y_n]$ be an ideal. The following two statements are equivalent.*

a.) The leading terms of a Gröbner basis with respect to a term order which eliminates $y_{d+1}, \dots, y_n$ contain monic monomials $y_i^{a_i}$, $a_i > 0$ for all $i = d+1, \dots, n$, but no monomials x^α.

b.) The variables $x_1, \dots, x_d$ are algebraic independent in $K[x,y]/J$ and $K[x,y]/J$ is a finitely generated module over the subring $K[x_1, \dots, x_d]$.

Proof: Assume we have a Gröbner basis with the properties in a.). The standard monomials *Std* represent the quotient ring $K[x,y]/J$ uniquely as a K-vector space. Thus

$$K[x,y]/J = \bigoplus_{y^\alpha x^\beta \in Std} y^\alpha x^\beta \; K.$$

Since no leading terms x^γ appear all monomials in $K[x]$ are standard monomials. This means that $x_1, \dots, x_d$ are algebraically independent and $K[x,y]/J$ is a $K[x]$-module. Because for each $i = d+1, \dots, n$ there is a leading term $y_i^{a_i}$ the quotient ring $K[x,y]/J$ is a *finitely generated* module over $K[x]$.

For the opposite direction assume that $x_1, \dots, x_d$ are algebraically independent in $K[x,y]/J$ and $K[x,y]/J$ is a finitely generated $K[x]$-module. Then no term x^α appears as leading term in a Gröbner basis. Suppose it would. By the elimination property there exists a polynomial $p(x)$ in the Gröbner basis. This is a contradiction to $x_1, \dots, x_d$ being algebraic independent in $K[x,y]/J$.

Secondly, for each $i = d+1, \dots, n$ there is a monic monomial $y_i^{a_i}$ as a leading term in the Gröbner basis. If it where not then all $y_i, y_i^2, y_i^3, \dots$ would be standard monomials and thus $K[x,y]/J$ would not be finitely generated as a $K[x]$-module. □

Lemma 2.3.13 *Let $J = \langle p_1, \ldots, p_m \rangle \subset K[x_1, \ldots, x_d, y_{d+1}, \ldots, y_n]$ be a W-homogeneous ideal with respect to a grading $W(x_i) > 0, W(y_j) > 0$ of $K[x, y]$. The following two statements are equivalent.*

a.) The reduced Gröbner basis of J with respect to a term order which eliminates $y_{d+1}, \ldots, y_n$ and fulfills $y^\alpha > x^\beta y^\gamma\ \forall\, \alpha \neq 0, \beta \neq 0, \gamma$ with $\deg_W(y^\alpha) = \deg_W(x^\beta y^\gamma)$ has leading terms which are all in $K[y]$.

b.) The ring $K[x, y]/J$ is a free module over $K[x_1, \ldots, x_d]$.

Proof: Assume a reduced Gröbner basis with respect to a term order as in a.) such that all leading monomials are in $K[y]$. Then all monomials in $K[x]$ are standard monomials. The standard monomials Std consist of $y^\alpha x^\beta$ where $y^\alpha \in Std_y = Std \cap K[y]$ is a standard monomials and x^β is any monomial in $K[x]$. Thus

$$K[x, y]/J = \bigoplus_{y^\alpha x^\beta \in Std} y^\alpha x^\beta \cdot K = \bigoplus_{y^\alpha \in Std_y} y^\alpha \cdot K[x].$$

This shows that Std_y form a free $K[x]$-module basis.

For the converse direction assume that $K[x, y]/J$ is a free $K[x]$-module. We have to show that each leading term of a reduced Gröbner basis with respect to a term order fulfilling the requirements in a.) is in $K[y]$. Let us assume the contrary. There exists a polynomial p in the reduced Gröbner basis with leading term $y^\alpha x^\beta, \beta \neq 0$. Then p has a representation

$$p(x, y) = y^\alpha x^\beta + \sum_{y^\gamma x^\delta \in A \subset Std} c_{\gamma,\delta} \cdot y^\gamma x^\delta, \quad c_{\gamma,\delta} \in K.$$

By the second property of the term order $\delta = 0$ is not possible. (Then we would have a different leading term.) Since $p \equiv 0$ in $K[x, y]/J$ we have

$$y^\alpha x^\beta + \sum_{y^\gamma x^\delta \in A} c_{\gamma,\delta} \cdot y^\gamma x^\delta \equiv 0 \quad \text{in} \quad K[x, y]/J.$$

Division by an appropriate monomial x^a leads to an identity

$$c \cdot y^{\gamma'} + \sum c'_{\gamma,\delta} \cdot y^\gamma x^\delta \equiv 0 \quad \text{in} \quad K[x, y]/J.$$

There has to be a monomial $y^{\gamma'}$ since otherwise the quotient ring is not free over $K[x]$. This polynomial is an element of J and has leading term $y^{\gamma'}$ by the second property of the term order. But $y^{\gamma'}$ divides a monomial in p properly. This is a contradiction to the assumption that p is an element of a reduced Gröbner basis. □

The theory and algorithm above yield the following **interpretation** of Cohen-Macaulay rings: Assume an ideal $J \subset K[x, y]$ with K a subfield of $\mathbf{C}$ such that $x_1, \ldots, x_d$ form parameters and $K[x, y]/J$ is a finitely generated, free module over $K[x]$. Then the variety $V(J) \subset \mathbf{C}^n$ has a special form. For each value $x = a \in \mathbf{C}^d$ the reduced Gröbner basis $p_1(a, y), \ldots, p_m(a, y)$ is a minimal Gröbner basis of an ideal $I \subset K[y]$. Since there are leading terms $y_{d+1}^{a_{d+1}}, \ldots, y_n^{a_n}$ the ideal I is zero-dimensional. For zero-dimensional ideals the number of solutions (counted with multiplicity) equals $\text{codim}(I)$ in $K[y]$ which is equal to the number of standard monomials. This gives solutions $(a, b^j) \in \mathbf{C}^n, j = 1, \ldots, \dim(K[y]/I)$. What happens if we vary a? Because $K[x, y]/J$

Table 2.3: Relations of primary and secondary invariants of an action of D_3

$$
\begin{aligned}
&\underline{n_2{}^2} - t_3 t_0 n_2 + 3\, s_2{}^2 s_1{}^2 n_1 - 2\, t_1 s_2{}^2 s_1 t_3 - 2\, s_2 s_1{}^2 t_2 t_0 + t_3 s_2 s_3 t_0 s_1 \\
&\qquad -12\, s_2{}^3 s_1{}^3 + s_1{}^3 t_0{}^2 + t_3{}^2 s_2{}^3\,, \\
&\underline{n_1 n_2} - s_2 s_1 n_2 - t_1 s_1{}^2 t_0 - t_3 s_2{}^2 t_2 - t_3 s_2 t_0 s_1\,, \\
&\underline{t_1 n_2} + t_0 s_1 n_1 - t_0 t_1 t_3 - 3\, t_2 s_2{}^2 s_1 + s_3 s_2{}^2 t_3 - 3\, s_2 s_1{}^2 t_0\,, \\
&\underline{t_2 n_2} + t_3 s_2 n_1 - 3\, t_1 s_2 s_1{}^2 - t_3 t_2 t_0 + s_3 s_1{}^2 t_0 - 3\, s_2^2 s_1 t_3\,, \\
&\underline{s_3 n_2} + s_2 s_1 n_1 - t_1 t_3 s_2 - t_2 t_0 s_1 - 4\, s_2{}^2 s_1{}^2\,, \\
&\underline{n_1{}^2} - 5\, s_2 s_1 n_1 + t_1 t_3 s_2 + t_2 t_0 s_1 - s_3 t_3 t_0 + 4\, s_2{}^2 s_1{}^2\,, \\
&\underline{t_1 n_1} - s_2 t_1 s_1 - s_3 t_0 s_1 - t_3 s_2{}^2\,, \\
&\underline{t_2 n_1} - s_1 t_2 s_2 - s_3 t_3 s_2 - s_1{}^2 t_0\,, \\
&\underline{s_3 n_1} + n_2 - 4\, s_3 s_2 s_1 - t_3 t_0\,, \\
&\underline{t_1{}^2} + s_2 n_1 - t_2 t_0 - 4\, s_2{}^2 s_1\,, \\
&\underline{t_2 t_1} - n_2\,, \\
&\underline{t_2{}^2} + s_1 n_1 - t_1 t_3 - 4\, s_2 s_1{}^2\,, \\
&-\underline{t_2 s_2} - t_0 s_1 + t_1 s_3\,, \\
&-\underline{t_3 s_2} - t_1 s_1 + t_2 s_3\,, \\
&\underline{s_3{}^2} - n_1\,.
\end{aligned}
$$

is free no mixed leading terms appear. This means the initial ideal of I is independent of the choice of $a \in \mathbf{C}^d$ and thus the number of solutions (a, b^j). Comparing Examples 2.3.4 and 2.3.9 we get the impression that non-Cohen-Macaulay rings are exceptional.

Example 2.3.14 *(Example 2.1.9 continued) The polynomials in (2.2) generate a homogeneous ideal J with respect to the induced grading. Thus $\mathbf{C}[s_1, s_2, t_0, t_3, s_3, t_1, t_2]/J$ is a local ring. Recomputing a Gröbner basis with respect to $s_1 > s_2 > t_0 > t_3 > s_3 > t_1 > t_2$ and the matrix term order given by*

$$
\left[\begin{array}{ccccccc}
2 & 2 & 3 & 3 & 2 & 3 & 3 \\
-2 & -2 & -3 & -3 & 0 & 0 & 0 \\
 & O_{2,3} & & 0_2 & id_{2,2} & & 0_2 \\
 & id_{3,3} & & 0_2 & O_{3,2} & & 0_2
\end{array}\right],
$$

gives

$$
\begin{gathered}
-4\, s_3 s_2 s_1 + \underline{s_3{}^3} - t_3 t_0 + t_2 t_1,\ \ -4\, s_2{}^2 s_1 + s_3{}^2 s_2 - t_2 t_0 + \underline{t_1{}^2}, \\
-4\, s_2 s_1{}^2 + s_3{}^2 s_1 - t_1 t_3 + \underline{t_2{}^2},\ \ -t_2 s_2 - t_0 s_1 + \underline{t_1 s_3},\ \ -t_3 s_2 - t_1 s_1 + \underline{t_2 s_3},
\end{gathered}
$$

with leading terms $s_3{}^3$, $t_1{}^2$, $t_2{}^2$, $t_1 s_3$, $t_2 s_3$. The term order is eliminating s_3, t_1, t_2 because of the second row. Since no monic terms in s_1, s_2, t_0, t_3 appear these four variables are a system of homogeneous parameters of $\mathbf{C}[s, t]/J$. Moreover, the parameters do not appear at all in the leading terms. Consequently, $\mathbf{C}[s, t]/J$ is a free module over $\mathbf{C}[s_1, s_2, t_0, t_3]$. The generators are the standard monomials 1, s_3, s_3^2, t_1, t_2, $t_1 t_2$. The ring $\mathbf{C}[s, t]/J$ is Cohen-Macaulay. Since the ring is isomorphic to the invariant ring of a finite group action *this agrees with the theorem that invariant rings of finite groups are Cohen-Macaulay. In this case of the invariant ring the parameters are called* primary invariants *and the generators of the free module* secondary invariants. *In this example the two new secondary invariants are introduced by new variables n_1, n_2 and new relations $n_1 - s_3^2$, $n_2 - t_1 t_2$. Similarly to the term order in [176] p. 52 the relations with respect to $s_1 > s_2 > t_0 > t_3 > s_3 > t_1 > t_2 > n_1 > n_2$ and the matrix term order*

$$\begin{bmatrix} 2 & 2 & 3 & 3 & 2 & 3 & 3 & 4 & 6 \\ -2 & -2 & -3 & -3 & & & 0_{1,5} & & \\ & 0_{1,3} & & 0 & 1 & 1 & 1 & 1 & 1 \\ & 0_{3,3} & & 0_3 & & id_{3,3} & & 0_{3,2} & \\ & id_{3,3} & & 0_3 & & 0_{3,3} & & 0_{3,2} & \end{bmatrix},$$

are given in Table 2.3. The relations have the classical form where the product of two generators of the free module is a combination which involves the generators only linearly.

Cohen-Macaulay rings have a *Hironaka decomposition*

$$R = \bigoplus_{i=1}^{l} \sigma_i \mathbf{C}[\pi_1, \ldots, \pi_d].$$

So for each pair of generators σ_i, σ_j of the free module R we have a relation

$$\sigma_i\, \sigma_j = \sum_{k=1}^{l} \sigma_k\, p_k(\pi), \quad 1 \le i \le j \le l,$$

since $\sigma_i\, \sigma_j \in R$ and as a module over $\mathbf{C}[\pi]$ a different representation exists. Secondly, from the Hironaka decomposition the Hilbert series is easily determined ([176] Corollary 2.3.4):

$$\mathcal{HP}_R(\lambda) = \frac{\sum_{i=1}^{l} \lambda^{\deg(\sigma_i)}}{\prod_{j=1}^{d}(1 - \lambda^{\deg(\pi_j)})}.$$

The following theorem due to Hochster and Roberts [99] can be found in [22] (Thm. 6.5.1 p. 280) and Vasconcelos [186] (Thm. 7.4.3 p. 203). For finite groups see also Sturmfels [176] p. 40 Thm. 2.3.5).

Theorem 2.3.15 *Invariant rings of groups which are linear reductive are Cohen-Macaulay.*

From this it is clear that finding a Hilbert basis of an invariant ring can be done by finding a homogeneous system of parameters (the so-called primary invariants) and then finding the free module basis (secondary invariants). For finite groups there is a criterion for parameters: Since $\mathcal{N} = V(I_{\mathcal{N}}) = \{0\}$ one has $V(\langle \pi_1, \ldots, \pi_n \rangle) = \{0\}$ and $d = n$. This is the key for the well-known Algorithm 2.3.17. Secondly, the form of the Hilbert series deduced from the Hironaka decomposition is a great help since it gives the degrees of the secondaries.

Here it is reasonable to comment on common misunderstandings about invariant theory by scientists working in dynamical systems, see e.g. Appendix A in [27].

A form of the Molien series of type $\sum_i c_i\lambda^{e_i}/\prod_j(1-\lambda^j)$ does not necessarily correspond to a Hironaka decomposition, see the example in the paper by Sloane [173] due to Stanley. Secondly, the primary invariants nor their degrees are unique. Thirdly, the parameters are algebraic independent. But not every set of n algebraic independent homogeneous polynomials form a homogeneous system of parameters.

Example 2.3.16 *Let* $\mathbf{Z}_2 = \{id, s\}$ *act on* $\mathbf{C}^2$ *by* $(x_1, x_2) \to (-x_1, -x_2)$. *Although the invariants* x_1^2, x_1x_2 *are algebraic independent they do not form parameters. The relation in the invariants* $\pi_1 = x_1^2$, $\sigma_2 = x_1x_2$, $\pi_2 = x_2^2$ *is given by* $\pi_1\pi_2 = \sigma_2^2$ *and* $\pi_2^2 \notin \langle\pi_1, \sigma_2\rangle$.

The following algorithm appeared in [179] and in [176] p. 57 Algorithm 2.5.14.

Algorithm 2.3.17 *(Computation of invariants of a finite group)*
(Sturmfels [176] p. 57)
Input: *finite group* G, *given by matrices* $\vartheta(s), \forall\, s \in G$
Output: *primary and secondary invariants*

1. *Compute Molien series* $\mathcal{HP}_{K[x]^G}(\lambda) = \sum_{i=0}^{\infty} a_i\lambda^i$ # *Lemma 2.1.6*
2. *Compute primary invariants*
$\Pi := \{\}$ $S := \{\}$ # *lists of primary and secondary invariants*
$d := \min\{i \mid a_i > 0, i > 0\}$
$M_d = \{$ *monomials of degree* d $\}$, $b_d := 0$
while $|\Pi| < n$ *or* $V(\Pi) \neq \{0\}$ *do* # *Subroutine 2.3.19*
  if $|M_d| = 0$ *then* $d := \min\{i \mid a_i > 0, i > d\}$
    $M_d = \{$ *monomials of degree* d $\}$, $b_d := 0$
  choose $m \in M_d$, $M_d := M_d \setminus \{m\}$
  $p := \mathcal{R}(m)$ # *Reynolds projection 2.3*
  if $p \notin Rad(\Pi)$ *then* # *Kantorovich trick*
    $\Pi := \Pi \cup \{p\}$, $b_d := b_d + 1$
    if $b_d = a_d$ *then* $d := \min\{i \mid a_i > 0, i > d\}$
      $M_d = \{$ *monomials of degree* d $\}$, $b_d := 0$
while $|\Pi| > n$ *do*
  choose $\pi \in \Pi$
    if $V(\Pi \setminus \{\pi\}) = \{0\}$ *then* $\Pi := \Pi \setminus \{\pi\}$
      $S := S \cup \{\pi\}$
3. *Determine degrees of secondary invariants*

$$\mathcal{HP}_{K[x]^G}(\lambda) \cdot \prod_{i=1}^{n}(1 - \lambda^{\deg(\pi_i)}) = c_{e_1}\lambda^{e_1} + \cdots + c_{e_r}\lambda^{e_r}$$

4. *Search secondary invariants*
for each $\sigma \in S$ *do* $c_{\deg(\sigma)} := c_{\deg(\sigma)} - 1$
compute truncated Gröbner basis $\mathcal{GB}(\oplus_{i=0}^{\max e_j} H_i(\Pi(x))$ # *cheap order*
$\tilde{S} := \{normalf(\sigma, \mathcal{GB} \mid \sigma \in S)\}$ # *prepare test*
for $e \in \{e_1, \ldots, e_r\}$ *do*
  while $c_e > 0$ *do*
    choose $m \in M_e$, $M_e := M_e \setminus \{m\}$
    $\sigma := \mathcal{R}(m)$ # *Reynolds projection 2.3*
    $\tilde{\sigma} := normalf(\sigma, \mathcal{GB})$
    if $\tilde{\sigma}$ *is linear independent of* $\tilde{S}$ *then*

$S := S \cup \{\sigma\}$, $\tilde{S} := \tilde{S} \cup \{\tilde{\sigma}\}$
$c_e := c_e - 1$

Table 2.4: Performance of computation of fundamental invariants and equivariants by Algorithm 2.3.17 and Algorithm 2.3.20 for various group actions on a Sun

group	dim	use in dynamics	reference	invariants primaries	sec.	equivs
$D_3,\ \vartheta^2+\vartheta^3$	3	C_3-Hopf bif.	[77]	4.4 sec	4.4 sec	41.5 sec
$D_3,\ \vartheta^3+\vartheta^3$	4	TB-pkt.	[125, 138]	28.3 sec	20.9 sec	254.5 sec
$D_4,\ \vartheta^2+\vartheta^5$	3	C_4-Hopf bif.	[125]	8.1 sec	7.2 sec	91.2 sec
$D_4,\ \vartheta^3+\vartheta^5$	3	secondary bif.	[29, 125]	12.0 sec	3.5 sec	75.5 sec
$D_4,\ \vartheta^5+\vartheta^5$	4	TB-pkt.	[125]	42.3 sec	42.2 sec	475.9 sec
$Z_4 \cdot Z_2^4$	4	sym. chaos	[194]	240.7 sec	69.2 sec	1297.9 sec

Remark 2.3.18 *i.) The implementation details are taken from* Invar *[106]. ii.) Other implementations are available in Magma [30], Singular [96] and in Maple [73]. iii.) The key in step 2 is that the primary invariants have as common solution the zero point only. This means in algebraic notation $Rad(\langle\pi_1(x),\dots,\pi_n(x)\rangle) = \langle x_1,\dots,x_n\rangle$. So in order to decrease the variety by a new π_i one needs to test whether the new candidate π_i satisfies $\pi_i \in Rad(\langle\pi_1,\dots,\pi_{i-1}(x)\rangle)$. This can be done by the usual Kantorovich trick for testing radical membership. In [107] it is suggested to decrease the dimension of the variety which is an even stronger requirement. In order to assure this a factorized Gröbner basis of $\langle\pi_1(x),\dots,\pi_{i-1}(x)\rangle$ is computed which is close to a primary decomposition. This enables to test for decreasing dimension. In [47] it is suggested to compute the dimensions and compare them. iv.) In [107] another variant is suggested. After primaries have been found first secondary invariants for a subgroup are computed and then a module intersection gives the required secondaries. v.) In [108] improvements are suggested in order to achieve the lowest possible degrees of the primary invariants. (Implementations exist by Steel and Kemper in Magma [30]). Also [150] address this question. But the work in Section 4.3 shows that in the context of dynamical systems with symmetry the minimality might be unnecessary. vi.) In [109] another variant for the search of secondaries is presented. One first tries to use products of previously computed secondary invariants. This is a helpful approach in order to find a minimal Hilbert basis which is important for dynamical systems [163]. vii.) In [74] other possibilities for the module membership test (secondary invariants) are presented and tested. The presented variant turned out to be most efficient.*

The test $V(\Pi) = \{0\}$ could be done by multiple application of the Kantororich trick since the condition is equivalent to $Rad(\langle\pi_1,\dots,\pi_n\rangle) = \langle x_1,\dots,x_n\rangle$ where we denote by $Rad(I) = \{f \mid \exists r \text{ with } f^r \in I\}$ the *radical* of an ideal. The condition $f \in Rad(\langle g_1,\dots,g_s\rangle)$ is tested by computing the Gröbner basis of $\langle g_1,\dots,g_s, 1-zf\rangle$ with a slack variable z. If $\mathcal{GB} = \{1\}$ then $f \in Rad(\langle g_1,\dots,g_s\rangle)$, see [41] p. 177. Since in this context we have a special situation a more efficient variant is preferable.

Subroutine 2.3.19 *(Test $V(\langle \pi_1, \ldots, \pi_d \rangle) = \{0\}$) (Kemper [106])*
Input: *homogeneous polynomials $\pi_1, \ldots, \pi_d$*
Output: *Yes or No*

Compute a Gröbner basis $\mathcal{GB}$ of $\langle \pi_1(1, x_2, \ldots, x_n), \ldots, \pi_d(1, x_2, \ldots, x_n) \rangle$
if $\mathcal{GB} \neq \{1\}$ then OUTPUT(No)
  else $i = 2$
    Repeat
      compute a Gröbner basis $\mathcal{GB}$ of
      $\langle \pi_1(0, \ldots, 0, 1, x_{i+1}, \ldots, x_n), \ldots, \pi_d(0, \ldots, 0, 1, x_{i+1}, \ldots, x_n) >$
      $i := i + 1$
      if $\mathcal{GB} \neq \{1\}$ then OUTPUT(No) Stop
    until $i = n$

In [71] and [194] it was shown that for finite groups the module of equivariants is Cohen-Macaulay. The proof in [71] is easily derived from the fact that the modules of semi-invariants are free modules over the ring in the primary invariants as stated in [175]. Thus Algorithm 2.3.17 easily generalizes to equivariants. Applications are given in Section 3.2 and [76], [125].

Algorithm 2.3.20 *(Computation of equivariants of a finite group) ([71, 74])*
Input: *finite group G, given by matrices $\vartheta(s), \forall\, s \in G$*
  primary invariants $\pi_1, \ldots, \pi_n$
Output: *equivariants*

1. *Compute the Hilbert series $\mathcal{HP}_{K[x]^G_G}(\lambda) = \sum_{i=0}^{\infty} a_i \lambda^i$ # Formula (2.6)*

2. *Determine degrees of equivariants*

$$\mathcal{HP}_{K[x]^G_G}(\lambda) \cdot \prod_{i=1}^{n}(1 - \lambda^{\deg(\pi_i)}) = c_{e_1}\lambda^{e_1} + \cdots + c_{e_r}\lambda^{e_r}$$

3. *Prepare computation*
  $\Pi := \{\pi_1, \ldots, \pi_n\}$ *# list of primary invariants*
  $S := \{\}$ $\tilde{S} := \{\}$ *# lists of equivariants*
  compute truncated Gröbner basis $\mathcal{GB}(\oplus_{i=0}^{\max e_j} H_i(\Pi(x))$ # cheap order

4. *Search equivariants*
  for $e \in \{e_1, \ldots, e_r\}$ do
    $M_e := \{$ monomials of degree $e\}$
    $\tilde{M}_e := \{$ tuples consisting of zeros and one monomial in $M_e\}$
    while $c_e > 0$ do # vector space basis of $H_e(K[x]^n)$
      choose $m \in \tilde{M}_e$, $\tilde{M}_e := \tilde{M}_e \setminus \{m\}$
      $f := \mathcal{R}(m)$ # Reynolds projection 2.8
      $\tilde{f} :=$ normalf$(\sum_{j=1}^{n} f_j(x) z_j, \mathcal{GB})$
      if $\tilde{f}$ is linear independent of $\tilde{S}$ then
        $S := S \cup \{f\}$, $\tilde{S} := \tilde{S} \cup \{\tilde{f}\}$
        $c_e := c_e - 1$

In equivariant dynamics it has been an open question for a long time whether the module of equivariants is Cohen-Macaulay for non-finite groups as well. In general this is not true, see [20, 183]. If the module is Cohen-Macaulay (a free module over a ring generated by a system of parameters of the invariant ring) the Algorithm 2.3.20 for computation of equivariants of finite groups generalizes to equivariants of compact Lie groups which are algebraic groups. Only the computation of the Hilbert series and the Reynolds projection are different. But one requires a set of primary invariants whose computation from a given set of invariants is the goal of the next section.

Algorithm 2.3.21 *(Computation of equivariants of an algebraic and compact Lie group)*

Input: *data structure of a compact Lie group G, with two representations ϑ and ρ*

Hilbert series $\mathcal{HP}_{K[x]^{\vartheta}_{\rho}}(\lambda) = \sum_{i=0}^{\infty} a_i \lambda^i$ # *Formula (2.7)*

primary invariants $\pi_1, \ldots, \pi_d$

Assumption: *module of equivariants is Cohen-Macaulay*

Output: free *module basis of module of equivariants*

1. *Determine degrees of equivariants*

$$\mathcal{HP}_{K[x]^{\vartheta}_{\rho}}(\lambda) \cdot \prod_{i=1}^{d}(1 - \lambda^{\deg(\pi_i)}) = c_{e_1}\lambda^{e_1} + \cdots + c_{e_r}\lambda^{e_r}$$

if one c_{e_i} *is negative then STOP print(module is not CM)*

2. *Prepare computation*

$\Pi := \{\pi_1, \ldots, \pi_d\}$ # *list of primary invariants*

$S := \{\}$ $\tilde{S} := \{\}$ # *lists of equivariants*

compute truncated Gröbner basis $\mathcal{GB}(\oplus_{i=0}^{\max e_j} H_i(\Pi(x))$ # *cheap order*

3. *Search equivariants*

for $e \in \{e_1, \ldots, e_r\}$ *do*

  $\tilde{V} := \{$ *monomials in* $H_e^N(K[x])\}$

  for $j \in \{j_1, \cdots, j_{dim}\}$ *do* # *dim - dimension of weight space*

    $\hat{V}_j = \{w(x) \in K[x]^m \mid w_i(x) = 0, i = 1, \ldots, m,\ i \neq j,\ w_j(x) \in \tilde{V}\}$

  $\hat{V} := \hat{V}_1 \cup \cdots \cup \hat{V}_{dim}$

  $Q := \mathcal{R}(\hat{V})$ # *vector space basis* $Q = \{q_1, \ldots, q_l\}$ *of* $H_e^N(K[x]^{\vartheta}_{\rho}) \cap \hat{V}$

  while $c_e > 0$ *do* # *vector space basis of* $H_e(K[x]^m)$

    choose next equivariant q_i

    $\tilde{q}_i :=$ *normalf*$(\sum_{j=1}^{m}(q_i(x))_j z_j, \mathcal{GB})$

    if $\tilde{q}_i$ *is linear independent of* $\tilde{S}$ *then*

      $S := S \cup \{q_i\}$, $\tilde{S} := \tilde{S} \cup \{\tilde{q}_i\}$

      $c_e := c_e - 1$

The algorithm has been implemented and tested. Since it exploits more structure than the other algorithms it is reasonable that it performs best as Table 2.2 demonstrates.

2.4 Computing uniqueness

While the Hilbert basis for a finite group computed with Algorithm 2.3.17 guarantees that each invariant polynomial p has a unique representation

$$p(x) = \sum_{i=1}^{l} \sigma_i(x)\, p_i(\pi_1(x), \ldots, \pi_n(x)),$$

this is not true for the Hilbert bases computed by Algorithm 2.1.10 and Algorithm 2.2.2. The basis computed by Algorithm 2.2.2 is not even minimal. For equivariants this is analogously.

On the other hand invariants of linear reductive groups are known to possess a Hironaka decomposition (Theorem 2.3.15). Given any Hilbert basis an algorithmic Noether normalization respecting the weighted degree converts the Hilbert basis to a set of primary and secondary invariants. This algorithm was inspired by [132] where the algorithmic Noether normalization without degree distinction is presented. There linear changes of coordinates $y_1 -> y_1', \ldots, y_{n-1} -> y_{n-1}'$ are performed such that y_n then becomes integral (analogous to [8] p. 69). This is recalled in [186] p. 35pp. Our approach is analogous, but it allows nonlinear changes of coordinates.

Algorithm 2.4.1 *(Iterative Noether normalization of a W-homogeneous ideal)*

Input: *grading W on $K[y_1, \ldots, y_n]$ with $\deg_W(y_i) = w_i > 0, i = 1, \ldots, n$*
W-homogeneous polynomials $f_1, \ldots, f_m$

Output: *new homogeneous coordinates:*
parameters $p_j = P_j(y) \in K[y], j = 1, \ldots, d$,
integral elements $z_i = C_i(y), i = 1, \ldots, r$, and
Gröbner basis $g_1(p, z), \ldots, g_s(p, z)$

1.) *Check for parameters and integral elements:*
We assume the variables are blocked into groups $Y_1, \ldots, Y_s$ having the same degree $d_1 < \cdots < d_s$, respectively.
- **a.)** *Choose a term order which*
  - *eliminates groups of variables of the same degree*
  - *first eliminates $Y_2, \ldots, Y_s$, then $Y_1, Y_3, , \ldots, Y_s$ etc.*
  - *eliminates successively all variables.*
- **b.)** *Compute a Gröbner basis $\mathcal{GB}$ with respect to this term order.*
- **c.)** *Inspect the leading terms:*
  $\Pi = \{$ *variables not appearing* $\}$
  $Z = \{$ *variables y_i such that a monic leading term $y_i^{a_i}$ appears*$\}$
  $R = Y \setminus (\Pi \cup Z)$

2.) *while $R \neq \{\}$ do*
- **a.)** *Choose a change of coordinates $p = P(y) \in H_d(K[y])$ depending weighted homogeneous on the variables $y \in r$ of degree $d \leq lcm_{y \in R}(deg_W(y))$ (first try lowest possible degree)*
  - *add new variable $\{p\}$ of degree $W(p) := \deg_W(P(y))$: $Y = Y \cup \{p\}$*
  - *new set of polynomials $\mathcal{GB} \cup \{p - P(y)\}$.*
- **b.)** *Choose a term order which*
  - *eliminates $Z \cup R$ as a block*
  - *eliminates the variables in Z and $\Pi \cup \{p\}$ successively for W-homogeneous ideals*
  - *then break ties.*
- **c.)** *Compute a Gröbner basis $\widetilde{\mathcal{GB}}$ of $\mathcal{GB} \cup \{p - P(y)\}$ with respect to $<$.*
- **d.)** *Check for parameters and integral elements:*
  $\tilde{\Pi} = \{$ *variables not appearing* $\}$

$\tilde{Z} = \{$ *variables such that a monic leading term appears* $\}$

e.) *If* $\tilde{\Pi} = \Pi \cup \{p\}$ *and* $|\tilde{Z}| > |Z|$ *then accept this step and set*

$\Pi = \Pi \cup \{p\},\ Y = Y \cup \{p\},\ Z = \tilde{Z},\ R = Y \setminus (\Pi \cup Z)\ \mathcal{GB} = \widetilde{\mathcal{GB}}$

else *try another guess* P *in step 2.)a.).*

Lemma 2.4.2 *For sufficiently generic choices of polynomials* P *in step 2.) a.) the algorithm computes a set of homogeneous parameters of* $K[y]/J$.

Proof: The existence of a set of homogeneous parameters follows from Noether's lemma. Algorithm 2.3.11 explains the choice of the term order and the three sets of variables. Since in each step the number of parameters increases the algorithm stops after finitely many steps. □

Example 2.4.3 *(Example 2.2.4 continued)*
Recall the torus action associated to the $1:1:2$ *resonance from Example 2.2.4. The ring of invariants* $\mathbf{C}[x]^T$ *is generated by* 11 *invariants. In [155] the determination of primary invariants is discussed with the help of circuits. Unfortunately, we are interested in the real fundamental invariants of the real algebra which is not considered in [155]. From the given invariants we deduce* 11 *real generators of the real invariant ring* $\mathbf{R}[x]^T$:

$$\begin{aligned}
p_1(x) &= x_1x_4, & p_2(x) &= x_5x_2,\\
p_3(x) &= x_6x_3, & p_4(x) &= x_5x_1 + x_2x_4,\\
p_5(x) &= i\,(x_5x_1 - x_2x_4)\,, & p_6(x) &= {x_1}^2x_6 + x_3{x_4}^2,\\
p_7(x) &= i\,({x_1}^2x_6 - x_3{x_4}^2)\,, & p_8(x) &= {x_2}^2x_6 + {x_5}^2x_3,\\
p_9(x) &= i\,({x_2}^2x_6 - {x_5}^2x_3)\,, & p_{10}(x) &= x_4x_5x_3 + x_1x_2x_6,\\
p_{11}(x) &= i\,(x_1x_2x_6 - x_4x_5x_3)\,.
\end{aligned}$$

In the following we use the variables $\pi_j, j = 1, \ldots, 11$ *for representing these invariants. The relations* $f_1(\pi), \ldots, f_m(\pi)$ *are computed in the standard way by computation of a Gröbner basis of* $\pi_j - p_j(x), j = 1, \ldots, 11$ *eliminating* x *and restricting to the elements in* $K[\pi]$ *only. The relations are homogeneous with respect to the induced weighted grading* $W(\pi_j) = 2, j = 1, \ldots, 5$, $W(\pi_j) = 3, j = 6, \ldots, 11$. *(A more careful analysis shows that they are homogeneous with respect to* 3 *different gradings.) Another Gröbner basis computation of* $J = \langle f_1, \ldots, f_m\rangle$ *checks for possible parameters and integral elements by choosing a term order with matrix*

$$B = \left[\begin{array}{ccccccccccc}
2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3\\
-2 & -2 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0\\
\multicolumn{4}{c}{-\,id_{4,4}} & 0_4 & \multicolumn{5}{c}{0_{4,5}} & 0_4\\
\multicolumn{4}{c}{0_{5,4}} & 0_5 & \multicolumn{5}{c}{-\,id_{5,5}} & 0_5
\end{array}\right],$$

and the variable order $\pi_i, i = 1, \ldots, 11$. *The leading terms*

$$\begin{aligned}
[\ & \pi_6\pi_4\pi_9,\ \pi_4\pi_6\pi_8,\ {\pi_{11}}^2,\ \pi_{11}\pi_{10},\ {\pi_{10}}^2,\ \pi_{11}\pi_9,\ \pi_{10}\pi_9,\ {\pi_9}^2,\ \pi_{11}\pi_7,\ \pi_{10}\pi_7,\\
& \pi_9\pi_7,\ \pi_8\pi_7,\ {\pi_7}^2,\ \pi_{11}\pi_5,\ \pi_{10}\pi_5,\ \pi_9\pi_5,\ \pi_8\pi_5,\ \pi_7\pi_5,\ \pi_6\pi_5,\ \pi_{11}\pi_4,\ \pi_{10}\pi_4,\ {\pi_5}^2\ ],
\end{aligned}$$

yields that $\Pi = \{\pi_1, \pi_2, \pi_3\}$ *are parameters,* $Z = \{\pi_5, \pi_7, \pi_9, \pi_{10}, \pi_{11}\}$ *are integral over the rest, and for* $R = \{\pi_4, \pi_6, \pi_8\}$ *no decision can be made. The new variables are* $[\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, p_{68}, \pi_7, \pi_8, \pi_9, \pi_{10}, \pi_{11}]$ *with* $p_{68} = \pi_6 + \pi_8$ *which is simply substituted into the relations. One may choose whether one drops* π_6 *or* π_8*. We decided for* π_6*. A Gröbner basis computation with respect to* $\pi_1, \ldots, \pi_5, p_{68}, \pi_7, \ldots, \pi_{11}$ *and the matrix term order given by*

$$C = \left[\begin{array}{ccccccccccc} 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\ -2 & -2 & -2 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ \multicolumn{3}{c}{-\,id_{3,3}} & 0_3 & \multicolumn{4}{c}{0_{3,4}} & \multicolumn{3}{c}{0_{3,3}} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \multicolumn{3}{c}{0_{3,3}} & 0_3 & \multicolumn{4}{c}{0_{3,4}} & \multicolumn{3}{c}{id_{3,3}} \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

which eliminates $Z \cup \{\pi_4, \pi_8\}$ *by the second row yields the leading terms*

$$\begin{array}{l} [\ \ \pi_4{\pi_8}^2,\ {\pi_7}^2,\ \pi_9\pi_7,\ \pi_{10}\pi_7,\ \pi_{11}\pi_7,\ \pi_8\pi_7,\ {\pi_9}^2,\ \pi_{10}\pi_9,\ \pi_{11}\pi_9, \\ \quad \pi_9\pi_8,\ {\pi_{10}}^2,\ {\pi_{11}}^2,\ \pi_7\pi_5,\ \pi_9\pi_5,\ \pi_{10}\pi_5,\ \pi_{11}\pi_5,\ \pi_8\pi_5,\ \pi_7\pi_4,\ \pi_{10}\pi_4,\ \pi_{11}\pi_4,\ {\pi_5}^2\ ]. \end{array}$$

This confirms the set of parameters $\Pi = \{\pi_1, \pi_2, \pi_3, p_{68}\}$ *and the integral elements* $Z = \{\pi_5, \pi_7, \pi_9, \pi_{10}, \pi_{11}\}$*. So we are left with* $R = \{\pi_4, \pi_8\}$*. Since* $\deg_W(\pi_4) = 2 \neq \deg_W(\pi_8)$ *a nonlinear change of coordinates is required. We add the polynomial* $p_{48} - {\pi_4}^3 - {\pi_8}^2$ *and the new variable* p_{48}*. A Gröbner basis with respect to the matrix*

$$D = \left[\begin{array}{cccccccccccc} 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 6 \\ -2 & -2 & -2 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & -6 \\ \multicolumn{3}{c}{-\,id_{3,3}} & 0_3 & \multicolumn{4}{c}{0_{3,4}} & \multicolumn{3}{c}{0_{3,3}} & 0_3 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \multicolumn{3}{c}{0_{3,3}} & 0_3 & \multicolumn{4}{c}{0_{3,4}} & \multicolumn{3}{c}{id_{3,3}} & 0_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

and the ordering of variables $[\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, p_{68}, \pi_7, \pi_8, \pi_9, \pi_{10}, \pi_{11}, p_{48}]$ *has the following leading terms:*

$$\begin{array}{l} [\ \ {\pi_8}^4, \pi_8\pi_{11}\pi_{10}, {\pi_8}^2\pi_{10}, \pi_{11}{\pi_8}^2, \pi_4{\pi_8}^2, {\pi_7}^2, \pi_9\pi_7, \pi_{10}\pi_7, \\ \quad \pi_{11}\pi_7, \pi_8\pi_7, {\pi_9}^2, \pi_{10}\pi_9, \pi_{11}\pi_9, \pi_9\pi_8, {\pi_{10}}^2, {\pi_{11}}^2, {\pi_4}^3, \\ \quad \pi_7\pi_5, \pi_9\pi_5, \pi_{10}\pi_5, \pi_{11}\pi_5, \pi_8\pi_5, \pi_7\pi_4, \pi_{10}\pi_4, \pi_{11}\pi_4, {\pi_5}^2\ ]. \end{array}$$

This shows that $\Pi = \{\pi_1, \pi_2, \pi_3, p_{68}, p_{48}\}$ *are parameters and over* $\mathbf{R}[\Pi]$ *the elements* $\{\pi_4, \pi_5, \pi_7, \pi_8, \pi_9, \pi_{10}, \pi_{11}\}$ *are integral. So each invariant* $f(x)$ *has a unique representation*

$$f(x) = \sum_{\eta} A_\eta\left(p_1(x), p_2(x), p_3(x), \tilde{p}_{68}(x), \tilde{p}_{48}(x)\right) \cdot \eta(x),$$

where $\tilde{p}_{68}(x) = p_6(x) + p_8(x)$ *and* $\tilde{p}_{48}(x) = (p_4(x))^3 + (p_8(x))^2$ *and* A_η *are polynomials in* 5 *variables. The* η *are polynomials associated to the standard monomials in the non-parameter variables not appearing in the leading terms. There are* 18 *standard monomials. That means that for each choice* $\Pi = a \in \mathbf{C}^5$ *there are* 18 *possibilities* $Z = b^j \in \mathbf{C}^7, j = 1, \ldots, 18$ *such that* (a, b^j) *solves the relations. Of course there might be less than* 18 *leaves for the restriction to the real variety. In Example 4.3.6 this will be continued.*

Remark 2.4.4 *It remains to be discussed what are sufficiently generic coefficients in Step 2.) a.). In applied mathematics genericity means that the set of exceptional values has measure zero. In algebra non-generic means that the values are zeros of a special polynomial. For the special case of the natural grading one can restrict to linear combinations* $p_i = c_{i1}y_1 + \cdots + c_{in}y_n$ *and choose all parameters at once. In [53] a polynomial* $H(c)$ *in the coefficients* c_{ij}*, the so-called* Chow form, *is considered such that* $H(c) \neq 0$ *guarantees that* p_i *form parameters. Sturmfels and Eisenbud [53] also give a Greedy algorithm based on the primary decomposition for a sparse choice of the coefficients* c_{ij}*. For the case of different weights on the variables the idea of Dade's algorithm (see [176] p. 56) is to consider powers of the variables such that all powers have the same degree. This reduces to the case of linear combinations above. However, in Algorithm 2.4.1 the degree of the generic polynomial* P *is not fixed. For the investigation of the genericity the coefficients of* P^2 *or* P^3 *or ... have to be considered in the Chow form. One just exploits the fact that if* $p_1, \ldots, p_d$ *forms a system of parameters so does* $p_1^{a_1}, \ldots, p_d^{a_d}$*.*

Applications of the Noether normalization are shown in Section 4.3. Observe that the Noether normalization is useful as a preparing step of Algorithm 2.3.21.

For the equivariants the question of unique representations is more complicated. In many practical situations the module is a free module over the ring in the primary invariants. The key for the general question is the *Stanley decomposition* of rings.

Definition 2.4.5 *([179] p. 277) Let* $R = K[x]/I$ *be a quotient ring of an ideal* $I \subset K[x_1, \ldots, x_n]$*. Then a decomposition as direct sum of free modules*

$$R = \bigoplus_{\alpha \in F} x^\alpha K[X_\alpha],$$

where F *is a finite subset of* $\mathbf{N}^n$ *and each* X_α *is a subset of* $\{x_1, \ldots, x_n\}$ *is called a* Stanley decomposition.

Of course there may exist several Stanley decompositions of R. But once a Stanley decomposition is fixed each $r \in R$ has a unique representation. A Stanley decomposition can be algorithmically determined. As usual the computations are based on the initial ideal. Thus first a Gröbner basis needs to be computed.

Algorithm 2.4.6 *(Stanley decomposition, Sturmfels and White [179], see [186] p. 24)*
Input: *generators* $f_1, \ldots, f_m$ *of ideal* $I \subset K[x_1, \ldots, x_n]$
Output: *Stanley decomposition*
1.) *Compute a Gröbner basis* $\mathcal{GB}$ *of* $\langle f_1, \ldots, f_m \rangle$*.*
2.) *Choose* x_n *as special variable: there exists monomials* $m_1, \ldots, m_l$

in $K[x_1,\dots,x_{n-1}]$ *and degrees* $d_1 \le d_2 \le \cdots \le d_l$ *such that* $m_1x_n^{d_1},\dots,m_lx_n^{d_l}$ *are the leading terms of the Gröbner basis and thus* $init(I) = \langle ht(g) \mid g \in \mathcal{GB}\rangle = \langle m_1x_n^{d_1},\dots,m_lx_n^{d_l}\rangle$

3.) $SD(K[x]/\langle m_1x_n^{d_1},\dots,m_lx_n^{d_l}\rangle)$

Subroutine $SD(K[x_1,\dots,x_n]/\langle g_1,\dots,g_s\rangle)$
(g_i are expected to be monomials)
reduce to minimal monomial generators $m_1x_n^{d_1},\dots,m_lx_n^{d_l}$
with $d_1 \le d_2 \le \cdots \le d_l$ *and* $m_1,\dots,m_l \in K[x_1,\dots,x_{n-1}]$.
if $n = 1$ *then*
$K \oplus x_nK \oplus x_n^2K \oplus \cdots \oplus x_n^{d_1-1}K$
else
$\bigoplus_{j=0}^{d_l-1} x_n^j \cdot SD\left(K[x_1,\dots,x_{n-1}]/\langle\{m_i \mid d_i \le j\}\rangle\right)$
$\bigoplus x_n^{d_l} \cdot K[x_n] \cdot SD(K[x_1,\dots,x_{n-1}]/\langle m_1,m_2,\dots,m_l\rangle)$

Proof: Gröbner bases enable the unique computation in the residue ring $K[x]/I$. That means that $K[x]/I$ and $K[x]/init(I)$ are isomorphic as vector spaces. Since Stanley decompositions are invariant under this isomorphism we need to show that the recursive algorithm computes a Stanley decomposition of $K[x]/init(I)$ which restricts the discussion to the standard monomials.

The proof is done by induction on the number of variables.
If $n = 1$ then $l = 1, m_1 = 1$ and $init(I) = \langle x_1^{d_1}\rangle$. Then $K[x_1]/init(I)$ is isomorphic to the vector space spanned by the standard monomials $1, x_1, \dots, x_1^{d_1-1}$.

For $n \ge 2$ we have to show that each standard monomial $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ occurcs exactly ones. By the distinction of the degree α_n it is clear that x^α is either in the last summand (for $\alpha_n \ge d_l$) or in the summand $j = \alpha_n$ (for $\alpha_n < d_l$). Then existence and uniqueness follows from the induction hypothesis. □

Example 2.4.7 *In [164] Sanders and Wangrefers to the* **C***-algebra* R *generated by*

$$u_1^5u_4,\ uu_1^2u_4,\ u^3u_1u_4^2,\ u^5u_4^3,\ u^2u_2u_4,\ u_1^3u_3,\ uu_3,\ u_1u_2,\ uu_2^3.$$

These 9 generators are abbreviated by $y_1,\dots,y_9$. *The algebra* R *is isomorphic to the quotient ring* $\mathbf{C}[y]/I$ *where* I *is the ideal of relations. Observe that* I *is generated by binomials since it is a toric ideal. There are various ways of finding generators of* I *in this special case, see [177] Algorithm 7.2 and [2] Theorem 2.1. We used the traditional way of using slack variables and a Gröbner basis computation with respect to a term order with elimination property. A Gröbner basis of* I *with respect to a lex term order has been computed in Macaulay 2 (modulo 31991) yielding the leading terms*

$$y_6y_9,\ y_4y_9,\ y_4y_7y_8^3,\ y_3y_9,\ y_3y_7y_8^2,\ y_3y_5,\ y_3^2y_7y_8,\ y_3^3y_7,\ y_2y_9,$$
$$y_2y_7y_8,\ y_2y_5,\ y_2y_4,\ y_2y_3y_7,\ y_2^2y_7,\ y_1y_9,\ y_1y_7,\ y_1y_5,\ y_1y_4,\ y_1y_3.$$

Various choices of ordering of variables are possible. The variable order $y_9, y_8, \dots, y_1$ *gives* *in the beginning of the recursion*

$$\begin{aligned}
& 1 \cdot SD\,(K[y_9,\ldots,y_2]/\langle y_6y_9,\ldots,y_2^2y_7\rangle) \\
& \oplus y_1 \cdot K[y_1] \cdot SD\,(K[y_9,\ldots,y_2]/\langle y_6y_9,\ldots,y_2^2y_7,y_9,y_7,y_5,y_4,y_3\rangle) \\
= \; & 1 \cdot SD\,(K[y_9,\ldots,y_3]/\langle y_6y_9,\ldots,y_3^3y_7\rangle) \\
& \oplus y_2 \cdot SD\,(K[y_9,\ldots,y_3]/\langle y_6y_9,\ldots,y_3^3y_7,y_9,\ldots,y_3y_7\rangle) \\
& \oplus y_2^2 \cdot K[y_2] \cdot SD\,(K[y_9,\ldots,y_3]/\langle y_6y_9,\ldots y_3^3y_7,y_9,\ldots,y_3y_7,y_7\rangle) \\
& \oplus y_1 \cdot K[y_1] \cdot K[y_8,y_6,y_2] \\
= \; & 1 \cdot SD\,(K[y_9,\ldots,y_3]/\langle y_6y_9,\ldots,y_3^3y_7\rangle) \\
& \oplus y_2 \cdot SD\,(K[y_9,\ldots,y_3]/\langle y_6y_9,\ldots,y_3^3y_7,y_9,\ldots,y_3y_7\rangle) \\
& \oplus y_2^2 \cdot K[y_8,y_6,y_3,y_2] \\
& \oplus y_1 \cdot K[y_8,y_6,y_2,y_1]
\end{aligned}$$

and finally the Stanley decompostion

$$\begin{aligned}
& y_3\, y_7 \cdot K[y_7,y_6,y_4] \oplus y_4\, y_7 \cdot K[y_7,y_6,y_5,y_4] \oplus y_7\, {y_8}^2 y_4 \cdot K[y_7,y_6,y_5,y_4] \\
& \oplus y_3\, y_7\, y_8 \cdot K[y_7,y_6,y_4] \oplus {y_3}^3 \cdot K[y_8,y_6,y_4,y_3] \oplus y_6 \cdot K[y_8,y_7,y_6,y_5] \\
& \oplus {y_3}^2 y_7 \cdot K[y_7,y_6,y_4] \oplus y_7\, y_8\, y_4 \cdot K[y_7,y_6,y_5,y_4] \oplus 1 \cdot K[y_9,y_8,y_7,y_5] \\
& \oplus y_4 \cdot K[y_8,y_6,y_5,y_4] \oplus {y_3}^2 \cdot K[y_8,y_6,y_4] \oplus y_3 \cdot K[y_8,y_6,y_4] \\
& \oplus y_3\, y_2 \cdot K[y_8,y_6,y_3] \oplus y_2\, y_7 \cdot K[y_7,y_6] \oplus y_2 \cdot K[y_8,y_6] \\
& \oplus {y_2}^2 \cdot K[y_8,y_6,y_3,y_2] \oplus y_1 \cdot K[y_8,y_6,y_2,y_1]\,.
\end{aligned}$$

Definition 2.4.8 *Let $R = K[x]/I$ be a quotient ring of an ideal I in $K[x_1,\ldots,x_n]$. For the restriction to modules we define the Kronecker grading $W(x_j) = 0, j = 1,\ldots,n, W(y_i) = 1, i = 1,\ldots,s$. Let*

$$\{f = (f_1,\ldots,f_s) \in K[x]^s \mid \sum_{i=1}^{s} y_i f_i(x) \equiv 0\} \simeq J \subset H_1^W(K[x,y])$$

be a submodule and $M = H_1^W(K[x,y])/J$ the corresponding R-module. A Stanley decomposition *of M is a direct sum of free modules*

$$M = \bigoplus_{i=1}^{s} \bigoplus_{\alpha\in F_i} y_i x^\alpha K[X_\alpha],$$

where the F_i are finite subsets of $\mathbf{N}^n$ and each X_α is a subset of $\{x_1,\ldots,x_n\}$.

Theorem 2.4.9 *Let I be an ideal in $K[x]$ and $J \subset H_1^W(K[x,y])$ a submodule, where W is the Kronecker grading $W(x_j) = 0, W(y_i) = 1$. If $y_i f \in J$ for all $f \in I$ and $y_i, i = 1,\ldots,s$ then the $K[x]/I$-Module $M = H_1^W(K[x,y])/J$ has a Stanley decomposition.*

Proof: Consider the ring $K[x,y]/S$ where the ideal S is generated by I and J and $y_iy_j, 1 \le i \le j \le s$. By Algorithm 2.4.6 a Stanley decomposition of $K[x,y]/S$ or $K[x,y]/init(S)$ is computed. If we choose a special term order for the Gröbner basis this gives a Stanley decomposition of M. We are choosing an order which is eliminating y and then uses the Kronecker grading W. The corresponding Gröbner basis of S includes a Gröbner basis of I (by the elimination property) and includes the monomials $y_iy_j, 1 \le i \le j \le s$ (by the use of W) and some generating elements of J. In the algorithm we are choosing the variable order $x_1,\ldots,x_n,y_1,\ldots,y_s$. Then a Stanley decomposition of $K[x,y]/S$ is a direct sum of a Stanley decomposition of $K[x]/I$ and a Stanley decomposition of M. □

Example 2.4.10 *(Example 2.4.7 continued) The monomials*

$$uu_4^2,\ u_1u_3u_4,\ u_2^2u_4,\ u_2u_3^2,\ u_2^4u_3,\ u_1^2u_3^3,\ u_1^3u_4^2,\ u_2^7\,.$$

generate a module over the ring R from Example 2.4.7. We abbreviate the monomials by $b_1,\ldots,b_8$. One Stanley decomposition is computed to be

$$\begin{aligned}
&y_7\, b_1 \cdot K[y_4] \oplus {y_7}^2 b_1 \cdot K[y_4, y_7] \oplus y_5\, y_7\, b_1 \cdot K[y_4] \oplus y_3\, b_1 \cdot K[y_2, y_3]\\
&\oplus y_5\, b_1 \cdot K[y_2, y_3] \oplus y_8\, b_1 \cdot K[y_1, y_2] \oplus y_4\, b_1 \cdot K[y_3, y_4]\\
&\oplus y_5\, y_4\, b_1 \cdot K[y_3, y_4] \oplus b_1 \cdot K[y_1, y_2] \oplus y_4\, y_7\, b_2 \cdot K[y_3, y_4, y_7]\\
&\oplus y_6\, y_7\, b_2 \cdot K[y_2, y_6, y_7] \oplus y_5\, y_4\, b_2 \cdot K[y_3, y_4] \oplus y_5\, b_2 \cdot K[y_2, y_3]\\
&\oplus y_6\, b_2 \cdot K[y_1, y_2, y_6] \oplus y_4\, b_2 \cdot K[y_3, y_4] \oplus y_8\, b_2 \cdot K[y_1, y_2]\\
&\oplus y_7\, b_2 \cdot K[y_2, y_3, y_7] \oplus y_3\, b_2 \cdot K[y_2, y_3] \oplus b_2 \cdot K[y_1, y_2]\\
&\oplus y_3\, y_7\, b_3 \cdot K[y_5, y_2, y_3] \oplus y_4\, b_3 \cdot K[y_5, y_3, y_4] \oplus y_6\, y_1\, b_3 \cdot K[y_8, y_2, y_1]\\
&\oplus y_1\, b_3 \cdot K[y_8, y_2, y_1] \oplus y_6\, y_2\, b_3 \cdot K[y_8, y_5, y_2] \oplus y_6\, b_3 \cdot K[y_8, y_5]\\
&\oplus y_2\, b_3 \cdot K[y_8, y_5, y_2] \oplus y_2\, y_7\, b_3 \cdot K[y_5, y_2] \oplus y_4\, y_7\, b_3 \cdot K[y_5, y_3, y_4]\\
&\oplus y_3\, b_3 \cdot K[y_5, y_2, y_3] \oplus y_7\, b_3 \cdot K[y_8, y_9, y_5] \oplus b_3 \cdot K[y_8, y_9, y_5]\\
&\oplus y_6\, b_4 \cdot K[y_8, y_7, y_6, y_5] \oplus y_4\, b_4 \cdot K[y_7, y_5, y_3, y_4] \oplus y_2\, b_4 \cdot K[y_8, y_6, y_5, y_2]\\
&\oplus y_2\, y_7\, b_4 \cdot K[y_7, y_6, y_5, y_2] \oplus y_1\, b_4 \cdot K[y_8, y_6, y_2, y_1]\\
&\oplus y_3\, b_4 \cdot K[y_7, y_5, y_2, y_3] \oplus b_4 \cdot K[y_8, y_7, y_9, y_5] \oplus b_5 \cdot K[y_8, y_9]\\
&\oplus b_6 \cdot K[y_6, y_7] \oplus y_8\, b_7 \cdot K[y_1] \oplus b_7 \cdot K[y_1] \oplus b_8 \cdot K[y_8, y_9]\,.
\end{aligned}$$

Corollary 2.4.11 *Let $\vartheta : G \to GL(K^n), \rho : G \to GL(K^m)$ be two linear representations of a compact Lie group. Then the module of equivariants $K[x]_\rho^\vartheta$ has a Stanley decomposition.*

Proof: Let denote the generators of the invariant ring by $y_1 = \pi_1(x), \ldots, y_r = \pi_r(x)$ and the ideal of relations by $I \subset K[y]$. Analogously, denote the generators of the module of equivariants by $u_1 = f_1(x), \ldots, u_s = f_s(x)$ and $J \subset H_1^W(K[y,u])$ the module of relations. W denotes as usual the Kronecker grading. The module of equivariants is isomorphic to the $K[y]/I$-module $H_1^W(K[y,u])/J$ where the invariant ring is isomorphic to $K[y]/I$. Thus the previous theorem induces its Stanley decomposition

$$M = \bigoplus_{i=1}^{s} \bigoplus_{\alpha \in F_i} u_i \cdot y^\alpha \cdot K[Y_\alpha], \quad Y_\alpha \subset \{y_1, \ldots, y_r\},$$

to the Stanley decomposition

$$K[x]_\rho^\vartheta = \bigoplus_{i=1}^{s} \bigoplus_{\alpha \in F_i} f_i(x) \cdot \pi_1(x)^{\alpha_1} \cdots \pi_r(x)^{\alpha_r} \cdot K[\pi_{j_1}(x), \ldots, \pi_{j_{l_\alpha}}(x)].$$

with $y_{j_1}, \ldots, y_{j_{l_\alpha}} \in Y_\alpha$. This means that every equivariant f has a *unique* representation

$$f(x) = \sum_{i=1}^{s} \sum_{\alpha \in F_i} f_i(x) \cdot (\pi_1(x))^{\alpha_1} \cdots (\pi_r(x))^{\alpha_r} \cdot P_\alpha(\pi_{j_1}(x), \ldots, \pi_{j_{l_\alpha}}(x)),$$

where the P_α are polynomials in $l_\alpha = |Y_\alpha|$ variables. □

Additionally, we may distinguish primary invariants $y_1 = \pi_1(x), \ldots, y_d = \pi_d(x)$ and secondary invariants $z_1 = 1, z_2 = \sigma_2(x), \ldots, z_m = \sigma_m(x)$. In order to adjust to this situation we choose a term order $<$ which is

- eliminating u ($u^\alpha > y^\beta z^\gamma \ \forall\, \alpha \neq 1, \beta, \gamma$),
- uses the Kronecker grading W ($\deg_W(u^\alpha) > \deg_W(u^\beta) \Rightarrow u^\alpha > u^\beta, \ \forall\, \alpha, \beta$)
- and uses a Kronecker grading $U(y_i) = 0, U(z_j) = 1, U(u_k) = 0$.

Then a representation as a direct sum of free modules over subrings of $K[y]$ is the result

$$M = \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{m} \bigoplus_{\alpha \in F_i^j} u_i \cdot z_j \cdot y^\alpha \cdot K[Y_\alpha].$$

From this decomposition the Hilbert series can easily be computed.

Chapter 3

Symmetric bifurcation theory

"Hinter der Mathematik stecken die Zahlen. Wenn mich jemand fragen würde, was mich richtig gücklich macht, dann würde ich antworten: die Zahlen. Schnee und Eis und Zahlen. Und weißt du warum?" ... " Weil das Zahlensystem wie das Menschenleben ist. Zu Anfang hat man die natürlichen Zahlen. Das sind die ganzen und positiven Zahlen. Die Zahlen des Kindes. Doch das menschliche Bewußtsein expandiert. Das Kind entdeckt die Sehnsucht, und weißt du, was der mathematische Ausdruck für die Sehnsucht ist?" ... " Es sind die negativen Zahlen. Die Formalisierung des Gefühls, daß einem etwas abgeht. Und das Bewußtsein erweitert sich immer noch und wächst, das Kind entdeckt die Zwischenräume. Zwischen den Steinen, den Mosen auf den Steinen, zwischen den Menschen. Und zwischen den Zahlen. Und weißt du wohin das führt? Zu den Brüchen. Die ganzen Zahlen plus die Brüche ergeben die irrationalen Zahlen. Aber das Bewußtsein macht dort nicht halt. Es will die Vernunft überschreiten. ..."

Peter Høeg
Fräulein Smillas Gespür für Schnee

In this section I show how symbolic computations enter symmetric bifurcation theory. The aim is to demonstrate the usage of Computer Algebra in this area of research. Traditionally papers in this area are full of hand calculations and basic manipulations. My aim is to introduce a different working method. As a simple example I investigate secondary Hopf bifurcation with D_3-symmetry showing the reliability and efficiency of automatic manipolation of formulas. Here the results of Section 2.3 on algorithmic determination of invariants and equivariants are applied in order to find a generic equivariant vector field. In the presentation of this chapter we assume again that the reader is familiar with linear representation theory of groups.

3.1 Local bifurcation analysis

It is a common understanding that the symmetry of a bifurcation problem strongly determines the bifurcation scenario and structures the dynamics.

Symmetry is formally described by a faithful representation of a group operating on the system of differential equations. In the system

$$\dot{x} = f(x, \lambda), \quad f : \mathbf{R}^{n+l} \to \mathbf{R}^n, \tag{3.1}$$

where f is sufficiently differentiable and is even more equivariant

$$f(\vartheta(s)x, \lambda) = \vartheta(s) f(x, \lambda), \quad \forall s \in G,$$

with respect to a representation $\vartheta : G \to GL(\mathbf{R}^n)$.

The aim of symmetric bifurcation theory is the study of solutions of (3.1) (stationary or more complicated dynamical phenomena) and their stability depending on the values of the parameters λ.

The following definitions are standard in symmetric bifurcation theory.

Definition 3.1.1 *Let $\vartheta : G \to GL(\mathbf{R}^n)$ be a representation of a compact Lie group G.*

i.) Assume that H is a subgroup. Then

$$Fix(H) := \{x \in \mathbf{R}^n \mid \vartheta(s)x = x, \quad \forall s \in H\},$$

is called the fixed point space *of H.*

ii.) For any given $x \in \mathbf{R}^n$ the set

$$\mathcal{O}_x := \{\, y \in \mathbf{R}^n \mid \text{ exists } s \in G, \text{ such that } \vartheta(s)x = y\}$$

is called orbit.

iii.) For any given $x \in \mathbf{R}^n$ the subgroup

$$H_x := \{s \in G \mid \vartheta(s)x = x\},$$

is called isotropy group *of x.*

iv.) The isotropy group H_x together with its conjugates sH_xs^{-1}, $s \in G$ is called orbit type.

Of course the structure of a group orbit $\mathcal{O}_x$ depends on the isotropy group of x. All conjugates sH_xs^{-1} of H_x appear as isotropy groups of elements of $\mathcal{O}_x$.

For finite groups it is possible to determine the isotropy subgroup lattice for a representation of a group G if the inequivalent irreducible representations of all finite groups being isomorphic to a subgroup H of G are known. It uses the fact that a fixed point space is the isotypic component corresponding to the trivial irreducible representation in the isotypical decomposition. The dimensions of an isotypic decomposition are easily computed by a trace formula.

Algorithm 3.1.2 *(Computation of isotropy subgroup lattice, Symcon [69])*
Input: *linear faithful representation $\vartheta : G \to GL(\mathbf{R}^n)$, given by group table of G and matrices $\vartheta(s)$, $\forall\ s \in G$*

Output: *list of isotropy subgroups*

1.) *Compute subgroup lattice*
2.) *For each subgroup H determine* $\dim(Fix(H)) = \frac{1}{|H|}\sum_{s\in H} trace(\vartheta(s))$
3.) *for each subgroup H in G do*
for each subgroup K of G such that H is a proper subgroup of K do
if dim(Fix(K))= dim(Fix(H)) then
H is not an isotropy group for ϑ

In Symcon additionally conjugate groups and normalizers of groups are determined since these have a meaning for the bifurcation scenario.

The first and basic result of symmetric bifurcation theory is the *equivariant branching lemma* by Vanderbauwhede [185] for one-parameter problems ($l = 1$). Bifurcation of steady states from a stationary solution (x_0, λ_0) is dominated by a zero-eigenvalue of the Jacobian $f_x(x_0, \lambda_0)$. Since the matrix $f_x(x_0, \lambda_0)$ commutes with the isotropy group G_{x_0} of x_0 it is generic for one-parameter problems that the eigenspace of the eigenvalue zero of $f_x(x_0, \lambda_0)$ defines an absolutely irreducible representation $\vartheta^i_{G_{x_0}}$ of G_{x_0}. Restriction to subgroups H of G_{x_0} with fixed point space with respect to $\vartheta^i_{G_{x_0}}$ of dimension one enables the application of standard functional analytic results. Generically branches of stationary solutions with isotropy H bifurcate[1].

Definition 3.1.3 *Let G be a group and ϑ^i an absolutely irreducible representation of G. A subgroup H of G is called a* bifurcation subgroup of type ϑ^i*, if the fixed point subspace of H has dimension one and H is a maximal proper subgroup of G with this property.*

Algorithm 3.1.4 *(Computation of relevant bifurcation subgroups, Symcon [69])*
Input: *linear faithful representation $\vartheta : G \to GL(\mathbf{R}^n)$, given by group table of G and matrices $\vartheta(s)$, $\forall\ s \in G$*

Data Basis: *for finite groups H:*
group table of H
characters ψ^i_H of inequivalent irreducible representations ϑ^i_H

Output: *list of relevant bifurcation subgroups*

1.) *Compute all subgroups*
2.) *For each subgroup K of G*
determine a group isomorphism $\varphi : K \to H$ to a subgroup H in the data basis.
3.) *For all absolutely irreducible representations $\vartheta^i_{\varphi(G)}$ of G do*
find all bifurcation subgroups H of type $\vartheta^i_{\varphi(G)}$
- searching through all subgroups
- check $\frac{1}{|H|}\sum_{s\in H} trace(\vartheta^i(\varphi(s))) = 1$
- check whether there exist no subgroup K such that H is a proper subgroup of K and $\frac{1}{|K|}\sum_{s\in K} trace(\vartheta^i(\varphi(s))) = 1$
4.) *For all previously found bifurcation subgroups H do*
For all absolutely irreducible representations $\vartheta^i_{\varphi(H)}$ of H do

[1] Of course this does not exclude the existence of other branches with other isotropy.

find all bifurcation subgroups of type $\vartheta^i_{\varphi(H)}$

5.) *Erase all bifurcation subgroups which are not isotropy subgroups with respect to* ϑ

The computation of the bifurcation subgroups is one topic of the Computer Algebra part of Symcon. It prepares the automatic symmetry exploitation of the numerical computations. In the numerical part of Symcon the bifurcation points and bifurcating branches are computed numerically taking into account all possibilities due to the equivariant branching lemma. Motivated by Symcon the Computer Algebra system GAP provides an operator for the computation of bifurcation subgroups.

But there are still other phenomena in the bifurcation analysis to be investigated. The analysis typically uses reduction techniques such that the essential phenomena can be demonstrated for a small sized system but are valid for the original system.

There are three standard ways of how a low-dimensional vector field may be derived:

i.) Liapunov Schmidt reduction,

ii.) Center manifold reduction,

iii.) Symmetry adapted ansatz for a solution of a PDE by Fourier modes, see e.g [115].

This yields a low-dimensional vector field $f : \mathbf{R}^n \times \mathbf{R}^r \to \mathbf{R}^n$ with $f \in C^\infty$ and $\dot{x} = f(x)$ such that the Jacobian has zero-eigenvalues or complex eigenvalues on the imaginary axis only. Once f is obtained the general line of idea is to restrict to the investigation of local bifurcation phenomena which leads to the investigation of germs (classes of C^∞-functions) and use of singularity theory. For an introduction to the application of singularity theory to bifurcation theory see [86, 87]. The idea of identifying similar bifurcation phenomena leads to the notion of *contact equivalence classes*. Representatives of these classes may be taken to be polynomial vector fields (at least if the codimension - number of unfolding parameters - is finite). It remains to check at which order the Taylor expansion may be truncated.

So the bifurcation phenomena of polynomial vector fields show the typical bifurcation phenomena of huge sized systems and (to some extent) of partial differential equations.

3.2 An example of secondary Hopf bifurcation

In this section I present a simple example in order to demonstrate the argumentation in the previous section and the appropriate usage of Computer Algebra within this theory. This result on secondary Hopf bifurcation first appeared in [77].

Theorem 3.2.1 *Let $\vartheta : D_3 \to Gl(\mathbf{R}^n)$ be a linear representation such that both non-trivial irreducible representations appear at least once in the isotypic decomposition. Let Z_3 denote the cyclic subgroup of order 3. Let $x^0 \in Fix(D_3) = \{x \in \mathbf{R}^n \mid \vartheta(s)x = x \ \forall\ s \in D_3\}$ and $\lambda^0 \in \mathbf{R}^2$ define our point of interest. Assume $f : U(x^0, \lambda^0) \to \mathbf{R}^n$ is an equivariant function which means*

$$f(\vartheta(s)x, \lambda) = \vartheta(s)f(x, \lambda), \qquad \forall\, s \in D_3 \quad \forall\, (x, \lambda) \in U(x^0, \lambda^0),$$

where $U(x^0,\lambda^0)$ is an open surrounding of (x^0,λ^0) in $\mathbf{R}^{n+2}$. Moreover, we assume that f is C^∞ in its domain of definition and $f(x^0,\lambda^0)=0$ and that the D_3-invariant kernel of the Jacobian $f_x(x^0,\lambda^o)$ is a subspace W of $\mathbf{R}^n$ such that the subrepresentation ϑ^W decomposes into the alternate and the two-dimensional irreducible representation of D_3. Then generically *there exists locally a branch $(x(t),\lambda(t))$ of Z_3-invariant Hopf bifurcation points such that $(x(0),\lambda(0))=(x^0,\lambda^0)$, $(x(t),\lambda(t))\in U(x^0,\lambda^0)$ for all t and $(x(-t),\lambda(-t))=(\vartheta(s)x(t),\lambda(t))$ for all t and some reflection $s\in D_3$. For all $t\neq 0$ the point $x(t)$ has isotropy Z_3 and is a Hopf point, i.e. there exists a vector $\alpha\in\mathbf{R}^2$ such that $(x(t),0)$ is a Hopf bifurcation point of*

$$\dot{x}=g(x,\beta), \tag{3.2}$$

where $g(x,\beta)=f(x,\lambda(t)+\beta\alpha)$ is defined for all $(x,\beta)\in\mathbf{R}^{n+1}$ such that $(x,\lambda(t)+\beta\alpha)\in U(x^0,\lambda^0)$. For $t\neq 0$ a branch of periodic orbits of (3.2) emanates from $(x(t),0)$.

Proof: By the Liapunov-Schmidt reduction it is sufficient to investigate a generic germ in $\vec{\epsilon}_{x,\lambda}$ being equivariant with respect to a representation of D_3 decomposing into the alternate and two-dimensional irreducible representation. Assume the generators of D_3 are operating as

$$[\_s=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix},\ \_r=\begin{bmatrix}1&0&0\\0&-1/2&\frac{\sqrt{3}}{2}\\0&-\frac{\sqrt{3}}{2}&-1/2\end{bmatrix}].$$

The primary invariants π_1,π_2,π_3 and secondary invariant σ_2 of the Hironaka decomposition are computed to be

$$\mathit{primary\_invs}=[x1^2,x2^2+x3^2,x2^3-3\,x2\,x3^2],$$

$$\mathit{secondary\_invs}=[1,x1\,x2^2x3-\frac{x1\,x3^3}{3}]$$

while the generators $b_i, i=1,\ldots,6$ of the module of equivariants are computed as

$$\begin{aligned}\mathit{equivariants}\ =\ [\ & [x1,0,0],[0,x2,x3],[0,x1\,x3,-x1\,x2],\\ & [0,x2^2-x3^2,-2\,x2\,x3],[0,x1\,x2\,x3,\tfrac{x1\,x2^2}{2}-\tfrac{x1\,x3^2}{2}],\\ & [x2^2x3-\tfrac{x3^3}{3},0,0]\ ].\end{aligned}$$

By the Theorems of Schwarz and Poénaru (see [87] p. 46 and p. 51) each equivariant germ has a representation $\sum_{i=1}^{6}A_i(\pi(x),\sigma_2(x))b_i(x)$ where the A_i are germs in 4 variables. By a standard argumentation it suffices to study Taylor series expansions up to a certain degree. A generic equivariant polynomial vector field has a unique representation

$$f_{generic}=A_1(\pi_1,\pi_2,\pi_3)b_1+A_2(\pi)b_2+A_3(\pi)b_3+A_4(\pi)b_4+A_5(\pi)b_5+A_6(\pi)b_6,$$

where A_i are polynomials in three variables. A generic polynomial equivariant $f(x)$ of degree 3 has a representation

$$c_{1,1}b_1+c_{1,2}b_2+c_{2,3}b_3+c_{3,1,1}b_1\pi_1+c_{3,1,2}b_1\pi_2+c_{3,2,1}b_2\pi_1+c_{3,2,2}b_2\pi_2,$$

where the coefficients $c_{i,j,k}$ are arbitrary real numbers. If we are choosing $\lambda_1 = c_{1,1}$, $\lambda_2 = c_{1,2}$ as unfolding parameters then $f(x, \lambda)$ has the properties that $f(0, \lambda) \equiv 0$ and that the Jacobian $f_x(0, \lambda)$ has the structure

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}.$$

By the generalized equivariant branching lemma one gets a branch of equilibria with isotropy Z_3. The restriction of the equivariant $f(x, \lambda)$ on $Fix(Z_3) = \{(x_1, 0, 0) \mid x_1 \in \mathbf{R}\}$ is given by $[\lambda_1 x1 + c_{3,1,1} x1^3, 0, 0]$ with solutions

$$[\{x1 = 0\}, \left\{x1 = \frac{\sqrt{-c_{3,1,1}\lambda_1}}{c_{3,1,1}}\right\}, \left\{x1 = -\frac{\sqrt{-c_{3,1,1}\lambda_1}}{c_{3,1,1}}\right\}].$$

The last two correspond to the pitchfork bifurcation of Z_3-equilibria. For Hopf bifurcation the eigenvalues of the Jacobian needs to be inspected. On $Fix(Z_3)$ the Jacobian $f_x((x_1, 0, 0), \lambda)$ has the form

$$\begin{bmatrix} \lambda_1 + 3\, c_{3,1,1} x1^2 & 0 & 0 \\ 0 & \lambda_2 + c_{3,2,1} x1^2 & c_{2,3} x1 \\ 0 & -c_{2,3} x1 & \lambda_2 + c_{3,2,1} x1^2 \end{bmatrix}.$$

The second block has complex conjugate eigenvalues. If the two entries on the diagonal are zero the eigenvalues are on the imaginary axis. This condition together with the property of being a solution yields a branch $\left\{x1 = x1, \lambda_2 = -c_{3,2,1} x1^2, \lambda_1 = -c_{3,1,1} x1^2\right\}$ of potential Hopf bifurcation points. In order to proof the bifurcation of periodic orbits one needs to show that there are no other eigenvalues on the imaginary axis ($\lambda_1 + 3\, c_{3,1,1} x1^2 \neq 0$) and that the real part of the complex eigenvalue crosses the imaginary axis with nonzero speed. Choosing a direction $\alpha \in \mathbf{R}^2 \setminus \{0\}$ in the λ-plane the derivative in this direction of the real part of the eigenvalue pair is given by

$$-\frac{\alpha_1 c_{3,2,1}}{c_{3,1,1}} + \alpha_2.$$

If $c_{3,1,1} \neq 0$ it is possible to choose α such that this expression is unequal zero and thus periodic orbits bifurcate. □

The formulas in the proof have been computed with the Maple commands given below and by the command *latex* they have been converted to the form presented in the text above. More examples of usage of the symmetry package within dynamical systems theory are given in [125], but deeper results concerning singularity theory, Computer Algebra and symmetric bifurcation theory are presented in [76].

Remark 3.2.2 *i.) In order to make the proof complete one still needs to argue that a truncation of the vector field at degree 3 is no restriction. An argumentation would use the concept of contact equivalence classes and show that each other vector field $h(x, \lambda)$ with the same properties is contact equivalent to $f(x, \lambda)$ (existence of S, X, Λ with $S(x, \lambda)\cdot$*

$h(X(x,\lambda),\Lambda(\lambda)) \equiv f(x,\lambda)$). In [81] an argumentation in a similar case is performed using Maple. ii.) The period (or frequency) of the periodic orbits close to the Hopf bifurcation point is determined by the pair of complex conjugate eigenvalues. Periodic orbits close to the mode interaction point (x^0,λ^0) have almost period infinity.

```
# Maple 5.5 worksheet
# secondary Hopf bifurcation caused by mode interaction
# load packages first
> with(linalg):  # Maple built in package
> read(moregroebner);   # tool for algebraic computation
# see:  http://www.zib.de/gatermann/moregroebner.html
> read(symmetry);   # tool for computation of invariants + equivariants
# see:  http://www.zib.de/gatermann/symmetry.html
> ?symmetry
> ?finitegroup
# What is a generic equivariant vector field for D3?
> D3:=dihedral(3,[2,3]);
> latex(D3[generators]);
> varias:=[x1,x2,x3];
> cmb:=CMbasis(D3,varias); # compute primary and secondary invariants
> latex(cmb[1]); latex(cmb[2]);
> prims:=rhs(cmb[1]):
> aequivarias:=equis(D3,D3,varias,prims); # compute equivariants
> latex(aequivarias[1]); latex(aequivarias[2]);
# define equivariant up to degree 3
> p1:=prims[1]; p2:=prims[2]; p3:=prims[3];
> for i from 1 to 6  do b.i:=rhs(aequivarias[2])[i]; od;
> fgeneric:=A[1](pi[1],pi[2],pi[3])*b[1]+A[2](pi)*b[2]+A[3](pi)*b[3]
>          +A[4](pi)*b[4]+A[5](pi)*b[5]+A[6](pi)*b[6];
> latex(fgeneric);
> f:= c[1,1]*b[1]+c[1,2]*b[2] +                        # degree 1
>       c[2,3]*b[3] +                                  # degree 2
>       c[3,1,1]*b[1]*pi[1]+ c[3,1,2]*b[1]*pi[2]     # degree 3
>       + c[3,2,1]*b[2]*pi[1] + c[3,2,2]*b[2]*pi[2];
> latex(f);
> fsub:={pi[1]=p1,pi[2]=p2,b[1]=evalm(b1),b[2]=evalm(b2),b[3]=evalm(b3)}
> ;
> f:=subs(fsub,f);
> f:=map(expand,evalm(f));
# We assume that the truncation at degree 3 is okay.
# fixed point space of D_3
> fixD3:={x1=0,x2=0,x3=0};
> fD3:=subs(fixD3,evalm(f));
# linearization at fixed point space of D3
> jacf:=matrix(3,3,[
>                 [diff(f[1],x1), diff(f[1],x2), diff(f[1],x3)],
>                 [diff(f[2],x1), diff(f[2],x2), diff(f[2],x3)],
>                   [diff(f[3],x1), diff(f[3],x2), diff(f[3],x3)]]);
> subs(fixD3,evalm(jacf));
```

```
# The coefficients c_1_1 and c_1_2 are unfolding parameters
> c[1,1]:=lambda[1];c[1,2]:=lambda[2];
> f:=map(eval,f);
> jacf:=map(eval,jacf);
> latex(subs(fixD3,evalm(jacf)));
# A generalization of the equivariant branching lemma states that
# at (x1,x2,x3,lambda_1, lambda_2)=(0,0,0,0,0) bifurcation of solutions
# with Z_3-isotropy happens. Obviously, Fix(Z_3)=<(1,0,0)>.
> fixZ3:={x2=0,x3=0};
# vector field restricted to Fix(Z_3)
> fZ3:=subs(fixZ3,evalm(f));
> latex(fZ3);
> solsZ3:=[solve(fZ3[1],{x1})];
> latex(solsZ3);
# We see the trivial solution x_1=0 and two branches beeing conjugate
# to each other. Depending on whether the generic coefficient c_3,1,1
# s positive or negative both solutions branch in the direction
# of positive or negative lambda_1.
# Linearization in Fix(Z_3)
> A:=subs(fixZ3,evalm(jacf));
> latex(evalm(A));
# For  c_2,3  <> 0  there are a pair of pure complex eigenvalues.
# If the two entries on the diagonal are zero, the necessary condition
# for the Hopf bifurcation is fullfilled.
# search Hopf bifurcation points with Z_3 Isotropy
> Hopfs:=[solve({fZ3[1],A[2,2]},{x1,lambda[1],lambda[2]})];
> latex(Hopfs[2]);
# The case x1=0 refers to the steady states with D_3 isotropy and
# bifurcation of type of 2-dimensional irreducible representation
# leading
# to steady states with Z_2-isotropy. The other case defines a curve of
# of (possible Hopf points).
# But we still need to check the transversality condition.
# transversality condition, Hopf points
# (Parametrization of Z_3 branch by lambda1)
> eigenvalue:=simplify(A[2,2],
>    {normal(fZ3[1]/x1)},{x1}) + I*simplify(A[2,3],{normal(fZ3[1]/x1)},
>       {x1});
> realval:=simplify(A[2,2],{normal(fZ3[1]/x1)},{x1}) ;
# derivative in direction of alpha[1]*lambda[1]+alpha[2]*lambda[2]
> transcond:=alpha[1]*diff(realval,lambda[1])
>                     +alpha[2]*diff(realval,lambda[2]);
> latex(transcond);
# Other eigenvalue has to be nonzero
> eigenvalue0:=simplify(A[1,1],{normal(fZ3[1]/x1)},{x1}) ;
# Let's consider the 1-parameter bifurcation problem defined by
# (lambda_1,lambda_2)=(-c_3,1,1 x1^2, -c_3,2,1 x1^2)
#                                   + (alpha[1],alpha[2])*beta.
# For a special value of x1 there is one possible Hopf point.
# For alpha[1],alpha[2] such that  transcond<>0 and eigenvalue0 <> 0
# there is a bifurcation of periodic orbits.
```

Chapter 4

Orbit space reduction

> "But if I have a weather system that I start up with a certain temperature and a certain wind speed and a certain humidity- and if I then repeat it with almost the same temperature, wind, and humidity-the second system will not behave almost the same. It'll wander off and rapidly will become very different from the first. Thunderstorms instead of sunshine. That's nonlinear dynamics. They are sensitive to initial conditions: tiny differences become amplified."
> "I think I see." Gennaro said.
> "The shorthand is the 'butterfly effect.' A butterfly flaps its wings in Peking, and weather in New York is different." ...
> "There is a problem with that island. It is an accident waiting to happen."
>
> Michael Crichton
> Jurassic Park

The idea of the orbit space reduction is the investigation of the system of differential equations modulo the group action. Dividing out the group action the system of differential equations to be studied is transported to another system on a space with a complicated structure. The stratification of this set is deduced from the structure of orbit types of the group action. The orbit space reduction is performed with the help of a Hilbert basis of the invariant ring. The new idea in this chapter is the use of a special Hilbert basis reflecting the orbit type structure as best as possible. Since steady state solutions are the most fundamental objects in dynamics we first recall the influence of symmetry and the symbolic methods on the exact solution of symmetric polynomial systems using invariant theory.

In Section 4.2 and Section 4.3 we will study a low-dimensional system of differential equations

$$\dot{x} = f(x, \lambda), \quad f \in \epsilon_{x,\lambda}$$

close to $(0,0)$ where the right hand side f is a C^∞-germ and λ are parameters. Additionally, we assume that f is equivariant with respect to an orthogonal faithful representation of a compact Lie group G. The restriction to orthogonal representations is natural since a lot of group actions in nature or engineering science are orthogonal anyway. Secondly, whenever a G-invariant inner product exists (e.g. for finite groups and compact Lie groups) the coordinate system can be chosen such that the representation matrices are orthogonal.

Following a standard argumentation based on singularity theory we are left with a polynomial system

$$\dot{x} = f(x, \lambda), \quad f_i \in \mathbf{R}[\lambda][x], i = 1, \ldots, n, \tag{4.1}$$

where $f \in \mathbf{R}[\lambda][x]_\vartheta^\vartheta$ of some degree $\leq d$. The first goal is the computation of equilibria.

4.1 Exact computation of steady states

For the discussion of orbit space reduction it is helpful to know the ideas of exact solution of symmetric polynomial systems

$$f(x) = 0, \quad f \in \mathbf{R}[x]_\rho^\vartheta, \tag{4.2}$$

$\vartheta : G \to GL(\mathbf{R}^n)$, $\rho : G \to GL(\mathbf{R}^m)$ orthogonal representations of a compact Lie group G, ϑ being faithful. In exact computations we will assume that the coefficients are elements of a subfield $K \subset \mathbf{R}$.

The most basic idea for the solution, the *restriction to fixed point spaces*, has been mentioned in Chapter 3 already. The aim of this section is to examine the role of invariants for the solution of systems. Let us first consider the case that each f_i is G-invariant. Denote a Hilbert basis by $\pi_1, \ldots, \pi_r$. By the theorem by Hilbert a representation

$$f_i(x) = g_i(\pi(x)), \quad i = 1, \ldots, n,$$

exists and decomposes the system (4.2) into the *'easier'* system $g(y) = 0$ and the resolution of orbits into its elements $\pi(x) = y$. The advantage is that the g_i have smaller degrees than the f_i. The polynomials g_i are easily found using Gröbner bases (computation in a subring, see e.g. [14] p. 269).

Definition 4.1.1 *Given a Hilbert basis $\pi_1, \ldots, \pi_r$ of the invariant ring the mapping*

$$\pi : \mathbf{C}^n \to \mathbf{C}^r, \quad x \mapsto (\pi_1(x), \ldots, \pi_r(x))$$

is called the Hilbert mapping.

This mapping is contracting orbits as illustrated in Figure 4.1. The following well-known lemma means that the Hilbert mapping is distinguishing group orbits.

Lemma 4.1.2 *For finite groups the Hilbert mapping is an isomorphism between the set of orbits $\mathbf{C}^n/\vartheta(G)$ and the image $\pi(\mathbf{C}^n) \subseteq V(J) \subseteq \mathbf{C}^r$ where J is the ideal of relations. For orthogonal representations of compact Lie groups the Hilbert mapping is an isomorphism between the real orbits $\mathbf{R}^n/\vartheta(G)$ and the image $\pi(\mathbf{R}^n) \subseteq V^R(J) \subseteq \mathbf{R}^r$.*

Proof: The first statement means that for finite groups the nullcone consists of 0 only. Assume $a, b \in \mathbf{C}^n$ are two points such that no $s \in G$ exists with $\vartheta(s)a = b$ and $\pi(a) = \pi(b)$. Since G is finite we can find a polynomial $p \in \mathbf{C}[x]$ with $p(a) = 1, p(b) = 0$ and $p(\vartheta(s)b) = 0$, $\forall\, s \in G$ and $p(\vartheta(s)a) = 0$ for all $s \in G$ with $\vartheta(s)a \neq a$. Then $P(x) = \mathcal{R}(p(x)) = \frac{1}{|G|}\sum_{s \in G} p(\vartheta(s)x)$ is an invariant polynomial with $P(a) = \frac{1}{|G|}, P(b) = 0$. Since $\pi(a) = \pi(b)$ there is no representation $P(x) = g(\pi(x))$. This is a contradiction to the fact that $\pi_1, \ldots, \pi_r$ form a Hilbert basis.

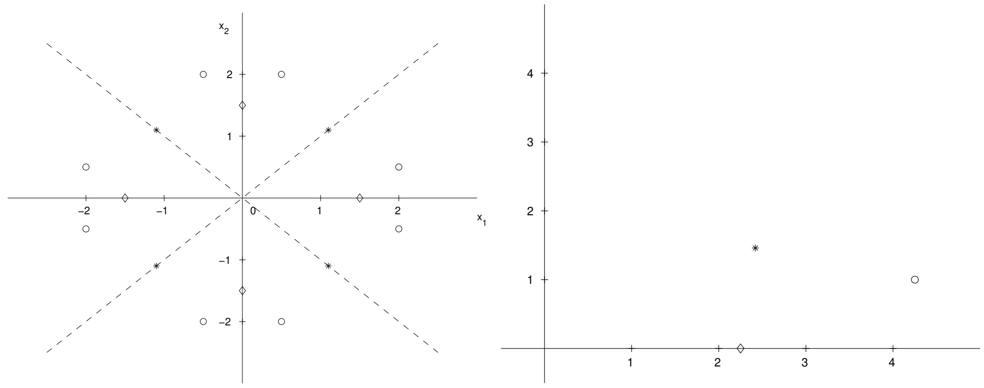

Figure 4.1: Three D_4-orbits of different types mapped by the Hilbert mapping onto the orbit space. Here the invariants $\pi_1(x) = x_1^2 + x_2^2$ and $\pi_2(x) = x_1^2 x_2^2$ have been used

For the second statement assume $a, b \in \mathbf{R}^n$ are real and $a \neq 0$ and $\pi(a) = \pi(b)$. Since ϑ is orthogonal the norm is invariant. Thus we may assume $\pi_1(x) = ||x||_2^2 = x_1^2 + \cdots + x_n^2$.

We distinguish two cases:

a.) The set $\{\vartheta(s)a$ for all s in the torus T of $G\}$ generates $\mathbf{R}^n$ as a vector space. Then for each $b \in \mathbf{R}^n$ with $\pi_1(b) = \pi_1(a)$ there exists $t \in T$ with $\vartheta(t)b = a$.

b.) The set $\vartheta(s)a$ for all $s \in T$ generates a proper subvector space $V \in \mathbf{R}^n$. Then there are finitely many group elements $s_1, \ldots, s_m \in G$ modulo T and modulo G_a such that $\vartheta(s_i)a \neq a$ and $\vartheta(s_i)a \notin V$. Let $p \in \mathbf{R}[x]$ be a polynomial with $p(a) = 1, p(\vartheta(s_i)a) = 0$ and $p(\vartheta(s)b) = 0$ for all $s \in G$. Then $P(x) := \mathcal{R}(p(x))$ is a G-invariant polynomial with $P(a) \neq 0$, but $P(b) = 0$. This yields the same contradiction than above. □

An alternative proof of the second statement may be found in [144] p. 50.

If the f_i are not invariant one nevertheless wants to use invariant theory. Obviously, $I = \langle f_1, \ldots, f_n \rangle$ is G-invariant and $I^G = I \cap K[x]^\vartheta$ is an ideal of invariant polynomials in $K[x]^\vartheta$. But I^G generates an ideal I_o^G in $K[x]$ as well. In general one expects $I \neq I_o^G$.

Lemma 4.1.3 *Assume G is a compact Lie group and ϑ, ρ are two linear representations of G. Furthermore let the polynomial mapping f be ϑ-ρ-equivariant. Let $I = \langle f_1, \ldots, f_m \rangle$ denote the G-invariant ideal and $I^G = I \cap \mathbf{R}[x]^\vartheta$ and $I_o^G = \langle I^G \rangle \subset \mathbf{R}[x]$. If G is a finite group then the complex solutions are the same: $V(I) = V(I_o^G) \subset \mathbf{C}^n$ (equivalently $Rad(I) = Rad(I_o^G)$ in $\mathbf{C}[x]$). If $\vartheta : G \to GL(\mathbf{R}^n)$ is an orthogonal group action then the real solutions are the same: $V^R(I) = V^R(I_o^G) \subset \mathbf{R}^n$.*

The proof of the finite group part may be found in [176] p. 61. Each $f \in I^G$ is represented as $f(x) = g(\pi_1(x), \ldots, \pi_r(x))$ giving rise to an ideal $\mathcal{I} \subset \mathbf{R}[x]/J$ where J denotes the ideal of relations. Since $\mathcal{I}$ carries the information on solutions it is appropriate to compute a set of generators of $\langle \mathcal{I}, J \rangle \subset \mathbf{R}[y]$. In Algorithm 2.6.2 p. 59 [176] the Gröbner basis of

$$\mathcal{J} = \langle f_1(x), \ldots, f_m(x), y_1 - \pi_1(x), \ldots, y_r - \pi_r(x) \rangle \subset K[x, y] \tag{4.3}$$

is computed with respect to a term order eliminating x. (For efficiency reasons one might want to precompute the Gröbner basis of the last set of polynomials.) The Gröbner basis $g_1(y), \ldots, g_m(y)$ of $\mathcal{J} \cap K[y]$ yields a generating system $g_1(\pi(x)), \ldots, g_m(\pi(x))$ of I_o^G or I^G, respectively. The obvious disadvantage of (4.3) is the doubling of variables.

A more recent variant was derived as a remark by Derksen [48].

Lemma 4.1.4 *(Derksen [48] Remark 3.3) Let $I = \langle f_1(x), \dots, f_m(x)\rangle \subseteq K[x]$ be a G-invariant ideal, where G is an algebraic group given by $A(z) \in K[x]^{n,n}$ modulo $J_G = \langle h_1(z), \dots, h_s(z)\rangle$. Consider the ideal*

$$I_\Gamma = \left\langle h_1(z), \dots, h_s(z), y_1 - \sum_{j=1}^{n} a_{1j}(z)x_j\,, \ \dots\ , y_n - \sum_{j=1}^{n} a_{nj}(z)x_j \right\rangle,$$

in $K[x,y,z]$ and

$$b = \{g(x,y) \in K[x,y] \mid g(x, \vartheta(s)x) = 0, \forall\, s \in G,\ \forall\, x\} = I_\Gamma \cap K[x,y].$$

Then

$$(I + b) \cap K[x] = I_o^G.$$

Proof: Let $g(x) \in I_o^G$. Since I_o^G is generated by invariant polynomials we may assume that $g(x)$ is invariant. Then $g(x) = (g(x) - g(y)) + g(y)$. Since $I \subseteq I_o^G$ we have $g(y) \in I$. Because g is invariant $(g(x) - g(y))(x, \vartheta(s)\,x) = 0$. This means $g \in I + b$.

For the other direction let $g(x) \in K[x]$ and even more $g(x) \in I + b \subseteq K[x,y]$. Then one has a representation

$$g(x) = \sum_i c_i(x) f_i(y) + b(x,y),$$

with $c_i(x) \in K[x]$ and $b(x,y) \in b$. Consider the linear representation $Id + \vartheta : G \to GL(K^{2n})$, $(x,y) \mapsto (x, \vartheta(s)y)$. The corresponding Reynolds projection $\mathcal{R}^{Id+\vartheta}$ is a $K[x]$-module homomorphism which gives

$$\begin{array}{rcl} \mathcal{R}^{Id+\vartheta}(g(x)) \ = \ g(x) & = & \mathcal{R}^{Id+\vartheta}\left(\sum_i c_i(x) f_i(y) + b(x,y)\right) \\ & = & \sum_i c_i(x)\mathcal{R}(f_i(y)) + \mathcal{R}^{Id+\vartheta}(b(x,y)) \\ & = & \sum_i c_i(x)\mathcal{R}(f_i(y)). \end{array}$$

Substituting $y = x$ yields

$$g(x) = \sum_i c_i(x)\mathcal{R}(f_i(x)).$$

Since the $\mathcal{R}(f_i(x))$ are invariant this means $g(x) \in I_o^G$. □

This proof is essentially the proof of Theorem 3.1 in [48]. The role of $\langle y_1, \dots, y_n\rangle$ has been replaced by I. That's why this proof also proves the correctness of Algorithm 2.2.2 in Section 2.2. Concerning the ideal b a comment is in favor. Knowing a Hilbert basis $\pi_1, \dots, \pi_r$ the ideal $\langle \pi_1(x) - \pi_1(y), \dots, \langle \pi_r(x) - \pi_r(y)\rangle$ is a subideal of b. But in general the inclusion is proper.

Lemma 4.1.4 suggests an algorithm for the solution of symmetric systems. I like to thank H. Derksen for pointing out to me in an email that his work might be interesting for symmetric system solving.

Algorithm 4.1.5 *(Solution of symmetric systems)*
Input: *hom. $f_1, \dots, f_m$ generating a G-invariant ideal $I \subset K[x]$*
generators $k_1(x,y), \dots, k_s(x,y)$ of
$b = \{g(x,y) \mid g(x, \vartheta(s)x) = 0,\ \forall s \in G\}$
homogeneous invariants $\pi_1(x), \dots, \pi_r(x)$ generating $K[x]^\vartheta$
Gröbner basis $\mathcal{GB}$ of ideal of relations $\langle u_1 - \pi_1(x). \dots, u_r - \pi_r(x)\rangle$
with respect to a term order $<$ eliminating x.

1.) *Compute a Gröbner basis GB of*

$$\langle f_1(y), \ldots, f_m(y), k_1(x,y), \ldots, k_s(x,y), \rangle \subset K[x,y],$$

with respect to a term order eliminating x.

2.) $J := \{\}$
for $f \in GB \cap K[y]$ do
$g(u) = normalf_<(\mathcal{R}(f)(x), \mathcal{GB})$ **or**
$g(x,u) = normalf_<(f(x), \mathcal{GB}),\ g(u) := g(0,u)$
$J = J \cup \{g\}$

3.) *Compute a Gröbner basis of $\langle J \rangle \subset K[u]$ with respect to a lex term order.*

4.) *For each solution $u \in \mathbf{C}^n$ the resolution of orbits is obtained with the Gröbner basis $\mathcal{GB}$.*

Proof of Correctness: The algorithm first computes an ideal basis of I_o^G as shown in Lemma 4.1.4. Since I is a homogeneous ideal a Reynolds projection of the generators yield an ideal basis of $I^G \subset K[y]^G$ by Nakayama's lemma. Then these invariant polynomials are rewritten in terms of the Hilbert basis. These steps may be realized in several ways. Either one uses the Reynolds projection and performs the rewriting with the help of Gröbner bases (or with the subduction algorithm if a SAGBI basis is available) or one uses Gröbner bases for the projection right away. □

The bottleneck of this algorithm is that an ideal basis of b is required. Typically it is computed by an elimination ideal computation from I_Γ. A speed up can be obtained by using additionally $\langle \pi_1(x) - \pi_1(y), \ldots, \pi_r(x) - \pi_r(y) \rangle \subset b$ as input of the elimination computation.

In case the input polynomials $f_1, \ldots, f_m$ are not homogeneous one needs to use homogenization of a Gröbner basis with respect to grevlex and dehomogenization as a final step.

The second ideal in the invariant ring is derived with equivariants and still carries the information on real solutions. The following lemma has been used in [194] and [71] for the exact solution of symmetric systems.

Lemma 4.1.6 *(Jaric, Michel, Sharp [102]) Let $\vartheta : G \to GL(\mathbf{R}^n)$, $\rho : G \to GL(\mathbf{R}^m)$ be two orthogonal linear representations of a compact Lie group G, as usual ϑ being faithful, and $f \in \mathbf{R}[x]_\rho^\vartheta$ a ϑ-ρ equivariant vector field ($f(\vartheta(s)\,x) = \rho(s)\,f(x)$, $\forall\, s \in G$). Denote the ϑ-invariant ideal $\langle f_1, \ldots, f_m \rangle \subset \mathbf{R}[x]$ by I. Assume $b_1(x), \ldots, b_s(x) \in (\mathbf{R}[x])^m$ generate the module of ϑ-ρ equivariants $\mathbf{R}[x]_\rho^\vartheta$. Then the polynomials*

$$b_1^t f, \ldots, b_s^t f \in \mathbf{R}[x]^\vartheta,$$

are invariant (with respect to ϑ) and the ideal $\tilde{I}_o^g \subset \mathbf{R}[x]$ generated by them has the same set of real solutions than I.

Proof: Let $a \in \mathbf{R}^n$ be a common zero of $f_1, \ldots, f_m$. Then obviously $(b_i(a))^t f(a) = 0$ for all $i = 1, \ldots, r$.

As the module of equivariants is finitely generated the vector field f has a representation

$$f(x) = \sum_{i=1}^{r} p_i(\pi(x))\, b_i(x)$$

Table 4.1: The first set of polynomials is a Gröbner basis of the ideal in the invariant ring of the cyclo hexane problem. The second set is the resolution of orbits

$$\begin{aligned}[\ & 81\,\pi_3{}^2 - 21654\,\pi_3 - 956880\,\pi_2 - 11815656\,\pi_1 - 775404\,\pi_1{}^2 + 478098\,\pi_1{}^3 \\ & \quad +77727320 - 60156\,\pi_1{}^4 + 3564\,\pi_1{}^5 - 81\,\pi_1{}^6, \\ & 27\,\pi_3\pi_2 - 1089\,\pi_3 + 28455\,\pi_2 + 287496\,\pi_1 + 45969\,\pi_1{}^2 - 13293\,\pi_1{}^3 - 2325620 \\ & \quad +990\,\pi_1{}^4 - 27\,\pi_1{}^5, \\ & 9\,\pi_1\pi_3 - 99\,\pi_3 - 1080\,\pi_2 - 3286\,\pi_1 - 2262\,\pi_1{}^2 + 297\,\pi_1{}^3 + 89870 - 9\,\pi_1{}^4, \\ & 9\,\pi_2{}^2 - 1686\,\pi_2 - 3432\,\pi_1 - 1702\,\pi_1{}^2 + 264\,\pi_1{}^3 + 77440 - 9\,\pi_1{}^4, \\ & -22\,\pi_1 + 29\,\pi_1{}^2 + 15\,\pi_2 - 3\,\pi_1{}^3 + 3\,\pi_1\pi_2 - 1210\ ]\end{aligned}$$

$$\begin{aligned}[\ & -\pi_1 + x_1 + x_2 + x_3, \\ & 2\,x_2{}^2 + 2\,x_3x_2 + 2\,x_3{}^2 - 2\,x_2\pi_1 - 2\,x_3\pi_1 + \pi_1{}^2 - \pi_2, \\ & 6\,x_3{}^3 - 6\,x_3{}^2\pi_1 + 3\,x_3\pi_1{}^2 - 3\,x_3\pi_2 - \pi_1{}^3 + 3\,\pi_1\pi_2 - 2\,\pi_3\ ]\end{aligned}$$

where $\pi_1, \ldots, \pi_r$ is a Hilbert basis and the p_i are polynomials in $\mathbf{R}[y_1, \ldots, y_r]$.

Then $||f(a)||_2^2 = (f(a))^t f(a) = \sum_{i=1}^r p_i(\pi(x)) b_i^t(a) f(a)$. If $a \in R^n$ is a common zero of $b_i^t f, i = 1, \ldots, s$, then $||f(a)||_2^2 = \sum_{i=1}^m (f_i(a))^2 = 0$. Since a is real $f(a) = 0$ follows immediately. □

Remark 4.1.7 *i.) In case the representation matrices $\rho(g)$ are not orthogonal, but a G-invariant inner product $(\cdot,\cdot)$ exists an analogous result is still valid with the polynomials $(b_1, f), \ldots, (b_s, f)$. ii.) For $\vartheta = \rho$* Jaric et al. *even show a sharper variant. It is sufficient to use the equivariants $b_i = \nabla\pi_i, i = 1, \ldots, r$, the gradients of a Hilbert basis. iii.) From the proof it is not clear that the complex solutions remain the same. Since $\tilde{I}_o^g \subseteq I_o^G$ it remains the question whether $\tilde{I}_o^g$ might have more complex solutions than I even in the case of finite groups.*

Algorithm 4.1.8 *(Solution of symmetric systems)*
Input: *ϑ-ρ-equivariant $f(x)$ generating a G-invariant ideal $I = \langle f_1, \ldots, f_m\rangle$*
homogeneous invariants $\pi_1(x), \ldots, \pi_r(x)$ generating $\mathbf{R}[x]^\vartheta$
homogeneous equivariants $b_1(x), \ldots, b_s(x)$ generating $\mathbf{R}[x]_\rho^\vartheta$
Gröbner basis $\mathcal{GB}$ of ideal of relations $\langle u_1 - \pi_1(x). \ldots, u_r - \pi_r(x)\rangle$
with respect to a term order $<$ eliminating x.

1.) *Compute $g_1 = normalf_<(b_1^t f, \mathcal{GB}), \ldots, g_s = normalf_<(b_s^t f, \mathcal{GB})$*

2.) *Compute a Gröbner basis of $\langle g_1, \ldots, g_s\rangle \subset K[u]$ with respect to a lexicographical term order.*

3.) *For each solution $u \in \mathbf{R}^r$ of $\langle g_1, \ldots, g_s\rangle$ the resolution of orbits is obtained with the Gröbner basis $\mathcal{GB}$.*

In contrast to the approach in [176] the doubling of variables appears in this algorithm only in the problem independent part, in the computation of relations which is done once for one group action.

Table 4.2: Nested ideal quotient computations of the first ideal in Table 4.1 give three ideals with Gröbner basis in upper triangular form. Since the coordinates x_i have to be positive and thus the invariants π_j have to be positive the first basis is irrelevant

$[18\,\pi_3 + 135\,\pi_2 - 875, -562\,\pi_2 + 4475 + 3\,{\pi_2}^2, 5 + \pi_1]$
$[(9\,\pi_3 - 1075)\,(9\,\pi_3 - 1331)\,, 3\,\pi_2 - 121, \pi_1 - 11]$
$\left[-3\,{\pi_1}^3 + 66\,{\pi_1}^2 - 388\,\pi_1 + 3\,\pi_3 - \frac{250}{3}, -{\pi_1}^2 + \frac{44\,\pi_1}{3} - \frac{242}{3} + \pi_2\right]$

Example 4.1.9 *We compare the three different approaches for the simple example of the cyclo hexane with D_3-symmetry which has already been investigated in Example 1.2.17. (The cyclo heptane has been studied by Levelt [130].) Recall that the polynomials are defined with determinants of matrices*

$$\begin{array}{ll} f_4(x) = \det(B), & f_3(x) = \det(A), \\ f_1(x_1,x_2,x_3) = f_3(x_2,x_3,x_1), & f_2(x_1,x_2,x_3) = f_3(x_3,x_1,x_2). \end{array}$$

This surely defines an equivariant system with respect to $D_3 = S_3$ acting by permutation of variables. The invariants are $\pi_1(x) = x_1+x_2+x_3$, $\pi_2(x) = x_1^2+x_2^2+x_3^2$, $\pi_3(x) = x_1^3+x_2^3+x_3^3$ and the fundamental equivariants are $(1,0,0,0)^t$, $(0,1,1,1)^t$, $(0,x_1,x_2,x_3)^t$, $(0,x_1^2,x_2^2,x_3^2)^t$.

All three approaches yield the same Gröbner basis (Table 4.1), but the timings differ (Table 4.3). In order to use Derksen's approach one needs the ideal b which is computed by elimination technique from I_Γ. The permutation group D_3 is written as algebraic group with the matrix $A(z)$ equal to

$$\begin{bmatrix} \frac{1+z_2}{2} & \frac{1-z_2}{2} & 0 \\ \frac{1-z_2}{2} & \frac{1+z_2}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2\,z_1 + 2\,{z_1}^2 + 2 & a_{12} & \tilde{a} \\ \tilde{a} & 2\,z_1 + 2\,{z_1}^2 + 2 & a_{12} \\ a_{12} & \tilde{a} & 2\,z_1 + 2\,{z_1}^2 + 2 \end{bmatrix}$$

$$\text{with } a_{12} := 2 - i\sqrt{3}{z_1}^2 - {z_1}^2 + i\sqrt{3}z_1 - z_1, \quad \tilde{a} := 2 + i\sqrt{3}{z_1}^2 - {z_1}^2 - i\sqrt{3}z_1 - z_1,$$

and $h_1(z_2) = z_2^2 - 1$, $h_2(z_1) = z_1^3 - 1$. Although I first computed a Gröbner basis of $\langle \pi_1(x) - \pi_1(y), \pi_2(x) - \pi_2(y), \pi_3(x) - \pi_3(y), h_1(z), h_2(z)\rangle$ and then a basis of the ideal generated by this plus I_Γ, the computation needed 14013 sec. As a consequence of the time comparison I recommend the approach using equivariants. Here no doubling of the number of variables appears while in both the two other algorithms this doubling happens.

Example 4.1.10 *Kotsireas gives on the webpage [118] the following system $G(x) = 0, G:$*

Table 4.3: Comparision of three different methods for solving symmetric systems. The timing has been taken in a way such that the steps depending on the group only have not been taken into account. Only the steps depending on the problem contributed to the time given here

problem	G	n	Alg. 2.6.2 in [176] Sturmfels	Alg. 4.1.8 Jaric	Alg. 4.1.5 Derksen
cyclo hexane [143]	D_3	3	85 s	7 s	12 s
celestial mechanics (Kotsireas [118])	Z_2	6	> 2 days	45675 s	3 days
Lotka Volterra (Noonburg [153])	D_3	3	42 s	8 s	2505 s

$\mathbf{R}^6 \to \mathbf{R}^6$ *with variables* $x = (B, D, F, b, d, f)$ *related to celestial mechanics.*

$$
\begin{aligned}
G_1(x) &:= (b-d)(B-D) - 2F + 2 = 0,\\
G_2(x) &:= (b-d)(B+D-2F) + 2(B-D) = 0,\\
G_3(x) &:= (b-d)^2 - 2(b+d) + f + 1 = 0,\\
G_4(x) &:= B^2b^3 - 1 = 0,\\
G_5(x) &:= D^2d^3 - 1 = 0,\\
G_6(x) &:= F^2f^3 - 1 = 0.
\end{aligned}
$$

The system is equivariant with respect to $Z_2 = \{id, s\}$. *While the reflection operates on the variables as* $\vartheta(s)(B, D, F, b, d, f) = (D, B, F, d, b, f)$ *on the image of* G *it operates as* $\rho(s)(G_1, \ldots, G_6) = (G_1, -G_2, G_3, G_5, G_4, G_6)$. *The methods described above apply and their timings are given in Table 4.3. For the approach in Algorithm 4.1.5 we need to write* Z_2 *as algebraic group*

$$
A(z) = \begin{bmatrix}
\frac{1}{2}(z_1+1) & \frac{1}{2}(1-z_1) & 0 & 0 & 0 & 0\\
\frac{1}{2}(1-z_1) & \frac{1}{2}(z_1+1) & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{2}(z_1+1) & \frac{1}{2}(1-z_1) & 0\\
0 & 0 & 0 & \frac{1}{2}(1-z_1) & \frac{1}{2}(z_1+1) & 0\\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

modulo $h_1(z) = z_1^2 - 1$. *Algorithm 4.1.8 requires the knowledge of invariants and equivariants which are computed by Algorithm 2.3.17 and Algorithm 2.3.20. For these* Z_2 *actions there are seven invariants and* 12 *fundamental equivariants.*

Example 4.1.11 *Noonburg investigates in [153] a Lotka-Volterra system. First the steady states needs to be found yielding the* D_3*-equivariant system*

$$
\begin{aligned}
1 - cx_1 - x_1x_2^2 - x_1x_3^2 &= 0,\\
1 - cx_2 - x_2x_3^2 - x_2x_1^2 &= 0,\\
1 - cx_3 - x_3x_1^2 - x_3x_2^2 &= 0.
\end{aligned}
$$

The computations are more complicated since they depend on a parameter c requiring computations over the ring $K[c]$. Since the D_3 action on this system is similiar than in Example 4.1.9 the computed invariants and equivariants are used again. The timings of the three solution methods are given in Table 4.3.

In the end of this section I like to make a few more comments on the different approaches of the solution of symmetric systems.

i.) The most basic idea is the *restriction to fixed point spaces*. One solves $f_{|\mathrm{Fix}(H)}(x) = 0$, for $x \in \mathrm{Fix}(H)$, $f : \mathbf{R}^n \to \mathbf{R}^m$ for a subgroup H of G and finds all solutions with isotropy group H or higher. For example this can be done by computing the Gröbner basis of $\langle (f_{|\mathrm{Fix}(H)})_1, \ldots, (f_{|\mathrm{Fix}(H)})_r \rangle$ with $r = \dim \mathrm{Fix}(H)$. Searching for solutions of isotropy less than H means the consideration of the original system (4.2). Except one computes the ideal quotient $\langle f_1, \ldots, f_n \rangle : \langle (f_{|\mathrm{Fix}(H)})_1, \ldots, (f_{|\mathrm{Fix}(H)})_r \rangle$ or even the saturation.

ii.) Another idea leads to a splitting of the system into smaller subsystems. As described in [67] the case of a group generated by reflections with respect to hyperplanes leads to polynomials having linear factors.

iii.) In the same paper a transformation of coordinates $y = Ax$ with respect to symmetry adapted bases is suggested for finite Abelian groups. By linear combination one finds polynomials which are elements of the isotypic components of the isotypic composition $K[x] = \sum_{i=1}^{h} K[x]_i$. For Abelian groups these polynomials have a dense representation in terms of monomials. Since this property is automatically exploited in the Buchberger algorithm a system in the transformed form requires less computing time.

iv.) Consider the k-th elimination ideal $I_k = I \cap K[x_{k+1}, \ldots, x_n]$ of a G-invariant ideal. Assume there exists a subgroup H of G such that the action decomposes into two blocks, operating on $x_1, \ldots, x_k$ and $x_{k+1}, \ldots, x_n$, respectively. Then I_k is H-invariant.

v.) If the polynomials $f_1, \ldots, f_m$ are invariant itself one may even more use the structure of the invariant ring or moreover of the *field* of invariants being generated by the primary invariants and one additional element. In [38] computations are carried out for the n-cyclic root problem.

4.2 Differential equations on the orbit space

The idea of Lemma 4.1.6 on the steady states carries further on to the dynamics of a system. For a Hilbert basis $\pi_1(x), \ldots, \pi_r(x)$ the computation of derivatives with respect to time yields

$$\dot{\pi}_i(x(t)) = (\nabla \pi_i(x(t)))^t \dot{x}(t) = (\nabla \pi_i)^t f(x(t), \lambda) = g_i(\pi(x(t)), \lambda), \quad i = 1, \ldots, r.$$

The first equality is given by the chain rule. The second exploits that the trajectory $x(t)$ satisfies the differential equation (4.1). Since $(\nabla \pi_i)^t f$ is invariant for orthogonal representations there exists a polynomial $g_i \in \mathbf{R}[\lambda][y_1, \ldots, y_r]$ with $(\nabla \pi_i)^t f(x, \lambda) = g_i(\pi(x), \lambda)$. This yields a differential equation

$$\dot{y} = F(y, \lambda), \quad \text{for } y \in \pi(\mathbf{R}^n) \subseteq \mathbf{R}^r. \tag{4.4}$$

The dynamics of this system is closely related to the dynamics of (4.1). The relation of steady states has been investigated in Section 4.1. But there is no correspondence of periodic orbits or homoclinic orbits of (4.1) and of (4.4). For example a drift along a group orbit is contracted to an equilibrium by the Hilbert mapping. A second difficulty is the determination of stability. Nevertheless orbit space reduction has been applied successfully in [32, 33, 34, 127, 129, 184]. In [1] p. 336 an old example with S_3-symmetry is presented.

First we will investigate the structure of the domain of (4.4), the image $\pi(\mathbf{R}^n)$. Denoting by $I \subset \mathbf{R}[y_1, \ldots, y_r]$ the ideal of relations of $\pi_1(x), \ldots, \pi_r(x)$ and by $V^R(I) \subset \mathbf{R}^r$ the corresponding real variety the inclusion $\pi(\mathbf{R}^n) \subset V^R(I)$ is trivially satisfied. Procesi and Schwarz give in [157] an even sharper characterization.

Theorem 4.2.1 *([157] p. 541) Let $\pi_1, \ldots, \pi_r$ be a Hilbert basis of an orthogonal, n-dimensional representation of a compact Lie group G and $V^R(I)$ the real variety of the ideal of relations. The matrix $B(y) = (b_{ij}(y))_{i,j=1,\ldots,r}$ with entries $b_{ij} \in \mathbf{R}[y_1, \ldots, y_r]$ is defined by*

$$(\nabla \pi_i(x))^t \nabla \pi_j(x) = b_{ij}(\pi_1(x), \ldots, \pi_r(x)).$$

Then

$$\pi(\mathbf{R}^n) = \{y \in \mathbf{R}^r \mid y \in V^R(I), B(y) \geq 0\},$$

where the sign $\geq$ means that the matrix is positive semidefinite.

The proof is based on complexification, the existence of a closed orbit and the slice theorem.

Next we recall more refined structures of the image of the Hilbert mapping, the subdivision into manifolds.

Definition 4.2.2 *A* stratification *of a set S is a finite collection of subsets $\{S_1, \ldots, S_m\}$ such that*

a.) each stratum *S_i is a smooth manifold;*

b.) $S = S_1 \cup \cdots \cup S_m$;

c.) If $S_i \cap \bar{S}_j \neq \{\}$ for some $i \neq j$ then $S_i \subset \bar{S}_j$ and $\dim S_i < \dim S_j$;

On $\mathbf{R}^n/\vartheta(G)$ there exists a natural stratification by orbit type. There exists a unique stratum of maximal dimension which is called *principal stratum*. But in this natural stratification there may be some strata which decomposes into non-connected sets. The stratification into connected smooth manifolds is called *secondary stratification*.

On the other hand also the image $\pi(\mathbf{R}^n) \subset \mathbf{R}^r$ processes a stratification. The strata are *semi-algebraic sets*, that means they are part of a real variety which is bounded by polynomial inequalities or is a finite collection of such sets.

The main result on the structure of the stratification of the orbit space is given by a Lemma by Bierstone.

Theorem 4.2.3 *(Bierstone [15] Thm. A p. 246)*
The semi-analytic stratification of the orbit space $\pi(\mathbf{R}^n)$ coincides with the stratification of $\mathbf{R}^n$ by components of submanifolds of given orbit type.

In the next section a systematic way is gained to describe the strata by equalities and inequalities.

4.3 Using Noether normalization

The description of the orbit space reduction in the last section focuses on one group G only. Since fixed point spaces Fix(H) are flow-invariant often the restriction of (4.1) to Fix(H) is studied first. Thus one also likes to study the corresponding restriction of (4.4). That's why the aim of this section is to exploit the isotropy group lattice and to choose symmetry adapted coordinates according to this lattice, or at least for a chain of subgroups $G \supset H_1 \supset H_2 \supset \cdots \supset H_l$.

Lemma 4.3.1 *Let $\vartheta : G \to GL(\mathbf{R}^n)$ be a linear representation of a compact Lie group G and $\pi_1, \ldots, \pi_r$ a Hilbert basis. Denote by I the ideal of relations in $\mathbf{R}[y_1, \ldots, y_r]$. Suppose H is an isotropy subgroup of G. The relations of $\pi_{1|Fix(H)}, \ldots, \pi_{r|Fix(H)}$ define a second ideal J in $\mathbf{R}[y]$. Then*

$$I \subset J, \quad V(J) \subset V(I), \quad V^R(J) \subset V^R(I)$$

$$\pi(\mathbf{R}^n) \subset V^R(I), \quad \pi(Fix(H)) \subset V^R(J), \quad \pi(Fix(H)) \subset \pi(\mathbf{R}^n).$$

The lemma has a meaning for the description of the (primary) semi-algebraic stratification. The principal stratum is given by

$$\begin{aligned} S = \{y \in \mathbf{R}^r \mid \quad & f_i(y) = 0, i = 1, \ldots, m, \ \ m_j(y) \geq 0, j = 1, \ldots, r, \\ & \text{for all minimal isotropy subgroups } H \\ & \text{exists } k \in \{1, \ldots, m_H\} \text{ with } g_k^H(y) \neq 0 \ \}, \end{aligned}$$

where $f_1, \ldots, f_m$ is an ideal basis of I and $m_j(y)$ are the determinants of the principal minors of $B(y)$. For an isotropy subgroups H denote by $g_1^H, \ldots, g_{m_H}^H$ a basis of the ideal of relations of $\pi_{1|\mathrm{Fix}(H)}, \ldots, \pi_{r|\mathrm{Fix}(H)}$ modulo I. For the other strata S_H corresponding to orbit types H the description is analogously. The role of I is replaced by J and the role of the minimal isotropy subgroups H of G by isotropy subgroups K with $H \subset K$ which are minimal with this property. Doing some simple manipulations one easily sees that each stratum is defined by a set of equalities and a set of inequalities. Observe that for the strata of the secondary stratification the distinction into *connected* manifolds still needs to be done.

Example 4.3.2 *In [116] Koenig investigates the standard example of a D_3 action on the plane which is generated by*

$$\vartheta(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \vartheta(r) = \begin{bmatrix} -1/2 & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -1/2 \end{bmatrix}.$$

The invariant ring is polynomial and is generated by $\pi_1(x) = x_1^2 + x_2^2$, $\pi_2(x) = x_1^3 - 3x_1x_2^2$. Then

$$B(y) = \begin{bmatrix} 4\, y_1 & 6\, y_2 \\ 6\, y_2 & 9\, {y_1}^2 \end{bmatrix} \quad \textit{with} \quad \det(B) = 36\, {y_1}^3 - 36\, {y_2}^2.$$

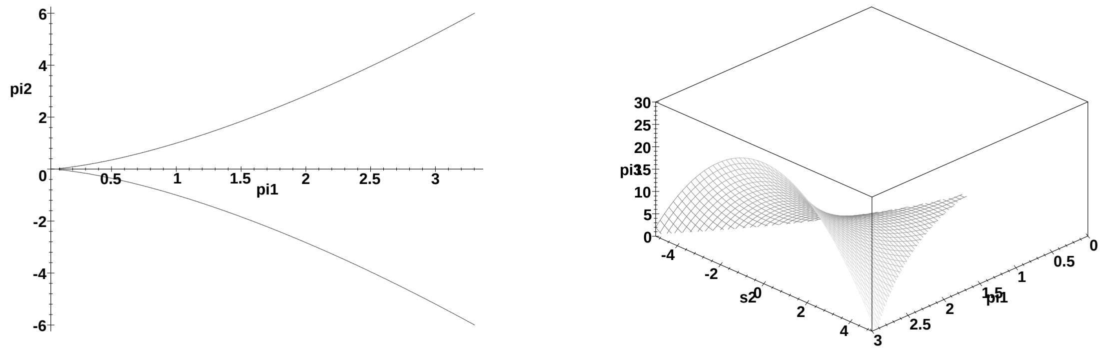

Figure 4.2: For the two-dimensional standard action of D_3 in the plane one may choose $\pi_1 = x_1^2 + x_2^2$, $\pi_2 = x_1^3 - 3x_1x_2^2$ or $\tilde{\pi}_1, = \pi_1$, $\tilde{\pi}_2(x) = (x_1^2 + x_2^2)^3 - (x_1^3 - 3x_1x_2^2)^2$, $\tilde{\sigma}_2 = \pi_2$ as invariants. In both cases the stratification has 4 strata. The principal stratum is either given by $\{\pi_1^3 - \pi_2^2 > 0\}$ or in the second case by $\{\tilde{\pi}_1 > 0, \tilde{\pi}_2 > 0, \tilde{\sigma}_2 = \pm\sqrt{\tilde{\pi}_1^3 - \tilde{\pi}_2}\}$. The fixed point space Fix(Z_2) decomposes into three strata and corresponds to $\{(\pi_1, \pi_2) \in \mathbf{R}^2 \,|\, \pi_1^3 - \pi_2^2 = 0\}$ or $\{(\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\sigma}_2) \in \mathbf{R}^3 \,|\, \tilde{\pi}_2 = 0, \tilde{\sigma}_2 = \pm\sqrt{\tilde{\pi}_1^3}\}$

There is an isotropy subgroup Z_2 and its conjugates. The restriction to Fix$(Z_2) = \{(x_1, 0)\}$ gives the relation $y_1^3 - y_2^2$. Thus the strata as illustrated in Figure 4.2 are given by

$$\begin{aligned}
S_{Id} &= \{y \in \mathbf{R}^2 \mid y_1 \geq 0,\ y_1^3 - y_2^2 \geq 0,\ y_1^3 - y_2^2 \neq 0\} \\
&= \{y \in \mathbf{R}^2 \mid y_1 > 0,\ y_1^3 - y_2^2 > 0\}, \\
S_{Z_2} &= \{y \in \mathbf{R}^2 \mid y_1^3 - y_2^2 = 0,\ y_1 \geq 0,\ y_1^3 - y_2^2 \geq 0,\ y_1 \neq 0,\ y_2 \neq 0\} \\
&= \{y \in \mathbf{R}^2 \mid y_1^3 - y_2^2 = 0,\ y_1 > 0\} \\
&= \{y_1 > 0, y_1^3 - y_2^2 = 0,\ y_2 > 0\} \cup \{y_1 > 0, y_1^3 - y_2^2 = 0,\ y_2 < 0\}, \\
S_{D_3} &= \{(0,0)\}.
\end{aligned}$$

Our aim is to investigate the structure which is imposed on the polynomials f_i, g_k^H by the Cohen-Macaulayness of the invariant ring.

Lemma 4.3.3 *Let the homogeneous invariants $\pi_1, \ldots, \pi_r \in K[x_1, \ldots, x_n]$ form a Hilbert basis of a faithful representation ϑ of a compact Lie group G. Assume H is an isotropy subgroup of G with respect to ϑ and Fix(H) the corresponding fixed point space. Then there exists invariants*

$$\begin{aligned}
\tilde{\pi}_1(x) = g_1(\pi(x)), \ldots, \tilde{\pi}_d(x) = g_d(\pi(x)); \\
\tilde{\sigma}_2(x) = h_2(\pi(x)), \ldots, \tilde{\sigma}_t(x) = h_t(\pi(x));
\end{aligned}$$

with $g_i, h_j \in K[y_1, \ldots, y_r]$ such that

i.) g_i, h_j are homogeneous with respect to the grading
$W(y_i) = \deg(\pi_i(x)), i = 1, \ldots, r$;

ii.) $\tilde{\pi}_1, \ldots, \tilde{\pi}_d$ are algebraic independent;

iii.) $\tilde{\pi}_{i|Fix(H)} \not\equiv 0,\ i = 1, \ldots, c$ *and* $\tilde{\pi}_{i|Fix(H)} \equiv 0,\ i = c+1, \ldots, d$;

iv.) $\tilde{\pi}_{1|Fix(H)}, \ldots, \tilde{\pi}_{c|Fix(H)}$ *are algebraic independent;*

v.) $\tilde{\pi}_1, \ldots, \tilde{\pi}_d$ *form a system of hom. parameters and* $\tilde{\sigma}_1 = 1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_t$ *form the secondary invariants, i.e. the invariant ring has a Hironaka decomposition* $\mathbf{R}[x]^G = \bigoplus_{i=1}^t \tilde{\sigma}_i \mathbf{R}[\tilde{\pi}]$.

vi.) $\tilde{\sigma}_{i|Fix(H)} \not\equiv 0, i = 2, \ldots, s$ *and* $\tilde{\sigma}_{i|Fix(H)} \equiv 0, i = s+1, \ldots, t$. *The number* $t - s$ *is the maximal number of secondaries vanishing on* $Fix(H)$.

Proof: : Basically this is Noether's normalization. Let $I \subset K[y_1, \ldots, y_r]$ be the ideal of relations. The polynomials $\pi_{i|\mathrm{Fix}(H)}, i = 1, \ldots, r$ fulfill relations $f \in \mathbf{R}[y_1, \ldots, y_r]$ with $f(\pi_{1|\mathrm{Fix}(H)}, \ldots, \pi_{r|\mathrm{Fix}(H)}) \equiv 0$ as well. Let this ideal of relations be denoted by J. Of course $I \subseteq J$. Since H is an isotropy subgroup $I \neq J$. So there is a non-trivial ideal $\mathcal{J} \subset \mathbf{R}[y]/I$. By Noether's Lemma 2.3.6 there exists elements $u_1, \ldots, u_d \in \mathbf{R}[y]/I$ given by $u_1 = g_1(y), \ldots, u_d = g_d(y)$ where g_i are representatives of rest classes in $\mathbf{R}[y]/I$ such that $\tilde{\pi}_1(x) = g_1(\pi(x)), \ldots, \tilde{\pi}_d(x) = g_d(\pi(x))$ are algebraic independent. Moreover, $\mathbf{R}[y]/I$ is integral over $\mathbf{R}[u_1, \ldots, u_d]$ and $\mathcal{J} \cap \mathbf{R}[u_1, \ldots, u_d] = \langle u_{c+1}, \ldots, u_d \rangle$. But $u_i \in \mathcal{J}$ means $\tilde{\pi}_{i|\mathrm{Fix}(H)} \equiv 0$ for $i = c+1, \ldots, d$ and $\mathcal{J} \cap \mathbf{R}[u] = \langle u_{c+1}, \ldots, u_d \rangle$ means that $\tilde{\pi}_{1|\mathrm{Fix}(H)}, \ldots, \tilde{\pi}_{c|\mathrm{Fix}(H)}$ are algebraic independent. Of course $u_1, \ldots, u_d$ may be chosen to be homogeneous which means that the g_i are weighted homogeneous with respect to W. Since $\mathbf{C}[y]/I$ is Cohen-Macaulay there exists elements $z_1 = 1, z_2, \ldots, z_t \in \mathbf{R}[y]/I$ given by $z_2 = h_2(y), \ldots, z_t = h_t(y)$ which generate $\mathbf{R}[y]/I$ as a free module over $\mathbf{R}[u_1, \ldots, u_d]$. Equivalently, $\hat{\sigma}_1 = 1, \hat{\sigma}_2(x) = h_2(\pi(x)), \ldots, \hat{\sigma}_t(x) = h_t(\pi(x))$ form a module basis of $\mathbf{R}[x]^G$ over $\mathbf{R}[\tilde{\pi}(x)]$ which generates $\mathbf{R}[x]^G$ as a free module. Moreover, h_j may be chosen to be homogeneous (with respect to W) which enables the special choice in vi.) by the following construction. The idea is to climb up degree by degree. Let $\tilde{\sigma}_2(x), \ldots, \tilde{\sigma}_l(x)$ be the secondaries of degree $< k$. The vector space $H_k(\mathbf{R}[x]^G)$ of homogeneous invariants of degree k is partially generated by $\tilde{\sigma}_2(x), \ldots, \tilde{\sigma}_l(x)$ over $\mathbf{R}[\tilde{\pi}_1(x), \ldots, \tilde{\pi}_d(x)]$. A vector space basis of a direct complement in $H_k(\mathbf{R}[x]^G)$ gives a choice of secondaries of degree k. The condition $\tilde{\sigma}_{|\mathrm{Fix}(H)} \equiv 0$ defines a subvector space of this direct complement. Choosing a vector space basis of this subvector space and a vector space basis of a complement gives the maximal number of secondaries of degree k which vanish on $\mathrm{Fix}(H)$.

Once all secondaries are found there are relations

$$f_{ij}(u, z) = z_i z_j - \sum_{k=2}^{t} z_k B_k^{ij}(u_1, \ldots, u_d) - B^{ij}(u), \quad 1 < i \leq j \leq t \tag{4.5}$$

with $f_{ij} \in \mathbf{R}[u_1, \ldots, u_d, z_2, \ldots, z_t]$ such that $f_{ij}(g(y), h(y)) \equiv 0$ in $\mathbf{R}[y]/I$ and as well $f_{ij}(\tilde{\pi}(x), \tilde{\sigma}(x)) \equiv 0$ in $\mathbf{R}[x]$. They are weighted homogeneous with respect to $U(u_i) = \deg(\tilde{\pi}_i(x)), i = 1, \ldots, c, U(z_j) = \deg(\tilde{\sigma}_j(x)), j = 2, \ldots, t$. The ideal $\tilde{I} \subset \mathbf{R}[u, z]$ of relations of $\tilde{\pi}$ and $\tilde{\sigma}$ is generated by the polynomials f_{ij} in (4.5). In these coordinates the ideal $\tilde{J}$ of relations for the restriction to $\mathrm{Fix}(H)$ is generated by $u_{c+1}, \ldots, u_d, z_{s+1}, \ldots, z_t$, the representatives of relations $f_{ij}(u, z)$ for $1 < i \leq j \leq s$ of type

$$z_i z_j - \sum_{k=2}^{t} z_k B_k^{ij}(u_1, \ldots, u_c, 0) - B^{ij}(u_1, \ldots, u_c, 0)\,,$$

and those for $1 < i \le t, s < j \le t$ of type

$$\sum_{k=2}^{s} z_k B_k^{ij}(u_1,\ldots,u_c,0) + B^{ij}(u_1,\ldots,u_c,0)\,,$$

being linear in z and may be some other elements $F_\nu \in \mathbf{R}[u_1,\ldots,u_c,z_2,\ldots,z_t]$ which are homogeneous with respect to U and linear in z. Observe that the third condition of Noether's lemma $\tilde{\mathcal{J}} \cap (\mathbf{R}[y,u]/\tilde{I}) = \langle u_1,\ldots,u_c\rangle$ is satisfied. □

Remark 4.3.4 *It is not guaranteed that the module generated by* $\tilde{\sigma}_1, \tilde{\sigma}_{2|Fix(H)},\ldots,\tilde{\sigma}_{s|Fix(H)}$ *over* $\mathbf{R}[\tilde{\pi}_{1|Fix(H)},\ldots,\tilde{\pi}_{c|Fix(H)}]$ *is free. But in many practical situations this will be the case. In turn this gives a simpler structure of the relations in* $\tilde{J}$. *Observe that in many cases there is still the group* $N_G(H)/H$ *acting on* $Fix(H)$. *The stratum corresponding to the orbit type* H *decomposes in the secondary stratification into its connected components. The group* $N_G(H)/H$ *is permuting these connected components.*

Now we like to discuss the special case that already sets of primary and secondary invariants $\pi_1(x),\ldots,\pi_d(x),\sigma_2(x),\ldots,\sigma_m(x)$ are given. Then the choice of the polynomials g_i, h_j is more special. Denote by $W(y_i) = \deg(\pi_i(x)), i = 1,\ldots,d, W(z_j) = \deg(\sigma_j(x)), j = 2,\ldots,m$ the weighted grading and by $U(y_i) = 0, i = 1,\ldots,d, U(z_j) = 1, j = 2,\ldots,m$ the Kronecker grading. Then

$$g_i, h_j \in H_0^U(K[y,z]) + H_1^U(K[y,z]),$$

and they are homogeneous with respect to W each. The restriction on the polynomials g_i is that

$$\mathrm{Rad}(\langle \pi_1(x),\ldots,\pi_d(x)\rangle) = \mathrm{Rad}(\langle \tilde{\pi}_1(x),\ldots,\tilde{\pi}_d(x)\rangle)\,,$$

since the parameters have the nullcone as common solutions. This imposes conditions on the coefficients of $g_1,\ldots,g_d$ which are easily checked by Gröbner bases since the radical containment is done by the Kantorovich trick (see [41] p. 177).

Example 4.3.5 *(Example 4.3.2 continued) For the* D_3*-action in Example 4.3.2 one may as well choose the invariants* $\tilde{\pi}_1 = \pi_1$, $\tilde{\sigma}_2 = \pi_2$, $\tilde{\pi}_2 = \pi_1^3 - \pi_2^2$ *with relation* $\tilde{\sigma}_2^2 - (\tilde{\pi}_1^3 - \tilde{\pi}_2)$. *Applying the rules above the strata are given by*

$$\begin{aligned}
S_{id} &= \{(\tilde{\pi}_1,\tilde{\pi}_2,\tilde{\sigma}_2) \in \mathbf{R}^3 \mid \ \sigma_2^2 - (\tilde{\pi}_1^3 - \tilde{\pi}_2) = 0,\ \tilde{\pi}_1 \ge 0,\ \tilde{\pi}_2 \ge 0,\ \tilde{\pi}_2 \ne 0\} \\
&= \{(\tilde{\pi}_1,\tilde{\pi}_2,\tilde{\sigma}_2) \in \mathbf{R}^3 \mid \ \tilde{\sigma}_2 = \pm\sqrt{\tilde{\pi}_1^3 - \tilde{\pi}_2},\ \tilde{\pi}_1 > 0,\ \tilde{\pi}_2 > 0\}, \\
S_{Z_2} &= \{(\tilde{\pi}_1,\tilde{\pi}_2,\tilde{\sigma}_2) \in \mathbf{R}^3 \mid \ \sigma_2^2 - (\tilde{\pi}_1^3 - \tilde{\pi}_2) = 0, \tilde{\pi}_1 \ge 0, \tilde{\pi}_2 \ge 0, \\
&\qquad\qquad\qquad\qquad\qquad\quad \tilde{\pi}_2 = 0, \pi_1 \ne 0, \tilde{\sigma}_2 \ne 0\ \} \\
&= \{(\tilde{\pi}_1,\tilde{\pi}_2,\tilde{\sigma}_2) \in \mathbf{R}^3 \mid \ \tilde{\sigma}_2 = \pm\sqrt{\tilde{\pi}_1^3},\ \tilde{\pi}_2 = 0,\ \tilde{\pi}_1 > 0\}, \\
S_{D_3} &= \{(0,0,0)\}.
\end{aligned}$$

Figure 4.2 is showing the stratification in these coordinates in comparison to the original coordinates. The symmetry adapted coordinates correspond to a subdivision of the symmetry cell in $\mathbf{R}^2$.

Example 4.3.6 *(Examples 2.2.4 and 2.4.3 continued) In Example 2.4.3 we found the five primary invariants $\pi_1, \pi_2, \pi_3, p_{68}, p_{48}$ and the integral elements $\pi_4, \pi_5, \pi_7, \pi_8, \pi_9, \pi_{10}, \pi_{11}$ of a torus action which are real generators of the real invariant ring.*

In order to describe the image of the Hilbert mapping we need the matrix B of Theorem 4.2.1 which is in this case a 12×12 matrix whose entries are polynomials in π, p_{68}, p_{48}. In Example 2.2.4 the invariants are computed in the 'complex' *coordinates $(x_1, x_2, x_3, \bar{x}_1, \bar{x}_2, \bar{x}_3)$. Changing to the* 'real' *coordinates $x_j = xr_j + i \cdot xi_j$ enables the computation of the gradients and thus $b_{ij}(x) = (\nabla \pi_i)^t \nabla \pi_j$. As usual a Gröbner basis of the ideal $\langle \pi_1 - \pi_1(x), \ldots, \rangle$ gives the representation of the b_{ij} in terms of invariants by the division algorithm. I have chosen the matrix term order with order of variables $xr_1, \ldots, xi_3, \pi_1, \pi_2, \pi_3, p_{68}, p_{48}, \pi_4, \pi_5, \pi_7, \pi_8, \pi_9, \pi_{10}, \pi_{11}$ and matrix*

$$\left[\begin{array}{cccccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 6 & 2 & 2 & 3 & 3 & 3 & 3 & 3 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & -2 & -3 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\multicolumn{6}{c}{\mathbf{0}_{4,6}} & \multicolumn{4}{c}{-Id_{4,4}} & \multicolumn{8}{c}{\mathbf{0}_{4,8}} \\
\multicolumn{11}{c}{\mathbf{0}_{6,11}} & \multicolumn{6}{c}{Id_{6,6}} & \mathbf{0}_6 \\
\mathbf{0}_5 & \multicolumn{5}{c}{-Id_{5,5}} & \multicolumn{12}{c}{\mathbf{0}_{5,12}}
\end{array}\right]$$

since it is eliminating $xr_1, \ldots, xi_3$ and has the properties on $K[\pi]$ as required in Lemma 2.3.13 distinguishing between parameters and integral elements. Restriction to degree 10 *with respect to the induced weighted grading may be used (9274 sec). The computation of derivatives and normal forms takes 13270 sec yielding*

$$B = \begin{pmatrix} B_{1,4} & B^1 & B^2 \\ {B^1}^t & B_{5,8} & B^3 \\ {B^2}^t & {B^3}^t & B_{9,12} \end{pmatrix},$$

with

$$B_{1,4} = \begin{bmatrix} \pi_1 & 0 & 0 & -\pi_8 + p_{68} \\ 0 & \pi_2 & 0 & \pi_8 \\ 0 & 0 & \pi_3 & \frac{p_{68}}{2} \\ -\pi_8 + p_{68} & \pi_8 & \frac{p_{68}}{2} & 4\,\pi_1\pi_3 + 2\,\pi_2\pi_1 + {\pi_1}^2 + 4\,\pi_2\pi_3 + {\pi_2}^2 - {\pi_5}^2 \end{bmatrix}$$

and $B_{9,12}$ equal to

$$\begin{bmatrix} 4\,\pi_2\pi_3 + {\pi_2}^2 & 0 & \pi_4\pi_3 + \frac{\pi_4\pi_2}{2} & \pi_5\pi_3 + \frac{\pi_5\pi_2}{2} \\ 0 & 4\,\pi_2\pi_3 + {\pi_2}^2 & -\pi_5\pi_3 - \frac{\pi_5\pi_2}{2} & \pi_4\pi_3 + \frac{\pi_4\pi_2}{2} \\ \pi_4\pi_3 + \frac{\pi_4\pi_2}{2} & -\pi_5\pi_3 - \frac{\pi_5\pi_2}{2} & \pi_1\pi_3 + \pi_2\pi_1 + \pi_2\pi_3 & 0 \\ \pi_5\pi_3 + \frac{\pi_5\pi_2}{2} & \pi_4\pi_3 + \frac{\pi_4\pi_2}{2} & 0 & \pi_1\pi_3 + \pi_2\pi_1 + \pi_2\pi_3 \end{bmatrix}.$$

Then

$$\begin{aligned} \pi(\mathbf{R}^6) = \{ y \in \mathbf{R}^{12} \mid \quad & y_1 \geq 0, y_2 \geq 0, y_3 \geq 0 + \textit{conditions}, \\ & y_i = g_i^j(y_1, \ldots, y_5), i = 6, \ldots, 12, \ \textit{for a } j \in \{1, \ldots, 18\} \}. \end{aligned}$$

Since $Fix(Z_2) = \{(0,0,x_3,0,0,\bar{x}_3) \in \mathbf{C}^6\} \simeq \{(0,0,0,0,xr_3,xi_3) \in \mathbf{R}^6\}$ *the stratification is given by*

$$\begin{array}{rcll} S_{Id} &=& \{y \in \mathbf{R}^{12} \,| & y_1 > 0, y_2 > 0, y_3 \geq 0, y_4 \neq 0, y_5 \neq 0, +conditions, \\ & & & y_i = g_i^j(y_1,\ldots,y_5), i = 6,\ldots,12, j = 1,\ldots,18\}, \\ S_{Z_2} &=& \{y \in \mathbf{R}^{12} \,| & y_i = 0, i \in \{1,\ldots,12\} \setminus \{3\},\, y_3 > 0\}, \\ S_T &=& \{y \in \mathbf{R}^{12} \,| & y = 0\}. \end{array}$$

Since Birkhoff normal forms of Hamiltonian systems in resonance have the symmetry of the associated torus action the Hamiltonian function H *is assumed to be invariant with respect to this torus action as well. Intersections of* $\pi(\mathbf{R}^6)$ *with* $H \equiv const.$ *are sets with equal energy or analog. Since solutions of Hamiltonian systems preserve the Hamiltonian (e.g. energy) it is interesting to study all possible intersections for arbitrary invariant Hamiltonians. This classifies the dynamics of Hamiltonian systems close to resonance.*

Example 4.3.7 *In [32] Chossat investigates the group* $O(2)$ *acting on the vector space* $\mathbf{R}^4 \simeq \{z \in \mathbf{C}^4 \mid z_3 = \bar{z}_1,\, z_4 = \bar{z}_2\}$ *given by*

$$\begin{array}{rcl} \vartheta(\phi)(z_1,z_2) &=& (e^{i\phi}z_1, e^{2i\phi}z_2), \\ \vartheta(\kappa)(z_1,z_2) &=& (\bar{z}_1,\bar{z}_2). \end{array}$$

The Molien series is computed in 23 *sec to be* $1/(1-\lambda^3)(1-\lambda^2)^2$. *By Algorithm 2.1.10 the Hilbert basis* $\pi_1(z) = z_1\bar{z}_1, \pi_2(z) = z_2\bar{z}_2, \pi_3(z) = z_1^2\bar{z}_2 + \bar{z}_1^2 z_2$ *is computed in* 7.5 *sec. The isotropy subgroups are* $Z_2(\pi), Z_2(\kappa), Id$ *with* $Fix(Z_2(\pi)) = \{(0,z_2,0,\bar{z}_2)\}$ *and the fixed point space* $Fix(Z_2(\kappa)) = \{(a,b,a,b) \,|\, a,b \in \mathbf{R}\}$. *The relations for the restriction to* $Fix(Z_2(\pi))$ *are generated by* π_1 *and* π_3. *For* $Fix(Z_2(\kappa))$ *the generator is* $4\pi_1^2\pi_2 - \pi_3^2$. *The subgroup* $Z_2 \times Z_2$ *is an isotropy subgroup as well. Since the fixed point spaces of the conjugates fill all of* $Fix(Z_2(\pi))$ *this subgroup is not considered.*

For the subgroup tower $Id \subset Z_2(\pi) \subset O(2)$ *the invariants are symmetry adapted according to Lemma 4.3.3, namely the relations for the restriction to* $Fix(Z_2(\pi))$ *are generated by* $\langle \pi_1, \pi_3 \rangle$.

Since the matrix B *is given by*

$$B(\pi) = \begin{bmatrix} \pi_1 & 0 & \pi_3 \\ 0 & \pi_2 & \frac{\pi_3}{2} \\ \pi_3 & \frac{\pi_3}{2} & {\pi_1}^2 + 4\,\pi_1\pi_2 \end{bmatrix}$$

$$with \quad \det(B) = -\frac{(\pi_1 + 4\,\pi_2)\,(-4\,\pi_2{\pi_1}^2 + {\pi_3}^2)}{4},$$

the strata in these coordinates are

$$\begin{array}{rcl} S_{Id} &=& \{0 \leq \pi_1, 0 \leq \pi_2, 0 \leq -(\pi_1 + 4\,\pi_2)\,(-4\,\pi_2{\pi_1}^2 + {\pi_3}^2), \\ & & \pi_1 \neq 0, \pi_3 \neq 0, 4\,\pi_2{\pi_1}^2 - {\pi_3}^2 \neq 0\} \\ &=& \{0 < \pi_1, 0 < \pi_2, 0 < 4\,\pi_2{\pi_1}^2 - {\pi_3}^2\}, \\ S_{Z_2(\pi)} &=& \{\pi_1 = 0, \pi_3 = 0, \pi_2 > 0\}, \\ S_{Z_2(\kappa)} &=& \{\pi_1 > 0, \pi_2 > 0, 4\pi_1^2\pi_2 - \pi_3^2 = 0\}, \\ S_{O(2)} &=& \{(0,0,0)\}. \end{array}$$

The Hilbert series of the module of equivariants is $(2\lambda + 2\lambda^2)/(1-\lambda^3)(1-\lambda^2)^2$ *which is derived in* 19 *sec. The four equivariants*

$$\begin{pmatrix} z_1 \\ 0 \\ \bar{z}_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ z_2 \\ 0 \\ \bar{z}_2 \end{pmatrix}, \quad \begin{pmatrix} \bar{z}_1 z_2 \\ 0 \\ z_1 \bar{z}_2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ z_1^2 \\ 0 \\ \bar{z}_1^2 \end{pmatrix},$$

are computed in 298 *sec by Algorithm 2.1.18. The equation with generic equivariant* $f = \sum_{i=1}^4 A_i(\pi) b_i$ *is projected on the orbit space to*

$$\begin{aligned} \dot{\pi}_1 &= 2A_1\pi_1 + A_3\pi_3\,, \\ \dot{\pi}_2 &= 2A_2\pi_2 + A_4\pi_3\,, \\ \dot{\pi}_3 &= 2A_4\pi_1^2 + 4A_3\pi_1\pi_2 + (2A_1 + A_2)\pi_3\,. \end{aligned}$$

The restriction to $Fix(Z_2(\pi))$ *is just*

$$\dot{\pi}_2 = 2A_2(0,\pi_2,0)\,\pi_2 \qquad \textit{and} \qquad \pi_1 = 0,\, \pi_3 = 0.$$

For the restriction to $Fix(Z_2(\kappa))$ *of the differential equations on the orbit space one needs to use a parameterization. Or one chooses a symmetry adapted set of invariants according to the subgroup tower* $Id \subset Z_2(\kappa) \subset O(2)$. *For the invariants* $\tilde{\pi}_1 = \pi_1, \tilde{\pi}_2 = \pi_2, \tilde{\pi}_3 = 4\pi_1^2\pi_2 - \pi_3^2, \tilde{\sigma}_2 = \pi_3$ *we have the relation* $\tilde{\sigma}_2^2 = -\tilde{\pi}_3 + 4\tilde{\pi}_1^2\tilde{\pi}_2$. *Then the strata are written as*

$$\begin{aligned} S_{Id} &= \{\tilde{\pi}_1,\tilde{\pi}_2,\tilde{\pi}_3,\tilde{\sigma}_2) \in \mathbf{R}^4 \mid \tilde{\pi}_1 > 0, \tilde{\pi}_2 > 0, \tilde{\pi}_3 > 0, \tilde{\sigma}_2 = \pm\sqrt{4\tilde{\pi}_1^2\tilde{\pi}_2 - \tilde{\pi}_3}\,\}, \\ S_{Z_2(\pi)} &= \{\tilde{\pi}_1,\tilde{\pi}_2,\tilde{\pi}_3,\tilde{\sigma}_2) \in \mathbf{R}^4 \mid \tilde{\pi}_1 = 0, \tilde{\pi}_2 > 0, \tilde{\pi}_3 = 0, \tilde{\sigma}_2 = 0\}, \\ S_{Z_2(\kappa)} &= \{\tilde{\pi}_1,\tilde{\pi}_2,\tilde{\pi}_3,\tilde{\sigma}_2) \in \mathbf{R}^4 \mid \tilde{\pi}_1 > 0, \tilde{\pi}_2 > 0, \tilde{\pi}_3 = 0, \tilde{\sigma}_2 = \pm\sqrt{4\tilde{\pi}_1^2\tilde{\pi}_2}\,\}, \\ S_{O(2)} &= \{0\}. \end{aligned}$$

For the generic equivariant $f = \sum_{i=1}^4 \tilde{A}_i(\tilde{\pi},\tilde{\sigma}) b_i$ *the equations on the orbit space read*

$$\begin{aligned} \dot{\tilde{\pi}}_1 &= 2\tilde{A}_1\tilde{\pi}_1 + \tilde{A}_3\tilde{\sigma}_2\,, \\ \dot{\tilde{\pi}}_2 &= 2\tilde{A}_2\tilde{\pi}_2 + \tilde{A}_4\tilde{\sigma}_2\,, \\ \dot{\tilde{\pi}}_3 &= (4\tilde{A}_1 + 2\tilde{A}_2)\tilde{\pi}_3\,, \\ \dot{\tilde{\sigma}}_2 &= 2\tilde{A}_4\tilde{\pi}_1^2 + 4\tilde{A}_3\tilde{\pi}_1\tilde{\pi}_2 + (2\tilde{A}_1 + \tilde{A}_2)\tilde{\sigma}_2\,, \\ \tilde{\sigma}_2 &= \pm\sqrt{-\tilde{\pi}_3 + 4\tilde{\pi}_1^2\tilde{\pi}_2}\,. \end{aligned}$$

Then the equations on $Fix(Z_2(\kappa))$ *are easily derived as*

$$\begin{aligned} \dot{\tilde{\pi}}_1 &= 2\tilde{A}_1\tilde{\pi}_1 + \tilde{A}_3\tilde{\sigma}_2\,, \\ \dot{\tilde{\pi}}_2 &= 2\tilde{A}_2\tilde{\pi}_2 + \tilde{A}_4\tilde{\sigma}_2\,, \\ \dot{\tilde{\sigma}}_2 &= 2\tilde{A}_4\tilde{\pi}_1^2 + 4\tilde{A}_3\tilde{\pi}_1\tilde{\pi}_2 + (2\tilde{A}_1 + \tilde{A}_2)\tilde{\sigma}_2\,, \\ \tilde{\sigma}_2 &= \pm\sqrt{4\tilde{\pi}_1^2\tilde{\pi}_2}\,, \end{aligned}$$

where $\tilde{A}_i = \tilde{A}_i(\tilde{\pi}_1,\tilde{\pi}_2,0,\tilde{\sigma}_2)$ *do not depend on* $\tilde{\pi}_3$. *The stability of an equilibrium* x *with isotropy* $Z_2(\kappa)$ $((\tilde{\pi}_1,\tilde{\pi}_2,\tilde{\pi}_3,\tilde{\sigma}_2) = (\tilde{\pi}_1(x),\tilde{\pi}_2(x),0,\tilde{\sigma}_2(x)))$ *outside of* $Fix(Z_2(\kappa))$ *is determined by the expression* $(4\tilde{A}_1 + 2\tilde{A}_2)$.

This example clearly illustrates the advantage of choosing appropriate coordinates of the invariant ring. The restriction to the flow-invariant fixed point spaces is obvious on the orbit space. The second advantage is the transparance of the description of the strata. They are mainly described by linear equalities and linear inequalities in the parameters. The secondaries are expressions in the parameters. The third advantage concerns the computation of stability. If an equilibrium has isotropy H then one distinguishes between stability inside of $\mathrm{Fix}(H)$ and outside of $\mathrm{Fix}(H)$. On the orbit space the restriction to $\mathrm{Fix}(H)$ corresponds to the restriction to $\pi_1, \ldots, \pi_c, \sigma_2, \ldots, \sigma_s$ and the corresponding differential equations $\dot{\pi}_j = F_j(\pi, \sigma), j = 1, \ldots, c,\ \dot{\sigma}_j = F_j^s(\pi, \sigma), j = 2, \ldots, s$. The restricted Jacobian

$$\begin{pmatrix} \frac{\partial F_j}{\partial \pi_i} & \frac{\partial F_j}{\partial \sigma_l} \\ \frac{\partial F_k^s}{\partial \pi_i} & \frac{\partial F_k^s}{\partial \sigma_l} \end{pmatrix}, \quad i, j = 2, \ldots, c,\ k, l = 1, \ldots, s,$$

and their eigenvalues determine the stability in $\mathrm{Fix}(H)$. Choosing a subgroup $H_2 \subset H$ and the associated invariants one finds the stability in $\mathrm{Fix}(H_2)$, but outside of $\mathrm{Fix}(H)$. Here one profits from the fact that the right hand side F has a special form.

Changing the coordinates $\tilde{\pi}_i = g_i(\pi, \sigma), \tilde{\sigma}_j = h_j(\pi, \sigma)$ (linear in σ) also changes the representation of the vector field

$$\begin{aligned} f(x) &= \sum_{i=1}^{s} A_i(\pi(x), \sigma(x)) \cdot b_i(x) \\ &= \sum_{i=1}^{s} \left(A_i^1(\pi(x)) + \sum_{k=2}^{\tau} A_i^k(\pi(x)) \cdot \sigma_k(x) \right) b_i(x), \end{aligned}$$

to

$$\begin{aligned} f(x) &= \sum_{i=1}^{s} \tilde{A}_i(\tilde{\pi}(x), \tilde{\sigma}(x)) \cdot b_i(x) \\ &= \sum_{i=1}^{s} \left(\tilde{A}_i^1(\pi(x)) + \sum_{k=2}^{t} \tilde{A}_i^k(\tilde{\pi}(x)) \cdot \tilde{\sigma}_k(x) \right) b_i(x), \end{aligned}$$

by $\tilde{A}_i(\tilde{\pi}, \tilde{\sigma}) = A_i(\theta(\tilde{\pi}, \tilde{\sigma}), \rho(\tilde{\pi}, \tilde{\sigma}))$ where θ, ρ expresses the inverse nonlinear change of coordinates. For a unique representation of the vector field see the Stanley decomposition in Section 2.4.

With this change of coordinates also the differential equations on the orbit space changes. Either one evaluates

$$\dot{\tilde{\pi}}_j = \sum_{i=1}^{s} \tilde{A}_i(\tilde{\pi}, \tilde{\sigma}) (\nabla \tilde{\pi}_j)^t b_i = \sum_{i=1}^{s} \tilde{A}_i(\tilde{\pi}, \tilde{\sigma}) b_{ij}(\tilde{\pi}, \tilde{\sigma}),$$

$$\dot{\tilde{\sigma}}_j = \sum_{i=1}^{s} \tilde{A}_i(\tilde{\pi}, \tilde{\sigma}) (\nabla \tilde{\sigma}_j)^t b_i = \sum_{i=1}^{s} \tilde{A}_i(\tilde{\pi}, \tilde{\sigma}) b_{ij}^s(\tilde{\pi}, \tilde{\sigma}),$$

by first computing $\nabla \tilde{\pi}_j, \nabla \tilde{\sigma}_j$, then the inner products with the equivariants b_i, applying the division algorithm in order to receive representations in the invariants, multiplication with the coefficients $\tilde{A}_i$ and finally using the relations again gives unique descriptions.

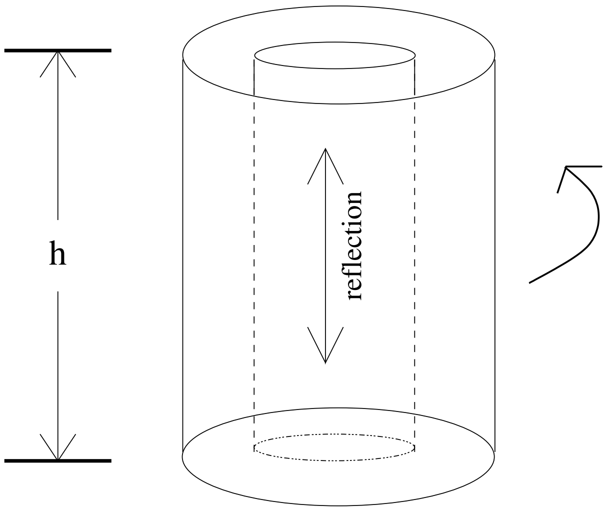

Figure 4.3: In the Taylor-Couette experiment the velocity of a liquid between two centered cylinders is studied. Depending on the velocity of the rotating cylinder different pattern are observed

Or one uses the already computed differential equations $\dot{\pi}_j = F_j(\pi, \sigma), j = 1, \ldots, d$, $\dot{\sigma}_j = F_j^s(\pi, \sigma), j = 2, \ldots, \tau$. By the chain rule

$$\begin{aligned}
\dot{\tilde{\pi}}_i &= \sum_{j=1}^{d} \frac{\partial g_i}{\partial \pi_j} \dot{\pi}_j + \sum_{j=2}^{\tau} \frac{\partial g_i}{\partial \sigma_j} \dot{\sigma}_j \;=\; \sum_{j=1}^{d} \frac{\partial g_i}{\partial \pi_j} F_j + \sum_{j=2}^{\tau} \frac{\partial g_i}{\partial \sigma_j} F_j^s \\
&= k_i(\pi, \sigma) \;=\; k_i(\theta(\tilde{\pi}, \tilde{\sigma}), \rho(\tilde{\pi}, \tilde{\sigma})), \\
\dot{\tilde{\sigma}}_i &= \sum_{j=1}^{d} \frac{\partial h_i}{\partial \pi_j} \dot{\pi}_j + \sum_{j=2}^{\tau} \frac{\partial h_i}{\partial \sigma_j} \dot{\sigma}_j \;=\; \sum_{j=1}^{d} \frac{\partial h_i}{\partial \pi_j} F_j + \sum_{j=2}^{\tau} \frac{\partial h_i}{\partial \sigma_j} F_j^s \\
&= k_i^s(\pi, \sigma) \;=\; k_i^s(\theta(\tilde{\pi}, \tilde{\sigma}), \rho(\tilde{\pi}, \tilde{\sigma})),
\end{aligned}$$

for $i = 1, \ldots, d$ and $i = 1, \ldots, t$ is obtained. Observe that the derivatives $\frac{\partial g_i}{\partial \pi_j}, \frac{\partial g_i}{\partial \sigma_j}, \frac{\partial h_i}{\partial \sigma_j}, \frac{\partial h_i}{\partial \sigma_j}$ are needed only. Use of relations implies the unique representation once more.

There is also a second aspect concerning the elimination of lower order terms analog to Birkhoff normal form. If the symmetry adapted coordinates of Lemma 4.3.3 (for a tower of subgroups) are not unique the coordinates may be chosen such that some lower order terms in the differential equations vanishes.

Finally, I give an example with relevant application.

Example 4.3.8 *The Taylor-Couette experiment is the topic of the book [35]. It is described in [87] pp. 485ff as well. A liquid is between two centered cylinders. While the outer cylinder is rotating the velocity field of the liquid shows interesting pattern varying with the velocity of the cylinder. The velocity field satisfies the Navier-Stokes equations on the cross section times an interval $[0, h]$ fulfilling no-slip boundary conditions on the cylinders and periodic boundary conditions in axial direction. Thus the problem has the symmetry of $O(2) \times SO(2)$ where the reflection is the reflection with respect to a cross section, see Figure 4.3. In order to study bifurcations the linearization of the PDE around the trivial Couette flow is considered and a Fourier expansion is performed.*

Table 4.4: Correspondence of steady states and periodic orbits as solutions of the differential equations on the orbit space to the solutions of the system of differential equations obtained by center manifold reduction

name	meaning	on orbit space	isotropy	strata
Couette flow	steady state	equilibria	$O(2) \times S_1$	$S_{O(2)\times S_1}$
Taylor vortex	steady state	equilibria	$Z_2(\kappa) \times S_1$	$S_{Z_2(\kappa)\times S_1}$
spiral wave flow	periodic orbit (rotating wave)	equilibria	$\widetilde{SO(2)}$	$S_{\widetilde{SO(2)}}$
ribbon flow	periodic orbit (standing wave)	equilibria	$Z_2(\kappa) \times Z_2(\pi, \pi)$	$S_{Z_2\times Z_2}$
wavy vortex flow	periodic orbit	equilibria	$Z_2(\kappa\pi, \pi)$	$S_{Z_2(\kappa\pi,\pi)}$
twisted vortex flow	periodic orbit	equilibria	$Z_2(\kappa)$	$S_{Z_2(\kappa)}$
mod. spiral wave	quasiperiodic	equilibria	Id	S_{Id}
quasiperiodic drift	quasiperiodic	equilibria	$Z_2(\pi, \pi)$	$S_{Z_2(\pi,\pi)}$
2-tori + drift	3-tori	periodic orbit		$S_{Z_2(\kappa)}$

A center manifold is a manifold being tangent to the generalized eigenspace of eignevalues on the imaginary axis attracting the dynamics in a neighborhood. Thus a small dynamical system on the center manifold is representing the essential dynamics. The symmetry of the eigenspace inherits to equivariance of the dynamical system. Taylor expansions of the center manifold mapping and the dynamical system may be computed. However, often one starts with a generic polynomial equivariant vector field on the center manifold and studies its typical behavior. Thus one can read off bifurcation behavior of the Navier-Stokes equations once one has calculated a Taylor expansion of the reduced dynamical system on the center manifold. If the group action on the generalized eigenspace (with respect to eingenvalues 0 and $\pm\omega i$) is the sum of two real irreducible representations one calls this situation mode interaction.

The equations on the center manifold for a mode interaction problem has been investigated in [144, 163, 168] with the method of orbit space reduction. The group action of this problem is a real representation of $O(2) \times S_1$ on $\{(z_0, z_1, z_2, z_3, z_4, z_5) \in \mathbf{C}^6 \mid z_3 = \bar{z}_0, z_4 = \bar{z}_1, z_5 = \bar{z}_2\}$ given by

$$\begin{array}{rcl} \vartheta(\kappa)(z_0, z_1, z_2) & = & (\quad \bar{z}_0, \quad z_2, \quad z_1 \ ), \\ \vartheta(\varphi)(z_0, z_1, z_2) & = & (\ e^{i\varphi} z_0, \ e^{i\varphi} z_1, \ e^{-i\varphi} z_2 \ ), \\ \vartheta(\theta)(z_0, z_1, z_2) & = & (\quad z_0, \ e^{i\theta} z_1, \ e^{i\theta} z_2 \ ), \end{array}$$

where $SO(2) \simeq S_1$ describes the geometry of the apparatus (two cylinders) and $O(2)$ comes in by assuming a cylinder of infinite length with periodic boundary conditions. The isotropy groups (orbit types) and the corresponding fixed point spaces are given in Figure 4.4 and Table 4.5.

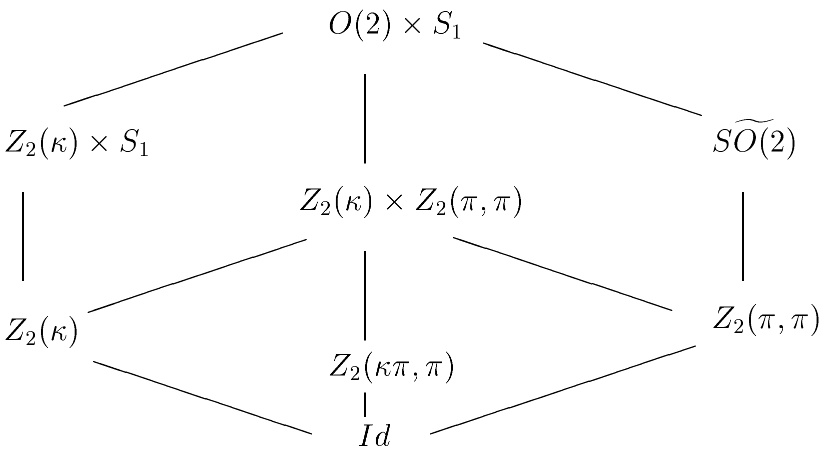

Figure 4.4: Isotropy subgroup lattice of the representation of $O(2) \times S_1$

The question of the generic equivariant vector field has been treated with the methods in Chapter 2. The Molien series

$$\frac{\lambda^4 - \lambda^2 + 1}{(1-\lambda^4)(1-\lambda^2)^3} = \frac{\lambda^6 + 1}{(1-\lambda^2)^2 (1-\lambda^4)^2} ,$$

has been computed within 217 sec. The Hilbert basis is derived by Algorithm 2.1.10 within 6309 sec to be

$$[\bar{z}_0 z_0, \bar{z}_1 z_1 + \bar{z}_2 z_2, z_2 \bar{z}_1 {z_0}^2 + \bar{z}_2 z_1 \bar{z}_0^2, \bar{z}_2 z_2 \bar{z}_1 z_1, \bar{z}_0^2 {z_1}^2 \bar{z}_1 \bar{z}_2 + \bar{z}_1 {z_0}^2 {z_2}^2 \bar{z}_2].$$

The first 4 *are real functions which we denote by* $\pi_1, \pi_2, \pi_3, \pi_4$.
With $\pi_5 = i\,(z_2 \bar{z}_1 {z_0}^2 - \bar{z}_2 z_1 \bar{z}_0^2)\,(\bar{z}_2 z_2 - \bar{z}_1 z_1)$ *derived from the last invariant we have a Hilbert basis of the invariant ring over the reals which differs from the Hilbert basis given in [87] p. 459 in the third and fourth element.*

There is only one relation $({\pi_2}^2 - 4\,\pi_4)\,(4\,{\pi_1}^2 \pi_4 - {\pi_3}^2) - {\pi_5}^2$ *derived in 4081 sec. This computation is an example of a computation over an algebraic extension as discussed in Section 1.2.4. The extension can be avoided by first converting to real coordinates. But then the computation of the relation needs 13484 sec. Obviously,* $\pi_1, \pi_2, \pi_3, \pi_4$ *form a homogeneous system of parameters and* π_5 *can be chosen as secondary invariant.*

Using the fixed point spaces as in Table 4.5 and computing the relations of the restricted invariants one sees that these invariants are symmetry adapted with respect to $\widetilde{SO}(2)$ *and* $Z_2(\pi, \pi)$. *The relations for* $Z_2(\pi, \pi)$ *are generated by* π_1, π_3, π_5. *For* $Z_2(\pi, \pi) \times Z_2(\kappa)$ *the generators are* $\pi_1, \pi_3, \pi_5, \pi_2^2 - 4\pi_4$. *For* $\widetilde{SO}(2)$ *the set* $\pi_1, \pi_3, \pi_4, \pi_5$ *form generators. Some special strata are given by*

$$\begin{aligned}
S_{Z_2(\pi,\pi)} &= \{\pi \in \mathbf{R}^5 \mid \pi_1 = 0, \pi_3 = 0, \pi_5 = 0, \pi_4 \neq 0, \pi_2^2 - 4\pi_4 \neq 0\}, \\
S_{Z_2 \times Z_2} &= \{\pi \in \mathbf{R}^5 \mid \pi_1 = 0, \pi_3 = 0, \pi_5 = 0, \pi_2^2 - 4\pi_4 = 0, \pi_2 \neq 0\}, \\
S_{\widetilde{SO}(2)} &= \{\pi \in \mathbf{R}^5 \mid \pi_1 = 0, \pi_3 = 0, \pi_4 = 0, \pi_5 = 0, \pi_2 \neq 0\}
\end{aligned}$$

In order to find the generic equivariant vector field the Hilbert series of the module of equivariants $\frac{3\,\lambda}{(1-\lambda^2)^4} = \frac{3\,\lambda^5 + 6\,\lambda^3 + 3\,\lambda}{(1-\lambda^2)^2 (1-\lambda^4)^2}$ *is derived within 340 sec.*

Table 4.5: Isotropy groups of the repr. of $O(2) \times S_1$ and their fixed point spaces

Isotropy	Fixed point space	dim Fix
$O(2) \times S_1$	$\{(0,0,0)\}$	0
$Z_2(\kappa) \times S_1$	$\{(z_0,0,0) \mid z_0 = a \in \mathbf{R}\}$	1
$\widetilde{SO}(2)(\varphi = -\theta)$	$\{(0,0,z_2)\}$	2
$Z_2(\kappa) \times Z_2(\pi,\pi)$	$\{(0,z_1,z_2) \mid z_1 = z_2\}$	2
$Z_2(\kappa)$	$\{(z_0,z_1,z_2) \mid z_0 = a \in \mathbf{R},\ z_1 = z_2\}$	3
$Z_2(\pi,\pi)$	$\{(0,z_1,z_2)\}$	4
$Z_2(\kappa\pi,\pi)$	$\{(z_0,z_1,z_2) \mid z_0 = ib,\ b \in \mathbf{R},\ z_1 = z_2\}$	3
Id	$\{(z_0,z_1,z_2)\} = \mathbf{R}^6$	6

Thus Algorithm 2.3.21 yields the equivariants of Table 4.6 in 188 sec. A coordinate transformation $z = Ax$ to the normal embedding of $\mathbf{R}^6$ in $\mathbf{C}^6$ yields 12 equivariants $b_i(x) = A^{-1}f_i(Ax)$ with real coefficients and the invariants $\pi_j(x) = \pi_j(Ax)$. The module of equivariants is a free module over $\mathbf{R}[\pi_1(x), \pi_2(x), \pi_3(x), \pi_4(x)]$ and thus a generic equivariant polynomial vector field has a unique representation

$$f(x) = \sum_{i=1}^{12} A_i(\pi(x)) \cdot b_i(x).$$

The reduction onto orbit space is given by $\dot{\pi_j} = \nabla\pi_j f = g_j(\pi)$, $j = 1, \ldots, 5$. The polynomials are easily determined by use of Gröbner bases. Once the Gröbner basis of $\langle y_1 - \pi_1(x), \ldots, y_5 - \pi_5(x)\rangle$ with respect to an elimination order (which also considers that π_5 is a secondary invariant as in [176] Alg. 2.5.6) is known the polynomials g_j are derived by the division algorithm. (Observe that one can use the Hilbert series driven version 1.2.19 and the truncation with respect to degree 10 *with respect to the induced weighted grading). The computation of the* 5 *gradients, scalar products, and normal forms is done within 2642 sec yielding*

$$\begin{aligned}
\dot{y}_1 &= 2\,A_1 y_1 + A_5 y_3 + A_{10} y_5 \\
\dot{y}_2 &= 2\,A_9 {y_2}^2 + A_{12} y_2 y_3 + 2\,A_3 y_2 + 2\,A_7 y_3 - 4\,A_9 y_4 - A_{11} y_5 \\
\dot{y}_3 &= 2\,A_7 {y_1}^2 y_2 + 4\,A_{12} {y_1}^2 y_4 + 4\,A_5 y_1 y_4 + A_9 y_2 y_3 + 2\,A_1 y_3 + 2\,A_3 y_3 \\
&\quad - (2\,A_4 - A_8)\, y_5, \\
\dot{y}_4 &= A_7 y_2 y_3 + 2\,A_9 y_2 y_4 + 2\,A_{12} y_3 y_4 + 4\,A_3 y_4 + A_6 y_5 \\
\dot{y}_5 &= 2\,A_6 {y_1}^2 {y_2}^2 - 4\,A_{11} {y_1}^2 y_2 y_4 - 16\,A_6 {y_1}^2 y_4 + 4\,A_{10} y_1 {y_2}^2 y_4 - 16\,A_{10} y_1 {y_4}^2 \\
&\quad - A_8 {y_2}^2 y_3 + 2\,A_4 {y_2}^2 y_3 + A_{11} y_2 {y_3}^2 + 2\,A_6 {y_3}^2 - 8\,A_4 y_3 y_4 + 4\,A_8 y_3 y_4 \\
&\quad + (3\,A_9 y_2 + A_{12} y_3 + 4\,A_3 + 2\,A_1)\, y_5\ .
\end{aligned}$$

By the choice of the special term order the invariant π_5 appears only linearly. Since the coordinates are symmetry adapted according to $Z_2(\pi,\pi)$ and $\widetilde{SO}(2)$ the restrictions to

Table 4.6: Equivariants of $O(2) \times S_1$

$$
\begin{aligned}
[\ & [z_0, \bar{z}_0, 0, 0, 0, 0],\\
& [0, 0, i\, z_1, -i\, \bar{z}_1, i\, z_2, -i\, \bar{z}_2],\\
& [0, 0, z_1, \bar{z}_1, z_2, \bar{z}_2],\\
& [i\, z_0 \bar{z}_1 z_1 - i\, z_0 \bar{z}_2 z_2, -i\, \bar{z}_0 \bar{z}_1 z_1 + i\, \bar{z}_0 \bar{z}_2 z_2, 0, 0, 0, 0],\\
& [\bar{z}_2 z_1 \bar{z}_0, z_2 \bar{z}_1 z_0, 0, 0, 0, 0],\\
& [0, 0, i\, z_2 {z_0}^2, -i\, \bar{z}_2 \bar{z}_0^2, i\, z_1 \bar{z}_0^2, -i\, \bar{z}_1 {z_0}^2],\\
& [0, 0, z_2 {z_0}^2, \bar{z}_2 \bar{z}_0^2, z_1 \bar{z}_0^2, \bar{z}_1 {z_0}^2],\\
& [0, 0, i\, \bar{z}_1 {z_1}^2, -i\, \bar{z}_1^2 z_1, i\, {z_2}^2 \bar{z}_2, -i\, \bar{z}_2^2 z_2],\\
& [0, 0, \bar{z}_1 {z_1}^2, \bar{z}_1^2 z_1, {z_2}^2 \bar{z}_2, \bar{z}_2^2 z_2],\\
& [i\, \bar{z}_0 \bar{z}_2 \bar{z}_1 {z_1}^2 - i\, \bar{z}_0 \bar{z}_2^2 z_2 z_1, -i\, z_0 z_2 \bar{z}_1^2 z_1 + i\, z_0 \bar{z}_2 {z_2}^2 \bar{z}_1, 0, 0, 0, 0],\\
& [0, 0, i\, {z_0}^2 z_1 z_2 \bar{z}_1, -i\, \bar{z}_0^2 \bar{z}_1 z_1 \bar{z}_2, i\, \bar{z}_0^2 \bar{z}_2 z_2 z_1, -i\, {z_0}^2 \bar{z}_2 z_2 \bar{z}_1],\\
& [0, 0, {z_0}^2 z_1 \bar{z}_1 z_2, \bar{z}_1 z_1 \bar{z}_2 \bar{z}_0^2, \bar{z}_2 z_2 z_1 \bar{z}_0^2, \bar{z}_2 z_2 \bar{z}_1 {z_0}^2]\ \ ],
\end{aligned}
$$

$Fix(Z_2(\pi,\pi)) = \{\pi_1 = 0, \pi_3 = 0, \pi_5 = 0\}$ and $Fix(\widetilde{SO}(2)) = \{\pi_1 = \pi_3 = \pi_4 = \pi_5 = 0\}$ on the orbit space are easily given as

$$
\begin{aligned}
\dot{y}_2 &= 2A_9 y_2^2 + 2A_3 y_2 - 4A_9 y_4\\
\dot{y}_4 &= 2A_9 y_2 y_4 + 4A_3 y_4
\end{aligned}
\qquad\qquad
\dot{y}_2 = 2A_9 y_2^2 + 2A_3 y_2\,.
$$

The Jacobian of the full system have for points with orbit type $Fix(Z_2(\pi,\pi))$ or $\widetilde{SO}(2)$ the special structures

$$
\begin{pmatrix}
* & 0 & * & 0 & *\\
* & * & * & * & *\\
* & 0 & * & 0 & *\\
* & * & * & * & *\\
* & 0 & * & 0 & *
\end{pmatrix}
\quad \textit{and} \quad
\begin{pmatrix}
* & 0 & * & 0 & *\\
* & * & * & * & *\\
* & 0 & * & 0 & *\\
* & 0 & * & * & *\\
* & 0 & * & 0 & *
\end{pmatrix}
$$

respectively. For example the stability of a point with isotropy $\widetilde{SO}(2)$ within $Fix(Z_2(\pi,\pi))$ is determined by the expression $2A_9y_2 + 4A_3$.

The fifth differential equation for π_5 has a special meaning. For given $\pi_1, \pi_2, \pi_3, \pi_4$ the value of π_5 is already determined by the relation. So the fifth differential equation means that a trajectory stays on the orbit space. Consequently, the derivative of the relation with respect to time should yield zero once the F_i are substituted and the relation is exploited. A computation verified the result is zero. The Maple Code is included, see last pages.

A second choice of symmetry adapted coordinates is derived by the principles described in the beginning of this section. In order to receive symmetry adapted coordinates with *respect to* $Z_2(\kappa)$ *and* $Z_2(\kappa)\times S_1$ *we choose*

$$u_1=\pi_1,\; u_2=\pi_2^2-4\pi_4,\; u_3=\pi_1\pi_2-\pi_3,\; u_4=\pi_4,\;\; \sigma_2=\pi_2,\; \sigma_3=\pi_5.$$

Obviously $\langle u_1(x),u_2(x),u_3(x),u_4(x)\rangle$ *and* $\langle \pi_1(x),\pi_2(x),\pi_3(x),\pi_4(x)\rangle$ *have the same radical and thus* u_1,u_2,u_3,u_4 *qualify as parameters. The secondary invariants are given by* σ_2,σ_3 *and the product* $\sigma_2\sigma_3$ *which we will not use. Then the ideal of relations is generated by*

$$u_2+4\,u_4-{\sigma_2}^2,\; u_2\left(-{u_1}^2u_2+2\,\sigma_2u_3u_1-{u_3}^2\right)-{\sigma_3}^2.$$

The strata are derived in these coordinates in a systematic way by changing the inequalities in $B\geq 0$ *and the relations on the fixed point space by the division algorithm with respect to a Gröbner basis in* $K[\pi,u,\sigma]$.

$$\begin{array}{rcll}
S_{Id} & = & \{(u,\sigma)\in \mathbf{R}^6 \mid & u_1>0,u_2>0,u_3>0,u_4>0,\sigma_2=\sqrt{u_2+4u_4},\\
 & & & \sigma_3=\pm\sqrt{u_2\left(-{u_1}^2u_2+2\,s_2u_3u_1-{u_3}^2\right)}\;\},\\
S_{Z_2(\kappa)} & = & \{(u,\sigma)\in \mathbf{R}^6 \mid & u_1>0,u_2=0,u_3=0,u_4>0,\sigma_2=2\,\sqrt{u_4},\\
 & & & \sigma_3=0\},\\
S_{Z_2(\kappa\pi,\pi)} & = & \{(u,\sigma)\in \mathbf{R}^6 \mid & u_1>0,u_2=0,u_3=2u_1\sigma_2,u_4>0,\\
 & & & \sigma_2=2\,\sqrt{u_4},\sigma_3=0\},\\
S_{Z_2(\pi,\pi)} & = & \{(u,\sigma)\in \mathbf{R}^6 \mid & u_1=0,u_2>0,u_3=0,u_4>0,\\
 & & & \sigma_2=\sqrt{u_2+4u_4},\sigma_3=0\},\\
S_{Z_2(\kappa)\times S_1} & = & \{(u,\sigma)\in \mathbf{R}^6 \mid & u_1>0,u_2=0,u_3=0,u_4=0,\sigma_2=0,\sigma_3=0\},\\
S_{Z_2\times Z_2} & = & \{(u,\sigma)\in \mathbf{R}^6 \mid & u_1=0,u_2=0,u_3=0,u_4>0,\sigma_2=2\,\sqrt{u_4},\\
 & & & \sigma_3=0\},\\
S_{\widetilde{SO(2)}} & = & \{(u,\sigma)\in \mathbf{R}^6 \mid & u_1=0,u_2>0,u_3=0,u_4=0,\sigma_2=\sqrt{u_2},\\
 & & & \sigma_3=0\},\\
S_{\widetilde{O(2)\times S_1}} & = & \{(u,\sigma)\in \mathbf{R}^6 \mid & u=\sigma=0\}\,.
\end{array}$$

The differential equations in these new coordinates are computed from the old differential equations by use of chain rule and the division algorithm. Observe that the module of equivariants is free over $\mathbf{R}[u]$ *as well, but the basis is* b_j,σ_2b_j. *This introduces the arbitrary polynomial functions* $A_j=A_{j,1}+\sigma_2A_{j,2}$ *with* $A_{j,i}$ *depending on* u.

$$\begin{array}{rcl}
\dot u_1 & = & A_{5,2}u_2u_1+A_{10,2}\sigma_3\sigma_2+A_{10,1}\sigma_3-A_{5,2}u_3\sigma_2+2\,A_{1,1}u_1-A_{5,1}u_3+\\
 & & (2\,A_{1,2}+A_{5,1})\,u_1\sigma_2+4\,A_{5,2}u_4u_1,\\
\dot u_2 & = & 16\,A_{9,2}u_4u_2-2\,A_{11,2}u_2\sigma_3-8\,A_{11,2}u_4\sigma_3+(-2\,A_{11,1}-4\,A_{6,2})\,\sigma_3\sigma_2\\
 & & -4\,A_{6,1}\sigma_3+(4\,A_{3,2}+4\,A_{9,1})\,u_2\sigma_2+4\,A_{9,2}{u_2}^2-2\,A_{12,2}u_2u_3\sigma_2\\
 & & +4\,A_{3,1}u_2+2\,A_{12,1}u_2u_1\sigma_2+8\,A_{12,2}u_4u_2u_1+2\,A_{12,2}u_1{u_2}^2\\
 & & -2\,A_{12,1}u_3u_2,\\
\dot u_3 & = & -A_{12,2}u_3u_1u_2+(A_{9,1}+A_{5,1})\,u_2u_1-2\,A_{7,1}u_3u_1+A_{12,2}{u_1}^2\sigma_2u_2\\
 & & +A_{10,2}u_2\sigma_3-4\,A_{12,2}u_3u_1u_4+4A_{10,2}u_4\sigma_3\\
 & & +(A_{10,1}-A_{8,2}+2A_{4,2})\,\sigma_3\sigma_2+(-A_{8,1}+2A_{4,1})\,\sigma_3-A_{11,1}\sigma_3u_1\\
 & & +(-A_{5,1}+2\,A_{3,2}+2\,A_{1,2}+A_{9,1})\,u_3\sigma_2+(-2\,A_{7,2}-A_{12,1})\,\sigma_2u_3u_1\\
 & & +(2\,A_{1,1}+2\,A_{3,1})\,u_3+(A_{9,2}+A_{5,2})\,u_2u_1\sigma_2+A_{12,1}{u_1}^2u_2\\
 & & -A_{11,2}\sigma_3u_1\sigma_2+(-A_{5,2}+A_{9,2})\,u_3u_2+(-4\,A_{5,2}+4\,A_{9,2})\,u_4u_3,
\end{array}$$

$$\begin{aligned}
\dot{u}_4 &= A_{7,1}u_2u_1 + (4\,A_{3,2} + 2\,A_{9,1})\,u_4\sigma_2 - 2\,A_{12,2}u_4u_3\sigma_2 + 2\,A_{9,2}u_4u_2 \\
&\quad +8\,A_{9,2}{u_4}^2 + A_{6,2}\sigma_3\sigma_2 + A_{6,1}\sigma_3 + (2\,A_{12,1} + 4\,A_{7,2})\,u_4u_1\sigma_2 \\
&\quad -A_{7,1}u_3\sigma_2 + A_{7,2}u_2u_1\sigma_2 + 8\,A_{12,2}{u_4}^2u_1 + 4\,A_{3,1}u_4 + 2\,A_{12,2}u_4u_2u_1 \\
&\quad +4\,A_{7,1}u_4u_1 + (-2\,A_{12,1} - 4\,A_{7,2})\,u_4u_3 - A_{7,2}u_3u_2, \\
\dot{\sigma}_2 &= (2A_{7,2} + A_{12,1})\,u_2u_1 + 2A_{9,2}u_2\sigma_2 + 4A_{9,2}u_4\sigma_2 - A_{11,2}\sigma_3\sigma_2 + 2\,A_{3,1}\sigma_2 \\
&\quad -A_{11,1}\sigma_3 + 4\,A_{12,2}u_4u_1\sigma_2 + (-2\,A_{7,2} - A_{12,1})\,u_3\sigma_2 \\
&\quad +(2\,A_{3,2} + 2\,A_{9,1})\,u_2 - 2\,A_{7,1}u_3 + A_{12,2}u_2u_1\sigma_2 + 2\,A_{7,1}u_1\sigma_2 \\
&\quad +(4\,A_{9,1} + 8\,A_{3,2})\,u_4 + (8\,A_{7,2} + 4\,A_{12,1})\,u_4u_1 - A_{12,2}u_3u_2 \\
&\quad -4\,A_{12,2}u_4u_3, \\
\dot{\sigma}_3 &= (4A_{3,1} + 2A_{1,1})\,\sigma_3 + 2A_{6,1}{u_3}^2 - A_{11,2}{\sigma_3}^2 + 4A_{6,1}{u_1}^2u_2 + 12A_{9,2}u_4\sigma_3 \\
&\quad -A_{12,1}\sigma_3u_3 + 3A_{9,2}u_2\sigma_3 + (2A_{6,2} + A_{11,1})\,{u_3}^2\sigma_2 + 4A_{11,2}u_4{u_3}^2 \\
&\quad +(4A_{3,2} + 2A_{1,2} + 3A_{9,1})\,\sigma_3\sigma_2 + (A_{8,1} - 2\,A_{4,1})\,u_3u_2 \\
&\quad +(2\,A_{4,2} - A_{8,2})\,u_1{u_2}^2 + 4\,A_{11,2}u_4{u_1}^2u_2 + (-A_{8,1} + 2\,A_{4,1})\,u_2u_1\sigma_2 \\
&\quad -A_{12,2}\sigma_3u_3\sigma_2 + (-2\,A_{4,2} + A_{8,2})\,u_2u_3\sigma_2 \\
&\quad +(8\,A_{4,2} - 4\,A_{8,2} + 4\,A_{10,1})\,u_4u_2u_1 + 4\,A_{12,2}u_4\sigma_3u_1 - 4\,A_{6,1}\sigma_2u_3u_1 \\
&\quad +4\,A_{10,2}u_2u_4u_1\sigma_2 + A_{12,1}\sigma_3u_1\sigma_2 + A_{12,2}u_2\sigma_3u_1 \\
&\quad +(-8\,A_{11,1} - 16\,A_{6,2})\,u_3u_1u_4 - 8\,A_{11,2}\sigma_2u_3u_1u_4 \\
&\quad +(4\,A_{6,2} + A_{11,1})\,{u_1}^2\sigma_2u_2 + (-2\,A_{11,1} - 4\,A_{6,2})\,u_3u_1u_2\,.
\end{aligned}$$

The equations look more complicated, but there is some obvious structure. The restriction to Fix($Z_2(\kappa) \times S_1$) simply is $\dot{u}_1 = 2\,A_{1,1}u_1$, the restriction to Fix($Z_2 \times Z_2$) is

$$\begin{aligned}
\dot{u}_4 &= (4\,A_{3,2} + 2\,A_{9,1})\,u_4\sigma_2 + 8\,A_{9,2}{u_4}^2 + 4\,A_{3,1}u_4, \\
\dot{\sigma}_2 &= 4\,A_{9,2}u_4\sigma_2 + 2\,A_{3,1}\sigma_2 + (4\,A_{9,1} + 8\,A_{3,2})\,u_4, \\
\sigma_2 &= 2\sqrt{u_4}\,,
\end{aligned}$$

while the restriction to Fix($Z_2(\kappa)$) is easily given by

$$\begin{aligned}
\dot{u}_1 &= 2\,A_{1,1}u_1 + (2\,A_{1,2} + A_{5,1})\,u_1\sigma_2 + 4\,A_{5,2}u_4u_1, \\
\dot{u}_4 &= (4\,A_{3,2} + 2\,A_{9,1})\,u_4\sigma_2 + 8\,A_{9,2}{u_4}^2 + (2\,A_{12,1} + 4\,A_{7,2})\,u_4u_1\sigma_2 \\
&\quad +8\,A_{12,2}{u_4}^2u_1 + 4\,A_{3,1}u_4 + 4\,A_{7,1}u_4u_1, \\
\dot{\sigma}_2 &= 4\,A_{9,2}u_4\sigma_2 + 2\,A_{3,1}\sigma_2 + 4\,A_{12,2}u_4u_1\sigma_2 + 2\,A_{7,1}u_1\sigma_2 \\
&\quad +(4\,A_{9,1} + 8\,A_{3,2})\,u_4 + (8\,A_{7,2} + 4\,A_{12,1})\,u_4u_1, \\
\sigma_2 &= 2\sqrt{u_4}\,.
\end{aligned}$$

The second structural property is that the Jacobian evaluated at a point in a stratum which has symmetry adapted coordinates has a special shape.

Rumberger [163] has investigated a Hopf bifurcation in Fix($Z_2(\kappa)$) of this Taylor-Couette problem by first choosing a certain parameterization of the differential equations on Fix($Z_2(\kappa)$). In turn this means a bifurcation of periodic orbits to 2-tori in the original system.

The advantages of Noether normalization are demonstrated in this context. The flow-invariance of the orbit types is clearly present in the differential equations on the orbit space. Secondly, the stratification is much more transparent. Moreover, the Jacobians have a special structure according to the fixed point spaces.

So far we have discussed the consequences of Cohen-Macaulayness and the Noether normalization for the structure of the differential equations on the orbit space. Of course there are many more aspects. From the literature we cite the following results. Krupa decomposes in [121] the vector field $f(x)$ into one part being tangent to the orbit and the relevant part being normal to the orbit. This implies some theoretical results on the structure of the dynamics. Rumberger [163] relates the Poincaré mappings of periodic orbits of the vector field and the projected vector field to each other. In case the orbit space reduction becomes too complicated one also uses a local version of it, see for example [34]. In [116] Koenig investigates the relation of the linearization $D_x f(0)$ of the equations at the origin on the original space and the linearization $D_y F(0)$ of the equations on the orbit space. The eigenvalues of $D_y F(0)$ are linear combinations with positive coefficients of the eigenvalues of $D_x f(0)$, if the Hilbert basis is chosen in an appropriate way. In order to define hyperbolicity of relative equilibria a variant of the tangent cone for semi-algebraic sets is introduced. Tangent cones of varieties (or a basis of their corresponding ideal) are computed by the tangent cone algorithm by Mora [148], see also [42] p. 170. Note that the computation of standard bases as in [89] is intermediate between Gröbner bases and tangent cones. Also Rumberger [163] discusses the question of relation of eigenvalues of $D_x f(0)$ and $D_y F(0)$. He exploits that the Hilbert basis is minimal. A stable equilibrium of $\dot{x} = f(x, \lambda)$ relates to a stable equilibrium of $\dot{y} = F(y, \lambda)$. An unstable equilibrium of $\dot{x} = f(x, \lambda)$ relates to an unstable equilibrium of $\dot{y} = F(y, \lambda)$. This gives two cases where bifurcation may be studied on the orbit space. The symmetry adapted coordinates as discussed in this section enable a more distinguished investigation of bifurcation phenomena on the orbit space.

```
> # load packages
> with(linalg): read(moregroebner): read(symmetry):
> infolevel[symmetry]:=1; infolevel[moregroebner]:=1; # print infos
----------------------------------------------------------------------
# group of Taylor-Couette example
----------------------------------------------------------------------
> go2:=S1([1,1,-1]);
> C:=matrix(6,6,0):
> C[1,1]:=1:C[2,2]:=-1:C[3,5]:=1:C[4,6]:=1:C[5,3]:=1:C[6,4]:=1:
> evalm(C);
> Z2:=mkfinitegroup({_s=evalm(C)},_Z2);
> go2[_fgnoncommute]:=op(Z2);
> s1:=S1([0,1,1]);
> g:=ProductCG(go2,s1);
----------------------------------------------------------------------
> # compute invariants
> mol:=Molien(g,lambda); # Molien series of invariant ring
> expand(simplify(mol*(1-lambda^2)^2*(1-lambda^4)^2)); # degrees
> series(mol,lambda=0,11);
> complexvars:=[z0,cz0,z1,cz1,z2,cz2];
> Hominvs(g,complexvars,4,_coords=complex);
> iis:=Invariants(g,complexvars,mol,6,_coords=complex);
> konjugate:={z0=cz0,cz0=z0,z1=cz1,cz1=z1,z2=cz2,cz2=z2}:
> realinvs:=[iis[1..4], I*(iis[5]-subs(konjugate,iis[5]))];
----------------------------------------------------------------------
> # compute relations
> rel:=invrelations([seq(pi[i]=realinvs[i],i=1..5)],complexvars);
> rel:=factor(rel+pi[5]^2)-pi[5]^2;
> realvs:=[x0,y0,x1,y1,x2,y2];
> invs:=map(convertinv,realinvs,complexvars,realvs,g,_coords=real);
> relationpireal:=invrelations([seq(pi[i]=invs[i],i=1..5)],realvs);
> slacks:=[seq(pi[i],i=1..5)];  vs:=[op(slacks),op(realvs)];
> MM:=matrix(3,11,0):
> for i from 1 to 6 do MM[1,5+i]:=1: od: # elimination property
> for i from 1 to 5 do MM[2,i]:=degree(invs[i],{op(realvs)}); od:
> MM[3,5]:=1:  # matrix term order showing Hironaka form
> evalm(MM);
> totaldeg:=mktermorder(vs,tdeg):
> eliorder:=mktermorder(vs,mat,evalm(MM),op(totaldeg)):
> eligrad:=table([_Hseriesvar=lambda,minint=0,maxint=10]):
> for i from 1 to 5 do eligrad[pi[i]]:=degree(invs[i],{op(realvs)});
> od:
> for i from 1 to 6 do eligrad[realvs[i]]:=1: od:
> gls:=[seq(slacks[j]-invs[j],j=1..5)]: hp:=1/((1-lambda)^6);
> gb:=homgroebner(gls,op(eliorder),{op(eligrad)},[op(eligrad)],hp);
----------------------------------------------------------------------
```

```
> # compute strata
> for i from 1 to 5 do
>    grad[i]:=vector(6):  # gradients
>    for j from 1 to 6 do grad[i][j]:=diff(invs[i],realvs[j]) od;
> od:
> B:=matrix(5,5);  # semi-positive matrix
> for i from 1 to 5 do
>     for j from i to 5 do
>         B[i,j]:=innerprod(grad[i],grad[j]):
>         B[i,j]:=normalform(B[i,j],gb,vs,op(eliorder));
>         B[j,i]:=B[i,j]:
>     od;
> od;
> evalm(B);
> bs:=[seq(det(submatrix(B,1..i,1..i)),i=1..5)];
> bs:=map(normalform,bs,{rel},vs,op(eliorder));
> bs:=map(factor,bs);
             ...   ...   ...   ...   ...   ...   ...
-----------------------------------------------------------------------
> # find generic equivariant vector field
> emol:=Equimolien(g,g,lambda); # Hilbert series
> Equivariants(g,g,complexvars,realinvs,emol,5,_coords=complex);
> Homequis(g,g,[z0,cz0,z1,cz1,z2,cz2],3,_coords=complex);
> paras:=[realinvs[1..4]];
> equivs:=CMEquivariants(g,g,complexvars,paras,emol,_coords=complex);
> f:=vector(6,0);
> for j from 1 to nops(equivs) do
>     f:=evalm(f+A[j]*equivs[j]);
> od;
> # conversion to real coordinates
> ff:=convertvectorfield(f,complexvars,realvs,g,g,_coords=real);
-----------------------------------------------------------------------
> diffgl:=[]; # compute differential equations
> for j from 1 to 5 do
>    p:=innerprod(evalm(grad[j]),evalm(ff));
>    diffgl:=[op(diffgl),normalform(p,gb,eliorder[vars],op(eliorder))]
> od:
> diffgl; # right hand side of differential eq.
> dglfixZ2pi:=subs({seq(rels[Z2pi][i]=0,i=1..nops(rels[Z2pi]))},diffgl);
> dglfixSO2:=subs({seq(rels[SO2][i]=0,i=1..nops(rels[SO2]))},diffgl);
-----------------------------------------------------------------------
> # derivative of relation
> define(di,binary, forall([x,y,t], di(x+y,t)=di(x,t)+di(y,t)),
>     forall([integer(a),x,t], di(a*x,t)=a*di(x,t)),
>     forall([x,y,t], di(x*y,t)=di(x,t)*y+x*di(y,t)),
>     forall([x,y,z,t], di(x*y*z,t)=di(x*y,t)*z+x*di(y*z,t)+y*di(z*z)),
>     forall([integer(n),x,t], di(x^n,t)=n*x^(n-1)*di(x,t)));
> h:=di(rel,t);
> hh:=expand(subs({seq(di(pi[i],t)=diffgl[i],i=1..5)},h));
> normalform(hh,{rel},eliorder[vars],op(eliorder));
```

Further Reading

A lot of people have explored how Computer Algebra methods are useful for the qualitative study of differential equations. Of course qualitative study means something different than looking for closed form solutions as done in symbolic computations based on differential Galois theory. Lie symmetry analysis is the second traditional branch of Computer Algebra methods for partial differential equations. In contrast to these algebraic computations the combination of Computer Algebra in the qualitative study within dynamical systems often restricts to the use of linear algebra techniques although in some cases advanced Lie group theory is involved. However, we mention the symbolic manipolations in the context of *Birkhoff normal forms*, *center manifold reductions*, and related topics [28, 43, 44, 45, 60, 64, 65, 82, 119, 126, 140], [144, 151, 158, 159, 172, 181, 189]. This list is far from being complete.

The most important link of dynamics to Computer Algebra is that to Computational Algebraic Geometry. Obviously, the steady state solutions $x(t) \equiv x_0 \in \mathbf{R}^n$ of $\dot{x} = f(x)$ where the right hand side f is polynomial form a real variety. Thus the Gröbner basis method of Chapter 1 is applicable. Many other questions indirectly lead to the study of varieties.

i.) How many limit cycles may a polynomial vector field in the plane have? One attempt to this question is the study of bifurcation of this special trajectories at a point where the linearization has eigenvalues $\pm i$. Typically the degenerate situation is studied with the help of *Liapunov functions* which help to investigate the stability of this special point. Requiring multiple bifurcation of periodic orbits leads to a system of polynomial equations in the coefficients of the vector field and the coefficients in the ansatz of the Liapunov function. Because of the nature of the question the only way to solve this system of equations is by Computer Algebra, see [131]. For more computations in this direction see [51, 93, 156, 169].

ii.) In [101] normal forms of *reversible* discrete systems are investigated exploiting the combination of Gröbner bases with a factorizer. With factorization the ideal is decomposed as intersection of other ideals which almost gives the primary decomposition.

iii.) In [124] a bifurcation problem being equivariant with respect to a group G is investigated for cases where the equivariant branching lemma by Vanderbauwhede [185] does not apply because the fixed point space with respect to a subgroup does not have dimension 1. For multidimensional fixed point spaces a *blow up* is performed leading to a different polynomial system. Blow up technique is a method from algebraic geometry in order to study singularities. The introduction of another variable transports the study of the affin object into projective space. Solvability and nonsingular Jacobian (application of the Theorem of Implicit Functions) guarantees the existence of bifurcating branches. Since the system is small, Gröbner bases are well suited. Especially the degeneracy conditions are computed, see [124].

The second point I like to mention is the combination of numerics and symbolic computation.

i.) The most elementary point is to use the C-Code generator in order to produce the differential equations as numerical source Code, see [120] for multibody system. The same group of people came up with a symbolic-numerical method for the detection and computation of bifurcation of special trajectories of $\dot{x} = f(x)$ called *periodic orbits* $(x(t+T) = x(t), \forall\ t)$, especially of higher codimension. The bifurcation depends on the coefficients of the Taylor polynomial approximation of the *Poincaré map*. The Poincaré map is a mapping in the phase space. A surrounding of a point on the orbit which is a transverse intersection with the orbit is mapped onto this surrounding. The coefficients of the Taylor expansion of the Poincaré map satisfy itself a system of differential equations which is solved numerically. But the explicit form of the differential equation for the coefficients involve higher order derivatives of the right hand side f of the original system. Setting up this system is a typical task of Computer Algebra, see [113, 114]. Of course this method is restricted to rather small systems.

ii.) Much more algebraic computations are performed in *Symcon* [75, 68, 69] where the group-theoretical computations automate the numerical investigation of steady states and their bifurcations of parameter dependent equivariant systems. Since the group action forces eigenvalues to be multiple the symmetry must be exploited in the numerical determination of bifurcation points. Computer Algebra *automates* the exploitation of symmetry. The underlying data basis of Symcon consists of group tables for some small finite groups (Z_2, D_3, D_4, A_4, S_4 etc.) and their irreducible representations (group actions which cannot be decomposed). The problem independent part computes the subgroup lattice, group homomorphisms, conjugate groups, normalizers and the *bifurcation subgroups* which describe the symmetry of generically bifurcating branches of equilibria. Then the equivariant system is prepared for the numerical algorithm. The *isotropy subgroups* describe the possible symmetry of equilibria. They are determined using the trace formula for multiplicities of irreducible representations [54, 171]. This algorithmic determination of isotropy subgroups is given in Algorithm 3.1.2. The numerical treatment necessitates even more functions such as the restriction to the *fixed point subspaces* and the evaluation of the Jacobian blocks, monitor functions, offset directions for handling of bifurcation points. They are generated by the Computer Algebra part in a C-Code file. This means that the symbolic part produces the numerical algorithm. Nevertheless, Symcon has often been misunderstood as a purely numerical algorithm.

In Chapters 3 and 4 we have seen that algorithmic invariant theory is a powerful tool in equivariant dynamics. In [27] algorithmic invariant theory is used in symmetric bifurcation theory, especially in [125] problems of mode interactions are studied. Another example is [76].

Finally, I like to present literature on using singularity theory within bifurcation theory and algorithmic approaches therein. The unfolding of bifurcation problems of higher codimension is investigated by inspecting the linearization of the family of bifurcation problems, the so-called *tangent space*. See [86, 87] for an introduction to singularity theory and bifurcation theory. The codimension of the tangent space in the full space of germs means the number of parameters which are required in order to unfold the bifurcation problem to the generic situation. With Gröbner bases the codimension of an ideal in the ring is easily determined. This fact was first used in [6]. A different approach has been used in [76]. Instead of starting with a bifurcation problem one starts with spaces which might be possible tangent spaces and deduces from this the associated bifurcation problems. Gröbner bases help to classify the possible tangent spaces.

However, instead of Gröbner bases the *standard bases* for ideals in local rings (such as the ring of formal power series) are much more suitable for bifurcation theory. Standard bases bases for $K[x_1, \ldots, x_n]_{\langle x_1,\ldots,x_n\rangle}$ and other local rings are implemented in Singular [90]. A nice introduction is given in Chapter 4 of [42]. For example in [21] this is used for studying unfoldings of Hamiltonian systems. In [133] Lunter improves the techniques of standard bases since in his study of a singularity in a Hamiltonian system he needs to compute the codimension of a direct sum of an ideal, an algebra, and a module. Thus he needed to generalize SAGBI-bases of algebras to the local case. Secondly, he adjusts the theory of Gröbner bases, standard bases, SAGBI bases altogether to this particular case.

I am looking forward to learning more on useful application of Computer Algebra to the theory of dynamical systems.

Bibliography

[1] M.J. Ablowitz and P.A. Clarkson. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, volume 149 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge UK, 1991.

[2] W.W. Adams, S. Hosten, P. Loustaunau, and J.L. Miller. SAGBI- and SAGBI-Gröbner bases over principal ideal domains. *J. Symb. Comp.*, 27:31 – 47, 1999.

[3] W.W. Adams and Ph. Loustaunau. *An Introduction to Gröbner bases*, volume 3 of *Graduate Studies in Mathematics*. AMS, Providence, RI, 1994.

[4] B. Amrhein and O. Gloor. The Gröbner walk, 1997. HTML/Java interface at http://www-sr.informatik.uni-tuebingen.de/~www-sr/java_gui/walk.html.

[5] B. Amrhein and O. Gloor. The fractal walk. In B. Buchberger and F. Winkler, editors, *Gröbner bases and Applications*, volume 251 of *London Mathematical Society Lecture Notes Series*, pages 305–322. Cambridge University Press, 1998.

[6] D. Armbruster. Bifurcation theory and computer algebra: An initial approach. In B. F. Caviness, editor, *Eurocal '85*, pages 126 – 136, Berlin, 1985. Springer.

[7] P. Ashwin and I. Melbourne. Symmetry groups of attractors. *Arch. Rat. Mech. Anal.*, 126:59–78, 1994.

[8] M.F. Atiyah and I.G. MacDonald. *Introduction to Commutative Algebra*. Addison-Wesley Publishing Company, Reading, Massachusetts, 1969.

[9] E. Barany, M. Dellnitz, and M. Golubitsky. Detecting the symmetry of attractors. *Physica D*, 67:66–87, 1993.

[10] R. Baumgarten and E. Kreuzer. Analyse nichtlinearer Schwingungen mit Computer-Algebra und der Methode der gewichteten Residuen. *ZAMM - Z. Angew. Math. Mech.*, 75:S45–S46, 1995. (In German).

[11] D. Bayer and I. Morrison. Gröbner bases and geometric invariant theory I. *J. Symb. Comp.*, 6:209 – 217, 1988.

[12] D. Bayer and M. Stillman. Macaulay: A system for computation in algebraic geometry and commutative algebra. Available via anonymous ftp from math.harvard.edu.

[13] D. Bayer and M. Stillman. Computation of Hilbert functions. *J. Symb. Comp.*, 14:31 – 50, 1992.

[14] Th. Becker and V. Weispfenning. *Gröbner Bases, a Computational Approach to Commutative Algebra*, volume 141 of *Graduate Texts in Mathematics.* Springer, New York, 1993.

[15] E. Bierstone. Lifting isotopies from orbit spaces. *Topology*, 14:145–252, 1975.

[16] A. Bigatti. Computation of Hilbert-Poincaré series. *J. Pure Appl. Algebra*, 119:237 – 253, 1997.

[17] L. Billera and L.L. Rose. Gröbner basis methods for multivariate splines. In T. Lyche and L. Schumacher, editors, *Mathematical Models in Computer Aided Geometric Design*, pages 93 – 104. Academic Press, 1989.

[18] L. Billera and L.L. Rose. A dimension series for multivariate splines. *Discrete Comput. Geom.*, 6:107 – 128, 1991.

[19] T. Bröcker and T. tom Dieck. *Representation of Compact Lie Groups.* Springer, New York, 1985.

[20] B. Broer. Classification of Cohen-Macaulay modules of covariants for systems of binary forms. *Proc. Am. Math. Soc.*, 120:37–45, 1994.

[21] H.W. Broer, I. Hoveijn, G.A. Lunter, and G. Vegter. Resonances in the spring pendulum: a case study in equivariant singularity theory. *Nonlinearity*, 11:1569–1605, 1998.

[22] W. Bruns and J. Herzog. *Cohen-Macaulay Rings.* Cambridge University Press, Cambridge, U.K., 1993.

[23] B. Buchberger. *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal, (An algorithm for finding a basis for the residue class ring of a zero-dimensional polynomial ideal).* PhD thesis, Math. Inst. Universität Innsbruck, Austria, 1965.

[24] B. Buchberger. Gröbner bases, an algorithmic method in polynomial ideal theory. In N.K. Bose, editor, *Multidimensional Systems Theory*, chapter 6, pages 184–232. Reidel Publishing Company, Dodrecht, 1985.

[25] M. Caboara. A dynamic algorithm for Gröbner basis computation. In M. Bronstein, editor, *ISSAC 93, Proceedings of the 1996 International Symposium on Symbolic and Algebraic Computation*, pages 275 – 283, New York, 1993. ACM Press.

[26] M. Caboara, G. De Dominicis, and L. Robbiano. Multigraded Hilbert functions and Buchberger algorithm. In Y.N. Lakshman, editor, *ISSAC 96, Proceedings of the 1996 International Symposium on Symbolic and Algebraic Computation*, pages 72 – 78. ACM, 1996.

[27] T.K. Callaham and E. Knobloch. Symmetry-breaking bifurcations on cubic lattices. *Nonlinearity*, 10:1179–1216, 1997.

[28] S.A. Campbell, J. Bélair, T. Ohira, and J. Milton. Limit cycles, tori, and complex dynamics in a second-order differential equation with delayed negative feedback. *Journal of Dynamics and Differential Equations*, 7:213–236, 1995.

[29] S.A. Campbell and Ph. Holmes. Heteroclinic cycles and modulated travelling waves in a system with D_4-symmetry. *Physica D*, 59:52–78, 1992.

[30] J.J. Cannon. Magma Release V2.3, 1996. Available by request, for more information see URL: http://www.maths.usyd.edu.au:8000/comp/magma/Overview.html.

[31] A. Capani, M. Niesi, and L. Robbiano. CoCoa, a system for doing computations in commutative algebra, 1995. Available via anonymous ftp from lancelot.dima.unige.it.

[32] P. Chossat. Forced reflectional symmetry breaking of an $O(2)$-symmetric homoclinic cycle. *Nonlinearity*, 6:723–731, 1993.

[33] P. Chossat and F. Dias. The 1:2 resonance with $O(2)$ symmetry and its applications in Hydrodynamics. *Nonlinear Science*, 5:105–129, 1995.

[34] P. Chossat, F. Guyard, and R. Lauterbach. Generalized heteroclinic cycles in spherically invariant systems and their perturbations. Technical Report Preprint No. 303, Weierstraß-Institut für Angewandte Analysis und Stochastik, 1996.

[35] P. Chossat and G. Ioos. *The Taylor-Couette Problem.*, volume 102 of *Applied Mathematical Sciences*. Springer, New York, 1994.

[36] V. Chvátal. *Linear Programming.* Freeman, New York, 1983.

[37] F. Chyzak. Mgfun, the Maple packages Ore-Algebra, Groebner and Holonomy. Available by www at http://www-rocq.inria.fr/algo/chyzak, to be distributed with Maple V.5, 1997.

[38] A. Colin. Solving a system of algebraic equations with symmetries. *J. Pure Appl. Algebra*, 117-118(1-3):195–215, 1997. Special issue for MEGA '96, Editors: Cohen, A.M. and Roy, M.-F.

[39] S. Collart, M. Kalkbrener, and D. Mall. Converting bases with the Gröbner walk. *J. Symb. Comp.*, 24:465–469, 1997.

[40] R.G. Cowell and F.J. Wright. CATFACT: Computer Algebraic Tools for Applications to Catastrophe Theory. In J.H. Davenport, editor, *EUROCAL '87 Europian Conference on Computer Algebra, Leipzig June 2-5, 1987, Proceedings*, volume 378 of *Lecture Notes in Computer Science*, pages 71–80, Berlin, 1987. Springer.

[41] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms, An Introduction to Computational Algebraic Geometry and Commutative Algebra.* Undergraduate Texts in Mathematics. Springer, New York, 1992.

[42] D. Cox, J. Little, and D. O'Shea. *Using Algebraic Geometry.* Springer Verlag, New York, 1998.

[43] W. Craig and P.A. Worfolk. An integrable normal form for water waves in infinite depth. *Physica D*, 84(3-4):513–531, 1995.

[44] R. Cushman and J.A. Sanders. Nilpotent normal forms and representation theory of $sl(2,r)$. In M. Golubitsky and J.M. Guckenheimer, editors, *Multiparameter Bifurcation Theory*, volume 56 of *Contemporary Mathematics*, pages 31–51, Providence, Rhode Island, 1986. AMS.

[45] R. Cushman and J.A. Sanders. A survey of invariant theory applied to normal forms of vector fields with nilpotent linear part. In D. Stanton, editor, *Invariant Theory and Tableaux*, volume 19 of *the IMA Volumes in Mathematics and its Applications*, pages 82–106, New York, 1987. Springer.

[46] W. Decker and T. de Jong. Gröbner bases and Invariant theory. In B. Buchberger and F. Winkler, editors, *Gröbner bases and Applications*, volume 251 of *London Mathematical Society Lecture Notes Series*, pages 305–322. Cambridge University Press, 1998.

[47] W. Decker, A.E. Heydtmann, and F.-O. Schreyer. Generating a Noetherian normalization of the invariant ring of a finite group. *J. Symb. Comp.*, 25:727–731, 1998.

[48] H. Derksen. Computation of invariants for reductive groups. *Advances in Mathematics*, 141:366–384, 1999.

[49] P. Deuflhard. *Newton Methods for Nonlinear Problems. Affine Invariance Properties and Adpative Algorithms.* Academic Press, 20xx. In preparation.

[50] J. Dieudonné and J.B. Carrell. *Invariant Theory, Old and New.* Academic Press, 1971.

[51] V.F. Edneral. Computer evaluation of cyclicity in planar cubic system. In W.W. Küchlin, editor, *Proc. ISSAC '97*, pages 305–309. ACM Press, New York, 1997.

[52] D. Eisenbud. *Commutative Algebra with a view Toward Algebraic Geometry*, volume 150 of *Graduate Texts in Mathematics.* Springer, New York, 1995.

[53] D. Eisenbud and B. Sturmfels. Finding sparse systems of parameters. *J. Pure Appl. Algebra*, 94:143–157, 1994.

[54] A. Fässler and E. Stiefel. *Group Theoretical Methods and Their Applications.* Birkhäuser, Boston, 1992.

[55] J.C. Faugère. GB, a system for Gröbner basis computations, 1990-1996. Available by anonymous ftp from posso.ibp.fr.

[56] J.C. Faugère, P. Gianni, D. Lazard, and D. Mora. Efficient computation of zero-dimensional Gröbner bases by change of ordering. *J. Symb. Comp.*, 13:117–131, 1992.

[57] M. Field. *Lectures on bifurcations, dynamics and symmetry*, volume 356 of *Pitman Research Notes in Mathematics Series.* Longman, Boston, Dordrecht, London, 1996.

[58] M. Field and M. Golubitsky. *Symmetry in Chaos.* Oxford University Press, Oxford, 1992.

[59] M. Forger. Invariant polynomials and Molien functions. *J. Math. Phys.*, 39(2):1107–1141, 1998.

[60] E. Freire, E. Gamero, E. Ponce, and L.G. Franquelo. An algorithm for symbolic computation of center manifolds. In P. Gianni, editor, *Symbolic and Algebraic Computation, International Symposium ISSAC '88, Rome, Italy, July 4-8, 1988*, volume 358 of *Lecture Notes in Computer Science*, pages 218–230. Springer, Berlin, 1988.

[61] R. Fröberg. *An Introduction to Gröbner bases.* John Wiley, 1998.

[62] R. Fröberg and J. Hollman. Hilbert series for ideals generated by generic forms. *J. Symb. Comp.*, 17:149–157, 1994.

[63] W. Fulton and J. Harris. *Representation Theory. A first Course*, volume 120 of *Graduate Texts in Mathematics.* Springer, New York, 1991.

[64] E. Gamero, E. Freire, and E. Ponce. Normal forms for planar systems with nilpotent linear part. In R. Seydel, F.W. Schneider, T. Küpper, and H. Troger, editors, *Bifurcation and Chaos: Analysis, Algorithms, Applications*, volume 97 of *ISNM*, pages 123–128, Basel, 1991. Birkhäuser.

[65] V.G. Ganzha, V.M. Rudenko, and E.V. Vorozhtsov, editors. *CAAM-90 International Seminar on Computer Algebra and Its Applications to Mechanics*, New York, 1993. Nova Science Publisher, Inc.

[66] K. Gatermann. The moregroebner package Version 2.1 - an improvement of the grobner package. A Maple V.3 program. Available by www by address http://-www.zib.de/gatermann/moregroebner.html. Updated to Version 3.1 for Maple V.5 1998-99.

[67] K. Gatermann. Symbolic solution of polynomial equation systems with symmetry. In Sh. Watanabe and M. Nagata, editors, *ISSAC 90, Proceedings of the 1990 International Symposium on Symbolic and Algebraic Computation*, pages 112–119, New York, 1990. ACM.

[68] K. Gatermann. Mixed symbolic-numeric solution of symmetrical nonlinear systems. In St.M. Watt, editor, *ISSAC 91, Proceedings of the 1991 International Symposium on Symbolic and Algebraic Computation*, pages 431–432. ACM, 1991.

[69] K. Gatermann. Computation of Bifurcation Graphs. In E. Allgower, K. Georg, and R. Miranda, editors, *Exploiting Symmetry in Applied and Numerical Analysis*, AMS Lectures in Applied Mathematics Vol. 29, pages 187–201. AMS, 1993.

[70] K. Gatermann. A remark on the detection of symmetry of attractors. In P. Chossat, editor, *Dynamics, Bifurcation and Symmetry. New Trends and New Tools*, volume 437 of *NATO ASI Series C: Mathematical and Physical Sciences*, pages 241–257, Dordrecht, Boston, London, 1994. Kluwer Academic Publishers.

[71] K. Gatermann. Semi-invariants, equivariants and algorithms. *Applicable Algebra in Engineering, Communications and Computing*, 7:105–124, 1996.

[72] K. Gatermann. Testing for S_n-symmetry with a recursive detective. In H.W. Broer, S.A. van Gils, I. Hoveijn, and F. Takens, editors, *Nonlinear Dynamical Systems and Chaos*, pages 61–78, Basel, Boston, Berlin, 1996. Birkhäuser.

[73] K. Gatermann and F. Guyard. The symmetry package in Maple V.3, Version 2.0. Available by www by address http://www.zib.de/gatermann/symmetry.html. Updated to Version 3.1 in Maple V.5 1998-99.

[74] K. Gatermann and F. Guyard. Gröbner bases, invariant theory and equivariant dynamics. *J. Symb. Comp.*, 27, 1999. In Press.

[75] K. Gatermann and A. Hohmann. Symbolic exploitation of symmetry in numerical pathfollowing. *Impact of Computing in Science and Engineering*, 3:330–365, 1991.

[76] K. Gatermann and R. Lauterbach. Automatic classification of normal forms. *Nonlinear Analysis, Theory, Methods, and Applications*, 34:157–190, 1998.

[77] K. Gatermann and B. Werner. Secondary Hopf bifurcation caused by steady-state steady-state mode interaction. In J. Chaddam, M. Golubitsky, W. Langford, and B. Wetton, editors, *Pattern Formations: Symmetry Methods and Applications*, volume 5 of *Fields Institute Communications*, pages 209–224. AMS, Providence, R.I., 1996.

[78] J. von zur Gathen and J. Gerhard. *Modern Computer Algebra.* To appear, 1999.

[79] R. Gebauer and H.M. Möller. On an installation of Buchberger's algorithm. *J. Symb. Comp.*, 6:275–286, 1988.

[80] K.O. Geddes, St. R. Czapor, and G. Labahn. *Algorithms for Computer Algebra.* Kluwer Academic Publishers, Boston, Dordrecht, London, 1992.

[81] C. Geiger, G. Dangelmayr, J.D. Rodriguez, and W. Güttinger. Symmetry breaking bifurcations in spherical Bénard convection part I: Results from singularity theory. In J. Chaddam, M. Golubitsky, W. Langford, and B. Wetton, editors, *Pattern Formations: Symmetry Methods and Applications*, volume 5 of *Fields Institute Communications*, pages 225–237. AMS, Providence, R.I., 1996.

[82] C. Geiger, G. Dangelmayr, J.D. Rodriguez, and W. Güttinger. Symmetry breaking bifurcations in spherical Bénard convection part II: Numerical results. In J. Chaddam, M. Golubitsky, W. Langford, and B. Wetton, editors, *Pattern Formations: Symmetry Methods and Applications*, volume 5 of *Fields Institute Communications*, pages 239–253. AMS, Providence, R.I., 1996.

[83] A. Giovini, T. Mora, G. Niesi, L. Robbiano, and C. Traverso. One sugar cube, please, or selection strategies in the Buchberger algorithm. In S.M. Watt, editor, *Proc. International Symposium on Symbolic and Algebraic Computation (ISSAC 91)*, pages 49 – 54, New York, 1991. ACM Press.

[84] M. Göbel. Computing bases for rings of permutation-invariant polynomials. *J. Symb. Comp.*, 19:285–291, 1995.

[85] M. Göbel. A constructive description of Sagbi bases for polynomial invariants of permutation groups. *J. Symb. Comp.*, 26:261–272, 1998.

[86] M. Golubitsky and D. G Schaeffer. *Singularities and Groups in Bifurcation Theory I*, volume 51 of *Applied Mathematical Sciences.* Springer, New York, 1985.

[87] M. Golubitsky, I. Stewart, and D. G Schaeffer. *Singularities and Groups in Bifurcation Theory II*, volume 69 of *Applied Mathematical Sciences.* Springer, New York, 1988.

[88] D. Grayson and M. Stillman. Macaulay 2, 1996. Available via www by http://www.math.uiuc.edu/Macaulay2.

[89] G.-M. Greuel and G. Pfister. Advances and improvements in the theory of standard bases and syzygies. *Archiv Math.*, 63, 1995.

[90] G.-M. Greuel, G. Pfister, and H. Schönemann. Singular version 1.2 User Manual . In *Reports On Computer Algebra*, number 21. Centre for Computer Algebra, University of Kaiserslautern, June 1998. http://www.mathematik.uni-kl.de/ zca/Singular.

[91] P. Gritzmann and B. Sturmfels. Minkowski addition of polytopes: computational complexity and applications to Gröbner bases. *SIAM J. Disc. Math.*, 6:246 – 269, 1993.

[92] J. Guckenheimer and Ph. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, volume 42 of *Applied Mathematical Sciences.* Springer, New York, 1983.

[93] J. Guckenheimer, R. Rand, and D. Schlomiuk. Degenerate homoclinic cycles in perturbations of quadratic Hamiltonian systems. *Nonlinearity*, 2:405–418, 1989.

[94] W. Hereman. Review of symbolic software for the computation of Lie symmetries of differential equations. *Euromath Bull.*, 2:45 – 82, 1994.

[95] W. Hereman. Symbolic software for Lie symmetry analysis. In N.H. Ibragimov, editor, *CRC Handbook of Lie Group Analysis of Differential Equations*, volume 3 of *New Trends in Theoretical Development and Computational Methods*, chapter 13. CRC Press, Boca Raton, Florida, 1995.

[96] A.E. Heydtmann. Finvar.lib, a Singular library to compute invariant rings and more, 1997. Available at http://www.mathematik.uni-kl/ zca/Singular. Demonstration at workshop Algorithmic Invariant Theory, Dagstuhl, May 1996.

[97] D. Hilbert. Über die Theorie der algebraischen Formen. *Math. Ann.*, 36:473–534, 1890.

[98] D. Hilbert. Über die vollen Invariantensysteme. *Math. Ann.*, 42:313–373, 1893.

[99] M. Hochster and J. Roberts. Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay. *Adv. in Math.*, 13:115–175, 1974.

[100] I. Hoveijn. The reduced phase space of a Hamiltonian system. draft, 1996.

[101] A. Jacquemard and M.A. Teixeira. Effective algebraic geometry and normal forms of reversible mappings. Preprint available at URL http://www.u-bourgogne.fr/-topologie/jacquemard/preprints.html, 1996.

[102] M.V. Jaric, L. Michel, and R.T. Sharp. Zeros of covariant vector fields for the point groups: invariant formulation. *J. Physique*, 45:1–27, 1984.

[103] A. Kandri-Rody and V. Weispfenning. Non-commutative Gröbner bases of solvable type. *J. Symb. Comp.*, 1:1–26, 1990.

[104] D. Kapur and K. Madlener. A completion procedure for computing a canonical basis for a k-algebra. In *Computers and Mathematics*, pages 1–11. Springer, 1989.

[105] C.T. Kelley. *Iterative Methods for Linear and Nonlinear Equations*, volume 16 of *Frontiers in Applied Mathematics*. SIAM, Philadelphia, 1995.

[106] G. Kemper. The *invar* package for calculating rings of invariants. Available by ftp from the electronic share library of Maple neptune@inf.ethz.ch, 1993.

[107] G. Kemper. Calculating invariant rings of finite groups over arbitrary fields. *J. Symb. Comp.*, 21:351–366, 1996.

[108] G. Kemper. An algorithm to calculate optimal homogeneous systems of parameters. *J. Symb. Comp.*, 27:171–184, 1999.

[109] G. Kemper and A. Steel. Some algorithms in invariant theory of finite groups. In P. Dräxler, G.O. Michler, and C.M. Ringel, editors, *Proceedings of the Euro-conference on Computational Methods for Representations of Groups and Algebras*, Progress in Mathematics. Birkhäuser, 1998.

[110] G. Kempf. Computing invariants. In S.S. Koh, editor, *Invariant Theory*, volume 1278 of *Lecture Notes in Mathematics*. Springer, Berlin, Heidelberg, New York,Tokio, 1987.

[111] I. Kersten. Noether's problem and normalization. In A. Krieg, editor, *DMV Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 100. B.G. Teubner, Stuttgart, 1998.

[112] A.I. Khibnik. LINLBF: A program for continuation and bifurcation anaylis of equilibria up to codimension three. In D. Roose, B. de Dier, and A. Spence, editors, *Continuation and Bifurcations: Numerical Techniques and Applications*, volume 313 of *NATO ASI Series C: Mathematical and Physical Sciences*, pages 283 – 296. Kluwer Academic Publishers, 1990.

[113] W. Kleczka, E. Kreuzer, and C. Wilmers. Combined analytical-numerical analysis of nonlinear dynamical systems. In R. Seydel, F.W. Schneider, T. Küpper, and H. Troger, editors, *Bifurcation and Chaos: Analysis, Algorithms, Applications*, pages 199 – 203. Birkhäuser, 1991.

[114] W. Klezka and E. Kreuzer. Systematic computer-aided analysis of dynamic systems. In St.M. Watt, editor, *ISSAC 91, Proceedings of the 1991 International Symposium on Symbolic and Algebraic Computation*, pages 429 – 430. ACM, 1991.

[115] E. Knobloch and M. Silber. Hopf bifurcation with $Z_4 \times T^2$-Symmetry. In E. Allgower, K. Böhmer, and M. Golubitsky, editors, *Bifurcation and Symmetry*, volume 104 of *ISNM*, pages 241–252, Basel, 1992. Birkhäuser.

[116] M. Koenig. Linearization of vector fields on the orbit space of the action of a compact Lie group. *Math. Proc. Cambridge Phil. Soc.*, 121:401–424, 1997.

[117] U. Koppenhagen and E.W. Mayr. An optimal algorithm for constructing the reduced Gröbner basis of binomial ideals. In Y.N. Lakshman, editor, *ISSAC 96, Proceedings of the 1996 International Symposium on Symbolic and Algebraic Computation*, pages 55 – 62. ACM, 1996.

[118] I. Kotsireas. Polynomial systems arising in the study of central configurations in the N-body problem of celestial mechanics, 1998. Webpage http://www-calfor.lip6.fr/ ilias/ccNbody.html.

[119] E. Kreuzer. *Computerized Symbolic Manipulation in Mechanics.*, volume 343 of *CISM Courses and Lectures.* Springer, Wien, 1994.

[120] E. Kreuzer and W. Schiehlen. NEWEUL–Software for the generation of symbolical equations of motion. In W. Schiehlen, editor, *Multibody Systems Handbook*, pages 181 – 202. Springer, 1990.

[121] M. Krupa. Bifurcations of relative equilibria. *SIAM J. Math. Anal.*, 21:1453–1486, 1990.

[122] Y.A. Kuznetsov. *Elements of Applied Bifurcation Theory*, volume 112 of *Applied Mathematical Sciences.* Springer, New York, 1995.

[123] Y.N. Lakshman. A single exponential bound on the complexity of computing Gröbner bases of zero-dimensional ideals. In T. Mora and C. Traverso, editors, *Effective Methods in Algebraic Geometry*, volume 94 of *Progress in Mathematics*, pages 227 – 234. Birkhäuser, 1990.

[124] A. Lari-Lavassani, W. Langford, and K. Huseyin. Symmetry-breaking bifurcations on multidimensional fixed point subspaces. *Dynamics and Stability of Systems*, 9:345–373, 1994.

[125] A. Lari-Lavassani, W.F. Langford, K. Huseyin, and K. Gatermann. Steady-state mode interactions for D_3 and D_4-symmetric systems. Technical Report Preprint FI-NP 1997-009, The Fields Institute, 1997. To appear in Dynamics of Continuous, Discrete and Impulsive Systems.

[126] J. Laure and Y. Demay. Symbolic computations and equation on the center manifold: application to the Couette-Taylor problem. *Computers and Fluids*, 16:229–238, 1988.

[127] R. Lauterbach and J.A. Sanders. Bifurcation analysis for spherically symmetric systems using invariant theory. *J. Dynam. Differential Equations*, 9(4):535–560, 1997.

[128] D. Lazard. Solving zero-dimensional algebraic systems. *J. Symb. Comp.*, 13:117–131, 1992.

[129] Ch. Leis. *Hopf-Verzweigung in Systemen mit sphärischer Symmetrie: Invariante Tori und verbindende Orbits.* PhD thesis, Freie Universität Berlin, Fachbereich Mathematik und Informatik, 1995. (in German).

[130] A.H.M. Levelt. The cycloheptane molecule, 1997. Invited lecture, ISSAC'97, Maui, Hawaii, USA.

[131] N.G. Lloyd and J.M. Pearson. REDUCE and the bifurcation of limit cycles. *J. Symb. Comput.*, 9:215–224, 1990.

[132] A. Logar. A computational proof of the Noether normalization lemma. In T. Mora, editor, *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes 6th International Conferences AAECC-6*, volume 357 of *Lecture Notes in Computer Science*, pages 259 – 273. Springer, 1988.

[133] G. Lunter. *Hamiltonian normal forms, an algorithmic approach.* PhD thesis, Department of Mathematics, Groningen, 1999. (in preparation).

[134] F.S. Macaulay. *The Algebraic Theory of Modular Systems*, volume 19 of *Cambridge Tracts in Mathematics and Mathematical Physics.* Cambridge University Press, Cambridge, 1916.

[135] F.S. Macaulay. Some properties of enumeration in the theory of modular systems. *Proc. London Math. Soc.*, 26:531 – 555, 1927.

[136] D. Manocha. Numerical methods for solving polynomial equations. In D.A. Cox and B. Sturmfels, editors, *Applications of Computational Algebraic Geometry*, volume 53 of *Proceedings of Symposia in Applied Mathematics*, pages 41–66. AMS, 1998.

[137] H. Matsumura. *Commutative Algebra.* W.A. Benjamin Inc. Publishers, New York, 1970.

[138] K. Matthies. A subshift of finite type in the Bogdanov-Takens bifurcation with D_3-symmetry. Technical Report Preprint 30/97, DFG-Schwerpunktprogramm Dynamik: Analysis, effiziente Simulation und Ergodentheorie, 1997.

[139] E.W. Mayr and A. Meyer. The complexity of the word problem for commutative semigroups and polynomial ideals. *Adv. Math.*, 46:305–329, 1982.

[140] Zhen Mei and A.J. Roberts. Equations for the turbulent flood waves. In A. Mielke and K. Kirchgässner, editors, *Proceedings of IUTAM/ISIMM Symposium on structure and Dynamics of Nonlinear Waves*, pages 242 – 252. World Scientific Publishing Co., 1995.

[141] I. Melbourne. Instantaneous symmetry and symmetry on average in the Couette-Taylor and Faraday experiments. In P. Chossat, editor, *Dynamics, Bifurcation and Symmetry. New Trends and New Tools*, volume 437 of *NATO ASI Series C: Mathematical and Physical Sciences*, pages 241–257, Dordrecht, Boston, London, 1994. Kluwer Academic Publishers.

[142] H. Melenk, H. M. Möller, and W. Neun. Groebner: A Groebner Basis package. Available with REDUCE 3.6.

[143] H. Melenk, H. M. Möller, and W. Neun. Symbolic solution of large stationary chemical kinetics problems. *Impact of Computing in Science and Engineering*, 1:138–167, 1989.

[144] J. Menck. *Analyse nichthyperbolischer Gleichgewichtspunkte in dynamischen Systemen unter Ausnutzung von Symmetrien, mit Anwendungen von Computeralgebra.* PhD thesis, Fachbereich Mathematik der Universität Hamburg, 1992. (in German).

[145] B. Mishra. *Algorithmic Algebra*, volume 141 of *Texts and Monographs in Computer Science*. Springer, New York, 1993.

[146] H.M. Möller and T. Mora. The computation of the Hilbert function. In *EUROCAL '83*, volume 162 of *Lecture Notes in Computer Science*, pages 157 – 167. Springer, 1983.

[147] H.M. Möller and T. Mora. Upper and lower bounds for the degree of Gröbner bases. In *Proc. EUROSAM '84*, volume 174 of *Lecture Notes in Computer Science*, pages 172 – 183. Springer, 1984.

[148] T. Mora. An algorithm to compute the equations of tangent cones. In J. Calmet, editor, *Computer Algebra, EUROCAM '82, European Computer Algebra Conference, Marseille, France, 5.-7. April 1982*, volume 144 of *Lecture Notes in Computer Science*, pages 158–165. Springer, 1982.

[149] T. Mora and L. Robbiano. The Gröbner fan of an ideal. *J. Symb. Comp.*, 6:183 – 208, 1988.

[150] J. Müller-Quade and Th. Beth. Homogeneous systems of parameters of minimal degree. Submitted to J. Symb. Comp. 1996.

[151] N.S. Namachchivaya, M.M. Doyle, W.F. Langford, and N.W. Evans. Normal form for generalized Hopf bifurcation with non-semisimple 1:1 resonance. *Z. angew. Math. Phys.*, 45:312–335, 1994.

[152] E. Noether. Der Endlichkeitssatz der Invarianten endlicher Gruppen. *Math. Ann.*, 77:89–92, 1916.

[153] V.W. Noonburg. A neural network modeled by an adaptive Lotka-Volterra system. *SIAM J. Appl. Math.*, 49:1779–1792, 1989.

[154] P.J. Olver. *Classical Invariant Theory*, volume 44 of *London Mathematical Society Student Texts*. Cambridge University Press, 1999.

[155] D. Pasechnik. On the nullcone and Hironaka decompositions of the invariants of a torus. Preprint, 1997.

[156] J.M. Pearson, N.G. Lloyd, and C.J. Christopher. Algorithmic derivation of centre conditions. *SIAM Rev.*, 38:619–636, 1996.

[157] C. Procesi and G. Schwarz. Inequalities defining orbit spaces. *Inventiones mathematicae*, 81:539–554, 1985.

[158] R.H. Rand. *Topics in Nonlinear Dynamics with Computer Algebra*, volume 1 of *Computation in Education: Mathematics, Science and Engineering.* Gordon and Breach Science Publishers, USA, 1994.

[159] R.H. Rand and D. Armbruster. *Perturbation Methods, Bifurcation Theory and Computer Algebra.* Springer, New York, 1987.

[160] A. Reeves. A parallel implementation of Buchberger's algorithm over Z_p for $p \leq 31991$. *J. Symb. Comp.*, 26:229 – 244, 1998.

[161] L. Robbiano. Term orderings on the polynomial ring. In *EUROCAL 85, European Conference on Computer Algebra Vol. II*, volume 204 of *LNCS*, pages 513 – 517. Springer, 1985.

[162] L. Robbiano and M. Sweedler. Subalgebra bases. In W. Bruns and A. Simis, editors, *Commutative Algebra Salvador*, volume 1430 of *LNM*, pages 61 – 87. Springer, 1988.

[163] M. Rumberger. *Symmetrische dynamische Systeme: Differenzierbarkeit und linearisierte Stabilität.* PhD thesis, Fachbereich Mathematik der Universität Hamburg, 1997. (in German).

[164] J. Sanders and J.P. Wang. Combining Maple and Form to decide on integrability questions. *Computer Physics Communications*, 115:447–459, 1998.

[165] G. Sartori. Geometric invariant theory: a model-independent approach to spontaneous symmetry and/or supersymmetry breaking. *Rivista del Nuovo Cimento*, 14:1–120, 1991.

[166] D.H. Sattinger. *Group theoretic methods in bifurcation theory.*, volume 762 of *Lectures Notes in Mathematics.* Springer, New York, 1979.

[167] T. Sauer. Polynomial interpolation of minimal degree and Gröbner bases. In B. Buchberger and F. Winkler, editors, *Gröbner bases and Applications*, volume 251 of *London Mathematical Society Lecture Notes Series*, pages 483–494. Cambridge University Press, 1998.

[168] J. Scheurle. Some aspects of successive bifurcations in the Couette-Taylor problem. In J. Chaddam, M. Golubitsky, W. Langford, and B. Wetton, editors, *Pattern Formations: Symmetry Methods and Applications*, volume 5 of *Fields Institute Communications*, pages 335–345. AMS, Providence, R.I., 1996.

[169] D. Schlomiuk, J. Guckenheimer, and R. Rand. Integrability of plane quadratic vector fields. *Expo. Math.*, 8:3–25, 1990.

[170] F. Schwarz. SPDE: A package for finding symmetry groups of PDE's. In: REDUCE User's Manual 3.6.

[171] J.-P. Serre. *Linear Representations of Finite Groups.* Springer, New York, 1977.

[172] M. Singer. *Differential Equations and Computer Algebra.* Computational Mathematics and Applications. Academic Press, London, 1991.

[173] N.J.A. Sloane. Error-correcting codes and invariant theory: new approach of a nineteenth-century technique. *Amer. Math. Monthly.*, 84:82–107, 1977.

[174] T.A. Springer. *Invariant Theory*, volume 585 of *Lecture Notes in Mathematics.* Springer Verlag, 1977.

[175] R.P. Stanley. Invariants of finite groups and their applications to combinatorics. *Bulletin Am. Math. Soc.*, 1:475–511, 1979.

[176] B. Sturmfels. *Algorithms in Invariant Theory*, volume 1 of *Texts and Monographs in Symbolic Computation.* Springer, Wien, 1993.

[177] B. Sturmfels. *Gröbner Bases and Convex Polytopes*, volume 8 of *University Lecture Series.* AMS, Providence, R.I., 1996.

[178] B. Sturmfels. Introduction to resultants. In D.A. Cox and B. Sturmfels, editors, *Applications of Computational Algebraic Geometry*, volume 53 of *Proceedings of Symposia in Applied Mathematics*, pages 25–39. AMS, 1998.

[179] B. Sturmfels and N. White. Computing combinatorial decompositions of rings. *Combinatorica*, 11:275–293, 1991.

[180] B. Sturmfels and M. Wiegelmann. Structual Gröbner basis detection. *Applicable Algebra in Engineering, Communications and Computing*, 8:257 – 263, 1997.

[181] E. Tournier. *Computer Algebra and Differential Equations.* Computational Mathematics and Applications. Academic Press, London, 1990.

[182] C. Traverso. Hilbert functions and the Buchberger algorithm. *J. Symb. Comp.*, 22:355 – 376, 1996.

[183] M. Van den Bergh. Modules of Covariants. In *Proceedings of the International Congress of Mathematicians, Zürich, Switzerland, 1994*, pages 352–362. Birkhäuser, Basel, 1995.

[184] J.C. van der Meer, J.A. Sanders, and A. Vanderbauwhede. Hamiltonian structure of the reversible nonsemisimple 1:1 resonance. In P. Chossat, editor, *Dynamics, Bifurcation and Symmetry. New Trends and New Tools*, volume 437 of *NATO ASI Series C: Mathematical and Physical Sciences*, pages 221–240, Dordrecht, Boston, London, 1994. Kluwer Academic Publishers.

[185] A. Vanderbauwhede. *Local Bifurcation and Symmetry.* Pitman, Boston, 1982.

[186] W.V. Vasconcelos. *Computational Methods in Commutative Algebra and Algebraic Geometry*, volume 2 of *Algorithms and Computation in Mathematics*. Springer, Berlin, 1998.

[187] J. Verschelde and K. Gatermann. Symmetric Newton polytopes for solving sparse polynomial systems. *Adv. Appl. Math.*, 16:95–127, 1995.

[188] J. Verschelde, K. Gatermann, and R. Cools. Mixed volume computation by dynamic lifting applied to polynomial system solving. *Journal of Discrete & Computational Geometry*, 16:69–112, 1996.

[189] S. Walcher. On transformations into normal form. *J. Math. Anal. Appl.*, 180:617–632, 1993.

[190] V. Weispfenning. Admissible orders and linear forms. *ACM Sigsam Bulletin*, 21:16 – 18, 1987.

[191] V. Weispfenning. Comprehensive Gröbner bases. *J. Symb. Comp.*, 14:1–29, 1992.

[192] F. Winkler. *Polynomial Algorithms in Computer Algebra.* Texts and Monographs in Symbolic Computation. Springer, Wien, 1996.

[193] Th. Wolf and A. Brand. CRACK: Solving overdetermined systems of PDEs or ODEs. In: REDUCE User's Manual 3.6.

[194] P. Worfolk. Zeros of equivariant vector fields: Algorithms for an invariant approach. *J. Symb. Comp.*, 17:487–511, 1994.

[195] G.M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics.* Springer, New York, 1995.

[196] R. Zippel. *Effective Polynomial Computation.* Kluwer Academic Publishers, Boston, Dordrecht, London, 1993.

Index