Deciding Linear-Transcendental Problems

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Abstract

We present a decision procedure for linear-transcendental problems formalized in a suitable first-order language. The problems are formalized by formulas with arbitrary quantified linear variables and a block of quantifiers with respect to mixed linear-transcendental variables. Variables may range both over the reals and over the integers. The transcendental functions admitted are characterized axiomatically; they include the exponential function applied to a polynomial, hyperbolic functions and their inverses, and the arcustangent. The decision procedure is explicit and implementable; it is based on mixed real-integer linear elimination, the symbolic test point method, elementary analysis, and Lindemann's theorem. As a byproduct we obtain sample solutions for existential formulas and a qualitative description of the connected components of the satisfaction set wrt. a mixed linear-transcendental variable.

1 Introduction

The decision problem for the real numbers as ordered exponential field that was open for a long time (compare [vdD82, vdD86]) was conditionally solved in the positive sense in the monumental paper [MW96], compare also [Wil96, Wil97]. The positive solution relies on Schanuel's conjecture. For algorithmic or relative algorithmic solutions of decision problems of a more specialized form see [Ric92, Vor92, Ric93, Ric95, Ric98]; for related problems also [vdDMM94, vdDM94, Wil96]. Here too, Schanuel's conjecture often plays a decisive role.

Here we consider the decision problem for a fragment of the full first-order theory of the reals as ordered field with one specific transcendental function. Let F be a fixed subfield of the field $\mathbb{R}_a$ of real algebraic numbers. We consider two disjoint sets of variables $x_1, \dots, x_n$ and $y_1, \dots, y_m$. Terms are restricted to the form $p(x_1, \dots, x_n, y_1, \dots, y_m) + c \cdot \mathrm{trans}(q(y_1, \dots, y_m))$, where p and q are a linear polynomials with coefficients in F and $c \in F$. Atomic formulas are equations or inequalities between terms. Quantifier-free formulas are obtained from atomic formulas by means of the boolean operations $\wedge, \vee, \neg$. Normal formulas are closed formulas of the form

$$\exists y_1 \dots \exists y_m Q_1 x_1 \dots Q_n x_n (\psi)$$

with arbitrary quantifiers Q_i and a quantifier-free formula ψ.

We specify some requirements for the real transcendental function $\mathrm{trans}(x)$ axiomatically. These axioms are satisfied e.g. for exponential functions of type $\exp(p(x))$, where $p(x)$ is a polynomial with real algebraic coefficients, hyperbolic functions, and the arcustangent. Moreover the decision procedure can be extended to the case of inverse functions of the above, when quantifiers are restricted to the natural domains of these functions.

We present an unconditional decision procedure for normal formulas. In contrast to the general decision procedure for real exponential formulas in [MW96] our decision algorithm does not require Schanuel's conjecture. Its role is taken over by the classical theorem of Lindemann (see [Shi89], page 70) asserting that the values of the exponential function for algebraic arguments are transcendental. We also consider the decision problem for normal formulas, where quantifiers range over some intermediate field $F \subseteq G \subseteq \mathbb{R}_a$ of real algebraic numbers.

Finally we extend the decision procedure for normal formulas to extended normal formulas, where terms may involve the integer-part operation. By means of this additional operation we can code the integers inside the reals and thus also quantifiers ranging over the integers. Hence we refer to this case as the *mixed real-integer case*. In this case we require the argument

$(q(y_1,\dots,y_m)$ of $\text{trans}(q(y_1,\dots,y_m))$ to be bounded by a rational constant s.

The proofs for the decision methods involve

- real and mixed real-integer linear quantifier elimination [Wei88, Wei90b, LW93, Wei99b] for a reduction to the case of a single mixed linear-transcendental variable,
- convexity arguments and elementary analysis for the univariate case,
- the evaluation of the sign of mixed linear-transcendental expressions at symbolic test points,
- Lindemann's theorem from transcendence theory for the decision of relations between transcendental and real algebraic constants.

For the purely real case and a transcendental function of type $\exp(p(x))$, where $p(x)$ is a polynomial with real algebraic coefficients, our decision problem may be considered as a special case of the problem solved in [Vor92] in combination with algorithmic quantifier elimination for Tarski algebra. In contrast to the highly sophisticated machinery of real algebraic geometry and non-standard analysis required there and in [MW96], our method is quite elementary in character and well-suited for implementation (e.g. on top of the REDLOG-package of REDUCE [DS97]). The case of other admissible transcendental functions such as the arcustangent is not covered by [Vor92]. The mixed real-integer case cannot be subsumed under any previous work. For the special case of the exponential function as $\text{trans}(x)$ our results were announced in [Wei99a].

2 The formal framework

Let F be an arbitrary subfield F of the field $\mathbb{R}_a$ of real algebraic numbers. We consider the following fragment L of a first-order language for the real numbers: *Linear terms* $s(y_1,\dots,y_m,x_1,\dots,x_n)$ are expressions of the form $s := \sum_{i=1}^m b_i y_i + \sum_{i=1}^n c_i x_i + c_0$ with $b_i, c_i \in F$. Fix a real function $\text{trans}(x)$ and a linear term $q(y_1,\dots,y_m)$ involving only the variables $y_1,\dots,y_m$. Then a *transcendental term* is of the form $c \cdot \text{trans}(q(y_1,\dots,y_m))$ with $c \in F$. Arbitrary *terms* are sums of linear and transcendental terms. *Atomic formulas* are equations or inequalities between terms. *Quantifier-free formulas* are obtained from atomic formulas by means of the boolean operations $\wedge, \vee, \neg$. *Normal formulas* are formulas without free variables of the form

$$\exists y_1 \dots \exists y_m Q_1 x_1 \dots Q_n x_n(\psi)$$

with arbitrary quantifiers Q_i and a quantifier-free formula ψ.

Our goal is a *decision procedure* for normal formulas in L i. e. an algorithm that takes normal formulas in L as inputs and outputs the truth value of the input formula over the reals. This goal is analogous to that in [Wei90a] for almost linear integer problems and to [AW00] for linear-trigonometric problems. We will also consider analogous decision procedures for normal formulas, where the quantifiers are regarded as ranging over some fixed intermediate field $F \subseteq G \subseteq \mathbb{R}_a$. In contrast to the purely linear case the decision of a normal formula in $\mathbb{R}$ and in G may differ.

In order for our decision procedures to work we need some requirements on the real function $\operatorname{trans}(x)$ that we specify axiomatically. In the following we call a real number *manageable* if it is either algebraic, given as a root of a squarefree rational polynomial together with an isolating interval with rational endpoints, or transcendental with a recursive binary expansion. Notice that by interval bisection and Sturm's theorem, equalities and inequalities between two algebraic manageable numbers are decidable (see [BWK98]). Moreover inequalities between an algebraic and a transcendental manageable number are decidable by iterated bisection of the isolating interval for the algebraic real and computation of a large enough initial segment of the transcendental manageable number. In the following we will frequently need these decisions. By way of contrast we will never run into the problem of deciding equalities or inequalities between two transcendental manageable numbers, a problem that is known to be undecidable.

The *axioms on trans*(x) are as follows:

1. $\operatorname{trans}(x)$ is a real analytic function defined on all of $\mathbb{R}$ (see [KP92]). There is a recursive function $\delta(n)$ such for all integers n, $\delta(n)$ is a modulus of continuity (see [BB85]) for $\operatorname{trans}(x)$ and its derivative on the interval $[n, n+1]$. Moreover for every rational number q, $\operatorname{trans}(x)$ and its derivative are given by recursive power series centered at q with an explicit rate of convergence. Finally $\lim_{x \longrightarrow \infty} \operatorname{trans}(x)$ and $\lim_{x \longrightarrow -\infty} \operatorname{trans}(x)$ are given as a manageable real number or $\pm\infty$. We collect all these properties by saying that $\operatorname{trans}(x)$ is a *well-behaved* real analytic function.

2. There exist finitely many manageable reals $r_1 < \ldots < r_k$, such that all zeros of the second derivative $\operatorname{trans}''(x)$ of trans are among the r_i. The values $\operatorname{trans}(r_i)$ are manageable real numbers and if r_i is transcendental then $\operatorname{trans}(r_i)$ and $\operatorname{trans}'(r_i)$ are algebraic. We call these points the *critical points* of $\operatorname{trans}(x)$.

3. For any real $r \neq r_1, \ldots, r_k$ it is never the case that both r and $\text{trans}(r)$ are algebraic.

4. The graph of $\text{trans}(x)$ has no tangent lines with real algebraic coefficients except possibly at the critical points r_i.

The last axiom is a consequence of axiom (3) and the following axiom:

5. For all reals $r \neq r_1, \ldots, r_k$ the value of $\text{trans}(r)$ is algebraically dependent on the value of the derivative $\text{trans}'(r)$ over the field $\mathbb{R}_a$.

Indeed if the line $cx+d$ is a tangent to $\text{trans}(x)$ at point r and $\text{trans}'(r) = c$ is algebraic, then by (5) $\text{trans}(r)$ is algebraic, and so by (3) r is transcendental. Since $d = \text{trans(r)}-cr$, it follows that d must be transcendental.

We call a function $\text{trans}(x)$ satisfying the axioms an *admissible function.* Using the remark above, [BB85], and Lindemann's theorem that asserts the transcendence of $\exp(x)$ for algebraic $x \neq 0$ (see [Shi89], p. 51) it is not difficult to verify that the following transcendental functions $\text{trans}(x)$ satisfy the axioms and hence are admissible:

1. $e^{p(x)}$ for any polynomial $p(x)$ with real algebraic coefficients.

2. $\sinh(x)$, $\cosh(x)$ and $\tanh(x)$. More generally, every real function of the form $ae^{mx} + be^{nx}$ for integers m, n and real algebraic numbers a, b.

3. $\arctan(x)$

By way of contrast trigonometric functions fail to satisfy axiom (2). The analogous problems for these functions have been solved in [AW00].

Our goal is a *decision procedure* for normal formulas in L i. e. an algorithm that takes normal formulas in L as inputs and output the truth value of the input formula over the reals.

In fact we will consider also formulas in a more general sense, where we admit besides the "ordinary" quantifiers $\exists, \forall$ also the "counting quantifiers" $\exists^{\leq n}, \exists^{n}, \exists^{\geq n}, \exists^{\leq \infty}, \exists^{\infty}$ interpreted as "there exist at most n, exactly n, at least n, only finitely many, infinitely many", respectively for $n \in \mathbb{N}$. Formulas, closed formulas, normal formulas in this extended sense will be refered to as formulas, closed formulas, normal formulas with counting quantifiers. Notice that the first three types of counting quantifiers are definable using ordinary quantifiers, while the last two are not definable in this way.

In the following we describe two *reduction steps* that will reduce the decision problem for normal formulas to normal formulas with a single mixed linear-transcendental variable.

Reduction step 1:
Consider a normal formula

$$\varphi := \exists y_1 \ldots \exists y_m Q_1 x_1 \ldots Q_n x_n(\psi)$$

with arbitrary quantifiers Q_i and a quantifier-free formula ψ.

Then φ is equivalent over the reals to the following normal formula

$$\varphi' := \exists y \exists y_1 \ldots \exists y_m Q_1 x_1 \ldots Q_n x_n(y = q(y_1, \ldots, y_m) \wedge \psi'),$$

where ψ' results from ψ by replacing everywhere in ψ the transcendental term $\mathrm{trans}(q(y_1, \ldots, y_m))$ by $\mathrm{trans}(y)$.

Notice that φ' is again a normal formula, but contains only y as mixed linear-transcendental variable; all other variables are linear in φ', moreover the transcendental term in φ' is of the simple form $\mathrm{trans}(y)$.

Reduction step 2:
Consider a normal formula with a single mixed linear-transcendental variable y and $\mathrm{trans}(y)$ as transcendental term:

$$\varphi' := \exists y Q_1 x_1 \ldots Q_n x_n(\psi).$$

Let ψ_1 result from ψ by replacing the transcendental term $\mathrm{trans}(y)$ by a new variable z. Then ψ_1 is a linear formula in the sense of [Wei88, LW93]. So by linear real quantifier elimination, $Q_1 x_1 \ldots Q_n x_n(\psi_1)$ is equivalent to a quantifier-free linear formula ψ_2 with y and z as the only linear variables. Let ψ_3 result from ψ_2 by backsubstitution of the transcendental term $\mathrm{trans}(y)$ for z. Then $\varphi'' := \exists y(\psi_3)$ is a formula with only one quantifier that is equivalent to φ'. Notice that linear combinations of terms with coefficients in F can be rewritten as a single term. Hence the formula φ'' can be rewritten as a normal formula $\exists y(\psi_4)$.

Both reduction steps work similarly if the given formula involves counting quantifiers by virtue of linear elimination by test points. So for example a counting quantifier $\exists^{\infty} x_i(\sigma(x_i))$ can be eliminated by restricting the disjunction over test terms used for the elimination of the quantifier $\exists x_i(\sigma(x_i))$ to test terms of the form $\pm\infty$ and $s \pm \epsilon$, where s is a linear term (compare [LW93]). Moreover if all quantifiers $Q_i x_i$ are existential, then extended linear elimination produces sample points for solutions x_i in dependence of $\mathrm{trans}(y)$.

So the problem of deciding the truth value of normal formulas is reduced to normal input formulas of the form $\exists y(\psi)$, where ψ is quantifier-free and has trans(y) as transcendental term. So it suffices to decide univariate normal formulas of this type over the reals. In fact we will do more: In case the truth value of $\exists y(\psi)$ is 'true', then we will provide a sample point $r \in \mathbb{R}$ for y. In fact if M is the set of reals satisfying $\psi(y)$, then we will also determine the number of connected components of M and their types as a non-empty intervals $[b, c], (b, c), [b, c)$ or $(b, c]$.

As a consequence we can also in an obvious way decide a normal formula $Qy(\psi)$ with quantifier-free ψ, where Q is a counting quantifier.

3 Geometry of admissible functions and straight lines

In this section we will solve the decision problem for univariate normal formulas φ of type $\exists x(\psi(x))$ with quantifier-free ψ and trans(x) as transcendental term. It will turn out that this is essentially a study of the geometry of the graph of trans(x) in relation to a finite system of straight lines. In order to carry out this study algorithmically, we require the axioms on the admissibility of the function trans(x).

The atomic subformulas of ψ can be written in the form $\mathrm{trans}(x) + cx + d \ \rho\ 0$, or $cx + d\ \rho\ 0$, where c, d are real algebraic numbers in F and ρ is one of the relations $=, <, >, \leq, \geq$. Let $r_1 < \ldots < r_k$, be the critical points of the function trans(x) specified in axiom (2). We call the intervals

$$I_{-\infty} = (-\infty, r_1),\ I_\infty = (r_k, \infty),\ I_1 = (r_1, r_2), \ldots, I_{k-1} = (r_{k-1}, r_k)$$

the *basic intervals* of ψ. Let ϵ be a symbol for a positive infinitesimal in an elementary extension field $\mathbb{R}^*$ of $\mathbb{R}$. Equivalently ϵ may be regarded as a positive real small enough for the given situation.

In order to decide φ it suffices

- to evaluate the truth value of $\psi(r_i)$ for $1 \leq i \leq k$,
- to decide the satisfiability of $\psi(x)$ on each basic interval of ψ.

The *first task* amounts to deciding all atomic subformulas of ψ at the critical points r_i i. e. the order relation between manageable reals of the form $r := \mathrm{trans}(a)$ and b. Since by axiom (2) not both r and b are transcendental, this is algorithmically possible.

For the *second task* we may assume that all atomic formulas in $\psi(x)$ are among $c_i x + d_i\ \rho_i\ 0$ and $\mathrm{trans}(x) + c_i x + d_i\ \rho_i\ 0$ for $1 \leq i \leq m$. We refer to the straight lines $y = c_i x + d_i,\ \ 1 \leq i \leq m$ as the *lines* of ψ.

Suppose now we can solve the following subtasks on each basic interval $I := (r, r')$:

1. Decide whether two lines $y = c_i x + d_i$ and $y = c_{i'} x + d_{i'}$ intersect on I, and if yes evaluate the sign of another line $y = c_j x + d_j$ of ψ and decide the order relation between the graph of $c_j x + d_j$ and of $\mathrm{trans}(x)$ at the intersection point ξ and infinitesimally to the right of this point.

2. Compute the number of intersection points between a line $y = c_i x + d_i$ of ψ and the graph of the function $\mathrm{trans}(x)$ on the given basic interval I.

3. For each intersection point ξ of a line $y = c_i x + d_i$ and the graph of the function $\mathrm{trans}(x)$ on I and for each other line $y = c_j x + d_j$ of ψ evaluate the sign of $c_j x + d_j$ and decide the order relation between the graph of $c_j x + d_j$ and of $\mathrm{trans}(x)$ at the point ξ and infinitesimally to the right of this point.

4. Determine the order relation between test points ξ arising in (1), (2), (3) for different atomic subformulas of ψ.

5. Evaluate the truth value an atomic subformula of ψ infinitesimally to the right of a critical point r_i.

6. For each line $y = c_j x + d_j$ of ψ decide the order relation between the graph of $c_j x + d_j$ and of $\mathrm{trans}(x)$ at $\pm\infty$.

We refer to the points r_i in the first task and the points ξ mentioned in subtasks (1) and (3) as the *proper test points* for $\psi(x)$. Point of the form $\xi + \epsilon$ and $\pm\infty$ are referred to as *improper test points* for $\psi(x)$.

Obviously $\psi(x)$ is satisfiable on a basic interval I iff $\psi(x)$ holds for at least one proper or improper test point in I. Hence $\psi(x)$ is satisfiable on $\mathbb{R}$ iff it is satisfied at some point r_i or it is satisfiable on some basic interval of $\psi(x)$.

From the evaluation of the atomic formulas of $\psi(x)$ at all test points we can moreover in a finite combinatorial manner determine the number of connected components of the satisfaction set of ψ on I and the type of interval of each such component. This involves in addition the subtask to determine the order relation between test points on the same basic interval of ψ arising from different atomic subformulas of ψ. By piecing together this information on each basic interval together with the evaluation of ψ at all points r_i we obtain the number of connected components of the global satisfaction set of

ψ and the type of interval of each such component. As a consequence we can now evaluate the existential or counting quantifier over the variable x, and thus decide the given normal formula φ. Moreover, any proper or improper test point at which $\psi(x)$ holds in $\mathbb{R}$ is a sample solution of $\psi(x)$. Together with the sample solutions provided by extended linear quantifier elimination, this provides sample solutions for arbitrary multivariate existential normal formulas.

For the rest of this section we will be concerned with *solutions of the subtasks* listed above.

The *first subtask* involves only easy computations with algebraic real numbers and the approximate evaluation of $\operatorname{trans}(a)$ at a real algebraic point ξ. We call the intersection points ξ of two lines arising in this subtask a point of *type 0.*

The *last subtask* requires the knowledge of an approximation of $\lim_{x \longrightarrow -\infty} \operatorname{trans}(x)$ provided by axiom (1).

For the *second subtask* let $I = (r, r')$ and recall that by axiom (2) $\operatorname{trans}(x)$ is either strictly convex or strictly concave on I. Moreover the decision between convexity and concavity of $\operatorname{trans}(x)$ on I can be made by the order relation between $\operatorname{trans}'(r)$ and $\operatorname{trans}'(r')$.

Suppose to begin with that $\operatorname{trans}(x)$ is strictly convex. Then $\operatorname{trans}'(x)$ is strictly increasing on I. Hence $y = c_i x + d_i$ has at exactly one intersection point with the graph of $\operatorname{trans}(x)$ on I, iff

$$(\operatorname{trans}(r) < c_i r + d_i \text{ and } c_i r' + d_i < \operatorname{trans}(r')) \text{ or}$$

$$(\operatorname{trans}(r) > c_i r + d_i \text{ and } c_i r' + d_i > \operatorname{trans}(r')).$$

In the first case we call the intersection point of type 1, in the second case of type 2. For an intersection point ξ of type 1 we have $\operatorname{trans}(\xi + \epsilon) > c_i \xi + d_i$, and for an intersection point ξ of type 2 we have $\operatorname{trans}(\xi + \epsilon) < c_i \xi + d_i$.

Notice that by axiom (4) a tangent situation between the line $y = c_i x + d_i$ and the graph of $\operatorname{trans}(x)$ on I is excluded.

Two intersection points between the line $y = c_i x + d_i$ and the graph of $\operatorname{trans}(x)$ on I occur iff

$$\operatorname{trans}'(r) < c_i < \operatorname{trans}'(r') \text{ and}$$

$$\operatorname{trans}(s) < c_i s + d_i \text{ for the unique point } s \in I \text{ such that } \operatorname{trans}'(s) = c_i.$$

The point s can be computed to arbitrary precision by iterated bisection of intervals and evaluation of the order relation between $\operatorname{trans}'(x)$ and c_i. This is algorithmically possible by axiom (1); moreover the first condition can be decided by axiom (2). Hence all of the condition above can be tested effectively. If it is satisfied then we call the smaller of the two intersection points a point of type 3 and the larger one a point of type 4. Notice that for an intersection point ξ of type 3 we have $\mathrm{trans}(\xi+\epsilon) < c_i\xi + d_i$, and for an intersection point ξ of type 4 we have $\mathrm{trans}(\xi+\epsilon) > c_i\xi + d_i$.

In the remaining case there is no intersection point between the line $y = c_ix + d_i$ and the graph of $\mathrm{trans}(x)$ on I.

The case, where $\mathrm{trans}(x)$ is strictly concave on I is treated analogously.

The *third subtask:*

Suppose we have identified the intersection point ξ of a line $y = c_ix + d_i$ and the graph of the function $\mathrm{trans}(x)$ by its basic interval I, the information on its type 1-4, and the information whether $\mathrm{trans}(x)$ is strictly convex or strictly concave on I. Then we know whether $\mathrm{trans}(x) - c_ix + d_i$ is increasing or decreasing at the intersection point ξ.

Let $y = c_jx + d_j$ be another straight line. Again we restrict to the case where $\mathrm{trans}(x)$ is strictly convex on I; The concave case is handled analogously.

In order to decide the sign of $c_j\xi + d_j$, it suffices to determine the order relation between ξ and the intersection point $-d_j/c_j$ of the line $y = c_jx + d_j$ with the x-axis, in case this point is in I. (Notice that the exceptional cases $c_j = 0$ or $-d_j/c_j \notin I$ are trivial.) This in turn can be done once one knows the type of the point ξ and the order relation between $c_i(-d_j/c_j) + d_i$ and $\mathrm{trans}(-d_j/c_j)$ and the order relation between $c_i(-d_j/c_j) + \epsilon + d_i$ and $\mathrm{trans}(-d_j/c_j + \epsilon)$

Finally we need to decide the order relation between $c_j\xi + d_j$ and $\mathrm{trans}(\xi)$. If the line $y = c_jx + d_j$ has no intersection point with the graph of $\mathrm{trans}(x)$ on I, then this order relation is the same as that between $c_j(r + \epsilon) + d_j$ and $\mathrm{trans}(r + \epsilon)$, and hence can be handled by the fifth subtask. Otherwise the line has one intersection point η or two intersection points $\eta_1 < \eta_2$ with the graph of $\mathrm{trans}(x)$ on I. Then our problem is reduced to knowing the types of ξ and of η, or η_1 and η_2 and the order relation between these points. Notice that we know for sure by axiom (3) that ξ can not coincide with one of the other points unless the lines $y = c_ix + d_i$ and $y = c_jx + d_j$ are identical. If these two lines have no intersection point on I, then the order relation between ξ and η, or η_1, or η_2 is obvious. If they have an intersection point p on I, then the order relation between $c_ip + d_i$ and $\mathrm{trans}(p)$ will determine the order relation between ξ and η, or η_1, or η_2.

Notice that all tests required in this subtask are again algorithmic by our axioms. The decisions are easily visualized by drawing pictures of the possible geometric situations; they are formally verified by elementary analysis.

For the *fourth subtask* let ξ_1 and ξ_2 be two test points of given type 0 - 4 as in subtasks (1) - (3) arising from different atomic subformulas of ψ. Then equality or an order relation between ξ_1 and ξ_2 is easily decided by using subtasks (1) - (3) together with the type information on the points.

The *fifth subtask* is handled similar to the first task taking into account also the evaluation of the derivative $\mathrm{trans}'(r_i)$.

This completes the evaluation of all atomic formulas of $\psi(x)$, and hence of $\psi(x)$ at all test points. As a corollary we obtain an overview over the number and types of connected components of the satisfaction set of the total formula ψ. Thus we have proved all the claims at the end of section 2.

Finally we remark that our decision procedure can also be construed as a quantifier elimination procedure in a liberal sense: If we admit parameters in the coefficients of terms in normal formulas φ, then the result of our decision procedure would turn into a boolean combination φ' of equations and inequalities about these parameters. The terms of φ' are, however, more complicated expressions than those allowed in normal formulas.

4 Inverses of admissible functions

In this section we consider the decision problem for normal formulas, where the role of an admisible function $\mathrm{trans}(x)$ is replaced by the inverse function $\mathrm{trans}^{-1}(x)$. In general such an inverse may not exist or it may be defined only on some real interval. So we have to make a further restriction in the fomation of normal formulas involving such fucntions, in order to guarantee that the truth-value of it is well-defined.

Let $\mathrm{trans}(x)$ be an admissible function, let I be an interval such that $\mathrm{trans}(x)$ is strictly monotonic on I, and let the interval J be the image of I under $\mathrm{trans}(x)$. Then the inverse function $\mathrm{trans}^{-1} : J \longrightarrow I$ exists and is also strictly monotonic. For fixed inverse function trans^{-1} and linear term $q(y_1, \ldots, y_m)$ we define now a *transcendental term* as an expression of the form $c \cdot \mathrm{trans}^{-1}(q(y_1, \ldots, y_m))$ with $c \in F$.

Arbitrary terms are again sums of linear and transcendental terms. *Atomic formulas* are equations or inequalities between terms. *Quantifier-free formulas* are obtained from atomic formulas by means of the boolean operations $\wedge, \vee, \neg$ as before. *Normal formulas* are now of the form

$$\exists y_1 \ldots \exists y_m Q_1 x_1 \ldots Q_n x_n (q(y_1, \ldots, y_m) \in J \ \wedge \ \psi)$$

with arbitrary quantifiers Q_i and a quantifier-free formula ψ. Here the expression $z \in J$ is of course only a short hand for the inequality or conjunction of two inequalities defining J.

We are going to show that normal formulas in this sense can be equivalently reduced to normal formulas in the previous sense. Hence all the results proved for normal formulas in the previous sense apply also to normal formulas in the current sense.

The *first reduction step* is as in section 1:

Consider a normal formula

$$\varphi := \exists y_1 \ldots \exists y_m Q_1 x_1 \ldots Q_n x_n (q(y_1, \ldots, y_m) \in J \ \wedge \ \psi)$$

with arbitrary quantifiers Q_i and a quantifier-free formula ψ.

Then φ is equivalent over the reals to the following normal formula

$$\varphi' := \ \exists y \exists y_1 \ldots \exists y_m Q_1 x_1 \ldots Q_n x_n (y \in J \ \wedge \ y = q(y_1, \ldots, y_m) \ \wedge \ \psi'),$$

where ψ' results from ψ by replacing everywhere in ψ the transcendental term $\mathrm{trans}^{-1}(q(y_1, \ldots, y_m))$ by $\mathrm{trans}^{-1}(y)$.

In a next reduction step we see that this formula is equivalent to

$$\varphi'' := \ \exists x \exists y_1 \ldots \exists y_m Q_1 x_1 \ldots Q_n x_n (x \in I \ \wedge \ \mathrm{trans}(x) = q(y_1, \ldots, y_m) \ \wedge \ \psi''),$$

where ψ'' results from ψ' by replacing everywhere in ψ' the transcendental term $\mathrm{trans}^{-1}(y)$ by x. Here again the expression $z \in I$ is only a short hand for the inequality or conjunction of two inequalities defining I. Hence φ'' is indeed a normal formula in the previous sense equivalent to φ. Hence it can be decided algorithmically.

So this decision procedure applies e. g. to linear-transcendental problems involving the real functions $\ln : (0, \infty) \longrightarrow \mathbb{R}$, or $\mathrm{Arsinh} : \mathbb{R} \longrightarrow \mathbb{R}$, or $\mathrm{Arcosh} : [1, \infty) \longrightarrow \mathbb{R}$, or $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow \mathbb{R}$. The more general case of linear-transcendental problems involving the tangent function on $\mathbb{R} \setminus \{\frac{\pi}{2} + \pi\mathbb{Z}\}$ has been discussed in [AW00].

5 The mixed real-integer case

In this section we turn to a considerable extension of linear-transcendental problems treated in sections 2 and 3 given by an extension of the concept of term used so far. We admit as new operation symbol in our extension language the integer-part operation (compare [Wei99b]). Similar as in the language L'' considered in [Wei99b], extended terms are now expressions obtained from terms by means of addition, scalar multiplication by rational constants and the integer-part operation. Atomic formulas in the extended sense are then of the form $t = 0$, $t > 0$, $t \geq 0$, for extended terms t. Formulas, closed formulas and normal formulas in the extended sense are obtained from atomic formulas as before with the following important *restriction*:

The argument $q(y_1, \ldots, y_m)$ of the transcendental term

$$c \cdot \mathrm{trans}(q(y_1, \ldots, y_m))$$

with $c \in F$ has to be bounded in absolute value by a rational constant s.

By means of the integer-part operation we are now able to code also quantifiers ranging over the integers. So the problems that can be expressed by normal formulas in the extended sense are now *mixed real-integer problems.*

We are now going to describe a decision procedure for normal formulas in the extended sense:

By an argument similar to the one in section 1 we can use reduction step 1 to reduce the input formula to a normal formula with at most one mixed linear-transcendental variable. Next the mixed real-integer linear quantifier elimination of [Wei99b] can be applied to reduce the decision problem for normal formulas with one mixed linear-transcendental variable to the case of normal formulas with just one bounded existential quantifier $\exists x(-s < x < s \ \wedge \ \psi(x))$, where $\psi(x)$ is quantifier-free. The bound s provides an obvious bound for all integer parts of terms in $\psi(x)$. So every occurence of an integer part operation can be eliminated by a finite disjunction over all integer constants within the corresponding bound. So the evaluation of the truth value of $\exists x(\psi(x)$ boils down to the evaluation of a finite disjunction of existentially quantified conjunctions of the form

$$\exists x(\bigwedge_{i=1}^{k} [t_i(x)] = m_i \ \wedge \ \psi'(x))$$

with integer constants m_i, and a non-extended quantifier-free formula $\psi'(x)$. Replacing each equation $[t_i(x)] = m_i$ equivalently by $m_i \leq t_i(x) < m_i + 1$, we reduce this extended normal formula to a normal formula in the original sense, and can now decide the result by the decision method of the previous sections.

6 Decision procedure over a subfield

Let as before F be a subfield of the field $\mathbb{R}_a$ of real algebraic numbers. We suppose as before that all linear terms have coefficients in F. In addition we assume that all the points $r_1, \ldots, r_k$ occuring in the axioms for the function trans(x) are in F. Normal formulas meeting these conditions are called *normal formulas defined over F.*

We consider the modified decision problem for normal formulas defined over F, where we interpret all quantifiers as ranging not over $\mathbb{R}$, but over some intermediate field G with $F \subseteq G \subseteq \mathbb{R}_a$. Notice that both the real linear elimination in [LW93] and the mixed real-integer linear elimination in [Wei99b] are valid in arbitrary subfields of the reals. This situation changes, however, radically for univariate normal formulas $\exists x(\psi(x))$: Consider e. g. the normal formula $\varphi := \exists x(e^x - 2 = 0)$. This formula is true over $\mathbb{R}$, but false over G due to Lindemann's theorem.

In contrast to this situation, we will show now that the truth-value of a normal formula defined over F is the same in all intermediate fields $F \subseteq G \subseteq \mathbb{R}_a$. Moreover there is again a decision procedure for normal formulas over such a field G.

It suffices to modify the original decision procedure in such a way that we drop all proper test terms arising from points that are not real algebraic. Thus we keep proper test terms arising from intersections of lines, but drop the test terms arising from intersections of lines with trans(x) on some basic interval. Notice that all improper test point can be regarded as representing points in F, since F is dense and cofinal in $\mathbb{R}$.

7 Examples

Example 7.1 *Put trans$(x) := e^x$ and consider the normal formula $\exists x(cx + \frac{1}{2} > e^x \ \wedge \ -x < e^x)$ for a rational constant c. Then our decision procedure will say that this formula holds in the reals iff*

$$c \leq -1 \ \vee \ (c > -1 \ \wedge \frac{1}{2(c+1)} > e^{\frac{-1}{2(c+1)}}) \ \vee \ (c > 1 \ \wedge c \leq e^{1-\frac{1}{2c}})$$

By Lindemann's theorem the equations

$$\frac{1}{2(c+1)} = e^{\frac{-1}{2(c+1)}}, \quad c = e^{(1-\frac{1}{2c})}$$

are impossible for rational c. Hence the inequalities above can be decided by computing enough of the expansion of $e^{\frac{-1}{2(c+1)}}$ and of $e^{(1-\frac{1}{2c})}$. So the formula holds e.g. for $c = -1, \ c = -1/4, \ c = 2,$ and fails for $c = -0.1$ and for $c = 1$.

Example 7.2 *The next example involves two variables and illustrates the role of linear elimination in the decision procedure. We put trans$(x) :=$*

arctan(x), *and observe that* arctan(x) *has unique inflection point* 0 *and basic intervals* $I_1 = (-\infty, 0)$, $I_2 := (0, \infty)$; *furthermore* arctan(x) *is strictly convex on* I_1 *and strictly concave on* I_2.

Let φ *be the normal formula* $\exists y \exists x (x \geq \arctan(y) \ \wedge \ 10x \leq -10y - 1 \ \wedge cy > \arctan(y) \ \wedge \ -y - 1 < \arctan(y))$. *Linear elimination applied to the formula* $\exists x (x \geq z \ \wedge \ 10x \leq -10y - 1)$ *yields the equivalent formula* $10y \leq 10z - 1$. *Hence* φ *is equivalent to* $\exists y (10y \leq 10 \arctan(y) - 1 \ \wedge \ cy > \arctan(y) \ \wedge \ -y - 1 < \arctan(y))$. *Our decision procedure applied to this formula will yield "true" iff* $c > -1 \ \wedge \ \arctan(-1/(c+1)) < -c/(c+1)$. *So* φ *holds e. g. for* $c \geq -0.5$ *and fails for* $c = -0.9$.

Example 7.3 *The final example illustrates the decision of extended normal formulas involving the integer-part operation. The transcendental function here is the Gaussian error function erf*$(x) := e^{-x^2}$. *It satisfies the axioms with critical points* $-1/\sqrt{2}, 0, 1/\sqrt{2}$. *It is strictly concave on the interval* $(-1/\sqrt{2}, 1/\sqrt{2})$, *and strictly convex on the intervals* $(-\infty, -1/\sqrt{2})$ *and* $(1/\sqrt{2}, \infty)$. *Consider the extended normal formula* $\varphi := \exists x(-10 < x < 10 \ \wedge \ [10 \ \mathit{erf}(x)] = x + c)$. *Then elimination of the integer-part operation yields the equivalent normal formula* $\bigvee_{i=0}^{9} \exists x(-10 < x < 10 \ \wedge \ i1 \leq 10 \ \mathit{erf}(x) < (i+1)1 \ \wedge \ x + c = i1)$, *which is in turn is equivalent to* $\bigvee_{0, c-10 \leq i \leq 9} i1 \leq 10 \ \mathit{erf}(i1 - c) < (i+1)1$. *So* φ *fails for e. g.* $c > 18$, *and holds for* $c = 9.5$.

8 Conclusions

We have presented an explicit, unconditional and implementable decision procedures for large classes of mixed linear-transcendental real and mixed real-integer problems formulated by first-order formulas. As a by-product we have obtained the structure of the satisfaction set of formulas with a single non-linear variable, and sample points for existential formulas. The main restriction on the transcendental part is that only a single, "well behaved" transcendental function is allowed in formulas. This may e.g. be the exponential function applied to a polynomial with real algebraic coefficients, hyperbolic functions, and the arcustangent. The case of real formulas involving several transcendental functions obtained by superposition of exponential functions and arithmetic operations has been solved under the condition of Schanuel's conjecture in [MW96]. The proof is, however, very indirect using a powerful model-theoretic machinery; so it is far away from a possible implementation in a computer algebra system.

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