On block–skip–free transition matrices

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Abstract: Starting from an abstract setting which extends the property "skip–free to the left" for transition matrices to a partition of the state space, we develop bounds for the mean hitting time to an arbitrary subset from an arbitrary initial law. We illustrate our theory with an asymptotic analysis of a non–reversible Markovian star network with loss.

Key Words: Markov chains on discrete state spaces; hitting times; stationary distributions; invariant measures; skip–free Markov chains; aggregation of states; balanced partitions; Markovian analysis of queueing networks

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1 Introduction

There is a growing interest in applications of Markov chains with a discrete state space. Reinforcement Learning, Stochastic Optimization, Queueing Systems, Game Theory and, Communication Protocols are examples of such applications.

One examines, among other parameters, the mean first passage time to an interesting subset. For random walks on a finite state space, there exist closed forms for the mean first passage time (cf. [9] or [11]) which render the computation of this parameter more tractable. In more general cases finding analytical expressions for the mean first passage times involves matrix multiplications. These calculations are potentially difficult if the state space is large (possibly infinite) or/and the transition probabilities are parametric expressions. One may have to be satisfied with finding bounds on this quantitiy. Estimates of general interest for the mean hitting time are one of the subjects of the present paper. Although the given bounds are also of interest for numerical applications, our focus are transitions matrices with parametric entries. For example the one describing the stochastic behaviour of a multiclass service system with loss and with which we illustrate our theory.

Our starting point, cf. Subsection 2.1, is an abstract model for transition matrices on a finite or denumerably infinite state space. In this setup the set of states is partitioned in such a way that *forward* transitions are only admissible between two consecutive sets of the partition. Moreover, these transitions are mandatory. This property is an extension to sets of *skip–freeness* to the left for transition matrices, cf. Example 2.23, p.78, [7]. Further, the considered transition matrices have a unique stationary distribution. For this model we derive bounds for the quotient of two suitable equilibrium probabilities, cf. Section 3.

An application of this setting is the following situation: Let A be a nonempty, proper subset of the state space S of an irreducible and positive recurrent transition matrix K. Our interest is the mean first passage time to the set A from an arbitrary initial law. We first construct a partition of the state space according to the number of steps needed to reach the set A, cf. Subsection 2.3. Further, in Section 5, we represent the mean first passage time to the set A as a quotient of equilibrium probabilities of an associated Markov chain defined in terms of the initial law and the transition kernel K. Finally, we show that this associated transition matrix together with the constructed partition satisfies the requirements of our abstract model. Therefore, the given bounds for the mean first passage time are a special case of the estimates derived for the quotient of suitable equilibrium probabilities of our abstract model.

We finally concretize our application in Section 6 with an asymptotic analysis of a non–reversible Markovian star network with loss.

1.1 General notation

Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ be the set of natural numbers, $\mathbb{N}_> = \{1, 2, \cdots\}$ be the set of natural numbers strictly greater than 0, $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the union of the natural numbers with infinity and $\overline{\mathbb{N}}_> = \mathbb{N}_> \cup \{\infty\}$ be the union of the natural numbers strictly greater than 0 with infinity. Given $N \in \overline{\mathbb{N}}$ we write $a < N + 1$ to mean $a \leq N$ if N is finite and $a < \infty$ otherwise. Let S be an at most countable set.

(a) For any matrix $K \in \mathbb{R}^{S \times S}$ and any subset $D \subseteq S$, we denote by $K_{|_{D\times D}}$ the restriction of the matrix K to the set $D \times D$.

(b) Let γ be a distribution on S and let D be a subset of E such that $\gamma(D) > 0$. We denote by γ_D the distribution $(\frac{\gamma(d)}{\gamma(D)})_{d\in D}$. The context defines if γ_D is a probability vector defined on S with support D or if it is defined on D.

(c) For a subset D of S, we denote by $\complement D$ the complement of the set D.

(d) The symbol $\mathbf{1}$ denotes a column vector with all entries equal to the scalar 1.

(e) The symbol δ_x denotes the Dirac distribution at an element $x \in S$.

(f) For a transition matrix $K \in \mathbb{R}^{S\times S}$, we denote by K^n the n–step transition matrix.

(g) We write $f(x) = \mathcal{O}(g(x))$ $(x \to \infty)$ (or shorter $f = \mathcal{O}(g)$), if there exist constants $C, c > 0$ such that $c|g(x)| \le f(x) \le C|g(x)|$, for x arbitrarily large.

2 Block–skip–freeness

Let $K \in \mathbb{R}^{S\times S}$ be a transition matrix on the state space S with *unique* stationary distribution γ.

2.1 Definition

Let $M \in \overline{\mathbb{N}}_{>}$. Let $\tilde{\mathcal{B}} = \{\tilde{B}_0\} \cup \{B_a | 1 \le a < M+1\}$ be a partition of the state space S. Further, let

$$(1) \quad B_0 := \{x \in \tilde{B}_0 | K(x, \bigcup_{1\le a<M+1} B_i) > 0\}.$$

We will call the set B_i, $0 \le i < M+1$, the ith *block*. For all $0 \le a < b < M+1$, let $B_a^b := \bigcup_{j=a}^b B_j$ and $\tilde{B}_0^a := \tilde{B}_0 \cup B_1^a$. Further, we denote by $\mathcal{B}$ the partition $\{B_a | 0 \le a < M+1\}$ of the set B_0^M.

We say that the pair $(K, \tilde{\mathcal{B}})$ satisfies the conditions $(\star)$, if

(i) K is *block-skip-free to the left* (with respect to $\mathcal{B}$), i.e,

$$\sup_{x\in B_a} K(x, \tilde{B}_0^{a-2}) = 0,$$

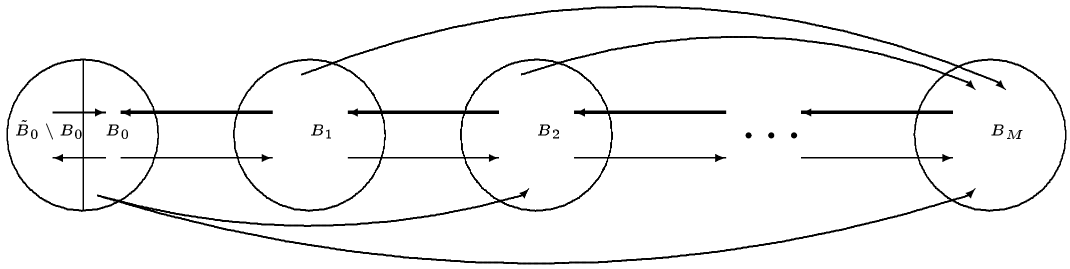

Figure 1: Transition graph of K for the case $M \in \mathbb{N}_>$. The thick arrows represent mandatory transitions whereas the thin arrows represent optional transitions.

for all $2 \leq a < M+1$. Graphically,

$$K = \begin{pmatrix} K_{|_{\tilde{B}_0\times\tilde{B}_0}} & K_{|_{\tilde{B}_0\times B_1}} & K_{|_{\tilde{B}_0\times B_2}} & K_{|_{\tilde{B}_0\times B_3}} & \cdots \\ K_{|_{B_1\times\tilde{B}_0}} & K_{|_{B_1\times B_1}} & K_{|_{B_1\times B_2}} & K_{|_{B_1\times B_3}} & \cdots \\ 0 & K_{|_{B_2\times B_1}} & K_{|_{B_2\times B_2}} & K_{|_{B_2\times B_3}} & \cdots \\ 0 & 0 & K_{|_{B_3\times B_2}} & K_{|_{B_3\times B_3}} & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix};$$

(ii) For all $1 \leq a < M+1$ and for all $x \in B_a$, we have $K(x, B_{a-1}) > 0$;

(iii) For all $x \in B_1$ we have $K(x, \tilde{B}_0 \setminus B_0) = 0$; and

(iv) There exists a number N, $1 \leq N < M+1$, such that for all $1 \leq a < N+1$, $\gamma(B_a) > 0$ and $\gamma(S \setminus \tilde{B}_0^N) = 0$.

Note that, under the conditions $(\star)$, $\gamma(B_0) > 0$. Indeed, by (iv) we have $\gamma(B_1) > 0$. Therefore, there exists an element $b \in B_1$ such that $\gamma(b) > 0$. This last assertion together with the balance equation $(\gamma K)(B_0) = \gamma(B_0)$ and the condition (ii) implies

(2) $\quad \gamma(B_0) \geq \gamma(b)K(b, B_0) > 0.$

2.2 Remarks

(a) In analogy to block-skip-freeness to the left we say that K is *block-skip-free to the right* (with respect to $\mathcal{B}$), if

$$\sup_{x \in B_a} K(x, B_{a+2}^M) = 0,$$

for all $0 \leq a < M - 1$. If K is both block-skip-free to the right and block-skip-free to the left then we say that it is *block-skip-free* (with respect to $\mathcal{B}$). Clearly, these definitions may be extended in a natural way to restrictions of the matrix K.

(b) Block–skip–freenes is an extension of skip–freeness, cf. Example 2.23, p.78, [7] or [8].

(c) Suppose that $B_a = \{x_a\}$ for all $0 \leq a < M + 1$ and $\tilde{B}_0 = B_0$. It is well known that the stationary distribution γ satisfies the equalities

$$\gamma(x_{a+1}) = \frac{\gamma(x_a)(1 - K(x_a, x_a)) - \sum_{b=0}^{a-1} \gamma(x_b) K(x_b, x_a)}{K(x_{a+1}, x_a)}, \qquad 0 \leq a < N.$$

If the entries of the matrix K are real numbers and S is a finite set then there exist algorithms to compute γ.

Let $F \subseteq S$ be the irreducible and positiv recurrent component of K. The goal of this paper is to develop bounds for the quotient $q_{\mathcal{B}} := \frac{\gamma(B_1^N)}{\gamma(B_0)}$ which do not depend on the knowledge of F.

2.3 Definition

Let A be a nonempty, proper subset of S and let $B := S \setminus A$. Let the pair $(K_{|_{B \times S}}, A)$ satisfy the condition

$(\star\,\star)$ for all $x \in B$ there exists a path from x to A.

The aim of this definition is to show how to construct in a natural way a partition $\tilde{B}$ such that $(K, \tilde{B})$ satisfies the conditions $(\star)$.

Let us denote by $G_K = (S, E_K)$ the oriented graph on the state space S associated with K, i.e.,

$$E_K = \{(x, y) \in S \times S | K(x, y) > 0\}.$$

For all $x \in S$, let us denote by $l_{x,A}$ the distance between x and A, i.e., the lenght $l_{x,A} = \min\{n \in \mathbb{N} | K^n(x, A) > 0\}$ of the shortest path from x to A. Let $\tilde{B}_{0,A} := A$. Let $M := \sup\{l_{x,A} | x \in S\} \in \overline{N}_>$ and, for all $1 \leq k < M+1$, let $B_{k,A} = \{x \in B | l_{x,A} = k\} \subseteq S$. Let finally $B_{0,A} := \{x \in \tilde{B}_{0,A} | K(x, B_{1,A}^M) > 0\}$.

Therefore, the given definition divides the state space S according to the minimal number of steps needed to reach the set A. Moreover, a simple induction argument shows that $B_{1,A} := \{x \in B | K(x, A) > 0\} \neq \emptyset$, and for all $2 \leq b < M + 1$, we have

$$(3) \quad B_{b,A} := \{x \in B \setminus \bigcup_{i=1}^{b-1} B_{i,A} | K(x, B_{b-1,A}) > 0\} \neq \emptyset,$$

where $\bigcup_{i=1}^{0} B_{i,A} := \emptyset$. Further, the condition $(\star\,\star)$ implies that $\tilde{B}_A := \{\tilde{B}_{0,A}\} \cup \{B_{b,A} | 0 < b < M+1\}$ is a partition of the state space S.

2.4 Lemma

Consider Definition 2.3 and let $\gamma(B) > 0$.

(a) There exists a number N, $1 \leq N < M+1$, such that $\gamma(B_{b,A}) > 0$ for all $0 \leq b < N+1$ and $\gamma(S \setminus \tilde{B}_{0,A}^N) = 0$.

(b) If, for all $x \in B_{1,A}$, $K(x, \tilde{B}_{0,A} \setminus B_{0,A}) = 0$ then the pair $(K, \tilde{\mathcal{B}}_A)$ satisfies the conditions $(\star)$.

Proof. In order to prove Part (a) note that since $\{B_{b,A} | 1 \leq b < M+1\}$ is a partition of the set B and $\gamma(B) > 0$, there exists a number b, $1 \leq b < M+1$, such that $\gamma(B_{b,A}) > 0$. Let

$$(4) \quad N := \sup\{1 \leq i < M+1 | \gamma(B_{i,A}) > 0\} \ (\geq 1).$$

Moreover, using a similar argument as the one used in order to prove (2) one may show that if $\gamma(B_{b+1,A}) > 0$, $0 \le b < N+1$ then $\gamma(B_{b,A}) > 0$. These arguments imply plainly the claim of Part (a).

We now prove Part (b). By 2.3 the conditions 2.1(i) and (ii) are clearly satisfied. By assumption, 2.1(iii) holds. Finally, 2.1(iv) is a consequence of Part (a). $\square$

2.5 Remarks and Examples

(a) Except for the condition $(\star\,\star)$ and the uniqueness of the stationary distribution γ, there is no structural assumption about the underlying transition matrix K in the Definition 2.3. The fact that the states can be labelled so that the transition matrix can be put into a block–skip–free to the left form is not an assumption but a *result* of the application of the Algorithm 2.3.

(b) It is important to note that the definition of the partition $\mathcal{B}_A$ depends only on the restriction $K_{|_{B\times S}}$ of the transition matrix K to the set $B \times S$. Not until we compute the set $B_{0,A}$ does the restriction of K to the set $A \times S$ play a role.

(c) In Section 5 we use the partition $\tilde{\mathcal{B}}_A$ of the state space together with the results of Section 4 in order to derive bounds for the mean first passage time to the set A.

(d) The following two examples show that the parameter N may equal M or it may be strictly smaller than M. Let $S = 0..2$, $A = \{0\}$, and $B = 1..2$. Morever, let

$$K_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } K_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The (unique) stationary distribution of K_1 is given by $\gamma_1 = (\frac{1}{2}, \frac{1}{2}, 0)$ whereas the (unique) stationary distribution of K_2 equals $\gamma_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Moreover, since the restriction of both transition matrices to the set $B \times A$ are equal, Definition 2.3 yields in both cases $B_{i,A} := \{i\}$ for all $i \in 0..2$. In both cases we have $M = 2$. Nevertheless, $N_1 = 1$ whereas

$N_2 = 2$. In the first situation there exists a nonempty set, namely $B_{2,A}$, for which the corresponding stationary probability is 0.

Consider the Background 2.1. As already written, the aim of this paper is to derive estimates for the quotient $\frac{\gamma(B_1^N)}{\gamma(B_0)}$. The developed method uses the concept of *balanced partitions* introduced in [5]. That's why we first give a résumé of the main results concerning this concept. The interested reader may find in [5] a detailed exposition of the subject.

3 Balanced partitions

As already mentioned, the concept of balanced partitions plays a central role in our theory. Loosly speaking, a partition of the state space of a Markov process is balanced if the flow in both directions between any two of its classes are equal in equilibirum. This property is a "macro" version of the detailed balance property of reversible processes, i.e., reversibility implies balancedness of *each* partition of the state space. Let $Q \in \mathbb{R}^{S\times S}$ be a transition matrix with stationary distribution σ and let $N \in \overline{\mathbb{N}}$. Let $\mathcal{Z} = \{S_k | 0 \leq k < N+1\}$ be a partition of the state space S consisting of sets S_k such that $\sigma(S_k) > 0$ for all $0 \leq k < N+1$. We will occasionally need the canonical projection $\pi : S \to \mathcal{Z}$. The stochastic matrix $Q_{\mathcal{Z}} \in \mathbb{R}^{\mathcal{Z}\times\mathcal{Z}}$ defined by

$$Q_{\mathcal{Z}}(S_k, S_l) = \frac{1}{\sigma(S_k)} \sum_{x\in S_k} \sigma(x) Q(x, S_l), \qquad S_k, S_l \in \mathcal{Z},$$

is called the ideal aggregation of Q with respect to $\mathcal{Z}$ (and σ), cf. [6], p.140, [12], and [2]. It is well known that $Q_{\mathcal{Z}}$ is an irreducible transition matrix with stationary distribution σ_π.

The following concept was introduced in [5]. We call the partition $\mathcal{Z}$ *balanced* (with respect to Q) if the stationary distribution σ_π of the ideal aggregation $Q_{\mathcal{Z}}$ is reversible. Thus, balancedness of $\mathcal{Z}$ means

$$\sigma(S_k) Q_{\mathcal{Z}}(S_k, S_l) = \sigma(S_l) Q_{\mathcal{Z}}(S_l, S_k)$$

or, equivalently,

$$\sum_{x\in S_k}\sigma(x)Q(x,S_l)=\sum_{x\in S_l}\sigma(x)Q(x,S_k).$$

This equality can be interpreted as "balance of flow" between S_k and S_l in equilibrium. Finally, if $Q_{\mathcal{Z}}(S_k,S_l)>0$ or, equivalently, $Q_{\mathcal{Z}}(S_l,S_k)>0$ then balancedness of $\mathcal{Z}$ implies

$$\frac{\sigma(S_k)}{\sigma(S_l)}=\frac{\sum_{x\in S_l}\sigma_{S_l}(dx)Q(x,S_k)}{\sum_{x\in S_k}\sigma_{S_k}(dx)Q(x,S_l)}. \tag{5}$$

Note that any splitting into *two* measurable subsets defines a balanced partition. The main interest in balanced partitions is their ability to furnish lower and upper estimates of the stationary distribution σ_π. The basis of them is the following result for balanced partitions due to [5], cf. Proposition 3.1. It deals with upper bounds of $\sigma(S_k)/\sigma(S_l)$. The corresponding lower bounds follow easily from them by swapping k and l. Let us abbreviate $S_{k,l}=\{x\in S_k|Q(x,S_l)>0\}$, $S_k,S_l\in\mathcal{Z}$.

3.1 Proposition

Let $\mathcal{Z}$ be balanced. If $S_k,S_l\in\mathcal{Z}$ are such that $Q_{\mathcal{Z}}(S_k,S_l)>0$ or, equivalently, $Q_{\mathcal{Z}}(S_l,S_k)>0$ then for all $S_{k,l}\subseteq T_{k,l}\subseteq S_k$ and $S_{l,k}\subseteq T_{l,k}\subseteq S_l$ we have $\sigma(T_{k,l})>0$ and $\sigma(T_{l,k})>0$.

(a) We have

$$\frac{\sigma(T_{k,l})}{\sigma(T_{l,k})}=\frac{\sum_{x\in T_{l,k}}\sigma_{T_{l,k}}(x)Q(x,S_k)}{\sum_{x\in T_{k,l}}\sigma_{T_{k,l}}(x)Q(x,S_l)}. \tag{6}$$

(b) The number $\sup_{x\in T_{l,k}}Q(x,S_k)$ is strictly positive and we have the estimate

$$\frac{\sigma(S_{k,l})}{\sigma(T_{l,k})}\le\frac{\sup_{x\in T_{l,k}}Q(x,S_k)}{\inf_{x\in S_{k,l}}Q(x,S_l)}=\frac{\sup_{x\in S_l}Q(x,S_k)}{\inf_{x\in S_{k,l}}Q(x,S_l)}\qquad\text{(the bound may be infinite.)} \tag{7}$$

Moreover, the number $\sup_{x\in T_{k,l}}Q(x,S_l)$ is strictly positive and we have the estimate

$$\frac{\sigma(T_{k,l})}{\sigma(S_{l,k})}\ge\frac{\inf_{x\in S_{l,k}}Q(x,S_k)}{\sup_{x\in T_{k,l}}Q(x,S_l)}=\frac{\inf_{x\in S_{l,k}}Q(x,S_k)}{\sup_{x\in S_k}Q(x,S_l)}\qquad\text{(the bound may be zero.)} \tag{8}$$

In the next section we develop bounds for $q_{\mathcal{B}}$. Our method uses the principle of *divide and conquer*. We also discuss the cases where our estimates are exact.

4 Bounds for $q_{\mathcal{B}}$

The exposition of our method needs some preliminaries. The following definition and lemma will be useful. They are simple generalizations of well–known results and definitions (see e.g. pp. 114–16 of [6]) and we shall omit their proofs.

4.1 Notation and explanation

Let $Q \in \mathbb{R}^{S\times S}$ be a transition matrix with a single irreducible and positive recurrent set F and (unique) stationary distribution σ. For every nonempty subset D of S such that $\sigma(D) > 0$, let

$$(9) \quad Q_D := Q_{|_{D\times D}} + Q_{|_{D\times \complement D}} \sum_{n\geq 0} (Q_{|_{\complement D\times \complement D}})^n Q_{|_{\complement D\times D}} \in \mathbb{R}^{D\times D}.$$

It is well known that $Q_D \in \mathbb{R}^{D\times D}$ is a stochastic matrix on D with a single positive recurrent set $D \cap F$ and stationary distribution σ_D, cf. [1], p. 153, Proposition 3.4. Moreover, let $Y_n : (\Omega, P) \to S$, $n \in \mathbb{N}$, be a Markov chain with transition matrix Q and initial distribution P_{Y_0}. It is well known that the transition matrix Q_D describes the motion of the chain $(Y_n)_{n\in\mathbb{N}}$ in D.

The following lemma gives componentwise lower bounds for Q_D, namely through matrices Q_C's for subsets C's that contain D.

4.2 Lemma

Let $D \subseteq C$ such that $\sigma(D) > 0$. Then

$$(10) \quad Q_D(x,y) \geq Q_C(x,y), \qquad x, y \in D.$$

Proof. Let $c, d \in D$. An induction argument implies that for all $z_1, z_2 \in \complement C$ and $n \in \mathbf{N}$ we have $(Q_{|_{\complement D\times \complement D}})^n(z_1,z_2) \geq (Q_{|_{\complement C\times \complement C}})^n(z_1,z_2)$. Moreover, by definition of the matrices Q_D and

Q_C together with the inclusion $\complement D \supseteq \complement C$ and the assertion above we have

$$
\begin{aligned}
& Q_D(c,d) \\
= \; & Q_{|_{D\times D}}(c,d) + Q_{|_{D\times \complement D}} \sum_{n\geq 0} (Q_{|_{\complement D\times \complement D}})^n Q_{|_{\complement D\times D}}(c,d) \\
= \; & Q(c,d) + \sum_{z_1,z_2\in \complement D} Q(c,z_1) \sum_{n\geq 0} (Q_{|_{\complement D\times \complement D}})^n (z_1,z_2) Q(z_2,d) \\
\geq \; & Q(c,d) + \sum_{z_1,z_2\in \complement C} Q(c,z_1) \sum_{n\geq 0} (Q_{|_{\complement D\times \complement D}})^n (z_1,z_2) Q(z_2,d) \\
= \; & Q(c,d) + \sum_{n\geq 0} \sum_{z_1,z_2\in \complement C} Q(c,z_1) (Q_{|_{\complement D\times \complement D}})^n (z_1,z_2) Q(z_2,d) \\
\geq \; & Q(c,d) + \sum_{n\geq 0} \sum_{z_1,z_2\in \complement C} Q(c,z_1) (Q_{|_{\complement C\times \complement C}})^n (z_1,z_2) Q(z_2,d) \\
= \; & Q_C(c,d).
\end{aligned}
$$

This is the claim. □

Let now $K \in \mathbb{R}^{S\times S}$ be a transition matrix with unique stationary distribution γ. Moreover, let $\tilde{\mathcal{B}} = \{\tilde{B}_0\} \cup \{B_a | 1 \leq a < M+1\}$ be a partition of the state space S such that the pair $(K, \tilde{\mathcal{B}})$ satisfies the conditions $(\star)$, cf. 2.1. The aim of this section is to develop bounds for the quotient $q_{\mathcal{B}} = \frac{\gamma(B_1^N)}{\gamma(B_0)}$. For this sake, we write first this parameter as

$$(11) \quad q_{\mathcal{B}} = \frac{\gamma(B_1^N)}{\gamma(B_0)} = \sum_{a=0}^{N-1} \prod_{b=0}^{a} \frac{\gamma(B_{b+1})}{\gamma(B_b)}.$$

Next, we develop lower and upper bounds for each of the quotients $\frac{\gamma(B_{b+1})}{\gamma(B_b)}$, $0 \leq b < N$, cf. Lemma 4.3.

Since for all $0 \leq a < N+1$ we have $\gamma(B_a^N) > 0$, the transition matrix $K_{B_a^N}$ given by

$$(12) \quad K_{B_a^N} = K_{|_{B_a^N\times B_a^N}} + K_{|_{B_a^N\times \complement B_a^N}} \sum_{n\geq 0} (K_{|_{\complement B_a^N\times \complement B_a^N}})^n K_{|_{\complement B_a^N\times B_a^N}} \in \mathbb{R}^{B_a^N\times B_a^N},$$

cf. (9), is well defined.

4.3 Lemma

(a) We have

$$(13)\quad \frac{\gamma(B_1)}{\gamma(B_0)} = \frac{\sum_{x\in B_0}\gamma_{B_0}(x)K(x,B_1^N)}{\sum_{x\in B_1}\gamma_{B_1}(x)K(x,B_0)}.$$

Moreover, for all $1\le a<N$ we have

$$(14)\quad \frac{\gamma(B_{a+1})}{\gamma(B_a)} = \frac{\sum_{x\in B_a}\gamma_{B_a}(x)K_{B_a^N}(x,B_{a+1}^N)}{\sum_{x\in B_{a+1}}\gamma_{B_{a+1}}(x)K(x,B_a)}.$$

(b) We have

$$(15)\quad L(0) := \frac{\inf_{x\in B_0}K(x,B_1^N)}{\sup_{x\in B_1}K(x,B_0)} \le \frac{\gamma(B_1)}{\gamma(B_0)} \le \frac{\sup_{x\in B_0}K(x,B_1^N)}{\inf_{x\in B_1}K(x,B_0)} =: U(0),$$

and, for all $1\le a<N$, we have

$$(16)\quad \frac{\gamma(B_{a+1})}{\gamma(B_a)} \le \frac{1-\inf_{x\in B_a}K(x,B_a)}{\inf_{x\in B_{a+1}}K(x,B_a)} =: U(a)$$

and

$$(17)\quad \frac{\gamma(B_{a+1})}{\gamma(B_a)} \ge \frac{\inf_{x\in B_a}K(x,B_{a+1}^N)}{\sup_{x\in B_{a+1}}K(x,B_a)} =: L(a).$$

(c) (B_0 *singleton*). Let the set B_0 be a singleton and let $\tilde{B}_0\setminus B_0=\emptyset$. Let us identify the unique element of the set B_0 with itself. We have the following estimates

$$(18)\quad L(0) := \frac{K(B_0,B_1^N)}{\sup_{x\in B_1}K(x,B_0)} \le \frac{\gamma(B_1)}{\gamma(B_0)} \le \frac{K(B_0,B_1^N)}{\inf_{x\in B_1}K(x,B_0)} =: U(0);$$

and

$$(19)\quad \begin{aligned} L(1) &:= \frac{\inf_{x\in B_1}\left(K(x,B_2^N)+\frac{K(x,B_0)K(B_0,B_2^N)}{K(B_0,B_1^N)}\right)}{\sup_{x\in B_2}K(x,B_1)} \\ &\le \frac{\gamma(B_2)}{\gamma(B_1)} \\ &\le \frac{\sup_{x\in B_1}\left(K(x,B_2^N)+\frac{K(x,B_0)K(B_0,B_2^N)}{K(B_0,B_1^N)}\right)}{\inf_{x\in B_2}K(x,B_1)} =: U(1); \end{aligned}$$

and for all $2\le a<N$ we have

$$L(a) \le \frac{\gamma(B_{a+1})}{\gamma(B_a)} \le U(a),$$

where $L(a)$ and $U(a)$ are given by (17) and (16) respectively.

(d) (*The block–skip–free case*). Let the restriction $K_{|B_1^N \times B_0^N}$ of K to the set $B_1^N \times B_0^N$ be block–skip–free (with respect to $\mathcal{B}$), cf. 2.2(a). We have the following estimates

$$L(0) \leq \tfrac{\gamma(B_1)}{\gamma(B_0)} \leq U(0),$$

where $L(0)$ and $U(0)$ are given by (15), and

$$(20) \qquad L(a) := \tfrac{\inf_{x \in B_a} K(x, B_{a+1})}{\sup_{x \in B_{a+1}} K(x, B_a)} \leq \tfrac{\gamma(B_{a+1})}{\gamma(B_a)} \leq \tfrac{\sup_{x \in B_a} K(x, B_{a+1})}{\inf_{x \in B_{a+1}} K(x, B_a)} =: U(a),$$

for all $1 \leq a < N$.

Proof. Let $F \subseteq S$ be the (unique) irreducible and positiv recurrent subset of S. Clearly, $F \subseteq \tilde{B}_0^N$. Consider first the transition matrix $K_{B_0^N}$ given by (12). A moment of reflection shows that

$$(21) \quad K_{B_0^N}(x, B_0) = K(x, B_0), x \in B_1^N \cap F, \text{ and } K_{B_0^N}(x, B_1) = K(x, B_1), x \in B_0 \cap F.$$

Let now, in Proposition 3.1, $S = B_0^N$, $Q = K_{B_0^N}$, $\mathcal{Z} = \{B_0, B_1^N\}$, $S_k = B_1^N$, and $S_l = B_0$. Therefore, by the definition of the set $S_{k,l}$ together with the left hand side assertion of (21) and 2.1(i)–(ii) we have

$$S_{k,l} \cap F = \{x \in B_1^N \cap F | K_{B_0^N}(x, B_0) > 0\} = \{x \in B_1^N \cap F | K(x, B_0) > 0\} = B_1 \cap F.$$

Using this time the assertion on the right hand side of (21) and 2.1(i)–(ii) we have

$$S_{l,k} \cap F = \{x \in B_0 \cap F | K_{B_0^N}(x, B_1^N) > 0\} = \{x \in B_0 \cap F | K(x, B_1^N) > 0\} = B_0 \cap F.$$

Now, since by assumption $\gamma(B_1) > 0$, there exists an $x \in B_1 \cap F$ such that $\gamma(x) > 0$. Moreover, by 2.1(ii) we have $K(x, B_0) > 0$ and, hence, by the first assertion in (21) we have

$$\begin{aligned} & (K_{B_0^N})_{\mathcal{Z}}(B_1^N, B_0) \\ = & \sum_{y \in B_1^N} \gamma_{B_1^N}(y) K_{B_1^N}(y, B_0) \\ \geq & \gamma(x) K_{B_1^N}(x, B_0) = \gamma(x) K(x, B_0) > 0. \end{aligned}$$

Let further $T_{k,l} = B_1$ and $T_{l,k} = B_0$. Finally, since the partition $\mathcal{Z}$ has two elements, it is balanced with respect to $K_{B_0^N}$. Therefore, the hypotheses of Proposition 3.1 are satisfied and by (21) together with (6) we have next

$$\begin{aligned}
&\frac{\gamma(B_1)}{\gamma(B_0)}\\
= &\frac{\sum_{x\in B_0}\gamma_{B_0}(x)K_{B_0^N}(x,B_1^N)}{\sum_{x\in B_1}\gamma_{B_1}(x)K_{B_0^N}(x,B_0)}\\
= &\frac{\sum_{x\in B_0\cap F}\gamma_{B_0}(x)K_{B_0^N}(x,B_1^N)}{\sum_{x\in B_1\cap F}\gamma_{B_1}(x)K_{B_0^N}(x,B_0)}\\
= &\frac{\sum_{x\in B_0\cap F}\gamma_{B_0}(x)K_{B_0^N}(x,B_1^N)}{\sum_{x\in B_1\cap F}\gamma_{B_1}(x)K(x,B_0)}\\
= &\frac{\sum_{x\in B_0}\gamma_{B_0}(x)K_{B_0^N}(x,B_1^N)}{\sum_{x\in B_1}\gamma_{B_1}(x)K(x,B_0)}.
\end{aligned}$$

This is the claim (13). Let now $1 \le a < N+1$ and let in 3.1 $S = B_a^N$, $Q = K_{B_a^N}$, $\mathcal{Z} = \{B_a, B_{a+1}^N\}$, $S_k = B_{a+1}^N$, and $S_l = B_a$. Reasoning in a similar way as in the case $a = 0$ but taking this time into account the assertion

$$(22)\quad K_{B_a^N}(x,B_a) = K(x,B_a),$$

for all $x \in B_{a+1}^N \cap F$, we obtain $S_{k,l}\cap F = B_{a+1}\cap F$ and $S_{l,k}\cap F \subseteq B_a\cap F$. Similarly, for the ideal aggregation $(K_{B_{a+1}^N})_{\mathcal{Z}}$ of $K_{B_{a+1}^N}$ with respect to $\mathcal{Z}$ we have

$$(K_{B_a^N})_{\mathcal{Z}}(B_{a+1}^N, B_a) = \sum_{x\in B_{a+1}^N}\gamma_{B_{a+1}^N}(x)K_{B_{a+1}^N}(x,B_a) > 0.$$

Further, since the partition $\mathcal{B}_{a,a+1}$ has two elements, it is balanced with respect to $K_{B_a^N}$. Thus, the hypotheses of Proposition 3.1 are satisfied and from (22) together with similiar arguments as those used for the case $a = 0$ we have that (14) is (6) applied to this situation. This is the Claim (a).

Clearly, (15) follows from (13) together with the fact that both, numerator and denominator of (13) are convex combinations. Let now $1 \le a < N+1$. Note first that $K_{B_a^N}(x,B_{a+1}^N) = 1 - K_{B_a^N}(x,B_a)$, $x \in B_a^N\cap F$. Moreover, by Lemma 4.2 applied to the case $Q = K_{B_a^N}$, $D = B_a^N$

and $C = S$ we have $K_{B_a^N}(x, B_a) \geq K(x, B_a)$, $x \in B_a^N$. These last assertions together with (14) imply

$$\frac{\gamma(B_{a+1})}{\gamma(B_a)} = \frac{1 - \sum_{x \in B_a \cap F} \gamma_{B_a}(x) K_{B_a^N}(x, B_a)}{\sum_{x \in B_{a+1}} \gamma_{B_a}(x) K(x, B_a)} \leq \frac{1 - \sum_{x \in B_a} \gamma_{B_a}(x) K(x, B_a)}{\sum_{x \in B_{a+1}} \gamma_{B_a}(x) K(x, B_a)} \leq \frac{1 - \inf_{x \in B_a} K(x, B_a)}{\inf_{x \in B_{a+1}} K(x, B_a)},$$

i.e., (16). Similar arguments make possible to derive (17) from (14).

We now prove Part (c). Clearly, (18) follows from (15) together with the fact that the set B_0 is a singleton. In order to proof (19) note that by (9) applied to the case $Q = K$ and $D = B_1^N$ together with the fact that the set $\complement B_1^N = B_0$ is a singleton we have for all $x, y \in B_1^N \cap F$

$$\begin{aligned} & K_{B_1^N}(x, y) \\ = \;& K(x, y) + K(x, B_0)(1 - K(B_0, B_0))^{-1} K(B_0, y) \\ = \;& K(x, y) + \frac{K(x, B_0) K(B_0, y)}{K(B_0, B_1^N)}. \end{aligned}$$

Taking into account the equality above, the claim follows from (14).

We finally prove Part (d). Let then $1 \leq a < N$. Clearly, block–skip–freeness of $K_{|B_1^N \times B_0^N}$ implies $K_{|B_0^{a-1} \times B_a^N}(y, B_{a+1}^N) = 0$ for all $x \in B_0^{a-1}$ and $K(x, B_{a+1}^N) = K(x, B_{a+1})$. In turn, these assertions imply

$$\begin{aligned} & K_{B_a^N}(x, B_{a+1}^N) \\ = \;& K(x, B_{a+1}^N) + K_{|B_0^{a-1} \times B_0^{a-1}} \sum_{n \geq 0} (K_{|B_0^{a-1} \times B_0^{a-1}})^n K_{|B_0^{a-1} \times B_a^N}(x, B_{a+1}^N) \\ = \;& K(x, B_{a+1}). \end{aligned}$$

Therefore, using (13) and (14) together with the equality above, we obtain

$$(23) \quad \frac{\gamma(B_{a+1})}{\gamma(B_a)} = \frac{\sum_{x \in B_a} \gamma_{B_a}(x) K(x, B_{a+1})}{\sum_{x \in B_{a+1}} \gamma_{B_{a+1}}(x) K(x, B_a)}.$$

Similar arguments as the one used in the proof of (15) make possible the derivation of (20). □

We are now ready to establish the main result of this section.

4.4 Theorem

We have

$$(24)\quad q_{\mathcal{B}}^{L} := \sum_{a=0}^{N-1}\prod_{b=0}^{a} L(b) \leq q_{\mathcal{B}} \leq \sum_{a=0}^{N-1}\prod_{b=0}^{a} U(b) =: q_{\mathcal{B}}^{U},$$

where $L(b)$ and $U(b)$, $0 \leq b < N$, are given by (15), (17) and (16). In the case 4.3(c), $L(b)$ and $U(b)$, $0 \leq b < N$, are given by (18), (19), (17) and (16), whereas in the case 4.3(d) the estimates $L(b)$ and $U(b)$, $0 \leq b < N$, are given by (20) and (15). The lower bound $q_{\mathcal{B}}^{L}$ may be trivial, whereas the upper bound $q_{\mathcal{B}}^{U}$ may be infinite.

Proof. The claim follows from the expression (11) together with the bounds given in the Lemma 4.3. □

Clearly, if S is finite then both bounds are nontrivial. Further, one may use Lemma 4.3 in order to estimate $\frac{\gamma(B_1^a)}{\gamma(B_0)}$ for each $1 \leq a < N+1$. The next proposition discusses the exactness of the bounds given in the theorem above.

4.5 Proposition (Exactness of the bounds)

Consider Theorem 4.4. Assume that the restriction of the transition matrix K to the set $B_0^N \times B_0^N$ is lumpable with respect to the partition $\mathcal{B}$, i.e., $K(x, B_b) = K(y, B_b)$, for all $x, y \in B_a$ and for all $0 \leq a, b < N+1$. If any one of the cases (i) 4.3(c) together with $N = 2$ or (ii) 4.3(d) or (iii) $N = 1$ meet then $q_{\mathcal{B}}^{L} = q_{\mathcal{B}} = q_{\mathcal{B}}^{U}$.

Proof. Consider first the case 4.3(c) together with $N = 2$ and let $0 \leq a < N+1$. Clearly, the given hypothesis of lumpability implies the lumpability of the transition matrix $K_{B_a^N}$ with respect to the partition $\{B_b | a \leq b < N+1\}$. Hence, we may simplify the quotients (13) and (14) as

$$(25)\qquad \frac{\gamma(B_1)}{\gamma(B_0)} = \frac{K(B_0, B_1^N)}{K(B_1, B_0)} \qquad \text{and} \qquad \frac{\gamma(B_{a+1})}{\gamma(B_a)} = \frac{K_{B_a^N}(B_a, B_{a+1}^N)}{K(B_{a+1}, B_a)},$$

where $K(B_0, B_1^N)$, $K(B_1, B_0)$, $K_{B_a^N}(B_1, B_{a+1}^N)$ and $K(B_{a+1}, B_a)$ denote the common transition probabilities over the corresponding sets. It is then straightforward that the value $\frac{\gamma(B_1)}{\gamma(B_0)}$ coincides with the given bounds $L(0)$ and $U(0)$, cf. (18), (19). Moreover, since in 4.3(b) we have computed exactly the transition matrix $K_{B_1^N}$, the discussion above implies plainly that both bounds, $L(1)$ and $U(1)$ given in (19) equal $\frac{\gamma(B_2)}{\gamma(B_1)}$. Therefore, the additional condition $N = 2$ assures the asserted equality.

Let now K be block–skip–free, cf. 4.3(d). Then the hypothesis of lumpability together with (23) and similar arguments as those used in the previous case, imply the asserted claim. Finally, the case $N = 1$ is a special case of the previous one. □

If a set $G \subset S$ containing the single irreducible and positiv recurrent component F can be determined then we may improve our given bounds by replacing the sets B_a, $0 \leq a < N+1$, with their intersection with G, throughout.

5 Bounds for the hitting time to the set A

Let $K \in \mathbb{R}^{S\times S}$ be an irreducible and positive recurrent transition matrix with unique stationary distribution γ. Let A be a nonempty, proper subset of S such that $\gamma(A) > 0$, and let $B := S \setminus A$. Further, let $\theta \in \mathbb{R}^S$ be a distribution on S. In this section, we first define a transition matrix on S which depends on K, A and θ and whose stationary distribution allows us to represent the mean first passage time to the set A as a quotient of the equilibrium probabilities of two suitable events. Next, we use the partition of the state space defined in 2.3 together with this representation of the mean first hitting time to the set A and the main result of Section 4 in order to develop bounds for this quantity.

5.1 The hitting times to the set A

Let $X_n : (\Omega, P) \to S$, $n \in \mathbb{N}$, be an homogeneous Markov chain with transition matrix K and initial distribution $P_{X_0} := \theta$. The random variable $T_A : (\Omega, P) \to \overline{\mathbb{N}}$ (resp. $S_A : (\Omega, P) \to \overline{\mathbb{N}}_>$)

defined by

$$T_A := \inf\{n \in \mathbb{N} | X_n \in A\}, \qquad S_A := \inf\{n \in \mathbb{N}_> | X_n \in A\},$$

where $\inf \emptyset := \infty$, is called the *first hitting time* to the set A, including time 0 (resp. not including time 0), cf. [10], Subsection 3.4. We also refer to T_A as the *first passage time* to the set A and to S_A as the *first return time* to the set A.

Our main interest are the expectations $E^\theta[T_A]$ and $E^\theta[S_A]$. Since $E^\theta[T_A] = \theta(B)E^{\theta_B}[T_A]$, we may assume that $\theta(B) = 1$. It is well known that, for the mean first passage time to the set A, we have

$$(26) \quad E^\theta[T_A] = \theta_{|B} \sum_{n \geq 0} (K_{|B\times B})^n \mathbf{1},$$

whereas for the mean first return time to the set A we have

$$(27) \quad E^\theta[S_A] = \theta(A) + \theta_{|A} K_{|A\times B} \sum_{n \geq 0} (K_{|B\times B})^n \mathbf{1} + \theta(B) E^{\theta_B}[T_A].$$

We restrict us to those θ's in the set $\mathcal{P}_A := \{\theta \in \mathbb{R}^S | E^\theta[S_A] < \infty\}$. It is therefore worthwile before we continue the exposition of the theory of this section to fix some ideas concerning the dependence of mean first return time to the set A with the initial distribution θ of the Markov chain X_n, $n \in \mathbb{N}$.

5.2 On the initial distribution

(a) Although K is irreducible and positive recurrent, it may be the case that $E^\theta[T_A] = \infty$ (and hence $E^\theta[S_A] = \infty$) for some θ's. Let, for example, $S = \mathbb{N}$ and let $K \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ be the transition matrix given by

$$K(x,y) = \begin{cases} 1, & x = 0, y = 1, \\ 1 - \frac{1}{x+1}, & x \geq 1, y = x - 1, \\ \frac{1}{x+1}, & x \geq 1, y = x + 1. \end{cases}$$

Clearly, K is irreducible. We now prove that K is positive recurrent by means of the classical Lyapunov–Foster Criterion, cf. [1], p. 18, Proposition 5.3. Let $E = 0..2$ and let $L : S \to \mathbb{R}$ be defined by $L(x) = x^2$, $x \in \mathbb{N}$. First, note that

$$(_EKL)(0) = \textstyle\sum_{x\geq 3} K(0,x)L(x) = 0 < \infty;$$

$$(_EKL)(1) = \textstyle\sum_{x\geq 3} K(1,x)L(x) = 0 < \infty;$$

$$(_EKL)(2) = \textstyle\sum_{x\geq 3} K(2,x)L(x) = K(2,3)L(3) = 9 < \infty.$$

Moreover, simple algebraic manipulations show that for all $x \geq 3$ we have

$$\begin{aligned} & (_EKL)(x) - L(x) + 1 \\ \leq\ & K(x,x-1)L(x-1) + K(x,x+1)L(x+1) - x^2 + 1 \\ =\ & (1 - \frac{1}{x+1})(x-1)^2 + \frac{1}{x+1}(x+1)^2 - x^2 + 1 \\ =\ & -2\frac{x^2 - 2x - 1}{x+1} < 0. \end{aligned}$$

Hence, both conditions of the Lyapunov–Foster Criterion are satisfied and K is positive recurrent. Clearly, for all $x \in \mathbb{N}_>$, we have $E^x[T_0] \geq x$. Let now $\theta \in \mathbb{R}^{\mathbb{N}}$ be given by $\theta(0) = 0$ and $\theta(x) = \frac{6}{\pi^2}\frac{1}{x^2}$, $x \in \mathbb{N}_>$. By the formula of total expectation we have

$$E^\theta[T_0] = \sum_{x\geq 1} \theta(x)E^x[T_0] = \frac{6}{\pi^2}\sum_{x\geq 1}\frac{1}{x^2}E^x[T_0] \geq \frac{6}{\pi^2}\sum_{x\geq 1}\frac{1}{x^2}x = \infty.$$

(b) Clearly, if $E^\theta[S_A] < \infty$ then $E^\theta[T_A] < \infty$. In order to see that the opposite direction is not always true, consider the example given in Part (a) and the following situation: Let $S = \mathbb{N}$ and $Q \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ the transition matrix given by

$$Q(x,y) = \begin{cases} \theta(y), & x = 0, y \in \mathbb{N}, \\ K(x,y), & \text{otherwise.} \end{cases}$$

Clearly, Q is irreducible. A moment of reflection shows that, in both situations, the corresponding mean first passage times to the set 0 coincide. A second moment of reflection shows that, in this situation, the mean first return time to the set 0 from the Dirac distribution at 0 and the mean first passage time to the set 0 from the law $Q(x, \cdot\,) = \theta$

differ by 1. Therefore, using the result of Part(a) together with these last assertions, we obtain $E^0[S_0] = \infty$. But $E^0[T_0] = 0$.

(c) $\mathcal{P}_A$ contains all distributions with finite support. Indeed, using [10], Corollary 5.4, which states that the irreducibility and positive recurrence of K implies that for all $x \in S$, $E^x[S_A] < \infty$ together with the formula of total expectation we have

$$E^\theta[S_A] = \sum_{x \in S_\theta} \theta(x) E^x[S_A] < \max_{x \in S_\theta} E^x[S_A] < \infty.$$

Therefore, if the state space S is finite then each distribution θ on S belongs to the set $\mathcal{P}_A$.

(d) A sufficient condition for the finiteness of $E^\theta[S_A]$ *for all* $\theta \in \mathbb{R}^S$ is the concept of *uniform recurrence*, cf. [10], p. 92, Definition 5.8. For a discrete Markov chain this notion is equivalent to the requirement

(28) $\quad \sup_{x \in S} E^x[S_y] < \infty,$

for all $y \in S$, cf. [10], p. 93, Examples 5.6(a). Indeed, let $X_n : (\Omega, P) \to S$, $n \in \mathbb{N}$, be uniformly recurrent and let $\theta \in \mathbb{R}^S$. By the formula of total expectation we have

$$E^\theta[S_A] = \sum_{x \in S} \theta(x) E^x[S_A] < \sup_{x \in S} E^x[S_A] < \infty.$$

The lemma 5.3 shows that the mean first passage time to the set A is a quotient of equilibrium probabilities of an associated chain defined in terms of the initial law and the transition kernel of the given Markov chain. This idea, and part of the proof of the lemma are borrowed from [6], Theorem 6.2.2. We extend this result to the case of an at most countable state space S. and improve it by reducing the original state space S in a suitable way.

5.3 Lemma

Let $\mathcal{Z}_A := \{A\} \cup \{\{x\} | x \in B\}$ and let $\pi_A : S \to \mathcal{Z}_A$ be the corresponding canonical projection. Let $\theta \in \mathcal{P}_A$. Let $K^{\theta,A} \in \mathbb{R}^{\mathcal{Z}_A \times \mathcal{Z}_A}$ be the transition matrix

$$(29) \quad K^{\theta,A} = \begin{pmatrix} \theta(A) & \theta_{|_B} \\ & \\ K_{|_{B\times A}}\mathbf{1} & K_{|_{B\times B}} \end{pmatrix}.$$

The following statements hold.

(a) Let $\theta_{\pi_A} := (\theta(A), \theta_{|_B}) \in \mathbb{R}^{\mathcal{Z}_A}$. Let $\mathbf{J}^{\theta,A}$ be the set of all states to which the process can go starting from a state in the support of the distribution θ_{π_A}. Then $\mathbf{J}^{\theta,A}$ is the unique irreducible and positive recurrent set and $A \in \mathbf{J}^{\theta,A}$.

(b) Let us denote by $\gamma^{\theta,A}$ the (unique) stationary distribution of the matrix $K^{\theta,A}$. Then $\gamma^{\theta,A}(A) > 0$ and

$$(30) \quad E^{\theta}[T_A] = \frac{\gamma^{\theta,A}(B)}{\gamma^{\theta,A}(A)}.$$

Proof. We first prove irreducibilty of $\mathbf{J}^{\theta,A}$. Clearly, from the state A we can go only to states in $\mathbf{J}^{\theta,A}$. By irreducibility of K, we can go from any state in the set B to the state A. Therefore, $A \in \mathbf{J}^{\theta,A}$, all the states in the set $S \setminus \mathbf{J}^{\theta,A}$ are transient, and the set $\mathbf{J}^{\theta,A}$ is irreducible.
We now prove that $K^{\theta,A}$ has a unique stationary distribution $\gamma^{\theta,A}$. Uniqueness follows from the fact that $\mathbf{J}^{\theta,A}$ is the single irreducible set. In order to show that this set is positive recurrent, it must be shown that $E^A[S_A^{\theta,A}] < \infty$, where $S_A^{\theta,A} : (\Omega, P) \to \overline{\mathbb{N}}_>$ is the first return time to the state A with respect to the Markov chain defined by the transition matrix $K^{\theta,A}$ and the inital distribution δ_A. A direct application of formulas (27) and (26) to this situation yields

$$(31) \quad E^A[S_A^{\theta,A}] = 1 + K^{\theta,A}(A,\cdot\,)_{|_B} \sum_{n\geq 0} (K_{|_{B\times B}})^n \mathbf{1} = 1 + \theta_{|_B} \sum_{n \geq 0} (K_{|_{B\times B}})^n \mathbf{1} = 1 + E^{\theta}[T_A] < \infty,$$

where the last assertion follows from the assumption $\theta \in \mathcal{P}_A$. The well–known assertion $E^A[S_A^{\theta,A}] = \frac{1}{\gamma^{\theta,A}(A)}$ together with the equality $E^A[S_A^{\theta,A}] = 1 + E^{\theta}[T_A]$, cf. (31), implies (30). $\square$

5.4 Remarks

(a) We thus see that the single vector $\gamma^{\theta,A}$ furnishes us with the mean first passage time to the set A. This representation is one of the building blocks used to develop bounds for this parameter. The set $\mathbf{J}^{\theta,A}$ may be too difficult to determine. For this reason, the given bounds for $E^{\theta}[S_A]$, cf. 5.5, do not depend on the knowledge of this set. Clearly, if the support of the distribution θ_{π_A} is the state space $\mathcal{Z}_A$ then $K^{\theta,A}$ is irreducible.

(b) Let $\theta \in \mathcal{P}_A$ such that $\theta(B) > 0$. As a corollary of 5.3(a) we have $\mathbf{J}^{\theta,A} \cap B \neq \emptyset$ and $\gamma^{\theta,A}(B) > 0$. Clearly, under the assumption of irreducibility of K, the transition matrix $K^{\theta,A}$ satisfies the condition $(\star\,\star)$. An application of Definition 2.3 to the case $K = K^{\theta,A}$ yields a partition $\mathcal{B}_A = \{\{A\}\} \cup \{B_{b,A} | 1 \leq b < M+1\}$ of the state space $\mathcal{Z}_A$. Let now $N := \sup\{1 \leq i < M+1 | \gamma(B_{i,A}) > 0\}$. By Lemma 2.4, the pair $(K^{\theta,A}, \mathcal{B}_A)$ satisfies the conditions $(\star)$, cf. 2.1.

(c) Using results due to Freidlin and Wentzell [4] in the large deviation theory of random dynamical systems Catoni, cf. Lemma 3.4 [3], derives a formula which expresses the expectation of the first passage time of A as a ratio of two finite sums of positive terms. The corresponding indexes of summation are acyclic paths joining a state in the set B with the set A whereas the corresponding weights are the products of the transition probabilities over the different paths. Although our approach also uses the concept of paths from a state in the set B to A, cf. 2.3, in order to derive a closed form for the mean first passage time to the set A, the way this formula is obtained differs substantially from the one given in [3]. First, we use *only* the path from x to A with *minimal* lenght. Second, the selected paths are used to construct (in the first place) a *partition* of the set B. Finally, the weights of the different paths do not (at least explicitly) play a role.

In the next subsection we apply the results of Section 4 together with the representation (30) and the Remark 5.4(b) in order to develop bounds for $E^{\theta}[T_A]$.

5.5 Theorem

Let $\theta \in \mathcal{P}_A$ such that $\theta(B) > 0$ and consider the partition $\mathcal{B}_A$ of the state space $\mathcal{Z}_A$, cf. 5.4(b). We have

(a)

$$(32)\quad L^{\theta,A} := \sum_{a=0}^{N-1} \prod_{b=0}^{a} L^{\theta,A}(b) \le E^\theta[T_A] \le \sum_{a=0}^{N-1} \prod_{b=0}^{a} U^{\theta,A}(b) =: U^{\theta,A},$$

where

$$(33)\qquad L^{\theta,A}(0) := \frac{\theta(B)}{\sup_{x\in B_{1,A}} K(x,A)} \qquad \text{and} \qquad U^{\theta,A}(0) := \frac{\theta(B)}{\inf_{x\in B_{1,A}} K(x,A)};$$

$$(34)\quad L^{\theta,A}(1) := \frac{\inf_{x\in B_{1,A}} \left(K(x, B^N_{2,A}) + \frac{K(x,A)\theta(B^N_{2,A})}{\theta(B)} \right)}{\sup_{x\in B_{2,A}} K(x, B_{1,A})};$$

$$(35)\quad U^{\theta,A}(1) := \frac{\sup_{x\in B_{1,A}} \left(K(x, B^N_{2,A}) + \frac{K(x,A)\theta(B^N_{2,A})}{\theta(B)} \right)}{\inf_{x\in B_{2,A}} K(x, B_{1,A})};$$

and for all $2 \le a < N$, we have

$$(36)\quad L^{\theta,A}(a) := \frac{\inf_{x\in B_{a,A}} K(x, B^N_{a+1,A})}{\sup_{x\in B_{a+1,A}} K(x, B_{a,A})} \text{ and } U^{\theta,A}(a) := \frac{1 - \inf_{x\in B_{a,A}} K(x, B_{a,A})}{\inf_{x\in B_{a+1,A}} K(x, B_{a,A})}.$$

The upper bound $U^{\theta,A}$ may be infinite, whereas the lower bound $L^{\theta,A}$ may be trivial.

(b) If $K_{|B^N_{1,A}\times B^N_{0,A}}$ is block–skip–free with respect to the partition $\{B_{a,A} | 1 \le a < N+1\}$ then

$$(37)\quad L^{\theta,A} := \sum_{a=0}^{N-1} \prod_{b=0}^{a} L^{\theta,A}(b) \le E^\theta[T_A] \le \sum_{a=0}^{N-1} \prod_{b=0}^{a} U^{\theta,A}(b) =: U^{\theta,A},$$

where $L^{\theta,A}(0)$ and $U^{\theta,A}(0)$ are given by (33), and for all $2 \le a < N$, we have

$$(38)\quad L^{\theta,A}(a) := \frac{\inf_{x\in B_{a,A}} K(x, B_{a+1,A})}{\sup_{x\in B_{a+1,A}} K(x, B_{a,A})} \text{ and } U^{\theta,A}(a) := \frac{\sup_{x\in B_{a,A}} K(x, B_{a+1,A})}{\inf_{x\in B_{a+1,A}} K(x, B_{a,A})}.$$

The upper bound $U^{\theta,A}$ may be infinite, whereas the lower bound $L^{\theta,A}$ may be trivial.

Proof. By (30) we have

$$E^\theta[T_A] = \frac{\gamma^{\theta,A}(B)}{\gamma^{\theta,A}(A)} = \frac{\gamma^{\theta,A}(B^N_{1,A})}{\gamma^{\theta,A}(A)} = q_{\mathcal{B}_A}.$$

Let now, in Theorem 4.4, $K = K^{\theta,A} \in \mathbb{R}^{\mathcal{Z}_A \times \mathcal{Z}_A}$. We have $B_0 = \{A\}$ and $K^{\theta,A}(B_0, \cdot\,) = \theta(\cdot)$. The claim now follows as a direct application of this theorem together with the Remark 5.4(b). □

The following corollary extends the formulae for the mean first passage time of a homogeneous Markov chains on the *finite* state space $0..N$ with general tridiagonal transition matrices given in [9], Section 3, or [11] to a finite or denumerably infinite state space. The methods of the authors mentioned above are different and the structure of the formulae are different too.

5.6 Corollary

Let $S = 0..N$ for a $N \in \overline{\mathbb{N}}_>$. Let $K \in \mathbb{R}^{S\times S}$ be an irreducible and positive recurrent transition matrix. Moreover, let the restriction $K_{|(S\setminus\{0\})\times S}$ of K to the set $(S \setminus \{0\}) \times S$ is skip–free with respect to the trivial partition of the set $S \setminus \{0\}$. For all $\theta \in S$ such that $E^\theta[S_0] < \infty$ we have

$$(39)\quad E^\theta[T_0] = \frac{1-\theta(0)}{K(1,0)}\left[1 + \sum_{a=0}^{N-1}\prod_{b=1}^{a}\frac{K(b,b+1)}{K(b+1,b)}\right].$$

Proof. Let in the discussion 4.5 of the exactness of the bounds, $A = \{0\}$ and $K = K^{\theta,A}$. The required lumpability condition follows from the fact that the chosen partition is the trivial one and, clearly, the case 4.3(d) meets here. □

If a set $G \subset \mathcal{Z}_A$ containing $\mathbf{J}^{\theta,A}$ can be determined then we may improve our given bounds by replacing the sets $B_{a,A}$, $0 \le a < N+1$, with their intersection with G, throughout.

6 An application to systems analysis

We now apply the foregoing theory in order to analyze the mean first passage times to different subsets in a non–reversible Markovian system discussed by communication and queueing theorists. A fairness property for this queueing network was already derived in [5], Section 4. Many of the preliminary results given here, were already postulated in this latter paper.

Therefore, we refer to them when necessary without a proof. We next described briefly the system. For a detailed description, see [5], 4.1.

6.1 A multiclass service system with loss

The queuing network is a discrete–time service system with customer classes and class–dependent loss and it operates as follows. It serves $r \geq 2$ *types* of customers. At time $t \in \mathbb{N}$ there arrive a random number $A_{k,t} : (\Omega, P) \to \mathbb{N}$ of clients of class $k \in 1..r$. The distributions of the $A_{k,t}$'s are summable and the r processes $(A_{k,t})_{t\in\mathbb{N}}$ are all i.i.d., and independent of one another. The system can hold *only* one class of customers at a time. When customers of one class are served, any arrivals of other classes are rejected. Moreover, when the system is empty and customers of different classes arrive simultaneously then a random and uniformly distributed "coin" decides which class enters the system. The system has, for each type of customer, m servers: if there are $x \in \mathbb{N}_>$ customers of type k in the system, then $x \wedge m$ of them are served at the same time. The service times of all customers are independent and geometrically distributed with rate q_k. In order to describe the departure process, let us introduce the independent and binomial$_{x\wedge m, q_k}$–distributed random variables $D^{(x)}_{k,t} : (\Omega, P) \to 0..(x \wedge m)$, $x \in \mathbb{N}_>, t \in \mathbb{N}$, $k \in 1..r$, independent of the $A_{k,t}$'s. If, at time $t \in \mathbb{N}$, there are x customers of class k on the system then the number of customers leaving the system at time t is $D^{(x\wedge m)}_{k,t}$. For all $k \in 1..m$, let e_k denote the kth unit vector in $\mathbb{N}^m$, let $R_k := \mathbb{N}_> e_k$, and let $S = \{0\} \cup \bigcup_{k=1}^r R_k$. Clearly, the customer class and number of customers on the system is represented (univocally) by a point in the union $\bigcup_{k=1}^r R_k$. Let $X_0 : (\Omega, P) \to S$ be the initial random state of the system. It is assumed to be independent of all the random element given above. Therefore, the random process $X_t : (\Omega, P) \to S$, $t \geq 0$, describing the number and type of customer in the system is a homogeneous Markov chain with state space S. It cannot transfer from a state xe_k to a state ye_l, $l \neq k$, $x, y > 0$, unless it passes through 0. Let us denote by $K \in \mathbb{R}^{S\times S}$ the corresponding transition matrix. The digraph $G_K = (S, \{(x, y) \in S \times S | K(x, y) > 0\})$ is a star with rays R_k, $k \in 1..r$, and center 0. It consists of forward arrows of possibly arbitrary lengths (depending on the distribution of $A_{k,0}$) and backward arrows of

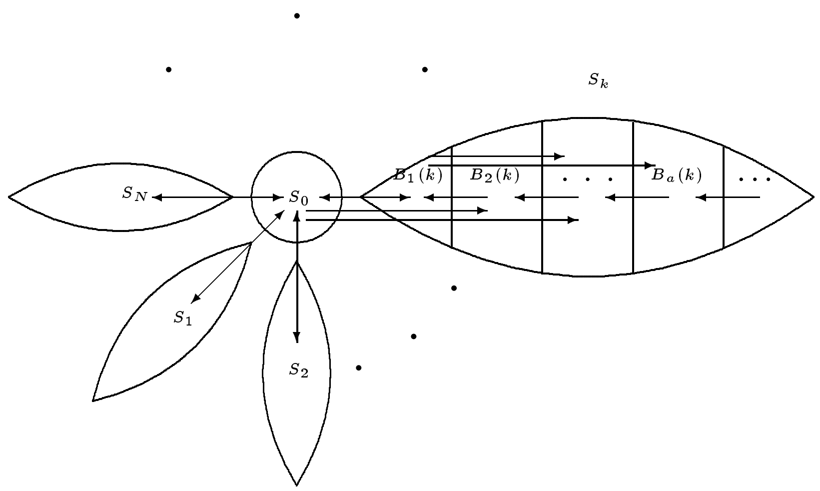

Figure 2: The configuration star.

lengths at most m in each set $\{0\} \cup R_k$. Let us define the blocks

$$B_0 = B_0(k) = \{0\}, \quad B_a(k) = \{((a-1)m+1)\, e_k, \ldots, am\, e_k\},$$

$k \in 1..r$, $a \geq 0$.

Some entries of the transition matrix are

$$K(0,0) = \prod_{k=1}^{r} P[A_{k,0} = 0] \quad \text{and}$$

(40) $K(xe_k, ye_k) = P[A_{k,0} - D_{k,0}^{(x \wedge m)} = y - x], \qquad y \geq 0,$

for all $x \geq 1$ and all $k \in 1..r$. Moreover, for these x, k and all $a \geq 0$, we have

(41) $K(xe_k, B_a(k)) = P[A_{k,0} - D_{k,0}^{(x \wedge m)} \in B_a(k) - x]$

and

(42) $K(xe_k, B_a^{\infty}(k)) = P[A_{k,0} - D_{k,0}^{(x \wedge m)} \in B_a^{\infty}(k) - x].$

Finally, if $a \geq 1$ then

(43)

$$K(0,B_a^\infty(k)) = P[A_{k,0}\in B_a^\infty(k)]\left[\sum_{l=0}^{r-1}\frac{1}{l+1}\left(\sum_{J\in\binom{1..r\setminus\{k\}}{l}}\prod_{j\in J}P[A_{j,0}>0]\prod_{j\notin J\cup\{k\}}P[A_{j,0}=0],\right)\right].$$

For abbreviation, let us denote the factor in brackets on the right hand side of the expression above by $p(r,k)$, $r\in\mathbb{N}_>$, $k\in 1..r$. It is the conditional probability for arriving custumers of class k to enter the system. As usual, we write $\rho_k = EA_{k,0}/ED_{k,0}^{(m)} = EA_{k,0}/(mq_k)$, $k\in 1..r$. If

(44) $0 < P[A_{k,0}=0] < 1 \quad \text{and} \quad 0<q_k<1 \quad \text{for all } k\in 1..r$

then K is *irreducible*. If, in addition,

(45) $\rho_k < 1, \quad k\in 1..r,$

then K is positive recurrent. Let us denote by γ the corresponding stationary distribution.

In order to apply our method, we first compute some minima and maxima of the transition probabilities. The following lemma is a simple generalization of [5], Lemma 4.2, and we omit its proof.

6.2 Lemma

Let $\emptyset\neq K\subseteq 1..r$ and let $a\geq 1$.

(a) We have

(46) $$\max_{x\in\cup_{k\in K}B_1(k)} K(x,B_0) = \max_{k\in K} P[A_{k,0}-D_{k,0}^{(1)}=-1] = \max_{k\in K} P[A_{k,0}=0]q_k;$$

(47) $$\max_{x\in\cup_{k\in K}B_{a+1}(k)} K(x,B_a(k)) = \max_{k\in K} P[A_{k,0}-D_{k,0}^{(m)}\in(-m)..(-1)];$$

and

$$(48)\qquad \begin{aligned} & \min_{x\in\cup_{k\in K}B_{a+1}(k)} K(x,B_a(k)) \\ = & \min_{x\in\cup_{k\in K}B_1(k)} K(x,B_0(k)) \\ = & \min_{k\in K} P[A_{k,0}-D_{k,0}^{(m)}\in -m] \\ = & \min_{k\in K} P[A_{k,0}=0]q_k^m. \end{aligned}$$

(b) For all $1\le a<b$ we have

$$(49)\qquad \min_{x\in\cup_{k\in K}B_a(k)} K(x,\cup_{k\in K}B_b^\infty(k)) = \begin{cases} \min_{k\in K} P[A_{k,0}-D_{k,0}^{(1)}\ge (b-1)m], & a=1, \\ \min_{k\in K} P[A_{k,0}-D_{k,0}^{(m)}\ge (b-a)m], & a\ge 2; \end{cases}$$

and

$$(50)\qquad \max_{x\in\cup_{k\in K}B_a(k)} K(x,\cup_{k\in K}B_b^\infty(k)) = \max_{k\in K} P[A_{k,0}-D_{k,0}^{(m)}\ge (b-a-1)m+1].$$

The aim of this section is to apply the method given in the Section 4 in order to estimate the mean first passage time to *interesting* subsets. Further, we use these bounds in order to study the asymptotic behavior of these parameters as the number of classes tends to infinity. Interesting events are the state where the system is empty and the sets where the system is serving clients of each type.

6.3 The mean first return time to the set $\{0\}$

We first deal with the mean first passage time to the set $\{0\}$ when the initial distribution is $K(0,\cdot\,)$. Since this parameter and the mean return time to this set differ by 1, i.e. $E^0[S_0]=1+E^{K(0,\cdot\,)}[T_0]$, we also gain information on the mean time till the system is empty again. Moreover, the assumptions of irreducibility, cf. (44), and positive recurrence, cf.(45), together with the equality above assure that $E^{K(0,\cdot\,)}[T_0]<\infty$.

By the formula of total expectation we have

$$(51)\qquad E^{K(0,\cdot\,)}[T_0]=\sum_{k=1}^r K(0,R_k)E^{\theta_k}[T_0]=\sum_{k=1}^r p(r,k)P[A_{k,0}>0]E^{\theta_k}[T_0],$$

where the last equality follows from (43) and $\theta_k := \left(\frac{K(0,xe_k)}{K(0,R_k)}\right)_{x\in\mathbb{N}_>}$ may be viewed as a distribution on S having support included in the ray R_k. Let now $k \in 1..r$ be arbitrary and let $A_k := B_0 \cup \bigcup_{j\neq k} R_k$. A moment of reflection shows that

(52) $\quad E^{\theta_k}[T_0] = E^{\theta_k}[T_{A_k}].$

Therefore, reproducing the procedure given in 5.4(b), it is meaningful to set in Definition 2.3 $A = A_k$ and $K = K^{\theta_k, A_k}$. We obtain the partition $\tilde{\mathcal{B}}_{A_k} = \mathcal{B}_{A_k} = \{\{A_k\}\} \cup \{B_a(k)|a \in \mathbb{N}_>\}$ consisiting of the sets $B_{0,A_k} = \{A_k\}$ and $B_{a,A_k} = B_a(k)$, for all $a \in \mathbb{N}_>$. A moment of reflection shows that $\gamma^{\theta_k,A_k}(B_{a,A_k}) > 0$, for all $a \in \mathbb{N}_>$, i.e., in the notation of 2.3 we have $N = \infty$.

The next proposition is the result of the application of the Theorem 5.5 to the situation described above.

6.4 Proposition

(a) For all $k \in 1..r$, let

(53)
$$C(k) := \frac{P[A_{k,0} > 0]}{P[A_{k,0} = 0]q_k}\left[1 + \frac{\min_{x\in 1..m}\left(P[A_{k,0} - D_{k,0}^{(x)} \geq m+1-x] + \frac{P[A_{k,0}\geq m+1]}{P[A_{k,0}\geq 1]}P[A_{k,0}=0]q_k^x\right)}{P[A_{k,0} - D_{k,0}^{(m)} \in -m..-1] - P[A_{k,0} - D_{k,0}^{(m)} \geq m]}\right].$$

For the mean first passage time to the set $\{0\}$ we have

(54) $\quad E^0[S_0] \geq \sum_{k=1}^r p(r,k)C(k) + 1.$

(b) (*The block–skip–free case*) For all $k \in 1..m$, let the support of the distribution of the random variable $A_{k,0} : (\Omega, P) \to \mathbb{N}$ be included in the set $0..m$. For all $k \in 1..r$, let

(55)
$$C(k) := \frac{P[A_{k,0} > 0]}{P[A_{k,0} = 0]q_k}\left[1 + \frac{P[A_{k,0} - D_{k,0}^{(1)} \geq m]}{P[A_{k,0} - D_{k,0}^{(m)} \in -m..-1] - P[A_{k,0} - D_{k,0}^{(m)} \geq m]}\right].$$

For the mean first return time to the set $\{0\}$ we have

(56) $\quad E^0[S_0] \geq \sum_{k=1}^r p(r,k)C(k) + 1.$

Let, in addition, for all $k \in 1..r$, $P[A_{k,0} - D^{(m)}_{k,0} \geq 1] < P[A_{k,0} = 0]q_k^m$ and let

$$(57)\quad D(k) := \frac{P[A_{k,0} > 0]}{P[A_{k,0} = 0]q_k^m - P[A_{k,0} - D^{(m)}_{k,0} \geq 1]}.$$

Then

$$(58)\qquad E^0[S_0] \leq \textstyle\sum_{k=1}^r p(r,k)D(k) + 1.$$

Proof. We prove first (a). Clearly, by (51) together with (52) it is enough to show that

$$(59)\quad E^{\theta_k}[T_{A_k}] \geq \frac{C(k)}{P[A_{k,0} > 0]},$$

for all $k \in 1..r$. For this sake, let in Theorem 5.5(a) $A = A_k$ and $\theta = \theta_k$. Hence, $B = R_k$. We obtain

$$(60)\quad E^{\theta_k}[T_{A_k}] \geq \sum_{a=0}^{\infty}\prod_{b=0}^{a} L^{\theta_k, A_k}(b),$$

where

$$L^{\theta_k, A_k}(0) = \frac{\theta_k(R_k)}{\max_{x\in B_{1,A_k}} K(x, A_k)} = \frac{1}{\max_{x\in B_1(k)} K(x, B_0)} = \frac{1}{P[A_{k,0} = 0]q_k},$$

and the first equality follows from the left hand side of (33), the second equality follows from the fact $B_{1,A_k} = B_1(k)$ together with the impossibility to transfer between different rays unless the state 0 is visited first, and the last one follows from (46). Reasoning in a similar way, we have

$$L^{\theta_k, A_k}(1) = \frac{\min_{x\in 1..m}\left(P[A_{k,0} - D^{(x)}_{k,0} \geq m+1-x] + \frac{P[A_{k,0}\geq m+1]}{P[A_{k,0}\geq 1]}P[A_{k,0} = 0]q_k^x\right)}{P[A_{k,0} - D^{(m)}_{k,0} \in -m..-1]};$$

and

$$L^{\theta_k, A_k}(a) := \frac{P[A_{k,0} - D^{(m)}_{k,0} \geq m]}{P[A_{k,0} - D^{(m)}_{k,0} \in -m..-1]},$$

for all $a \geq 2$. Substituing these last three equalities in (60), noting that for all $a \geq 2$ the quantity $L^{\theta_k, A_k}(a)$ does not depend on a, and making simple algebraic manipulations we obtain (59).

For the proof of Part (b), let $k \in 1..r$ and note that under the given assumption the matrix $K^{\theta_k, A_k}{}_{|B_1^\infty(k)\times(\{A_k\}\cup B_1^\infty(k))}$ is block–skip–free with respect to the partition $\{B_a(k) | a \in \mathbb{N}_>\}$. Our claim follows now as an application of the Part (b) of Theorem 5.5 together with similar arguments as those used in the proof of Part (a) above. □

6.5 The mean first passage time to a ray

We now deal with the mean first passage time to the different rays when the system is initially empty. Let $k \in 1..r$ and let in Definition 2.3 $A = R_k$ and $K = K^{\delta_0, R_k}$. The irreducibility of K assures that condition $(\star\,\star)$ is satisfied. We obtain the partition

$$\mathcal{B}_{R_k} := \{\{R_k\}\} \cup \{\{0\}, \bigcup_{i \neq k} B_a(i) | a \in \mathbb{N}_>\}$$

of the state space $\mathcal{Z}_{R_k}$. A moment of reflection shows that $\gamma^{\delta_0, R_k}(B_{a,R_k}) > 0$ for all $a \in \mathbb{N}$ and, therefore, $N = \infty$.

An application of Theorem 5.5 together with similar arguments as those used in order to prove 6.4 yields the following bounds for $E^{\delta_0}[T_{R_k}]$.

6.6 Proposition

Let $k \in 1..r$.

(a) For the mean first passage time to the ray R_k when the system is initially empty, we have the following lower estimate

$$E^{\delta_0}[T_{R_k}] \geq \frac{1}{p(r,k)P[A_{k,0} > 0]} \left\{ 1 + \frac{\sum_{j \neq k} p(r,j) P[A_{j,0} > 0]}{\max_{j \neq k} P[A_{j,0} = 0] q_j} \times \left[1 + \frac{\min_{j \neq k} P[A_{j,0} - D_{j,0}^{(1)} \geq m]}{\max_{j \neq k} P[A_{j,0} - D_{j,0}^{(m)} \in -m..-1] - \min_{j \neq k} P[A_{j,0} - D_{j,0}^{(m)} \geq m]} \right] \right\}. \tag{61}$$

(b) (*The block–skip–free case*) For all $j \in 1..m$, let the support of the distribution of the random variable $A_{j,0} : (\Omega, P) \to \mathbb{N}$ be included in the set $0..m$. For the expected passage time to the ray R_k when the system is initially empty we have

$$E^{\delta_0}[T_{R_k}] \geq \frac{1}{p(r,k)P[A_{k,0} > 0]} \left\{ 1 + \frac{\sum_{j \neq k} p(r,j) P[A_{j,0} > 0]}{\max_{j \neq k} P[A_{j,0} = 0] q_j} \times \left[1 + \frac{\min_{j \neq k} P[A_{j,0} = m](1 - q_j)}{\max_{j \neq k} P[A_{j,0} - D_{j,0}^{(m)} \in -m..-1] - \min_{j \neq k} P[A_{j,0} = m](1 - q_j)^m} \right] \right\}. \tag{62}$$

Let, in addition,

$$\min_{j\in 1..r} P[A_{j,0}=0]q_j^m > \max_{j\in 1..r} P[A_{j,0}-D_{j,0}^{(m)}\geq 1].$$

Then

$$E^{\delta_0}[T_{R_k}] \leq \frac{1}{p(r,k)P[A_{k,0}>0]}\left[1+\frac{\sum_{j\neq k} p(r,j)P[A_{j,0}>0]}{\min_{j\neq k} P[A_{j,0}=0]q_j^m - \max_{j\neq k} P[A_{j,0}-D_{j,0}^{(m)}\geq 1]}\right]. \tag{63}$$

We give now some results concerning the asymptotics of the parameters $E^0[S_0]$ and $E^{\delta_0}[T_{R_k}]$, $k\in 1..r$, as the number of classes r tends to infinite. For the sake of brevity we omit the dependence of these parameters with the number r of classes.

We assume further that

$$\begin{aligned} p_{\text{inf}} &:= \inf\nolimits_{k\in\mathbb{N}_>} P[A_{k,0}>0] > 0; \quad p_{\text{sup}} := \sup\nolimits_{k\in\mathbb{N}_>} P[A_{k,0}>0] < 1; \\ q_{\text{inf}} &:= \inf\nolimits_{k\in\mathbb{N}_>} q_k > 0; \quad q_{\text{sup}} := \sup\nolimits_{k\in\mathbb{N}_>} q_k < 1; \end{aligned} \tag{64}$$

and

$$\sup_{k\in\mathbb{N}_>} \rho_k < 1.$$

In order to accomplish our goal we need first a preliminary result.

6.7 Lemma

(a) For all $r\in\mathbb{N}_>$ and all $h,k\leq r$, we have $p(r,k)\leq 2\,p(r,h)$.

(b) For all $r\in\mathbb{N}_>$ and all $k\leq r$, we have

$$\frac{1}{r}\leq p(r,k)\leq \frac{2}{r+1}\frac{1}{p_{\text{inf}}}. \tag{65}$$

Particularly, $\frac{1}{p(r,k)}=\mathcal{O}(r)$, $k\in 1..r$.

Proof. Let $r\in\mathbb{N}_>$. The Claim (a) and the left hand side inequality of Part (b) is Lemma 4.8 of [5]. In order to prove the right hand side inequality of Part (b) let $k\in 1..r$ be arbitrary. By (a), we have

$$p(r,h)P[A_{h,0}>0] \geq \frac{1}{2}p(r,k)P[A_{h,0}>0] \geq \frac{1}{2}p(r,k)\min_{k\in 1..r} P[A_{k,0}>0],$$

for all $h \in 1..r$, $h \neq k$. Therefore,

$$1 \geq \sum_{h=1}^{r} p(r,h)P[A_{h,0}>0] \geq \frac{r+1}{2}p(r,k)\min_{h\in 1..r} P[A_{h,0}>0].$$

This is the claim. □

The following first main result of this section deals with the asymptotics of the mean return time to the set $\{0\}$.

6.8 Theorem

Consider the situation 6.3.

(a) Let $C := \inf_{k\in\mathbb{N}_>} C(k)$, where $C(k)$, $k \in 1..r$, is given by (53). Then $C > 0$ and

$$\inf_{r\in\mathbb{N}_>} E^0[S_0] \geq C + 1.$$

(b) For all $k \in 1..m$, let the support of the distribution of the random variable $A_{k,0} : (\Omega, P) \to \mathbb{N}$ be included in the set $0..m$. Let $C := \inf_{k\in\mathbb{N}_>} C(k)$, where $C(k)$, $k \in 1..r$, is given by (55). Then $C > 0$ and

$$\inf_{r\in\mathbb{N}_>} E^0[S_0] \geq C + 1. \tag{66}$$

Let, in addition,

$$\sup_{k\in\mathbb{N}_>} P[A_{k,0} - D^{(m)}_{k,0} \geq 1] < \inf_{k\in\mathbb{N}_>} P[A_{k,0} = 0]q_k^m. \tag{67}$$

Further, let $D := \sup_{k\in\mathbb{N}_>} D(k)$, where $D(k)$, $k \in 1..r$, is given by (57). Then,

$$\sup_{r\in\mathbb{N}_>} E^0[S_0] \leq \frac{2D}{p_{\inf}} + 1 < \infty. \tag{68}$$

Proof. We prove first the assertion (a). The claim $C > 0$ follows immediatly from the assumptions (64). Clearly, the second claim follows from (54) together with the left hand side of (65) and the definition of C. The proof of the first three assertions of Part (b) is similar as the one given above. The corresponding last claim follows from (58) together with the right hand side of (65) and the definition of D. $\square$

The second main result of this section deals with the asymptotics of the mean first passage time to the ray R_k, $k \in 1..r$.

6.9 Theorem

Consider the situation 6.5 and let $k \in \mathbb{N}_>$.

(a) Let

$$p := \frac{1}{\sup_{j\in\mathbb{N}_>} P[A_{j,0} = 0]q_j} \frac{\inf_{j\in\mathbb{N}_>} P[A_{j,0} - D^{(1)}_{j,0} \geq m]}{\sup_{j\in\mathbb{N}_>} P[A_{j,0} - D^{(m)}_{j,0} \in -m.. - 1] - \inf_{j\in\mathbb{N}_>} P[A_{j,0} - D^{(m)}_{j,0} \geq m]}.$$

For the mean first passage time to the ray R_k when the system is initially empty we have

$$\inf_{r\in\mathbb{N}_>} \frac{1}{r} E^{\delta_0}[T_{R_k}] \geq \frac{p_{\inf}}{2p_{\sup}}(1 + p).$$

(b) For all $k \in 1..m$, let the support of the distribution of the random variable $A_{k,0} : (\Omega, P) \to \mathbb{N}$ be included in the set $0..m$. Let

$$p_{\mathrm{bsf}} := \frac{1}{\sup_{j\in\mathbb{N}_>} P[A_{j,0} = 0]q_j} \frac{\inf_{j\in\mathbb{N}_>} P[A_{j,0} - D^{(1)}_{j,0} = m](1 - q_j)}{\sup_{j\in\mathbb{N}_>} P[A_{j,0} - D^{(m)}_{j,0} \in -m.. - 1] - \inf_{j\in\mathbb{N}_>} P[A_{j,0} - D^{(m)}_{j,0} = m](1 - q_j)^m}.$$

We have

$$\inf_{k\in\mathbb{N}_>} \frac{1}{r} E^{\delta_0}[T_{R_k}] \geq \frac{p_{\inf}}{2p_{\sup}}(1 + p_{\mathrm{bsf}}). \tag{69}$$

Let, in addition,

$$\inf_{j\in\mathbb{N}_>} P[A_{j,0} = 0]q_j^m > \sup_{j\in\mathbb{N}_>} P[A_{j,0} - D^{(m)}_{j,0} \geq 1].$$

Define

$$q_{\text{bsf}} := \frac{1}{\inf_{j\in\mathbb{N}_>} P[A_{j,0}=0]q_j^m - \sup_{j\in\mathbb{N}_>} P[A_{j,0}-D_{j,0}^{(m)} \geq 1]}.$$

We have

$$\sup_{r\in\mathbb{N}_>} \frac{1}{r} E^{\delta_0}[T_{R_k}] \leq \frac{1+2q_{\text{bsf}}p_{\text{sup}}}{p_{\text{inf}}}. \tag{70}$$

Particularly, $E^{\delta_0}[T_{R_k}] = \mathcal{O}(r)$.

Proof. In order to prove Part (a), we first write the lower bound (61) as

$$\frac{1}{p(r,k)P[A_{k,0}>0]} + \frac{\sum_{j\neq k} \frac{p(r,j)P[A_{j,0}>0]}{p(r,k)P[A_{k,0}>0]}}{\max_{j\neq k} P[A_{j,0}=0]q_j} \left[1 + \frac{\min_{j\neq k} P[A_{j,0}-D_{j,0}^{(1)} \geq m]}{\max_{j\neq k} P[A_{j,0}-D_{j,0}^{(m)} \in -m..-1] - \min_{j\neq k} P[A_{j,0}-D_{j,0}^{(m)} \geq m]}\right].$$

Next, we bound from below the three components of the expression above. By the right hand side of (65) we have

$$\frac{1}{p(r,k)P[A_{k,0}>0]} \geq \frac{r+1}{2}\frac{p_{\text{inf}}}{p_{\text{sup}}}.$$

By 6.7(a) we have

$$\frac{\sum_{j\neq k} \frac{p(r,j)P[A_{j,0}>0]}{p(r,k)P[A_{k,0}>0]}}{\max_{j\neq k} P[A_{j,0}=0]q_j} \geq (r-1)\frac{p_{\text{inf}}}{2p_{\text{sup}}}\frac{1}{\sup_{j\in\mathbb{N}_>} P[A_{j,0}=0]q_j}.$$

Finally, the lower bound for the last component is reasonable. Simple algebraic manipulations imply Claim (a). The proof of (69) is similar as the one given above. In order to prove (70) note that by the left hand side of (65) we have

$$\frac{1}{p(r,k)P[A_{k,0}>0]} \leq \frac{r}{p_{\text{inf}}}.$$

Using the same procedure as the already used but this time using (63) we obtain the desired assertion. □

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