The quantization dimension of self–similar probabilities

by

S. Graf and H. Luschgy

Abstract. The quantization dimensions of self–similar probabilities satisfying the open set condition are determined.

1. Introduction

Given a Borel probability P on $\mathbb{R}^d$, a natural number $n \geq 1$, and a real number $r > 0$ let

$$e_{n,r} = \inf\{(\int d(x,\alpha)^r dP(x))^{1/r} | \alpha \subset \mathbb{R}^d, \ \mathrm{card}(\alpha) \leq n\},$$

where $d(x, \alpha)$ denotes the distance of the point x to the set α with respect to a given norm $\| \ \|$ on $\mathbb{R}^d$. For probabilities P on $\mathbb{R}^d$ which satisfy $\int \|x\|^{r+\delta} dP(x) < +\infty$ for some $\delta > 0$ and have a non–vanishing absolutely continuous part $\lim_{n\to\infty} n e_{n,r}^d$ exists and is finite and strictly positive (see [1]). This fact implies that the quantization dimension $D_r(P) := \lim_{n\to\infty} \frac{\log n}{-\log e_{n,r}}$ of such a P equals d. For self–similar probabilities P satisfying the strong separation condition it was shown in [2] that there is a unique $D_r \in (0, \infty)$ such that the sequence $(n e_{n,r}^{D_r})_{n\in\mathbb{N}}$ is bounded and bounded away from 0 and hence that P has quantization dimension D_r. In the present note we show that for self–similar probabilities satisfying the open set condition the quantization dimension

$$D_r = \lim_{n\to\infty} \frac{\log n}{-\log e_{n,r}}$$

exists in $(0, \infty)$. But it remains an open question whether the sequence $(n e_{n,r}^{D_r})_{n\in\mathbb{N}}$ is bounded away from 0.

2. Basic notation and definitions

In what follows N is always a natural number ≥ 2 and $S_1, \ldots, S_N$ are contractive similitudes from $\mathbb{R}^d$ into itself. Let s_i be the contraction number of S_i, i.e. $s_i \in (0,1)$ and $\|S_i x - S_i y\| = s_i \|x - y\|$ for all $x, y \in \mathbb{R}^d$. Without loss of generality we assume $s_1 \geq \ldots \geq s_N$. Sometimes the N–tuple $(S_1, \ldots, S_N)$ is called an *iterated function system* (IFS). Its *attractor* is the unique non–empty compact set A in $\mathbb{R}^d$ with

$$A = S_1(A) \cup \ldots \cup S_N(A).$$

For every probability vector $p = (p_1, \ldots, p_N)$ there exists a unique Borel probability P on $\mathbb{R}^d$ which satisfies the equation

$$P = \sum_{i=1}^{N} p_i P \circ S_i^{-1}.$$

If each component p_i of p is strictly positive then the support of P equals A. P is called the *self–similar probability* corresponding to $(S_1, \ldots, S_N; p)$.
$(S_1, \ldots, S_N)$ is said to satisfy the *open set condition* (OSC) iff there exists a non–empty open set U in $\mathbb{R}^d$ with $S_i(U) \subset U$ and $S_i(U) \cap S_j(U) = \emptyset$ for all i, j with $i \neq j$. According to a result of Schief [4] U can be chosen to be bounded and such that $U \cap A \neq \emptyset$.
Let $\{1, \ldots, N\}^*$ be the set of finite words over the alphabet $\{1, \ldots, N\}$ including the empty word $\emptyset$. For $\sigma \in \{1, \ldots, N\}^*$ the length of σ is denoted by $|\sigma|$. For $n \in \mathbb{N}$, $\{1, \ldots, N\}^n$ is the set of all words of length n. A word $\sigma = \sigma_1 \ldots \sigma_n$ is said to be a predecessor of a word $\tau = \tau_1 \ldots \tau_m$, in symbols $\sigma \prec \tau$ iff $n \leq m$ an $\sigma_i = \tau_i$ for $i = 1, \ldots, n$. The empty word is the predecessor of every word. σ and τ are called incomparable if neither $\sigma \prec \tau$ nor $\tau \prec \sigma$. For $\sigma \in \{1, \ldots N\}^*$ set

$$
\begin{aligned}
S_\sigma &= \begin{cases} id_{\mathbb{R}^d} & , \ \sigma = \emptyset \\ S_{\sigma_1} \circ \ldots \circ S_{\sigma_n} & , \ \sigma = \sigma_1 \ldots \sigma_n, \end{cases} \\
A_\sigma &= S_\sigma(A), \\
s_\sigma &= \begin{cases} 1 & , \ \sigma = \emptyset \\ s_{\sigma_1} \cdot \ldots \cdot s_{\sigma_n} & , \ \sigma = \sigma_1 \ldots \sigma_n, \end{cases}
\end{aligned}
$$

and

$$
p_\sigma = \begin{cases} 1 & , \ \sigma = \emptyset \\ p_{\sigma_1} \cdot \ldots \cdot p_{\sigma_n} & , \ \sigma = \sigma_1 \ldots \sigma_n. \end{cases}
$$

If $(S_1, \ldots, S_N)$ satisfies the OSC then

$$
P(A_\sigma \cap A_\tau) = 0
$$

if σ and τ are incomparable and moreover, $P(A_\sigma) = p_\sigma$ (see [2], Lemma 3.3).

3. Statement of the main result

For $r \in (0, +\infty)$ there exists a unique $D_r \in (0, +\infty)$ satisfying

$$
\sum_{i=1}^{N} (p_i s_i^r)^{\frac{D_r}{r+D_r}} = 1
$$

(see [3], Lemma 14.4).

3.1 Theorem
Let $(S_1,\ldots,S_N)$ satisfy the OSC and let P be the self–similar probability corresponding to $(S_1,\ldots,S_N;p)$ where $p_i>0$ for $i=1,\ldots N$. Let D_r be as above. Then
$$\lim_{n\to\infty}\frac{\log n}{-\log e_{n,r}}=D_r.$$

4. Proof of the main result

In what follows $q_i=(p_i s_i^r)^{\frac{D_r}{r+D_r}}$. Let Q be the self–similar probability corresponding to $(S_1,\ldots,S_N;q)$ where $q=(q_1,\ldots,q_N)$. Let U be a bounded open set with $A\cap U\neq\emptyset$ and such that $S_i(U)\subset U$ and $S_i(U)\cap S_j(U)=\emptyset$ for $i\neq j$ holds. Then it is easy to show that there is a $\tau\in\{1,\ldots,N\}^*$ with $A_\tau\subset U$ (see for instance [2], proof of Lemma 3.3).

4.1 Lemma
Let τ be as above and $c=\frac{1}{2}+\frac{1}{2}(1-q_\tau)$. For very $V\subset A$ which is open with respect to the relative topology on A there exists an $n\in\mathbb{N}$ and $(\sigma^{(k)})_{1\le k\le n}$ in $\{1,\ldots,N\}^*\backslash\{\emptyset\}$ such that the sets $A_{\sigma^{(1)}},\ldots,A_{\sigma^{(n)}}$ are pairwise disjoint, $\subset V$, and satisfy
$$Q(V\backslash\bigcup_{k=1}^{n}A_{\sigma^{(k)}})\le cQ(V).$$

Proof. It is easy to see that there is a finite or infinite sequence $(\zeta^{(k)})$ in $\{1,\ldots,N\}^*$ whose elements are pairwise incomparable with
$$V=\bigcup_k A_{\zeta^{(k)}}.$$

Set $\varepsilon=\frac{1}{2}-\frac{1}{2}(1-q_\tau)$. Then we have $\varepsilon>0$. Hence there is an $n\in\mathbb{N}$ with
$$Q(V\backslash\bigcup_{k=1}^{n}A_{\zeta^{(k)}})<\varepsilon Q(V).$$

Let $\sigma^{(k)}=\zeta^{(k)}\tau$. Since the sets $S_{\zeta^{(k)}}(U)$ are pairwise disjoint and
$$A_{\sigma^{(k)}}=S_{\sigma^{(k)}}(A)=S_{\zeta^{(k)}}(S_\tau(A))\subset S_{\zeta^{(k)}}(U)$$

the sets $A_{\sigma^{(k)}}$ are also pairwise disjoint. This implies

$$\begin{aligned} Q(V\setminus\bigcup_{k=1}^{n}A_{\sigma^{(k)}}) &= Q(V)-\sum_{k=1}^{n}Q(A_{\sigma^{(k)}})\\ &= Q(V)-\sum_{k=1}^{n}q_{\tau}q_{\zeta^{(k)}}\\ &= Q(V)-\sum_{k=1}^{n}q_{\zeta^{(k)}}+(1-q_{\tau})\sum_{k=1}^{n}q_{\zeta^{(k)}}\\ &\leq Q(V\setminus\bigcup_{k=1}^{n}A_{\zeta^{(k)}})+(1-q_{\tau})Q(V)\\ &\leq (\varepsilon+1-q_{\tau})Q(V)\\ &= cQ(V). \end{aligned}$$

4.2 Proposition
There exists a finite or infinite sequence $(\sigma^{(k)})_k$ in $\{1,\dots,N\}^*\setminus\{\emptyset\}$ such that $(A_{\sigma^{(k)}})_k$ is pairwise disjoint and satisfies

$$\sum_k Q(A_{\sigma^{(k)}})=1.$$

Proof. Let $c\in(0,1)$ be as in Lemma 4.1. According to Lemma 4.1 there are $\sigma^{(1)},\dots,\sigma^{(n_1)}\in\{1,\dots,N\}\setminus\{\emptyset\}$ with

$$Q(A\setminus\bigcup_{i=1}^{n_1}A_{\sigma^{(i)}})\leq cQ(A)=c.$$

Set $V_1=A\setminus\bigcup_{i=1}^{n_1}A_{\sigma^{(i)}}$. Then V_1 is open with respect to the relative topology on A. Assume $n_1,\dots,n_k\in\mathbb{N}$, $(A_{\sigma^{(i)}})_{1\leq i\leq n_k}$ and $V_1,\dots,V_k$ have already been defined so that

$$V_l=V_{l-1}\setminus\bigcup_{i=n_{l-1}+1}^{n_l}A_{\sigma^{(i)}}$$

and $Q(V_l)\leq cQ(V_{l-1})$ for $l=2,\dots,k$. If $V_k=\emptyset$ then the claim of the proposition is proved. Otherwise, by Lemma 4.1, there are pairwise disjoint $A_{\sigma^{(n_k+1)}},\dots,A_{\sigma^{(n_{k+1})}}\subset V_k$ with

$$Q(V_k\setminus\bigcup_{i=n_k+1}^{n_{k+1}}A_{\sigma^{(i)}})\leq cQ(V_k).$$

If the construction does not terminate after finitely many steps then we get

$$Q(A \setminus \bigcup_{i=1}^{\infty} A_{\sigma^{(i)}}) \le Q(V_k) \le c^k$$

for all k, hence $Q(A \setminus \bigcup_{i=1}^{\infty} A_{\sigma^{(i)}}) = 0$. Thus the proposition is proved.

4.3 Proof of the main theorem 3.1

Let $(\sigma^{(k)})_k$ be a finite or infinite sequence in $\{1, \ldots, N\}^*$ such that the sets $A_{\sigma^{(k)}}$ are pairwise disjoint with $1 = \sum_k Q(A_{\sigma^{(k)}}) = \sum_k (p_{\sigma^{(k)}} s^r_{\sigma^{(k)}})^{\frac{D_r}{r+D_r}}$. Let $s < D_r$ be arbitrary. Then $\frac{s}{s+r} < \frac{D_r}{r+D_r}$. Hence there is an $n \in \mathbb{N}$ with

$$\sum_{k=1}^{n} (p_{\sigma^{(k)}} s^r_{\sigma^{(k)}})^{\frac{s}{s+r}} \ge 1.$$

Therefore one can find a $t \ge s$ with

$$\sum_{k=1}^{n} (p_{\sigma^{(k)}} s^r_{\sigma^{(k)}})^{\frac{t}{t+r}} = 1.$$

Since the sets $A_{\sigma^{(1)}}, \ldots, A_{\sigma^{(n)}}$ are compact and pairwise disjoint we have

$$\delta = \min\{d(A_{\sigma^{(i)}}, A_{\sigma^{(j)}}) | 1 \le i, j < n, i \ne j\} > 0.$$

There exisits a set $\alpha_k \subset \mathbb{R}^d$ with $\mathrm{card}(\alpha_k) \le k$ and

$$e_{k,r} = \left(\int d(x, \alpha_k)^r dP(x) \right)^{\frac{1}{r}}$$

(see [3], Theorem 4.12).
Set $\delta_k = \sup_{x \in A} d(x, \alpha_k)$. Then $\lim_{k \to \infty} \delta_k = 0$ (see [3], Lemma 13.8 and Lemma 6.1). Thus there exists an $k_0 \in \mathbb{N}$ with $\delta_k < \frac{1}{2}\delta$ for all $k \ge k_0$. For $k \ge k_0$ and $i \in \{1, \ldots, n\}$ let

$$\alpha_{k,i} = \{a \in \alpha_k | d(a, A_{\sigma^{(i)}}) \le \delta_k\}.$$

and $n_i(k) = \mathrm{card}(\alpha_{k,i})$.
Then $n_i(k) \geq 1$, the sets $\alpha_{k,i}$ $(i = 1, \ldots, n)$ are pairwise disjoint and we get

$$\begin{aligned}
e^r_{k,r} &= \int d(x,\alpha_k)^r dP(x) \\
&\geq \int\limits_{\bigcup_{i=1}^{n} A_{\sigma(i)}} d(x,\alpha_k)^r dP(x) \\
&= \sum_{i=1}^{n} \int\limits_{A_{\sigma(i)}} d(x,\alpha_k)^r dP(x) \\
&= \sum_{i=1}^{n} \int\limits_{A_{\sigma(i)}} d(x,\alpha_{k,i})^r dP(x) \\
&= \sum_{i=1}^{n} p_{\sigma(i)} s^r_{\sigma(i)} \int\limits_{A} d(x, S^{-1}_{\sigma(i)}(\alpha_{k,i}))^r dP(x) \\
&\geq \sum_{i=1}^{n} p_{\sigma(i)} s^r_{\sigma(i)} e^r_{n_i(k),r}
\end{aligned}$$

Let $b = \min\{ke^t_{k,r} | k \leq k_0\}$. Then $b > 0$. Assume $b \leq le^t_{l,r}$ for all $l \leq k-1$ and $k - 1 \geq k_0$. Then it follows from Hölder's inequality that

$$\begin{aligned}
e^r_{k,r} &\geq \sum_{i=1}^{n} p_{\sigma(i)} s^r_{\sigma(i)} b^{\frac{r}{t}} n_i(k)^{-\frac{r}{t}} \\
&\geq \left(\sum_{i=1}^{n} \left(p_{\sigma(i)} s^r_{\sigma(i)} \right)^{\frac{t}{r+t}} \right)^{1+\frac{r}{t}} \left(\sum_{i=1}^{n} \left(b^{\frac{r}{t}} n_i(k)^{-\frac{r}{t}} \right)^{-\frac{t}{r}} \right)^{-\frac{r}{t}} \\
&= b^{\frac{r}{t}} k^{-\frac{r}{t}}
\end{aligned}$$

hence,

$$ke^t_{k,r} \geq b.$$

By induction this yields

$$\inf\{ke^t_{k,r} | k \in \mathbb{N}\} \geq b > 0,$$

in particular,

$$\liminf_{k\to\infty} ke^t_{k,r} > 0.$$

It follows from [3], Prop. 11.3 that

$$\liminf_{k\to\infty} \frac{\log k}{-\log e_{k,r}} \geq t \geq s.$$

Since $s < D_r$ was arbitrary we deduce that

$$\liminf_{k\to\infty} \frac{\log k}{-\log e_{k,r}} \geq D_r.$$

The inequality

$$\limsup_{k\to\infty} \frac{\log k}{-\log e_{k,r}} \leq D_r$$

was proved in [3], Prop. 14.5.

References

[1] J.A. Bucklew – G.L. Wise, *Multidimensional asymptotic quantization with r–th power distortion measures*, IEEE Trans. Inform. Theory **28**(1982), 239–247

[2] S. Graf, *On Bandt's tangential distribution for self–similar measures*, Mh. Math. **120**(1995), 223–246

[3] S. Graf – H. Luschgy, *Foundations of Quantization for probability distributions*, Lect. Notes Math. 1730, Springer 2000

[4] A. Schief, *Separation properties of self–similar sets*, Proc. Amer. Math. Soc. **122**(1994), 111–116