Model states revisited

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Abstract

Logic programs posess a unique smallest Herbrand model, the initial model semantics. In addition, the smallest Herbrand model of a logic program can be constructed as least fixpoint of an appropriate closure operator on Herbrand structures. This is the initial model semantics of the logic program.

Replacing logic programs by the more general disjunctive logic programs where we admit disjunctions in the head of rules we have the following situation. However, a disjunctive fact has several different minimal Herbrand models. Hence, disjunctive logic programs posess in general a set of minimal Herbrand models, the minimal model semantics.

To obtain an analoguous fixpoint characterization one has to collect the Herbrand structures into one object. Therefore Lobo, Minker, Rajasekar (cf.[5]) have introduced the notion of a model state of a disjunctive logic program where disjunctions of ground goals replace the ground goals. — However, a small counterexample shows that their definition needs to be corrected.

In this paper we introduce an improved definition and develop carefully the correspondence between states and their saet of structures. As a pay-off we obtain a generalization of the results to states for 3-valued Herbrand structures. Since 3-valued Herbrand structures are described by negated and unnegated ground goals, the generalized Herbrand states consist of disjunctions of negated and unnegated ground goals. The 3-valued minimal model semantics yields a minimal-maximal model semantics of Herbrand structures.

1 The minimal model semantics

For the following we fix a clause theory i.e. a set of clauses P over a given first order language $\mathcal{L}$. By $ground(P)$ we denote the set of ground instances of P. $HB(\mathcal{L})$, the *Herband base* over $\mathcal{L}$, is the set $At_0(\mathcal{L})$ of ground atoms (=goals) over $\mathcal{L}$. As usual, $HStr(\mathcal{L})$, the set of *Herbrand structures* over $\mathcal{L}$, consists of subsets of $HB(\mathcal{L})$.

Using disjunctions of atoms rather than single atoms we obtain the following notions. $DHB(\mathcal{L})$, the *disjunctive Herband base* over $\mathcal{L}$, is the set $\bigvee HB(\mathcal{L})$ of positive ground clauses (i.e. disjunctions of ground atoms without repetitions) over $\mathcal{L}$. We use $\leq$ for the inclusion between clauses and $\vee$ for the union of clauses. The empty clause is denoted by $\Box$.

1.1 Herbrand states and their minimal Herbrand models

Now $DHStr(\mathcal{L})$, the set of *Herbrand states (= disjunctive Herbrand structures)* over $\mathcal{L}$, consists of subsets of $DHB(\mathcal{L})$. The *canonical Herbrand state* M_P for P is given by the set $\{C \in DHB(\mathcal{L}) : P \Vdash C\}$. Beside the inclusion ordering $\subseteq$ for Herbrand states we use also the finer partial order $\sqsubseteq$ which is defined as follows. $M_1 \sqsubseteq M_2$ whenever for every $C_1 \in M_1$ there exists $C_2 \in M_2$ with $C_2 \leq C_1$. The corresponding equivalence relation is denoted by $\approx$. On Herbrand structures $\sqsubseteq$ coincides with the inclusion ordering.

The expansion resp. canonical part of a Herbrand state is defined as follows. $exp(M) := \{C \in DHB(\mathcal{L}) : C' \leq C \text{ for some } C' \in M\}$ resp. $can(M) := \{C \in DHB(\mathcal{L}) : \text{ for all } C' \in M C' \leq C \text{ implies } C' = C\}$ is the largest resp. smallest representative of the equivalence class of M. The former is useful in proofs whereas the latter is useful in computations. It should be noted that *exp* is a closure operator on Herbrand states which commutes with arbitrary unions.

The extension of the validity relation to Herbrand states is obtained as follows. Let M be a Herbrand state and $C \in DHB(\mathcal{L})$ be a positive disjunction. $M \models C$ i.e. *C holds in M* whenever $C \in exp(M)$. The relation $M \models C$ depends only on $M/\approx$ since the validity relation is monotonic in both arguments. Moreover, for Herbrand structures the definition reduces to the usual definition. For a Herbrand structure N we have $N \models C$ if $A \in N$ for some $A \in C$.

The Herbrand structures associated with a Herbrand state M are obtained as follows. The set $HMod(M)$ of *Herbrand models (= realizations)* of M consists of those Herbrand structures N which satisfy the set M of positive ground clauses i.e. $N \models C$ for all $C \in M$. Note that $N \models M$ is equivalent to $M \sqsubseteq N$. Now the set $MHMod(M)$ of *minimal Herbrand models* of M is the set of Herbrand models associated with M.

The lemma below shows that there is a minimal Herbrand model below any given Herbrand model of a model state M. The proof makes use of Zorn's lemma to produce the required model.

Lemma 1 *1. If $N \in MHMod(M)$ and $N \models L$ then there exists $C \in M$ with $L \in C$, $(C - \{L\}) \cap N = \emptyset$*

2. For every $N \in HMod(M)$ there exists $N' \in MHMod(M)$ with $N' \subseteq N$

The following proposition shows that the notion of a Herbrand state behaves as wanted. The same positive ground clauses hold in a Herbrand state and its associated set of minimal Herbrand models. It is based on lemma 1 and the observation that each structure K contains a unique Herbrand structure K_0 which is equivalent to K with respect to the validity of elements of $DHB(\mathcal{L})$.

Proposition 2 *For $M \in DHStr(\mathcal{L})$ and $C \in DHB(\mathcal{L})$ the following statements are equivalent:*

1. $M \models C$
2. $HMod(M) \models C$
3. $MHMod(M) \models C$
4. $M \Vdash C$

Modifying the proof of lemma 1 we obtain the extension of lemma 1 to the nonpositive clause set $ground(P)$.

Lemma 3 *For each $N \in HMod(ground(P))$ there exists $N' \in MHMod(M)$ with $N' \subseteq N$.*

1.2 Implication pairs and model states

In general, the clause set $ground(P)$ is not a Herbrand state, since its clauses may contain negated atoms. To accomodate our formalism to these clauses we replace everywhere ground atoms by ground literals (= negated or unnegated ground atoms). We obtain $HB^*(\mathcal{L})$, the *generalized Herband base* over $\mathcal{L}$, as the set $At_0(\mathcal{L}) \cup \neg(At_0(\mathcal{L}))$ of ground literals over $\mathcal{L}$. Similarly, we obtain $DHB^*(\mathcal{L})$, the *generalized disjunctive Herband base* over $\mathcal{L}$, as the set $\bigvee HB^*(\mathcal{L})$ of ground clauses (= disjunctions of ground literals without repetitions) over $\mathcal{L}$. Thus $ground(P)$ becomes a subset of $DHB^*(\mathcal{L})$.

It should be noted that each $C \in DHB^*(\mathcal{L})$ can be written in the form $\bar{D} \vee D' := \neg A_1 \vee \ldots \vee \neg A_k \vee D$ with $D = \{A_1, \ldots, A_k\}, D' \in DHB(\mathcal{L})$. Hence arbitrary ground clauses are described by pairs of positive clauses. Now the validity relation for a Herbrand state M can be extended to an *implications pair* $D \Rightarrow D'$ with $D, D' \in DHB(\mathcal{L})$ with $D = \{A_1, \ldots, A_k\}$ as follows:

$$M \models D \Rightarrow D' \text{ i.e. } D \Rightarrow D' \textit{ holds in } M$$

$$\text{if for all } C_1, \ldots, C_k \in DHB(\mathcal{L})$$

$$M \models A_1 \vee C_1, \ldots, M \models A_k \vee C_k \text{ implies } M \models D' \vee C_1 \vee \ldots \vee C_k$$

For a Herbrand structure N this definition reduces to $N \models D \Rightarrow D'$ iff $N \models A_1, \ldots, N \models A_k$ implies $N \models D'$. In this case $D \Rightarrow D'$ corresponds to $\bar{D} \vee D' := \neg A_1 \vee \ldots \vee \neg A_k \vee D'$ as expected.

It should be admitted that the definition of the validity of an implication pair $D \Rightarrow D'$ was derived from the following definition. The direct implication operator $T_{D \Rightarrow D'}$ on $DHStr(\mathcal{L})$ is defined by:

$$T_{D \Rightarrow D'}(M) := \{D' \vee C_1 \vee \ldots \vee C_k : A_i \vee C_i \text{ for } i = 1, \ldots, k\}$$

It is obtained by applying positive hyperresolution with repect to $\bar{D} \vee D'$ to the clauses in M (cf.J.A.Robinson [9]).

The proposition below extends proposition 2 to implication pairs.

Proposition 4 *For $M \in DHStr(\mathcal{L})$ and $D, D' \in DHB(\mathcal{L})$ the following statements are equivalent:*

1. $M \models D \Rightarrow D'$
2. $MHMod(M) \models D \Rightarrow D'$
3. $T_{D \Rightarrow D'}(M) \leq M$

It should be noted that in statement (2) $MHMod(M)$ may not be replaced by $HMod(M)$. In addition, we may not add $M \Vdash \bar{D} \vee D'$ as another equivalent statement. Take $M := \{p(a) \vee p(b)\}$ with $MHMod(M) = \{N_1, N_2\}$ where $N_1 = \{p(a)\}, N_2 = \{p(b)\}$ and $N \in (MHMod(M) - HMod(M))$ where $N = \{p(a), p(b)\}$. Now we have $M \models (p(a) \vee p(b)) \Rightarrow \Box$ and hence $N_i \models \neg p(a) \vee \neg p(b)$ for $i = 1, 2$ but $N \not\models \neg p(a) \vee \neg p(b)$ and hence $M \nVdash \neg p(a) \vee \neg p(b)$.

The clauses in $ground(P)$ can now be represented as implication pairs. Hence a Herbrand state M is called a *model state* for P if $M \models C \Rightarrow D$ holds for all implication pairs in $ground(P)$. This is our version of the notion of a model state.

In addition, we can define the operator T_P of positive hyperresolution by $T_P(M) := \bigcup\{T_{C \Rightarrow D}(M) : C \Rightarrow D \in ground(P)\}$. As a corollary of proposition 4 we obtain the following characterization of the model states of P.

Corollary 5 *For $M \in DHStr(\mathcal{L})$ the following statements are equivalent:*

1. $M \models ground(P)$
2. $MHMod(M) \subseteq HMod(ground(P))$
3. $T_P(M) \leq M$

Now we are in the position to show that the canonical model state M_P has the usual fixpoint characterization as expected. It generalizes the classical result for logic programs.

Proposition 6 *M_P is a minimal fixpoint of T_P which is unique up to $\approx$ and M_P is the smallest fixpoint of $T_P \cdot exp$.*

The combination of corollary 5 and proposition 6 yields $MHMod(M_P) \subseteq HMod(ground(P))$. Moreover, by definition we have $ground(P) \Vdash M_P$ and hence $HMod(ground(P)) \subseteq HMod(M_P)$. Applying lemma 1 and lemma 3 we obtain the following result.

Proposition 7 *$MHMod(ground(P)) = MHMod(M_P)$ i.e. M_P represents the minimal model semantics of P.*

Below we shall exhibit two nonequivalent clause theories P and P' which satisfy $M_P = M_{P'}$ i.e. they have the same minimal model semantics. Hence this semantics looses some information contained in the theory. Later on we shall consider a finer semantics, the minimal-maximal model semantics.

2 The counterexample

A Herbrand state M is model state in the sense of Lobo, Minker, Rajasekar (cf.[5]) of a disjunctive logic program P if:

$$HMod(M) \subseteq HMod(P)$$

$$MHMod(P) \subseteq down(MHMod(M))$$

where $N \in down(MHMod(M))$ if $N \subseteq N' \in MHMod(M)$ for some N'.

Let P be the following set of clauses: $\{r(Y) \vee q(Y) \leftarrow p(Y), p(a) \vee p(b)\}$. The computation of M_P, the canonical Herbrand state for P, yields:

$$exp(\{p(a) \vee p(b), p(a) \vee r(b) \vee q(b), p(b) \vee r(a) \vee q(a), r(a) \vee q(a) \vee r(b) \vee q(b)\})$$

$$MHMod(M_P) = \{N_1, N_2, N_3, N_4\}$$

where $N_1 = \{p(a), r(a)\}$, $N_2 = \{p(a), q(a)\}$, $N_3 = \{p(b), r(b)\}$, $N_4 = \{p(b), q(b)\}$. In addition, we know $MHMod(P) = MHMod(M_P)$. This yields the second condition in the above definition. However, the Herbrand structure $N_0 = \{p(a), p(b), r(a)\}$ belongs to $HMod(M_P)$ but not to $HMod(P)$. This contradicts the first condition in the above definition. – However, the results of the previous section show that our version of the notion of model state works as required.

Now extend P to P' by the clause $\Box \leftarrow (q(X) \wedge r(X))$. It can be verified that one has $M_P = M_{P'}$ i.e. P and P' have the same minimal model semantics.

3 The minimal-maximal model semantic

One obtains a finer semantics if negated ground atoms are included in the definition of Herbrand states. Replacing everywhere ground atoms by ground literals i.e. negated or unnegated ground atoms we arrive at the following generalization of our previous results, where 3-valued Herbrand structures replace the Herbrand structures.

$HB^*(\mathcal{L})$, the *generalized Herband base* over $\mathcal{L}$, is the set $At_0(\mathcal{L}) \cup \neg(At_0(\mathcal{L}))$ of ground literals over $\mathcal{L}$. As before, $HStr^*(\mathcal{L})$, the set of *generalized Herbrand structures* over $\mathcal{L}$, consists of the subsets of $HB^*(\mathcal{L})$.

Note that a generalized Herbrand structure N corresponds to a unique *3-valued Herbrand structure* (N_1, N_0) in the sense of Fitting [1] by means of $\neg(N_0) = N \cap \neg(At_0(\mathcal{L}))$, $N_1 = N \cap At_0(\mathcal{L})$ and $N = N_1 \cup \neg(N_0)$ (cf. also [8],[7]). A generalized Herbrand structure N is said to be *consistent* resp. *complete* if N satisfies $N_1 \cap N_0 = \emptyset$ resp. $N_1 \cup N_0 = HB(\mathcal{L})$.

For every Herbrand structure N there is a complete generalized Herbrand structure $N^* := N \cup \neg(HB(\mathcal{L}) - N)$. Note that we have $(N_1, N_0) \leq K^*$ iff $N_1 \subseteq K \subseteq (HB(\mathcal{L}) - N_0)$. Hence the Herbrand structures K above a minimal 3-valued Herbrand structures (N_1, N_0) have N_1 resp. $HB(\mathcal{L}) - N_0$ as as a lower resp. upper bound. This shows that the set of minimal 3-valued Herbrand models yields a *minimal-maximal semantics*.

Continuing the generalization to ground literals rather than ground atoms most of the results in section 1 can be generalized. The details will have to be worked out in a later version of this paper. In particular, the relation of the minimal-maximal model semantics to other semantics in the literature should be clarified.

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