The quantization dimension of self–similar sets

by

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Abstract

Let μ be the normalized α–dimensional Hausdorff measure on a self–similar set A satisfying the open set condition, where α denotes the Hausdorff dimension of A. Upper and lower bounds for the quantization errors of μ are given and the quantization dimension of μ is calculated.

1 Introduction

Given a Borel probability measure μ on $\mathbb{R}^d$ and a natural number $n \geq 1$ let $\mathcal{F}_n$ be the set of all Borel measurable maps $f : \mathbb{R}^d \to \mathbb{R}^d$ with $card(f(\mathbb{R}^d)) \leq n$. The elements of $\mathcal{F}_n$ are called **n–quantizers** and the number

$$V_n = V_n(\mu) = \inf_{f \in \mathcal{F}_n} \int \|x - f(x)\|^2 d\mu(x)$$

is called the n–th **quantization error** for μ. Here $\| \; \|$ stands for the euclidian norm in $\mathbb{R}^d$. An n–quantizer f is called **optimal** for μ if

$$V_n = \int \|x - f(x)\|^2 d\mu(x).$$

For probability measures μ on $\mathbb{R}^d$ which satisfy $\int \|x\|^{2+\delta} d\mu(x) < \infty$ for some $\delta > 0$ and have a non–vanishing absolutely continuous part $\lim\limits_{n\to\infty} n^{\frac{2}{d}} V_n(\mu)$ exists and is finite and strictly positive. (see [2]). For the Cantor measure μ on $\mathbb{R}$ the sequence $(n^{\frac{2}{\beta}} V_n(\mu))_{n\in\mathbb{N}}$ is still bounded and bounded away from 0, where β denotes the Hausdorff dimension of the classical Cantor set C which is the support of μ. (see[6]). It is the purpose of this paper to generalize this last result to normalized Hausdorff measures on self–similar sets satisfying the open set condition (OSC). It is shown that if μ is such a normalized Hausdorff measure then the sequence $(n^{\frac{2}{\beta}} V_n(\mu))_{n\in\mathbb{N}}$ is still bounded and bounded away from 0, where β is the Hausdorff dimension of the support of μ. This result implies the dimension formula

$$\lim_{n\to\infty} \frac{\log n}{-\frac{1}{2}\log V_n(\mu)} = \beta.$$

The value $\lim\limits_{n\to\infty} \frac{\log n}{-\frac{1}{2}\log V_n(\mu)}$ is called the **quantization dimension** of μ if it exists.

The paper is organized as follows:

Section 2 contains the basic definitions and facts about self–similar measures.

In Section 3 we show that for a self–similar measure μ on $\mathbb{R}^d$ and any affine subspace V of $\mathbb{R}^d$ the zero–one law

$$\mu(V) = 0 \text{ or } \mu(V) = 1$$

holds.

In Section 4 we prove that for a self–similar measure every sphere (with respect to the euclidean metric) has measure 0.

Section 5 gives a lower bound for the quantization errors of a measure on $\mathbb{R}^d$ which vanishes on spheres.

Section 6 and 7 contain the main result and its proof.

2 Basic notation and definitions

In what follows N is always a natural number ≥ 2 and $S_1, \ldots, S_N$ are contractive similitudes from $\mathbb{R}^d$ into itself. r_i is the contraction number of S_i, i.e., $r_i \in]0,1[$ and $\|S_i x - S_i y\| = r_i \|x - y\|$ for all $x, y \in \mathbb{R}^d$. Without loss of generality we assume $r_1 \geq \ldots \geq r_N$. Sometimes the N–tuple $(S_1, \ldots, S_N)$ is called an **iterated function system** (IFS). Its **attractor** is the unique non–empty compact set A in $\mathbb{R}^d$ with

$$A = S_1(A) \cup \ldots \cup S_N(A). \tag{2.1}$$

For a probability vector $p = (p_1, \ldots, p_N)$ there exists a unique Borel probability μ_p on $\mathbb{R}^d$ which satisfies the equation

$$\mu_p = \sum_{i=1}^{N} p_i \mu_p \circ S_i^{-1}. \tag{2.2}$$

If each component p_i of p is strictly positive then the support of μ_p equals A. μ_p is called the **self–similar measure** induced by $(S_1, \ldots, S_N; p)$. μ_p can be constructed as follows: Let $\{1, \ldots, N\}^{\mathbb{N}}$ be equipped with the product of the discrete topology on $\{1, \ldots, N\}$. Let ν_p be the product probability on $\{1, \ldots, N\}^{\mathbb{N}}$ obtained from the probability on $\{1, \ldots, N\}$ induced by p. Let $\pi : \{1, \ldots, N\}^{\mathbb{N}} \to A$ be the function which maps a sequence $\eta = (\eta_n)_{n \in \mathbb{N}}$ from $\{1, \ldots, N\}^{\mathbb{N}}$ to $\hat{\eta} \in \mathbb{R}^d$, where $\hat{\eta}$ is the unique element in $\bigcap_{n \in \mathbb{N}} S_{\eta_1} \circ \ldots \circ S_{\eta_n}(A)$. Then π is continuous and onto. Moreover,

$$\mu_p = \nu_p \circ \pi^{-1}. \tag{2.3}$$

There is a unique real number D with

$$\sum_{i=1}^{N} r_i^D = 1. \tag{2.4}$$

D is called the **similarity dimension** of $(S_1, \ldots, S_N)$. If $p = (r_1^D, \ldots, r_N^D)$ the corresponding probability μ_p is denoted by μ. $(S_1, \ldots, S_N)$ satisfies the **open set condition** (OSC) iff there exists an non–empty open subset U of $\mathbb{R}^d$ with $S_i(U) \subset U$ and $S_i(U) \cap S_j(U) = \emptyset$ for all i, j with $i \neq j$. Hutchinson[7] has shown that, under this condition, $0 < \mathcal{H}^D(A) < \infty$, $H\!-\!dim(A) = D$ and $\mu = \frac{1}{\mathcal{H}^D(A)} \mathcal{H}^D_{|A}$. Here $\mathcal{H}^D$ denotes the D–dimensional Hausdorff measure, $\mathcal{H}^D_{|A}$ is its restriction to A, and $H\!-\!dim(A)$ is an abbreviation for the Hausdorff dimension of A. In this context μ is often called the **equidistribution** on A.

Let $\{1, \ldots, N\}^*$ be the set of finite words over the alphabet $\{1, \ldots, N\}$ including the empty word $\emptyset$. For $\sigma \in \{1, \ldots, N\}^*$ the length of σ is denoted by $\mid \sigma \mid$. For $n \in \mathbb{N}, \{1, \ldots, N\}^n$ is the set of all words of length n. A word $\sigma = \sigma_1 \ldots \sigma_n$ is said to be a predecessor of a word $\tau = \tau_1 \ldots \tau_m$, in symbols $\sigma \prec \tau$, iff $n \leq m$ and $\sigma_i = \tau_i$ for $i = 1, \ldots, n$. The empty word is the predecessor of every word. If τ is infinite then $\sigma \prec \tau$ is defined similarly. For $n \geq 1$ and $\sigma = \sigma_1 \ldots \sigma_n \in \{1, \ldots, N\}^n$ set

$$\sigma^- = \begin{cases} \emptyset & , \quad n = 1 \\ \sigma_1 \ldots \sigma_{n-1} & , \quad n > 1. \end{cases}$$

For $\sigma \in \{1, \ldots, N\}^*$ set

$$S_\sigma = \begin{cases} id_{\mathbb{R}^d} & , \quad \sigma = \emptyset \\ S_{\sigma_1} \circ \ldots \circ S_{\sigma_n} & , \quad \sigma = \sigma_1 \ldots \sigma_n, \end{cases}$$

$$r_\sigma = \begin{cases} 1 & , \quad \sigma = \emptyset \\ r_{\sigma_1} \cdot \ldots \cdot r_{\sigma_n} & , \quad \sigma = \sigma_1 \ldots \sigma_n, \end{cases}$$

and

$$p_\sigma = \begin{cases} 1 & , \quad \sigma = \emptyset \\ p_{\sigma_1} \cdot \ldots p_{\sigma_n} & , \quad \sigma = \sigma_1 \ldots \sigma_n. \end{cases}$$

A subset $\Gamma \subset \{1, \ldots, N\}^*$ is called a **minimal covering** (of $\{1, \ldots, N\}^{\mathbb{N}}$) iff, for every $\eta \in \{1, \ldots, N\}^{\mathbb{N}}$, there exists a unique $\sigma \in \Gamma$ with $\sigma \prec \eta$. It is easy to check that

$$\sum_{\sigma \in \Gamma} p_\sigma = 1 \tag{2.5}$$

for every minimal covering Γ. In the following we always assume that $(S_1, \ldots, S_N)$ satisfies the OSC. Then $\mu_p(S_\sigma(A) \cap S_\tau(A)) = 0$ if $\sigma \not\prec \tau$ and $\tau \not\prec \sigma$ and

$$\mu_p(S_\sigma(A)) = p_\sigma \tag{2.6}$$

for all $\sigma, \tau \in \{1, \ldots, N\}^*$ (see, for instance [4], Lemma 3.3).
For $a \in A$ let $i \in \{1, \ldots, N\}$ be the first index with $a \in S_i(A)$. Set $f(a) = S_i^{-1}(a)$. Then $f : A \to A,\ a \to f(a)$ is a Borel measurable μ_p–measure preserving transformation. It is well–known that f is ergodic (see, for instance, Barnsley – Demko[1], p. 261 for the case of pairwise disjoint $S_i(A)$). Thus, if $B, C \subset A$ are Borel sets with $\mu_p(B) > 0$ and $\mu_p(C) > 0$ then there exists a $k \geq 0$ with $\mu(f^{-k}(B) \cap C) > 0$ (see, for instance, Krengel [8], Proposition 1.8). Using the definition of f this last statement means that there is a $\tau \in \{1, \ldots, N\}^*$ with

$$\mu(S_\tau(B) \cap C) > 0. \tag{2.7}$$

3 The self–similar measure of affine subspaces

Let $(S_1, \ldots, S_N)$ satisfy the OSC and let $p = (p_1, \ldots, p_N)$ be a probability vector with $p_i > 0$ for every $i \in \{1, \ldots, N\}$.

3.1 Lemma

Let $(\varphi_n)_{n \in \mathbb{N}}$ be a sequence of non–trivial linear functionals on $\mathbb{R}^d$ which converges *to a non–trivial linear $\varphi : \mathbb{R}^d \to \mathbb{R}$ (in the operator norm). Let $(c_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ converging to $c \in \mathbb{R}$. Let $H = \{x \in \mathbb{R}^d \mid \varphi(x) = c\}$ and $H_n = \{x \in \mathbb{R}^d \mid \varphi_n(x) = c_n\}$. Then*

$$H = \bigcap_{m\in\mathbb{N}} \overline{\bigcup_{n\geq m} H_n}.$$

Proof

"$\supset$": Let $x \in \bigcap_{m\in\mathbb{N}} \overline{\bigcup_{n\geq m} H_n}$ be arbitrary. There exists a sequence $(x_k)_{k\in\mathbb{N}}$ and a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $x_k \in H_{n_k}$ and $(x_k)_{k\in\mathbb{N}}$ converges to x. Then we get $\varphi_{n_k}(x_k) = c_{n_k}$ and, therefore, $\lim_{k\to\infty} \varphi_{n_k}(x_k) = c$. Since

$$\|\varphi(x) - \varphi_{n_k}(x_k)\| \leq \|\varphi(x) - \varphi(x_k)\| + \|\varphi(x_k) - \varphi_{n_k}(x_k)\|$$

and the right hand side converges to 0 we obtain $\varphi(x) = c$, hence $x \in H$.

"$\subset$": Let $x \in H$, $m \in \mathbb{N}$, and $\varepsilon > 0$ be arbitrary. By the open mapping theorem there are $a, b \in \mathbb{R}$ with $a < b$ and $\varphi(B(x,\varepsilon)) =]a,b[$ where $B(x,\varepsilon)$ denotes the open ball with center x and radius ε. Set $r := \frac{1}{2}\min(c-a, b-c)$. Then there is an $n \geq m$ with $\mid \varphi_k(y) - \varphi(y) \mid < r$ for all $k \geq n$ and all $y \in B(x,\varepsilon)$. Again the open mapping theorem implies that there exist $a_k, b_k \in \mathbb{R}$ with $a_k < b_k$ and

$$\varphi_k(B(x,\varepsilon)) =]a_k, b_k[.$$

For $k \geq n$ and every $t \in]a,b[$ the distance of t to $]a_k,b_k[$ is less than r and, for every $s \in]a_k,b_k[$, the distance of s to $]a,b[$ is less than r. This implies

$$a \geq a_k - r \text{ and } b \leq b_k + r$$

and, hence,

$$c \in]a+r, b-r[\ \subset \ ]a_k, b_k[.$$

If, in addition, k is so large that $c_k \in]a+r, b-r[$ then there exists a $y \in B(x,\varepsilon)$ with $\varphi_k(y) = c_k$, hence $y \in H_k$. Thus

$$B(x,\varepsilon) \cap \bigcup_{k\geq m} H_k \neq \emptyset.$$

Since this holds for every $\varepsilon > 0$ and every $m \in \mathbb{N}$ we get

$$x \in \bigcap_{m\in\mathbb{N}} \overline{\bigcup_{k\geq m} H_k}.$$

3.2 Lemma

Suppose that there is no proper affine subspace of $\mathbb{R}^d$ containing A. Then $\mu_p(H) = 0$ for all affine hyperspaces H.

Proof If $\mu_p(H) = 0$ for all H then there is nothing to show. Otherwise we claim that there is an affine hyperspace M with $\mu_p(M) = \sup\{\mu_p(H) \mid H$ affine hyperspace of $\mathbb{R}^d\}$. Let $(H_n)_{n\in\mathbb{N}}$ be a sequence of affine hyperspaces with $\mu_p(H_n)$ converging to the above supremum a and $\mu_p(H_n) > 0$ for all n. Then there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of linear functionals $\varphi_n : \mathbb{R}^d \to \mathbb{R}$ with $\|\varphi_n\| = 1$ and a sequence (c_n) in $\mathbb{R}$ such that

$$H_n = \{x \in \mathbb{R}^d \mid \varphi_n(x) = c_n\}.$$

Since $H_n \cap A \neq \emptyset$ we have

$$| c_n | \leq \sup\{\|x\| : x \in A\} < \infty.$$

Thus there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $(\varphi_{n_k})_{k\in\mathbb{N}}$ and $(c_{n_k})_{k\in\mathbb{N}}$ converge. Let φ and c be the limits. Since $\|\varphi\| = 1$, φ is onto. Set $M = \{x \in \mathbb{R}^d \mid \varphi(x) = c\}$. Then Lemma 2.1 implies

$$M = \bigcap_{l\in\mathbb{N}} \overline{\bigcup_{k\geq l} H_{n_k}}.$$

Hence

$$\mu_p(M) \geq \lim_{l\to\infty} \mu_p(H_{n_l}) = a$$

and our claim is proved.
For every $n \in \mathbb{N}$, it follows from 2.2 that

$$\mu_p(M) = \sum_{\sigma\in\{1,\ldots,N\}^n} p_\sigma \mu_p \circ S_\sigma^{-1}(M).$$

Since $S_\sigma^{-1}(M)$ is an affine hyperspace we have $\mu_p(S_\sigma^{-1}(M)) \leq \mu_p(M)$. Thus the above equality implies $\mu_p(S_\sigma^{-1}(M)) = \mu_p(M)$ for all $\sigma \in \{1,\ldots,\mathbb{N}\}^*$. Since $\mu_p(M) > 0$ this implies $A \cap S_\sigma^{-1}(M) \neq \emptyset$ for all σ, hence $S_\sigma(A) \cap M \neq \emptyset$. Now let $x \in A$ be arbitrary. Then there is an $\eta \in \{1,\ldots,N\}^{\mathbb{N}}$ with $x = \hat{\eta}$. We have $S_{\eta|n}(A) \cap M \neq \emptyset$ for every $n \in \mathbb{N}$. Since $\{x\} = \bigcap_{n\in\mathbb{N}} S_{\eta|n}(A)$ and since $S_{\eta|n}(A) \cap M$ is a decreasing sequence of compact sets we deduce $x \in M$. Since $x \in A$ was arbitrary this implies $A \subset M$, a contradiction.

3.3 Lemma

Let W be the affine subspace of $\mathbb{R}^d$ generated by A. Then $S_i(W) \subset W$ for every $i \in \{1,\ldots,N\}$.

Proof

This follows from the fact that the maps S_i are affine with $S_i(A) \subset A$ for every $i \in \{1, \ldots, N\}$.

3.4 Proposition

Let V be an affine subspace of $\mathbb{R}^d$. Then

$$\mu_p(V) = 0 \text{ or } \mu_p(V) = 1.$$

Proof

Let W be the affine subspace of $\mathbb{R}^d$ generated by A. Let l be the dimension of W. Then there exists a bijective affine map $\Phi : W \to \mathbb{R}^l$ which preserves the norm. For $i \in \{1, \ldots, N\}$ set $S_i' = \Phi \circ S_i \circ \Phi^{-1}$. Using Lemma 3.3 we see that $S_1', \ldots, S_n' : \mathbb{R}^l \to \mathbb{R}^l$ are contractive similitudes. We have

$$\begin{aligned} \bigcup_{i=1}^N S_i'(\Phi(A)) &= \bigcup_{i=1}^N \Phi \circ S_i \Phi^{-1}(\Phi(A)) \\ &= \Phi(\bigcup_{i=1}^N S_i(A)) \\ &= \Phi(A), \end{aligned}$$

so that $\Phi(A)$ is the attractor of $(S_1', \ldots, S_n')$. Similarly we have

$$\mu_p \circ \Phi^{-1} = \sum_{i=1}^N p_i (\mu_p \circ \Phi^{-1}) \circ (S_i')^{-1}.$$

Let V be an affine subspace of $\mathbb{R}^d$. Then $\Phi(V \cap W)$ is an affine subspace of $\mathbb{R}^l$. Since $\Phi(A)$ generates $\mathbb{R}^l$ as an affine space Lemma 3.2 implies $\mu_p \circ \Phi^{-1}(\Phi(V \cap W)) = 0$ if $\Phi(W)$ is a proper affine subspace of $\mathbb{R}^l$ and $\mu_p \circ \Phi^{-1}(\Phi(V \cap W)) = 1$ otherwise. Since

$$\mu_p \circ \Phi^{-1}(\Phi(V \cap W)) = \mu_p(V \cap W) = \mu_p(V)$$

this implies the statement of the proposition.

3.5 Corollary

If V is an affine subspace of $\mathbb{R}^d$ with $\mu_p(V) > 0$ then $A \subset V$.

Proof. By Proposition 3.4 we have $\mu_p(A \cap V) = 1$. Since $A \cap V$ is closed and A is the support of μ_p this implies $A \cap V \supset A$, hence $A \subset V$.

4 The self–similar measure of spheres

The general assumptions are the same as in Section 3, $N \geq 2$. For $r \geq 0$ and $a \in \mathbb{R}^d$ let $S(a, r) = \{x \in \mathbb{R}^d \mid \|x - a\| = r\}$.

4.1 Proposition

For every $a \in \mathbb{R}^d$ and every $r \geq 0$,

$$\mu_p(S(a,r)) = 0.$$

Proof.

For $r = 0$ we have $S(a,r) = \{a\}$, hence $\mu(S(a,r)) = 0$ for $a \notin A$. For $a \in A$ there is an $\eta \in \{1, \ldots, N\}^{\mathbb{N}}$ with $a = \hat{\eta}$. Then $\mu_p(\{a\}) = \lim_{n\to\infty} \mu_p(S_{\eta_1} \circ \ldots \circ S_{\eta_n}(A))$. By the OSC $\mu_p(S_{\eta_1} \circ \ldots \circ S_{\eta_n}(A)) = p_{\eta_1} \cdot \ldots \cdot p_{\eta_n}$. Since $\max\{p_i \mid i = 1, \ldots, N\} < 1$ we have

$$\lim_{n\to\infty} p_{\eta_1} \cdot \ldots \cdot p_{\eta_n} = 0.$$

Now we will prove the proposition for $r > 0$. Assume that there is an $r > 0$ and an $a \in \mathbb{R}^d$ with $\mu_p(S(a,r)) > 0$. Then $\mu_p(S_1(S(a,r))) > 0$ since $\mu_p(S_1(S(a,r))) = \sum_{i=1}^{N} p_i \mu_p(S_i^{-1}(S_1(S(a,r))) \geq p_1 \mu_p(S(a,r))$. Hence by 2.7 there exists a $\tau \in \{1, \ldots, N\}^*$ with $\mu_p(S(a,r) \cap S_\tau(S_1(S(a,r)))) > 0$. Set $\sigma = \tau * 1$. Let V be the affine subspace of $\mathbb{R}^d$ generated by $S(a,r) \cap S_\sigma(S(a,r)) \cap A$. By Corollary 3.5 we get $A \subset V$. Since $S_i^{-1}(V)$ is an affine subspace of $\mathbb{R}^d$ with $\mu_p(S_i^{-1}(V)) \geq \mu_p(S_i^{-1}(A)) \geq \mu_p(A) = 1$ it follows from Corollary 3.5 that $A \subset S_i^{-1}(V)$ hence $V \subset S_i^{-1}(V)$ and, therefore,

$$S_i(V) = V \qquad \text{for} \quad i = 1, \ldots, \mathbb{N}.$$

Let $b \in V$ be the orthogonal projection of a onto V and let b' be the orthogonal projection of $S_\sigma(a)$ onto V. Since $S_\sigma(V) = V$ we have $S_\sigma(b) = b'$. It is easy to show that

$$S(a,r) \cap V = S(b, \sqrt{r^2 - \|a-b\|^2}) \cap V$$

and $S_\sigma(S(a,r)) \cap V = S_\sigma(S(a,r) \cap V) = S(S_\sigma(b), r_\sigma \sqrt{r^2 - \|a-b\|^2}) \cap V$. Thus, for $x \in S(a,r) \cap S_\sigma(S(a,r)) \cap V$ we have

$$(x - b \mid x - b) = r^2 - \|a-b\|^2,$$

where $(\quad | \quad)$ is the standard scalar product in $\mathbb{R}^d$,
and

$$(x - S_\sigma(b) \mid x - S_\sigma(b)) = r_\sigma^2 (r^2 - \|a-b\|^2),$$

hence

$$2(x \mid b - S_\sigma(b)) + \|S_\sigma(b)\|^2 - \|b\|^2 + (r^2 - \|a-b\|^2)(1 - r_\sigma^2) = 0.$$

For $x, y \in S(a,r) \cap S_\sigma(S(a,r)) \cap V$ this implies

$$2(x - y \mid b - S_\sigma b) = 0.$$

Thus $V - y$ is orthogonal to $(S(a,r) \cap S_\sigma(S(a,r)) \cap A) - y$, a set which generates $V - y$ as a vector space. On the other hand $b \in V$ and $S_\sigma b \in V$, hence $b - S_\sigma(b) \in V - y$. This yields

$$b = S_\sigma(b).$$

But since $S(b, \sqrt{r^2 - \|a-b\|^2}) \cap S(S_\sigma(b), r_\sigma \sqrt{r^2 - \|a-b\|^2}) \neq \emptyset$ this is a contradiction. Thus we get $\mu_p(S(a,r)) = 0$.

4.2 Corollary

For every $a \in \mathbb{R}^d$ the function $\mathbb{R}_+ \to \mathbb{R}_+, r \to \mu_p(B(a,r))$ is continuous.

5 A lower bound for the quantization error of certain Borel probabilities

In this section ν is a Borel probability in $\mathbb{R}^d$ which satisfies $\nu(S(a,r)) = 0$ for every $a \in \mathbb{R}$ and every $r \geq 0$.

5.1 Lemma

For every bounded Borel set $B \subset \mathbb{R}^d$ with $0 < \nu(B)$ and every $a \in \mathbb{R}^d$ there is an $r \geq 0$ with $\nu(B(a,r)) = \nu(B)$. Moreover, for such an r,

$$\int\limits_B \|x-a\|^2 d\nu(x) \geq \int\limits_{B(a,r)} \|x-a\|^2 d\nu(x). \tag{5.1}$$

Proof. Since B is bounded there exists an $R > 0$ with $B \subset B(a,R)$, hence $0 < \nu(B) \leq \nu(B(a,R))$. Since the map $\mathbb{R}_+ \to \mathbb{R}_+, r \to \nu(B(a,r))$ is continuous under the assumptions for ν the intermediate value theorem yields the existence of an $r > 0$ with $r \leq R$ and $\nu(B(a,r)) = \nu(B)$. Set $C = B \cup B(a,r)$. Then we have

$$\begin{aligned} \int\limits_B \|x-a\|^2 d\nu(x) &= \int\limits_{B(a,r)} \|x-a\|^2 d\nu(x) + \int\limits_{C \setminus B(a,r))} \|x-a\|^2 d\nu(x) \\ &- \int\limits_{C \setminus B} \|x-a\|^2 d\nu(x). \end{aligned}$$

Obviously

$$\int\limits_{C \setminus B(a,r))} \|x-a\|^2 d\nu(x) \geq r^2 \nu(C \setminus B(a,r))$$

and

$$\int\limits_{C \setminus B} \|x-a\|^2 d\nu(x) = \int\limits_{B(a,r) \setminus B} \|x-a\|^2 d\nu(x) \leq r^2 \nu(C \setminus B).$$

This implies

$$\begin{aligned}\int\limits_{B(a,r)\setminus B}\|x-a\|^2d\nu(x)-\int\limits_{C\setminus B}\|x-a\|^2d\nu(x) &\geq r^2\left[\nu(C\setminus B(a,r))-\nu(C\setminus B)\right]\\ &= r^2\left[\nu(C)-\nu(B(a,r))-\nu(C)+\nu(B)\right]\\ &= 0.\end{aligned}$$

Hence, the lemma is proved.

5.2 Proposition

Suppose ν has compact support $K = supp(\nu)$. Let $D > 0$ be arbitrary and set

$$c=\inf\{\frac{1}{\nu(B(a,r))^{\frac{2}{D}+1}}\int\limits_{B(a,r)}\|x-a\|^2d\nu(x)\mid a\in\mathbb{R}^d, r>d(a,K)\},$$

where $d(a,K)$ denotes the distance of a to K. Then, for every $n \in \mathbb{N}$,

$$n^{\frac{2}{D}}V_n(\nu)\geq c$$

(c may be equal to 0).

Proof. Let f be an optimal n–quantizer for ν and $\alpha_n = f(\mathbb{R}^d)$. For $a \in \alpha_n$ let $W_n(a)$ be the Voronai cell of a with respect to α_n, i.e. $W_n(a) = \{x \in \mathbb{R}^d \mid \|x-a\| = \min\limits_{b\in\alpha_n}\|x-b\|\}$. Then we have

$$V_n(\nu)=\sum_{a\in\alpha_n}\int\limits_{W_n(a)}\|x-a\|^2d\nu(x)$$

(see [5]). Let $a \in \alpha_n$ be fixed. By Lemma 5.1 there exists an $r > 0$ with $\nu(W_n(a)) = \nu(W_n(a)\cap K) = \nu(B(a,r))$ and

$$\int\limits_{W_n(a)}\|x-a\|^2d\nu(x)=\int\limits_{W_n(a)\cap K}\|x-a\|^2d\nu(x)\geq\int\limits_{B(a,r)}\|x-a\|^2d\nu(x).$$

Since $\nu(W_n(a)) > 0$ we have $r > d(a,K)$ and hence

$$\begin{aligned}\int\limits_{W_n(a)}\|x-a\|^2d\nu(x) &\geq \nu(W_n(a))^{\frac{2}{D}+1}\frac{1}{\nu(B(a,r))^{\frac{2}{D}+1}}\int\limits_{B(a,r)}\|x-a\|^2d\nu(x)\\ &\geq c\nu(W_n(a))^{\frac{2}{D}+1}.\end{aligned}$$

Thus we obtain

$$n^{\frac{2}{D}}V_n(\nu)\geq cn^{\frac{2}{D}}\sum_{a\in\alpha_n}\nu(W_n(a))^{\frac{2}{D}+1}.$$

By Hölder's inequality with $p=\frac{D+2}{D}$ and $q=\frac{D+2}{2}$ we deduce

$$\begin{aligned}n^{\frac{2}{D}}\sum_{a\in\alpha_n}\nu(W_n(a))^{\frac{2}{D}+1} &= \left(\sum_{\nu=1}^{n}1^q\right)^{\frac{p}{q}}\sum_{a\in\alpha_n}\nu(W_n(a))^p\\ &\geq \left(\sum_{a\in\alpha_n}\nu(W_n(a))\right)^p.\end{aligned}$$

Using Theorem 2.6 of [5] we obtain

$$\sum_{a\in\alpha_n} \nu(W_n(a)) = 1$$

and the proposition is proved.

6 A lower bound for the quantization error of the equidistriubtion on a self–similar set

In this section $(S_1, \ldots, S_N)$ satisfies the open set condition. Since $(S_1, \ldots, S_N)$ satisfies the OSC there exists a compact set J with $\overset{\circ}{J} \neq \emptyset$, $S_i(J) \subset J$ and $S_i(\overset{\circ}{J}) \cap S_j(\overset{\circ}{J}) = \emptyset$ for $i \neq j$. For $\sigma \in \{1, \ldots, N\}^*$ let $J_\sigma = S_\sigma(J)$. Let μ be as in Section 2.

6.1 Lemma

There exists a constant $c_1 > 0$ such that

$$\mu(B(a,r)) \leq c_1 r^D$$

for every $a \in \mathbb{R}^d$ and $r > 0$.

Proof. For $0 < r \leq diam(J)$ and $a \in \mathbb{R}^d$ set

$$\mathcal{C}_r(a) = \{\sigma \in \{1, \ldots, N\}^* \mid J_\sigma \cap B(a,r) \neq \emptyset,\ diam(J_{\sigma^-}) \geq r > diam(J_\sigma)\}.$$

Then it is a well–known fact that there is a constant $c' > 0$ with $card(\mathcal{C}_r(a)) \leq c' = \frac{\lambda^d(B(0,1))2^d diam(J)^d}{r_N^d \lambda^d(\overset{\circ}{J})}$ for every $a \in \mathbb{R}^d$ and $0 < r \leq diam(J)$.
Now we have

$$\begin{aligned} \mu(B(a,r)) &\leq \sum_{\sigma\in\mathcal{C}_r(a)} \mu(A_\sigma) \\ &= \sum_{\sigma\in\mathcal{C}_r(a)} r_\sigma^D \\ &= \sum_{\sigma\in\mathcal{C}_r(a)} \left(\frac{diam(J_\sigma)}{diam(J)}\right)^D \\ &\leq card(\mathcal{C}_r(a)) \frac{r^D}{diam(J)^D} \\ &\leq \frac{c'}{diam(J)^D} r^D. \end{aligned}$$

Set $c_1 = diam(J)^{-D} max(1, c')$.
Then it is easy to check that the conclusion of the lemma holds for this c_1.

6.2 Lemma

For every $a \in \mathbb{R}^d$ and every $r > d(a, A)$ with $A \not\subset B(a, r)$,

$$\int\limits_{B(a,r)} \|x - a\|^2 d\mu(x) \geq \left[\frac{\min(r_1, \ldots, r_N)}{diam(A)}\right]^{D+2} V(\mu)(r - d(a, A))^{D+2}$$

Proof. Choose $b \in A$ with $\|a - b\| = d(a, A)$. Let $\eta \in \{1, \ldots, N\}^{\mathbb{N}}$ be such that $\hat{\eta} = b$. Then there is an $n \in \mathbb{N}$ with $S_{\eta_1} \circ \ldots \circ S_{\eta_n}(A) \subset B(a, r)$ and $S_{\eta_1} \circ \ldots \circ S_{\eta_{n-1}}(A) \not\subset B(a, r)$ (for $n = 1$ we set $S_{\eta_1} \circ \ldots \circ S_{\eta_{n-1}}(A) = A$). For this n we deduce

$$\begin{aligned} \int\limits_{B(a,r)} \|x - a\|^2 d\mu(x) &\geq \int\limits_{S_{\eta_1} \circ \ldots \circ S_{\eta_n}(A)} \|x - a\|^2 d\mu(x) \\ &= r_{\eta_1}^D \ldots . r_{\eta_n}^D \int \|S_{\eta_1} \circ \ldots \circ S_{\eta_n}(x) - a\|^2 d\mu(x) \\ &= r_{\eta_1}^{D+2} \cdot \ldots \cdot r_{\eta_n}^{D+2} \int \|x - S_{\eta_n}^{-1} \circ \ldots \circ S_{\eta_1}^{-1}(a)\|^2 d\mu(x) \end{aligned}$$

Since $S_{\eta_1} \circ \ldots \circ S_{\eta_{n-1}}(A) \not\subset B(a, r)$ we have $r_{\eta_1} \cdot \ldots \cdot r_{\eta_{n-1}} diam(A) = diam(S_{\eta_1} \circ \ldots \circ S_{\eta_{n-1}}(A)) \geq r - d(a, A)$.
This implies

$$\int\limits_{B(a,r)} \|x - a\|^2 d\mu(x) \geq \left[\frac{r - d(a, A)}{diam(A)} \min(r_1, \ldots, r_N)\right]^{2+D} V(\mu),$$

i.e. the conclusion of the lemma.

6.3 Proposition

There exists a constant $c > 0$ such that

$$(1) \quad \int\limits_{B(a,r)} \|x - a\|^2 d\mu(x) \geq c\mu(B(a, r))^{\frac{2+D}{D}}$$

for every $a \in \mathbb{R}^d$ and every $r > 0$.

Proof. If $\mu(B(a, r)) = 0$ the inequality (1) obviously holds for any constant c. Thus, we may assume that $\mu(B(a, r)) > 0$. This implies $B(a, r) \cap A \neq \emptyset$, hence $r > d(a, A)$. Since $\mu(S(a, r)) = 0$, we have $r > d(a, A)$. If $d(a, A) < r \leq 2d(a, A)$ then by Lemma 6.1

$$\begin{aligned} \int\limits_{B(a,r)} \|x - a\|^2 d\mu(x) &\geq d(a, A)^2 \mu(B(a, r)) \\ &\geq d(a, A)^2 \mu(B(a, r)) \frac{\mu(B(a,r))^{\frac{2}{D}}}{(c_1 r^D)^{\frac{2}{D}}} \\ &= \left(\frac{d(a,A)}{r}\right)^2 c_1^{-\frac{2}{D}} \mu(B(a, r))^{\frac{2+D}{D}} \\ &\geq \tfrac{1}{4} c_1^{-\frac{2}{D}} \mu(B(a, r))^{\frac{2+D}{D}}. \end{aligned}$$

If $2d(a,A) < r < \infty$ and $A \not\subset B(a,r)$ then Lemmas 6.1 and 6.2 yield, for $c_3 = \left[\frac{\min(r_1,\ldots,r_N)}{diam(A)}\right]^{D+2} V(\mu)$,

$$\begin{aligned} \int_{B(a,r)} \|x-a\|^2 d\mu(x) &\geq c_3(r-d(a,A))^{D+2} \\ &\geq c_3\left(1-\tfrac{d(a,A)}{r}\right)^{D+2} r^{D+2} \tfrac{\mu(B(a,r))}{(c_1 r^D)}^{\frac{2+D}{D}} \\ &\geq c_3(\tfrac{1}{2})^{D+2} c_1^{-\frac{2+D}{D}} \mu(B(a,r))^{\frac{2+D}{D}} \end{aligned}$$

If $2d(a,A) < r < \infty$ and $A \subset B(a,r)$ then

$$\begin{aligned} \int_{B(a,r)} \|x-a\|^2 d\mu(x) &= \int_A \|x-a\|^2 d\mu(x) \geq V(\mu) \\ &\geq V(\mu)\mu(B(a,r))^{\frac{2+D}{D}} \end{aligned}$$

Thus, for $c = \min(\frac{1}{4}c_1^{-\frac{2}{D}}, c_3(\frac{1}{2})^{D+2}c_1^{-\frac{2+D}{D}}, V(\mu))$, inequality (1) holds.

6.4 Theorem

There exists a constant $c > 0$ such that

$$n^{\frac{2}{D}} V_n(\mu) \geq c$$

for every $n \in \mathbb{N}$.

Proof. The theorem follows from combining Proposition 6.3 and Proposition 5.2

6.5 Corollary

$$\liminf_{n\to\infty} n^{\frac{2}{D}} V_n(\mu) > 0.$$

7 An upper bound for the quantization error of the equidistribution on a self–similar set

Let the assumptions on $(S_1, \ldots, S_N)$ and μ be as in the last section.

7.1 Proposition

For all $n_1, \ldots, n_N \in \mathbb{N}\backslash\{0\}$,

$$V_{n_1+\ldots+n_N}(\mu) \leq \sum_{i=1}^{N} r_i^{D+2} V_{n_i}(\mu).$$

Proof. For $n \in \mathbb{N}$ let $f_n : \mathbb{R}^d$ be an n–optimal quantizer for μ and let $\alpha_n = f_n(\mathbb{R}^d)$. For $n_1, \ldots, n_N \in \mathbb{N}\backslash\{0\}$ and $i \in \{1, \ldots, N\}$ we have

$$\begin{aligned} V_{n_i}(\mu) &= \int \min_{a\in\alpha_{n_i}} \|x-a\|^2 d\mu(x) \\ &= r_i^{-2} \int \min_{a\in\alpha_{n_i}} \|S_i(x) - S_i(a)\|^2 d\mu(x) \end{aligned}$$

Since $\mu = \sum_{i=1}^{N} r_i^D \mu \circ S_i$ we deduce

$$\begin{aligned}
\sum_{i=1}^{N} r_i^{2+D} V_{n_i}(\mu) &= \sum_{i=1}^{N} r_i^D \int \min_{b \in \bigcup_{j=1}^{N} S_j(\alpha_{n_j})} \|S_i(x) - b\|^2 d\mu(x) \\
&= \int \min_{b \in \cup S_j(\alpha_{n_j})} \|x - b\|^2 d\mu(x) \\
&\geq V_{n_1 + \ldots + n_N}(\mu).
\end{aligned}$$

7.2 Corollary

Let $\Gamma \subset \{1, \ldots, N\}^*$ be a minimal covering. For $\sigma \in \Gamma$ let n_σ be a natural number, ≥ 1. Set $n = \sum_{\sigma \in \Gamma} n_\sigma$. Then

$$V_n(\mu) \leq \sum_{\sigma \in \Gamma} r_\sigma^{D+2} V_{n_\sigma}(\mu)$$

Proof. The corollary follows from Proposition 7.1 by induction on $\max\{|\sigma| : \sigma \in \Gamma\}$.

7.3 Theorem

$$\limsup_{n \to \infty} n^{\frac{2}{D}} V_n(\mu) \leq [\min(r_1, \ldots, r_N)]^{-2} \inf_{m \in \mathbb{N} \setminus \{0\}} m^{\frac{2}{D}} V_m(\mu).$$

Proof. For $\varepsilon < \min(r_1, \ldots, r_N) = r_N$ define $\Gamma(\varepsilon) = \{\sigma \in \{1, \ldots, N\}^* \mid r_{\sigma^-} \geq \varepsilon > r_\sigma\}$. Then $\Gamma(\varepsilon)$ is a minimal covering. Let $m \in \mathbb{N} \setminus \{0\}$ be arbitrary. For $n \geq m\, card(\Gamma(\varepsilon))$ there exists a family $(n_\sigma)_{\sigma \in \Gamma(\varepsilon)}$ in $\mathbb{N}$ with $n_\sigma \geq m$ and $n = \sum_{\sigma \in \Gamma(\varepsilon)} n_\sigma$. By Corollary 7.2 we get

$$\begin{aligned}
V_n(\mu) &\leq \sum_{\sigma \in \Gamma(\varepsilon)} r_\sigma^{D+2} V_{n_\sigma}(\mu) \\
&\leq \varepsilon^2 V_m(\mu) \sum_{\sigma \in \Gamma(\varepsilon)} r_\sigma^D \\
&= \varepsilon^2 V_m(\mu).
\end{aligned}$$

Since

$$1 = \sum_{\sigma \in \Gamma(\varepsilon)} r_\sigma^D \geq r_N^D \sum_{\sigma \in \Gamma(\varepsilon)} r_{\sigma^-}^D \geq r_N^D \varepsilon^D card(\Gamma(\varepsilon))$$

we have $card(\Gamma(\varepsilon)_{\leq} r_N^{-D} \varepsilon^{-D}$.
If $n \geq m\, card(\Gamma(\delta))$ for some $\delta < r_N$ then

$$\begin{aligned}
V_n(\mu) &\leq \min\{\varepsilon^2 V_m(\mu) \mid \varepsilon > 0 \text{ and } n \geq m\, card(\Gamma(\varepsilon))\} \\
&\leq \min\{\varepsilon^2 V_m(\mu) \mid \varepsilon > 0 \text{ and } \tfrac{n}{m} \geq r_n^{-D} \varepsilon^{-D}\} \\
&= \min\{\varepsilon^2 V_m(\mu) \mid \varepsilon > 0 \text{ and } \varepsilon \geq r_n^{-1} (\tfrac{n}{m})^{-\frac{1}{D}}\} \\
&= r_N^{-2} (\tfrac{m}{n})^{\frac{2}{D}} V_m(\mu)
\end{aligned}$$

hence

$$n^{\frac{2}{D}}V_n(\mu) \le r_N^{-2} m^{\frac{2}{D}} \cdot V_m(\mu).$$

Since this holds for all large enough n the statement of the proposition follows.

7.4 Problems

An interesting question is under what conditions $\lim_{n\to\infty} n^{\frac{2}{D}}V_n(\mu)$ exists. It is know that this limit exists if μ satisfies the moment condition $\int \|x\|^{2+\delta} d\mu(x) < \infty$ for some $\delta > 0$ and the absolutely continuous part of μ does not vanish (see [2]). On the other hand, for the classical Cantor distribution, the above limit does not exist (see [6]). A related problem is to calculate the values of $\limsup_{n\to\infty} n^{\frac{2}{D}}V_n(\mu)$ and $\liminf_{n\to\infty} n^{\frac{2}{D}}V_n(\mu)$. For the equidistribution μ on a square this has been done in [3]. For the Cantor distribution these values are computed in [6].

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