Computational Geometry Problems in REDLOG

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MIP-9708

April 16, 1997

Abstract. We solve algorithmic geometrical problems in real 3-space or the real plane arising from applications in the area of CAD, computer vision, and motion planning. The problems include parallel and central projection problems, shade and cast shadow problems, reconstruction of objects from images, offsets of objects, Voronoi diagrams of a finite families of objects, and collision of moving objects. Our tools are real elimination algorithms implemented in the REDUCE package REDLOG. In many cases the problems can be solved uniformly in unspecified parameters. The power of the method is illustrated by examples many of which have been outside the scope of real elimination methods so far.

1 Introduction

This note is concerned with a number of computational problems in real 3-space or the real plane that arise from applications in the area of CAD, computer vision, and motion planning. The problems are of the following types:

Parallel projection Given an object A and a parametric light ray r, compute the parallel projection of A in direction r onto a plane perpendicular to r. Compute the shaded and lighted parts of A. Compute the shadow cast by A onto some other object B. Reconstruct object A, or determine features of A from its image under parallel projection.

Central projection We treat the same problems as with parallel projection, but with respect to central projection arising from a punctual light source.

Offsets Given a closed object A and a positive real r. Compute the r-offset of A: This is the set of points having distance exactly r from A. Here distance may be Euclidean distance or distance in some other norm.

Voronoi Diagrams Given finitely many pairwise disjoint closed objects A_1, $\dots$, A_n, compute their Voronoi diagram: This is the set of all points that have equal minimal distance to at least two different objects A_i and A_j. Again distances may be measured according to various norms.

Collision Problems Given two moving objects A and B, a starting position and velocity vector for A, and a parametric starting position and velocity vector for B. Determine the conditions on the parameters under which the objects A and B will eventually collide, and under these conditions the time and place, where the collision will happen.

Throughout the paper, we will assume that all inputs to our computations are objects in the real plane or in real 3-space given by a "low-degree" semi-algebraic description, i.e. a Boolean combination of polynomial equations and inequalities in two or three variables, respectively. These equations and inequalities will be quadratic in most cases. Frequently we will allow unspecified real parameters in such a description; then we obtain a solution for a whole parametric family of problems. Our methods do not impose any restrictions on the degree of parameters. Similarly, the computed output objects will also be given in such a description, where, however, the polynomial degrees may and will exceed 2 as a rule.

We will show that all the problem types listed above can be modeled as *extended quantifier elimination* problems in real algebra. Roughly speaking, extended quantifier elimination differs from ordinary quantifier elimination by the fact that the procedure provides—besides a quantifier-free equivalent to the input formula—sample parametric answers for values of the variables in the outmost existential quantifier block. In the most frequent case of a purely existential input formula extended quantifier elimination can be construed as parametric constraint solving. Extended quantifier elimination is implemented in the REDLOG [DS96a, DS96b] package of REDUCE for input formulas with low-degree quantified variables in several variants.

The variant that we will use almost exclusively is *generic quantifier elimination*. Here the system will automatically exclude a measure zero set of degenerate real parameter values wherever this supports the elimination process. Since REDLOG does, however, explicitly output its non-degeneracy assumptions, the user can easily rerun the quantifier elimination for specific degenerate parameter values that may be of interest to him.

Many of the examples computed below involve a significant number of parameters besides the variables to be eliminated. Examples of this kind have been inaccessible to other implemented real elimination methods.

The plan of the paper is as follows: In Section 2 we give a general survey on the REDLOG package. In Section 3 we roughly sketch the REDLOG algorithms for quantifier elimination, extended quantifier elimination, generic quantifier elimination, and Gröbner simplification. The following sections are devoted to the different types of geometrical problems mentioned above. For each problem class we first describe its modeling by first-order formulas. Then we discuss specific instances in this problem class, and their automatic solution using the REDLOG package. All computations have been performed on a SUN SPARC-4 workstation.

2 The REDUCE package REDLOG

REDLOG [DS96a, DS96b] stands for REDUCE logic system. It extends the computer algebra system REDUCE by symbolic algorithms on first-order formulas wrt. temporarily fixed first-order languages and theories. For the purpose of this paper we are interested in the theory of real closed fields over the language of ordered rings. In contrast to constraint logic programming systems, [Col90], the algebraic component is not only used for supporting the logical engine but the largest part of the logical algorithms is defined and implemented in terms of algebraic algorithms.

The algorithms implemented in REDLOG include the following:

- Several techniques for the *simplification* of quantifier-free formulas. The simplifiers do not only operate on the Boolean structure of the formulas but also discover algebraic relationships (see Section 3.4). For the notion of simplification and a detailed description of the implemented techniques cf. [DS95].
- *Quantifier elimination*
  - For formulas obeying certain degree restrictions for the quantified variables, we use a technique based on elimination set ideas [Wei88, LW93, Wei97b] (see Section 3).
  - In addition, there is an interface to Hoon Hong's QEPCAD [HCJE93] package implementing a complete quantifier elimination.
- *Generic quantifier elimination* (see Section 3.3).
- *Extended* variants of both classical and generic quantifier elimination (see Section 3.2).
- Linear *optimization* using quantifier elimination techniques [Wei94].
- CNF/DNF computation including both Boolean and algebraic simplification [DS95].

– Several other *normal form* computations, e.g., prenex normal form computation minimizing the number of quantifier changes.
– A lot of useful tools for constructing, decomposing, and analyzing formulas.

REDLOG has been applied successfully for solving non-academic problems, mainly for the simulation and error diagnosis of physical networks [Wei96].

Applications inside the scientific community include the following:

– Control theory [ADL+96].
– Stability analysis for PDE's [HLS96].
– Geometric reasoning [DSW96].
– Parametric scheduling.
– Non-convex parametric linear and quadratic optimization [Wei94], transportation problems [LW93].
– Real implicitization of algebraic surfaces.
– Computation of comprehensive Gröbner bases.
– Implementation of guarded expressions for coping with degenerate cases in the evaluation of algebraic expressions [CJ92, DS97].

For non-commercial use the REDLOG source code is freely available on the www.[1]

3 REDLOG Procedures Used in this Paper

We consider polynomial equations and inequalities $f\ R\ 0$, where f is a multivariate polynomial with rational coefficients and $R \in \{=, \geq, \leq, >, <, \neq\}$. A *quantifier-free formula* ψ is a Boolean combination of such equations and inequalities obtained by applying negation "$\neg$," conjunction "$\wedge$," and disjunction "$\vee$." We call ψ of degree d in a variable x if all polynomials occurring in ψ have an x-degree of at most d.

3.1 Quantifier elimination

Suppose that a quantifier-free formula ψ is quadratic, i.e. of degree 2, in some variable x, and denote $\exists x\bigl(\psi(x, u_1, \dots, u_n)\bigr)$ by $\varphi(u_1, \dots, u_n)$. Then the algorithm given in [Wei97b] computes from φ a quantifier-free formula $\varphi'(u_1, \dots, u_n)$ not containing x such that over the ordered field of the reals we have the equivalence

$$\varphi(u_1, \dots, u_n) \longleftrightarrow \varphi'(u_1, \dots, u_n).$$

In other words, for arbitrary values $a_1, \dots, a_n \in \mathbb{R}$ of the u_i, the assertion $\varphi'(a_1, \dots, a_n)$ holds in $\mathbb{R}$ iff there exists $b \in \mathbb{R}$ such that $\psi(b, a_1, \dots, a_n)$ holds in $\mathbb{R}$. The elimination of a universal quantifier can be reduced to that of an existential quantifier using the equivalence

$$\forall x \psi \longleftrightarrow \neg\exists x \neg\psi.$$

[1] `http://www.fmi.uni-passau.de/~redlog/`

Several quantifiers can be eliminated one by one starting with the innermost one provided that the elimination result for some inner quantifier still obeys the degree restrictions. The process sketched above is referred to as *quantifier elimination.* Note that the implementation in REDLOG includes various heuristics for coping with formulas violating the degree restrictions, e.g. polynomial factorization, cf. [DSW96] for details.

3.2 Extended Quantifier elimination

For eliminating a quantifier $\exists x$ from $\exists x\psi$, our elimination algorithm proceeds as follows: All equations and inequalities $f(x, u_1, \dots, u_n)\ R\ 0$ contained in ψ are renormalized wrt. x obtaining, e.g. in the case of a quadratic constraint:

$$a(u_1, \dots, u_n)x^2 + b(u_1, \dots, u_n)x + c(u_1, \dots, u_n)\ R\ 0.$$

From these renormalized constraints we compute, using the well-known solution formulas for quadratic and linear equations, a finite set T of *test terms* not containing x such that

$$\exists x\psi \longleftrightarrow \bigvee_{t \in T} \psi[x /\!\!/ t].$$

Here $[x /\!\!/ t]$ denotes a modified substitution of t for x with the following features (cf. [Wei97b] for details):

- We can, semantically correct, substitute terms t involving square roots such that the substitution result is a well-formed formula not containing any square root.
- We can, semantically correct, substitute infinite elements $t = \pm\infty$ and terms $t = s \pm \varepsilon$ involving infinitesimal elements without these non-standard elements occurring in the substitution result.
- Besides substituting, we also add conditions and case distinctions wrt. the validity of substitution terms. For instance, with terms arising from the quadratic solution formula, we add the condition $a \neq 0 \land b^2 - 4ac \geq 0$. That is, the constraint is actually quadratic and the discriminant is non-negative.

By keeping track of the terms t substituted during the elimination process, we obtain—instead of the quantifier-free equivalent $\psi' = \bigvee_{i=1}^{k} \psi[x /\!\!/ t_i]$—a scheme

$$\begin{bmatrix} \psi[x /\!\!/ t_1] & x = t_1 \\ \vdots & \vdots \\ \psi[x /\!\!/ t_k] & x = t_k \end{bmatrix}$$

including satisfying sample points. This process of *extended quantifier elimination* can also be repeated for several existential quantifiers. The result then is a set of conditions each associated with an answer for each eliminated variable obtained by resubstitution.

Note that the sample points t_i can include the symbols ∞ and ε. The former has to be read as "a real number x_0 such all $x \geq x_0$ satisfy ψ." In an analogous way, ε stands for a "small enough" positive real. Unfortunately, nothing can be said about the order between several non-standard symbols in the output. For identifying equal non-standard symbols after resubstitution, all such symbols are indexed.

3.3 Generic Quantifier elimination

Very much of the complexity of our quantifier elimination procedure arises from the case distinctions wrt. parametric coefficient expressions being zero or not. *Generic quantifier elimination* assumes for purely parametric expressions to be non-zero wherever this supports the elimination process. The assumptions are collected and, besides the elimination result, returned to the user. It has turned out that in most cases these assumptions are *non-degeneracy conditions* (ND-conditions), cf. [DSW96]. Generic quantifier elimination can, of course, be combined with extended quantifier elimination.

Besides the automatic generation of assumptions, all REDLOG procedures used throughout this paper allow to pass a set of polynomial equations and inequalities as an optional *background theory* argument.

3.4 Gröbner Simplification

During the quantifier elimination procedures, REDLOG applies a simplification procedure to the obtained (sub)results. Besides this fast, though still sophisticated, *standard simplifier*, REDLOG provides several advanced simplifiers that can be explicitly applied to final elimination results. Of particular importance for our purposes is a simplifier using *Gröbner basis* [Buc65] methods.

We illustrate the technique by means of a very simple example: Consider as input formula $xy + 1 \neq 0 \lor yz + 1 \neq 0 \lor x - z = 0$. It can be rewritten as

$$xy + 1 = 0 \land yz + 1 = 0 \longrightarrow x - z = 0.$$

Reducing the conclusion modulo the Gröbner basis $\{x-z, yz+1\}$ of the premises, we obtain the equivalent formula $xy + 1 = 0 \land yz + 1 = 0 \longrightarrow 0 = 0$, which can in turn be easily simplified to "true." For details on the algorithm actually used cf. [DS95].

4 Parallel Projection

We consider semi-algebraic objects A, B, ... in real 3-space $S = \mathbb{R}^3$ given by corresponding defining quantifier-free formulas $\alpha(u, v, w)$, $\beta(u, v, w)$, ...

The *boundary* $\mathrm{b}(A)$ of A can be described by a quantifier-free formula α'_{b} equivalent to

$$\alpha_{\mathrm{b}}(x, y, z) \equiv \alpha(x, y, z) \land \forall\varepsilon\big[\varepsilon > 0 \longrightarrow \exists u \exists v \exists w\big(\neg\alpha(u, v, w) \land \\ -\varepsilon < x - u < \varepsilon \land -\varepsilon < y - v < \varepsilon \land -\varepsilon < z - w < \varepsilon\big)\big].$$

In a similar way, β'_{b} denotes a quantifier-free formula describing the boundary $\mathrm{b}(B)$ of object B, and so on.

Let now $r = (k, l, m)$ be a non-zero vector in S. Interpreting r as the direction of light coming in parallel rays from an infinitely far light source, we can model various concepts of image and shadow as follows: A point $(x, y, z) \in S$ is the image of a point $(u, v, w) \in S$ under parallel projection along r iff

$$\exists t(t > 0 \wedge x = u + kt \wedge y = v + lt \wedge z = w + mt).$$

Hence the region $\mathrm{SR}(A)$ in space S that is shaded by object A can be described by the formula

$$\begin{aligned}(x, y, z) \in \mathrm{SR}(A) \iff \alpha_{\mathrm{SR}}(x, y, z) \equiv \exists u \exists v \exists w \exists t\big(\alpha(u, v, w) \wedge t > 0 \wedge{}& \\ x = u + kt \wedge y = v + lt \wedge z = w + mt\big).&\end{aligned}$$

So by determining a quantifier-free equivalent $\alpha'_{\mathrm{SR}}(x, y, z)$ to the formula α_{SR} above, we get a semi-algebraic description of the region $\mathrm{SR}(A)$.

Similarly, we can describe the shaded part $\mathrm{Sh}(A)$ and the lighted part $\mathrm{Li}(A)$ of the boundary of object A:

$$\begin{aligned}(x, y, z) \in \mathrm{Sh}(A) \iff{}& \alpha_{\mathrm{b}}(x, y, z) \wedge \exists u \exists v \exists w \exists t\big(\alpha(u, v, w) \wedge t > 0 \wedge \\ & x = u + kt \wedge y = v + lt \wedge z = w + mt\big) \\ (x, y, z) \in \mathrm{Li}(A) \iff{}& \alpha_{\mathrm{b}}(x, y, z) \wedge \neg\exists u \exists v \exists w \exists t\big(\alpha(u, v, w) \wedge t > 0 \wedge \\ & x = u + kt \wedge y = v + lt \wedge z = w + mt\big)\end{aligned}$$

For strictly convex objects C, we can also define the boundary $\mathrm{SL}(C)$ between $\mathrm{Sh}(C)$ and $\mathrm{Li}(C)$. This definition works even for convex, but not strictly convex objects, with a non-degenerate light direction r:

$$\begin{aligned}(x, y, z) \in \mathrm{SL}(C) \iff{}& \gamma_{\mathrm{b}}(x, y, z) \wedge \neg\exists u \exists v \exists w \exists t\big(\gamma(u, v, w) \wedge t \neq 0 \wedge \\ & x = u + kt \wedge y = v + lt \wedge z = w + mt\big).\end{aligned}$$

Suppose now that B is a second object in S. Then the shadow $\mathrm{CS}(A, B)$ cast by object A onto the boundary of B can be described by the following formula:

$$\begin{aligned}(x, y, z) \in \mathrm{CS}(A, B) \iff{}& \beta_{\mathrm{b}}(x, y, z) \wedge \exists u \exists v \exists w \exists t\big(\alpha(u, v, w) \wedge t \geq 0 \wedge \\ & x = u + kt \wedge y = v + lt \wedge z = w + mt\big).\end{aligned}$$

This applies in particular to the case when B is a plane perpendicular to the light ray r given by the equation $kx + ly + mz = p$. In this case the shadow cast onto B by A is identical to the image of A viewed in direction r. This image is called the *aspect* of A. For simplicity, we may take B as a plane through the origin and drop the condition $t \geq 0$. Then we get the following formula for the aspect of A in direction $r = (k, l, m)$:

$$\begin{aligned}(x, y, z) \in \mathrm{Asp}(r, A) \iff{}& kx + ly + mz = 0 \wedge \exists u \exists v \exists w \exists t\big(\alpha(u, v, w) \wedge \\ & x = u + kt \wedge y = v + lt \wedge z = w + mt\big).\end{aligned}$$

Finally we may use extended quantifier elimination to reconstruct features of the object A and its orientation relative to the projection plane from its aspect. To this end we assume that our object belongs to a parametric class of objects, e.g. cuboids, balls, or ellipsoids, described by a quantifier-free formula $\alpha(u, v, w, a_1, \dots, a_n)$. The real parameters $a_1, \dots, a_n$ describe the dimensions and orientation of the object.

In general, we can put our coordinate system into such a position that the light ray projects the object along the Z-axis onto the X-Y-plane. Since the image of the object is invariant under translation of the object along the light ray, we may assume in addition that some reference point of the object is placed in the origin of the coordinate system. Suppose now that we have a quantifier-free formula $\alpha_0(x, y)$ which we assume to describe $\operatorname{Asp}\bigl((0,0,1), A\bigr)$. Then an application of extended quantifier elimination to the formula

$$\exists w \exists a_1 \dots \exists a_n \forall x \forall y \bigl(\alpha(x, y, w, a_1, \dots, a_n) \longleftrightarrow \alpha_0(x, y)\bigr)$$

will yield the result "false" if α_0 does not describe the aspect of an object in the given class; otherwise it will yield "true" together with a list of possible real values for $a_1, \dots, a_n$.

Among the problems described in this section, feature reconstruction is by far the hardest one. In order to obtain suitable results within a tolerable amount of time, one has to put more intelligence into the coding of the problem than with the general technique described above.

For recovering, e.g., the dimensions of a *rectangular solid* A and its orientation in space from its aspect, we proceed as follows: We consider A to be generated by pairwise perpendicular vectors

$$e_1 = (e_{11}, e_{12}, e_{13}), \quad e_2 = (e_{21}, e_{22}, e_{23}), \quad e_3 = (e_{31}, e_{32}, e_{33})$$

with one corner located at the origin. Its corners $x = (x_1, x_2, x_3)$ are described by the formula

$$e_1 e_2 = e_1 e_3 = e_2 e_3 = 0 \land \exists \lambda_1 \exists \lambda_2 \exists \lambda_3 \bigl(\gamma(\underline{\lambda}, \underline{e})\bigr),$$

where $\gamma(\underline{\lambda}, \underline{e}) \equiv \lambda_1, \lambda_2, \lambda_3 \in \{0,1\} \land x = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$. The direction of the light is along the x_3-axis. The projection plane P is the x_1-x_2-plane.

Provided that A is in generic position, i.e., not parallel to any of the axes, its aspect is the convex hull of 6 points with two possible choices for points which cannot be observed:

1. All e_{i3} coordinates have the same sign. Then the images of the origin and of its opposite are in the interior of the aspect.
2. Up to a permutation of the e_i, the sign of e_{13} differs from that of e_{23} and e_{33}. Then the image of e_1 and that of its opposite $e_2 + e_3$ are in the interior of the aspect.

The concrete coordinate system is chosen by the observer of the aspect, who will exclude the first of the two cases above by placing one aspect corner into the origin.

We generate by quantifier elimination (255 ms) a quantifier-free generic aspect description ι' from the following formula:

$$\iota(e_{11}, e_{12}, \ldots, e_{33}, i_1, i_2) \equiv e_1e_2 = e_1e_3 = e_2e_3 = 0 \wedge \exists x_1 \exists x_2 \exists x_3 \exists \lambda_1 \exists \lambda_2 \exists \lambda_3 \big(\gamma(\underline{\lambda}, \underline{e}) \wedge \neg(\lambda_1 \neq \lambda_2 = \lambda_3) \wedge i_1 = x_1 \wedge i_2 = x_2 \big)$$

For a given semi-algebraic description α_0 of aspect corners we can now reconstruct the original cuboid by applying extended quantifier elimination to the following *reconstruction* formula:

$$\varrho(\alpha_0) \equiv \exists e_{11} \exists e_{12} \exists e_{13} \exists e_{21} \exists e_{22} \exists e_{23} \exists e_{31} \exists e_{32} \exists e_{33} \forall i_1 \forall i_2 (\iota' \longleftrightarrow \alpha_0).$$

For correct generic aspect descriptions α_0 this elimination will yield "true" together with suitable vectors e_1, e_2, e_3 as answer. If α_0 does not describe the aspect of a cuboid in generic position the elimination will yield "false."

Examples computed with REDLOG

For our first Examples 1–5, we consider *a parametric quadric* in space of the form $\{\, (u, v, w) \in S \mid au^2 + bv^2 + cw^2 \leq 1 \,\}$ and a light ray (k, l, m). Depending on the signs of the parameters a, b, c, this quadric is an ellipsoid, a cylinder, or a hyperboloid. Similar results as below have been obtained with other parametric quadrics.

Example 1 (Shadowed region). The following formula describes the region shadowed by the ellipsoid:

$$\alpha_{\mathrm{SR}}(x, y, z, k, l, m) \equiv \exists u \exists v \exists w \exists t (t > 0 \wedge au^2 + bv^2 + cw^2 \leq 1 \wedge u + tk = x \wedge v + tl = y \wedge w + tm = z).$$

We obtain after 442 ms a quantifier-free equivalent containing 6 atomic formulas, and the ND-condition $ak^2 + bl^2 + cm^2 \neq 0$. Fig. 1 illustrates how such a computation looks like in REDLOG.

Example 2 (Shaded part of the surface). The shaded part of the surface is described by the following formula:

$$\sigma(x, y, z, a, b, c, k, l, m) \equiv ax^2 + by^2 + cz^2 = 1 \wedge \exists u \exists v \exists w \exists t (t > 0 \wedge au^2 + bv^2 + cw^2 \leq 1 \wedge u + tk = x \wedge v + tl = y \wedge w + tm = z).$$

We obtain after 629 ms a quantifier-free description containing 9 atomic formulas, valid under the ND-condition $ak^2 + bl^2 + cm^2 \neq 0$.

```
9: ells1:= ex({u,v,w,t},t>0 and a*u**2+b*v**2+c*w**2 <= 1 and
9: u + t*k = x and v + t*l = y and w + t*m = z))))$

10: rlgqe(ells1);
---- (ex u v w t) [BFS: depth 4]
-- left: 4
[1lg]
-- left: 3
[1lg]
-- left: 2
[1lg]
-- left: 1
[1e#q!] [DEL:0/4]
{{a*k**2 + b*l**2 + c*m**2 <> 0},
a*k**2 + b*l**2 + c*m**2 <= 0 or
a*x**2 + b*y**2 + c*z**2 - 1 < 0 or
a**2*b*k**4*y**2 - 2*a**2*b*k**3*l*x*y +
a**2*b*k**2*l**2*x**2 + a**2*c*k**4*z**2 -
2*a**2*c*k**3*m*x*z + a**2*c*k**2*m**2*x**2 -
a**2*k**4 + a*b**2*k**2*l**2*y**2 -
2*a*b**2*k*l**3*x*y + a*b**2*l**4*x**2 +
2*a*b*c*k**2*l**2*z**2 - 2*a*b*c*k**2*l*m*y*z +
2*a*b*c*k**2*m**2*y**2 - 2*a*b*c*k*l**2*m*x*z -
2*a*b*c*k*l*m**2*x*y + 2*a*b*c*l**2*m**2*x**2 -
2*a*b*k**2*l**2 + a*c**2*k**2*m**2*z**2 -
2*a*c**2*k*m**3*x*z + a*c**2*m**4*x**2 -
2*a*c*k**2*m**2 + b**2*c*l**4*z**2 -
2*b**2*c*l**3*m*y*z + b**2*c*l**2*m**2*y**2 -
b**2*l**4 + b*c**2*l**2*m**2*z**2 -
2*b*c**2*l*m**3*y*z + b*c**2*m**4*y**2 -
2*b*c*l**2*m**2 - c**2*m**4 <= 0 and
a**2*k**3*x + a*b*k**2*l*y + a*b*k*l**2*x +
a*c*k**2*m*z + a*c*k*m**2*x + b**2*l**3*y +
b*c*l**2*m*z + b*c*l*m**2*y + c**2*m**3*z > 0 or
a*k*x + b*l*y + c*m*z > 0 and
a*x**2 + b*y**2 + c*z**2 - 1 = 0}$

Time: 442 ms plus GC time: 68 ms

11: rlatnum(second(ws));

6$
```

Fig. 1. A generic quantifier elimination in REDLOG, showing the successive elimination of quantified variables and the elimination technique used in each case. The first entry of the output list contains the ND-conditions assumed during the elimination; the second entry is the quantifier-free formula equivalent to the input formula under the ND-conditions.

Example 3 (Lighted part of the surface). We use the following formula to describe the lighted part of the surface of our quadric:

$$\begin{aligned}\lambda(x,y,z,a,b,c,k,l,m) \equiv{}& ax^2+by^2+cz^2=1 \wedge \\ & \neg\exists u\exists v\exists w\exists t(t>0 \wedge au^2+bv^2+cw^2 \le 1 \wedge \\ & u+tk=x \wedge v+tl=y \wedge w+tm=z).\end{aligned}$$

We obtain after 561 ms a quantifier-free description in 5 atomic formulas valid under the ND-condition $ak^2+bl^2+cm^2 \neq 0$.

Example 4 (Boundary between shaded and lighted part). Describing the boundary between the lighted and the shaded part of the quadric by

$$\begin{aligned}\beta(x,y,z,a,b,c,k,l,m) \equiv{}& ax^2+by^2+cz^2=1 \wedge \\ & \neg\exists u\exists v\exists w\exists t(t\neq 0 \wedge au^2+bv^2+cw^2\leq 1 \wedge \\ & u+tk=x \wedge v+tl=y \wedge w+tm=z)\end{aligned}$$

we obtain within 476 ms the quantifier-free equivalent

$$ak^2+bl^2+cm^2\geq 0 \wedge akx+bly+cmz=0 \wedge ax^2+by^2+cz^2-1=0$$

valid under the ND-condition $ak^2+bl^2+cm^2\neq 0$.

Example 5 (Aspect). The aspect of our quadric can be described by the following formula:

$$\begin{aligned}\varphi(x,y,z,a,b,c,k,l,m) \equiv{}& \exists u\exists v\exists w\exists t(au^2+bv^2+cw^2\leq 1 \wedge u+tk=x \wedge \\ & v+tl=y \wedge w+tm=z \wedge kx+ly+mz=0).\end{aligned}$$

We obtain after 442 ms a quantifier-free equivalent with 6 atomic formulas valid under the ND-condition $ak^2+bl^2+cm^2\neq 0$.

Example 6 (Intersection aspect). We compute the aspect of the intersection line of two parametric tubes crossing perpendicularly but not necessarily centrally:

$$\begin{aligned}\tau(x,y,z,a,b,c,k,l,m) \equiv{}& \exists u\exists v\exists w\exists t\big((u-a)^2+(v-b)^2=d_1^2 \wedge \\ & u^2+w^2=d_2^2 \wedge x=u+kt \wedge y=v+lt \wedge \\ & z=w+mt \wedge kx+ly+zw=0\big).\end{aligned}$$

The result obtained after 357 ms is the following quantifier-free description valid under the ND-conditions $m\neq 0$ and $z\neq 0$:

$$\begin{aligned}& a^2m^2z^2+2ak^2mxz+2aklmyz+2akmz^3-2am^2xz^2+b^2m^2z^2+ \\ & \quad 2bklmxz+2bl^2myz+2blmz^3-2bm^2yz^2-d_1^2m^2z^2+k^4x^2+ \\ & \quad 2k^3lxy+2k^3xz^2+k^2l^2x^2+k^2l^2y^2+2k^2lyz^2-2k^2mx^2z+ \\ & k^2z^4+2kl^3xy+2kl^2xz^2-4klmxyz-2kmxz^3+l^4y^2+2l^3yz^2- \\ & \quad 2l^2my^2z+l^2z^4-2lmyz^3+m^2x^2z^2+m^2y^2z^2=0 \wedge \\ & \quad d_2^2m^2z^2-k^4x^2-2k^3lxy-2k^3xz^2-k^2l^2y^2-2k^2lyz^2- \\ & k^2m^2x^2+2k^2mx^2z-k^2z^4-2klm^2xy+2klmxyz+2kmxz^3- \\ & \quad l^2m^2y^2-m^2x^2z^2=0.\end{aligned}$$

Example 7 (Solid reconstruction). As an example for recovering a rectangular solid, consider the aspect given in Fig. 2. Its corners have the following semi-algebraic description:

$$\begin{aligned}\alpha_0 \equiv{}& (13i_1-1124=0 \wedge 13i_2-1960=0) \vee \\ & (13i_1+176=0 \wedge 13i_2-1050=0) \vee \\ & (13i_1+436=0 \wedge 13i_2-530=0) \vee (i_1-120=0 \wedge i_2-110=0) \vee \\ & (i_1-100=0 \wedge i_2-70=0) \vee (i_1=0 \wedge i_2=0).\end{aligned}$$

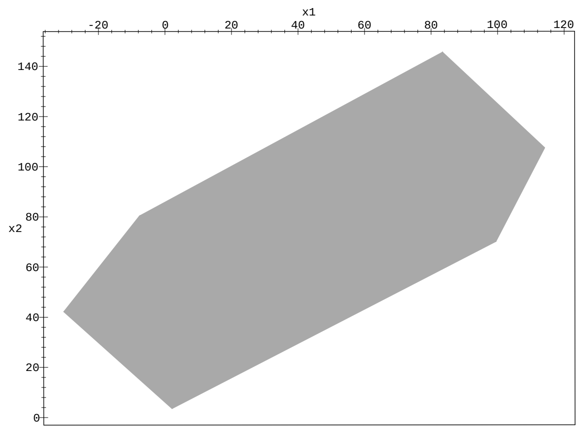

Fig. 2. A sample aspect of a rectangular solid in generic position.

Extended quantifier elimination applied to $\varrho(\alpha_0)$ yields after 15 674 ms "true" together with the correct reconstruction pictured in Fig. 3:

$$e_1 = (20, 40, -96), \quad e_2 = \left(-\frac{436}{13}, \frac{530}{13}, 10\right), \quad e_3 = (100, 70, 50).$$

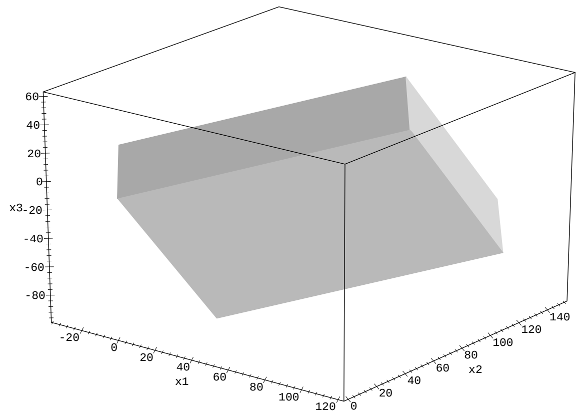

Fig. 3. Rectangular solid reconstructed from the top aspect given in Fig. 2.

5 Central Projection

In this section we consider the same problems as in the previous one but for central projection: We again consider, objects A, B, $\ldots$ described by quantifier-free formulas α, β, $\ldots$; their boundaries $\mathrm{b}(A)$, $\mathrm{b}(B)$, $\ldots$ are described by formulas α_{b}, β_{b}, $\ldots$

Let $(q, r, s) \in S$ denote the position of the punctual light source in space. Then a point $(x, y, z) \in S$ is the image of a point $(u, v, w) \in S$ under central projection from point (q, r, s) iff

$$\exists t\big(t > 0 \wedge x = t(u - q) \wedge y = t(v - r) \ \wedge z = t(w - s)\big).$$

Hence the region $\mathrm{SR}(A)$ in space that is shaded by object A under central projection from (q, r, s) is described by the formula

$$\begin{aligned}(x, y, z) \in \mathrm{SR}(A) \iff \alpha_{\mathrm{SR}}(x, y, z) \equiv \exists u \exists v \exists w \exists t\big(&\alpha(u, v, w) \wedge t > 0 \wedge \\ &x = t(u - q) \wedge y = t(v - r) \wedge \\ &z = t(w - s)\big).\end{aligned}$$

By determining a quantifier-free equivalent α'_{SR} to the formula α_{SR} above, we get a semi-algebraic description of the region $\mathrm{SR}(A)$.

In a similar way our descriptions given in the previous section of the shaded part, the lighted part, and of the boundary between these parts can be adapted to central projection. For the latter we again have to exclude degenerate situations, where the surface of the object contains more than one point on some light ray:

$$\begin{aligned}(x, y, z) \in \mathrm{Sh}(A) \iff\ &\alpha_{\mathrm{b}}(x, y, z) \wedge \exists u \exists v \exists w \exists t\big(\alpha(u, v, w) \wedge t > 0 \wedge \\ &x = t(u - q) \wedge \ y = t(v - r) \wedge z = t(w - s)\big) \\ (x, y, z) \in \mathrm{Li}(A) \iff\ &\alpha_{\mathrm{b}}(x, y, z) \wedge \neg \exists u \exists v \exists w \exists t\big(\alpha(u, v, w) \wedge t > 0 \wedge \\ &x = t(u - q) \wedge y = t(v - r) \wedge z = t(w - s)\big) \\ (x, y, z) \in \mathrm{SL}(A) \iff\ &\alpha_{\mathrm{b}}(x, y, z) \wedge \neg \exists u \exists v \exists w \exists t\big(\alpha(u, v, w) \wedge t \neq 0 \wedge \\ &x = t(u - q) \wedge y = t(v - r) \wedge z = t(w - s)\big).\end{aligned}$$

Let B be another object in S. Then the shadow $\mathrm{CS}(A, B)$ cast by object A onto the boundary of B can be described by the following formula:

$$\begin{aligned}(x, y, z) \in \mathrm{CS}(A, B) \iff\ &\beta_{\mathrm{b}}(x, y, z) \wedge \exists u \exists v \exists w \exists t\big(\alpha(u, v, w) \wedge t \geq 0 \wedge \\ &x = t(u - q) \wedge y = t(v - r) \wedge z = t(w - s)\big).\end{aligned}$$

This applies in particular to the case when B is a plane given by the equation $kx + ly + mz = p$. In this case the shadow cast onto B by A is identical to the *image* $\mathrm{Im}(A, B)$ of A on B under central projection from (q, r, s). The following formula describes this image:

$$\begin{aligned}(x, y, z) \in \mathrm{Im}(A, B) \iff\ &kx + ly + mz = p \wedge \exists u \exists v \exists w \exists t\big(\alpha(u, v, w) \wedge t \geq 0 \wedge \\ &x = t(u - q) \wedge y = t(v - r) \wedge z = t(w - s)\big).\end{aligned}$$

In the previous section, we have recovered a rectangular solid from its aspect. We now turn to a more difficult problem. We wish to recover a cuboid from a central projection. To make to problem easier, we consider a wire frame instead of a solid avoiding visibility considerations. Still the cuboid will not be uniquely determined by one wire frame image.

We modify our aspect case model as follows: The cuboid generated by the vectors e_1, e_2, e_3 is shifted from the origin by some translation vector $v = (v_1, v_2, v_3)$. The projection plane ist the x_1-x_2 plane. The focal point is, e.g., $(0, 0, 5)$. Our idea is that the focal point lies between the image and the projection plane modeling photography: We take photos along the x_3-axis. The 3-space origin is the middle of our photo.

A generic image description ι' is derived by applying quantifier elimination (765 ms) to the following formula:

$$\begin{aligned}\iota \equiv e_1e_2 = e_1e_3 = e_2e_3 = 0 \wedge \exists x_1 \exists x_2 \exists x_3 \exists \lambda_1 \exists \lambda_2 \exists \lambda_3 \exists k \bigl(\lambda_1, \lambda_2, \lambda_3 \in \{0, 1\} \wedge \\ x = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + v \wedge i_1 = kx_1 \wedge i_2 = kx_2 \wedge 0 = 5 + k(5 - x_3)\bigr).\end{aligned}$$

For a given semi-algebraic image description π_0, we can recover information about the original wire frame by applying extended quantifier elimination to the following reconstruction formula:

$$\exists e_{11} \exists e_{12} \exists e_{13} \exists e_{21} \exists e_{22} \exists e_{23} \exists e_{31} \exists e_{32} \exists e_{33} \exists v_1 \exists v_2 \exists v_3 \forall i_1 \forall i_2 (\iota' \longleftrightarrow \pi_0).$$

In practice, the elimination can be supported by fixing the image of the shifted origin v. Compare Example 10 for details.

With two images taken simultaneously from different locations, a cuboid can be reconstructed uniquely provided that the choices for the v, e_1, e_2, e_3 are consistent. Unfortunately, in a corresponding reconstruction formula for a stereo camera one cannot play the trick of fixing the images of v indicated above, because one cannot automatically determine which points correspond in two different projection images.

Examples computed with REDLOG

Example 8. This example is inspired by an example discussed in [KM88]. We consider the central projection of the line

$$a_1u + b_1v + c_1w = d_1 \wedge a_2u + b_2v + c_2w = d_2$$

from a punctual light source located at $(0, 0, f)$ onto the projection plane $w = 0$:

$$\begin{aligned}\varphi_1(x, y, a_1, \ldots, d_2, f) \equiv \exists u \exists v \exists w \bigl(x(f - w) = fu \wedge y(f - w) = fv \wedge \\ a_1u + b_1v + c_1w = d_1 \wedge a_2u + b_2v + c_2w = d_2\bigr).\end{aligned}$$

Generic quantifier elimination with the theory $\{f > 0\}$ yields after 221 ms

$$\begin{aligned}&a_1c_2fx - a_1d_2x - a_2c_1fx + a_2d_1x + \\ &b_1c_2fy - b_1d_2y - b_2c_1fy + b_2d_1y + c_1d_2f - c_2d_1f = 0\end{aligned}$$

subject to the input theory plus the ND-condition $a_1x + b_1y - c_1f \neq 0$. Given another line

$$a_3u + b_3v + c_3w = d_3 \wedge a_4u + b_4v + c_4w = d_4$$

in space, we can read off from the elimination result above the necessary and sufficient condition for the images of the two lines to be parallel under the corresponding ND-conditions, viz.:

$$\begin{aligned}(a_1c_2f - a_1d_2 - a_2c_1f + a_2d_1)(b_3c_4f - b_3d_4 - b_4c_3f + b_4d_3) =\\ (a_3c_4f - a_3d_4 - a_4c_3f + a_4d_3)(b_1c_2f - b_1d_2 - b_2c_1f + b_2d_1)\end{aligned}$$

provided all bracketed items are non-zero.

Example 9. We compute the central projection of the unit ball in arbitrary position. Again, the light source is located at $(0, 0, f)$, and the projection plane is given by $w = 0$:

$$\begin{aligned}\varphi_2(x, y, a, b, c, f) \equiv \exists u \exists v \exists w \big(x(f - w) = fu \wedge y(f - w) = fv \wedge\\ (u - a)^2 + (v - b)^2 + (w - c)^2 = 1\big).\end{aligned}$$

Generic quantifier elimination with the theory $\{f > 0\}$ yields after 187 ms:

$$\begin{aligned}a^2f^2 + a^2y^2 - 2abxy + 2acfx - 2af^2x + b^2f^2 + b^2x^2 + 2bcfy -\\ 2bf^2y + c^2x^2 + c^2y^2 - 2cfx^2 - 2cfy^2 + f^2x^2 + f^2y^2 - f^2 - x^2 - y^2 \leq 0\end{aligned}$$

without any additional ND-condition.

Example 10 (Recovering a cuboid). Fig. 4 shows a photo taken of the wire frame of the cuboid in Fig. 3 shifted from the origin by $v = (100, 200, 300)$. The semi-algebraic image description is as follows:

$$\begin{aligned}\pi_0 \equiv\ & (3367i_1 - 12120 = 0 \wedge 3367i_2 - 22800 = 0) \vee\\ & (923i_1 - 2164 = 0 \wedge 923i_2 - 4040 = 0) \vee\\ & (2717i_1 - 5620 = 0 \wedge 2717i_2 - 18250 = 0) \vee\\ & (793i_1 - 864 = 0 \wedge 793i_2 - 3130 = 0) \vee\\ & (249i_1 - 1100 = 0 \wedge 249i_2 - 1550 = 0) \vee\\ & (69i_1 - 200 = 0 \wedge 23i_2 - 90 = 0) \vee\\ & (199i_1 - 600 = 0 \wedge 199i_2 - 1200 = 0) \vee\\ & (59i_1 - 100 = 0 \wedge 59i_2 - 200 = 0).\end{aligned}$$

In the reconstruction formula we support the elimination procedure by fixing the image point $\big(\frac{100}{59}, \frac{200}{59}, 0\big)$ to be the image the shifted origin:

$$\begin{aligned}&\exists e_{11}\exists e_{12}\exists e_{13}\exists e_{21}\exists e_{22}\exists e_{23}\exists e_{31}\exists e_{32}\exists e_{33}\exists v_1\exists v_2\exists v_3\forall i_1\forall i_2\\ &\Big[(\iota' \longleftrightarrow \pi_0) \wedge \exists k\Big(\frac{100}{59} = kv_1 \wedge \frac{200}{59} = kv_2 \wedge 0 = 5 + k(5 - v_3)\Big)\Big].\end{aligned}$$

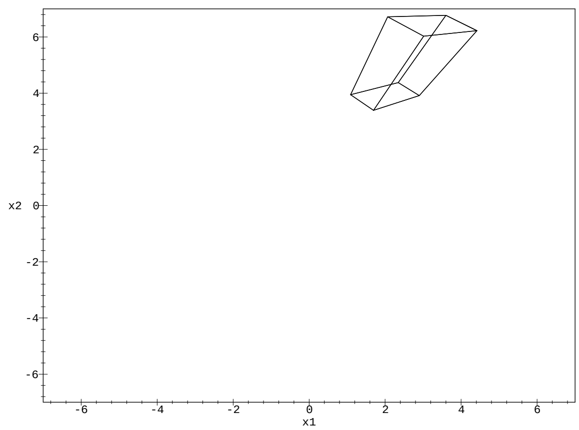

Fig. 4. A photo of the wire frame of the cuboid in Fig. 3.

We obtain after 1952 671 ms (33 minutes) the quantifier-free equivalent "true" together with the following answer:

$$v = \Big(\frac{5\infty_1}{2}, 5\infty_1, \frac{59\infty_1 + 40}{8}\Big), \quad e_1 = \Big(\frac{5\infty_1}{2}, \frac{7\infty_1}{4}, \frac{5\infty_1}{4}\Big),$$

$$e_2 = \Big(\frac{\infty_1}{2}, \infty_1, \frac{-12\infty_1}{5}\Big), \quad e_3 = \Big(\frac{-109\infty_1}{130}, \frac{53\infty_1}{52}, \frac{\infty_1}{4}\Big).$$

For $\infty_1 = 40$ this is our original cuboid.

Without fixing the image of v, i.e., using

$$\exists e_{11}\exists e_{12}\exists e_{13}\exists e_{21}\exists e_{22}\exists e_{23}\exists e_{31}\exists e_{32}\exists e_{33}\exists v_1\exists v_2\exists v_3\forall i_1\forall i_2(\iota' \longleftrightarrow \pi_0)$$

as reconstruction formula, the elimination takes 42998 933 ms (approximately 12 h) yielding a correct reconstruction with different choices for the vectors.

6 Offsets

Let $A \subseteq S$ be an object that constitutes a non-empty closed set. Fix a metric $d : S \to \mathbb{R}$ that induces the usual topology on S. Then the distance of a point $(x, y, z) \in S$ from A is defined as

$$d\big((x, y, z), A\big) := \min\{\, d\big((x, y, z), (u, v, w)\big) \mid (u, v, w) \in A \,\}.$$

The existence of the minimum is guaranteed by the assumption that A is closed in S and that d is continuous. Let now r be a non-negative real number. Then the r-*offset* of A is defined as

$$A_r = \{\, (x, y, z) \in S \mid d\big((x, y, z), A\big) = r \,\}.$$

We generalize the notion of an r-offset: The *$\leq$-r-offset* $A_{\leq,r}$ of A and the *$\geq$-r-offset* $A_{\geq,r}$ are defined by $d\big((x,y,z),A\big) \leq r$ and $d\big((x,y,z),A\big) \geq r$, respectively. We then obviously have $A_r = A_{\leq,r} \cap A_{\geq,r}$.

For a fixed metric d on S, suppose we have formulas

$$\delta_{\leq}(x,y,z,u,v,w,r) \quad \text{and} \quad \delta_{\geq}(x,y,z,u,v,w,r)$$

describing the facts that $d\big((u,v,w),(x,y,z)\big) \leq r$ and $d\big((u,v,w),(x,y,z)\big) \geq r$, respectively. Then the following formulas describe various offsets of A:

$$\begin{aligned}
\alpha_{\leq,r}(x,y,z) &\equiv \exists u \exists v \exists w \big(\alpha(u,v,w) \wedge \delta_{\leq}(x,y,z,u,v,w,r)\big)\\
\alpha_{\geq,r}(x,y,z) &\equiv \forall u \forall v \forall w \big(\alpha(u,v,w) \longrightarrow \delta_{\geq}(x,y,z,u,v,w,r)\big)\\
\alpha_r(x,y,z) &\equiv \alpha_{\leq,r}(x,y,z) \wedge \alpha_{\geq,r}(x,y,z).
\end{aligned}$$

We give the relevant formulas $\delta_{\geq}(x,y,z,u,v,w,r)$ for some common metrics on S. The formulas $\delta_{\leq}(x,y,z,u,v,w,r)$ are formed similarly:

- for the Euclidean (L_2) metric:

$$(x-u)^2 + (y-v)^2 + (z-w)^2 \geq r^2$$

- for the "Manhattan" (L_1) metric:

$$\begin{aligned}
&\exists k \exists l \exists m \Big(\big(k \geq 0 \wedge (k = x-u \vee k = u-x)\big) \wedge\\
&\big(l \geq 0 \wedge (l = y-v \vee l = v-y)\big) \wedge\\
&\big(m \geq 0 \wedge (m = z-w \vee m = w-z)\big) \wedge k+l+m \geq r\Big)
\end{aligned}$$

- for the maximum (L_∞) metric:

$$r \leq x-u \vee r \leq u-x \vee r \leq y-v \vee r \leq v-y \vee r \leq z-w \vee r \leq w-z.$$

Notice that the formulas α_r for the r-offset involve 3 existential and 3 universal quantifiers and hence pose a non-trivial problem for quantifier elimination.

For certain smooth algebraic surfaces $A : f(u,v,w) = 0$ without singularities in S we have much simpler formulas for A_r: Suppose the minimal radius of the curvature of A is less than or equal to r. Then the point $(u,v,w) \in A$ of minimal distance to a given point $(x,y,z) \in S$ is determined by the fact that the vector $(x-u, y-v, z-w)$ is perpendicular to the tangent plane of A at (u,v,w). In other words this vector is a real multiple of the gradient vector ∇f. As a consequence we may in this case define α_r^* instead of α_r:

$$\begin{aligned}
\alpha_r^*(x,y,z) \equiv \exists u \exists v \exists w \exists t \Big(& f(u,v,w) = 0 \wedge \delta(x,y,z,u,v,w,r) \wedge\\
& t\frac{\partial f(u,v,w)}{\partial u} = x-u \wedge\\
& t\frac{\partial f(u,v,w)}{\partial v} = y-v \wedge\\
& t\frac{\partial f(u,v,w)}{\partial w} = z-w\Big).
\end{aligned}$$

Here $\delta \equiv \delta_{\leq} \wedge \delta_{\geq}$ describes the fact that $d\big((u,v,w),(x,y,z)\big) = r$. For a well-behaved algebraic surface with singularities, α_r^* can be combined with α_r restricted to the singularities.

The idea of using the gradient vectors can be extended to smooth algebraic curves $A : f(u,v,w) = 0 \wedge g(u,v,w) = 0$ without singularities in S, where we assume again that the radius of the curvature of A is less than or equal to r. We then have the following description α_r^{**} of the r-offset:

$$\begin{aligned}\alpha_r^{**}(x,y,z) \equiv \exists u \exists v \exists w \exists s \exists t \Big(& f(u,v,w) = 0 \wedge \delta(x,y,z,u,v,w,r) \wedge \\ & s\frac{\partial f(u,v,w)}{\partial u} + t\frac{\partial g(u,v,w)}{\partial u} = x - u \wedge \\ & s\frac{\partial f(u,v,w)}{\partial v} + t\frac{\partial g(u,v,w)}{\partial v} = y - v \wedge \\ & s\frac{\partial f(u,v,w)}{\partial w} + t\frac{\partial g(u,v,w)}{\partial w} = z - w \Big).\end{aligned}$$

For certain special objects in 3-space there are even simpler descriptions available: Consider, e.g., the r_2-offset of a circle $x^2 + y^2 = r_1^2$ wrt. the Euclidean metric, which is a torus T. It can be defined by the following formula τ with only one quantifier:

$$\tau(x,y,z) \equiv \exists u \big(u^2 + z^2 = r_2^2 \wedge (r_1 + u)^2 = x^2 + y^2\big).$$

The idea behind this description is to regard T as a union of circles in planes parallel to the X-Y-plane.

Example 14 below suggests to use iterated offset computations for the problem of rounding corners and edges, cf. [Hof96].

The concept of an r-offset in the plane instead of 3-space is defined analogously. It should be obvious, how to modify the formulas given above for the planar case. Compare [PP96] and [Wan92] for a discussion of offsets and examples, using other methods.

Examples computed with REDLOG

Example 11. We compute a quantifier-free description of the torus given by

$$\tau(x,y,z,r_1,r_2) \equiv \exists u \big(u^2 + z^2 = r_2^2 \wedge (r_1 + u)^2 = x^2 + y^2\big).$$

Quantifier elimination with subsequent Gröbner basis simplification yields within 1241 ms the following quantifier-free description:

$$\begin{aligned}& r_1^4 - 2r_1^2 r_2^2 - 2r_1^2 x^2 - 2r_1^2 y^2 + 2r_1^2 z^2 + r_2^4 - 2r_2^2 x^2 - \\ & \quad 2r_2^2 y^2 - 2r_2^2 z^2 + x^4 + 2x^2y^2 + 2x^2z^2 + y^4 + 2y^2z^2 + z^4 = 0 \wedge r_2^2 - z^2 \geq 0.\end{aligned}$$

Example 12. We compute r-offsets of the cross consisting of the X-axis and the Y-axis in S. Its offset wrt. the Euclidean metric and the L_1 metric is described by the following formulas φ_2 and φ_1, respectively:

$$\begin{aligned}
\varphi_2(x, y, z, r) \equiv{} & \exists u \exists v\big((u = 0 \vee v = 0) \wedge (x - u)^2 + (y - v)^2 + z^2 = r^2\big) \wedge \\
& \forall u \forall v \neg\big((u = 0 \vee v = 0) \wedge (x - u)^2 + (y - v)^2 + z^2 < r^2\big)
\end{aligned}$$

$$\begin{aligned}
\varphi_1(x, y, z, r) \equiv{} & \exists u \exists v\big((u = 0 \vee v = 0) \wedge \exists k \exists l \exists m(\\
& k \geq 0 \wedge (k = x - u \vee k = u - x) \wedge \\
& l \geq 0 \wedge (l = y - v \vee l = v - y) \wedge \\
& m \geq 0 \wedge (m = z - w \vee m = w - z) \wedge k + l + m = r)\big) \wedge \\
& \forall u \forall v \neg\big((u = 0 \vee v = 0) \wedge \exists k \exists l \exists m(\\
& k \geq 0 \wedge (k = x - u \vee k = u - x) \wedge \\
& l \geq 0 \wedge (l = y - v \vee l = v - y) \wedge \\
& m \geq 0 \wedge (m = z - w \vee m = w - z) \wedge k + l + m < r)\big).
\end{aligned}$$

Quantifier elimination with subsequent Gröbner basis simplification applied to φ_2 yields after 1292 ms:

$$\begin{aligned}
&(r^2 - x^2 - z^2 = 0 \wedge x^2 - y^2 < 0 \wedge x \neq 0 \wedge y \neq 0) \vee \\
&(r^2 - x^2 - z^2 = 0 \wedge x^2 - y^2 < 0 \wedge x = 0) \vee \\
&(r^2 - y^2 - z^2 = 0 \wedge x^2 - y^2 \geq 0 \wedge x \neq 0 \wedge y \neq 0) \vee \\
&(r^2 - y^2 - z^2 = 0 \wedge x^2 - y^2 \geq 0 \wedge y = 0).
\end{aligned}$$

The offset thus consists of segments of the tubes $r^2 - x^2 - z^2 = 0$ and $r^2 - y^2 - z^2 = 0$ separated by the planes $X = Y$ and $X = -Y$ at which they exactly fit together. Fig. 5 pictures this for $r = 4$.

For φ_1 we obtain without Gröbner simplification after 5.3 s a quantifier-free formula with 632 atomic subformulas. The Gröbner simplifier does not yield a simplified equivalent for this result. It fails in performing an initial Boolean normal form, i.e. CNF or DNF, computation step, which is necessary with the current version of REDLOG, cf. [DS95].

Example 13. We compute, wrt. the Euclidean metric, the r-offset of the parabola in S given by the equations $v = u^2$ and $w = 0$:

$$\begin{aligned}
\varphi(x, y, z, r) \equiv{} & \exists u \exists v \exists s(v = u^2 \wedge \\
& (x - u)^2 + (y - v)^2 + z^2 = r^2 \wedge 2su = x - u \wedge -s = y - v).
\end{aligned}$$

The generic elimination can eliminate only v and leaves a formula in which u occurs in a conjunction of a quartic equation $f_1(u) = 0$ and a cubic equation $f_2(u) = 0$. We conjunctively add $h(u) = 0$, where $h(u)$ is the quadratic remainder of $f_1(u)/f_2(u)$. The generic elimination of u can then be completed automatically, yielding a formula with 7 atomic subformulas under the ND-condition $2y - 1 \neq 0$. The total time is 5.8 s.

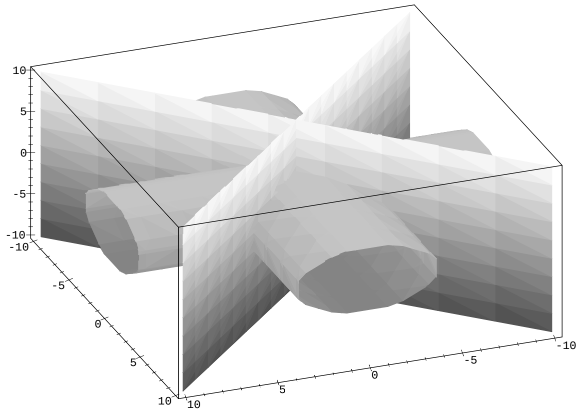

Fig. 5. 4-Offset of the X-axis and the Y-axis together with the planes $X = Y$ and $X = -Y$ separating the relevant segments of the tubes.

Example 14. Consider the rectangle A of side lengths 1 and 2 described by the formula
$$\alpha(u, v) \equiv 0 \leq u \leq 1 \wedge 0 \leq v \leq 2.$$
We want to round the corners of A by circular segments of radius $0 < r < \frac{1}{2}$.

For this purpose, let B be the closure of the complement of A. It is described by $\beta(u, v) \equiv \neg(0 < u < 1 \wedge 0 < v < 2)$. The $\geq$-r-offset $B_{\geq,r}$ of B is described by the formula

$$\beta_{\geq,r}(u, v, r) \equiv \forall u \forall v \big(\beta \longrightarrow (x-u)^2 + (y-v)^2 \geq r^2)\big).$$

Quantifier elimination using the theory $\{0 < r, r < \frac{1}{2}\}$ with subsequent Gröbner simplification yields after 153 ms the following quantifier-free description:

$$\begin{aligned}\beta'_{\geq,r}(x, y, r) \equiv\ & r^2 - x^2 + 2x - 1 \leq 0 \wedge r^2 - x^2 - y^2 + 4y - 4 \leq 0 \wedge \\ & r^2 - x^2 - y^2 \leq 0 \wedge r^2 - x^2 \leq 0 \wedge r^2 - y^2 + 4y - 4 \leq 0 \wedge \\ & r^2 - y^2 \leq 0 \wedge x - 1 < 0 \wedge x > 0 \wedge y - 2 < 0 \wedge y > 0.\end{aligned}$$

Under our theory $\{0 < r, r < \frac{1}{2}\}$ we can transform this by some easy hand computations including completion of squares into the following simpler form:

$$\beta''_{\geq,r}(x, y, r) \equiv x \geq r \wedge 1 - r \geq x \wedge y \geq r \wedge 2 - r \geq y.$$

The correctness of our simplification can be proved automatically in less than 17 ms by applying quantifier elimination to

$$\forall x \forall y \forall r \left(0 < r < \frac{1}{2} \longrightarrow (\beta'_{\geq,r} \longleftrightarrow \beta''_{\geq,r})\right).$$

The final result of rounding is obtained as the $\leq$-r-offset

$$(B_{\geq,r})_{\leq,r}$$

of $B_{\geq,r}$. The corresponding quantifier elimination, again with the theory $\{0 < r, r < \frac{1}{2}\}$, delivers after 323 ms the following quantifier-free description of the rounded rectangle:

$$\begin{aligned}
&(r^2 + 2rx + 2ry - 6r + x^2 - 2x + y^2 - 4y + 5 \leq 0) \vee \\
&\ (r^2 + 2rx - 2ry - 2r + x^2 - 2x + y^2 + 1 \leq 0) \vee \\
&\ (r^2 - 2rx + 2ry - 4r + x^2 + y^2 - 4y + 4 \leq 0) \vee \\
&\ (r^2 - 2rx - 2ry + x^2 + y^2 \leq 0) \vee \\
&\ (r + x - 1 \leq 0 \wedge r - x \leq 0 \wedge r + y - 2 \leq 0 \wedge r - y \leq 0) \vee \\
&\ (2rx - 2r + x^2 - 2x + 1 \leq 0 \wedge r + y - 2 \leq 0 \wedge r - y \leq 0) \vee \\
&\ (2rx - x^2 \geq 0 \wedge r + y - 2 \leq 0 \wedge r - y \leq 0) \vee \\
&\ (2ry - 4r + y^2 - 4y + 4 \leq 0 \wedge r + x - 1 \leq 0 \wedge r - x \leq 0) \vee \\
&\ (2ry - y^2 \geq 0 \wedge r + x - 1 \leq 0 \wedge r - x \leq 0).
\end{aligned}$$

It contains 20 atomic formulas. The shortest description we are able to produce by hand contains 12 atomic formulas. For fixed values of r the equivalence between the automatic result and our hand formulation can be shown by quantifier elimination in less than one minute.

7 Voronoi Diagrams and Equi-Distance Surfaces

Let $F = \{A_1, \ldots, A_n\}$ be a finite family of objects that constitute non-empty closed sets in space. Fix a metric $d : S \to \mathbb{R}$ that induces the usual topology on S. Then the *Voronoi diagram* $\mathrm{Vor}(F)$ wrt. d of F is defined as the set of all $(x, y, z) \in S$ for which there are $1 \leq i < j \leq n$ such that

$$d\big((x,y,z), A_i\big) = d\big((x,y,z), A_j\big) = \min\big\{\, d\big((x,y,z), A_k\big) \;\big|\; 1 \leq k \leq n \,\big\}.$$

In the special case $F = \{A_1, A_2\}$, the Voronoi diagram is the *equi-distance surface* of the objects A_1 and A_2.

A similar definition applies to Voronoi diagrams in the plane. Compare [PS85] for a discussion of Voronoi diagrams in the plane.

Suppose that we have quantifier-free formulas $\alpha_i(u, v, w)$ describing the objects A_i for $1 \leq i \leq n$. As sketched in the previous section, there are formulas $\delta(x, y, z, u, v, w, r)$ and $\delta_{\geq}(x, y, z, u, v, w, r)$ describing, under the given metric d, the fact that $d\big((x, y, z), (u, v, w)\big) = r$ and $d\big((x, y, z), (u, v, w)\big) \geq r$, respectively. From these we can construct formulas

$$\delta_i(x, y, z, r) \quad \text{and} \quad \delta_{\geq,i}(x, y, z, r)$$

describing the fact that $d\big((x, y, z), A_i\big) = r$ and $d\big((x, y, z), A_i\big) \geq r$, respectively. Based on these intermediate formulas, the Voronoi diagram $\mathrm{Vor}(F)$ can be described by the following formula:

$$\bigvee_{\substack{i,j=1\\ i\neq j}}^{n} \exists r \Big(\delta_i(x,y,z,r) \wedge \delta_j(x,y,z,r) \wedge \bigwedge_{\substack{k=1\\ k\neq i,j}}^{n} \delta_{\geq,k}(x,y,z,r)\Big).$$

Examples computed with REDLOG

Example 15. In the real plane, the Voronoi diagram of the three points $(-1, 0)$, $(1, 0)$, $(1, 1)$ is described by the formula

$$\begin{aligned}\varphi(x, y) \equiv \exists r\big(&(r^2 = (x-1)^2 + y^2 = (x+1)^2 + y^2 \wedge r^2 \leq (x-1)^2 + (y-1)^2) \vee \\ &(r^2 = (x-1)^2 + y^2 = (x-1)^2 + (y-1)^2 \wedge r^2 \leq (x+1)^2 + y^2) \vee \\ &(r^2 = (x+1)^2 + y^2 = (x-1)^2 + (y-1)^2 \wedge r^2 \leq (x-1)^2 + y^2)\big).\end{aligned}$$

Quantifier elimination with subsequent Gröbner simplification yields the following quantifier-free description (170 ms):

$$\begin{aligned}&(4x + 2y - 1 = 0 \wedge 4y^2 - 4y + 5 \geq 0 \wedge 2y - 1 \geq 0) \vee \\ &(4x^2 - 8x + 5 \geq 0 \wedge x \geq 0 \wedge 2y - 1 = 0) \vee (x = 0 \wedge 2y - 1 \leq 0).\end{aligned}$$

Example 16. The Voronoi diagram of the unit circle and the lines $X = -2$ and $X = 2$ in the plane is described by the formula

$$\begin{aligned}\varphi(x, y) \equiv \exists r \exists u \exists v\big[&u^2 + v^2 = 1 \wedge vx = uy \wedge \\ &\big((r^2 = (x+2)^2 = (x-2)^2 \wedge r^2 \leq (x-u)^2 + (y-v)^2) \vee \\ &(r^2 = (x+2)^2 = (x-u)^2 + (y-v)^2 \wedge r^2 \leq (x-2)^2) \vee \\ &(r^2 = (x-2)^2 = (x-u)^2 + (y-v)^2 \wedge r^2 \leq (x+2)^2)\big)\big].\end{aligned}$$

Generic quantifier elimination yields after 1122 ms a formula with 16 atomic subformulas plus the ND-condition $x \neq 0$. Subsequent Gröbner simplification produces the following result (153 ms):

$$\begin{aligned}&(6x - y^2 + 9 = 0 \wedge x < 0) \vee (2x - y^2 + 1 = 0 \wedge x < 0) \vee \\ &(6x + y^2 - 9 = 0 \wedge x > 0) \vee (2x + y^2 - 1 = 0 \wedge x > 0).\end{aligned}$$

The following two examples concern the computation of equi-distance surfaces in space, cf. [Hof96].

Example 17. Find a description of the surface defined by equal distance to the unit ball and a parametric plane $X = a$:

$$\begin{aligned}\varphi_1(x, y, z, a) \equiv \exists u \exists v \exists w\big[&(x-a)^2 = (x-u)^2 + (y-v)^2 + (z-w)^2 \wedge \\ &u^2 + v^2 + w^2 = 1 \wedge uy = vx \wedge uz = wx \wedge vz = wy \wedge \\ &\big(xu > 0 \vee (x = 0 \wedge u = 0)\big) \wedge \big(yv > 0 \vee (y = 0 \wedge v = 0)\big) \wedge \\ &\big(zw > 0 \vee (z = 0 \wedge w = 0)\big)\big].\end{aligned}$$

Generic quantifier elimination produces after 1037 ms a formula φ_1' with 8 atomic subformulas under the ND-condition $x \neq 0$. Fixing $a = 5$ and applying the Gröbner simplifier, we obtain the following φ_1'':

$$\begin{aligned}&(x^2 - 6x \geq 0 \wedge 12x + y^2 + z^2 - 36 = 0 \wedge x < 0) \vee \\ &(x^2 - 4x \leq 0 \wedge 8x + y^2 + z^2 - 16 = 0 \wedge x < 0) \vee \\ &(x^2 - 6x \leq 0 \wedge 12x + y^2 + z^2 - 36 = 0 \wedge x - 6 \neq 0 \wedge x > 0) \vee \\ &(x^2 - 4x \geq 0 \wedge 8x + y^2 + z^2 - 16 = 0 \wedge x - 4 \neq 0 \wedge x > 0).\end{aligned}$$

We suspect that one of the two contained equations describes the Voronoi diagram: Applying quantifier elimination to

$$\forall x \forall y \forall z \big(x \neq 0 \longrightarrow (\varphi_1'' \longleftrightarrow 12x + y^2 + z^2 - 36 = 0)\big)$$

we in fact obtain "true" after 170 ms. Fig. 6 pictures our special case $a = 5$.

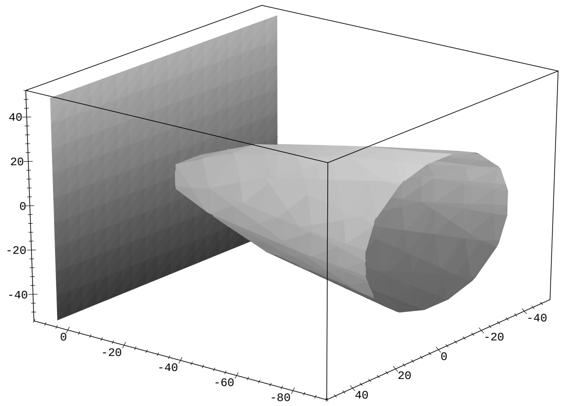

Fig. 6. The plane $X = 5$ and its equi-distance surface with the unit ball.

Example 18. We compute the equi-distance surface for the X-axis and the line $X = Y$, $Z = 1$:

$$\varphi_2(x, y, z) \equiv \exists u\big(y^2 + z^2 = (x - u)^2 + (y - u)^2 + (z - 1)^2 \wedge x - u = -(y - u)\big).$$

Elimination yields $x^2 - 2xy - y^2 - 4z + 2 = 0$ after 17 ms. This surface is pictured in Fig. 7.

8 Collision Problems

Let A, B be objects moving in space from their initial position with time. The question is, whether A and B will collide, and—if yes—when and where is a collision going to happen?

In the easiest case, A and B move along straight lines with constant velocities. Suppose $\alpha(x, y, z)$ and $\beta(x, y, z)$ describe the objects A and B at time $t = 0$, and (k, l, m), (p, q, r) are the velocity vectors for the motion of A and B, respectively. Then a collision situation of A and B is characterized by both objects having a point in common. This is described by the formula $\psi \equiv \exists t\big(t > 0 \wedge \varphi(t)\big)$, where

$$\varphi(t) \equiv \exists x \exists y \exists z\big(\alpha(x - kt, y - lt, z - mt) \wedge \beta(x - pt, y - qt, z - rt)\big).$$

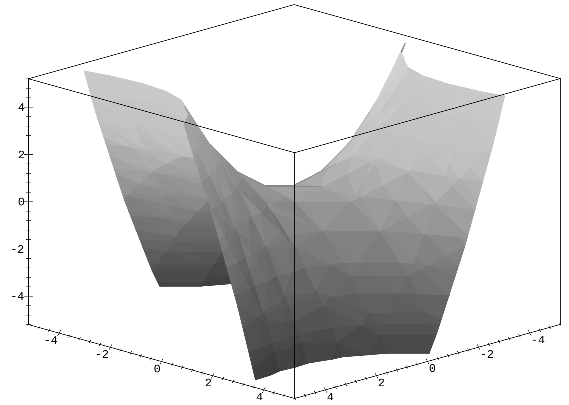

Fig. 7. The equi-distance surface of X-axis and the line $X = Y$, $Z = 1$.

If we eliminate from ψ the four quantifiers by extended quantifier elimination, we get either "false," i.e., A and B will never collide, or "true" together with a common point and the corresponding time. In general, this does not describe the first collision of A and B. In order to obtain time and location of the first contact of the two objects, we have to apply quantifier elimination with answer to the formula $\psi^* \equiv \exists t\bigl[t > 0 \land \varphi(t) \land \forall t'\bigl(0 < t' < t \longrightarrow \neg\varphi(t')\bigr)\bigr]$.

Examples computed with REDLOG

Example 19. Consider the unit disk moving with velocity 1 along the X-axis, and a square with center at $(0, -8)$ moving with velocity vector (a, b) from its starting position. A collision situation is described by

$$\varphi(t, a, b) \equiv \exists x \exists y(-1 \leq x - at \leq 1 \land -9 \leq y - bt \leq -7 \land (x - t)^2 + y^2 \leq 1).$$

For obtaining necessary and sufficient conditions on the parametric velocity vector (a, b) for a collision to take place, we apply generic quantifier elimination to the formula

$$\psi(a, b) \equiv \exists t\bigl(t > 0 \land \varphi(t, a, b)\bigr).$$

This yields after 1292 ms a quantifier-free formula $\psi'(a, b)$ with 69 atomic subformulas under the ND-condition $a^2 - 2a + b^2 + 1 \neq 0$, which obviously excludes only the degenerate velocity vector $(a, b) = (0, 0)$.

This example is generalized from a decision problem in [CH91], where the parameters a, b were both fixed to $\frac{17}{16}$. We plug into φ these original values for a and b yielding

$$\varphi_0(t) \equiv \varphi\Bigl(t, \frac{17}{16}, \frac{17}{16}\Bigr).$$

The result of quantifier elimination with answer applied to the corresponding

$$\psi_0 \equiv \exists t(t > 0 \land \varphi_0)$$

is "true" together with the answer $t = \frac{144}{17}$ and $(x, y) = (\frac{144}{17}, 0)$. This extended quantifier elimination takes 85 ms.

Next we check whether this describes the first collision by eliminating quantifiers with answer from

$$\psi_0^* \equiv \exists t\big[t > 0 \wedge \varphi_0(t) \wedge \forall t'\big(0 < t' < t \longrightarrow \neg\varphi_0(t')\big)\big].$$

The result obtained after 629 ms is "true" together with the answer $t = \frac{96}{17}$ and $(x, y) = (\frac{96}{17}, -1)$ showing the true first collision.

The next example is a very strong 3-dimensional generalization of the previous one.

Example 20. Starting with its center located at the origin, a solid parametric quadric $a_1x^2 + b_1y^2 + c_1z^2 \le 1$ with main axes parallel to the coordinate axes moves with velocity 1 along the X-axis. A cube with side length 2 moves starting with its center at (a, b, c) with constant parametric velocity (k, l, m) through space: The input formula is $\psi \equiv \exists t(t > 0 \wedge \varphi)$, where

$$\begin{aligned}\varphi(t, a_1, b_1, c_1, a, b, c, k, l, m) \equiv \exists x\exists y\exists z(&a_1(x - t)^2 + b_1y^2 + c_1z^2 \le 1 \wedge\\ &a - 1 \le x - kt \le a + 1 \wedge\\ &b - 1 \le y - lt \le b + 1 \wedge\\ &c - 1 \le z - mt \le c + 1).\end{aligned}$$

Note that $\psi(a_1, b_1, c_1, a, b, c, k, l, m)$ has 9 parameters. Generic quantifier elimination yields a quantifier-free formula with 2792 atomic subformulas under 7 ND-conditions in 43.4 s.

We fix the dimensions of the quadric and the starting position for the cube yielding

$$\varphi_0(t, k, l, m) \equiv \varphi(t, 3, 2, 1, 1, -10, 15, k, l, m)$$

and $\psi_0(k, l, m) \equiv \exists t\big(t > 0 \wedge \varphi_0(t, k, l, m)\big)$ as corresponding input formula. This reduces the timing for generic quantifier elimination to 9.4 s yielding only 336 atomic formulas under 5 ND-conditions. For this specialized situation, we ask for a velocity vector such that a collision takes place not later than at the time $t = 10$:

$$\psi_1 \equiv \exists k\exists l\exists m\exists t\big(0 < t \le 10 \wedge \varphi_0(t, k, l, m)\big).$$

Applying extended quantifier elimination, we obtain after 272 ms "true" together with the answer $(k, l, m) = (1, \frac{11}{10}, -\frac{7}{5})$, $t = 10$, and $(x, y, z) = (10, 0, 0)$.

Example 21. We consider two planes circling in descending spirals in a cylinder above a circle with radius r. Each plane has a given position and constant horizontal and vertical velocity. The question is, whether the two planes are going to crash within a given number of laps.

All distances are given in meters, velocities are given in meters per second. For velocities and positions we take coordinates (C_1, C_2), where C_1 is the position on the circumference, i.e. the distance on the circumference of the circle from a fixed reference point, and C_2 is the height. We consider two models of the situation:

Model 1 In the simpler model planes are modeled as points and a collision occurs if the two points coincide. Table 1 summarizes the data involved in this model (and also in the other one). A collision is then described by the following formula:

$$\varphi_1 \equiv p_2 + tv_2 = q_2 + tw_2 \wedge \bigvee_{k,l=0}^{n} \big(2\pi rk \leq tv_1 < 2\pi r(k+1) \wedge$$
$$2\pi rl \leq tw_1 < 2\pi r(l+1) \wedge p_1 + tv_1 - 2\pi rk = q_1 + tw_1 - 2\pi rl\big).$$

Model 2 In the second, more realistic, model we prescribe, in addition, horizontal and vertical safety distances, cf. Table 2. A "collision" occurs, if the points representing the planes violate both safety distances. The description of a collision then reads as follows:

$$\varphi_2 \equiv -h \leq p_2 + tv_2 - (q_2 + tw_2) \leq h \wedge$$
$$\bigvee_{k,l=0}^{n} \big(2\pi rk \leq tv_1 < 2\pi r(k+1) \wedge 2\pi rl \leq tw_1 < 2\pi r(l+1) \wedge$$
$$-w \leq p_1 + tv_1 - 2\pi rk - (q_1 + tw_1 - 2\pi rl) \leq w\big).$$

Table 1. Data in both models for plane collision; k, l, and n are not first-order variables but concrete integers.

t	time
r	radius of the circle
(p_1, p_2)	starting position of plane 1
(q_1, q_2)	starting position of plane 2
(v_1, v_2)	velocity of plane 1
(w_1, w_2)	velocity of plane 2
k	number of complete laps of plane 1
l	number of complete laps of plane 2
n	a bound on the number of complete laps of both planes

Table 2. Additional data for Model 2.

w	horizontal safety distance
h	vertical safety distance

One may be tempted to replace the disjunctions by an unbounded existential quantification $\exists k \exists l$. This is, however, not possible in out context, because these quantifiers would range over integers. In principle mixed real/integer elimination is possible, cf. [Wei90, Wei97a].

For obtaining a well-formed input formula we have to fix n to a natural number. Setting $n = 3$, φ_1 has 81 atomic subformulas in 10 parameters plus the symbol π. Our algorithms do not know anything about π. It plays the role of just another parameter.

Eliminating the time by applying generic quantifier elimination to $\psi_1 \equiv \exists t \varphi_1$, we obtain after 1870 ms necessary and sufficient conditions

$$\psi_1'(r, p_1, p_2, q_1, q_2, v_1, v_2, w_1, w_2, \pi)$$

in the other parameters for a collision to take place. This ψ_1' is valid under the obtained ND-condition $v_2 - w_2 \neq 0$. It contains 80 atomic formulas.

Proceeding the same way for the second model, we obtain after 48790 ms a quantifier-free formula

$$\psi_2'(r, p_1, p_2, q_1, q_2, v_1, v_2, w_1, w_2, h, w, \pi)$$

without any ND-condition. This ψ_2 contains 4597 atomic formulas.

Next we fix in φ_1 and φ_2 the following values for the radius of the circle, the starting positions, both velocities of plane 1, the horizontal velocity of plane 2, and the horizontal and vertical safety distances yielding $\varphi_1^*(t, w_2, \pi)$ and $\varphi_2^*(t, w_2, \pi)$, respectively:

$$\begin{aligned} r &= 10000 \\ (p_1, p_2) &= (0, 9000) \\ (q_1, q_2) &= (2000, 10000) \\ (v_1, v_2) &= (100, -3) \\ w_1 &= 50 \\ h &= 50 \\ w &= 500. \end{aligned}$$

Applying extended quantifier elimination to $\exists t \varphi_1^*$ we ask for vertical velocities w_2 of plane 2 that will lead to a collision within at most 3 laps of each plane. We obtain after 391 ms two possible conditions on w_2 together with corresponding answers for the collision times:

$$\begin{bmatrix} 10\pi w_2 + 30\pi + w_2 + 28 = 0 \land 5\pi - 1 > 0 & t = 400\pi + 40 \\ 5\pi - 1 > 0 \land w_2 + 28 = 0 & t = 40 \end{bmatrix}.$$

This means in rounded numerical values that a collision occurs within the first 3 laps if either $w_2 \approx -3.77$ or $w_2 = -28$. In the first case the collision time is $t \approx 1296.64$, in the second case it is $t = 40$.

The corresponding call applied to $\exists t \varphi_2^*$ takes 1683 ms yielding 6 conditions involving w_2 and π together with suitable answers for t. At least two of these cases have conditions γ which cannot hold for the true π. This is automatically found out by deciding $\exists w_2 \exists \pi (3.1 < \pi < 3.2 \wedge \gamma)$ via quantifier elimination, which takes 8.7 s for all 6 cases together.

Finally, we check the danger of a collision for concrete vertical speed ranges: Deciding the formula

$$\exists w_2 \exists \pi \exists t (3.1 < \pi < 3.2 \wedge w_2 \geq -3.8 \wedge \varphi_2^*)$$

by quantifier elimination yields "true" after 799 ms, while for

$$\exists w_2 \exists \pi \exists t (3.1 < \pi < 3.2 \wedge w_2 \geq -3.7 \wedge \varphi_2^*)$$

we obtain "false" after 835 ms. Taking these results together, some vertical speed $-3.8 \leq w_2 < -3.7$ will—modulo the approximate π—lead to a collision during the first 3 laps. The prescription $w_2 \geq -3.7$, in contrast, is collision-safe for this period.

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