Reasoning over Networks by Symbolic Methods

Thomas Sturm
Universität Passau, D-94030 Passau, Germany
sturm@uni-passau.de

MIP-9719
17 December 1997

Abstract

Effective quantifier elimination procedures for the reals allow to solve problems that can be encoded into corresponding first-order formulas including ordering constraints. In contrast to constraint logic programming, which is often used in similar areas, this encoding is straightforward. A quantifier elimination procedure using test point ideas is part of the REDUCE package REDLOG by the author et. al, which is freely available to the scientific community. We discuss how this method can be used for analysis, sizing, and error diagnosis of physical networks. The focus is on electrical networks.

1 Introduction

Effective quantifier elimination procedures for the reals allow to solve problems that can be encoded into first-order formulas over the real numbers using the relations "$=$," "$\leq$," "$<$," and "$\neq$." In contrast to constraint logic programming, cf. [Col90], which is often used in similar areas, this encoding is straightforward.

A *quantifier elimination procedure* is an algorithm that, given a first-order formula, computes an equivalent quantifier-free formula. Such a procedure for the reals has been devised firstly by Tarski around 1930 but remained unpublished until 1948, cf. [Tar48]. The first implementation of a quantifier elimination procedure by Arnon, cf. [Arn81], used the *cylindrical algebraic decomposition* (CAD) method of Collins, cf. [Col75, ACM84]. This approach has been improved considerably by Collins and Hong resulting in *partial* CAD, cf. [CH91], which has been implemented by Hong et al. in the QEPCAD package.

Besides the CAD approach, the implementation of quantifier elimination procedures using *test point* ideas of Weispfenning, cf. [Wei88, LW93, Wei94, Wei97a], is under development since 1992. The latest implementation is part of the REDUCE package REDLOG developed by the author together with A. Dolzmann, cf. [DS96, DS97a]. REDLOG is freely available to the scientific community.[1]

In general, the test point method can cope better than CAD with input involving many free variables (parameters). On the other hand, it is less general: The input formulas have to obey certain degree restrictions: considering the involved terms as multivariate polynomials, their total degree in the quantified variables must be at most two. Moreover, quantifiers are eliminated one by one and with the elimination of one quantifier the degree in the other quantified variables can increase. There are several heuristic techniques to overcome the degree restriction, e.g. factorization of the terms, which are surprisingly successful with practical problems, cf. [DSW96].

In addition to the pure quantifier elimination, REDLOG provides two important variants of quantifier elimination. *Quantifier elimination with answer* (also called *extended quantifier elimination*, cf. [Wei97b]) provides with the elimination of an existential quantifier block a set of *guarded points*

$$\{(\psi_1, S_1), \dots, (\psi_n, S_n)\}$$

such that whenever an interpretation of the parameters satisfies some ψ_i, the original $\exists$-quantified formula holds for this interpretation, and the S_i are

[1] `http://www.fmi.uni-passau.de/~redlog/`

some possibly parametric sample values for the quantified variables. The S_i can contain non-standard symbols such as ∞.

Generic quantifier elimination, cf. [DSW96], is a weaker but more efficient counterpart to quantifier elimination. Here, the procedure is allowed to make certain assumptions of the form $t \neq 0$ for terms t containing only parameters. These assumptions support the elimination process leading to a much smaller result, which is correct on the assumptions. The list of assumptions is, of course, returned to the caller. It has turned out that for the majority of practical applications, the assumptions made are "harmless." They describe some degenerate cases, which are actually not relevant. There is also generic quantifier elimination with answer available.

There are numerous applications of quantifier elimination in mathematics and engineering, cf. [DSW96, DSW97, DYA97, HLS97, Jir97, SW97a, SW97b, Wei97b]. Here we are going to discuss how REDLOG can be used for simulation, design, and error diagnosis of physical networks.

The plan of this note is as follows: Section 2 gives an outline of the test point method. By means of an academic example, Section 3 introduces relevant notions, gives an idea of possible questions on networks, and examines how to apply quantifier elimination to answer these questions. Section 4 points on the importance of our method working over the *real* numbers: it avoids all problems concerned with *parasitic solutions*. Section 5 applies quantifier elimination to the sizing of a BJT amplifier including transistors and capacities. In Section 6 we briefly indicate the applicability of our methods to hydraulic networks. In Section 7 we finally summarize and evaluate our results.

All computations have been performed on a SUN Ultra 1 Model 140 workstation using 32 MB of memory.

2 The Test Point Method

The elimination method we use dates back to a theoretical paper by Weispfenning in 1988, cf. [Wei88]. During the last five years a lot of theoretical work has been done to improve the method, cf. [LW93, Wei94, DSW96, DS97b, Wei97a]. After promising experimental implementations by Burhenne in 1990, cf. [Bur90], and by the author in 1992, the method has been efficiently reimplemented within the REDUCE package REDLOG by the author together with A. Dolzmann. REDLOG is in fact a *computer logic system* providing not only quantifier elimination but a sophisticated working environment for first-order logic over various languages and theories, cf. [DS96, DS97a].

The applicability of the method is currently restricted to formulas in which the quantified variables occur at most quadratically. Moreover quantifiers are eliminated one by one, and the elimination of one quantifier can increase the degree of other quantified variables. On the other hand there are various heuristic methods built in for decreasing the degrees during elimination. One obvious example for such methods is polynomial factorization.

For eliminating the quantifiers from an input formula

$$\varphi(u_1, \dots, u_m) \equiv \mathsf{Q}_1 x_1 \dots \mathsf{Q}_n x_n \psi(u_1, \dots, u_m, x_1, \dots, x_n), \quad \mathsf{Q}_i \in \{\exists, \forall\}$$

the elimination starts with the innermost quantifier regarding the other quantified variables within ψ as extra parameters. Universal quantifiers are handled by means of the equivalence $\forall x \psi \longleftrightarrow \neg \exists x \neg \psi$. We may thus restrict our attention to a formula of the form

$$\varphi^*(u_1, \dots, u_k) \equiv \exists x \psi^*(u_1, \dots, u_k, x),$$

where the $u_{m+1}, \dots, u_k$ are actually x_i quantified from further outside.

We fix real values $a_1, \dots, a_k$ for the parameters $u_1, \dots, u_k$. Then all polynomials occurring in ψ^* become linear or quadratic univariate polynomials in x with real coefficients. So the set

$$M = \{\, b \in \mathbb{R} \mid \psi^*(a_1, \dots, a_k, b) \,\}$$

of all real values b of x satisfying ψ^* is a finite union of closed, open, and half-open intervals on the real line. The endpoints of these intervals are among $\pm\infty$ together with the real zeros of the linear and quadratic polynomials occurring in ψ^*. Candidate terms $\alpha_1, \dots, \alpha_r$ for the zeros can be computed uniformly in $u_1, \dots, u_k$ by the solution formulas for linear and quadratic equations.

If all inequalities in ψ^* are weak, then all the intervals constituting M will, into each direction, be either unbounded or closed. In the latter case, such an interval will contain its real endpoint. Thus M is non-empty if and only if the substitution of $\pm\infty$ or of one of the candidate solutions α_j for x satisfies ψ^*. The substitution of $\pm\infty$ into a polynomial equation or inequality is evaluated in the obvious sense. The substitution of expressions in $u_1, \dots, u_k$ of the form $(a + b\sqrt{c})/d$ among the α_j can be rewritten in such a way that all denominators involving the u_i and all square-root expressions are removed from the result, cf. [Wei97a]. By disjunctively substituting all candidates into ψ^* we obtain a quantifier-free formula equivalent to $\exists x \psi^*$.

If ψ^* contains also strict inequalities, we need to add to our candidates for points in M expressions of the form $\alpha \pm \varepsilon$, where α is candidate solution for some left-hand side polynomial occurring in a strict inequality. The symbol

ε stands for a positive infinitesimal number. Again the substitution of these expressions into a polynomial equation or inequality can be rewritten in such a form that there occur neither denominators involving any of the u_i, nor any square root expressions, nor the symbol ε in the result, cf. [Wei97a]. Again this yields a quantifier-free formula equivalent to $\exists x \psi^*$. For practical applications this method, of course, has to be refined by a combination with powerful simplification techniques for quantifier-free formulas, cf. [DS97b] for details.

Recall that the well-known solution formula for quadratic equations

$$ax^2 + bx + c = 0$$

requires $a \neq 0$. In our situation a is a term in $u_1, \ldots, u_k$, so $a \neq 0$ can in general not be decided uniformly but depends on the interpretation of the u_i. Thus a quadratic polynomial $ax^2 + bx + c$ does not only deliver two square-root expressions α_1 and α_2 as candidate solutions but also $\alpha_3 = -c/b$, which in turn requires $b \neq 0$. Let t_1, t_2, and t_3 be the candidate points for M obtained from α_1, α_2, and α_3, respectively, by possibly adding or subtracting ε. With the substitution of the t_i into ψ^*, it is necessary to add the conditions on the non-vanishing of a and b. Formally, we obtain

$$(a \neq 0 \wedge \Delta \geq 0 \wedge (\psi^*[x/t_1] \vee \psi^*[x/t_2])) \vee (a = 0 \wedge b \neq 0 \wedge \psi^*[x/t_3]),$$

where Δ denotes the discriminant of the equation $ax^2 + bx + c = 0$. If, however, a is a rational constant, then the case distinction is superfluous. In particular, if a is non-zero, the second case can be dropped.

Suppose we have eliminated an existential quantifier. Then we have in general obtained a disjunction $\psi'_1 \vee \cdots \vee \psi'_r$. If the next quantifier to be eliminated is also an existential one, then we make use of the equivalence

$$\exists x_{n-1}(\psi'_1 \vee \cdots \vee \psi'_r) \longleftrightarrow \exists x_{n-1}(\psi'_1) \vee \cdots \vee \exists x_{n-1}(\psi'_r)$$

eliminating all $\exists x_{n-1}(\psi'_j)$ independently. As a consequence, no candidate solutions obtained from, say, ψ'_1 are substituted into the other ψ_j. This decreases the complexity of our procedure for single quantifier blocks from doubly exponential to singly exponential in the number of quantifiers, cf. [Wei88].

Dramatic improvements of the general procedure sketched up to now can be obtained by reducing the number of test candidates for M depending on the structure of the formula ψ^*, cf. [LW93, Wei97a]. One simple instance for such an improvement is the following natural extension of *Gauss elimination*: Suppose ψ^* is of the form

$$bx + c = 0 \wedge \psi^{**},$$

where at least one of the coefficient terms b, c is a rational non-zero constant. Then we know that under any interpretation of the u_i the equation is *non-trivial*, i.e. different from $0 = 0$. Hence the only test candidate required is $-c/b$ substituted, of course, with the condition $b \neq 0$. No additional test candidates arising from equations or inequalities in the remainder ψ^{**} of ψ^* need be considered. This idea can easily be extended to a quadratic equation instead of a linear one, taking into account again the discriminant.

An extended quantifier elimination can be straightforwardly derived from this method by not constructing a disjunction at the end. Instead, all the quantifier-free substitution results are kept separately together with the candidate terms yielding them.

Generic quantifier elimination avoids the case distinctions wrt. the vanishing of the leading coefficients sketched above assuming $a \neq 0$ for all formally quadratic left hand sides $ax^2 + bx + c$, and $b \neq 0$ for all formally linear left hand sides $bx + c$. For quadratic constraints, the condition that the discriminant be positive is still added such that all assumptions made are non-zero assumptions. For details on generic quantifier elimination, cf. [DSW96].

The complexity of the test point methods depends on the number of quantified variables and, even more, on the number of quantifier changes. In theory, the parameters play a very minor role for the complexity. They turn in fact out to be uncritical in practice.

3 A Simple Resistor Network

We consider electrical networks of the layout given in Figure 1, cf. [Wei97b]: For each of the resistors r_{jk} there is some maximal wattage p_{jk} specified. We translate this generic network into a quantifier-free formula: First, we have Ohm's law describing the effect of the resistances on the currents and voltages:

$$\omega \equiv \bigwedge_{(j,k)\in N} (u_k - u_j) = r_{jk} i_{jk}, \quad N = \{(0,1),(0,2),(1,2),(1,3),(2,3)\}$$

Next, Kirchhoff's laws state that the sum of all currents at each connecting node be zero:

$$\begin{aligned}\kappa \equiv {} & -i_0 + i_{01} + i_{02} = 0 \land \\ & -i_{01} + i_{12} + i_{13} = 0 \land -i_{02} - i_{12} + i_{23} = 0 \land -i_{13} - i_{23} + i_3 = 0.\end{aligned}$$

Finally, we write down the wattage restrictions for the resistors:

$$\rho \equiv \bigwedge_{(j,k)\in N} (u_k - u_j) i_{jk} \leq p_{jk}.$$

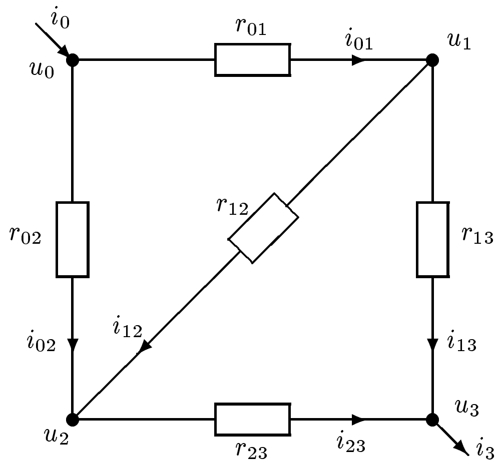

Figure 1: A simple example network consisting of resistors

This yields an *algebraic translation* $\nu = \omega \wedge \kappa \wedge \rho$ of our network. Finally, we may normalize $u_0 = 0$, and we may certainly assume that $i_3 = i_0$. Since our network is intended to be a 12 V circuit, we also set $u_3 = 12$. These settings yield an instance ν_0 of ν on which we will base our computations. Equations and inequalities are called *atomic formulas.* Our ν_0 contains 19 such atomic formulas.

3.1 Analysis

The first questions on our network are quite simple. We are going to verify concrete network designs. Consider for example the selection of 0.25 W resistors with the following resistances (in Ω):

$$r_{01} = 1000, \quad r_{02} = 100, \quad r_{12} = 300, \quad r_{13} = 47000, \quad r_{23} = 200.$$

The network obtained by substituting into ν_0 these values for the resistances and maximum wattages is called γ_1. It contains the following variables:

$$i_0, \quad i_{01}, \quad i_{02}, \quad i_{12}, \quad i_{13}, \quad i_{23}, \quad u_1, \quad u_2.$$

Let us now check if this setting will work, i.e., if none of the 0.25 W restrictions is violated. For this, we quantify all the variables existentially, and then eliminate them. The elimination result obtained after 120 ms is “false.” In other words: there are no choices for the voltages and amperages that obey both the physical laws and the wattage restrictions of the resistors.

For some reason, we may now suspect that the resistance $r_{23} = 200$ is chosen too small. We change it to $r_{23} = 400$ and eliminate again. This time, we obtain "true" after 60 ms. So in this case there are valid choices for the amperages and voltages. We would certainly like to know them but up to now the quantifier elimination procedure has only verified their existence. For obtaining the amperages and voltages we apply quantifier elimination with answer. This yields, besides the result "true," the following values (in A and V, respectively):

$$i_0 = \frac{20007}{815500}, \quad i_{01} = \frac{1467}{815500}, \quad i_{02} = \frac{927}{40775}, \quad i_{12} = \frac{129}{81550},$$
$$i_{13} = \frac{177}{815500}, \quad i_{23} = \frac{1983}{81550}, \quad u_1 = \frac{2934}{1631}, \quad u_2 = \frac{3708}{1631}.$$

Note that we always obtain exact, i.e., unrounded results, which can for convenience, of course, be displayed as rounded floats of arbitrary accuracy. The above elimination with answer takes 70 ms.

We have determined the behavior of two networks from the physical properties of the involved components. This is usually referred to as *network simulation* or *network analysis*.

3.2 Sizing

We would still prefer a smaller resistance r_{23}. For finding out the smallest possible value, we construct from ν_0 the network γ_2 in the same way as γ_1 but leaving r_{23} unspecified. Then we eliminate the quantifiers from the following formula, where r_{23} is a parameter:

$$\exists i_0 \exists i_{01} \exists i_{02} \exists i_{12} \exists i_{13} \exists i_{23} \exists u_1 \exists u_2 (\gamma_2).$$

The elimination result γ_2' obtained after 150 ms is a quantifier-free formula containing 13 atomic formulas in r_{23}. We never have to look at γ_2'. Instead, we ask for the minimal r_{23} satisfying γ_2' by eliminating the quantifiers with answer from

$$\mu \equiv \exists r_{23}(0 < r_{23} \land \gamma_2' \land \forall z(0 < z < r_{23} \longrightarrow \neg\gamma_2'[r_{23}/z])).$$

The degree of r_{23} in γ_2' and hence the degree of both r_{23} and z in μ is 4. Using its implicit factorization facility, however, our extended elimination procedure succeeds in eliminating the quantifiers from μ within 180 ms. The elimination result is "true," the answer is

$$r_{23} = \frac{31728\sqrt{347966} + 21296612}{109561} \approx 365.207871.$$

We decide that this is not tolerable: r_{23} should be at most 300 Ω. We thus have to take a resistor that can cope with a higher wattage than 0.25 W. For finding out the minimum suitable wattage, we produce γ_3 from ν_0 just like γ_1 but setting $r_{23} = 300$ and leaving p_{23} unspecified. In analogy to the previous paragraph, we eliminate without answer the quantifiers from the formula

$$\exists i_0 \exists i_{01} \exists i_{02} \exists i_{12} \exists i_{13} \exists i_{23} \exists u_1 \exists u_2 (\gamma_3)$$

yielding $\gamma_3' \equiv 42250000 p_{23} - 11796867 \geq 0$ after 60 ms. In contrast to γ_2' above, γ_3' already states the answer to our question in a comprehensible form:

$$p_{23} \geq \frac{11796867}{42250000} \approx 0.279216.$$

We have determined features of network elements (resistances, maximum wattages) in such a way that certain restrictions on the amperages and voltages are met. This is referred to as *network design* or *network sizing*.

3.3 Error Diagnosis

Let us return to the $r_{23} = 400\ \Omega/0.25$ W design of Section 3.1. Assume now that r_{12} is not simply a resistor but a 300 Ω ammeter. Assume furthermore that the line $\overline{u_1 u_2}$ from u_1 to u_2 is the only part of the network that is easily accessible. We have computed that on normal conditions

$$i_{12} = \frac{129}{81550} \approx 0.001582.$$

If this amperage changes, say to -0.000181, this means that there is a malfunction somewhere in the network. Since all components are hard to access, we would like to infer where the problem is located from the measured amperage.

For each resistor r_{jk}, we construct a diagnosis formula δ_{jk} from ν_0 by fixing all resistances and maximum wattages except for r_{jk} and p_{jk} to their normal values. In addition, we set i_{12} to the measured value -0.000181. Then we existentially quantify the 9 remaining variables

$$i_0, \quad i_{01}, \quad i_{02}, \quad i_{13}, \quad i_{23}, \quad u_1, \quad u_2, \quad r_{jk}, \quad p_{jk}.$$

We eliminate with answer the quantifiers from each of the δ_{jk} being interested only in the answer for r_{jk}. The results are collected in Table 1, where the answers for the r_{jk} are given as rounded floats for convenience. According to these data, a broken line $\overline{u_2 u_3}$, a broken $\overline{u_0 u_1}$, or a short-circuited $\overline{u_1 u_3}$ are

δ_{jk}	result	r_{jk} (answer)	r_{jk} (original)	time
δ_{01}	true	$1.132888 \cdot 10^5$	1000	60 ms
δ_{02}	false	—	100	60 ms
δ_{13}	true	3596.985410	47000	60 ms
δ_{23}	true	$3.230974 \cdot 10^6$	400	70 ms

Table 1: Diagnosis results

possible reasons for the malfunction of the network. Note that any missing clearness in this concern is a physical fact but not a drawback of our method.

It is important to interpret the elimination result "false" for δ_{02} in the right way: This means that the measured amperage i_{12} cannot be caused by a malfunction of r_{02} *alone*. Still, a fused or short-circuited r_{23} can be the original reason. The network might be constructed in such a way that some malfunction of r_{23} immediately crashes other resistors. Such situations with more than one faulty resistor are not captured by our diagnosis formulas.

Let us analyze which other resistors would be affected by a failure of r_{02}. We are interested in the actual powers $(u_k - u_j)i_{jk}$—not to be confused with the p_{jk}—at the resistors r_{jk} in dependence on the value of r_{02}. For making these quantities accessible we add to ν_0 a conjunction

$$\bigwedge_{(j,k)\in N\setminus\{(0,2)\}} z_{jk} = (u_k - u_j)i_{jk}, \quad \text{where} \quad u_0 = 0 \quad \text{and} \quad u_3 = 12.$$

Since we are interested in the moment immediately after the crash of r_{02}, we existentially quantify the p_{jk}, i.e., we do not put any restrictions on them. We also existentially quantify all other variables except for r_{02} and the z_{jk}. This yields the formula Δ_{02} from which we eliminate the quantifiers obtaining an equivalent quantifier-free formula Δ'_{02} with 166 atomic formulas (3130 ms). The application of a simplifier based on Gröbner Basis methods built into REDLOG, cf. [DS97b], reduces the size of Δ'_{02} to 62 atomic formulas (9100 ms).

Whenever we substitute a value for r_{02} in Δ'_{02} we obtain a formula only in the z_{jk}. This formula states the values of the various powers. We are particularly interested in a short-circuited and in a fused r_{02} corresponding to extremely small and large resistances, respectively:

$$\beta_{02} = \Delta'_{02}[r_{02}/10^{10}], \quad \sigma_{02} = \Delta'_{02}[r_{02}/10^{-10}].$$

Unfortunately, β_{02} and σ_{02} do not contain the values for the z_{jk} in an explicit form. For instance, after some straightforward automatic simplification, we have

$$\beta_{02} = 555179625062500z_{01}^2 - 260245051250000z_{01}z_{23} - 94888055344500z_{01} + 30498006250000z_{23}^2 - 22239781245000z_{23} + 4054427900721 = 0 \wedge 23562250z_{01} - 5522500z_{23} - 2013561 \leq 0 \wedge 5551796250625z_{12}^2 - 814311360000z_{12}z_{23} - 281064655350z_{12} + 29859840000z_{23}^2 - 20612620800z_{23} + 3557287449 = 0 \wedge 2356225z_{12} - 172800z_{23} - 59643 \leq 0 \wedge 555179625062500z_{13}^2 - 5537128750000z_{13}z_{23} - 179402971500z_{13} + 13806250000z_{23}^2 - 894645000z_{23} + 14493249 = 0 \wedge 23562250z_{13} - 117500z_{23} - 3807 \geq 0 \wedge 253764077965125235622 5z_{23} - 49702512986100848241 = 0.$$

We know, however, that these z_{jk} are uniquely determined. On the other hand, we know that quantifier elimination with answer can yield *one* solution. We thus eliminate with answer the quantifiers from

$$\exists z_{01}\exists z_{12}\exists z_{23}\exists z_{23}(\sigma_{02}) \quad \text{and} \quad \exists z_{01}\exists z_{12}\exists z_{23}\exists z_{23}(\beta_{02})$$

obtaining the elimination result "true" and answers B'_{02} and S'_{02} after 3350 ms and 5270 ms respectively. The answer B'_{02} for the burn-through case reads as follows:

$$\begin{aligned}
z_{01} &= \tfrac{127985062564395000081}{2537640779651252356225 0} &\approx&\ 0.050435\\
z_{12} &= \tfrac{37276874936550000027}{2537640779651252356225} &\approx&\ 0.014690\\
z_{13} &= \tfrac{12954376924650071487}{25376407796512523562250} &\approx&\ 0.000510\\
z_{23} &= \tfrac{49702512986100848241}{2537640779651252356225} &\approx&\ 0.019586.
\end{aligned}$$

All these wattages are less than 0.25 and hence not critical. We may conclude that our measured $i_{12} = -0.000181$ cannot be caused by a fused r_{02}.

For the short-circuit case σ_{02} we have the following wattages S'_{02}:

$$\begin{aligned}
z_{01} &= \tfrac{1296000000103032000002047761}{376996000000247442000000040602250} &\approx&\ 0.000003\\
z_{12} &= \tfrac{43199999998984800000059643}{37699600000024744200000004060225} &\approx&\ 0.000011\\
z_{13} &= \tfrac{114379200000030794400000000020727}{376996000000247442000000040602250} &\approx&\ 0.003034\\
z_{23} &= \tfrac{13571856000002077776000000079524}{37699600000024744200000004060225} &\approx&\ 0.36.
\end{aligned}$$

This means that r_{23} will probably be destroyed. A next step would be to analyze if a combined malfunction of both r_{02} and r_{23} can cause the amperage $i_{12} = -0.000181$ in question.

In practice, network diagnosis is supported by the knowledge of probabilities for certain malfunctions.

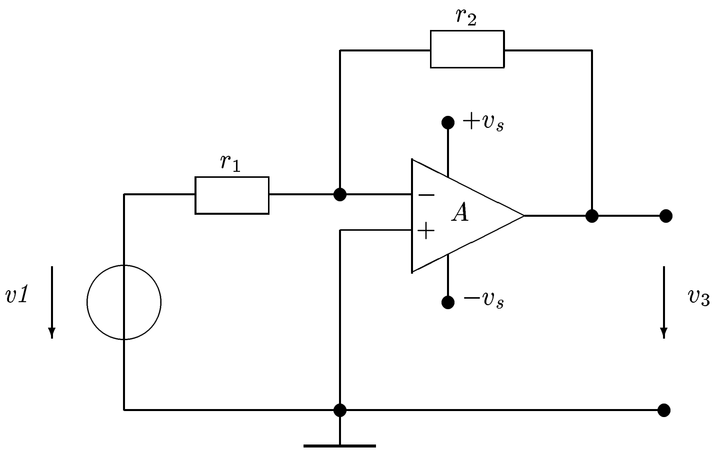

Figure 2: An inverting operation amplifier circuit

4 An Inverting Operation Amplifier

For the operation amplifier circuit shown in Figure 2, we want to determine the output voltage v_3 as a function of the input voltage *v1*. This is an illustrative example taken from [Hen95]. Its algebraic formulation ω is the conjunction over the following equations:

$$\begin{aligned}
v_1 &= \mathit{v1} \\
v_2 &= -v_{\mathrm{pm_op1}} \\
v_3 &= v_{\mathrm{og_op1}} \\
v_1 + i_{\mathrm{v0}} r_1 &= v_2 \\
v_2 r_1 + v_2 r_2 &= v_3 r_1 + v_1 r_2 + i_{\mathrm{pm_op1}} r_1 r_2 \\
v_3 + i_{\mathrm{og_op1}} r_2 &= v_2 \\
v_{\mathrm{og_op1}} &= v_{\mathrm{pm_op1}} x_{\mathrm{op1}}^2 \\
v_s^2 x_{\mathrm{op1}}^2 + A v_{\mathrm{og_op1}}^2 &= A v_s^2 \\
i_{\mathrm{pm_op1}} &= 0.
\end{aligned}$$

Using the symbolic system Analog Insydes, cf. [SH95], these equations are obtained automatically from a *net list*, i.e., a nested list description of Figure 2. The variables to be existentially eliminated are:

$$V := \{i_{\mathrm{og\_op1}}, v_2, i_{\mathrm{pm\_op1}}, v_1, i_{\mathrm{v0}}, v_{\mathrm{pm\_op1}}, v_{\mathrm{og\_op1}}, x_{\mathrm{op1}}\}.$$

The amplification factor A, the supply voltage v_s, and the resistances r_1 and r_2 are parameters.

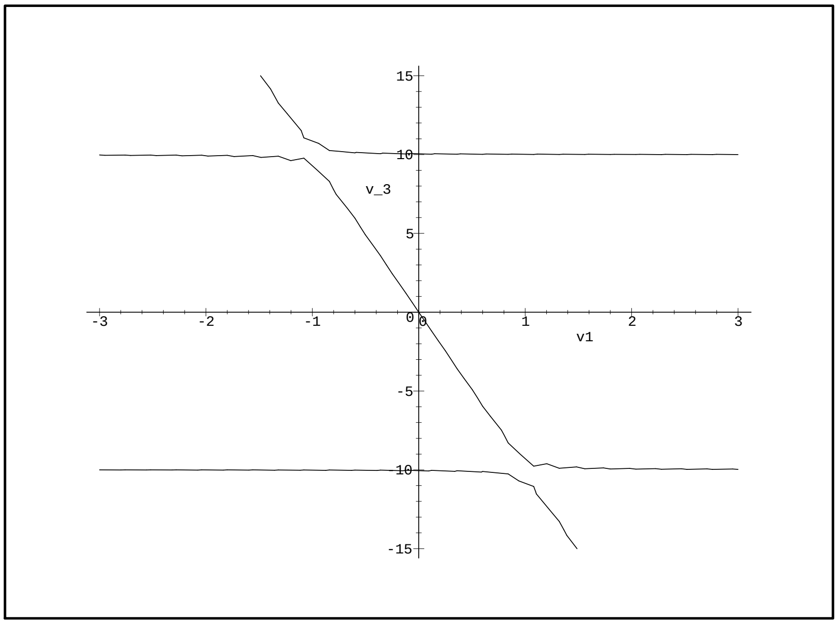

Figure 3: The characteristic curve of the inverting operation amplifier including parasitic solutions for $|v_3| > 10$.

A symbolic method that has been tried for eliminating the variables from these equations is the computation of an *elimination ideal basis*, cf. [Hen95]. For obtaining such a basis we proceed as follows: In the equations in ω we subtract all right hand sides to the corresponding left hand sides. Then we choose a lexicographic term order such that the variables V to be eliminated are greater than $v1$, v_3, and the parameters, and compute a Gröbner basis G, cf. [Buc65], of the left hand side polynomials (150 ms in REDUCE). The polynomials from G which do not contain any variables from V now form a Gröbner basis of the elimination ideal, cf. [BWK93] for details. There is in fact only one such polynomial, and the corresponding equation should be an implicit description of our desired function:

$$-Ar_1v_3^3 + Ar_1v_3v_s^2 - Ar_2v1\,v_3^2 + Ar_2v1\,v_s^2 + r_1v_3v_s^2 + r_2v_3v_s^2 = 0.$$

For concrete parameter values $A = 1000$, $v_s = 10$, $r_1 = 1000$, and $r_2 = 10000$ this amounts to

$$100v1\,v_3^2 - 10000v1 + 10v_3^3 - 1011v_3 = 0.$$

Figure 3 displays the set of *real* solutions of this equation. Only the part of the curve for $-10 \leq v_3 \leq 10$, however, is the desired solution for our problem. The other parts are *parasitic solutions* caused by the fact that the elimination ideal computation has taken place not over the reals but over an algebraically closed extension field such as $\mathbb{C}$. First fixing the parameter values and then computing the elimination ideal leads to the same result.

In general, it is a problem to distinguish between proper solutions and parasitic solutions. Since our quantifier elimination is a *real* method, it completely avoids them: The result with fixed parameter values obtained after 60 ms contains the correct constraint on the range of v_3 to exclude the parasitic solutions of the equation:

$$100\mathit{v1}\,v_3^2 - 10000\mathit{v1} + 10v_3^3 - 1011v_3 = 0 \wedge v_3^2 - 100 \leq 0.$$

Notice that the corresponding input formula does not contain any ordering constraint but is purely equational.

Next, we solve the original parametric problem with REDLOG: For $\exists V(\omega)$, we obtain 604 atomic formulas after 1110 ms. A result of this size might be useful for further automatic processing but it does not immediately contribute to understanding the network.

We thus apply generic quantifier elimination. The result then obtained for $\exists V(\omega)$ after 120 ms is:

$$\begin{aligned}Ar_1v_3^3 - Ar_1v_3v_s^2 + Ar_2\mathit{v1}\,v_3^2 - Ar_2\mathit{v1}\,v_s^2 - r_1v_3v_s^2 - r_2v_3v_s^2 = 0 \wedge \\ Av_3^2 - Av_s^2 \leq 0 \wedge v_s \neq 0\end{aligned}$$

valid on the following assumptions:

$$A \neq 0, \quad r_1 \neq 0, \quad r_2 \neq 0, \quad v_3 + v_s \neq 0, \quad v_3 - v_s \neq 0, \quad v_3 \neq 0.$$

None of these conditions is a problem: A is the amplification factor, r_1 and r_2 are resistors, the absolute value of the output voltage v_3 can certainly never get equal to the supply power v_s.

5 Sizing a BJT Amplifier

Hennig and Sommer have studied the applicability of symbolic methods to the sizing of a BJT amplifier, cf. [HS95]. After several non-trivial steps of symbolic analysis, they obtain a description of the circuit by the set E_1 of 40 operating point equations given in Figure 4 together with the set E_2 of 5 AC conditions given in Figure 5.

The entire system $E_1 \cup E_2$ has to be solved wrt. the *main* variables

$$M = \{r_1, \ldots, r_8, c_3\}$$

$$
\begin{aligned}
-v_{\mathrm{vin}} + v_{\mathrm{r1}} + v_{\mathrm{c1}} + v_{\mathrm{be\_q1}} &= 0 \\
v_{\mathrm{r5}} - v_{\mathrm{r4}} - v_{\mathrm{r2}} - v_{\mathrm{ce\_q1}} &= 0 \\
-v_{\mathrm{vin}} + v_{\mathrm{r8}} + v_{\mathrm{r7}} + v_{\mathrm{c1}} &= 0 \\
-v_{\mathrm{r7}} + v_{\mathrm{r6}} + v_{\mathrm{r2}} + v_{\mathrm{ce\_q2}} + v_{\mathrm{be\_q1}} &= 0 \\
-v_{\mathrm{r7}} + v_{\mathrm{r6}} - v_{\mathrm{ce\_q1}} + v_{\mathrm{be\_q2}} + v_{\mathrm{be\_q1}} &= 0 \\
-v_{\mathrm{vin}} + v_{\mathrm{r7}} - v_{\mathrm{r6}} + v_{\mathrm{c2}} + v_{\mathrm{c1}} &= 0 \\
v_{\mathrm{r3}} - v_{\mathrm{r2}} + v_{\mathrm{c3}} &= 0 \\
v_{\mathrm{v\_cc}} - v_{\mathrm{vin}} - v_{\mathrm{r4}} - v_{\mathrm{ce\_q1}} + v_{\mathrm{c1}} + v_{\mathrm{be\_q1}} &= 0 \\
-v_{\mathrm{vin}} + v_{\mathrm{r2}} + v_{\mathrm{izo}} + v_{\mathrm{c1}} + v_{\mathrm{be\_q1}} &= 0 \\
i_{\mathrm{r7}} - i_{\mathrm{c1}} + i_{\mathrm{be\_q1}} &= 0 \\
-i_{\mathrm{r4}} + i_{\mathrm{ce\_q1}} + i_{\mathrm{be\_q2}} &= 0 \\
i_{\mathrm{r3}} + i_{\mathrm{r2}} + i_{\mathrm{r1}} - i_{\mathrm{ce\_q1}} - i_{\mathrm{be\_q1}} &= 0 \\
i_{\mathrm{c3}} - i_{\mathrm{r3}} &= 0 \\
i_{\mathrm{r6}} - i_{\mathrm{ce\_q2}} + i_{\mathrm{c2}} - i_{\mathrm{be\_q2}} &= 0 \\
i_{\mathrm{r8}} - i_{\mathrm{r7}} - i_{\mathrm{r6}} &= 0 \\
i_{\mathrm{vin}} + i_{\mathrm{c1}} &= 0 \\
-i_{\mathrm{r5}} - i_{\mathrm{r2}} + i_{\mathrm{izo}} + i_{\mathrm{ce\_q2}} - i_{\mathrm{c3}} &= 0 \\
i_{\mathrm{v\_cc}} + i_{\mathrm{r5}} + i_{\mathrm{r4}} &= 0 \\
v_{\mathrm{vin}} &= v_1 \\
v_{\mathrm{v\_cc}} &= v_{\mathrm{cc}} \\
i_{\mathrm{izo}} &= i_0 \\
-i_{\mathrm{c1}},\ \ldots,\ -i_{\mathrm{c3}} &= 0 \\
i_{\mathrm{r1}} r_1 - v_{\mathrm{r1}},\ \ldots,\ i_{\mathrm{r8}} r_8 - v_{\mathrm{r8}} &= 0 \\
v_{\mathrm{ce\_q1}} &= 2.72 \\
v_{\mathrm{be\_q1}} &= 0.607 \\
v_{\mathrm{ce\_q2}} &= 6.42 \\
v_{\mathrm{be\_q2}} &= 0.698 \\
i_{\mathrm{ce\_q1}} &= 1.11 \cdot 10^{-4} \\
i_{\mathrm{be\_q1}} &= 5.75001 \cdot 10^{-7} \\
i_{\mathrm{ce\_q2}} &= 0.00401 \\
i_{\mathrm{be\_q2}} &= 1.26 \cdot 10^{-5}
\end{aligned}
$$

Figure 4: Operating point equations E_1 for the BJT amplifier

$$
\begin{aligned}
a_{\text{high}} &= (r_2r_3 + r_1r_2)/(r_1r_3 + r_1r_2) \\
a_{\text{low}} &= r_2/r_1 \\
p &= -1/(c_3r_3 + c_3r_2) \\
z_{\text{in}} &= r_7 \\
z_{\text{out}} &= (10976r_2r_7 + 2590336r_1r_2)/(891433631r_1).
\end{aligned}
$$

Figure 5: AC conditions E_2 for the BJT amplifier

in terms of the *parameter* variables $P = \{v_{\text{cc}}, a_{\text{high}}, a_{\text{low}}, p, z_{\text{in}}, z_{\text{out}}\}$ specifying the circuit. In addition to M and P, there is a set X of 38 *internal* variables to be completely eliminated. Hennig and Sommer point out that it is a crucial feature of any solver for such sizing problems to allow to distinguish between these three types of variables.

We start by eliminating the internal variables from the system applying generic quantifier elimination with all internal variables existentially quantified: $\exists X(\bigwedge E_1 \wedge \bigwedge E_2)$. After 1730 ms we obtain the conjunction over the following equations:

$$
\begin{aligned}
a_{\text{high}}r_1r_2 + a_{\text{high}}r_1r_3 - r_1r_2 - r_2r_3 &= 0 \\
a_{\text{low}}r_1 - r_2 &= 0 \\
c_3pr_2 + c_3pr_3 + 1 &= 0 \\
111575001r_1r_2 + 4022600000r_1r_6 + 575001r_1r_7 + 7027000000000r_1 & \\
+575001r_2r_7 - 4022024999r_2r_8 + 607000000000r_2 &= 0 \\
111575001r_1r_2 + 8442000000000r_1 - 4022600000r_2r_6 & \\
-4022024999r_2r_8 + 2022000000000r_2 &= 0 \\
2590336r_1r_2 - 891433631r_1z_{\text{out}} + 10976r_2r_7 &= 0 \\
123600000r_4 + 4022600000r_6 + 4022024999r_8 - 1000000000000v_{\text{cc}} & \\
+698000000000 &= 0 \\
r_7 - z_{\text{in}} &= 0.
\end{aligned}
$$

This is equivalent on the assumptions $r_1 + r_2 \neq 0$ and $r_5 \neq 0$ also obtained by the generic quantifier elimination. We conjunctively add these conditions to the system, and apply generic quantifier elimination with answer existentially quantifying the variables $r_1, \dots, r_8, c_3$. After 180 ms we obtain further assumptions:

$$
a_{\text{high}} - a_{\text{low}} \neq 0, \quad a_{\text{low}} + 1 \neq 0, \quad a_{\text{low}} \neq 0,
$$

a quantifier-free equivalent, i.e., a necessary and sufficient condition in the parameters for the solvability of the original system on the assumptions made so far:

$$10976 a_{\text{low}} z_{\text{in}} - 891433631 z_{\text{out}} \neq 0 \wedge a_{\text{low}} - 1 \neq 0 \wedge p \neq 0,$$

and sample solutions for the eliminated main variables $r_1, \ldots, r_8, c_3$ in terms of the parameters:

$$\begin{aligned}
c_3 &= (-2590336 a_{\text{high}} + 2590336 a_{\text{low}})/(10976 a_{\text{low}}^2 p z_{\text{in}} \\
&\quad - 10976 a_{\text{low}} p z_{\text{in}} - 891433631 a_{\text{low}} p z_{\text{out}} + 891433631 p z_{\text{out}}) \\
r_1 &= (-10976 a_{\text{low}} z_{\text{in}} + 891433631 z_{\text{out}})/(2590336 a_{\text{low}}) \\
r_2 &= (-10976 a_{\text{low}} z_{\text{in}} + 891433631 z_{\text{out}})/2590336 \\
r_3 &= (10976 a_{\text{high}} a_{\text{low}} z_{\text{in}} - 891433631 a_{\text{high}} z_{\text{out}} - 10976 a_{\text{low}} z_{\text{in}} \\
&\quad + 891433631 z_{\text{out}})/(2590336 a_{\text{high}} - 2590336 a_{\text{low}}) \\
r_4 &= (2590336000000000000 a_{\text{low}} v_{\text{cc}} + 1224647210976 a_{\text{low}} z_{\text{in}} \\
&\quad - 7045713920000000000 a_{\text{low}} - 99461708270258631 z_{\text{out}} \\
&\quad - 2186761651200000000)/(320165529600000 a_{\text{low}}) \\
r_5 &= \infty \\
r_6 &= (-575001 z_{\text{in}} + 1415000000000)/4022600000 \\
r_7 &= z_{\text{in}} \\
r_8 &= (264798579360 a_{\text{low}} z_{\text{in}} + 1572333952000000000 a_{\text{low}} \\
&\quad + 99461708270258631 z_{\text{out}} + 2186761651200000000)/ \\
&\quad (10418396147809664 a_{\text{low}}).
\end{aligned}$$

This final solution is essentially the same as that obtained by Hennig and Sommer using a heuristic-based special purpose solver.

6 An Outlook on Hydraulic Networks

The methods sketched above can also be applied to hydraulic networks. There Ohm's law $U = RI$ has to be replaced by a suitable instance of Bernoulli's law, such as

$$P = RF^2,$$

where the *pressure* P corresponds to the voltage U, and the *flow* F corresponds to the amperage I. The flow occurs quadratically leading to higher degrees in the input formulas. Moreover, the direction of such a flow has to be added explicitly. Anyway, first elimination experiments with such formulas have been very promising, cf. [Wei97b].

7 Conclusions

We have given an outline of a real quantifier elimination method using test point ideas. By means of a rather trivial example, we have illustrated the scope of such methods within the area of physical network analysis, sizing, and error detection. It has turned out that the use of a real elimination method avoids the problem of parasitic solutions. In combination with other symbolic methods, our approach is even applicable to an example recently discussed in the community of electrical engineers. Finally, we have indicated that our approach is by no means restricted to electrical networks.

References

[ACM84] Dennis S. Arnon, George E. Collins, and Scott McCallum. Cylindrical algebraic decomposition I: The basic algorithm. *SIAM Journal on Computing*, 13(4):865–877, November 1984.

[Arn81] Dennis S. Arnon. *Algorithms for the geometry of semi-algebraic sets.* Ph.D. dissertation, Computer Sciences Department, University of Wisconsin, Madison, 1981. Technical Report No. 436.

[Buc65] Bruno Buchberger. *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal.* Doctoral dissertation, Mathematical Institute, University of Innsbruck, Innsbruck, Austria, 1965.

[Bur90] Klaus-Dieter Burhenne. Implementierung eines Algorithmus zur Quantorenelimination für lineare reelle Probleme. Diploma thesis, Universität Passau, D-94030 Passau, Germany, December 1990.

[BWK93] Thomas Becker, Volker Weispfenning, and Heinz Kredel. *Gröbner Bases, a Computational Approach to Commutative Algebra*, volume 141 of *Graduate Texts in Mathematics.* Springer, New York, 1993.

[CH91] George E. Collins and Hoon Hong. Partial cylindrical algebraic decomposition for quantifier elimination. *Journal of Symbolic Computation*, 12(3):299–328, September 1991.

[Col75] George E. Collins. Quantifier elimination for the elementary theory of real closed fields by cylindrical algebraic decomposition. In H. Brakhage, editor, *Automata Theory and Formal Languages.*

2nd GI Conference, volume 33 of *Lecture Notes in Computer Science*, pages 134–183, Berlin, Heidelberg, New York, May 1975. Gesellschaft für Informatik, Springer-Verlag.

[Col90] Alain Colmerauer. Prolog III. *Communications of the ACM*, 33(7):70–90, July 1990.

[DS96] Andreas Dolzmann and Thomas Sturm. Redlog user manual. Technical Report MIP-9616, FMI, Universität Passau, D-94030 Passau, Germany, October 1996. Edition 1.0 for Version 1.0.

[DS97a] Andreas Dolzmann and Thomas Sturm. Redlog: Computer algebra meets computer logic. *ACM SIGSAM Bulletin*, 31(2):2–9, June 1997.

[DS97b] Andreas Dolzmann and Thomas Sturm. Simplification of quantifier-free formulae over ordered fields. *Journal of Symbolic Computation*, 24(2):209–231, August 1997.

[DSW96] Andreas Dolzmann, Thomas Sturm, and Volker Weispfenning. A new approach for automatic theorem proving in real geometry. Technical Report MIP-9611, FMI, Universität Passau, D-94030 Passau, Germany, May 1996. To appear in the Journal of Automated Reasoning.

[DSW97] Andreas Dolzmann, Thomas Sturm, and Volker Weispfenning. Real quantifier elimination in practice. Technical Report MIP-9720, FMI, Universität Passau, D-94030 Passau, Germany, December 1997.

[DYA97] Peter Dorato, Wei Yang, and Chaouki Abdallah. Robust multi-objective feedback design by quantifier elimination. *Journal of Symbolic Computation*, 24(2):153–159, August 1997. Special issue on applications of quantifier elimination.

[Hen95] Eckhard Hennig. Rechnergestützte Dimensionierung analoger Schaltungen auf der Basis symbolischer Analyseverfahren. Technical report, Zentrum für Mikroelektronik, Universität Kaiserslautern, December 1995. Annual report on a research project.

[HLS97] Hoon Hong, Richard Liska, and Stanly Steinberg. Testing stability by quantifier elimination. *Journal of Symbolic Computation*, 24(2):161–187, August 1997. Special issue on applications of quantifier elimination.

[HS95] Eckhard Hennig and Ralf Sommer. Application of computer algebra methods to analog circuit sizing. In *Proceedings of the 12th European Conference on Circuit Theory and Design (ECCTD), Istanbul, Turkey*. EURASIP, 1995.

[Jir97] Mats Jirtsrand. Nonlinear control system design by quantifier elimination. *Journal of Symbolic Computation*, 24(2):137–152, August 1997. Special issue on applications of quantifier elimination.

[LW93] Rüdiger Loos and Volker Weispfenning. Applying linear quantifier elimination. *The Computer Journal*, 36(5):450–462, 1993. Special issue on computational quantifier elimination.

[SH95] Ralf Sommer and Eckhard Hennig. *Analog Insydes V1.1 Benutzerhandbuch*. Institut für Techno- und Wirtschaftsmathematik e. V., Erwin-Schrödinger-Straße, 67663 Kaiserslautern, Germany, 1995.

[SW97a] Thomas Sturm and Volker Weispfenning. Computational geometry problems in Redlog. Technical Report MIP-9708, FMI, Universität Passau, D-94030 Passau, Germany, April 1997.

[SW97b] Thomas Sturm and Volker Weispfenning. Rounding and blending of solids by a real elimination method. In Achim Sydow, editor, *Proceedings of the 15th IMACS World Congress on Scientific Computation, Modelling, and Applied Mathematics (IMACS 97)*, volume 2, pages 727–732, Berlin, August 1997. IMACS, Wissenschaft & Technik Verlag.

[Tar48] Alfred Tarski. A decision method for elementary algebra and geometry. Technical report, RAND, Santa Monica, CA, 1948.

[Wei88] Volker Weispfenning. The complexity of linear problems in fields. *Journal of Symbolic Computation*, 5(1–2):3–27, February–April 1988.

[Wei94] Volker Weispfenning. Quantifier elimination for real algebra—the cubic case. In *Proceedings of the International Symposium on Symbolic and Algebraic Computation in Oxford*, pages 258–263, New York, July 1994. ACM Press.

[Wei97a] Volker Weispfenning. Quantifier elimination for real algebra—the quadratic case and beyond. *Applicable Algebra in Engineering Communication and Computing*, 8(2):85–101, February 1997.

[Wei97b] Volker Weispfenning. Simulation and optimization by quantifier elimination. *Journal of Symbolic Computation*, 24(2):189–208, August 1997. Special issue on applications of quantifier elimination.