A proof of Higman's Lemma by open induction

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Abstract

We use Raoult's principle of *open induction* to give a new proof of Higman's Lemma. In contrast to previous proofs, it directly uses the property that every infinite sequence has an infinite ordered subsequence. For a slightly more complex order we exchange a fairly straight inductive argument.

List of Keywords: Wellfounded order, Open induction, Complete partial order, Higman's Lemma, Kruskal's Tree Theorem.

1 Introduction

Program verification and automated theorem proving depend on the availability of powerful wellfounded orders. The crucial part are their proofs of wellfoundedness, often obtained as a corollary from Kruskal's Tree Theorem.

Kruskal's Tree Theorem and the related Higman's Lemma are strong properties that cannot be proved in relatively powerful arithmetic theories (see [Gal91] for an overview). Their proofs are classically performed by "minimal counterexample": One deduces the existence of a minimal counterexample, in a nonempty set of counterexamples, by Zorn's Lemma; and then one constructs a still smaller counterexample.

The unsatisfactory indirection of the minimal counterexample argument can in many proofs be replaced by a form of induction, for instance Noetherian induction. Noetherian induction requires that the used order admits no infinitely descending sequences. However Higman's Lemma, which is a property of infinite sequences, does not suggest such an order. The particularly strong *open induction* invented by Raoult [Rao88], in contrast, requires only that the order is *downward complete* and that the property is *open* w.r.t. the lower topology.

Higman's Lemma and Kruskal's Tree Theorem say that under certain conditions the embedding order on strings and on trees, respectively, is a well-quasiorder. There are three characterizations of well-quasiorders:

1. every infinite sequence contains an increasing pair,
2. every infinite sequence contains an infinite ordered subsequence,
3. every nonempty subset contains finitely many minimal elements.

All three variants are suited for inductive proofs as they are universal sentences. The classical proofs of Higman's Lemma, e.g. that of Nash-Williams [**?**], use the first variant. Raoult has suggested to use open induction to prove Kruskal's Tree Theorem. He propagates the lexicographic order on infinite sequences as a suitable downward complete order. This is certainly possible, however with a complex proof of inductiveness, the proper inductive argument. Reason is, the existence of an ordered pair does not point to an application of the inductive hypothesis. The same can be said about the third variant.

The second variant in contrast yields an infinite subsequence from which an application of the inductive hypothesis may be derived. As we will demonstrate, this actually leads to a very concise and easy inductiveness proof (Lemma 15).

On the other hand, the proof cannot be carried out with the lexicographic order. The problem is that to show its openness, one has to provide a copy of the whole inductive proof. We therefore chose an order stronger than the lexicographic order, which additionally requires that elements of the smaller sequence, concatenated with ascending strings form a subsequence of the larger sequence. For this order it is easy to show openness of the (Lemma 14). To show downward completeness, we use a diagonalization (Lemma 13) across the grid of subsequences.

The aim of our work is twofold: To show how a minimal counterexample proof is encoded in an inductive proof, and to provide an alternative access to theorems over well-quasiorders. The first is a step towards constructive proofs. The second should add one more bit of intuition to the difficult matter of well-quasiorders.

Constructive proofs of Higman's Lemma have been given by Richman and Stolzenberg [RS90], Murthy and Russell [MR90], and by Coquand [Coq94a].

2 Open induction

A property $P \subseteq D$ of elements of a given set D is called *inductive* if

$$\forall x \in D. \, (\forall y \sqsubset x. \, y \in P) \Longrightarrow x \in P$$

holds. Inductiveness means that the property P can be proven for x assuming the inductive hypothesis $y \in P$ for all $y \sqsubset x$. An order $\sqsupset$ on a set D is called *wellfounded* if there is no infinite strictly descending sequence

$$t_1 \sqsupset t_2 \sqsupset \cdots \ .$$

With that, the widely known *principle of Noetherian induction* (see e.g. [Coh81]) reads as follows.

Theorem 1 (Principle of Noetherian Induction). *If $\sqsupset$ is a wellfounded order on a set D, and $P \subseteq D$ is inductive, then $P = D$.*

In 1988, Raoult introduced the following induction principle, which is still stronger.

Definition 2. An order $\sqsupset$ on a set D is called *downward complete*, if every *nonempty* $\sqsupset$-chain has a greatest lower bound, $\sqcap X$, in D. A property $P \subseteq D$ is called *open*, if for every $\sqsupset$-chain $X \subset D$: If $\sqcap X \in P$ then $\exists x \in X. \, x \in P$.

Theorem 3 ([Rao88]). *Let $\sqsupset$ be a downward complete partial order on a set D. If $P \subseteq D$ is an open and inductive property then $P = D$.*

Openness can easily be established with the help of the following standard topological properties.

Definition 4. A function $f : D \to D'$ where $\sqsupset$ and $\sqsupset'$ are downward complete orders on D, D', respectively, is called *downward continuous*, if

1. f is monotonic: $x \sqsupseteq y \Longrightarrow f(x) \sqsupseteq' f(y)$
2. for each *nonempty* $\sqsupset$-chain $X \subset D$: $\sqcap'\{f(x) \mid x \in X\} = f(\sqcap X)$.

Proposition 5. *Let $\sqsupset$ and $\sqsupset'$ be downward complete partial orders on D and D', respectively. Then*

1. *arbitrary union preserves open sets,*

2. *finite intersection preserves open sets,*

3. *if $f : D \to D'$ is a downward continuous function then $f^{-1}(P)$ is open if P is open in D',*

4. *if $\sqsupset$ is wellfounded then every subset P of D is open.*

Moreover by (4) Noetherian induction is a special case. Open induction appears appropriate for proofs where Noetherian induction does not apply because the underlying order is not wellfounded.

Coquand has used open induction to give proofs to Higman's Lemma [Coq94a] and van der Waerden's Theorem [Coq94b]. Below we give another proof of Higman's Lemma.

3 Basic Notions

Let us briefly review some standard results. Rather than speaking about ascending chains, Noetherian, least upper bounds, etc., we will speak about descending chains, wellfoundedness, greatest lower bounds, etc.; we think it better reflects the way induction is used.

A $\sqsupset$-chain Y is said to *minorize* a chain X if for each $x \in X$ there is $y \in Y$ such that $y \sqsubseteq x$. Two chains $X, Y \subseteq D$ are called *cofinal* if X minorizes Y and Y minorizes X. By a straightforward argument, the lower bounds of cofinal X and Y coincide. In particular their greatest lower bounds coincide, provided one of X, Y has a greatest lower bound.

Let $>$ be an order on a set C. The set C^k of k-tuples with components from C, is ordered lexicographically by $>^k_{lex} \subseteq (C^k)^2$, defined by

$$s = (s_1, \ldots, s_k) >^k_{lex} (t_1, \ldots, t_k) = t \quad \Longleftrightarrow_{\text{def}} \quad \exists i.\ s >^k_i t \qquad \text{where}$$
$$s = (s_1, \ldots, s_k) >^k_i (t_1, \ldots, t_k) = t \quad \Longleftrightarrow_{\text{def}} \quad i \le k \wedge \forall j < i.\ s_j = t_j \wedge s_i > t_i$$

It is wellknown that $>^k_{lex}$ is wellfounded if $>$ is wellfounded.

Proposition 6. *Let $>$ be a wellfounded order on a set C. Then C^ω, ordered by $>_{lex}$, is downward complete.*

Proof. Let $X \subseteq D = C^\omega$ be a $>_{lex}$-chain. We define simultaneously objects $d_j \in C$ and subchains X_j of X by induction on $j \in \mathbb{N}$ as follows.

1. $X_j =_{\text{def}} \{x \in X \mid x = (x_1, x_2, \ldots) \wedge x_1 = d_1 \wedge \cdots \wedge x_j = d_j\}$,

2. Let $X'_j =_{\text{def}} \{x_j \mid (x_1, x_2, \dots) \in X_{j-1}\}$. With X_{j-1} a $>_{lex}$-chain, X'_j forms a $>$-chain. Define $d_j =_{\text{def}} \min X'_j$, well defined since $>$ is wellfounded.

Each X_j is nonempty and each X_j is cofinal to X_{j-1}, and so to $X = X_0$. Therefore their greatest lower bounds coincide, if they exist. If $\bigcap\{X_j \mid k \in \mathbb{N}\}$ is nonempty, it contains the only element $d = (d_1, d_2, \dots)$ by construction. Then d is the greatest lower bound of X.

If $\bigcap\{X_j \mid k \in \mathbb{N}\}$ is empty then $x >_j d$ holds for all $x \in X_{j-1} \setminus X_j$, by definition of d_j. But then $x >_{lex} d$ holds for all $x \in \bigcup\{X_{j-1} \setminus X_j \mid k \in \mathbb{N}\} = X \setminus \{d\}$, by definition of $>_{lex}$. So d is a lower bound of X.

To show that d is the greatest lower bound of X, suppose e is another lower bound of X, and j is such that $e_k = d_k$ holds for all $k < j$, and $e_j \neq d_j$. We claim that $d_j > d_j$. Suppose $x \in X_j$. Then $x_j = d_j \neq e_j$ by definition of X_j, and the lower bound property $x >_{lex} e$ of e implies that $d_j > e_j$. This proves that $d >_j e$ and so $d >_{lex} e$. So d is the greatest lower bound of X.

Altogether, every chain X has a greatest lower bound d. This finishes the proof. □

Proposition 7. *Let $>$ be a wellfounded order on a set C, and let C^k and C^ω be ordered by $>^k_{lex}$ and $>_{lex}$, respectively. Then the* projection *operator $\pi_k : C^\omega \to C^k$, is downward continuous.*

Proof. It is easy to show that π_k is monotonic. To show that it preserves greatest lower bounds of chains, let $X \subseteq C^\omega$ be a $>_{lex}$-chain on C^ω. By Prop. 6 X has a greatest lower bound $\sqcap X = d$. Let d_j and X_j be as in Prop. 6. Now we define sets $X'_j =_{\text{def}} \{\pi_k(x) \mid x \in X_j\}$. Then X'_{j-1} and X'_j are cofinal as X_{j-1} and X_j are cofinal and π_k is monotonic. It appears that $X'_k = \{(d_1, \dots, d_k)\} = \{\pi_k(d)\}$, so $\sqcap X'_k = \pi_k(d)$. This implies $\sqcap\{\pi(x) \mid x \in X\} = \pi(d)$. Hence π_k is continuous. □

4 An inductive proof of Higman's Lemma

Higman's Lemma is a property about embeddings of strings. We will use notation and formalism akin to Gallier's survey [Gal91].

Let Σ be a set, and let Σ^* denote the set of (finite) strings over Σ. Let ε denote the string of length 0, and $s \cdot s'$ the concatenation of strings s and s'.

Definition 8 (Embedding). Let $\preceq$ be a binary relation on Σ. The induced *embedding relation* $\preceq^*$ on Σ^* is defined by

$$a_1 \cdot a_2 \cdot \dots \cdot a_m \quad \preceq^* \quad b_1 \cdot b_2 \cdot \dots \cdot b_n$$

for letters $a_i, b_j \in \Sigma$, if there is a strictly monotonic function $\sigma : \{1, \dots, m\} \to \{1, \dots, n\}$ such that $a_i \preceq b_{\sigma(i)}$ holds for all $i \in \{1, \dots, m\}$.

The embedding quasiorder can also be characterized as the smallest relation which is

1. transitive,
2. satisfies the right factor property, $t \preceq^* a \cdot t$, and
3. $a \preceq b$ and $s \preceq^* t$ implies $a \cdot s \preceq^* b \cdot t$.

For $\preceq$ the equality $=$ one obtains a special case of the embedding order, the subsequence order. We will use it for sequences of strings below. The sequence $(s_{\sigma(1)}, s_{\sigma(2)}, \dots, s_{\sigma(m)})$ is called a *subsequence* of the sequence $(s_1, s_2, \dots, s_n)$ if $\sigma : \{1, \dots, m\} \to \{1, \dots, n\}$ is a strictly monotonic function. We will occasionally call σ the *enumeration function* of the subsequence. An infinite sequence $(s_{\sigma(1)}, s_{\sigma(2)}, \dots)$ is called a *subsequence* of the infinite sequence $(s_1, s_2, \dots)$ if $\sigma : \mathbb{N} \to \mathbb{N}$ is a strictly monotonic function.

Definition 9 (Good sequence; well-quasiorder).

1. An infinite sequence $(t_1, t_2, \dots) \in D$ is called ($\preceq$-)*good* if there are indices $i < j$ such that $t_i \preceq t_j$. Otherwise it is called *bad.*
2. An infinite sequence $(t_1, t_2, \dots) \in D$ is called ($\preceq$-)*pretty* if there is an infinite subsequence $t_{\sigma(1)} \preceq t_{\sigma(2)} \preceq \cdots$, $1 \leq \sigma(1) < \sigma(2) < \cdots$. Otherwise it is called *ugly.*
3. The quasiorder $\preceq$ is called a *well-quasiorder* on C if every infinite C-sequence is good.

A lemma of Gallier's [Gal91] states that the notion of well-quasiorder remains the same if "good" is replaced by "pretty".

Lemma 10 ([Gal91]). *Let $\preceq$ be a quasiorder on a set C. Then $\preceq$ is a well-quasiorder if and only if every infinite C-sequence is pretty.*

Higman's Lemma gives a sufficient condition for well-quasiorders on strings.

Theorem 11 (Higman's Lemma). *If $\preceq$ is a well-quasiorder on Σ then $\preceq^*$ is a well-quasiorder on Σ^*.*

For the proof let $D =_{\text{def}} (\Sigma^*)^\omega$ be the set of all ω-sequences of strings over the alphabet Σ. We are going to prove by open induction that every element of D is $\preceq^*$-pretty. So the property P is "prettiness". We will use the order $\sqsupset$ on D defined as follows.

> $s = (s_1, s_2, \dots) \sqsupset (u_1, u_2, \dots) = u \quad \Longleftrightarrow_{\text{def}}$
> there is i such that $s \rhd_i u$, and there are strings $c_i \preceq^* c_{i+1} \preceq^* c_{i+2} \preceq^* \cdots$ such that $(c_i \cdot u_i, c_{i+1} \cdot u_{i+1}, c_{i+2} \cdot u_{i+2}, \dots)$ is a subsequence of $(s_i, s_{i+1}, s_{i+2}, \dots)$.

Here $\rhd \subseteq (\Sigma^*)^2$ denotes the *right factor* order on strings, defined by $s \cdot t \rhd t$. The right factor order is wellfounded.

We will prove below that the premises of Theorem 3 hold, i.e. that $\sqsupset$ is downward complete and that prettiness is open and inductive.

Of course, the order $\rhd_{lex}$, too, fits to this proof, and is complete. So the reader may wonder why it is not used instead. We have good reason not to choose it: To show that prettiness for $\rhd_{lex}$ is open, amounts to prove prettiness everywhere. In other words, the inductive scheme is of no help to the goal.

Example 1. For each $k \in \mathbb{N}$ let $d_k, d'_k, e_k \in \Sigma^*$ such that $d'_k \rhd d_k$. We are going to demonstrate that from $d = (d_1, d_2, \dots)$ pretty one can infer $e = (e_1, e_2, \dots)$ pretty, provided that prettiness is $\rhd_{lex}$-open. Let X be the set of infinite sequences x_i of the form

$$x_i = (d_1, \dots, d_{i-1}, d'_i, e_{i+1}, e_{i+2}, \dots)$$

Then $x_i \rhd_{lex} x_{i+1}$ due to $d'_i \rhd d_i$. Hence X is a $\rhd_{lex}$-chain. The greatest lower bound of X is d which is pretty by premise. If prettiness is open for $\rhd_{lex}$ then one of the chain elements x_i is pretty. But then e is pretty, too.

Thus we have shown that under weak premises any successful openness proof already entails that every infinite sequence e is pretty.

So the openness proof would contain an alternative to our inductive proof. This is certainly useless. We need an order where the proof of openness is *strictly simpler* than the whole inductive proof. As we will demonstrate below, the stronger order $\sqsupset$ satisfies this requirement. To ease reasoning, we will occasionally decompose the order $\sqsupset$ as follows.

Lemma 12. *$(t_1, t_2, \dots) \sqsupset (u_1, u_2, \dots)$ is equivalent to the conjunction*

1. *$(t_1, t_2, \dots) \rhd_{lex} (u_1, u_2, \dots)$ and*
2. *there are strings $c_1 \preceq^* c_2 \preceq^* \cdots$ such that $(c_1 \cdot u_1, c_2 \cdot u_2, \dots)$ is a subsequence of $(t_1, t_2, \dots)$.*

Proof. "$\Longrightarrow$" is obvious by definition of $\rhd_{lex}$ and the setting $c_j = \varepsilon$ for $j < i$. For "$\Longleftarrow$" assume that $t = (t_1, t_2, \dots) \rhd_i (u_1, u_2, \dots) = u$ and there are strings $c_1 \preceq^* c_2 \preceq^* \cdots$ such that $(c_1 \cdot u_1, c_2 \cdot u_2, \dots)$ is a subsequence of $(t_1, t_2, \dots)$. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be the enumeration function of this subsequence. Because σ is strictly monotonic, $\sigma(i) \geq i$, and so $(c_i \cdot u_i, c_{i+1} \cdot u_{i+1}, \dots)$ is a subsequence of $(t_i, t_{i+1}, \dots)$. This finishes the proof. □

Lemma 13. *$\sqsupset$ is downward complete on D.*

Proof. Let X be a nonempty $\sqsupset$-chain. By Lemma 12 it is *a fortiori* a $\rhd_{lex}$-chain, whence by Prop. 6 it has a greatest lower bound d w.r.t. the order $\rhd_{lex}$. We claim that d is a lower bound of X even for the order $\sqsupset$. According to Lemma 12 this leaves to show that there are strings $c_1 \preceq^* c_2 \preceq^* \cdots$ such that $(c_1 \cdot u_1, c_2 \cdot u_2, \dots)$ is a subsequence of x for any $x \in X$, with enumeration function $\sigma : \mathbb{N} \to \mathbb{N}$, say. Below we are going to construct the c_i and $\sigma(i)$ by a diagonalization over the corresponding constituents c_i^k and $\sigma^k(i)$ gathered from an infinite sequence of elements $x = x^0 \sqsupseteq x^1 \sqsupseteq \cdots$ of X.

To this end let $x \in X$ be arbitrary. Recall the definition of X_k from Prop. 6. We choose infinite sequences of strings $x^k \in X_k$ by induction on k as follows.

$$x^0 = x, \qquad x^{k+1} \in \{x \mid x \in X_{k+1} \wedge x^k \sqsupseteq x\}$$

The axiom of choice is used to get the x^{k+1} from the sets $\{x \in X_{k+1} \mid x^k \sqsupseteq x\}$ which are nonempty by the following reasoning. Because X is a $\sqsupset$-chain, $x \rhd_{lex} y$ is equivalent to $x \sqsupset y$ for $x, y \in X$. So the X_k are cofinal subchains of X w.r.t. the order $\sqsupset$.

By construction $x^{k-1} \sqsupseteq x^k$. By definition of $\sqsupset$ one gets for each $k > 0$ an enumeration function σ^k and strings $c_1^k \preceq^* c_2^k \preceq^* \cdots$ such that $c_i^k \cdot x_i^k = x_{\sigma^k(i)}^{k-1}$

holds for all i. W.l.o.g. $\sigma^k(i) = i$ and $c_i^k = \varepsilon$ for $1 \le i \le k-1$. Now we define

$$\sigma(k) =_{\text{def}} \sigma^1(\sigma^2(\cdots(\sigma^k(k)))) \qquad \text{and} \qquad c_k =_{\text{def}} c_k^1 \cdot c_k^2 \cdot \cdots \cdot c_k^k \ .$$

We claim that $\sigma : \mathbb{N} \to \mathbb{N}$ is an enumeration function for the subsequence $(c_1 \cdot d_1, c_2 \cdot d_2, \dots)$ of $x_0 = x$, and that $c_1 \preceq^* c_2 \preceq^* \cdots$ holds.

σ is strictly monotonic: For $k > 0$

$$\begin{aligned}
\sigma(k-1) &= \sigma^1(\sigma^2(\cdots(\sigma^{k-1}(k-1)))) \\
&= \sigma^1(\sigma^2(\cdots(\sigma^{k-1}(\sigma^k(k-1))))) \\
&< \sigma^1(\sigma^2(\cdots(\sigma^{k-1}(\sigma^k(k))))) \\
&= \sigma(k)
\end{aligned}$$

by $\sigma^k(k-1) = k-1$ and the property that all $\sigma^1, \dots, \sigma^k$ are strictly monotonic.

The strings satisfy $c_{k-1} \preceq^* c_k$:

$$\begin{aligned}
c_{k-1} &= c_{k-1}^1 \cdot c_{k-1}^2 \cdot \cdots \cdot c_{k-1}^{k-1} \\
&= c_{k-1}^1 \cdot c_{k-1}^2 \cdot \cdots \cdot c_{k-1}^{k-1} \cdot c_{k-1}^k \\
&\preceq^* c_k^1 \cdot c_k^2 \cdot \cdots \cdot c_k^{k-1} \cdot c_k^k \\
&= c_k
\end{aligned}$$

by $c_{k-1}^k = \varepsilon$ and $c_{k-1}^i \preceq^* c_k^i$, and monotonicity of $\cdot$ in each argument.

Finally, $c_k \cdot d_k = x^0_{\sigma(k)}$ holds for each k:

$$\begin{aligned}
c_k \cdot d_k = c_k \cdot x_k^k = c_k^1 \cdot c_k^2 \cdot \cdots \cdot c_k^k \cdot x_k^k = c_k^1 \cdot c_k^2 \cdot \cdots \cdot c_k^{k-1} \cdot x^{k-1}_{\sigma^k(k)} = \cdots \\
= c_k^1 \cdot x^1_{\sigma^2(\sigma^3(\cdots(\sigma^k(k))))} = x^0_{\sigma^1(\sigma^2(\cdots(\sigma^k(k))))} = x^0_{\sigma(k)}
\end{aligned}$$

So we conclude that $d \sqsubseteq x_0 = x$; this finishes the proof. □

Lemma 14. *The set P of pretty sequences is open.*

Proof. We show that for every $\sqsupseteq$-chain $X \subseteq D$, if its greatest lower bound $d = \sqcap X$ is pretty, then some $x \in X$, too, is pretty. Let d be pretty, i.e. there is a subsequence $d_{\sigma(1)} \preceq^* d_{\sigma(2)} \preceq^* \cdots$. From $x \sqsupseteq d$, by Part 2 of Lemma 12 it follows that there are strings $c_1 \preceq^* c_2 \preceq^* \cdots$ such that $(c_1 \cdot d_1, c_2 \cdot d_2, \dots)$ is a subsequence of x. By transitivity of the subsequence relation, $(c_{\sigma(1)} \cdot d_{\sigma(1)}, c_{\sigma(2)} \cdot d_{\sigma(2)}, \dots)$, too, is a subsequence of x. Moreover this subsequence is $\preceq^*$-ordered, since for every i,

$$c_{\sigma(i)} \cdot d_{\sigma(i)} \preceq^* c_{\sigma(i)} \cdot d_{\sigma(i+1)} \preceq^* c_{\sigma(i+1)} \cdot d_{\sigma(i+1)}$$

by monotonicity of $\cdot$ in each argument. Hence x is pretty. $\square$

The reason why we prefered "pretty" to "good" is the following straightforward proof of inductiveness.

Lemma 15. *The set of pretty infinite sequences of strings is inductive.*

Proof. Suppose given an infinite sequence $(t_1, t_2, \dots) \in D$. Either ε appears infinitely often in this sequence; then it is pretty because $\varepsilon \preceq^* \varepsilon$, and we are finished. Or for infinitely many i the string t_i is of the form $a_i \cdot t'_i$ where $a_i \in \Sigma$ and $t'_i \in \Sigma^*$. By the property that $\preceq$ is a well-quasiorder on Σ, there is an infinite subsequence $(a_{\sigma_0(1)} \cdot t'_{\sigma_0(1)}, a_{\sigma_0(2)} \cdot t'_{\sigma_0(2)}, \dots)$ where $a_{\sigma_0(1)} \preceq a_{\sigma_0(2)} \preceq \cdots$ for $\sigma_0 : \mathbb{N} \to \mathbb{N}$ its enumeration function. The infinite sequence $(t_1, \dots, t_{\sigma_0(1)-1}, t'_{\sigma_0(1)}, t'_{\sigma_0(2)}, \dots)$ therefore satisfies

$$(t_1, \dots, t_{\sigma_0(1)-1}, t'_{\sigma_0(1)}, t'_{\sigma_0(2)}, \dots) \quad \sqsubset \quad (t_1, t_2, \dots)$$

by definition of $\sqsupset$. So by inductive hypothesis it is pretty. Let $\sigma_1 : \mathbb{N} \to \mathbb{N}$ be the enumeration function such that w.l.o.g. $t'_{\sigma_1(1)} \preceq^* t'_{\sigma_1(2)} \preceq^* \cdots$ holds. So $t_{\sigma_1(1)} \preceq^* t_{\sigma_1(2)} \preceq^* \cdots$ follows by definition of $\preceq^*$. This is, $(t_1, t_2, \dots)$ is pretty. $\square$

By the principle of open induction (Theorem 3) it follows from Lemmas 13, 14, and 15 that every infinite sequence is pretty. By Lemma 10 $\preceq^*$ is a well-quasiorder.

Conclusion

Higman's Lemma, a powerful theorem in recursion theory, is classically proven by a minimal counterexample argument that is difficult to replace by an inductive proof. We have given a proof of Higman's Lemma by Raoult's *open induction*, a particularly strong form of induction that replaces the wellfoundedness requirement of Noetherian induction by the weaker topological properties *downward completeness* and *openness*.

Unusually, we put the existence of infinite ascending subsequences in an infinite sequence as the inductive hypothesis of our proof, a choice that greatly alleviates the proof of inductiveness. On the other hand, this choice excludes Raoult's suggestion to use a lexicographic order since that would re-introduce the original proof obligation within the openness proof. The order we have chosen is a refinement of the lexicographic order that allows for a straightforward argument to openness. Downward completeness is a little more involved to prove than for the lexicographic order. We succeeded applying a diagonalization construction.

We feel that the effort paid for the downward completeness proof is acceptable in view of the greater clarity of the inductiveness proof. The inductiveness proof is unique for the proof at hand, whereas the downward completeness proof may be reused for all proofs that use the same order.

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