Omega-Termination is Undecidable for Totally Terminating Term Rewriting Systems

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Abstract

We give a complete proof of the fact that the following problem is undecidable:

Given: A term rewriting system, where the termination of its rewrite relation is provable by a total reduction order on ground terms,
Wanted: Does there exist a strictly monotonic interpretation in the positive integers that proves termination?

Keywords: term rewriting, termination, total termination, omega-termination, termination hierarchy, termination type

1 Introduction

Termination of a term rewriting system (TRS), i.e. the nonexistence of an infinite rewrite reduction, is one of the key notions in term rewriting. It is the basis of a number of decision algorithms for properties undecidable in the general case.

It is known that termination is undecidable for TRSs, even for the restricted case of TRS with only unary function symbols [7], and for the case of one-rule TRSs [2].

The following true hierarchy of TRSs is called the *termination hierarchy* [12].

$$\text{termination} \supsetneq \text{non-self-embedding} \supsetneq \text{simple termination} \supsetneq \text{total termination} \supsetneq \omega\text{-termination} \supsetneq \text{polynomial termination}$$

Each of the upper four properties is known to be undecidable [7, 10, 1, 13]. The present paper addresses the remaining question: Whether the same holds for ω-termination. We study the following problem:

Given: A totally terminating TRS R.
Wanted: Is R ω-terminating?

We prove by a reduction from Post correspondence problems that this problem is *undecidable*. As a consequence, ω-termination is an undecidable property of TRSs.

2 Basic Notions

We assume that the reader is familiar with termination of term rewriting systems (TRS) [3]. We use notation as summarized in Dershowitz/Jouannaud [4].

A term rewriting system (TRS) R defines a one-step reduction relation $\rightarrow_R$. Starting with a term t_1, one may develop a reduction $t_1 \rightarrow_R t_2 \rightarrow_R \cdots$ that may be finite or infinite. R is called terminating if no infinite reductions exist. Here and in the rest of the paper we will tacitly assume that TRS have finitely many rewrite rules.

The termination of a TRS R means precisely the existence of a well-founded order $>$ on terms, closed under contexts and substitution – a *reduction order* – such that every rule $l \rightarrow r$ in R is ordered $l > r$ [9].

Terminating TRSs may now be further distinguished by some additional properties $>$ may have. If $>$ orders subterms, i.e. $s > t$ holds if t is a proper subterm of s, then R is called *simply terminating.* If $>$ is *total on ground terms*, i.e. $s > t$ or $t > s$ holds for all ground terms $s \neq t$, then R is called *totally terminating.* If $>$ is induced by a strictly monotonic interpretation into the positive integers (the well-ordered set ω) then R is called ω-terminating. These specializations form a true hierarchy [12].

Post's well-known Correspondence Problem (PCP) is defined as follows. Given a finite alphabet Γ, a PCP *instance* is a finite binary relation $P \subseteq$

$\Gamma^+ \times \Gamma^+$ on proper words over Γ. The PCP instance P is said to *have a solution* $\gamma \in \Gamma^+$ if

$$\alpha_1\alpha_2 \cdots \alpha_n = \gamma = \beta_1\beta_2 \cdots \beta_n$$

holds where $(\alpha_i, \beta_i) \in P$ for all $1 \leq i \leq n$. If Γ has at least two elements then the problem: given P, has P a solution? is undecidable [11].

3 Plan

We start off from the standard example of a totally terminating, not ω-terminating TRS, due to Zantema.

Theorem 1 ([12]). *The one-rule TRS*

$$h(g(x)) \to g(h(h(x))) \tag{I}$$

is totally terminating but not ω-terminating.

Proof. To show total termination one gives the strictly monotonic interpretation $[h](m, n) = (m, m + n)$, $[g](m, n) = (2m + 1, n)$ in pairs of positive integer numbers, lexicographically ordered. To disprove ω-termination one shows that strictly monotonic interpretations into the positive integer numbers for $[g]$ and $[h]$ entails that the reduction order is finitely branching. This is in contradiction to the fact that $[g](x) > [h]^n(x)$ holds for all $n \in \mathbb{N}$, a fact that can be proven by straightforward induction on n. $\square$

We will use a PCP to switch between this system, and another system which is clearly ω-terminating,

$$h(g(x)) \to g'(h(h(x))) \ . \tag{II}$$

If P has a solution, then the resulting system will have a behaviour comparable to system (I), else to system (II).

However this is a strongly simplified picture; the detailed technical treatment of this idea is much more involved. Now we are going to put these ideas in a precise form.

To every letter $a \in \Gamma$ let there be a unique new barred letter $\bar{a}$. By small Greek letters we denote strings over Γ. For $\alpha = a_1a_2 \ldots a_n$ let $\bar{\alpha} =_{\text{def}} \bar{a_n}\bar{a_{n-1}} \ldots \bar{a_1}$, the string of letters of α barred and *in reversed order*. We will consider barred and unbarred letters also as unary function symbols, and strings of barred and unbarred letters as contexts. Apart from that we fix another unary function symbol h, two 5-ary function symbols f and g, and a constant c.

Definition 2. To every PCP P let a TRS R_P be assigned that contains exactly the rules

$$h(g(\alpha(x), c, \beta(z), c, u)) \to f(x, \bar{\alpha}(c), z, \bar{\beta}(c), u) \tag{1}$$

$$f(\alpha(x), y, \beta(z), w, u) \to f(x, \bar{\alpha}(y), z, \bar{\beta}(w), u) \tag{2}$$

for each $(\alpha, \beta) \in P$, and the rules

$$f(c, \bar{a}(y), c, \bar{a}(w), u) \to g(a(c), y, a(c), w, h(h(u))) \tag{3}$$

$$g(x, \bar{a}(y), z, \bar{a}(w), u) \to g(a(x), y, a(z), w, u) \tag{4}$$

for each $a \in \Gamma$.

The following claims will be proven each in one successive section.

Theorem 3. *R_P is totally terminating.*

Theorem 4. *If the PCP P has a solution then R_P is not ω-terminating.*

Theorem 5. *If the PCP P has no solution then R_P is ω-terminating.*

These three statements together suffice to establish a proof of our claim.

Corollary. *The following problem is undecidable.*

> **Given:** *A totally terminating TRS R.*
> **Wanted:** *Is R ω-terminating?*

Proof. If the problem were decidable, then we could particularly employ the putative procedure for TRSs of the form R_P where P is a PCP, thanks to Thm. 3. But then we could use the procedure via Thm. 4 and Thm. 5 to solve the problem whether, given a PCP P, it has a solution. Contradiction to the well-known undecidability of the PCP [11]. $\square$

4 Total Termination

Usually people prove total termination by a strictly monotonic interpretation in the positive integers, or by a lexicographic recursive path order (RPO). We surely cannot use the first method as we intend to prove that under certain circumstances (the PCP P has a solution) such an interpretation does not exist.

We cannot use an RPO either, for a different reason: To order Rule (2), f must have lexicographic status where argument 1 or 3 comes first in the comparison. Likewise by rule (4), g must have lexicographic status where argument 2 or 4 comes first in the lexicographic comparison. In this case Rule (1) can only be ordered in the intended direction, if $h \gtrsim_{prec} f$. But then Rule (3) can no longer be ordered.

To prove that R_P is totally terminating, no matter whether the PCP instance P has a solution or not, we employ the wellknown Knuth/Bendix order.

Definition 6 (Knuth/Bendix order [8]). Let $\gtrsim_{prec}$ be a total wellfounded quasiorder on $\mathcal{F}$, and $[\_]$ be a linear monotonic interpretation in the positive integers such that $[f](x_1, \ldots, x_n) = x_i$ has a solution for some i only if $n = 1$ and $f \gtrsim g$ holds for every function symbol $g \in \mathcal{F}$. Then $\gtrsim_{kbo} \subseteq \mathcal{GT}$ is defined by $s = f(s_1, \ldots, s_m) \gtrsim_{kbo} g(t_1, \ldots, t_n) = t$ if

- $[s] > [t]$, or
- $[s] = [t]$, and $f >_{prec} g$, or
- $[s] = [t]$, and $f \sim_{prec} g$, and $(s_{\pi(1)}, \ldots, s_{\pi(m)}) \gtrsim_{kbo,lex} (t_{\rho(1)}, \ldots, t_{\rho(n)})$,

where f and g have lexicographic status π and ρ respectively.

Theorem 7 ([8, 5]). *Under the conditions of Defn. 6, the Knuth/Bendix order $>_{kbo}$ is a reduction order.*

The Knuth/Bendix order is also suitable to prove total termination, by a result of Ferreira.

Theorem 8 ([6], Thm. 4.47). *Under the conditions of Defn. 6, the Knuth/Bendix order $>_{kbo}$ can be extended to a reduction order that is total on ground terms.*

Hence any term rewriting system R that satisfies $l\sigma >_{kbo} r\sigma$ for any ground instance $l\sigma \to r\sigma$ of a rule $l \to r$ in R is totally terminating.

In our setting we choose the interpretation ("weight function") $[\_]$ in $\mathbb{N}$ defined by

$$\begin{aligned}
&[c] = 1\\
&[0](x) = [1](x) = [\bar{0}](x) = [\bar{1}](x) = x + 1\\
&[f](x, y, z, w, u) = [g](x, y, z, w, u) = x + y + x + w + u + 1\\
&[h](x) = x
\end{aligned}$$

where the results are ordered by $>$ on the positive integers, and the precedence $h >_{prec} f >_{prec} g >_{prec} a >_{prec} \bar{a} >_{prec} c$, and status f left-to-right, and g right-to-left. With that, the Knuth/Bendix order orders each rule. By Thm. 8 it follows that R_P is totally terminating.

5 If P has a Solution

We show in this section that in case the PCP P has a solution, $\gamma \in \Gamma^+$ say, then the TRS R_P is not ω-terminating.

Let P have a solution γ. Then we have for all ground terms t a R_P-reduction

$$\begin{aligned}
h(g(\gamma(c), c, \gamma(c), c, t)) \to_{(1)}\to^*_{(2)} &\ f(c, \bar{\gamma}(c), c, \bar{\gamma}(c), c, t)\\
&\to_{(3)}\to^*_{(4)} g(\gamma(c), c, \gamma(c), c, h(h(t))) \ . \quad (5)
\end{aligned}$$

For the sake of contradiction assume now that there is a strictly monotonic interpretation $[g], [h] : N_+ \to N_+$ in the positive integer numbers, ordered by $>$ that orders R_P. The following proof is mimicking Zantema's for Theorem 1.

Lemma 9. *If $\gamma \in \Gamma^+$ is a solution on the PCP P, then for all $k \in \mathbb{N}$ and $t \in \mathcal{GT}$, we have $[g(\gamma(c), c, \gamma(c), c, t)] \geq [h^k(t)]$.*

Proof. Let $x = [\gamma(c)] = z$ and $y = [c] = w$ be fixed. We prove that for all $u =_{\text{def}} [t]$ we have $[g](x, y, z, w, u) \geq [h]^k(u)$, by induction on k. Since the order $>$ is total, $>$ orders subterms [12]. So $[h]([g](x, y, z, w, u)) >$

$[g](x, y, z, w, u) > u$ which solves the case $k = 0$. Otherwise we have

$$\begin{array}{lll} & [h]([g](x, y, z, w, u)) & \\ > & [g](x, y, z, w, [h]([h](u))) & (*) \\ \geq & [h]^{k+2}(u) & \text{(ind. hyp.)} \\ = & [h]([h]^{k+1}(u)) & \end{array}$$

where the step labelled by $(*)$ is justified because (5) holds, and $>$ is a reduction order that orders every ground instance of a rule in R_P. □

By Lemma 9 $[g(\gamma(c), c, \gamma(c), c, t)]$ is an upper bound to infinitely many numbers $[t] < [h(t)] < [h^2(t)] < \cdots$. This is impossible in $\mathbb{N}_+$.

So, if P has a solution, the assumption that R_P were ω-terminating, is false.

6 If P has No Solution

In this section finally, we are going to show that in the case P has no solution, R_P indeed can be ordered by a strictly monotonic interpretation in the positive integers.

Let P be a PCP instance that has no solution, and let $\Gamma = \{0, 1\}$.

6.1 Upper Bounds for Reduction Lenghts

In this case one gets an upper bound for reduction lengths which can then be used to encode part of the interpretation.

For a ground term t, let $|t|$ denote the number of barred or unbarred letters in t not below an f, g, or h symbol.

Lemma 10. *For every $k \in \mathbb{N}$ the following hold.*

1. *If a ground term $h^k(g(p, q, r, s, t))$ starts a R_P-reduction with at least*

$$\min\{|q|, |s|\} + 2 \cdot \min\{|p| + |q|, |r| + |s|\}$$

steps at the outermost f or g position, then P has a solution.

2. *If a ground term $h^k(f(p,q,r,s,t))$ starts a R_P-reduction with at least*

$$\min\{|p|,|r|\} + 2 \cdot \min\{|p|+|q|,|r|+|s|\}$$

steps at the outermost f or g position, then P has a solution.

Proof. For convenience let us ignore the h symbols. Then Rule (1) and (3) are simplified (back) to

$$g(\alpha(x),c,\beta(z),c,u) \to f(x,\bar{\alpha}(c),z,\bar{\beta}(c),u) \tag{6}$$
$$f(c,\bar{a}(y),c,\bar{a}(w),u) \to g(a(c),y,a(c),w,u) \tag{7}$$

and we may call the outermost f or g position the top position.

We show only the proof of claim 1 below; claim 2 is proven in the same spirit.

Let a reduction starting from $g(p,q,r,s,t)$ be given. By the form of the rules, the reduction must be a prefix of some reduction

$$\begin{array}{ll} & g(p,q,r,s,t) \\ \to^*_{P1} & g(\gamma(p),c,\gamma(r),c,t) \quad \to \quad f(\dots) \\ \to^*_{P2} & f(c,\bar{\alpha}_i\bar{\alpha}_{i-1}\cdots\bar{\alpha}_1(c),c,\bar{\beta}_i\bar{\beta}_{i-1}\cdots\bar{\beta}_1(c),t) \quad \to \quad g(\dots) \\ \to^*_{P3} & g(\delta(c),c,\delta(c),c) \quad \to \quad f(\dots) \\ \to & \cdots \end{array}$$

where during $\to^*_{P1}$ every term has top symbol g, during $\to^*_{P2}$ every term has top symbol f, during $\to^*_{P3}$ every term has top symbol g.

We distinguish the phases $P1$,$P2$,$P3$ of which we deduce upper bounds of their reduction length.

During $P1$ only Rule (4) can be applied at the top. Therefore $P1$ has at most $\min\{|q|,|s|\}-1$ top steps, since at each top step the length of the first and third argument of the outermost g is decreased each by one.

In order to arrive at $P2$, we must have $q = \bar{\gamma}(c) = s$. During $P2$ only Rule (2) is applicable at the top; at most $\min\{|p|+|q|,|r|+|s|\}-1$ times, since at each top (2) step at least one letter of $\gamma(p)$ and of $\gamma(r)$ is consumed.

Only if

$$\gamma(p) = \alpha_1\alpha_2\cdots\alpha_i(c)$$
$$\gamma(r) = \beta_1\beta_2\cdots\beta_i(c)$$

do we arrive at $P3$, provided that $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_i, \beta_i) \in P$. This phase takes at most $\min\{|p|+|q|, |r|+|s|\}-1$ steps again, since only Rule (4) is applicable at the top, which each time consumes one letter of $\gamma(p)$ and one of $\gamma(r)$.

If our given reduction has length at least $\min\{|q|, |s|\} + 2 \cdot \min\{|p| + |q|, |r| + |s|\}$, then it must have passed $P3$. But then

$$\alpha_1\alpha_2 \cdots \alpha_i = \delta = \beta_1\beta_2 \cdots \beta_i$$

is a solution of the PCP P. $\square$

Next we define mutually recursively two functions len_f,len_g on quadruples of ground terms.

$$\begin{aligned}
&\mathsf{len}_f(p, q, r, s) = \\
&\quad \max\{\mathsf{len}_g(c, q, c, s) + 1 \mid p = c = r \wedge q = \bar{a}(q') \wedge s = \bar{a}(s') \wedge a \in \Gamma\} \cup \\
&\qquad \{\mathsf{len}_f(p', \bar{\alpha}(q), r', \bar{\beta}(s)) + 1 \mid p = \alpha(p') \wedge r = \beta(r') \wedge (\alpha, \beta) \in P\} \\
&\mathsf{len}_g(p, q, r, s) = \\
&\quad \max\{\mathsf{len}_f(p, c, r, c) + 1 \mid q = c = s \wedge p = \alpha(p') \wedge r = \alpha(r') \wedge (\alpha, \beta) \in P\} \cup \\
&\qquad \{\mathsf{len}_g(a(p), q', a(r), s') + 1 \mid q = \bar{a}(q') \wedge s = \bar{a}(s') \wedge a \in \Gamma\}
\end{aligned}$$

Here we take the view that $\max \emptyset = 0$ whence for instance $\mathsf{len}_f(c, c, c, c) = 0$ holds. Termination, and hence welldefinedness of this recursive definition is ensured by Lemma 10. The following is then an immediate consequence of the recursive definition.

Proposition 11. *For every quintuple (p, q, r, s, t) of ground terms,*

1. $\mathsf{len}_f(p, q, r, s)$ *yields the maximum number of steps at the topmost position of f or g, of any reduction starting from a term $h^k(f(p, q, r, s, t))$, $k \geq 0$, and*

2. $\mathsf{len}_g(p, q, r, s)$ *yields the maximum number of steps at the topmost position of f or g, of any reduction starting from a term $h^k(g(p, q, r, s, t))$, $k \geq 0$.*

6.2 An Interpretation

Now we are going to extract an interpretation in the positive integers from this knowledge. First let us define a few useful auxiliary functions on $\mathbb{N}_+$.

$$\mathsf{conc}(x, y) =_{\text{def}} 10^{(\lfloor \log(y) \rfloor + 1)} \cdot x + y$$

$$\mathsf{revc}(x, y) =_{\text{def}} \mathsf{conc}(\mathsf{conc}(x, y), \underbrace{4 \dots 4}_{\lfloor \log(x) \rfloor + 1 \text{ digits}})$$

$$\mathsf{bound}(x, y, z, w) =_{\text{def}} 3 \cdot \min\{\lfloor \log(\mathsf{revc}(x, y)) \rfloor, \lfloor \log(\mathsf{revc}(z, w) \rfloor\}$$

Here "$\log(z)$" denotes the base-10 logarithm of z, and $\lfloor z \rfloor$ denotes the "floor" of z, i.e. the greatest integer at or below the real number z. For $z \in \mathbb{N}_+$, the expression $\lfloor \log(z) \rfloor$ gives the number of digits of z minus one. Intuitively, if

> x and y have the digit sequences without leading zeroes $x_1 x_2 \dots x_m$ and $y_1 y_2 \dots y_n$, respectively,

then

> $\mathsf{conc}(x, y)$ yields the sequence $x_1 x_2 \dots x_m y_1 y_2 \dots y_n$, and
> $\mathsf{revc}(x, y)$ yields the sequence $x_1 x_2 \dots x_m y_1 y_2 \dots y_n \underbrace{4 \dots 4}_{m}$.

A function $\pi : \mathbb{N} \to \mathcal{GT}$ from positive integers to ground terms is defined recursively as follows.

$$\begin{aligned}
\pi(10x' + 2) &= 0(\pi(x')), & \pi(10 \cdot \mathsf{conc}(2, x') + 4) &= \bar{0}(\pi(x')), \\
\pi(10x' + 3) &= 1(\pi(x')), & \pi(10 \cdot \mathsf{conc}(3, x') + 4) &= \bar{1}(\pi(x')), \\
\pi(x) &= c \text{ else}
\end{aligned}$$

With that the definitions of the interpretation functions read as follows.

$$\begin{aligned}
[c] &= 1 \\
[0](x) &= 10x + 2 & [\bar{0}](x) &= 10 \cdot \mathsf{conc}(2, x) + 4 \\
[1](x) &= 10x + 3 & [\bar{1}](x) &= 10 \cdot \mathsf{conc}(3, x) + 4 \\
[h](x) &= 10x + 5
\end{aligned}$$

$$[f](x, y, z, w, u) =$$
$$6 + 10u \cdot 156^{(\mathsf{len}_f(\pi(x), \pi(y), \pi(z), \pi(w)) + 1 + (\mathsf{bound}(x,y,z,w) + 1)(\mathsf{revc}(x,y) + \mathsf{revc}(z,w)))}$$

$$[g](x, y, z, w, u) =$$
$$7 + 10u \cdot 156^{(\mathsf{len}_g(\pi(x), \pi(y), \pi(z), \pi(w)) + 1 + (\mathsf{bound}(x,y,z,w) + 1)(\mathsf{revc}(x,y) + \mathsf{revc}(z,w)))}$$

A few words of explanation are in order. The interpretation mapping of a ground term t to the positive integer number $[t]$ is injective. More specifically, it is such that from the last digit of $[t]$, the top function symbol of t can be retrieved, unless that digit is 4, in which case the first digit, too, is needed to distinguish between $\bar{0}$ and $\bar{1}$. To exploit this phenomenon partially, we introduced the function $\pi : \mathbb{N} \to \mathcal{GT}$.

As on the one hand, $[\_]$ is not surjective on $\mathbb{N}_+$, and on the other hand, we are only interested in the leading string of $0,1,\bar{0},\bar{1}$, we have π map all numbers not ending in $0, \ldots, 4$, or ending in 4 but not beginning with 2 or 3, to c. Structural induction on ζ shows:

Proposition 12. *For all strings ζ of barred or unbarred letters, and for all ground terms t, $\pi([t]) = \zeta(c)$ is equivalent to the existence of a ground term t' not starting with 0, 1, $\bar{0}$, $\bar{1}$ such that $t = \zeta(t')$.*

6.3 The Interpretation Orders R_P

The strange exception of the interpretation of barred symbols is motivated to achieve the following effect.

Example 1. We have $[01c] > [10c]$ by

$$[01c] = (10 \cdot 1 + 3) \cdot 10 + 2 = 132$$
$$[10c] = (10 \cdot 1 + 2) \cdot 10 + 3 = 123$$

but $[\bar{0}1c] = [\bar{1}\bar{0}c] > [\bar{0}\bar{1}c] = [1\bar{0}c]$ by

$$[\bar{1}\bar{0}c] = 10 \cdot \mathsf{conc}(3, 10 \cdot \mathsf{conc}(2,1) + 4) + 4 = 32144$$
$$[\bar{0}\bar{1}c] = 10 \cdot \mathsf{conc}(2, 10 \cdot \mathsf{conc}(1,1) + 4) + 4 = 23144$$

The reader will easily find out that the crucial sequence of digits to compare is the same. Unbarred sequences are compared by their digits right-to-left, barred sequences left-to-right. This behaviour is intended, for we aim at the following general result.

Lemma 13. *For all $\alpha \in \Gamma^*$,*

$$\mathsf{revc}([\alpha](x), y) = \mathsf{revc}(x, [\bar{\alpha}](y)) \enspace .$$

Proof. By structural induction on α. If α is the empty string, then the claim is trivial. Else let $\alpha = a\alpha'$ and let $d = 2$ if $a = 0$ and $d = 3$ if $a = 1$. Then

$$\begin{array}{lll} & \mathsf{revc}([a]([\alpha'](x)), y) & \\ = & \mathsf{revc}(10 \cdot [\alpha'](x) + d, y) & (\text{defn. } [a]) \\ = & \mathsf{revc}([\alpha'](x), 10 \cdot \mathsf{conc}(d, y) + 4) & (\text{defn. } \mathsf{revc}) \\ = & \mathsf{revc}([\alpha'](x), [\bar{a}](y)) & (\text{defn. } [\bar{a}]) \\ = & \mathsf{revc}(x, [\bar{\alpha'}]([\bar{a}](y))) & (\text{ind. hyp.}) \end{array}$$

□

That the given interpretation orders R_P, is a consequence of the following lemma.

Lemma 14. *Let P be a PCP that has no solution. For all $F, G \in f, g$, and $k \in \{0, 1\}$, if*

$$h^k(F(p, q, r, s, t)) \to G(p', q', r', s', t')$$

is a ground instance of a rule in R_P, then

$$[F(p, q, r, s, t)] > [G(p', q', r', s', h(h(t)))]$$

holds.

Proof. From the definition of len_f and len_g, and Prop. 12, it follows immediately that for all ground terms p, q, r, and s,

$$\mathsf{len}_f(p, q, r, s) = \mathsf{len}_f(\pi([p]), \pi([q]), \pi([r]), \pi([s])), \tag{8}$$

$$\mathsf{len}_g(p, q, r, s) = \mathsf{len}_g(\pi([p]), \pi([q]), \pi([r]), \pi([s])) \ . \tag{9}$$

Because every rewrite step strictly decreases the derivation length, we have

$$\mathsf{len}_F(p, q, r, s) > \mathsf{len}_G(p', q', r', s') \tag{10}$$

by Prop. 11. By the form of the rules, and Lemma 13, we get

$$\mathsf{revc}([p], [q]) = \mathsf{revc}([p'], [q']) \tag{11}$$

$$\mathsf{revc}([r], [s]) = \mathsf{revc}([r'], [s']) \tag{12}$$

and so, by definition of bound,

$$\mathsf{bound}([p],[q],[r],[s]) = \mathsf{bound}([p'],[q'],[r'],[s']) \ . \tag{13}$$

For abbreviation let

$$\begin{aligned}
b &=_{\text{def}} (\mathsf{bound}([p],[q],[r],[s])+1)(\mathsf{revc}([p],[q])+\mathsf{revc}([r],[s])),\\
b' &=_{\text{def}} (\mathsf{bound}([p'],[q'],[r'],[s'])+1)(\mathsf{revc}([p'],[q'])+\mathsf{revc}([r'],[s'])),\\
lhs &=_{\text{def}} [F(p,q,r,s,t)], \quad \text{and}\\
rhs &=_{\text{def}} [G(p',q',r',s',h(h(t)))] \ .
\end{aligned}$$

With that we get $b = b'$ by (11), (12), and (13), and so

$$\begin{array}{lll}
& \lfloor\frac{lhs}{10}\rfloor & \\
= & [t]\cdot 156^{(\mathsf{len}_F(\pi([p]),\pi([q]),\pi([r]),\pi([s]))+1+b)} & \\
= & [t]\cdot 156^{(\mathsf{len}_F(p,q,r,s)+1+b)} & (8) \text{ or } (9)\\
\geq & [t]\cdot 156^{(\mathsf{len}_G(p',q',r',s')+2+b)} & (10)\\
= & [t]\cdot 156\cdot 156^{(\mathsf{len}_G(p',q',r',s')+1+b)} & \\
> & [t]\cdot 155\cdot 156^{(\mathsf{len}_G(p',q',r',s')+1+b)} & ([t]\geq 1)\\
= & (10\cdot(10\cdot[t]+5)+5)\cdot 156^{(\mathsf{len}_G(p',q',r',s')+1+b)} & \\
= & [h]([h]([t]))\cdot 156^{(\mathsf{len}_G(p',q',r',s')+1+b)} & (\text{defn. } [h])\\
= & [h]([h]([t]))\cdot 156^{(\mathsf{len}_G(p',q',r',s')+1+b')} & (b=b')\\
= & [h]([h]([t]))\cdot 156^{(\mathsf{len}_G(\pi([p']),\pi([q']),\pi([r']),\pi([s']))+1+b')} & (8) \text{ or } (9)\\
= & \lfloor\frac{rhs}{10}\rfloor &
\end{array}$$

and so $lhs > rhs$. □

6.4 The Interpretation is Strictly Monotonic

This leaves to show that every interpretation function $[0]$, $[1]$, $[\bar{0}]$, $[\bar{1}]$, $[f]$, $[g]$, $[h]$ is strictly monotonic in $\mathbb{N}_+$. The claim is trivial for $[0]$, $[1]$, and $[h]$. For $[\bar{0}]$ and $[\bar{1}]$ we use strict monotonicity of conc.

Proposition 15. conc *and* revc *are strictly monotonic in each parameter.* bound *is (non-strictly) monotonic in each parameter.*

This leaves $[f]$ and $[g]$ for which strict monotonicity in each parameter is a "little more" difficult to prove. We need an auxiliary result:

Lemma 16. *If P is a PCP instance that has no solution then for all $x, y, z, w \in N_+$ and ground terms t,*

$$\mathsf{len}_f(\pi(x), \pi(y), \pi(z), \pi(w)) < \mathsf{bound}(x, y, z, w), \textit{ and}$$
$$\mathsf{len}_g(\pi(x), \pi(y), \pi(z), \pi(w)) < \mathsf{bound}(x, y, z, w)$$

holds.

Proof. One easily establishes that

$$|\pi(x)| \leq \lfloor \log(x) \rfloor \tag{14}$$

holds for all $x \in \mathbb{N}_+$, and that

$$\lfloor \log(\mathsf{revc}(x, y)) \rfloor = 2\lfloor \log(x) \rfloor + \lfloor \log(y) \rfloor + 2 \tag{15}$$

holds for all $x, y \in \mathbb{N}_+$.

We therefore have:

$$\begin{array}{lll}
 & \mathsf{len}_f(\pi(x), \pi(y), \pi(z), \pi(w)) & \\
= & \max\{\text{length of a } R_P\text{-reduction starting from} & \\
 & \quad h^k(f(\pi(x), \pi(y), \pi(z), \pi(w), t)) \text{ for some } t \in \mathcal{GT}\} & \text{(Prop. 11)} \\
\leq & \min\{|\pi(x)|, |\pi(z)|\} + & \\
 & 2 \cdot \min\{|\pi(x)| + |\pi(y)|, |\pi(z)| + |\pi(w)|\} & \text{(Lemma 10)} \\
\leq & 3 \cdot \min\{|\pi(x)| + |\pi(y)|, |\pi(z)| + |\pi(w)|\} & (|\pi(\ldots)| \geq 0) \\
\leq & 3 \cdot \min\{\lfloor \log(x) \rfloor + \lfloor \log(y) \rfloor, \lfloor \log(z) \rfloor + \lfloor \log(w) \rfloor\} & (14) \\
< & 3 \cdot \min\{\lfloor \log(\mathsf{revc}(x, y)) \rfloor, \lfloor \log(\mathsf{revc}(z, w)) \rfloor\} & (15) \\
= & \mathsf{bound}(x, y, z, w) & \text{(defn. } \mathsf{bound}\text{)}
\end{array}$$

The proof for len_g works similarly. $\square$

Finally we arrive at:

Lemma 17. *$[f]$ and $[g]$ are strictly monotonic in every parameter.*

Proof. Obviously [f] and [g] are strictly monotonic in their last parameter. For the other parameters, let $x \leq x', y \leq y', z \leq z', w \leq w'$ where at least one of these inequalities is strict.

We observe that

$$\mathsf{revc}(x,y) + \mathsf{revc}(z,w) < \mathsf{revc}(x',y') + \mathsf{revc}(z',w') \tag{16}$$

holds, by strict monotonicity of revc, and that

$$\mathsf{bound}(x,y,z,w) \leq \mathsf{bound}(x',y',z',w') \tag{17}$$

by monotonicity of bound. Let $F, G \in \{f, g\}$, and let

$$\begin{aligned} lhs &=_{\text{def}} [F](x,y,z,w,u) \ , \\ rhs &=_{\text{def}} [G](x',y',z',w',u) \ . \end{aligned}$$

Then we have:

$$\begin{array}{cll}
& \log_{156}\lfloor \dfrac{lhs}{10u} \rfloor & \\
= & \mathsf{len}_F(\pi(x),\pi(y),\pi(z),\pi(w)) + 1 + & \\
& (\mathsf{bound}(x,y,z,w)+1)(\mathsf{revc}(x,y)+\mathsf{revc}(z,w)) & \\
< & \mathsf{bound}(x,y,z,w) + 1 + & \\
& (\mathsf{bound}(x,y,z,w)+1)(\mathsf{revc}(x,y)+\mathsf{revc}(z,w)) & \text{(Lemma 16)} \\
= & (\mathsf{bound}(x,y,z,w)+1)(\mathsf{revc}(x,y)+\mathsf{revc}(z,w)+1) & \\
\leq & (\mathsf{bound}(x,y,z,w)+1)(\mathsf{revc}(x',y')+\mathsf{revc}(z',w')) & (16) \\
\leq & (\mathsf{bound}(x',y',z',w')+1)(\mathsf{revc}(x',y')+\mathsf{revc}(z',w')) & (17) \\
\leq & \mathsf{len}_G(\pi(x'),\pi(y'),\pi(z'),\pi(w')) + 1 + & \\
& (\mathsf{bound}(x',y',z',w')+1)(\mathsf{revc}(x',y')+\mathsf{revc}(z',w')) & (\mathsf{len}_G(...) > 0) \\
= & \log_{156}\lfloor \dfrac{rhs}{10u} \rfloor &
\end{array}$$

So $\log_{156}\lfloor \frac{rhs}{10u} \rfloor < \log_{156}\lfloor \frac{rhs}{10u} \rfloor$, and so $lhs < rhs$. ☐

Conclusion

A term rewriting system is called *totally terminating* if there is a strictly monotonic interpretation into a wellordered set, and ω*-terminating*, if there is a strictly monotonic interpretation into the positive integer numbers. Both are special forms of termination of term rewriting systems, and the latter is a proper special case of the former. We have proven that the question whether a totally terminating term rewriting system is even ω-terminating is undecidable.

To this end we encoded Post correspondence problems P into term rewriting systems R_P in such a way that R_P is always totally terminating; and ω-terminating exactly if P has no solution.

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References

[1] CARON, A.-C. Linear bounded automata and rewrite systems: Influence of initial configurations on decision properties. In *Coll. Trees in Algebra and Programming* (Brighton, 1991), LNCS 493, Springer, pp. 74–89.

[2] DAUCHET, M. Simulation of Turing machines by a regular rewrite rule. *Theoret. Comput. Sci. 103* (1992), 409–420.

[3] DERSHOWITZ, N. Termination of rewriting. *J. Symb. Comput. 3*, 1&2 (Feb./April 1987), 69–115. Corrigendum: 4, 3, Dec. 1987, 409-410.

[4] DERSHOWITZ, N., AND JOUANNAUD, J.-P. Notations for rewriting. *Bull. EATCS 43* (1991), 162–172.

[5] DICK, A. J. J., KALMUS, J. R., AND MARTIN, U. Automating the Knuth-Bendix ordering. *Acta Inform. 28* (1990), 95–119.

[6] FERREIRA, M. C. F. *Termination or term rewriting – well-foundedness, totality, and transformations.* PhD thesis, University of Utrecht, NL, 1995.

[7] HUET, G., AND LANKFORD, D. S. On the uniform halting problem for term rewriting systems. Tech. Rep. 283, INRIA, Rocquencourt, FR, Mar. 1978.

[8] Knuth, D. E., and Bendix, P. B. Simple word problems in universal algebras. In *Computational Problems in Abstract Algebra*, J. Leech, Ed. Pergamon Press, 1970, pp. 263–297.

[9] Manna, Z., and Ness, S. On the termination of Markov algorithms. In *Proc. 3rd Intl. Conf. System Science* (Honolulu, HI, 1970), pp. 789–792.

[10] Plaisted, D. The undecidability of self-embedding for term rewriting systems. *Inform. Process. Lett. 20* (1985), 61–64.

[11] Post, E. A variant of a recursively unsolvable problem. *Bulletin of the American Mathematical Society 52* (1946).

[12] Zantema, H. Termination of term rewriting: interpretation and type elimination. *J. Symb. Comput. 17*, 1 (1994), 23–50.

[13] Zantema, H. Total termination of term rewriting is undecidable. Tech. Rep. UU-CS-1994-55, Utrecht University, December 1994.