The L^2-Metric on the Moduli Space of $SU(2)$-Instantons with Instanton Number 1 over the Euclidean 4-Space *

Lutz Habermann

August 13, 1992

Abstract

The L^2-metric $\widehat{g_0}$ on the moduli space $\mathcal{M}_1(Q)$ of self-dual $SU(2)$-connections with instanton number 1 over the Euclidean 4-space is described. It is shown that the Riemannian manifold $(\mathcal{M}_1(Q), \widehat{g_0})$ is isometric to $\mathbf{R}^+ \times \mathbf{R}^4$ with the Euclidean metric.

1 Introduction

Let G be a compact Lie group and consider a principal G-bundle P over a closed, oriented Riemannian 4-manifold (M, g). Suppose that the moduli space $\mathcal{M}(P)$ of irreducible self-dual connections on P is a smooth manifold. Identifying tangent vectors to $\mathcal{M}(P)$ with harmonic 1-forms on M with values in the vector bundle $\mathrm{Ad}P$ associated with P by means of the adjoint representation, we obtain an L^2-metric $\hat{g}$ on $\mathcal{M}(P)$. Results concerning the Riemannian metric $\hat{g}$ were obtained in [5]–[9].

The purpose of the present paper is to describe the analogue of the L^2-metric on the moduli space $\mathcal{M}_1(Q)$ of self-dual $SU(2)$-connections with instanton number 1 over $\mathbf{R}^4$ with the Euclidean metric $g_{\mathbf{R}^4}$, suppressing analytical questions which arise from the fact that $\mathbf{R}^4$ is not compact.

*Supported by Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 288

In Sect. 2 we consider a family of Riemannian metrics g_s, $s \in (0,1]$, on the 4-sphere S^4 which lie in the conformal class of the standard metric and satisfy

$$\lim_{s\to 0} g_s = g_{\mathbf{R}^4} \qquad \text{on} \quad \mathbf{R}^4 = S^4 \setminus \{\text{south pole}\}\,.$$

The family $\{g_s\}$ induces a family $\{\widehat{g_s}\}$ of L^2-metrics on the moduli space $\mathcal{M}(P) \equiv \mathcal{M}_1(Q)$ of self-dual connections on the principal $SU(2)$-bundle P with $c_2(P) = -1$ over the standard 4-sphere. Therefore, if $\{\widehat{g_s}\}$ converges to a Riemannian metric $\widehat{g_0}$ on $\mathcal{M}(P)$, we can consider $\widehat{g_0}$ as the L^2-metric with respect to the Euclidean metric $g_{\mathbf{R}^4}$. Before we state the main result of Sect. 2, we recall that $\mathcal{M}(P)$ is diffeomorphic to $\mathbf{R}^+ \times \mathbf{R}^4$ (see [1]).

Theorem. *In terms of the parametrization $\mathcal{M}(P) = \mathbf{R}^+ \times \mathbf{R}^4$ we have pointwise*

$$\widehat{g_0} = \lim_{s\to 0} \widehat{g_s} = \frac{\pi^2}{r}\, dr^2 + 2\pi^2\, g_{\mathbf{R}^4}\,.$$

In Sect. 3 we show that $\widehat{g_0}$ is in fact an L^2-metric. For this we explain how to identify tangent vectors to $\mathcal{M}_1(Q)$ with harmonic 1-forms over $\mathbf{R}^4$.

In the last section we state some further results. For example, we consider the family $\{\widehat{g_0}^s\}_{s\geq 0}$ of Riemannian metrics on $\mathcal{M}_1(Q)$ defined by

$$(\widehat{g_0}^s)_{[A]}(\eta_1, \eta_2) = ((1 + s\Delta_A)\eta_1, (1 + s\Delta_A)\eta_2)_{L^2}$$

for $[A] \in \mathcal{M}_1(Q)$ and $\eta_1, \eta_2 \in T_{[A]}\mathcal{M}_1(Q)$ and show that the Riemannian manifolds $(\mathcal{M}_1(Q), \widehat{g_0}^s)$ with $s > 0$ are complete.

2 The L^2-metric over $\mathbf{R}^4$ as the limit of a family of L^2-metrics over S^4

Let P be the principal $SU(2)$-bundle with $c_2(P) = -1$ over the 4-sphere S^4 with the standard metric g_{S^4}, and let $Q = \mathbf{R}^4 \times SU(2)$ be the trivial principal $SU(2)$-bundle over $\mathbf{R}^4$ equipped with the Euclidean metric $g_{\mathbf{R}^4}$. On the Lie algebra $\mathbf{su}(2)$ of $SU(2)$ we fix the Ad-invariant scalar product defined by

$$\langle X, Y\rangle = -\frac{1}{2}\mathrm{Tr}(XY)\,.$$

Denote by $\mathcal{SD}(P)$ the space of smooth self-dual connections on P and by $\mathcal{SD}_1(Q)$ the space of smooth self-dual connections A on Q with instanton number

$$k(A) = -\frac{1}{8\pi^2}\int_{\mathbf{R}^4} \mathrm{Tr}(F^A \wedge F^A) = 1\,.$$

Here F^A is the curvature of A. Let $\mathcal{G}(P)$ and $\mathcal{G}(Q)$ be the groups of smooth gauge transformations of P and Q. By Uhlenbeck's theorem on removable singularities in Yang-Mills fields (see[12]), we can identify the moduli spaces $\mathcal{M}(P) = \mathcal{SD}(P)/\mathcal{G}(P)$ and $\mathcal{M}_1(Q) = \mathcal{SD}_1(Q)/\mathcal{G}(Q)$.

Let $\mathbf{H}$ be the skew-field of quaternions. The group $SL(2,\mathbf{H})$, acting on $S^4 = \mathbf{H} \cup \{\infty\}$ via fractional linear transformations, is the double covering of the group $Conf_0(S^4)$ of orientation-preserving conformal transformations on S^4. As is well-known, the corresponding action of $SL(2,\mathbf{H})$ on $\mathcal{M}(P)$ is transitive, where $Sp(2)$ is the isotropy group of the gauge equivalence class $[A_1]$ of the so-called basic instanton A_1. Thus, the mapping $\Phi : (0,1)\times\mathbf{H} \to \mathcal{M}(P)$ given by

$$\Phi(t,x) = [A_1]\cdot c_t \cdot d_x$$

with

$$\begin{aligned} c_t &= \begin{pmatrix} t^{-1/4} & 0 \\ 0 & t^{1/4} \end{pmatrix} \in SL(2,\mathbf{H}) \qquad \text{and} \\ d_x &= \frac{1}{\sqrt{1+|x|^2}} \begin{pmatrix} 1 & x \\ -\bar{x} & 1 \end{pmatrix} \in Sp(2) \end{aligned}$$

yields a parametrization of $\mathcal{M}(P) \setminus \{[A_1]\cdot c_t \,:\, t \in [1,\infty)\}$.

Consider a Riemannian metric g on S^4 which lies in the conformal class of g_{S^4}. Then g induces an L^2-metric $\hat{g}$ on $\mathcal{M}(P)$ by identifying the tangent space $T_{[A]}\mathcal{M}(P)$ with $\{\eta \in \Omega^1(\mathrm{Ad}P) \,:\, d_A^{*,g}\eta = 0 \text{ and } p_- d_A \eta = 0\}$, where $\mathrm{Ad}P = P \times_{\mathrm{Ad}} \mathbf{su}(2)$, $d_A^{*,g} : \Omega^1(\mathrm{Ad}P) \to \Omega^0(\mathrm{Ad}P)$ is the formally adjoint operator of d_A with respect to the metric g, and p_- is the (conformally invariant) orthogonal projection onto the space of anti-self-dual 2-forms (for details see [7]–[9]). Now set

$$g_s = \frac{1}{4s}\,(c_{s^{-1}})^*\, g_{S^4} \qquad \text{for} \qquad s \in (0,1]\,.$$

Checking that

$$g_s = \frac{1}{(1+s|x|^2)^2}\, g_{\mathbf{R}^4} \qquad \text{on} \quad \mathbf{R}^4 = S^4 \setminus \{\text{south pole}\}\,,$$

one observes that

$$\lim_{s\to 0} g_s = g_{\mathbf{R}^4}$$

pointwise on $\mathbf{R}^4$. Therefore, if

$$\widehat{g_0} = \lim_{s\to 0} \widehat{g_s}$$

exists and defines a Riemannian metric on $\mathcal{M}(P)$, one can consider $\widehat{g_0}$ as the L^2-metric with respect to the Euclidean metric $g_{\mathbf{R}^4}$.

We are now going to compute the limit $\widehat{g_0}$. For this we use the following facts.

Lemma 2.1. *For $s \in (0,1]$ we have*

$$\widehat{g_s} = \frac{1}{4s} F_s^* \widehat{g_{S^4}} ,$$

where $F_s : \mathcal{M}(P) \to \mathcal{M}(P)$ is given by $F_s([A]) = [A] \cdot c_s$.

Proof. One verifies that

$$(\eta_1, \eta_2)_{L^2(g_s)} = \frac{1}{4s} (f_s^* \eta_1, f_s^* \eta_2)_{L^2(g_{S^4})} ,$$

where f_s is any bundle morphism of P which projects down to c_s. □

Proposition 2.2. (i) *The Riemannian metric $\widehat{g_{S^4}}$ is given by*

$$\Phi^* \widehat{g_{S^4}} = \alpha(t)\, dt^2 + \beta(t) \frac{4}{(1+|x|^2)^2}\, g_{\mathbf{R}^4}$$

with

$$\begin{aligned} \alpha(t) &= 4\pi^2 \left(\frac{t^2+10t+1}{t(t-1)^4} - \frac{6(t+1)}{(t-1)^5} \ln(t) \right) \qquad \textit{and} \\ \beta(t) &= 2\pi^2 \left(\frac{t^2-8t+1}{(t-1)^2} + \frac{12t^2}{(t+1)(t-1)^3} \ln(t) \right) , \end{aligned}$$

where $\mathbf{H}$ and $\mathbf{R}^4$ are identified as usual.
(ii) *Taking the limits for t tending to zero yields*

$$\lim_{t\to 0} (t\,\alpha(t)) = 4\,\pi^2 \quad \textit{and} \quad \lim_{t\to 0} \beta(t) = 2\,\pi^2 .$$

Proof. (i) was shown in [9] (see also [5],[7]). Assertion (ii) is immediate. □

For our goal it will be helpful to make use of the parametrization $\Psi : \mathbf{R}^+ \times \mathbf{H} \to \mathcal{M}(P)$ of $\mathcal{M}(P)$ defined by

$$\Psi(r,b) = [A_1] \cdot c_r \cdot h_b$$

with

$$h_b = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbf{H}).$$

Comparing the parametrizations Φ and Ψ, we see that $\Phi(t,x) = \Psi(r,b)$ if and only if

$$c_r \cdot h_b \cdot d_x^{-1} \cdot c_{t^{-1}} \in Sp(2).$$

Using $d_x^{-1} = \overline{d_x}^T = d_{-x}$, this implies the equations

$$\begin{aligned} -(1 - x\bar{b})(x+b) + rx &= 0, & (1) \\ \frac{1}{\sqrt{t}(1+|x|^2)}\left(\frac{1}{\sqrt{r}}|x+b|^2 + \sqrt{r}\right) &= 1. & (2) \end{aligned}$$

Under the assumption that $b = b_1 \in \mathbf{R}^-$, we conclude

Lemma 2.3. *The transition mapping*

$$\Phi \circ \Psi^{-1}{}_{|\mathbf{R}^+ \times \mathbf{R}^-} : \mathbf{R}^+ \times \mathbf{R}^- \longrightarrow (0,1) \times \mathbf{R}^+$$

is given by $\Phi \circ \Psi^{-1}(r,b_1) = (t(r,b_1), x_1(r,b_1))$, *where*

$$\begin{aligned} x_1(r,b_1) &= \lambda(r,b_1) + \sqrt{\lambda(r,b_1)^2 + 1} \in \mathbf{R}^+ \qquad \textit{with} \\ \lambda(r,b_1) &= \frac{1 - r - b_1^2}{2b_1} \qquad \textit{and} \\ t(r,b_1) &= \theta(r,b_1)^2 \in (0,1) \qquad \textit{with} \\ \theta(r,b_1) &= \frac{1}{\sqrt{r}} \cdot \frac{r + (x_1(r,b_1) + b_1)^2}{1 + x_1(r,b_1)^2}. \qquad \square \end{aligned}$$

Now we have

$$\begin{aligned} \frac{\partial}{\partial r}(r,b_1) &= \frac{\partial t}{\partial r}(r,b_1)\,\frac{\partial}{\partial t}(t,x_1) + \frac{\partial x_1}{\partial r}(r,b_1)\,\frac{\partial}{\partial x_1}(t,x_1) \qquad \text{and} \\ \frac{\partial}{\partial b_1}(r,b_1) &= \frac{\partial t}{\partial b_1}(r,b_1)\,\frac{\partial}{\partial t}(t,x_1) + \frac{\partial x_1}{\partial b_1}(r,b_1)\,\frac{\partial}{\partial x_1}(t,x_1) \end{aligned}$$

for $(r,b_1) \in \mathbf{R}^+ \times \mathbf{R}^-$. On the other hand, we get

Lemma 2.4. *The canonical vector fields* $\frac{\partial}{\partial b_j}$, $j = 2,3,4$, *are related to* $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x_i}\}$ *by*

$$\frac{\partial}{\partial b_j}(r,b_1) = \frac{1+x_1(r,b_1)^2}{2b_1\sqrt{\lambda(r,b_1)^2+1}}\,\frac{\partial}{\partial x_j}(t,x_1) \qquad \mathit{for} \qquad (r,b_1)\in \mathbf{R}^+\times\mathbf{R}^-\,.$$

Proof. Let $\{a_2,a_3,a_4\}$ be the standard basis of the imaginary quaternions. Then

$$\frac{\partial}{\partial b_j}(r,b_1) = \frac{d\gamma}{dv}(0) \quad \text{with} \quad \gamma(v) = \Psi(r, b_1 + a_j v)\,.$$

According to Equations (1) and (2), we have $\gamma(v) = \Phi(t(v), x(v))$ with

$$\begin{aligned} rx(v) &= [1 - x(v)\,(b_1 - a_j v)]\,[x(v) + b_1 + a_j v]\ , && (3)\\ \sqrt{t(v)}\left(1+|x(v)|^2\right) &= \frac{1}{\sqrt{r}}|x(v)+b_1+a_jv|^2 + \sqrt{r}\,. && (4)\end{aligned}$$

Differentiating Equation (3), we obtain

$$\left(r-1+b_1^2+2b_1x_1\right)\frac{dx}{dv}(0) = a_j\left(1+x_1^2\right)$$

which implies

$$\frac{dx_i}{dv}(0) = \frac{1+x_1^2}{r-1+b_1^2+2b_1x_1}\,\delta_{ij} \quad \text{for} \quad i=1,\dots,4\,.$$

Further, from (4) and

$$\frac{d}{dv}\left(|x(v)+b_1+a_jv|^2\right)_{|v=0} = \frac{d}{dv}\left(|x(v)|^2\right)_{|v=0} = 0$$

we get $\frac{dt}{dv}(0) = 0$. Now, using $1 - r - b_1^2 = 2b_1\lambda(r,b_1)$ and

$$\frac{\partial}{\partial b_j}(r,b_1) = \frac{dt}{dv}(0)\,\frac{\partial}{\partial t}(t,x_1) + \sum_{i=1}^{4}\frac{dx_i}{dv}(0)\,\frac{\partial}{\partial x_i}(t,x_1)\,,$$

the assertion follows. $\square$

Before determining the coefficients of $\widehat{g_0}$, we state

Lemma 2.5. *For* $(r, b_1) \in \mathbf{R}^+ \times \mathbf{R}^-$ *we have*

$$
\begin{aligned}
&\text{(i)} \quad \lim_{s\to 0} t(sr, \sqrt{s}\, b_1) = \lim_{s\to 0} x_1(sr, \sqrt{s}\, b_1) = 0 \\
&\text{(ii)} \quad \lim_{s\to 0} \left(\sqrt{s}\, \frac{\partial \theta}{\partial r}(sr, \sqrt{s}\, b_1)\right) = -\frac{1}{2\sqrt{r}} \qquad \textit{and} \\
&\qquad\;\; \lim_{s\to 0} \left(\sqrt{s}\, \frac{\partial x_1}{\partial r}(sr, \sqrt{s}\, b_1)\right) = 0 \\
&\text{(iii)} \quad \lim_{s\to 0} \frac{\partial \theta}{\partial b_1}(sr, \sqrt{s}\, b_1) = 0 \quad \textit{and} \quad \lim_{s\to 0} \frac{\partial x_1}{\partial b_1}(sr, \sqrt{s}\, b_1) = -1 \\
&\text{(iv)} \quad \lim_{s\to 0} \frac{1 + x_1(sr, \sqrt{s}\, b_1)^2}{2b_1\sqrt{s\lambda(sr, \sqrt{s}\, b_1)^2 + s}} = -1
\end{aligned}
$$

Proof. The assertions can be deduced from

$$
\lim_{s\to 0} \left(\sqrt{s}\, \lambda(sr, \sqrt{s}\, b_1)\right) = \frac{1}{2b_1}\,, \quad \lim_{s\to 0} \left(s\, \frac{\partial \lambda}{\partial b_1}(sr, \sqrt{s}\, b_1)\right) = -\frac{1}{2b_1^2}\,,
$$

and

$$
\begin{aligned}
\lim_{s\to 0} \frac{\sqrt{s}}{x_1(sr, \sqrt{s}\, b_1)} &= \lim_{s\to 0} \left[\sqrt{s}\left(\sqrt{\lambda(sr, \sqrt{s}\, b_1)^2 + 1} - \lambda(sr, \sqrt{s}\, b_1)\right)\right] \\
&= \sqrt{\left(\frac{1}{2b_1}\right)^2} - \frac{1}{2b_1} = -\frac{1}{b_1}\,. \qquad \square
\end{aligned}
$$

Now we are able to prove

Theorem 2.6. *The limit* $\widehat{g}_0$ *exists pointwise and is the flat Riemannian metric given by*

$$
\Psi^* \widehat{g}_0 = \frac{\pi^2}{r}\, dr^2 + 2\pi^2\, g_{\mathbf{R}^4}\,,
$$

where again $\mathbf{H}$ *and* $\mathbf{R}^4$ *are identified.*

Proof. Let $(q_1, q_2) \in Sp(1) \times Sp(1) \subset Sp(2)$. Then

$$
F_s([A] \cdot (q_1, q_2)) = F_s([A]) \cdot (q_1, q_2)\,.
$$

Further, we see that

$$
\begin{aligned}
\Psi(r, b) \cdot (q_1, q_2) &= [A_1] \cdot c_r \cdot h_b \cdot (q_1, q_2) \\
&= [A_1] \cdot c_r \cdot (q_1, q_2)^{-1} \cdot h_b \cdot (q_1, q_2) \\
&= [A_1] \cdot c_r \cdot h_{\overline{q_1} b q_2} = \Psi(r, \overline{q_1} b q_2)
\end{aligned}
$$

for $(r,b) \in \mathbf{R}^+ \times \mathbf{H}$, applying $[A_1] \cdot c_r \cdot (q_1,q_2)^{-1} = [A_1] \cdot c_r$. Recalling that $Sp(1) \times Sp(1)$ acts isometrically on $(\mathcal{M}(P), \widehat{g_{S^4}})$ and $Sp(1) \cdot \mathbf{R}^- \cdot Sp(1) = \mathbf{H}$, it is enough to show that

$$\begin{aligned} \widehat{g_0}(\frac{\partial}{\partial r}(r,b_1), \frac{\partial}{\partial r}(r,b_1)) &= \frac{\pi^2}{r}\,, && (5)\\ \widehat{g_0}(\frac{\partial}{\partial r}(r,b_1), \frac{\partial}{\partial b_i}(r,b_1)) &= 0\,, && (6)\\ \widehat{g_0}(\frac{\partial}{\partial b_i}(r,b_1), \frac{\partial}{\partial b_j}(r,b_1)) &= 2\pi^2\,\delta_{ij} && (7) \end{aligned}$$

for $i,j = 1,\ldots,4$ and $(r,b_1) \in \mathbf{R}^+ \times \mathbf{R}^-$. Because of $h_b \cdot c_s = c_s \cdot h_{\sqrt{s}\,b}$, we have $F_s \circ \Psi(r,b) = \Psi(sr, \sqrt{s}\,b)$ which yields

$$\begin{aligned} (F_s)_* \frac{\partial}{\partial r}(r,b) &= s\,\frac{\partial}{\partial r}(sr, \sqrt{s}\,b) \qquad \text{and}\\ (F_s)_* \frac{\partial}{\partial b_i}(r,b) &= \sqrt{s}\,\frac{\partial}{\partial b_i}(sr, \sqrt{s}\,b) \quad \text{for} \quad i = 1,\ldots,4\,. \end{aligned}$$

¿From Lemma 2.1, Proposition 2.2 and Lemma 2.5 we now conclude

$$\begin{aligned} &\widehat{g_0}(\frac{\partial}{\partial r}(r,b_1), \frac{\partial}{\partial r}(r,b_1))\\ =\ &\lim_{s\to 0}\left(\frac{1}{4s}\,\widehat{g_{S^4}}((F_s)_*\frac{\partial}{\partial r}(r,b_1), (F_s)_*\frac{\partial}{\partial r}(r,b_1))\right)\\ =\ &\lim_{s\to 0}\left(\frac{s}{4}\,\widehat{g_{S^4}}(\frac{\partial}{\partial r}(sr,\sqrt{s}\,b_1), \frac{\partial}{\partial r}(sr,\sqrt{s}\,b_1))\right)\\ =\ &\lim_{s\to 0}\Bigg(\left[\sqrt{s}\,\frac{\partial\theta}{\partial r}(sr,\sqrt{s}\,b_1)\right]^2\, t(sr,\sqrt{s}\,b_1)\,\alpha(t(sr,\sqrt{s}\,b_1))\\ &+\left[\sqrt{s}\,\frac{\partial x_1}{\partial r}(sr,\sqrt{s}\,b_1)\right]^2\,\frac{\beta(t(sr,\sqrt{s}\,b_1))}{(1+x_1(sr,\sqrt{s}\,b_1)^2)^2}\Bigg)\\ =\ &4\pi^2\left[-\frac{1}{2\sqrt{r}}\right]^2 = \frac{\pi^2}{r}\,. \end{aligned}$$

Analogously, using also Lemma 2.4, one gets (6) and (7). Finally, observe that

$$\Psi^*\widehat{g_0} = 2\pi^2\left(dl^2 + g_{\mathbf{R}^4}\right)$$

with $l = \sqrt{2r}$. $\square$

The following simple conclusion is an interesting analogue of results in the compact case (see [6]–[8]).

Corollary 2.7. *Let $\overline{\mathcal{M}_1(Q)}$ be the completion of $(\mathcal{M}_1(Q), \widehat{g}_0)$ as a metric space. Then*

(i) $\overline{\mathcal{M}_1(Q)} \setminus \mathcal{M}_1(Q)$ *is diffeomorphic to* $\mathbf{R}^4$.

(ii) *The inclusion of the boundary of* $\mathcal{M}_1(Q)$ *into* $\overline{\mathcal{M}_1(Q)}$ *is totally geodesic.* $\square$

3 A more analytical approach

In this section we will show that the Riemannian metric $\widehat{g}_0$ given in Sect. 2 is in fact an L^2-metric. For this we will identify $SU(2)$ with the group $Sp(1)$. Then the scalar product on the Lie algebra $\mathbf{sp}(1) = \mathrm{Im}(\mathbf{H})$ is given by

$$\langle v_1, v_2 \rangle = -\mathrm{Re}(v_1 v_2)\,.$$

Recall that, using $\mathcal{M}(P) \equiv \mathcal{M}_1(Q)$, the parametrization $\Psi : \mathbf{R}^+ \times \mathbf{H} \to \mathcal{M}_1(Q)$ is described by

$$\Psi(r, b) = [A_{r,b}] \quad \text{with} \quad A_{r,b} = h_b^* \, A_r\,,$$

where the connections A_r on the trivial bundle Q over $\mathbf{R}^4 = \mathbf{H}$ are defined by

$$A_r = c_r^* \, A_1 = \mathrm{Im}\left(\frac{x\, d\bar{x}}{r + |x|^2}\right)\,.$$

We mention that the elements $c_r, h_b \in SL(2, \mathbf{H})$ act on $\mathbf{H}$ by

$$c_r \, x = \frac{1}{\sqrt{r}}\, x \quad \text{and} \quad h_b \, x = x - b\,.$$

Now we want to identify, as in the compact case, tangent vectors to $\mathcal{M}_1(Q)$ at a point $[A_{r,b}]$ with 1-forms in the space

$$H(A_{r,b}) = \{\eta \in \Omega^1(\mathbf{H}, \mathbf{sp}(1)) \,:\, (d_{A_{r,b}})^* \eta = 0 \ \text{and} \ p_- d_{A_{r,b}} \eta = 0\} \cap L^2\,.$$

Here $(d_{A_{r,b}})^* : \Omega^1(\mathbf{H}, \mathbf{sp}(1)) \to \Omega^0(\mathbf{H}, \mathbf{sp}(1))$ is the formally adjoint operator of $d_{A_{r,b}}$ with respect to the Euclidean metric $g_{\mathbf{R}^4}$. The statements below give an explicit description of $H(A_r)$ for $r \in \mathbf{R}^+$.

Proposition 3.1. *The 1-forms*

$$\begin{aligned}
\psi_k^r &= \frac{d\gamma^k}{(r+|x|^2)^2} \qquad for \qquad k=1,\ldots,4,\\
\psi_5^r &= \mathrm{Im}\left(\frac{xd\bar{x}}{(r+|x|^2)^2}\right),\\
\psi_6^r &= \frac{1}{(r+|x|^2)^2}\left(\gamma^2dx_1-\gamma^1dx_2-\gamma^4dx_3+\gamma^3dx_4\right),\\
\psi_7^r &= \frac{1}{(r+|x|^2)^2}\left(\gamma^3dx_1+\gamma^4dx_2-\gamma^1dx_3-\gamma^2dx_4\right), \quad and\\
\psi_8^r &= \frac{1}{(r+|x|^2)^2}\left(\gamma^4dx_1-\gamma^3dx_2+\gamma^2dx_3-\gamma^1dx_4\right),
\end{aligned}$$

where γ_k*,* $k=1,\ldots,4$*, are determined by*

$$\mathrm{Im}(xd\bar{x}) = \sum_{k=1}^{4}\gamma^k dx_k\,,$$

form an L^2*-orthogonal basis of* $H(A_r)$*.*

Proof. By a result of Taubes (see [11], Lemma 8.4), we know that $\dim H(A_r) = 8$. To show that $\psi_i^r \in H(A_r)$ for $i=1,\ldots,8$ is straightforward. Since $\langle\psi_k^r,\psi_m^r\rangle$ for $k=1,\ldots,4$ and $m=5,\ldots,8$ are odd functions of x and $\langle d\gamma^k, d\gamma^l\rangle = 0$ for $k\neq l$, the forms ψ_i^r are L^2-orthogonal. □

Clearly, since $h_b:\mathbf{H}\to\mathbf{H}$ is an isometry with respect to $g_{\mathbf{R}^4}$, the set $\{h_b^*\psi_i^r : i=1,\ldots,8\}$ is an L^2-orthogonal basis of $H(A_{r,b})$.

Lemma 3.2. *For* $m=6,7,8$ *let*

$$\varphi_m^r = \frac{1}{2r\,(r+|x|^2)}\left([\gamma^1,\gamma^{m-4}] - a_{m-4}|x|^2\right),$$

where $\{a_2,a_3,a_4\}$ *is the standard basis of* $\mathrm{Im}(\mathbf{H})$*. Then* $\psi_m^r = d_{A_r}\varphi_m^r$*.*

Proof. The lemma is proved by a straightforward calculation. □

Lemma 3.2 states that the forms ψ_m^r, $m=6,7,8$, are fundamental vector fields of the action of the group $\mathcal{G}(Q)$ of gauge transformations. On the other hand, we have

Proposition 3.3. *For* $(r,b)\in\mathbf{R}^+\times\mathbf{H}$ *let* $\mu_{r,b}:\mathbf{H}\to Sp(1)$ *be the gauge transformation defined by*

$$\mu_{r,b} = \exp\left(\frac{\mathrm{Im}(b\bar{x})}{r+|x|^2}\right).$$

Then

$$\begin{aligned}\frac{\partial}{\partial r}\left(A_{r,b}\cdot\mu_{r,b}\right)_{|b=0} &= \frac{d}{dr}A_r = -\psi_5^r\,, \qquad \textit{and}\\ \frac{\partial}{\partial b_k}\left(A_{r,b}\cdot\mu_{r,b}\right)_{|b=0} &= 2r\,\psi_k^r \quad \textit{for} \quad k=1,\dots,4.\end{aligned}$$

Proof. Let $A(v)$ be a smooth curve of connections with $A(0)=A$, and let $\mu(v)$ be a smooth curve of gauge transformations with $\mu(0)=1$. Then

$$\frac{d}{dv}\left(A(v)\cdot\mu(v)\right)_{v=0} = \frac{dA}{dv}(0) + d_A\frac{d\mu}{dv}(0)\,.$$

Using this and $\gamma^k = -\mathrm{Im}(a_k\bar{x})$ for $k=1,\dots,4$ with $a_1=1$, one verifies the assertions. □

Note. For analytical questions we remark that the forms

$$\begin{aligned}\frac{\partial}{\partial b_k}\left(A_{r,b}\cdot\mu_{r,b}-A_{r,b}\right)_{|b=0} &= d_{A_r}\left(\frac{\partial}{\partial b_k}\mu_{r,b}\right)_{|b=0}\\ &= \frac{1}{(r+|x|^2)}\left(2x_k\,\mathrm{Im}(xd\bar{x}) + \left(|x|^2-r\right)d\gamma^k\right)\end{aligned}$$

for $k=1,\dots,4$ are not in L^2. □

Lemma 3.2 and Proposition 3.3 force to identify $T_{[A_{r,b}]}\mathcal{M}_1(Q)$ with the L^2-orthogonal complement $\tilde{H}(A_{r,b})$ of $d_{A_{r,b}}(\Omega^0(\mathbf{H},\mathbf{sp}(1))\cap H(A_{r,b})$ in $H(A_{r,b})$, setting

$$\begin{aligned}\frac{\partial}{\partial r}(r,b) &= -h_b^*\,\psi_5^r\\ \frac{\partial}{\partial b_k}(r,b) &= 2r\,h_b^*\,\psi_k^r \quad \text{for} \quad k=1,\dots,4\end{aligned}$$

with respect to the parametrization Ψ. Now, by means of the identification above, we obtain an L^2-metric on $\mathcal{M}_1(Q)$.

Theorem 3.4. *The L^2-metric on $\mathcal{M}_1(Q)$ described above coincides with the Riemannian metric $\widehat{g}_0$.*

Proof. Since h_b acts isometrically on $\mathbf{H}$, it is enough to verify that

$$\left(\frac{\partial}{\partial r}(r,0),\frac{\partial}{\partial r}(r,0)\right)_{L^2} = \frac{\pi^2}{r}\,, \tag{8}$$

$$\left(\frac{\partial}{\partial r}(r,0),\frac{\partial}{\partial b_k}(r,0)\right)_{L^2} = 0\,, \tag{9}$$

$$\left(\frac{\partial}{\partial b_k}(r,0),\frac{\partial}{\partial b_l}(r,0)\right)_{L^2} = 2\pi^2\,\delta_{kl} \tag{10}$$

for $k, l = 1, \ldots, 4$. We have

$$\begin{aligned}
\left(\frac{\partial}{\partial r}(r,0), \frac{\partial}{\partial r}(r,0)\right)_{L^2} &= (\psi_5^r, \psi_5^r)_{L^2} = \int_{\mathbf{H}} \frac{3|x|^2}{(r+|x|^2)^4}\, dx \\
&= vol(S^3) \int_0^\infty \frac{3v^5}{(r+v^2)^4}\, dv \\
&= \pi^2 \int_0^\infty \frac{3z^2}{(r+z)^4}\, dz = \frac{\pi^2}{r}\,.
\end{aligned}$$

This shows (8). Similarly, one checks (9) and (10). $\square$

4 Some further results

Applying the identification $T_{A_{r,b}}\mathcal{M}_1(Q) = \tilde{H}(A_{r,b})$ described in Sect. 3, we can define a family $\{\widehat{g_0}^s\}_{s\geq 0}$ of Riemannian metrics on $\mathcal{M}_1(Q)$ by

$$(\widehat{g_0}^s)_{[A_{r,b}]}(\eta_1, \eta_2) = ((1 + s\Delta_{A_{r,b}})\eta_1, (1 + s\Delta_{A_{r,b}})\eta_2)_{L^2}\,,$$

where $\Delta_{A_{r,b}} = (d_{A_{r,b}})^* d_{A_{r,b}} + d_{A_{r,b}}(d_{A_{r,b}})^*$ is the Laplacian associated with $A_{r,b}$.

Proposition 4.1. *The Riemannian metrics $\widehat{g_0}^s$, $s \geq 0$, on $\mathcal{M}_1(Q)$ are given by*

$$\Psi^* \,\widehat{g_0}^s = \delta^s(r)\, dr^2 + \rho^s(r)\, g_{\mathbf{R}^4}$$

with

$$\begin{aligned}
\delta^s(r) &= \pi^2 \left(\frac{1}{r} + \frac{8}{5\,r^2}\, s + \frac{64}{35\,r^3}\, s^2\right) \quad \textit{and} \\
\rho^s(r) &= 4\pi^2 \left(\frac{1}{2} + \frac{12}{5\,r}\, s + \frac{32}{7\,r^2}\, s^2\right)\,.
\end{aligned}$$

Proof. We have (see [2])

$$\Delta_A\, \eta = - * [F^A, \eta] \quad \text{for } A \in \mathcal{SD}_1(Q) \text{ and } \eta \in \tilde{H}(A)\,.$$

Further,

$$F^{A_r} = \frac{r}{(r+|x|^2)^2}\, dx \wedge d\bar{x}$$

and

$$
\begin{aligned}
*[dx\wedge d\bar{x},\operatorname{Im}(xd\bar{x})] &= -8\operatorname{Im}(xd\bar{x})\,,\\
*[dx\wedge d\bar{x},d\gamma^k] &= -8\,d\gamma^k \quad\text{for}\quad k=1,\ldots,4
\end{aligned}
$$

imply

$$
*[F^{A_r},\eta] = -\frac{8r}{(r+|x|^2)^2}\,\eta \quad\text{for}\quad \eta\in\tilde{H}(A_r)\,.
$$

Now the remainder of the proof is similar to the proof of Theorem 3.4. $\square$

As an analogue of results in the compact case (see [2], [10]) we have

Corollary 4.2. *The Riemannian manifolds* $(\mathcal{M}_1(Q),\widehat{g_0}^s)$ *with* $s>0$ *are complete.*

Proof. Consider $(\mathcal{M}_1(Q),\widehat{g_0}^s)$ with $s>0$. Then the curves $\{[A_r]\,:\,r\in(0,1)\}$ and $\{[A_r]\,:\,r\in(1,\infty)\}$ in $\mathcal{M}_1(Q)$ have infinite length. Thus, the assertion follows from the fact that a warped product $(\mathbf{R}\times F, dr^2+\kappa(r)\,h)$ is complete if and only if the fibre (F,h) is complete (see [4]). $\square$

Note. One can calculate the sectional curvatures k^s of $(\mathbf{R}^+\times\mathbf{R}^4,\Psi^*\,\widehat{g_0}^s)$ by means of the formulas

$$
\begin{aligned}
k^s(X,Y) &= -\frac{1}{4\,\delta^s(r)\rho^s(r)^2}\left(\frac{d\rho^s}{dr}(r)\right)^2 \qquad\text{and}\\
k^s(X,\frac{\partial}{\partial r}) &= \frac{1}{4\,\delta^s(r)\rho^s(r)^2}\left(\frac{d\rho^s}{dr}(r)\right)^2\\
&+\frac{1}{4\,\delta^s(r)^2\rho^s(r)}\left(\frac{d\delta^s}{dr}(r)\frac{d\rho^s}{dr}(r)-2\delta^s(r)\frac{d^2\rho^s}{dr^2}(r)\right)\,,
\end{aligned}
$$

where X and Y denote tangent vectors to $\mathbf{R}^4$. In particular, one deduces that

$$
\begin{aligned}
k^s(X,Y) &= \frac{-144\,s^2}{25\,\pi^2\,r^3}+\mathrm{O}(s^3) \qquad\text{and}\\
k^s(X,\frac{\partial}{\partial r}) &= \frac{-18\,s}{5\,\pi^2\,r^2}+\mathrm{O}(s^2)\,. \qquad\square
\end{aligned}
$$

We will conclude with the following observation. Let $A\in\mathcal{SD}_1(Q)$. As above, set

$$
H(A)=\{\eta\in\Omega^1(\mathbf{H},\mathbf{sp}(1))\,:\,d_A^*\eta=0\ \text{ and }\ p_-d_A\eta=0\}\cap L^2\,,
$$

and denote by $\tilde{H}(A)$ the L^2-orthogonal complement of $d_A(\Omega^0(\mathbf{H}, \mathbf{sp}(1)) \cap H(A)$ in $H(A)$. Then the distribution

$$A \in \mathcal{SD}_1(Q) \mapsto \tilde{H}(A) \subset T_{[A]}\mathcal{SD}_1(Q)$$

defines a connection ω on the $\mathcal{G}$-principal bundle $\mathcal{SD}_1(Q) \to \mathcal{M}_1(Q)$.

Proposition 4.3. *The connection ω on $\mathcal{SD}_1(Q) \to \mathcal{M}_1(Q)$ is not flat.*

Proof. Assume that ω is flat. Then $\pi_1(\mathbf{R}^+ \times \mathbf{H}) = 0$ and $\frac{d}{dr}A_r \in \tilde{H}(A_r)$ implies that there is a horizontal section

$$\sigma : \mathcal{M}_1(Q) \to \mathcal{SD}_1(Q)\,, \quad \sigma([A_{r,b}]) = Z_{r,b}$$

with $Z_{r,0} = A_r$ for $r \in \mathbf{R}^+$. Therefore, we have

$$\frac{\partial}{\partial b_k}\,(Z_{r,b})_{|b=0} = \frac{\partial}{\partial b_k}(r,0) = 2r\,\psi_k^r$$

for $k = 1, \ldots, 4$. By means of

$$\frac{d}{dv}\,\Big((d_{A(v)})^*\eta(v)\Big) = (d_{A(v)})^*\frac{d\eta}{dv}(v) - \sum_{k=1}^{4}\left[\frac{dA}{dv}(v)\left(\frac{\partial}{\partial x_k}\right), \eta(v)\left(\frac{\partial}{\partial x_k}\right)\right]$$

for a smooth curve $A(v)$ of connections and a smooth curve $\eta(v)$ of 1-forms, we see that

$$\begin{aligned}
&\frac{\partial}{\partial b_k}\left((d_{Z_{r,b}})^*\frac{\partial}{\partial r}Z_{r,b}\right)_{|b=0} \\
= \;& (d_{A_r})^*\frac{\partial}{\partial b_k}\left(\frac{\partial}{\partial r}Z_{r,b}\right)_{|b=0} - \sum_{k=1}^{4}\left[\frac{\partial}{\partial b_k}\,(Z_{r,b})_{|b=0}\left(\frac{\partial}{\partial x_k}\right), \frac{d}{dr}A_r\left(\frac{\partial}{\partial x_k}\right)\right] \\
= \;& \frac{\partial}{\partial r}\left((d_{A_r})^*\left(\frac{\partial}{\partial b_k}Z_{r,b}\right)_{|b=0}\right) \\
& -2\sum_{k=1}^{4}\left[\frac{\partial}{\partial b_k}\,(Z_{r,b})_{|b=0}\left(\frac{\partial}{\partial x_k}\right), \frac{d}{dr}A_r\left(\frac{\partial}{\partial x_k}\right)\right] \\
= \;& 2\sum_{k=1}^{4}\left[\frac{d}{dr}A_r\left(\frac{\partial}{\partial x_k}\right), 2r\,\psi_k^r\left(\frac{\partial}{\partial x_k}\right)\right] \\
= \;& -\frac{4r}{(r+|x|^2)^4}\,\gamma^k
\end{aligned}$$

which contradicts

$$(d_{Z_{t,b}})^* \frac{\partial}{\partial r} Z_{r,b} = 0 \quad \text{for all } (r,b) \in \mathbf{R}^+ \times \mathbf{H}. \qquad \square$$

References

[1] Atiyah, M.F.: Geometry of Yang-Mills fields. Scuola Normale Superiore, Pisa (1979)

[2] Babadshanjan, F., Habermann, L.: A family of metrics on the moduli space of BPST-instantons. Ann. Global Anal. Geom. **9**, 245-252 (1991)

[3] Bérard-Bergery, L.: Sur de nouvelles variétés riemannienes d'Einstein. Publications de l'Institut Elie Cartan, Nancy, **4**, 1-60 (1982)

[4] Bishop, R.L., O'Neill, B.: Manifolds of negative curvature. Trans. Am. Math. Soc. **145**, 1-49 (1969)

[5] Doi, H., Matsumoto, Y., Matumoto, T.: An explicit formula of the metric on the moduli space of BPST-instantons over S^4, A Fête of Topology. New York: Academic Press 1987

[6] Groisser, D.: The geometry of the moduli space of $\mathbf{CP}^2$ instantons. Invent. Math. **99**, 393-409 (1990)

[7] Groisser, D., Parker, T.H.: The Riemannian geometry of the Yang-Mills moduli space. Commun. Math. Phys. **112**, 663-689 (1987)

[8] Groisser, D., Parker, T.H.: The geometry of the Yang-Mills moduli space for definite manifolds. J. Diff. Geom. **29**, 499-544 (1989)

[9] Habermann, L.: On the geometry of the space of $Sp(1)$-instantons with Pontrjagin index 1 on the 4-sphere. Ann. Global Anal. Geom. **6**, 3-29 (1988)

[10] Habermann, L.: A family of metrics on the moduli space of $\mathbf{CP}^2$ instantons, Commun. Math. Phys. (to appear)

[11] Taubes, C.H.: Stability in Yang-Mills theories. Commun. Math. Phys. **91**, 235-263 (1983)

[12] Uhlenbeck, K.: Removable singularities in Yang-Mills fields. Commun. Math. Phys. **83**, 11-29 (1982)

Lutz Habermann
Humboldt-Universität Berlin
PF 1297
Unter den Linden 6
O-1086 Berlin
Germany