On a simple method to compute polygonal minimal surfaces

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Abstract

Given a closed polygonal contour with N+3 vertices in $\mathbb{R}^q$ ($q \geq 2$) there is a one to one correspondence between the zeros of $\nabla\theta$ and minimal surfaces spanned by Γ, where $\theta(\kappa) := \inf_{v \in X(\kappa)} D(v)$ ($\kappa \in T \subset \mathbb{R}^N$) denotes Shiffman's function for the polygon Γ and D denotes Dirichlet's Integral. We derive an explicit expression for $\nabla\theta_h$, where $\theta_h(\kappa) := \inf_{v \in X_h(\kappa)} D(v)$ with suitable finite element spaces $X_h(\kappa)$. Furthermore we assure that each zero of $\nabla\theta_h$ is also approximately a zero of $\nabla\theta$ and vice-versa. Using quasi-newton-methods and variants of Newton's Method, several examples of minimal surfaces are computed.

1 Introduction

The minimal surfaces spanning a closed polygon $\Gamma \subset \mathbb{R}^q$ $(q \geq 2)$ with finitely many corners correspond in a one to one manner to the critical points of Shiffmans function $\theta(\kappa) := \inf_{v \in X(\kappa)} D(v)$ $(\kappa \in T$, T defined through (1)) of finitely many variables. That means the minimal surface problem for a closed polygon with finitely many corners is embedded into a family of quasiminimal surfaces, i.e. solutions of (3) for fixed κ (see [2], [3] and [5]). In an earlier paper ([4]) we have been concerned with the numerical evaluation of θ. There we have approximated the solution of (7) with finite elements.

The present paper deals with the numerical approximation of minimal surfaces bounded by a polygon in the sense of Shiffman. It is mainly based on an earlier paper of the author [4].

The paper is organized as follows. In this chapter we introduce the basic notations and state the variational problems with their underlying partial differential systems. Moreover we cite some important results concerning polygonaly bounded minimal and quasiminimal surfaces. Chapter 2 deals with the discrete treatment of θ. Following Courant, we derive an expression for $\nabla\theta_h$ with the help of inner variations of the unit disc. Furthermore we estimate $\|\nabla\theta_h - \nabla\theta\|_{\infty,T'}$ where $T' \subset\subset T$ (see (1)). In chapter 3 we give some numerical examples.

Let $B = \{w = x_1 + ix_2; |w| < 1\}$ denote the unit disc and let $\Gamma \subset \mathbb{R}^q (q \geq 2)$ be a closed polygon with $N+3$ distinct vertices $a_1, \ldots, a_{N+3}$ arranged in cyclical order. With $\phi_k := \frac{a_{k+1}-a_k}{|a_{k+1}-a_k|}$ $(1 \leq k \leq N+3; a_{N+4} := a_1)$ we assume the pairs ϕ_k, ϕ_{k+1} linearly independent $(1 \leq k \leq N+2)$. Let

$$T := \{\kappa \in \mathbb{R}^N; 0 < \kappa_1 < \ldots < \kappa_N < \pi\} \tag{1}$$

and fix

$$\kappa_{N+i} := \frac{1+i}{2}\pi \ (i = 1,2,3), \ \ \kappa_{N+4} := \kappa_1.$$

Let

$$\gamma_k := \{w = e^{i\xi}; \kappa_k < \xi < \kappa_{k+1}\} \ (1 \leq k \leq N+3). \tag{2}$$

With $\beta_k := e^{i\kappa_k}$ $(1 \leq k \leq N+3)$ we donote the singular points. Consider the vector valued functions

$$X(\kappa) = \{v : B \longrightarrow \mathbb{R}^q; \ v \in H^1(B), \ v(\gamma_k) \subset a_k + \Gamma_k \ (1 \leq k \leq N+3)\}$$

and let $D(v) := \frac{1}{2}\int_B |\nabla v|^2 dx_1 dx_2$ denote Dirichlet's Integral. Here Γ_k stands for the line $t\phi_k$ $(t \in \mathbb{R})$ and $H^1(B)$ denotes the usual Sobolev space of functions having square integrable weak derivatives up to order 1.

For fixed κ the unique solution of

$$\begin{aligned} \Delta u &= 0 && in\ B \\ T_k u &= u && \text{on } \gamma_k \quad (1 \leq k \leq N+3) \\ (\partial_r u\ ,\ \phi_k) &= 0 && \text{on } \gamma_k \quad (1 \leq k \leq N+3) \end{aligned} \tag{3}$$

is called a **quasiminimal surface**. Here T_k denotes the reflection at the line $a_k + \Gamma_k$ and ∂_r stands for the radial derivative. $T_k u = u$ simply means that $u_{|\gamma_k} \subset a_k + \Gamma_k$ $(1 \leq k \leq N+3)$. A harmonic surface with

$$(u_{x_1}\ ,\ u_{x_2}) = 0, \quad |u_{x_1}| = |u_{x_2}| \tag{4}$$

is called a **minimal surface**. Any solution of (3) - (4) is called a minimal surface bounded by Γ in the sense of Shiffman.

Remark 1.1 *Every solution of Plateau's Problem for Γ is also a solution of (3) - (4)* $\square$

In [4] we have shown that the solution of (3) for fixed κ coincides with the unique solution of

$$\theta = \inf_{v \in X(\kappa)} D(v). \tag{5}$$

Compare also [2], [3] and [5]. Therefore, (5) defines a function $\theta : T \longrightarrow \mathbb{R}$. Let $u(\kappa)$ denote the solution of (5). It is a well known fact that $u(\kappa)$ depends analyticaly on κ. Moreover, for fixed κ, $u(\kappa)$ is an analytical function of $\overline{B} \setminus \{\beta_1, \ldots, \beta_{N+3}\}$ to $\mathbb{R}$ (see [2]).
Important is

Theorem 1.2 *The unique solution of (5) is a solution of (3) - (4)* $\Longleftrightarrow$

$$\nabla\theta(\kappa) = 0 \tag{6}$$

$\square$

For a proof of Theorem 1.2 we refer to [5], [3].
We finish this section with two remarks.

Remark 1.3 *In the case where Γ is an extreme polygon with geodesic curvature lower than 4π $(2 \leq q \leq 3)$ or $\frac{10}{3}\pi$ $(q \geq 4)$ there exits a unique κ^* with $\nabla\theta(\kappa^*) = 0$ (see [6])* $\square$

Remark 1.4 *It is not known till yet whether the set of solutions of (5) for an arbitrary closed polygon is finite or not* $\square$

2 The discrete approach

The discrete analogon to (5) is

$$\theta_h = \inf_{v \in X_h(\kappa)} D(v) \tag{7}$$

with a suitable finite element space $X_h(\kappa)$. Here h denotes the gridwidth of a regular triangulation $\tau = \tau(\kappa)$ of the unit disc (for a more detailed description see [4]). With B_h we denote the collection of triangles D of τ and $Y_h := \{v \in C^0(\overline{B})^q;\ v_{|D}\ \textit{linear}\ \forall\ D \in \tau\}$. With this we set

$$X_h(\kappa) := \{v \in Y_h;\ T_k v = v \text{ on } \gamma_k \cap \partial B_h\ (1 \le k \le N+3)\}.$$

For an arbitrary domain Ω, $\|\cdot\|_{1,\Omega}$ denotes the $H^1(B)$ norm, $|\cdot|_{1,p,\Omega}$ denotes the $H^{1,p}(\Omega)$ seminorm $(1 \le p \le \infty)$. We are now prepared to state

Lemma 2.1 *For every $\kappa \in T$ (7) has a unique solution $u_h(\kappa)$. Furthermore, with $u(\kappa)$ from (5) we are able to estimate*

$$\|u - u_h\|_{1,B_h} \le\ c\ h^{\frac{p-2}{p}}\ |u|_{1,p,B} \qquad \forall\ 2 < p < -\frac{2}{p_{min}} \tag{8}$$

with $p_{min} \in (-1,\ 0]$ and a positive constant c depending linearly on N $\square$

For a proof see [4]. From Lemma 2.1 we induce that θ_h is a function from T to $\mathbb{R}$. In a next step we ensure the differential dependence of $\theta_h(\kappa)$ with respect to its arguments. Let $S(\kappa_{k-1}, \kappa_k) := \{re^{i\xi};\ 0 \le r \le 1;\ \kappa_{k-1} < \xi < \kappa_k\}$ and $S_h(\kappa_{k-1}, \kappa_k) := S(\kappa_{k-1}, \kappa_k)\ \cap \overline{B}_h$.

Theorem 2.2 *$\theta_h \in C^\infty(T)$. For the partial derivatives with respect to κ_k we have*

$$\begin{aligned}\theta_{h_{\kappa_k}} = \frac{1}{2}\Big(&\frac{1}{\kappa_k - \kappa_{k-1}} \int\limits_{S_h(\kappa_{k-1},\kappa_k)} (u_{hx_1}^2 - u_{hx_2}^2)\frac{x_1^2 - x_2^2}{x_1^2 + x_2^2} + 4u_{hx_1}u_{hx_2}\frac{x_1 x_2}{x_1^2 + x_2^2} dx_1 dx_2 - \\ &\frac{1}{\kappa_{k+1} - \kappa_k} \int\limits_{S_h(\kappa_{k+1},\kappa_k)} (u_{hx_1}^2 - u_{hx_2}^2)\frac{x_1^2 - x_2^2}{x_1^2 + x_2^2} + 4u_{hx_1}u_{hx_2}\frac{x_1 x_2}{x_1^2 + x_2^2} dx_1 dx_2\Big)\end{aligned} \tag{9}$$

Proof : For fixed κ, every function $v_h \in X_h(\kappa)$ has a representation

$$v_h = \sum_{j=1}^{NV} v_h^j b_j,$$

where $\{b_1, \ldots, b_{NV}\}$ are usual continous, piecewise linear shape functions corresponding to the triangulation $\tau(\kappa)$ (note that every triangulation depends on κ because the singular points $\{\beta_1, \ldots, \beta_{N+3}\}$ have to be vertices of the triangulation),

v_h^j $(1 \leq j \leq NV)$ are the corresponding nodal vectors and NV denotes the number of nodes of τ (see [4]). In this context (7) is a minimization problem and the unique solution u_h of (7) corresponds to the unique solution $\overline{u}_h$ of a linear system, i. e.

$$A\overline{u}_h = a, \tag{10}$$

where the stiffness matrix A and the right hand side depend on κ and on the geometry of the problem. Cramers Rule now yields $\overline{u}_h = \overline{u}_h(\kappa) \in C^\infty(T)$ $\forall h > 0$ and therefore $u_h \in C^\infty(T)$.
To obtain (9) we procced as Courant in his famous investigations on unstable polygonal minimal surfaces [1], chap. 6. Our notation now takes pattern from Struwe [7]. Let $\phi \in C^{0,1}(\overline{B}, \mathbb{R}^2)$. For $\epsilon \in \mathbb{R}$ with

$$|\epsilon| \; \|\nabla \phi\|_{\infty, B} \; < 1$$

consider the maps

$$g_\epsilon := id + \epsilon\phi : B \longrightarrow g_\epsilon(B) =: B_\epsilon \cong B.$$

Then g_ϵ is a homeomorphism $\overline{B} \longrightarrow \overline{B}_\epsilon$. Given κ, κ_ϵ is obtained as $e^{i\kappa_{\epsilon_k}} := g_\epsilon(e^{i\kappa_k})$ $(1 \leq k \leq N+3)$. Thus we have

$$\begin{aligned}
\theta_h(\kappa_\epsilon) &= D_\epsilon(u_h(\kappa_\epsilon) := \frac{1}{2}\int_{B_\epsilon} |\nabla u_h(\kappa_\epsilon)|^2 dx_1 dx_2 \\
&\leq D_\epsilon(u_h(\kappa) \circ g_\epsilon^{-1}) \\
&= D(u_h(\kappa)) - \frac{1}{2}\epsilon \int_B (u_{hx_1}^2 - u_{hx_2}^2)(\phi_{x_1}^{(1)} - \phi_{x_2}^{(2)}) + \\
&\quad +2u_{h_{x_1}} u_{h_{x_2}} (\phi_{x_1}^{(2)} + \phi_{x_2}^{(1)}) dx_1 dx_2 + O(\epsilon^2)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{\epsilon \to 0} \frac{\theta_h(\kappa_\epsilon) - \theta_h(\kappa)}{\epsilon} &= -\frac{1}{2}\int_B (u_{hx_1}^2 - u_{hx_2}^2)(\phi_{x_1}^{(1)} - \phi_{x_2}^{(2)}) + \\
&\quad +2u_{h_{x_1}} u_{h_{x_2}} (\phi_{x_1}^{(2)} + \phi_{x_2}^{(1)}) dx_1 dx_2 \\
&= \frac{d}{d\epsilon} D_\epsilon(u_h(\kappa) \circ g_\epsilon^{-1})_{|\epsilon=0}.
\end{aligned}$$

We now choose a piecewise linear, continous ϕ which only affects $S(\kappa_{k-1}, \kappa_{k+1})$ and acts on the angle to obtain (9) (see [9], p. 66 and [1], chap. 6) □

Remark 2.3 *(9) is also valid if we replace θ_h by θ, u_h by u and S_h by S* □

We are now prepared to state the main Theorem of this paper. Let $\|\cdot\|_{1,\infty,\Omega}$ denote the $H^{1,\infty}(\Omega)$ norm.

Theorem 2.4 *For every compact subset $T' \subset\subset T$ we have*

$$\|\theta_h - \theta\|_{1,\infty,T'} \leq \qquad\qquad (11)$$
$$c(T') \quad h^{\frac{p-2}{p}} \qquad \sup_{\kappa \in T'} |u(\kappa)|_{1,p,B} \quad \forall\ 2 < p < -\frac{2}{p_{min}}$$

with a positive constant $c(T')$ and p_{min} from Lemma 2.1.

The proof is an immediate consequence of Lemma 2.1 together with Theorem 2.2 and the continous dependence of u with respect to κ under consideration of Remark 2.3.
Theorem 2.4 yields that every zero of $\nabla\theta_h$ is approximately a zero of $\nabla\theta$ and vice-versa. To compute a minimal surface bounded by Γ we have to seek for a solution κ^* of

$$\nabla\theta_h(\kappa) = 0. \qquad (12)$$

We can do this using Newton- and Quasi-Newton methods. The major disadvantage when using Newton's Method or variants of Newton's Method is that we have to involve the second derivative of θ. Using the Simplified Newton Method we only have to evaluate θ_h $O(N)$ times for a difference approximation of $D^2\theta$ based on the expression (9) for $\nabla\theta_h$ (note that every evaluation of θ_h amounts to solve a linear system of the form (10)).

3 Examples

At last we give four examples of computed polygonal minimal surfaces. In Figures 1, 2 we are concerned with Courants Example. The triangulation and the corresponding surface are shown. The contour has 18 vertices and a hight of 1.5 cm (compare Wagner's computations in [8]). Dirichlet's Integral has a value of 7.12313.
The second example shows a cloverknot-like contour with 18 vertices. Dirichlet's Integral has a value of 6.42447 (see Figures 3, 4).
Figures 5 and 6 show a triangulation and a surface in an Enneper-like contour with 12 vertices. Dirichlet's Integral has a value of 9.03082.
The last example is a cloverleaf-like contour in $\mathbb{R}^2$ with 20 vertices (r=0.5, Figures 7, 8). The energy of the mapping is 3.81654.

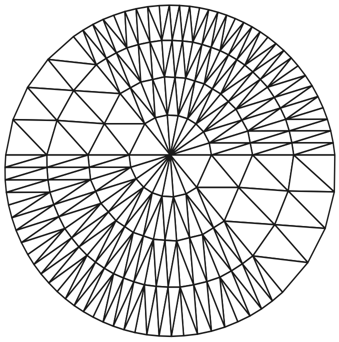

Figure 1:

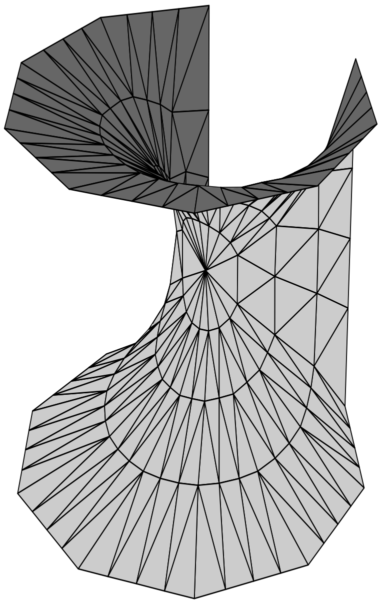

Figure 2:

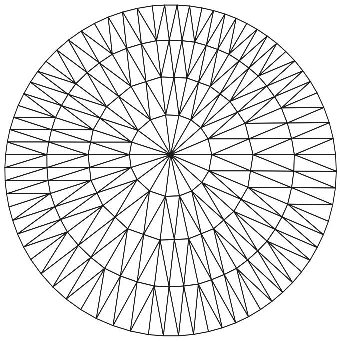

Figure 3:

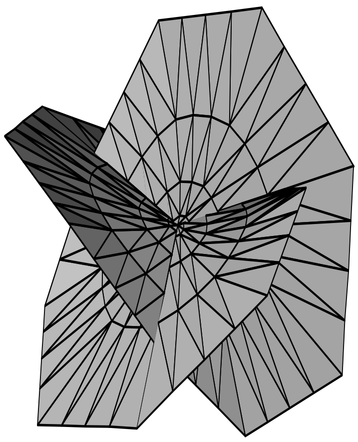

Figure 4:

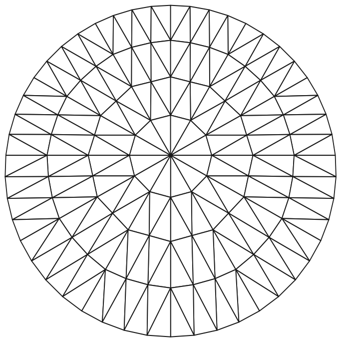

Figure 5:

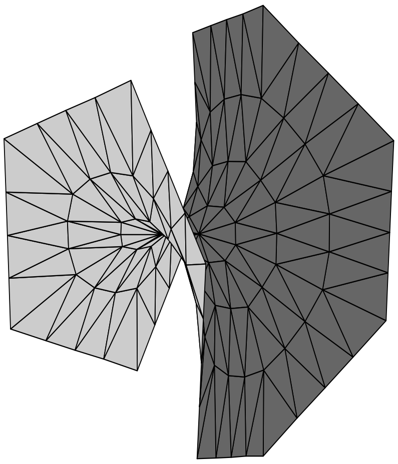

Figure 6:

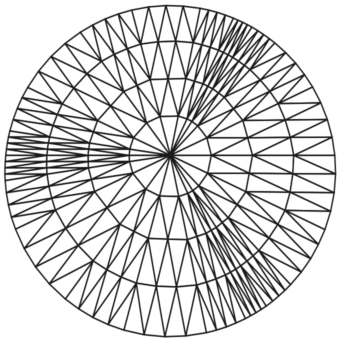

Figure 7:

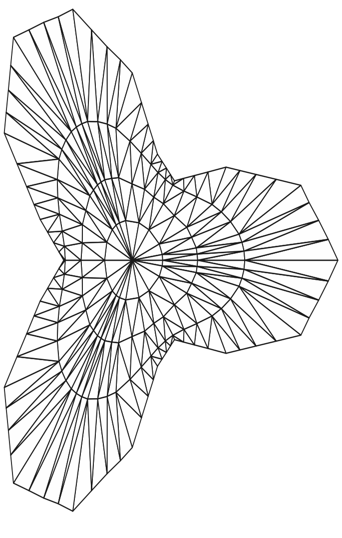

Figure 8:

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