# On a Ferromagnetic Spin Chain. Part II: Thermodynamic Limit.

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**Abstract**

We prove the existence of the thermodynamic limit of the free energy for the ferromagnetic spin chain connected with the Riemann zeta function.

The basic problem of equilibrium statistical mechanics is the analysis of phase transitions. In turn this leads to the question of the localization of the zeroes of the partition function.

Amongst the many techniques developed to that purpose, the Lee-Yang circle theorem is particularly interesting. In its basic form it states that all partition function zeroes of a ferromagnetic Ising model in the complex plane have unit modulus. Another version, due to Ruelle, predicts a zero-free half plane of inverse temperatures.

The proof of that remarkable theorem has been inspired by a paper of Pólya on the Riemann zeta function $\zeta(s)$ (see the comments of M. Kac in Pólya [6], pp. 424–426). Afterwards, Newman [5], Ruelle [8] and others asked a reverse question, namely, whether the Riemann hypothesis on the location of the nontrivial zeroes of $\zeta(s)$ could be proved by some version of the Lee-Yang circle theorem.

In [2] we showed that there is indeed a connection between $\zeta(s)$ and classical statistical mechanics. Namely, the quotient

$$Z(s) := \zeta(s-1)/\zeta(s)$$

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can be interpreted as the partition function of an infinite ferromagnetic spin chain with long-range multi-body interaction (but the Lee-Yang theorem cannot be applied because of the multi-body type of the interaction).

This interpretation is only literally true for inverse temperatures $s$ with $\mathrm{Re}(s) > 2$, where the chain is in a frozen state, the mean magnetization being equal to one. For smaller real parts, both the Dirichlet series and the Euler product representations

$$Z(s) = \sum_{n=1}^{\infty} \varphi(n) \cdot n^{-s} = \prod_{p \text{ prime}} \frac{1-p^{-s}}{1-p^{1-s}}$$

of $Z(s)$ diverge ($\varphi(n)$ being the Euler totient).

Number-theoretically, this divergence is a consequence of the prime number theorem which says that the mean density of prime numbers decays only logarithmically. In the statistical mechanics interpretation, one observes a occurrence of a phase transition, see [3].

As is well-known, such a phase transition can only occur for a spin chain if the interaction is long-ranged. But that long-range interaction leads in the low temperature regime to the above-mentioned frozen state and to the unusual fact that there the partition function of the infinite chain is finite.

From the point of view of physics it is now natural to consider the free energy per spin, since that quantity should be finite in the full temperature range. This is done in the present paper.

The mere existence of the thermodynamic limit of the free energy is of mathematical interest since it shows that $Z(s)$ has some self-similar properties. One natural question is now whether one can use these properties to approximate $Z(s)$ in its 'critical strip' $1 \leq \mathrm{Re}(s) \leq 2$.

It is clear that the pole at $s = 2$ is an obstacle to this aim. One method is to remove that pole by considering a function $\tilde{Z}(s)$ which is equivalent to $Z(s)$ in the sense of Bohr (see, e.g., Apostol [1]) but has no such pole. Numerically, such an approximation based on statistical mechanics seems to converge for $\mathrm{Re}(s) > 1.5$, but a mathematical demonstration of that phenomenon is missing, see [4].

In [2] we considered a classical spin chain of length $k \in \mathbf{N}_0$ whose canonical energy function $\mathbf{H}_k^C : \mathbf{G}_k \to \mathbf{R}_0^+$ on the additive group $\mathbf{G}_k := (\{0,1\})^k$ was defined as follows. We set $\mathbf{H}_k^C := \ln \mathbf{h}_k^C$, the coefficients $\mathbf{h}_k^C(\sigma)$, $\sigma =$

$(\sigma_1, \ldots, \sigma_k) \in \mathbf{G}_k$, being defined inductively by setting $\mathbf{h}_0^C(0) := 1$,

$$\mathbf{h}_{k+1}^C(\sigma, 0) := \mathbf{h}_k^C(\sigma) \text{ and } \mathbf{h}_{k+1}^C(\sigma, 1) := \mathbf{h}_k^C(\sigma) + \mathbf{h}_k^C(1-\sigma),$$

with $1 - \sigma \equiv (1 - \sigma_1, \ldots, 1 - \sigma_k)$.

Then on any half-plane of the form $\mathrm{Re}(s) > 2 + \varepsilon$, $\varepsilon > 0$, the partition functions

$$Z_k(s) := \sum_{\sigma \in \mathbf{G}_k} \exp\left(-s \cdot \mathbf{H}_k^C(\sigma)\right)$$

of the spin chains of lengths $k$ converge uniformly to the quotient

$$Z(s) = \zeta(s-1)/\zeta(s)$$

of Riemann zeta functions.

The free energy (per spin) of the spin chain of length $k$ equals

$$F_k(\beta) := -\frac{1}{\beta \cdot k} \ln\left(Z_k(\beta)\right).$$

In this paper we show that the thermodynamic limit

$$F(\beta) := \lim_{k \to \infty} F_k(\beta)$$

of the free energy exists for all $\beta > 0$.

The canonical interaction coefficients $j_k^C(t)$ appearing in the Fourier decomposition

$$\mathbf{H}_k^C = -2^{-k} \sum_{t \in \mathbf{G}_k^*} j_k^C(t) \cdot \chi_t$$

(with $\chi_t(\sigma) = (-1)^{t \cdot \sigma}$) of the canonical energy function are neither translation invariant nor in any sense independent of $k$.

So the standard theorems on the existence of the thermodynamic limit (like the ones in Ruelle's book [7]) do not apply in our situation.

In fact, rather precise estimates will be necessary in order to prove existence of the limit.

For example, we showed in Prop. 7.3 of [2] that the interaction coefficients are translation invariant up to variations which are exponentially small in the separation of the spins involved from the edges of the chain.



Such an exponential estimate will not suffice here, and we shall need an additional polynomial decay (not for the variation of the individual coefficients but in the mean).

Such estimates are necessary since the interaction itself is quite long-ranged (otherwise the spin chain would not exhibit a phase transition). As demonstrated in [3], the *non*-invariance of the interaction plays an important rôle in the description of the phases and is just a bit too weak to endanger the existence of the thermodynamic limit.

It is clear that in the low temperature region $\beta > 2$ the limit free energy $F(\beta)$ exists and equals zero since there the partition functions $Z_k(\beta)$ converge to the quotient $Z(\beta) = \zeta(\beta - 1)/\zeta(\beta)$. On the other hand, we know from [3] that at least for $\beta < \ln(2)/\ln(3/2)$ the free energy $F(\beta)$ is strictly negative. In a forthcoming paper we intend to analyze the location and nature of the phase transition.

In order to simplify notation we use the following conventions: Sums over empty sets equal zero, whereas products over empty sets equal one. For $k \in \mathbf{N}_0$ the symbol $0_k$ denotes the neutral element in $\mathbf{G}_k$ or $\mathbf{G}_k^*$, respectively.

We begin as usual by comparing the free energies of spin chains of lengths $k$ and $s \cdot k$ for $k, s \in \mathbf{N}$.

$$\begin{aligned}
&\beta \cdot |F_{sk}(\beta) - F_k(\beta)| = \\
&= \left| \frac{1}{sk} \ln \left[ \sum_{\sigma \in \mathbf{G}_{sk}} \exp(\beta \sum_{t \in \mathbf{G}_{sk}^*} j_{sk}^C(t) \chi_t(\sigma)) \right] - \frac{1}{k} \ln \left[ \sum_{\sigma \in \mathbf{G}_k} \exp(\beta \sum_{t \in \mathbf{G}_k^*} j_k^C(t) \chi_t(\sigma)) \right] \right| \\
&= \frac{1}{sk} \left| \ln \left[ \frac{\sum_{\sigma^1, \ldots, \sigma^s \in \mathbf{G}_k} \exp\left(\beta \sum_{t^1, \ldots, t^s \in \mathbf{G}_k^*} j_{sk}^C(t^1, \ldots, t^s) \prod_{m=1}^s \chi_{t^m}(\sigma^m)\right)}{\sum_{\sigma^1, \ldots, \sigma^s \in \mathbf{G}_k} \exp\left(\beta \sum_{t \in \mathbf{G}_k^*} j_k^C(t) \cdot \sum_{m=1}^s \chi_t(\sigma^m)\right)} \right] \right| \\
&\leq \frac{\beta}{sk} \sup_{\sigma^1, \ldots, \sigma^s \in \mathbf{G}_k} \left| \sum_{t^1, \ldots, t^s \in \mathbf{G}_k^*} j_{sk}^C(t^1, \ldots, t^s) \prod_{m=1}^s \chi_{t^m}(\sigma^m) - \sum_{t \in \mathbf{G}_k^*} j_k^C(t) \cdot \sum_{m=1}^s \chi_t(\sigma^m) \right| \\
&= \frac{\beta}{sk} \sup_{\sigma^1, \ldots, \sigma^s \in \mathbf{G}_k} \Big| j_{sk}^C(0) - s \cdot j_k^C(0) + \\
&\qquad + \sum_{m=1}^s \sum_{t \in \mathbf{G}_k^* \setminus \{0\}} \left( j_{sk}^C(0_{(m-1)k}, t, 0_{(s-m)k}) - j_k^C(t) \right) \chi_t(\sigma^m) \\
&\qquad + \sum_{1 \leq L < R \leq s} \sum_{t^L, t^R \in \mathbf{G}_k^* \setminus \{0\}} \sum_{t^{L+1}, \ldots, t^{R-1} \in \mathbf{G}_k^*}
\end{aligned}$$

$$
\begin{aligned}
& \left| j^C_{sk}\left(0_{(L-1)k}, t^L, \ldots, t^R, 0_{(s-R)k}\right) \prod_{m=L}^{R} \chi_{t^m}(\sigma^m) \right| \\
\leq\ & \frac{\beta}{s \cdot k}(C_1 + C_2 + C_3)
\end{aligned}
\tag{1}
$$

with $C_1 := \left| j^C_{sk}(0) - s \cdot j^C_k(0) \right|$,

$$
C_2 := \sup_{\sigma^1, \ldots, \sigma^s \in \mathbf{G}_k} \left| \sum_{m=1}^{s} \sum_{t \in \mathbf{G}^*_k \setminus \{0\}} \left( j^C_{sk}(0_{(m-1)k}, t, 0_{(s-m)k}) - j^C_k(t) \right) \chi_t(\sigma^m) \right|
$$

and

$$
C_3 := \sum_{1 \leq L < R \leq s} \sum_{t^L, t^R \in \mathbf{G}^*_k \setminus \{0\}} \sum_{t^{L+1}, \ldots, t^{R-1} \in \mathbf{G}^*_k} j^C_{sk}\left(0_{(L-1)k}, t^L, \ldots, t^R, 0_{(s-R)k}\right).
$$

In the last inequality in (1) we used the weak ferromagnetic property $j^C_l(t) \geq 0$ for $t \in \mathbf{G}_l \setminus \{0\}$ (Prop. 5.7 of [2]).

Roughly speaking, $C_3$ measures the long-range part of the interaction, whereas $C_2$ describes the lack of translation invariance. We first estimate $C_1$ which is small since the mean value $j^C_k(0)$ of the canonical energy function grows asymptotically linearly in $k$:

**Lemma 1** *For $n \in \mathbf{N}$*

$$
|j^C_{n+1}(0) - 2j^C_n(0) + j^C_{n-1}(0)| \leq 2^{-(n+1)} \ln 2.
$$

**Proof.** We remind the reader that besides the group $\mathbf{G}_k = (\mathbf{Z}_2)^k$ of spin configurations we also used the group $G_k = \mathbf{Z}_{2^k}$ whose elements are represented by the integers $0, \ldots, 2^k - 1$. Using that notation, we have

$$
\begin{aligned}
& j^C_{n+1}(0) - 2j^C_n(0) + j^C_{n-1}(0) = \\
& = -2^{-(n+1)} \sum_{a \in \{0,\ldots,2^{n+1}-1\}} H^C_{n+1}(a) + \\
& \quad +2^{-n+1} \sum_{a \in \{0,\ldots,2^n-1\}} H^C_n(a) - 2^{-n+1} \sum_{a \in \{0,\ldots,2^{n-1}-1\}} H^C_{n-1}(a) \\
& = -2^{-(n+1)} H^C_{n+1}(2^n) - 2^{-n} \sum_{a \in \{0,\ldots,2^n-1\}} H^C_{n+1}(a) \\
& \quad +2^{-n} H^C_n(2^{n-1}) + 2^{-n} \sum_{a \in \{0,\ldots,2^n-1\}} H^C_n(a) + 2^{-n+1} \sum_{a \in \{0,\ldots,2^{n-1}-1\}} H^C_n(a) \\
& \quad -2^{-n+1} \sum_{a \in \{0,\ldots,2^{n-1}-1\}} H^C_{n-1}(a) \\
& = 2^{-(n+1)} \ln 2 - 2^{-n} \sum_{a \in \{0,\ldots,2^n-1\}} (H^C_{n+1}(a) - H^C_{n+1}(2a)) \\
& \quad +2^{-n+1} \sum_{a \in \{0,\ldots,2^{n-1}-1\}} (H^C_{n+1}(2a) - H^C_{n+1}(4a)) \\
& = 2^{-(n+1)} \ln 2 + 2^{-n} \sum_{a \in \{0,\ldots,2^{n-1}-1\}} \Big[ (H^C_{n+1}(2a) - H^C_{n+1}(4a)) \\
& \qquad - (H^C_{n+1}(2a+1) - H^C_{n+1}(2(2a+1))) \Big].
\end{aligned}
$$

These equations follow from the identities $H^C_l(-b) = H^C_l(b)$, $H^C_l(0) = 0$, $H^C_{l+1}(2^l) = \ln 2$ and $H^C_l(b) = H^C_{l+1}(2b)$.

We know from the proof of Lemma 6.3 of [2] that $H^C_l(b) - H^C_l(2b)$ is monotone increasing w.r.t. $b \in \{0, \ldots, 2^{l-1}\}$ and takes values between 0 and $\ln 2$. So

$$
0 \geq \sum_{a \in \{0,\ldots,2^{n-1}-1\}} [(H^C_{n+1}(2a) - H^C_{n+1}(4a)) - (H^C_{n+1}(2a+1) - H^C_{n+1}(4a+2))] \geq -\ln 2
$$

which proves the lemma. □

**Lemma 2** *For $k, s \in \mathbf{N}$*

$$
C_1 \leq 2(s-1) \cdot \ln 2.
$$

**Proof.** We assume $s \geq 2$ since $C_1 = 0$ for $s = 1$. Since $j^C_0(0) = 0$, we have

$$
\begin{aligned}
j^C_{sk}(0) &= \sum_{m=0}^{k-1} \sum_{n=0}^{s-1} \left( j^C_{(s-n)k-m}(0) - j^C_{(s-n)k-m-1}(0) \right) \\
&= \sum_{m=0}^{k-1} \Big[ s \cdot \left( j^C_{k-m}(0) - j^C_{k-m-1}(0) \right) + \\
&\qquad \sum_{n=1}^{s-1} n \cdot \Big( (j^C_{(s-n+1)k-m}(0) - j^C_{(s-n+1)k-m-1}(0)) - \\
&\qquad\qquad (j^C_{(s-n)k-m}(0) - j^C_{(s-n)k-m-1}(0)) \Big) \quad \Big]
\end{aligned}
$$

so that

$$
\begin{aligned}
&j^C_{sk}(0) - s \cdot j^C_k(0)\\
&\quad = \sum_{m=0}^{k-1}\sum_{n=1}^{s-1} n \cdot \Big( (j^C_{(s-n+1)k-m}(0) - j^C_{(s-n+1)k-m-1}(0)) - \\
&\qquad\qquad (j^C_{(s-n)k-m}(0) - j^C_{(s-n)k-m-1}(0)) \Big)\\
&\quad = \sum_{m=0}^{k-1}\sum_{n=1}^{s-1} n \left( \sum_{r=1}^{k} (j^C_{(s-n)k-m+r}(0) - 2j^C_{(s-n)k-m+r-1}(0) + j^C_{(s-n)k-m+r-2}(0)) \right).
\end{aligned}
$$

But then by Lemma 1

$$
\begin{aligned}
C_1 &\leq \sum_{m=0}^{k-1}\sum_{n=1}^{s-1} n \cdot \sum_{r=1}^{k} 2^{-((s-n)k-m+r)} \ln 2\\
&< \ln 2 \cdot \sum_{n=1}^{s-1} n \cdot 2^{(n-s+1)k}\\
&< 2(s-1)\ln 2. \qquad \square
\end{aligned}
$$

We shall estimate $C_2$, controlling the signs of the different contributions with the following proposition.

**Proposition 3** *For* $n \in \mathbf{N}$ *and* $t \in \mathbf{G}^*_n \setminus \{0\}$

$$j^C_{n+1}(t, 0) \leq j^C_n(t) \leq j^C_{n+1}(0, t).$$

**Proof.** The first inequality is already stated in Prop. 5.7 of [2].

For $t = (t_1, \ldots, t_n)$ with $t_n = 1$ the second equality appears in Corollary 5.2 of [2].

If $t_n = 0$ we apply the identities

$$j^C_n(t) = j^C_n(t_1, \ldots, t_{n-1}, 1) + j^G_{n-1}(t_1, \ldots, t_{n-1})$$

and

$$j^C_{n+1}(0, t) = j^C_{n+1}(0, t_1, \ldots, t_{n-1}, 1) + j^G_n(0, t_1, \ldots, t_{n-1})$$

following from Lemma 4.2 of [2].

We just remarked that $j^C_{n+1}(0, t_1, \ldots, t_{n-1}, 1) \geq j^C_n(t_1, \ldots, t_{n-1}, 1)$. On the other hand, if $\#_{n-1}(t_1, \ldots, t_{n-1})$ is odd, then by Prop. 4.3 of [2]

$$j^G_n(0, t_1, \ldots, t_{n-1}) = j^G_{n-1}(t_1, \ldots, t_{n-1}) = 0.$$



If $\#_{n-1}(t_1, \ldots, t_{n-1})$ is even, then by Prop. 5.7 of [2]

$$j^G_n(0, t_1, \ldots, t_{n-1}) \geq j^G_{n-1}(t_1, \ldots, t_{n-1})$$

(observe that in that case $(t_1, \ldots, t_n) \neq (0, \ldots, 0)$ since by assumption $t \in \mathbf{G}^*_n \setminus \{0\}$ and $t_n = 0$). $\square$

**Lemma 4** *For* $k, s \in \mathbf{N}$

$$C_2 \leq s \cdot (5 + \ln k).$$

**Proof.**

$$
\begin{aligned}
C_2 &\leq \sum_{m=1}^{s} \sum_{t \in \mathbf{G}^*_k \setminus \{0\}} |j^C_{sk}(0_{(m-1)k}, t, 0_{(s-m)k}) - j^C_k(t)|\\
&\leq \tfrac{1}{2} \textstyle\sum_{m=1}^{s} \sum_{t \in \mathbf{G}^*_k \setminus \{0\}} \Big[ |j^C_{sk}(0_{(m-1)k}, t, 0_{(s-m)k}) - j^C_{mk}(0_{(m-1)k}, t)| + \\
&\qquad |j^C_{mk}(0_{(m-1)k}, t) - j^C_k(t)| + \\
&\qquad |j^C_{sk}(0_{(m-1)k}, t, 0_{(s-m)k}) - j^C_{(s-m+1)k}(t, 0_{(s-m)k})| + \\
&\qquad |j^C_{(s-m+1)k}(t, 0_{(s-m)k}) - j^C_k(t)| \Big]\\
&= \tfrac{1}{2} \textstyle\sum_{m=1}^{s} \sum_{t \in \mathbf{G}^*_k \setminus \{0\}} \Big[ j^C_{mk}(0_{(m-1)k}, t) - j^C_{sk}(0_{(m-1)k}, t, 0_{(s-m)k}) + \\
&\qquad j^C_{mk}(0_{(m-1)k}, t) - j^C_k(t) + \\
&\qquad j^C_{sk}(0_{(m-1)k}, t, 0_{(s-m)k}) - j^C_{(s-m+1)k}(t, 0_{(s-m)k}) + \\
&\qquad j^C_k(t) - j^C_{(s-m+1)k}(t, 0_{(s-m)k}) \Big],
\end{aligned}
$$

using Prop. 3. Thus

$$
\begin{aligned}
C_2 &\leq \sum_{m=1}^{s} \sum_{t \in \mathbf{G}^*_k \setminus \{0\}} \left( j^C_{mk}(0_{(m-1)k}, t) - j^C_{(s-m+1)k}(t, 0_{(s-m)k}) \right)\\
&= \sum_{m=1}^{s} \sum_{t \in \mathbf{G}^*_k \setminus \{0\}} \Big( 2j^C_{mk+1}(0_{(m-1)k}, t, 1) - j^G_{mk}(0_{(m-1)k}, t)\\
&\qquad - (2j^C_{(s-m+1)k+1}(t, 0_{(s-m)k}, 1) - j^G_{(s-m+1)k}(t, 0_{(s-m)k})) \Big)\\
&= 2 \sum_{m=1}^{s} \sum_{t \in \mathbf{G}^*_k \setminus \{0\}} \left( j^C_{mk+1}(0_{(m-1)k}, t, 1) - j^C_{mk+1}(0_{(m-1)k}, t, 1) \right)\\
&\leq 2 \sum_{m=1}^{s} \sum_{t \in \mathbf{G}^*_k \setminus \{0\}} j^C_{mk+1}(0_{(m-1)k}, t, 1). \qquad (2)
\end{aligned}
$$

To derive the first equality in (2) we used the identity $j_l^C(s) = 2j_{l+1}^C(s,1) + j_l^G(s)$ (Lemma 4.2 of [2]). The second equality follows from the inversion symmetry $j_l^G(s_l,\ldots,s_1) = j_l^G(s_1,\ldots,s_l)$ of the grand canonical ensemble (Prop. 4.9 of [2]), whereas the last inequality follows from the weak ferromagnetic property $j_l^C(s) \geq 0$ for $s \neq 0$.

Now

$$\begin{aligned}
&\sum_{t\in \mathbf{G}_k^*\setminus\{0\}} j_{mk+1}^C(0_{(m-1)k},t,1)\\
&= \sum_{l=(m-1)k+1}^{mk} \sum_{t\in \mathbf{G}_{mk-l}^*} j_{mk+1}^C(0_{l-1},1,t,1)\\
&= \sum_{l=(m-1)k+1}^{mk} A_{mk+1}^C(l,mk+1)\\
&\leq \sum_{l=(m-1)k+1}^{mk} \left(\frac{1}{(mk+1-l)^2} + \tfrac{1}{2}\frac{1}{mk+1-l}\right)\\
&= \sum_{l=1}^{k}\left(\frac{1}{l^2}+\frac{1}{2l}\right) < \frac{\pi^2}{6} + \tfrac{1}{2}(1+\ln k)\\
&< \tfrac{1}{2}(5+\ln k),
\end{aligned} \tag{3}$$

using Prop. 8.4 of [2].

By inserting (3) in (2) we prove the lemma. □.

**Lemma 5** *For $k, s \in \mathbf{N}$*

$$C_3 \leq s\cdot(3+2\ln k).$$

**Proof.**

$$\begin{aligned}
C_3 &= \sum_{1\leq L<R\leq s}\ \sum_{l=(L-1)k+1}^{Lk}\ \sum_{r=(R-1)k+1}^{Rk}\ \sum_{t\in \mathbf{G}_{r-l-1}^*} j_{sk}^C(0_{l-1},1,t,1,0_{sk-r})\\
&= \sum_{1\leq L<R\leq s}\ \sum_{l=(L-1)k+1}^{Lk}\ \sum_{r=(R-1)k+1}^{Rk} A_{sk}^C(l,r)\\
&\leq \sum_{1\leq L<R\leq s}\ \sum_{l=(L-1)k+1}^{Lk}\ \sum_{r=(R-1)k+1}^{Rk} \left(\left(\frac{1}{r-l}\right)^2 + \frac{2^{-(sk-r+1)}}{r-l}\right).
\end{aligned} \tag{4}$$



The first term in (4) is estimated as follows:

$$\begin{aligned}
&\sum_{1\leq L<R\leq s}\ \sum_{l=(L-1)k+1}^{Lk}\ \sum_{r=(R-1)k+1}^{Rk}\left(\frac{1}{r-l}\right)^2\\
&\leq \sum_{1\leq L<s}\ \sum_{l=(L-1)k+1}^{Lk}\ \sum_{r=Lk+1}^{\infty}\left(\frac{1}{r-l}\right)^2\\
&= (s-1)\sum_{l=1}^{k}\sum_{r=k+1}^{\infty}\left(\frac{1}{r-l}\right)^2 \leq (s-1)\sum_{n=1}^{k}\left(\frac{1}{n^2}+\frac{1}{n}\right)\\
&\leq (s-1)\cdot\left(\frac{\pi^2}{6}+1+\ln k\right)
\end{aligned} \tag{5}$$

For the second term in (4) we find

$$\begin{aligned}
&\sum_{1\leq L<R\leq s}\ \sum_{l=(L-1)k+1}^{Lk}\ \sum_{r=(R-1)k+1}^{Rk}\frac{2^{-(sk-r+1)}}{r-l}\\
&\leq \sum_{1\leq l<r\leq k\cdot s}\frac{2^{-(sk-r+1)}}{r-l} \leq \sum_{1<r\leq k\cdot s} 2^{-(sk-r+1)}\cdot(1+\ln r)\\
&\leq 1+\ln(sk) \leq 1+(s-1)\ln k,
\end{aligned} \tag{6}$$

since $\ln s \leq s-1$.

Adding (5) and (6), we obtain the result. □

**Proposition 6** *For inverse temperature $\beta > 0$ the thermodynamic limit*

$$F(\beta) = \lim_{k\to\infty} F_k(\beta)$$

*exists and is continuous in $\beta$. $\beta\cdot F(\beta)$ is a concave function. $F(\beta)\leq 0$,*

$$F(\beta) = 0 \quad \textit{for} \quad \beta \geq 2,$$

*and*

$$-\frac{\ln 2}{\beta} + \tfrac{1}{2}\ln 2 \leq F(\beta) \leq -\tfrac{\ln 2}{\beta} + \ln(3/2) \qquad \textit{for } 0<\beta\leq 2.$$

**Proof.** Adding the estimates from the Lemmata 2, 4 and 5, we obtain

$$|F_{sk}(\beta) - F_k(\beta)| \leq \frac{10+3\ln k}{k}$$

so that for $k_1, k_2 \in \mathbf{N}$

$$|F_{k_1}(\beta) - F_{k_2}(\beta)| \leq \frac{10 + 3\ln k_1}{k_1} + \frac{10 + 3\ln k_2}{k_2}.$$

Thus $F_k(\beta)$ is a Cauchy sequence.

Continuity of $F$ and concavity of $\beta F(\beta)$ follow from general thermodynamic principles. We already noted that $F(\beta) = 0$ for $\beta > 2$ since there $\lim_{k\to\infty} Z_k(\beta) = Z(\beta)$. But then by concavity $F(\beta) \leq 0$ for $\beta > 0$.

The inequality $F(\beta) \leq -\frac{\ln 2}{\beta} + \ln(3/2)$ was derived in [3] under the assumption of the existence of the thermodynamic limit.

The inequality $F(\beta) \geq -\frac{\ln 2}{\beta} + \frac{1}{2}\ln 2$ follows from $F(2) = 0$, $\lim_{\beta\searrow 0} \beta F(\beta) = -\ln 2$ (since $Z_k(0) = 2^k$) and concavity of $\beta F(\beta)$. □

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