1 The classical system

Consider a classical particle moving on a circle (parametrized by a coordinate $q \in S^1/2\pi\mathbb{Z}$) subject to a potential $V : S^1 \to \mathbb{R}$. The equation of motion will look like:

$$\ddot{q} = f(q) \text{ where } f = -V'. \tag{1}$$

Let's consider the mathematical pendulum, for which $V = -4k \cos q$, and $\ddot{q} = -4k \sin q$. The phase space of the system is a cylinder $M = S^1 \times \mathbb{R}$, parametrized by q and $p = \dot{q}$.

Here we will be concerned with time discrete systems. The most obvious way to discretize (1) is to look for a sequence $t \to q_t,\ t \in \mathbb{Z}$ satisfying

$$q_{t-1} - 2q_t + q_{t+1} = f(q_t) \tag{2}$$

The use of a lattice constant h different from 1 for the time discretization could be easily realized by multiplying f with h^2.

The main feature of (2) is that, as (1), it is still a Hamiltonian system: The whole sequence $..., q_{-1}, q_0, q_1, ...$is uniquely defined by giving (q_{t-1}, q_t) for any fixed $t \in \mathbb{Z}$. So, for any fixed $t \in \mathbb{Z},\ q_{t-1}, q_t$ and should be considered as the coordinate functions on the phase space M, which is therefore diffeomorphic to $T^2 = S^1 \times S^1$. The time evolution is then given by

$$\begin{array}{rccl} T: & M & \to & M \\ & (q_{t-1}, q_t) & \longmapsto & (q_t, q_{t+1}) = (q_t, 2q_t + f(q_t) - q_{t-1}). \end{array} \tag{3}$$

Observe, that for any function $g : S^1 \to \mathbb{R}$ the map $(q, \tilde{q}) \longmapsto (\tilde{q}, g(\tilde{q}) - q)$ preserves the standard symplectic form $dq \wedge d\tilde{q}$ on M.
Figure 1 shows some orbits of the map (3) with $f(q) = -4k \sin q,\ k = 0.7$.

Figure 1
Four copies of phase space

Pictures like those in Figure 1 are familiar from KAM theory, and in fact the present T is nothing else than the so-called "standard map" which is a well-studied example in chaos theory. The crucial point is, that this T does not admit any continous integral of the motion: the system is **not** integrable.

Inspired by the idea of looking at solutions of the pendulum equation as special solutions for the sine-gordon equation $\ddot{q} - q'' = k\sin q$, namely those for which $q(x,t) = q(t)$ we consider the discrete analog of the sine-gordon equation in light-cone coordinates:

$$q_{m+1,n+1} - q_{m+1,n} - q_{m,n+1} + q_{m,n} = 2\ \arg(1 + ke^{-iq_{m+1,n}}) + 2\ \arg(1 + ke^{-iq_{m,n+1}}). \quad (4)$$

This equation describes the angles of discrete surfaces with constant negative curvature and was derived and investigated in [BP].

The variables n, m are the cone variables. In order to reduce equation (4) to the discrete pendulum equation we consider space independent solutions $q_{m+1,n-1} = q_{m,n}$.

Then in the discrete time variable $t = m + n$ the discrete pendulum equation reads as follows:

$$q_{t+1} - 2q_t + q_{t-1} = 4\ \arg(1 + ke^{-iq_t}) \quad (5)$$

The corresponding new $T : M \to M$, for which the standard map can be regarded as a pertubation, is integrable, i.e. there exists an integral of the motion $H = 2(\cos q_n + \cos q_{n-1}) + k\cos(q_n + q_{n-1}) + k^{-1}\cos(q_n - q_{n-1})$.

Figure 2
Phase portrait of system (5)

Using $Q_n \stackrel{\text{def}}{=} e^{-iq_n}$ (5) can be written as

$$Q_{n+1}Q_{n-1} = \left(\frac{Q_n + k}{1 + kQ_n}\right)^2 \quad (6)$$

In the following we find quantized analogs of (6) and (4).

2 The quantum discrete Volterra model

Lets consider a stairway with matrices L and M on its edges (L and M are associated to the different kinds of edges) of the following form:

$$L = \begin{pmatrix} u^L & -\lambda k^{-1/2}(v^L)^{-1} \\ \lambda k^{-1/2} v^L & (u^L)^{-1} \end{pmatrix}, \quad M = \begin{pmatrix} u^M & -\lambda k^{1/2}(v^M)^{-1} \\ \lambda k^{1/2} v^M & (u^M)^{-1} \end{pmatrix} \tag{7}$$

Here the u and v's are unitary operators, k is a parameter of the model, λ is a spectral parameter.

The operators u, v of different edges commute and on the same edge they satisfy the following commutation relation:

$$uv = e^{-i\gamma} vu. \tag{8}$$

In their recent paper [FV] Faddeev and Volkov showed that this model is integrable by the quantum inverse transform method and described its discrete time evolution. They constructed operators $U_\pm$ (shift operators in cone directions) determining the evolution (see Figure 3)

$$\tilde{L} = U_+^{-1} L U_+, \ \tilde{M} = U_- M U_-^{-1}.$$

This evolution has the following properties:
1.) The matrices $\tilde{L}$, $\tilde{M}$ have the same form as L, M and the commutation relations for the u and v's are preserved (i.e. $\tilde{u}$, $\tilde{v}$ commute in the same way as u and v).
2.) The zero curvature condition is satisfied: $LM = \tilde{M}\tilde{L}$.

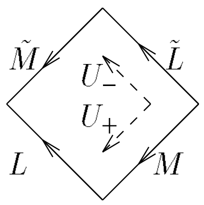

Figure 3

The last equation is equivalent to the following system:

$$\begin{aligned} u^L u^M &= \tilde{u}^M \tilde{u}^L, & (9)\\ (v^L)^{-1} v^M &= (\tilde{v}^M)^{-1} \tilde{v}^L, & (10)\\ k^{1/2}(u^L)^{-1} v^M + k^{-1/2} v^L u^M &= k^{-1/2}(\tilde{u}^M)^{-1}\tilde{v}^L + k^{1/2}\tilde{v}^M \tilde{u}^L & (11) \end{aligned}$$

3 Quantum discrete Sine-Gordon equation

Let's consider a bigger fragment of the lattice (see Figure 4). The initial stairway is

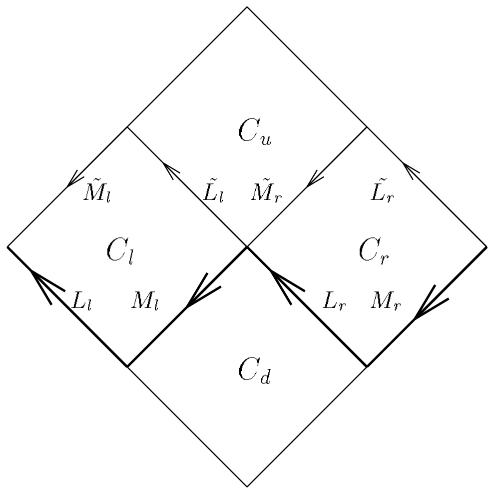

Figure 4

marked by the thick line. We denote the four faces in Fig. 4 by **u**p, **d**own, **l**eft, **r**ight and use these labels to distinguish fields on different edges and faces.

Starting with the discrete Volterra model we construct an algebra of operators on faces. First, we associate to each face the pair of commuting operators:

$$A = u^L u^M = \tilde{u}^M \tilde{u}^L, \;\; B = (v^L)^{-1} v^M = (\tilde{v}^M)^{-1} \tilde{v}^L.$$

Equation (11) can be written as follows:

$$(u^L)^{-1} v^L (e^{-i\gamma} A + kB) = (\tilde{u}^M)^{-1} (\tilde{v}^M)(e^{-i\gamma} kA + B),$$

or equivalently

$$(v^L)^{-1} u^L (\tilde{u}^M)^{-1} \tilde{v}^M = \frac{A + kBe^{i\gamma}}{kA + Be^{i\gamma}},$$

where we use the fraction notation for commuting operators. Combining two of these equations written for the faces **l** and **r** with the trivial observation following from (9) and (10)

$$(v_l^L)^{-1} u_l^L (\tilde{u}_l^M)^{-1} \tilde{v}_l^M = (u_l^M)^{-1} (v_l^M)^{-1} \tilde{v}_l^L \tilde{u}_l^L = A_d^{-1} B_d (v_r^L)^{-1} u_r^L (\tilde{u}_r^M)^{-1} \tilde{v}_r^M B_u^{-1} A_u,$$

we finally get for $C = BA^{-1} = (v^L)^{-1} v^M (u^L)^{-1} (u^M)^{-1}$

$$\frac{1 + ke^{i\gamma} C_l}{k + e^{i\gamma} C_l} = C_d \frac{1 + ke^{i\gamma} C_r}{k + e^{i\gamma} C_r} C_u^{-1}.$$

Only for the operators on neighbouring faces the commutation relations are non-trivial: $C_u C_l = e^{2i\gamma} C_l C_u, C_d C_l = e^{2i\gamma} C_l C_d$.
We prefer to change the sign of our "quantum angle" along the diagonals. For the combinations of four faces shown in Fig. 2 we get consequently the following redefinitions:

$$Q_u = C_u, Q_l = C_l, Q_r = C_r^{-1}, Q_d = C_d^{-1};$$
$$Q_u = C_u^{-1}, Q_l = C_l^{-1}, Q_r = C_r, Q_d = C_d.$$

In both cases the equation and the commutation relations for Q look as follows:

$$Q_u Q_d = \frac{k + e^{i\gamma} Q_l}{1 + e^{i\gamma} k Q_l} \frac{k + e^{i\gamma} Q_r}{1 + e^{i\gamma} k Q_r}, \tag{12}$$

$$Q_d Q_r = e^{-2i\gamma} Q_r Q_d, Q_d Q_l = e^{-2i\gamma} Q_l Q_d \tag{13}$$

(all others Q's along the stairway commute with Q_d). We call this equation the quantum discrete sine-Gordon equation.[1]

4 The discrete quantum pendulum

The simplest reduction, which is possible to impose to (12) and (13), is $Q_l = Q_r$. In this case we get a sequence of unitary operators Q_n satisfying

$$Q_{n+1} Q_{n-1} = (\frac{k + e^{i\gamma} Q_n}{1 + e^{i\gamma} k Q_n})^2, \; Q_{n+1} Q_n = e^{2i\gamma} Q_n Q_{n+1}. \tag{14}$$

For obvious reasons we call this system a (discrete) quantum pendulum.

In a standard way the hamiltonian of this quantum pendulum can be calculated as $H = \frac{d}{d\lambda} \mathrm{tr}(L_l M_l L_r M_r)_{|\lambda=0}$. It turns out, that H depends only on the Q's

$$\begin{aligned} H = k(e^{i\gamma} Q_{n-1} Q_n + e^{-i\gamma} Q_n^* Q_{n-1}^*) &+ k^{-1}(e^{i\gamma} Q_n Q_{n-1}^* + e^{-i\gamma} Q_{n-1} Q_n^*) \\ &+ 2(Q_n + Q_n^* + Q_{n-1} + Q_{n-1}^*). \end{aligned} \tag{15}$$

The evolution operator U is a unitary operator transforming Q_n into Q_{n+1}

$$Q_{n+1} = U Q_n U^*, \; \forall n. \tag{16}$$

In case of rational γ there is a finite dimensional representation of the algebra of Q_n's, and U can be calculated explicitely. As a matter of fact, let $h = e^{2i\gamma}$ be such that $h^N = 1$. Then, taking into account the commutation relation

[1] Independently this equation was obtained by L.D. Faddeev.

$Q_m Q_{m-1} = h Q_{m-1} Q_m$, the operators Q_m and Q_{m-1} can be represented in the following form:

$$Q_{m-1} = \begin{pmatrix} 0 & & & & 0 & 1 \\ 1 & 0 & & & & 0 \\ & 1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ 0 & & & & 1 & 0 \end{pmatrix}, \quad Q_m = \begin{pmatrix} 1 & & & & & 0 \\ & h & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & h^{N-1} \end{pmatrix}.$$

Let F be a Fourier transformation matrix

$$F_{kl} = \frac{1}{\sqrt{N}} h^{(k-1)(l-1)}. \text{ with } h = e^{2\pi i/N}$$

Then

$$F Q_{m-1} F^* = Q_m, \;\; F Q_m F^* = Q^*_{m-1}.$$

Comparing the first relation with (16), we have

$$U = DF,$$

where D is some diagonal matrix

$$D_{kl} = \delta_{kl} e^{id_k}.$$

To determine D we use the evolution equation

$$Q_{m+1} = U Q_m U^* = D F Q_m F^* D^* = D Q^*_{m-1} D^*$$

or, equivalently

$$\left(\frac{\sqrt{h} Q_m + k}{1 + \sqrt{h} k Q_m}\right)^2 = D Q^*_{m-1} D^* Q_{m-1}. \tag{17}$$

Both matrices in (17) are diagonal. For the elements of D this implies

$$e^{id_{j+1}} = e^{id_j} \left(\frac{1 + k h^{j-\frac{1}{2}}}{k + h^{j-\frac{1}{2}}}\right)^2,$$

which completes the calculation of U.

Figure 5 presents eigenstates of H (15) for $k = 0.5$, $h = e^{i\pi/16}$. Let us explain how these pictures were made. First we construct a coherent state Ψ_{00}, which is characterized by the fact that the probability distributions of both observables Q_n and Q_{n-1} are maximally localized around $Q = 1$. The state Ψ_{00} can be obtained as a solution of the following variational problem. First one should consider a set S_a of states with a fixed value of Re$< \Phi | Q_n | \Phi >$

$$S_a = \{ \Phi \in \mathcal{H} | < \Phi, \Phi >= 1, \mathrm{Re} Re < \Phi | Q_n | \Phi >= a \}.$$

Here $\mathcal{H}$ is the Hilbert space, which is $\mathcal{H} = \mathbb{C}^N$ in our case. The maximum

$$b = \max_{\Phi \in S_a} \mathrm{Re} < \Phi | Q_{n-1} | \Phi >$$

is achieved at some state Φ^a, which is equal to $\Psi_{00} := \Phi^a$, when $b = a$. In fact, Ψ_{00} is the ground state of $-(Q_n + Q_n^* + Q_{n-1} + Q_{n-1}^*)$.

Then we obtain states $\Psi_{kl} = Q_n^k Q_{n-1}^l \Psi_{00}$ which are localized around the point $(e^{2\pi ik/N}, e^{2\pi il/N})$ in the classical phase space $S^1 \times S^1$. To make a picture of an eigenstate $\Psi \in \mathbb{C}^N$ we then display the "Husimi function" $(k, l) \mapsto |\langle \Psi, \Psi_{kl} \rangle|^2$ by grey levels. Fig. 5 shows characteristic eigenstates out of 32, including the top and the bottom energy levels. The results presented in Fig. 5 were obtained numerically. Using the formulas of this section the operator H was represented as a $N \times N$ dimensional matrix, the eigenvector system of which was calculated using Mathematica [W]. The problem of analytical calculation of the spectrum and the eigenfunctions of H (QISM sometimes implies a possibility of this calculation) is now under consideration.

5 Other related results

At the end we would like to mention some important papers closely related to the present one.

First of all we should mention the papers [NCP], [NC], where the first examples of integrable quantum mappings appeared and where their quatum Yang-Baxter structure was investigated. These were examples of quantum mappings associated with the lattice analogues of the KdV and the MKdV equations. A first discrete version of the quantum sine-Gordon equation (discrete space, continous time) was obtained and investigated in [IK]. The Lax operator of that paper has a similar form to the one used by us.

We would also like to refer to two good surveys [V], [BRST] on integrable classical symplectic mappings.

References

[BP] A. I. Bobenko, U. Pinkall, "Discrete Surfaces with Constant Negative Gaussian Curvature and the Hirota Equation", *Preprint TU Berlin*, Berlin 1992

[BRST] M. Bruschi, O. Ragnisco, P.M. Santini, G.-Z. Tu, *Physica D***49** 273 (1991)

[FV] L. D. Faddeev, A. Yu. Volkov, "Quantum Inverse Transform Method with Discrete Space-Time", *Theor. Math. Phys.* **92:2**, 207–214 (1992) (in Russian)

[IK] A. G. Izergin, V. E. Korepin, *Lett. Math. Phys* **5**, 199 (1981)

[NC] F. W. Nijhoff, H. W. Capel, "Integrable Quantum Mappings and Quantization aspects of Integrable Discrete-Time Systems ", *Clarkson University Preprint* **INS 193** (1992) (to be published in Proceedings of the NATO ARW on Applications of Analytic and Geometric Methods to Nonlinear Differential Equations, Exeter 1992, ed. P. Clarkson)

[NCP] F.W. Nijhoff, H. W. Capel, V.G. Papageorgiu, *Phys. Rev. A* **46**, 2155 (1992)

[V] A. P. Veselov, *Russ. Math. Surv.* **46** No. **5** (1991)

[W] S. Wolfram, "Mathematica" Second Edition, Addison-Wesley (1991)

Figure 5