On the Cohomology and the Analytic Torsion of certain elliptic Complexes associated to Geometric Structures Examples

Ulrich Bunke*

February 11, 1993

Abstract

On a (G, X)-manifold we define an elliptic complex an study its cohomology and analytic torsion. We compute examples.

Contents

*Humboldt-Universität zu Berlin, Fachbereich für reine Mathematik (SFB288), Unter den Linden 6, Berlin O-1086. E-mail:bunke@informatik.hu-berlin.de

1 Introduction

Let $\mathbf{G}$ be a Lie group, $\mathbf{H}$ be a closed subgroup and $\mathbf{X} := \mathbf{G}/\mathbf{H}$. Assume that $(\mathbf{G}, \mathbf{X})$ satisfies the uniformization condition (see Section 2.1). A $(\mathbf{G}, \mathbf{X})$- manifold M^n is a manifold modelled locally on $\mathbf{X}$ such that the transition functions are restrictions of elements of $\mathbf{G}$. Well known examples are Riemannian manifolds with a metric of constant sectional curvature K. Depending on the sign of K we have

- $\mathbf{G} = SO(n, 1), \quad \mathbf{H} = SO(n), \quad \mathbf{X} = H^n, \quad K = -1$
- $\mathbf{G} = SO(n+1), \quad \mathbf{H} = SO(n), \quad \mathbf{X} = S^n, \quad K = 1$
- $\mathbf{G} = E(n), \quad \mathbf{H} = SO(n), \quad \mathbf{X} = \mathbf{R}^n, \quad K = 0.$

Here $E(n)$ is the group of euclidian motions of $\mathbf{R}^n$ and H^n is the Hyperbolic space of sectional curvature $K = -1$ of dimension n.

Another example are locally conformally flat manifolds. In this case

$$\mathbf{G} = SO(n+1, 1), \quad \mathbf{H} = \mathbf{P}, \quad \mathbf{X} = S^n$$

where $\mathbf{P}$ is the maximal parabolic subgroup and S^n is equipped with the standard conformal structure.

Other examples which have been studied in great detail are the geometric structures on 3-manifolds (see [13], [12]).

A $(\mathbf{G},\mathbf{X})$-structure on a manifold M determines a developing map

$$dev : \tilde{M} \to \mathbf{X}$$

where $\tilde{M} \to M$ is the universal cover. This map is unique up to translation by an element of $\mathbf{G}$. Moreover there is the holonomy map

$$dev_\sharp : \pi_1(M) \to \mathbf{G}$$

unique up to conjugation such that

$$dev_\sharp(\gamma) \circ\ dev = dev \circ \gamma \quad \forall \gamma \in \pi_1(M)$$

where $\pi_1(M)$ acts on $\tilde{M}$ by covering automorphisms.

If $N \to M$ is a local diffeomorphism there is a pull-back $(\mathbf{G},\mathbf{X})$-structure on N. Two $(\mathbf{G},\mathbf{X})$-manifolds M_i, $I = 1,2$ are equivalent if there is a diffeomorphism $f : M_1 \to M_2$ such that the induced structure on M_1 coincides with the original one. In general it is complicated to decide wether two $(\mathbf{G},\mathbf{X})$-structures on a manifold are equivalent. One possibility is to use the conjugacy class of $dev_\sharp$. But on one hand it remains to be complicated to distinguish such conjugacy classes of representations. On the other hand there are examples of different $(\mathbf{G},\mathbf{X})$-structures which give the same $dev_\sharp$.

Consider e.g. a compact manifold N^{n-1} with constant negative sectional curvature -1 and form the locally conformally flat manifold $M_a := S^1_a \times N$ where we equip S^1_a with the metric such that $vol(S^1_a) = a$ and M_a with the product metric. The fundamental group of M_a is $\pi_1(M_a) = \mathbf{Z} \times \pi_1(N)$. There is a $a \in \mathbf{R}$ such that $(1,e) \in \pi_1(M_a)$ is mapped to $e \in \mathbf{G}$ by $dev^a_\sharp$. Here the index a indicates the parameter of the conformally flat structure. But then $dev^{na}_\sharp = dev^a_\sharp$ for all $n \in \mathbf{N}$.

This motivates the search for numerical invariants of $(\mathbf{G},\mathbf{X})$-manifolds which potentially depend on more than the holonomy map. In the present paper we discuss the analytic torsion (a spectral invariant) of elliptic complexes which are canonically defined by the $(\mathbf{G},\mathbf{X})$-structure.

We introduce a locally constant sheaf $\mathcal{E}_M$ and a complex of sheafs $(\Gamma C^*_M, d^*)$. $\mathcal{E}_M$ is the subsheaf of the sheaf $\mathcal{T}_M$ of tangent vector fields such that $\mathcal{E}_M(U)$, $U \subset M$ is the Lie algebra of the infinitesimal automorphisms of the $(\mathbf{G},\mathbf{X})$-structure on U. The stalk of $\mathcal{E}_M$ is isomorphic to the Lie algebra $\mathbf{g}$ of $\mathbf{G}$. For small connected U the complex $(\Gamma C^*_M(U), d^*)$ is the standard complex computing the Lie algebra cohomology $H^*(\mathcal{E}_M(U), \mathcal{T}_M(U))$. An algebraic interpretation in a more general setting and the relation of these sheafs to deformation theory will be given in another paper. Here we focus on the analytic properties of these special sheafs. Alternatively ΓC^*_M is the sheaf of sections of a complex of vector bundles C^*_M and d^* is given by a first order differential operator such that (C^*_M, d^*) becomes an elliptic complex. Let M be compact. In order to produce spectral invariants we have to choose metrics on the C^*_M and a volume on M. Then we can define L^2-spaces of sections, duals $(d^\cdot)^*$ of $d^\cdot$ and the Laplacians $\Delta^\cdot = d^\cdot(d^\cdot)^* + (d^\cdot)^* d^\cdot$. In general spectral invariants of the Laplacians will depend on the choices. But if the dimension of M is odd and the complex is acyclic, i.e. has no cohomology, then a certain combination of the ζ-regularized determinants of the Laplacians is in fact independent. This is the analytic torsion [11]

$$\tau(C^{\cdot}_M) := \Pi_{i=0} det(\Delta^i)^{(-1)^i i}$$

where we omit the differential in the notation of the complex in order to simplify the sybolics. We show the independence of the analytic torsion from the choices in Section 4.1.

There is a whole world of elliptic complexes associated to a $(\mathbf{G}, \mathbf{X})$-structure. The reason for choosing this special complex is that we can show its acyclicity in interesting cases. By Theorem 3.1 we have $H^*(\Gamma C^{\cdot}_M(M)) = 0$ if $\mathbf{G}$ is semisimple of real rank ≥ 1 and H is parabolic. This includes locally conformally flat structures.

We will compute (C^*_M, d^*) and its analytic torsion for several $(\mathbf{G}, \mathbf{X})$-structures on S^1. The result will be in every case

$$\tau(C^{\cdot}_{S^1}) = 1.$$

In order to study the behaviour of the analytic torsion of C^*_M under certain deformations of the $(\mathbf{G}, \mathbf{X})$-structure we need information on the lower bound of the spectrum of the Laplacians Δ^*. This is related to the L^2-cohomology of the complex $(\mathbf{C}^*_M(M), d^*)$ on certain non-compact $(\mathbf{G}, \mathbf{X})$-manifolds. Thus we have included a section dealing with vanishing results on the L^2-cohomology. We apply these results in the case of locally conformally flat manifolds. We study the behaviour of the torsion under confomal connected sum. We can define a regularized analytic torsion for locally conformally flat manifolds with cylindrical ends $\mathbf{R}_+ \times S^{n-1}$ where we choose translationary invariant metrics on the ends. If we form a conformal connected sum of two such manifolds along the cylinder bases the torsion will be almost additive when the cylinder connecting the two pieces is (conformally) long.

There are a lot of different locally conformally flat tori. Using the fact that they can cover themselves conformally arbitrary many times we show that for odd-dimensional locally conformally flat tori

$$\tau(C^{\cdot}_{T^n}) = 1.$$

Thus we have no example of a $(\mathbf{G}, \mathbf{X})$-manifold where the torsion is different from 1. This leads to the following conjecture

Conjecture 1.1 *If M is an odd-dimensional compact locally conformally flat manifold then $\tau(C^{\cdot}_M) = 1$.*

Using the ideas of Burghelea/Friedlander/Kappeler [4] and $\tau(C^{\cdot}_{T^n}) = 1$ one can reduce the proof of this conjecture to a uniform positivity result as follows. Equip $C^*_{\mathbf{R}^n}$ with translationary invariant metrics and choose a translationary invariant volume. Let $f(x) = 1/2(\pm x_1^2 + \pm x_2^2 \ldots \pm x_n^2)$. Consider the Witten deformed complex [14]

$$\begin{aligned} d_t^{\cdot} &:= e^{-tf} d^{\cdot} e^{tf}, \quad t \in \mathbf{R} \\ \Delta_t^{\cdot} &:= d_t^{\cdot}(d_t^{\cdot})^* + (d_t^{\cdot})^* d_t^{\cdot}. \end{aligned}$$

Then the following fact would imply the conjecture.

$$spec(\Delta_t^{\cdot}) \geq c > 0, \quad \forall t \in \mathbf{R}.$$

Up till now we were not able to prove this since an explicit computation of $\Delta_t^{\cdot}$ is difficult because of the large dimension of the bundles (the smallest interesting dimension is 3 and then $dim\ C_M^{\cdot} = 3\ 2^{10}$).

The methods used in Bismut/Zhang [2] give possibly another method to localize the computation of the analytic torsion using the Witten deformation. But one has to invest new ideas in order to come up with the local computations.

This preprint reports on a research which has been carried out partly at the Max Planck Institut für Mathematik and represents a stage between a hope and a negative result which has been resisted further attempts for a while.

2 Definitions

2.1 $(\mathbf{G}, \mathbf{X})$-structures

Let $\mathbf{G}$ be a Lie group with Lie algebra $\mathbf{g}$, $\mathbf{H} \subset \mathbf{G}$ be a closed subgroup and $\mathbf{X} := \mathbf{G}/\mathbf{H}$. We suppose that $(\mathbf{G}, \mathbf{X})$ satisfies the *uniformization condition* [9],[5], i.e. $g \in \mathbf{G}$ is uniquely determined by it's action on any open subset of $\mathbf{X}$. A $(\mathbf{G}, \mathbf{X})$-structure on a manifold M is given by a *distinguished atlas* $\{U_\alpha, \psi_\alpha\}$ where $\{U_\alpha\}$ is an open cover of M and $\psi_\alpha : U_\alpha \to \mathbf{X}$ are diffeomorphisms on the image such that

$$\psi_\alpha \circ \psi_\beta^{-1} :\to \psi_\beta(U_\alpha \cap U_\beta) \to \psi_\alpha(U_\alpha \cap U_\beta)$$

is the restriction of the action of some $g \in \mathbf{G}$. A chart $\psi : U \to \mathbf{X}$ is *compatible* with the distinguished atlas if we can enlarge the atlas by this chart. There is a *maximal* atlas consisting of all charts which are compatible with the distinguished atlas. Two distinguished atlases define the *same* $(\mathbf{G}, \mathbf{X})$-structure if their maximal compatible atlases coincide.

2.2 The sheaf $\mathcal{E}_M$

Let $\mathbf{X} := \mathbf{G}/\mathbf{H}$. There is a sheaf of Lie algebras $\mathcal{E}_X$ over $\mathbf{X}$ given by

$$\mathcal{E}_X(U) = \{\text{restrictions of the fundamental vector fields of the } G\text{–action to } U\}.$$

This sheaf is a locally constant subsheaf of the sheaf of tangent vector fields $\mathcal{T}_X$. The stalks are isomorphic to $\mathbf{g}$.

Let now M be a $(\mathbf{G},\mathbf{X})$-manifold. We define $\mathcal{E}_M$ as the subsheaf of the sheaf of tangent vectors $\mathcal{T}_M$ given by

$$\mathcal{E}_M(U) := \{X \in \mathcal{T}_M(U) \mid \forall \text{ charts } (U_\alpha, \psi_\alpha) \ \ d\psi_\alpha(X_{|U\cap U_\alpha}) \in \mathcal{E}_X(\psi_\alpha(U \cap U_\alpha))\}$$

It is again a locally constant sheaf of Lie-algebras with stalks isomorphic to $\mathbf{g}$. There is a bundle of real Lie algebras $E_M \to M$ with fibre $\mathbf{g}$ with a flat connection ∇^E which is compatible with the Lie algebra structure of the fibre such that $\mathcal{E}_M$ is the sheaf of parallel sections. The sheaf of sections ΓE_M of E_M is $\mathcal{E}_M \otimes_{\mathbf{R}} \mathcal{O}_M$ where $\mathcal{O}_M$ is the sheaf of smooth functions.

There is a canonical bundle map $V : E_M \to TM$ given by evaluation and there is a commutator

$$\{X, Y\} := [X, Y] + \nabla^E_{V(X)} Y - \nabla^E_{V(Y)} X$$

on ΓE_M where $[,]$ is the fibre-wise commutator of E_M making ΓE_M into a sheaf of Lie algebras such that the induced map

$$V : \Gamma E_M \to \mathcal{T}_M$$

becomes a homomorphism of sheafs of Lie algebras.

2.3 The complex (C^*_M, d^*)

Let M be a $(\mathbf{G},\mathbf{X})$-manifold. We set

$$C^*_M := \Lambda^* E^v_M \otimes TM$$

where E^v_M is the dual bundle to E_M. We define the differential

$$d^* : \Gamma C^*_M \to \Gamma C^{*+1}_M$$

by the following formula. Let $U \subset M$ be some open set and $X_1, \ldots, X_n \in \Gamma E_M(U)$. Then a section $E \in \Gamma C^n_M(U)$ can be evaluated on the X_i giving a section in $\mathcal{T}_M(U)$. We have

$$\begin{aligned}
& d^n E(X_1, \ldots, X_{n+1}) \\
= \ & (-1)^{n+1} \sum_{1\le s<t\le n+1} (-1)^{s+t+1} E([X_s, X_t], X_1, \ldots, \hat{X}_s, \ldots, \hat{X}_t, \ldots, X_{n+1}) \\
+ \ & (-1)^{n+1} \sum_{1\le s\le n+1} (-1)^s [V(X_s), E(X_1, \ldots, \hat{X}_s, \ldots, X_{n+1})] \\
+ \ & (-1)^{n+1} \sum_{1\le s\le n+1} (-1)^s V(\nabla^E_{E(X_1,\ldots,\hat{X}_s,\ldots,X_{n+1})} X_s) \\
- \ & (-1)^{n+1} \sum_{1\le s\le n+1} (-1)^s \sum_{s\ne t} E(X_1, \ldots, \hat{X}_s, \ldots, \nabla^E_{V(X_s)} X_t, \ldots, X_{n+1}).
\end{aligned}$$

One can show the following by calculation.

Lemma 2.1 *1.* $d^{n+1} \circ d^n = 0$ *for any* $n \geq 0$.

2. *The symbol of* d^n *is* $\sigma(d^n)(\xi) = (-1)^{n+1} V^* \xi \wedge \quad \xi \in T^v M$.

3. *The complex* $(\Gamma C^*_M, d^*)$ *is elliptic.*

4. *If* $U \subset M$ *is a simply connected open subset then* $(C^*_M(U), d^*)$ *is the standard complex computing the Lie-algebra cohomology* $H^*(\mathbf{g}, \mathcal{T}_M(U))$ *where we identify* $\mathbf{g}$ *with* $\mathcal{E}(U)$.

Note that $C^*_M(U)$ is spanned by sections of the type $\omega \otimes A$ where $\omega \in \Gamma\Lambda^* E^v_M(U)$ is parallel and $A \in \mathcal{T}_M(U)$. Evaluating the above formula for the differential on such sections and assuming that the X_i's are parallel the last statement of the Lemma is obvious. The first statement of the Lemma follows from the verification that the r.h.s. of the formula is in fact tensorial in the X_i's. We will give more motivation for this construction in a more algebraic framework in a forthcoming paper. The choice of this special complex is simply due to the vanishing result for its cohomology which we will prove below.

3 The vanishing of the cohomology of $\Gamma C^*_M(M)$

The aim of this section is to show

Theorem 3.1 *If* $\mathbf{G}$ *is semisimple of real rank* ≥ 1 *and* $\mathbf{H}$ *is a parabolic subgroup then for any compact* $(\mathbf{G}, \mathbf{X})$*-manifold* M *we have* $H^*(C^{\cdot}_M) = 0$.

(The point "." should indicate that the object is a graded one. Usually we omit the differential in the notation of the complex).

The proof of this theorem splits into a localization result, an algebraic and an analytic part. Let M be a compact $(\mathbf{G}, \mathbf{X})$-manifold.

3.1 Localization

Since ΓC^*_M is a complex of fine sheafs we have

$$H^*(\Gamma C^{\cdot}_M(M)) = \mathbf{H}^*(\Gamma C^{\cdot}_M)$$

where $\mathbf{H}^*$ is the hypercohomology of the complex of sheafs. There is a spectral sequence converging to $\mathbf{H}^*(\Gamma C^{\cdot}_M)$ with second term

$$E_2^{pq} = H^p(\mathcal{H}^q(\Gamma C^{\cdot}_M)) \tag{1}$$

where $\mathcal{H}^q(\Gamma C^\cdot_M)$ is the q-th cohomology sheaf of $(\Gamma C^*_M, d^*)$. For $x \in M$ we have by Lemma 2.1 (4)

$$\begin{aligned}
\mathcal{H}^q_x(\Gamma C^\cdot_M) &= \lim_{\overrightarrow{x\in U}} \mathcal{H}^q(\Gamma C^\cdot_M)(U) \\
&= \lim_{\overrightarrow{x\in U}} H^q(\mathbf{g}, \mathcal{T}_M(U)) \\
&= H^q(\mathbf{g}, \lim_{\overrightarrow{x\in U}} \mathcal{T}_M(U)) \\
&= H^q(\mathbf{g}, \mathcal{T}_{M,x})
\end{aligned}$$

where we have identified $\mathcal{E}_M(U) = \mathbf{g}$ for small connected U and $\mathcal{T}_{M,x}$ is the stalk of the sheaf of $\mathcal{T}_M$ at x.

In the following we will show the vanishing of $H^*(\mathbf{g}, \mathcal{T}_{M,x})$. This is equivalent to the vanishing of the cohomology sheaf $\mathcal{H}^*(\Gamma C^\cdot_M)$ and hence of the second term of the spectral sequence (1). This proves the Theorem 3.1.

3.2 A first reduction

Let $\mathbf{G}$ be semisimple, $\mathbf{P} \subset \mathbf{G}$ be parabolic with Langlands decomposition $\mathbf{P} = \mathbf{MAN}^+$. Let $\mathbf{p} = \mathbf{m} \oplus \mathbf{a} \oplus \mathbf{n}^+$ be the corresponding decomposition of the Lie-algebras and Δ^+ be the system of weights of $(\mathbf{a}, \mathbf{n}^+)$. Then $\mathbf{g} = \mathbf{n}^- \oplus \mathbf{m} \oplus \mathbf{a} \oplus \mathbf{n}^+$ where $\mathbf{a}$ acts on $\mathbf{n}^-$ with the weights $-\Delta^+$. Set $\mathbf{N}^- := exp(\mathbf{n}^-)$. Then $\mathbf{X} = \mathbf{G}/\mathbf{P}$ and $T\mathbf{X} = \mathbf{G} \times_{\mathbf{P}} (\mathbf{g}/\mathbf{p})$. We can identify $\mathbf{g}/\mathbf{p} =: V$ with $\mathbf{n}^-$ as a $\mathbf{MA}$-modul. For every neighbourhood $1 \in U \subset \mathbf{N}^-$ we have

$$\mathcal{T}_X(U\mathbf{P}) = \{f \in C^\infty(U\mathbf{P}, V) \mid f(gman) = a^{-1}m^{-1}f(g)\ \forall m \in \mathbf{M}, a \in \mathbf{A}, n \in \mathbf{N}^+\}.$$

For $1 \in W \subset U \subset \mathbf{N}^-$ there is the restiction map $r^U_W : \mathcal{T}_X(U\mathbf{P}) \to \mathcal{T}_X(W\mathbf{P})$. Set

$$T := \mathcal{T}_{X,[1]} = \lim_{\overrightarrow{1\in U}} \mathcal{T}_X(U\mathbf{P}).$$

The $\mathbf{g}$ module T can be viewed as the standard model for $\mathcal{T}_{M,x}$.

The main tool in studying the cohomology of $\mathbf{g}$ with coefficients in T is the exact sequence of $\mathbf{g}$-modules (the author thanks J. Franke for this hint)

$$0 \to T_0 \xrightarrow{I} T \to Hom_{U(\mathbf{p})}(U(\mathbf{g}), V) \to 0$$

where T_0 are the germs in T with vanishing Taylor series and the last map is given by taking the Taylor series. $U(\mathbf{g}), U(\mathbf{p})$ are the universal enveloping algebras of $\mathbf{g}, \mathbf{p}$ and V is considered as a $\mathbf{p}$-module. There is an associated long exact sequence

$$\to H^p(\mathbf{g}, T_0) \xrightarrow{I_*} H^p(\mathbf{g}, T) \xrightarrow{J} H^p(\mathbf{g}, I^{\mathbf{g}}_{\mathbf{p}}(V)) \to$$

where we use the notation $I^{\mathbf{g}}_{\mathbf{p}}(V) := Hom_{U(\mathbf{p})}(U(\mathbf{g}), V)$ for the induced representation. The following discussion splits into the algebraic part: $H^*(\mathbf{g}, I^{\mathbf{g}}_{\mathbf{p}}(V)) = 0$ and the analytic part : $I_* = 0$. Both facts together imply $H^*(\mathbf{g}, T) = 0$.

3.3 Algebraic part

Lemma 3.2 $H^*(\mathbf{g}, I_{\mathbf{p}}^{\mathbf{g}}(V)) = 0$

Proof: First we apply the Shapiro Lemma:

$$H^*(\mathbf{g}, I_{\mathbf{p}}^{\mathbf{g}}(V)) = H^*(\mathbf{p}, V).$$

Next we use the Hochschild-Serre spectral sequence associated to the extension

$$0 \to \mathbf{n}^+ \to \mathbf{p} \to \mathbf{m} \oplus \mathbf{a} \to 0.$$

It has the second term $E_2^{p,q} = H^p(\mathbf{m} \oplus \mathbf{a}, H^q(\mathbf{n}^+, V))$. Here $H^q(\mathbf{n}^+, V)$ has a $\mathbf{m} \oplus \mathbf{a}$-module structure in a natural way. It decomposes into weight spaces under the $\mathbf{a}$-action

$$H^q(\mathbf{n}^+, V) = \oplus_{\mu \in \mathbf{a}^*} (H^q(\mathbf{n}^+, V))_\mu.$$

This decomposition also respects the $\mathbf{m}$-action. Let $\mathbf{R}_\mu$ be the one-dimensional representation of $\mathbf{a}$ with weight μ. Then we can write the $\mathbf{m} \oplus \mathbf{a}$ module $H^q(\mathbf{n}^+, V)$ as

$$H^q(\mathbf{n}^+, V) = \oplus_{\mu \in \mathbf{a}^*} (H^q(\mathbf{n}^+, V))_\mu \otimes \mathbf{R}_\mu.$$

We apply now the Künneth formula:

$$H^p(\mathbf{m} \oplus \mathbf{a}, H^q(\mathbf{n}^+, V)) = \oplus_{\mu \in \mathbf{a}^*} \oplus_{i+j=p} H^i(\mathbf{m}, (H^q(\mathbf{n}^+, V))_\mu) \otimes H^j(\mathbf{a}, \mathbf{R}_\mu). \quad (2)$$

Note that $H^j(\mathbf{a}, \mathbf{R}_\mu)$ vanishes for $\mu \neq 0$. But since $\mathbf{a}$ acts on $\mathbf{n}^+$ with strictly positive and on V with strictly negative weights it acts on $\Lambda^*(\mathbf{n}^+)^v \otimes V$ with strictly negative weights. Since the cohomology groups $H^*(\mathbf{n}^+, V)$ are factor spaces of subspaces of $\Lambda^*(\mathbf{n}^+)^v \otimes V$ all weights occuring in the sum (2) are strictly negative. Hence the right hand side of (2) and thus the second term of the spectral sequence vanishes. This proves the Lemma. $\square$

This algebraic result holds also in other situations. It *seems* likely that the vanishing theorem for $\mathcal{H}^*(\Gamma C_M^\cdot)$ is true quite often.

Lemma 3.3 *Let* $\mathbf{G}$ *be arbitrary and* $\mathbf{H}$ *be reductive such that there is no* $\mathbf{h}$*-invariant vector in* $\mathbf{g}/\mathbf{h}$. *Then* $H^*(\mathbf{g}, I_{\mathbf{h}}^{\mathbf{g}}(\mathbf{g}/\mathbf{h})) = 0$.

Proof: By the Shapiro Lemma

$$H^*(\mathbf{g}, I_{\mathbf{h}}^{\mathbf{g}}(\mathbf{g}/\mathbf{h})) = H^*(\mathbf{h}, \mathbf{g}/\mathbf{h}) = H^*(\mathbf{h}) \otimes (\mathbf{g}/\mathbf{h})^{\mathbf{h}} = 0.$$

$\square$

3.4 Analytic part

Lemma 3.4 $I_* = 0$

Proof: Let $f \in \Lambda^q \mathbf{g}^v \otimes \mathcal{T}_X(U\mathbf{P})$ represent a cycle in $H^q(\mathbf{g}, T_0)$. Thus f has vanishing Taylor series at $[1] \in \mathbf{X}$. We have to show that f represents a boundary in $\Lambda^q \mathbf{g}^v \otimes T$, i.e. there is a $h \in \Lambda^{q-1}\mathbf{g}^v \otimes \mathcal{T}_X(V\mathbf{P})$ such that $dh = f_{|V\mathbf{P}}$ for some $V \subset U$.

For $X \in \mathbf{g}$ let Θ_X be the operation on cochaines given by

$$\begin{aligned}(\Theta_X f)(X_1,\ldots,X_q) &= \sum_{i=1}^q f(X_1,\ldots,[X_i,X],\ldots,X_q) \\ &+ X f(X_1,\ldots,X_q) \quad X_1,\ldots,X_q \in \mathbf{g}\end{aligned}$$

using the $\mathbf{g}$-module structure of $\mathcal{T}_X(U)$. Moreover let i_X be given by

$$(i_X f)(X_1,\ldots,X_{q-1}) = f(X,X_1,\ldots,X_{q-1}), \quad X_1,\ldots,X_{q-1} \in \mathbf{g}.$$

Then $\Theta_X = d \circ i_X + i_X \circ d$ and

$$\Theta_X g_t f = \frac{d}{ds}_{s=t} g_s f, \quad g_s := exp(sX).$$

Here we employ the $\mathbf{G}$-action on $\Lambda^* \mathbf{g}^v \otimes \mathcal{T}_X$ given by the co-adjoint action on $\mathbf{g}^v$ and the action on $\mathbf{X}$. Let $a \in \mathbf{a}$ be in the positive Weyl chamber i.e. $\rho(a) > 0 \,\forall\, \rho \in \Delta^+$ and $a_t := exp(-ta)$. Since a acts on $\mathbf{n}^-$ with strictly negative weights we have for neighbourhoods $1 \in U \in \mathbf{N}^-$ of suitable shape

$$U \subset a_t U \quad \forall t \geq 0.$$

Since $a_t f$ is defined on $a_t U$ we can restrict it again to U. Thus we obtain a semigroup action of $\{a_t\}_{t\geq 0}$ on $\Lambda^q \mathbf{g}^v \otimes \mathcal{T}_X(U)$. Since f vanishes with arbitrary high order at $[1]$ we have

$$\begin{aligned}\lim_{t\to\infty} a_t f &= 0 \\ r_U \int_0^\infty \frac{\partial}{\partial t} a_t f \, dt &= -f \\ r_U d \int_0^\infty i_a a_t f \, dt &= r_U \int_0^\infty d\, i_a a_t f \, dt.\end{aligned}$$

Taking into account $df = 0$ we obtain

$$\begin{aligned}f &= -r_U \int_0^\infty \frac{\partial}{\partial t} a_t f \, dt \\ &= r_U \int_0^\infty \Theta_a a_t f \, dt \\ &= d r_U \int_0^\infty i_a a_t f \, dt.\end{aligned}$$

This proves the Lemma. □

In fact this proof shows $H^*(\mathbf{g}, T_0) = 0$.

4 Analytic torsion

4.1 Independence of the choice of metrics

We recall the definition of the analytic torsion of an elliptic complex. It requires the choice of hermitian metrics on the bundles and of a volume. If the complex is acyclic and the dimension of the underlying manifold is odd we show the well known fact that the analytic torsion does not depend on the choices.

Let M be compact, n-dimensional and

$$\mathcal{C}_* := (\Gamma C_*(M), d_*) = (\to \Gamma C_i(M) \stackrel{d_i}{\to} \Gamma C_{i+1}(M) \to)$$

(in abuse of usual notation we use the lower index in order to make the notation of the adjoints easier) be an acyclic elliptic complex. We assume that the d_i are of first order. Let N be the $\mathbf{Z}$-graduation and $z := e^{\pi \imath N}$ be the $\mathbf{Z}_2$-graduation operator. Choose smooth positive *duality maps* $J_i : C_i \to C_i^v$ such that $< X, Y >_{J_i} := J_i(X)(Y)$ defines a positive definite metric on C_i. Moreover choose a *smooth volume* vol on M. The pair (J_*, vol) defines a duality I_* on $\Gamma C_*(M)$ and a Hilbert space structure by

$$I_i(\psi)(\phi) := (\psi, \phi)_{I_i} := \int_M < \psi, \phi >_{J_i} vol, \quad \psi, \phi \in \Gamma C_i(M)$$

Let $\mathcal{H}_I = \oplus_i L^2(M, C_i)$ be the corresponding Hilbert space.

We can now define the adjoint of d_i by

$$\partial_i := d_i^* = I_i^{-1} d_i^v I_{i+1}$$

and the Laplacian $\Delta := (d + \partial)^2$. It induces a selfadjoint operator on $\mathcal{H}_I$. Define for $Re(s) > n/2$

$$F(s) = -\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr\ z\ N\ e^{-t\Delta}\ dt.$$

It is well known that $F(s)$ has an analytic continuation to $\mathbf{C}$ and is regular in $s = 0$. The *analytic* torsion of the complex is defined as

$$\tau := e^{\frac{d}{ds}_{s=0} F(s)}.$$

In terms of the ζ-regularized determinants we have

$$\tau = \Pi_{i=1}^{max} (det\ \Delta_i)^{(-1)^i i}$$

where Δ_i is the Laplacian restricted to $L^2(M, C_i)$.

The definition of τ involves the choice of (J_*, vol).

Theorem 4.1 *If M is odd-dimensional then τ does not depend on the choice of (J_*, vol).*

Proof: This fact is well known [11], [10]. It depends on the computation of a variation formula. We will only give the formal calculation. It is the standard to make it exact. First of all note that due to the compactness of M all Hilbert spaces $\mathcal{H}_I$ for different I coincide as topological vector spaces. Hence Tr is independent from I. We have

$$\begin{aligned}\delta F(s) &= -\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\, Tr\ z\ N\ \delta e^{-t\Delta}\, dt \\ &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s}\, Tr\ z\ N\ \delta(\Delta) e^{-t\Delta}\, dt.\end{aligned} \tag{3}$$

Next we compute

$$\begin{aligned}\delta(\Delta) &= \delta((d+\partial)^2) = \delta(\partial)(d+\partial) + (d+\partial)\delta(\partial) \\ \delta(\partial) &= \delta(I^{-1}d^v I) = -I^{-1}\delta(I)\partial + \partial I^{-1}\delta(I).\end{aligned}$$

Let $L := I^{-1}\delta(I)$. Then

$$\delta(\Delta) = -L\partial d + d\partial L - dL\partial + \partial L d.$$

Note that

$$[\Delta, d] = [\Delta, \partial] = 0, \quad [d, N] = -d, \quad [\partial, N] = \partial.$$

Using this and the cyclicity of the trace we have

$$\begin{aligned}Tr\ z\ N\ \delta(\Delta)\ e^{-t\Delta} &= Tr\ z\ N\ (-L\partial d + d\partial L - dL\partial + \partial L d)\ e^{-t\Delta} \\ &= Tr\ z\ L\ \Delta\ e^{-t\Delta} \\ &= -\frac{d}{dt} Tr\ z\ L\ e^{-t\Delta}\end{aligned}$$

Plugging this into equation (3) and integrating by parts we get

$$\begin{aligned}\delta F(s) &= -\frac{1}{\Gamma(s)}\int_0^\infty t^s \frac{d}{dt} Tr\ z\ L\ e^{-t\Delta}\, dt \\ &= \frac{s}{\Gamma(s)}\int_0^\infty t^{s-1}\, Tr\ z\ L\ e^{-t\Delta}\, dt.\end{aligned}$$

Again

$$I(s) := \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\, Tr\ z\ L\ e^{-t\Delta}\, dt$$

has an analytic continuation to $\mathbf{C}$. Let

$$Tr\ z\ L\ e^{-t\Delta} = \sum_{i=0}^{m} t^{-n/2+i}\, b_i \ +\ O(t^{m-n/2+1})$$

be the asymptotic expansion for small t. Then

$$\begin{aligned} I(s) &= \frac{1}{\Gamma(s)}\int_0^1 (\sum_{i=0}^m t^{s-1-n/2+i}\, b_i \;+\; O(t^{m-n/2+s}))\, dt \\ &+ \frac{1}{\Gamma(s)}\int_1^\infty t^{s-1} Tr\; z\; L\; e^{-t\Delta}\, dt \\ &= -\frac{1}{\Gamma(s)}\left(\sum_{i=0}^m \frac{b_i}{s-n/2+i} \;+\; \eta(s)\right) \end{aligned}$$

where $\eta(s)$ is holomorphic for $Re(s) > n/2 - m$. If we choose m large enough we see that $sI(s)$ has a zero of second order at $s = 0$. Thus $\delta \frac{d}{ds}|_{s=0} F(s) = 0$. This proves the Theorem. $\square$

If M is even-dimensional the variation involves the coefficient $b_{n/2}$. Variations with $b_{n/2} = 0$ leave the torsion invariant.

4.2 Local torsion

Let M be an odd-dimensional complete manifold and $\mathcal{C}^*$ be an elliptic complex as above. The analytic torsion $ln\tau(\mathcal{C}^*)$ can be expressed as the integral of the local torsion. Note that this local torsion does depend on the manifold globally.

Assume that $\Delta^\cdot$ is essentially selfadjoint on $C_c^\infty(M, C^\cdot)$ and that $spec(\Delta) \geq c > 0$. Then on one hand for all $x \in M$ there are $C < \infty$, $c > 0$ such that

$$\|tr_x z N e^{-t\Delta}\| \leq C e^{-ct}, \quad \forall t \geq 1.$$

Here tr_x is the local trace of the heat kernel at the point $x \in M$. On the other hand

$$\|tr_x z N e^{-t\Delta}\| \leq C t^{-n/2}.$$

Thus one can define for $s > n/2$

$$F(s,x) = -\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} tr_x\; z\; N\; e^{-t\Delta}\, dt.$$

Employing the asymptotic expansion of the heat kernel one can show that $F(s,x)$ has a meromorphic extension to C with respect to s and is regular in $s = 0$. Define the *local torsion* as

$$ln\tau(x) := \frac{d}{ds}_{s=0} F(s,x).$$

If M would be compact then

$$ln\tau = \int_M ln\tau(x).$$

We will employ the local torsion of homogeneus elliptic complexes in order to compare with the torsion on quotients with respect to a discrete group. The local torsion (defined with an invariant metric and volume) of a homogeneus elliptic complex is just a constant.

5 1-dimensional examples

5.1 General remarks

The simplest compact odd-dimensional manifold is of course S^1. There are several $(\mathbf{G},\mathbf{X})$-structures on S^1. The associated elliptic complexes have small dimensions such that it is possible to compute them explicitly. If $\mathbf{X}=\mathbf{R}$ and there is a domain of discontiniuity Ω of the holonomy representation $\rho:\mathbf{Z}\to\mathbf{G}$ we have

$$S^1=\Omega/\rho(\mathbf{Z}).$$

We will identify Ω again with $\mathbf{R}$. Assume that the translation group of $\mathbf{R}$ belongs to $\mathbf{G}$. Then one can choose translation invariant metrics and volumes and express the Laplacian as differential operator of second order with constant coefficients acting on periodic functions. In our examples the Laplacians will be obviously positive such that the cohomology of the complexes vanish. The analytic torsion can be computed explicitly using the formulas for the ζ-regularized determinants of [3].

5.2 The $(\mathbf{S}_2,\mathbf{R})$-structure (1)

Let $\mathbf{G}:=\mathbf{S}_2$ be the two-dimensional solvable group with Lie-algebra generated by X_0,X_1 with the relation

$$[X_0,X_1]=X_0.$$

Let $\mathbf{H}:=exp(\mathbf{R}X_1)$. Then $\mathbf{X}:=\mathbf{R}=\mathbf{S}_2/\mathbf{H}$. The fundamental vector fields corresponding to the X_i's are (using the same symbols)

$$X_0=\partial r,\quad X_1=r\partial r.$$

We consider the representation $\rho:\mathbf{Z}\to\mathbf{S}_2$, given by $n\in\mathbf{Z}\to exp(nX_0)\in\mathbf{S}_2$. It acts on $\mathbf{R}$ by $(n,r)\to r+n$ properly discontinuous. Thus we have a $(\mathbf{S}_2,\mathbf{R})$-structure on $S^1:=\mathbf{R}/\rho(\mathbf{Z})$. We compute the complex $(C^*_{\mathbf{R}},d^*)$ using the trivialization of the bundles given by the sections

$$\begin{aligned}\partial r &\in \Gamma C^0_{\mathbf{R}}(\mathbf{R}),\\ X_0^v\otimes\partial r, X_1^v\otimes\partial r &\in \Gamma C^1_{\mathbf{R}}(\mathbf{R}),\\ X_0^v\wedge X_1^v\otimes\partial r &\in \Gamma C^2_{\mathbf{R}}(\mathbf{R}).\end{aligned}$$

Then the differentials are

$$d^0=\begin{pmatrix}\partial\\ r\partial-1\end{pmatrix},\quad d^1=(r\partial,-\partial).$$

We will use the translation invariant base

$$\begin{aligned}\partial r &\in \Gamma C^0_{\mathbf{R}}(\mathbf{R}),\\ Y_0^v(r)\otimes\partial r &:= X_0^v(r)+rX_1^v(r)\in\Gamma C^1_{\mathbf{R}}(\mathbf{R}),\\ Y_1^v(r)\otimes\partial r &:= X_1^v(r)\in\Gamma C^1_{\mathbf{R}}(\mathbf{R}),\\ Y_0^v\wedge Y_1^v\otimes\partial r &:= X_0^v\wedge X_1^v\in\Gamma C^2_{\mathbf{R}}(\mathbf{R}).\end{aligned}$$

With respect to this new base the differentials have the form (we write $\tilde{d}^*$)

$$\tilde{d}^0 = \begin{pmatrix} \partial \\ -1 \end{pmatrix}, \quad \tilde{d}^1 = (-1, -\partial).$$

We choose the standard volume of $\mathbf{R}$ and define a translation invariant metric on C^*_M by declearing this invariant base to be orthonormal. Then the adjoints of the differentials are

$$(\tilde{d}^1)^* = \begin{pmatrix} -1 \\ \partial \end{pmatrix}, \quad (\tilde{d}^0)^* = (-\partial, -1).$$

We compute the Laplacians. They become differential operators with constant coefficients. The result is

$$\begin{aligned} \Delta^0 &= (\tilde{d}^0)^*\tilde{d}^0 = -\partial^2 + 1 \\ \Delta^1 &= \tilde{d}^0(\tilde{d}^0)^* + (\tilde{d}^1)^*\tilde{d}^1 = \begin{pmatrix} -\partial^2+1 & 0 \\ 0 & -\partial^2+1 \end{pmatrix} \\ \Delta^2 &= \tilde{d}^1(\tilde{d}^1)^* = -\partial^2 + 1. \end{aligned}$$

The complex $\Gamma C^*_{S^1}(S^1)$ is the subcomplex of functions with period 1 of $\Gamma C^*_{\mathbf{R}}(\mathbf{R})$. Thus we see

Proposition 5.1 *1.* $H^*(\Gamma C^{\cdot}_{S^1}(S^1)) = 0$,

2. $\tau(\Gamma C^{\cdot}_{S^1}(S^1)) = 1$.

Proof: Let $\Delta^{\cdot}_{S^1}$ be the Laplacian on S^1. It is obvious from the above formulas that $\Delta^{\cdot}_{S^1} \geq 1$. Thus (1) follows from Hodge theory. For the second assertion note that $F(s) = 0, \quad \forall s \in \mathbf{C}$ since $\Delta^1_{S^1} \cong \Delta^2_{S^1} \oplus \Delta^2_{S^1}$. $\square$

5.3 The $(\mathbf{S}_2, \mathbf{R})$-structure (2)

There is a 1-parameter family of $(\mathbf{S}_2, \mathbf{R})$-structures on S^1 given by the family of homomorphisms $\sigma_b : \mathbf{Z} \to \mathbf{S}_2$, $b \in \mathbf{R}_+$,

$$n \in \mathbf{Z} \to \sigma_b(n) = exp(bnX_1) \in \mathbf{S}_2.$$

It acts on $\mathbf{R}$ by $(n, r) \to e^{bn}r$. $\Omega := (0, \infty)$ is a domain of discontiniuity of $\sigma_b(\mathbf{Z})$. Let $S^1_b := \Omega/\sigma_b(\mathbf{Z})$ with the induced $(\mathbf{S}_2, \mathbf{R})$-structure. We use the coordinate $z := ln\, r$, $r \in \Omega$. The fundamental vector fields become

$$\begin{aligned} X_0 &= e^{-z}\partial z \\ X_1 &= \partial z. \end{aligned}$$

We compute the differentials of the complex $C^*_{\mathbf{R}}$ in the z-coordinate with respect to the trivialization

$$\begin{aligned} e^z\partial z &\in \Gamma C^0_{\mathbf{R}}(\Omega) \\ X_0^v \otimes e^z\partial z, X_1^v \otimes e^z\partial z &\in \Gamma C^1_{\mathbf{R}}(\Omega) \\ X_0^v \wedge X_1^v \otimes e^z\partial z &\in \Gamma C^2_{\mathbf{R}}(\Omega). \end{aligned}$$

The result is

$$d^0 = \begin{pmatrix} e^{-z}\partial \\ \partial - 1 \end{pmatrix}, \quad d^1 = (\partial, -e^{-z}\partial).$$

The subgroup $exp(\mathbf{R}X_1)$ acts on Ω by translations. We use the tranlation invariant base

$$Y_0^v := e^{-z}X_0^v, \quad Y_1^v := X_1^v.$$

With respect to the trivialization

$$\begin{aligned} \partial z &\in \Gamma C^0_{\mathbf{R}}(\Omega) \\ Y_0^v \otimes \partial z, Y_1^v \otimes \partial z &\in \Gamma C^1_{\mathbf{R}}(\Omega) \\ Y_0^v \wedge Y_1^v \otimes \partial z &\in \Gamma C^2_{\mathbf{R}}(\Omega). \end{aligned}$$

we have for the differentials (using again the notation $\tilde{d}^*$):

$$\tilde{d}^0 = \begin{pmatrix} \partial + 1 \\ \partial \end{pmatrix}, \quad \tilde{d}^1 = (\partial, -\partial - 1).$$

We introduce metrics by declearing the translation invariant base to be orthonormal and choose the volume dz. Then the adjoints are

$$(\tilde{d^1})^* = \begin{pmatrix} -\partial \\ \partial - 1 \end{pmatrix}, \quad (\tilde{d^0})^* = (-\partial + 1, -\partial).$$

The Laplacians are

$$\begin{aligned} \Delta^0 &= (\tilde{d^0})^*\tilde{d^0} = -\partial^2 + 1 \\ \Delta^1 &= \tilde{d^0}(\tilde{d^0})^* + (\tilde{d^1})^*\tilde{d^1} = \begin{pmatrix} -\partial^2 + 1 & 0 \\ 0 & -\partial^2 + 1 \end{pmatrix} \\ \Delta^2 &= \tilde{d^1}(\tilde{d^1})^* = -\partial^2 + 1. \end{aligned}$$

Thus as above we have

Proposition 5.2 *1.* $H^*(\Gamma C^{\cdot}_{S^1_b}(S^1_b)) = 0,$

2. $\tau(\Gamma C^{\cdot}_{S^1_b}(S^1_b)) = 1, \quad \forall b \in \mathbf{R}_+.$

5.4 The $(\mathbf{Sl}_2, \mathbf{R})$-structure (1)

Let $\mathbf{Sl}_2$ be the 3-dimensional semisimple group with Lie-algebra generated by X_0, X_1, X_2 with relations

$$[X_0, X_{-1}] = -X_{-1}, \quad [X_{-1}, X_1] = 2X_0, \quad [X_0, X_1] = X_1.$$

$\mathbf{Sl}_2$ acts transitively on $\mathbf{R}$ with fundamental vector fields (using the same symbols)

$$X_{-1} = \partial r, \quad X_0 = r\partial r, \quad X_1 = r^2\partial r.$$

We consider the homomorphism $\rho : \mathbf{Z} \to \mathbf{Sl}_2$, given by $n \in \mathbf{Z} \to exp(nX_{-1}) \in \mathbf{Sl}_2$. The $\mathbf{Z}$-action on $\mathbf{R}$ becomes $(n, r) \to r + n$. Let $S^1 := \mathbf{R}^1/\rho(\mathbf{Z})$ with the induced $(\mathbf{Sl}_2, \mathbf{R})$-structure. We compute the complex $(C^*_{\mathbf{R}}(\mathbf{R}), d^*)$ with respect to the trivialization

$$\begin{array}{rcl}
\partial r & \in & \Gamma C^0_{\mathbf{R}}(\mathbf{R}) \\
X^v_{-1} \otimes \partial r, \quad X^v_0 \otimes \partial r, \quad X^v_1 \otimes \partial r & \in & \Gamma C^1_{\mathbf{R}}(\mathbf{R}) \\
X^v_{-1} \wedge X^v_0 \otimes \partial r, \quad X^v_{-1} \wedge X^v_1 \otimes \partial r, \quad X^v_0 \wedge X^v_1 \otimes \partial r & \in & \Gamma C^2_{\mathbf{R}}(\mathbf{R}) \\
X^v_{-1} \wedge X^v_0 \wedge X^v_1 \otimes \partial r & \in & \Gamma C^3_{\mathbf{R}}(\mathbf{R}).
\end{array}$$

The result is

$$d^0 = \begin{pmatrix} \partial \\ r\partial - 1 \\ r^2\partial - 2r \end{pmatrix}, d^1 = \begin{pmatrix} r\partial & -\partial & 0 \\ r^2\partial - 2r & 2 & -\partial \\ 0 & r^2\partial - 2r & -r\partial + 2 \end{pmatrix}, d^2 = \begin{pmatrix} {}^2\partial - 2r \\ -r\partial + 1 \\ \partial \end{pmatrix}^t.$$

The subgroup $exp(\mathbf{R}X_{-1})$ acts on $\mathbf{R}$ by translations. We will employ the translation invariant base

$$\begin{array}{rcl}
Y^v_{-1}(r) & := & X^v_{-1}(r) + rX^v_0(r) + r^2X^v_1(r) \\
Y^v_0(r) & := & X^v_0(r) + 2rX^v_1(r) \\
Y^v_1(r) & := & X^v_1(r)
\end{array}$$

With respect to the new trivialization

$$\begin{array}{rcl}
\partial r & \in & \Gamma C^0_{\mathbf{R}}(\mathbf{R}) \\
Y^v_{-1} \otimes \partial r, \quad Y^v_0 \otimes \partial r, \quad Y^v_1 \otimes \partial r & \in & \Gamma C^1_{\mathbf{R}}(\mathbf{R}) \\
Y^v_{-1} \wedge Y^v_0 \otimes \partial r, \quad Y^v_{-1} \wedge Y^v_1 \otimes \partial r, \quad Y^v_0 \wedge Y^v_1 \otimes \partial r & \in & \Gamma C^2_{\mathbf{R}}(\mathbf{R}) \\
Y^v_{-1} \wedge Y^v_0 \wedge Y^v_1 \otimes \partial r & \in & \Gamma C^3_{\mathbf{R}}(\mathbf{R})
\end{array}$$

the differentials become (using the notation $\tilde{d^*}$):

$$\tilde{d^0} = \begin{pmatrix} \partial \\ -1 \\ 0 \end{pmatrix}, \tilde{d^1} = \begin{pmatrix} -1 & -\partial & 0 \\ 0 & 0 & -\partial \\ 0 & 0 & 2 \end{pmatrix}, \tilde{d^2} = \begin{pmatrix} 0 \\ 2 \\ \partial \end{pmatrix}^t.$$

We define metrics on the bundles by declearing the translation invariant base to be orthonormal and choose the volume dr. Then the adjoints of the differentials are

$$(\tilde{d^0})^* = \begin{pmatrix} -\partial \\ -1 \\ 0 \end{pmatrix}^t, (\tilde{d^1})^* = \begin{pmatrix} -1 & 0 & 0 \\ \partial & 0 & 0 \\ 0 & \partial & 2 \end{pmatrix}, (\tilde{d^2})^* = \begin{pmatrix} 0 \\ 2 \\ -\partial \end{pmatrix}.$$

The Laplacians are

$$\begin{aligned}
\Delta^0 &= (\tilde{d^0})^* \tilde{d^0} = -\partial^2 + 1 \\
\Delta^1 &= \tilde{d^0}(\tilde{d^0})^* + (\tilde{d^1})^* \tilde{d^1} = \begin{pmatrix} -\partial^2+1 & 0 & 0 \\ 0 & -\partial^2+1 & 0 \\ 0 & 0 & -\partial^2+4 \end{pmatrix} \\
\Delta^2 &= \tilde{d^1}(\tilde{d^1})^* + (\tilde{d^2})^* \tilde{d^2} = \begin{pmatrix} -\partial^2+1 & 0 & 0 \\ 0 & -\partial^2+4 & 0 \\ 0 & 0 & -\partial^2+4 \end{pmatrix} \\
\Delta^3 &= \tilde{d^2}(\tilde{d^2})^* = -\partial^2 + 4.
\end{aligned}$$

The restriction of these operators to the $\rho(\mathbf{Z})$-periodic sections are the Laplacians on $\Gamma C^*_{S^1}(S^1)$. We have

Proposition 5.3 *1.* $H^*(\Gamma C^{\cdot}_{S^1}(S^1)) = 0$,

2. $\tau(\Gamma C^{\cdot}_{S^1}(S^1)) = 1$.

Proof: Again it is obvious that the Laplacians are positive. Hence (1) follows by Hodge theory. It is easy to see that also in the present case $F(s) = 0$, $\forall s \in \mathbf{C}$. Hence (2). $\square$

5.5 The $(\mathbf{Sl}_2, \mathbf{R})$-structure (2)

There is also a 1-parameter family of $(\mathbf{Sl}_2, \mathbf{R})$-structures on S^1 similar to the family considered in Section 5.3. We describe the fundamental vector fields on the discontiniuity domain which can be identified with $\mathbf{R}$. We have

$$X_{-1} = e^{-r}\partial r, \quad X_0 = \partial r, \quad X_1 = e^r \partial r.$$

The family of $\mathbf{Z}$ actions is given by

$$n \in \mathbf{Z} \to \rho_a(n) := exp(anX_0) \in \mathbf{Sl}_2, \quad a \in \mathbf{R}.$$

Let $S^1_a := \mathbf{R}/\rho_a(\mathbf{Z})$ with the induced $(\mathbf{Sl}_2, \mathbf{R})$-structure. We compute the differentials of the complex $(C^*_{\mathbf{R}}, d^*)$ with respect to the trivialization given by the basis fields

$$\begin{array}{rcl}
\partial r & \in & \Gamma C^0_{\mathbf{R}}(\mathbf{R}) \\
X^v_{-1} \otimes \partial r, \quad X^v_0 \otimes \partial r, \quad X^v_1 \otimes \partial r & \in & \Gamma C^1_{\mathbf{R}}(\mathbf{R}) \\
X^v_{-1} \wedge X^v_0 \otimes \partial r, \quad X^v_{-1} \wedge X^v_1 \otimes \partial r, \quad X^v_0 \wedge X^v_1 \otimes \partial r & \in & \Gamma C^2_{\mathbf{R}}(\mathbf{R}) \\
X^v_{-1} \wedge X^v_0 \wedge X^v_1 \otimes \partial r & \in & \Gamma C^3_{\mathbf{R}}(\mathbf{R}).
\end{array}$$

The result is

$$d^0 = \begin{pmatrix} e^{-r}\partial + e^{-r} \\ \partial \\ e^{r}\partial - e^{r} \end{pmatrix}$$

$$d^1 = \begin{pmatrix} \partial + 1 & -e^{-r}\partial - e^{-r} & 0 \\ e^{r}\partial - e^{r} & 2 & -e^{-r}\partial - e^{-r} \\ 0 & e^{r}\partial - e^{r} & -\partial + 1 \end{pmatrix}$$

$$d^2 = (e^{r}\partial - e^{r},\ -\partial,\ e^{-r}\partial + e^{-r}).$$

Let $Y_0 := X_0$, $Y_{-1} := e^{r}X_{-1}$, $Y_1 := e^{-r}X_1$. We consider the translation invariant base

$$\begin{array}{rcl} \partial r & \in & \Gamma C^0_{\mathbf{R}}(\mathbf{R}) \\ Y^v_{-1} \otimes \partial r, \quad Y^v_0 \otimes \partial r, \quad Y^v_1 \otimes \partial r & \in & \Gamma C^1_{\mathbf{R}}(\mathbf{R}) \\ Y^v_{-1} \wedge Y^v_0 \otimes \partial r, \quad Y^v_{-1} \wedge Y^v_1 \otimes \partial r, \quad Y^v_0 \wedge Y^v_1 \otimes \partial r & \in & \Gamma C^2_{\mathbf{R}}(\mathbf{R}) \\ Y^v_{-1} \wedge Y^v_0 \wedge Y^v_1 \otimes \partial r & \in & \Gamma C^3_{\mathbf{R}}(\mathbf{R}) \end{array}$$

With respect to this base the differentials (notation $\tilde{d}^*$) are:

$$\begin{array}{rcl} \tilde{d}^0 & = & \begin{pmatrix} \partial + 1 \\ \partial \\ \partial - 1 \end{pmatrix} \\ \tilde{d}^1 & = & \begin{pmatrix} \partial & -\partial - 1 & 0 \\ \partial - 2 & 2 & -\partial - 2 \\ 0 & \partial - 1 & -\partial \end{pmatrix} \\ \tilde{d}^2 & = & (\partial - 2,\ -\partial,\ \partial + 2). \end{array}$$

We introduce metrics by declearing this invariant base to be orthonormal. These metrics descend to all factors S^1_a. The adjoints of the differentials are

$$\begin{array}{rcl} (\tilde{d}^0) & = & (-\partial + 1,\ -\partial,\ -\partial - 1) \\ (\tilde{d}^1) & = & \begin{pmatrix} -\partial & -\partial - 2 & 0 \\ \partial - 1 & 2 & -\partial + 1 \\ 0 & \partial - 2 & \partial \end{pmatrix} \\ (\tilde{d}^2) & = & \begin{pmatrix} -\partial - 2 \\ \partial \\ -\partial + 2 \end{pmatrix}. \end{array}$$

Then the Laplacians become

$$\begin{array}{rcl} \Delta^0 & = & (\tilde{d}^0)^*\tilde{d}^0 = -3\partial^2 + 2 \\ \Delta^1 & = & \tilde{d}^0(\tilde{d}^0)^* + (\tilde{d}^1)^*\tilde{d}^1 = \begin{pmatrix} -3\partial^2 + 5 & -2\partial - 4 & 2\partial + 3 \\ 2\partial - 4 & -3\partial^2 + 6 & -2\partial - 4 \\ -2\partial + 3 & 2\partial - 4 & -3\partial^2 + 5 \end{pmatrix} \end{array}$$

$$\begin{aligned}\Delta^2 &= \tilde{d}^1(\tilde{d}^1)^* + (\tilde{d}^2)^*\tilde{d}^2 = \begin{pmatrix} -3\partial^2+5 & -2\partial-2 & -2\partial-3 \\ 2\partial-2 & -3\partial^2+12 & -2\partial-2 \\ 2\partial-3 & 2\partial-2 & -3\partial^2+5 \end{pmatrix} \\ \Delta^3 &= \tilde{d}^2(\tilde{d}^2)^* = -3\partial^2+8.\end{aligned}$$

The Laplacians Δ^*_a on S^1_a are simply the restrictions to the functions with period a. We apply the formula of Burghelea/Friedlander/Kappeler [3] for the determinants of elliptic operators on S^1. It reduces to

$$det\ \Delta^i_a = -det(1 - P^i_a)$$

where P^i_a is the monodromy map. Let $\sigma(\Delta^i)(\xi)$ be the total symbol and $Z(i)$ be the set of solutions of $det\ \sigma(\Delta^i)(\xi) = 0$ counted with multiplicity. Then

$$det(1 - P^i_a) = \Pi_{\lambda\in Z(i)}(1 - e^{a\lambda}).$$

We have

$$\begin{aligned} Z(0) &= \{-\sqrt{2/3},\ \sqrt{2/3}\} \\ Z(1) &= \{\sqrt{2/3},\ \sqrt{2/3},\ -\sqrt{2/3},\ -\sqrt{2/3},\ \sqrt{8/3},\ -\sqrt{8/3}\} \\ Z(2) &= \{\sqrt{8/3},\ \sqrt{8/3},\ -\sqrt{8/3},\ -\sqrt{8/3},\ \sqrt{2/3},\ -\sqrt{2/3}\} \\ Z(3) &= \{\sqrt{8/3},\ -\sqrt{8/3}\} \end{aligned}$$

From this calculation one gets

Proposition 5.4 *1.* $H^*(\Gamma C^{\cdot}_{S^1_a}(S^1_a)) = 0$,

2. $\tau(\Gamma C^{\cdot}_{S^1_a}(S^1)) = 1$.

The first results follows since 0 is not in $Z(i)$ for all i. For the second one employs

$$\tau(\Gamma C^{\cdot}_{S^1_a}(S^1)) = \Pi^3_{i=1}\Pi_{\lambda\in Z(i)}(1 - e^{a\lambda})^{-1^i i} = 1.$$

□

6 L^2-Cohomology

Let $\mathbf{G}$ be semi-simple and $\mathbf{H} \subset \mathbf{G}$ be parabolic. Let $\mathbf{X} := \mathbf{G}/\mathbf{H}$. In this section we consider the L^2-cohomology of the complex $(\Gamma C^*_M(M), d^*)$ where M is a $(\mathbf{G}, \mathbf{X})$-manifold. It requires the choice of metrics on the bundles C^*_M and of a volume. There is a subpresheaf $\Gamma_2 C^*_M$ of ΓC^*_M defined by

$$\Gamma_2 C^*_M(U) := \{\psi \in \Gamma C^*_M(U) | \psi \in L^2(M, C^*_M), d^*\psi \in L^2(M, C^{*+1}_M\}, \quad U \subset M.$$

The L^2-cohomology $H_2^*(C^\cdot_M)$ is defined as the cohomology of $(\Gamma_2 C^*_M(M), d^*)$. There is a related presheaf $\bar\Gamma_2 C^*_M$ defined by

$$\bar\Gamma_2 C^*_M(U) = \{\psi \in L^2(U, C^*_M) | \psi \in dom\ \bar d^*\}$$

where $\bar d^*$ is the maximal extension of the differential operator d^*. The cohomology of $(\bar\Gamma_2 C^*_M(M), \bar d^*)$ is denoted by $\bar H_2^*(C^\cdot_M)$ It is clear the the L^2-cohomology does only depend on the quasiisometry class of the metrics and the volume.

6.1 Local vanishing of H_2^*

Let $a \in \mathbf{a}$ be in the positive Weyl chamber with respect to the Iwasawa decomposition $\mathbf{G} = \mathbf{KAN}^+$. Set $a_t := e^{ta}$. Let $U \subset X$ be a precompact open subset such that $a_s U \subset a_s \bar U \subset a_t U$ for all $s > t \geq 0$.

Theorem 6.1 $H_2^*(U, C^\cdot_{\mathbf{X}}) = 0$

Proof: Let $\psi \in \Gamma_2 C^*_{\mathbf{X}}(U)$, $d\psi = 0$. We show that there is some $\phi \in \Gamma_2 C^{*-1}_{\mathbf{X}}(U)$ with $d\phi = \psi$. Let

$$\rho := \int_0^1 r_U i_a a_t \psi \, dt$$

where we use the notation of section 3.4 and r_U is the restriction map. It is easy to see that $\rho \in L^2(U, C^{*-1}_{\mathbf{X}})$. Moreover $d\rho = r_U a_1 \psi - \psi$. But $a_1\psi \in \Gamma C^{*-1}_{\mathbf{X}}(a_{-1}U)$. Thus by Theorem 3.1 there is a smooth $\mu \in \Gamma C^{*-1}_{\mathbf{X}}(a_{-1}U)$ with $d\mu = a_1\psi$. It follows $\phi := r_U\mu - \rho \in \Gamma_2 C^{*-1}_{\mathbf{X}}(U)$ and $\psi = d\phi$. This proves the Theorem. $\Box$

6.2 The map $i : H_2^* \to \bar H_2^*$

Let M be compact. The inclusion of presheafs $\Gamma_2 C^*_M \subset \bar\Gamma_2 C^*_M$ induces a map of the cohomology $i_* : H_2^*(C^\cdot_M) \to \bar H_2^*(C^\cdot_M)$.

Theorem 6.2 *i_* is an isomorphism.*

Proof: It is enough to show this locally since there is are Mayer-Vietoris sequences for L^2-cohomology. We adopt the notation of Section 6.1. Let us first show the surjectivity of i_*. Let $\psi \in \bar\Gamma_2 C^*_{\mathbf{X}}(U)$, $\bar d\psi = 0$. We have to show that $[\psi]$ can be represented by some element of $\Gamma_2 C^*_X(U)$. Note that

$$\rho := \int_0^1 i_a r_U a_t \psi \, dt \in \bar\Gamma_2 C^{*-1}_{\mathbf{X}}(U)$$

and $\bar d\rho = r_U a_1 \psi - \psi$. Thus we can represent the class of ψ by $\chi := a_1\psi$ which extends to the larger set $a_{-1}U$. Next we apply a mollifier to χ. Let $r := dim\mathbf{X}$ and choose a base $n_1, \dots, n_r$ of $\mathbf{n}^-$. Define the diffeomorphism $F : \mathbf{R}^r \to \mathbf{N}^-$ by

$F(t_1,\dots,t_r) := e^{t_1 n_1}\dots e^{t_r n_r}$. Let δ be a mollifier kernel on $\mathbf{R}$ with support in a small enough neighbourhood of 0. Define

$$J\chi := \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\Pi_{i=1}^r \delta(t_i)F(t_1,\dots,t_r)\chi dt_r\dots dt_1.$$

Then $r_U J\chi \in \Gamma_2 C^*_{\mathbf{X}}(U)$ and $dr_U J\chi = r_U J\bar{d}\chi = 0$. By partial integration

$$\begin{aligned} J\chi &= \int_{-\epsilon}^{\epsilon}\delta(t_1)e^{t_1 n_1}\dots\int_{-\epsilon}^{\epsilon}\delta(t_{r-1})e^{t_{r-1}n_{r-1}}(e^{\epsilon n_r}\chi \\ &- \ d\int_{-\epsilon}^{\epsilon}\left(\int_{-\epsilon}^{s}\delta(\tau)d\tau\right) i_{n_r}e^{sn_r}\chi)ds)dt_{r-1}\dots dt_1 \end{aligned}$$

Here we have chosen ϵ small enough such that $F(t_1,\dots,t_r)\chi$ is defined on U if $|t_i|\le\epsilon$, $i=1,\dots,r$ and $supp\,\delta\subset[-\epsilon,\epsilon]$. Repeating this procedure we obtain the representation $r_U J\chi = r_U e^{\epsilon n_1}\dots e^{\epsilon n_r}\chi + d\sigma$ for some $\sigma\in\bar{\Gamma}_2 C^{*-1}_{\mathbf{X}}(U)$. Thus we have the equation

$$[\psi] = [r_U\chi] = [r_U e^{\epsilon n_1}\dots e^{\epsilon n_r}\chi] = [r_U J\chi]\in \bar{H}^*_2(U, C^{\cdot}_M)$$

This shows surjectivity. Let us now show the injectivity. If $\phi = \bar{d}\psi$ for some $\psi\in\bar{\Gamma}_2 C^*_X(U)$ then $J\phi = J\bar{d}\psi = dJ\psi$. Moreover $[\phi]=[J\phi]\in H^*_2(U,(C^{\cdot}_X, d^{\cdot}))$. These two equations show that $[\phi]=0\in H^*_2(U, C^{\cdot}_X)$. $\Box$

6.3 Harmonic forms

Assume that the metric we have chosen on C^*_M induces a complete Riemannian metric on M (note that $TM = C^0_M$) and that the symbols of $d^{\cdot}$ and $(d^{\cdot})^*$ are uniformly bounded. Then $D := d^{\cdot} + (d^{\cdot})^* : C^\infty_c(M, C^{\cdot}_M)\to C^\infty_c(M, C^{\cdot}_M)$ is essentially selfadjoint as an unbounded operator on $L^2(M, C^{\cdot}_M)$ (see Chernoff [7]). Let

$$\mathcal{H}^*_2(C^{\cdot}_M) := ker\ D = \{\psi\in\Gamma_2 C^*_M(M)|d^{\cdot}\psi = 0, (d^{\cdot})^*\psi = 0\}$$

be the space of harmonic sections. We have the orthogonal decomposition

$$L^2(M, C^*_M) = clo\ im\ d^{*-1}\oplus clo\ im\ (d^*)^*\oplus\mathcal{H}^*_2(C^{\cdot}_M)$$

("$*$" denotes the adjoint as well as the grading). Hence there is an exact sequence

$$0\to\mathcal{H}^*_2(C^{\cdot}_M)\stackrel{j}{\to}\bar{H}^*_2(C^{\cdot}_M).$$

If $spec_{ess}(D)$ does not contain 0 then j is an isomorphism (see [6] for a discussion of similar results).

6.4 Positivity of the Laplacians on cylinders

Let $(\mathbf{G},\mathbf{X})$ be such that a local vanishing result for $\mathcal{H}^*(\Gamma C^{\cdot}_{\mathbf{X}})$ is true. Let M be a compact $(\mathbf{G},\mathbf{X})$-manifold. Assume that $M = N \times S^1$ for some compact manifold N and that ∂x (the vector field along S^1) is a fundamental field $V(X)$ for some $X \in \mathcal{E}_M(M)$. Let $\tilde{M} = N \times \mathbf{R}$ be the covering with respect to the subgroup $\mathbf{Z} = \pi_1(S^1) \subset \pi_1(M)$. Let $\tilde{X} \in \mathcal{E}_{\tilde{M}}(\tilde{M})$ be the lift of X. Then there is an $\mathbf{R}$-action on $\tilde{M}$ by translations and automorphisms of $(\mathbf{G},\mathbf{X})$-manifolds. Hence this action lifts to $\mathcal{E}_{\tilde{M}}$ and $(C^*_{\tilde{M}}, d^*)$. We choose a volume and metrics on $C^*_{\tilde{M}}$ which are $\mathbf{R}$-invariant. Let $\Delta^{\cdot}$ be the associated Laplacians of the complex $L^2(\tilde{M}, C^{\cdot}_{\tilde{M}})$. They are $\mathbf{R}$-invariant.

We define a family of $(\mathbf{G},\mathbf{X})$-structures on M by $M_a = \tilde{M}/\rho_a(\mathbf{Z})$ where $\rho_a(n) = e^{an\tilde{X}}$ for $a \in \mathbf{R}$. The initial structure was M_1. The metrics and the Laplacians descend to the factors M_a. We will employ this in order to show the following

Proposition 6.3 *There is a constant $c > 0$ such that $\Delta^{\cdot} \geq c$. Especially $\mathcal{H}^*_2(\tilde{M}) = \bar{H}^*_2(C^{\cdot}_M) = 0$.*

Proof: Using the Fourier-transformation in the $\mathbf{R}$-direction we have an unitary equivalence of $\Delta^{\cdot}$ and the family of elliptic differential operators on N given by

$$\sigma(\Delta^{\cdot})(\xi) := \partial x - \text{zero} - \text{order part of } e^{-\imath x\xi}\Delta^{\cdot} e^{\imath x\xi}$$

acting on $L^2(\mathbf{R}_\xi, L^2(N, C^{\cdot}_M))$ where N is identified with some cross-section of $\tilde{M} = N \times \mathbf{R}$. Hence

$$inf\ spec\,\Delta^{\cdot} = inf_{\xi\in\mathbf{R}} inf\ spec\,\sigma(\Delta^{\cdot})(\xi).$$

Assume that there is a sequence $\xi_i \to \infty$ such that there is a sequence $\lambda_i \in spec\,\sigma(\Delta^{\cdot})(\xi_i)$ with $\lambda_i \to 0$. Then one can find a sequence $n_j \in \mathbf{Z}$ such that $n_j - \xi_{i_j} \to 0$ where i_j is a subsequence of $\mathbf{Z}$. Replacing ξ_i by ξ_{i_j} we can assume that $\xi_j - n_j \to 0$. Since $\sigma(\Delta^{\cdot})(\xi) = (\sigma(d^{\cdot} + (d^{\cdot})^*)(\xi))^2$ and $\sigma(d^{\cdot} + (d^{\cdot})^*)(\xi) = A + B\xi$ with differential operators A, B on N where A is elliptic and B is bounded it is easy to see that the families of eigenvalues $\lambda(\xi)$ with $\lambda(\xi_i) = \lambda_i$ are uniformly Lipschitz continuous for large i near ξ_i. Hence there is also a sequence $\mu_i \in spec\,\sigma(\Delta^{\cdot})(n_i)$ with $\mu_i \to 0$. But the corresponding generalized eigenfunctions descend to $M_{2\pi}$. Thus we have a sequence of eigenvalues of the Laplacian on $M_{2\pi}$ converging to zero. But this is inpossible since the the Laplacian on $M_{2\pi}$ is positive and has discrete spectrum. Thus if $inf_{\xi\in\mathbf{R}} inf\ spec\,\sigma(\Delta^{\cdot})(\xi) = 0$ then there is some $\xi_0 \in \mathbf{R}$ such that $0 \in spec\,\sigma(\Delta^{\cdot})(\xi_0)$. The corresponding generalized eigenfunction descends to $M_{2\pi/\xi_0}$ giving a nontrivial harmonic section there. This is again inpossible. Hence $inf_{\xi\in\mathbf{R}} inf\ spec\,\sigma(\Delta^{\cdot})(\xi) = c > 0$.

6.5 Locally conformally flat manifolds

Let $\mathbf{G} = SO(n+1, 1)$ and $\mathbf{H}$ be the maximal parabolic subgroup. Then $\mathbf{X} = S^n$ and the $(\mathbf{G},\mathbf{X})$-manifolds are exactly the locally conformally flat manifolds. For locally conformally flat manifolds the local vanishing of $\mathcal{H}^*(\Gamma C^{\cdot}_{\mathbf{X}})$ holds by Theorem 3.1.

A locally conformally flat structure on M can also be given by a conformal class of Riemannian metrics with vanishing Weyl-tensor. A standard example is $\mathbf{R}^n$. $\mathbf{R}^n$ acts on itself by conformal automorphisms. We choose translation-invariant metrics and volume as above. Let $\Delta^{\cdot}$ be the Laplacians over $\mathbf{R}^n$. An argument very similar to the one of Proposition 6.3 gives

Proposition 6.4 *1.* $\Delta^{\cdot}_{\geq} c > 0$

2. $\mathcal{H}^*_2(C^{\cdot}_{\mathbf{R}^n}) = \bar{H}^*_2(C^{\cdot}_{\mathbf{R}^n}) = 0$

If M is a locally conformally flat manifold with finitly many ends beeing conformally euclidian we choose the metrics and volume such that they coincide with the translation invariant ones on the ends. Then we have

Proposition 6.5 *1.* $\Delta^{\cdot}_M \geq c > 0$

2. $\mathcal{H}^*_2(C^{\cdot})M = \bar{H}^*_2(C^{\cdot}_M) = 0$

Proof The first result follows from the spectral decomposition theorem (see [8]) stating

$$spec_{ess}(\Delta_{\mathbf{R}^n}) = spec_{ess}(\Delta_M)$$

and the second statement. We use a Mayer-Vietoris sequence in order to show that if $\mathbf{R}^n = U \cup V$ with U beeing cocompact and V precompact then $\bar{H}^*_2(U, C^{\cdot}_{\mathbf{R}^n}) = 0$. Using again the Mayer-Vietoris sequence now for the decomposition of M into the ends and a precompact part one shows $\mathcal{H}^*_2(C^{\cdot}_M) = \bar{H}^*_2(C^{\cdot}_M) = 0$. $\square$

Let N be a compact manifold with a metric of constant sectional curvature. Then $M := N \times S^1$ is locally conformally flat (with the product metric) (see Besse [1]). The rotation in the S^1-factor is a conformal automorphism. We choose rotation invariant metrics and volume. For the covering $\tilde{M} := N \times R$ we have from Proposition 6.3

Proposition 6.6 *1.* $\Delta^{\cdot}_{\tilde{M}} \geq c > 0$

2. $\mathcal{H}^*_2(C^{\cdot}_{\tilde{M}}) = \bar{H}^*_2(C^{\cdot}_{\tilde{M}}) = 0$

Let M be a locally conformally flat manifold with finitly many ends beeing conformally equivalent to a cylinder $[0, \infty) \times N$ where N has a metric with constant sectional curvature. We fix a volume and metrics which are translation invariant at infinity. Then using again the spectral decomposition theorem and the Mayer-Vietoris sequence one can show

Proposition 6.7 *1.* $\Delta^{\cdot}_M \geq c > 0$

2. $\mathcal{H}^*_2(C^{\cdot}_M) = \bar{H}^*_2(C^{\cdot}_M) = 0$

6.6 Blowing up cylinders

Let N be a manifold with constant sectional curvature and M be a locally conformally flat manifold which has a (conformal) decomposition $M = M_- \cup [-1,1] \times N \cup M_+$. Then there is a family on conformally flat manifolds obtained by making the cylinder longer. Thus let

$$M_\lambda := M_- \cup [-1,0] \times N \cup [-\lambda, \lambda] \times N \cup [0,1] \times N \cup M_+.$$

We choose metrics and a volume such that they are translation invariant on the cylindrical part and induce there a product metric giving the metric of constant sectional curvature on N. We assume that the transverse metrics do not depent on λ and that the cylindrical middle part has length 2λ. Let $\Delta^\cdot_\lambda$ be the Laplacians associated to the complex $(C^\cdot_{M_\lambda}, d^\cdot)$. Then

Proposition 6.8 *There is a constant $c > 0$ such that for all $\lambda \geq 0$ holds $spec\, \Delta^\cdot_\lambda \geq c$.*

Proof: Since M_λ is compact and $\mathcal{H}^*_2(M_\lambda) = 0$ we have $spec \Delta_\lambda > 0$ for all λ. Thus we will argue by contradiction. Let $\lambda_i \to \infty$ be a sequence such that there is a sequence $\mu_i^2 \in spec\, \Delta_{\lambda_i}$ with $\mu_i \to 0$. Let ψ_i be the corresponding normalized eigenfunctions. Using separation of variables and the explicit form of the eigenfunctions on the cylinder it is standard to show that

$$1 = \|\psi_i\|_{L^2(M_{\lambda_i})} \leq C(\|\psi_i\|_{L^2(M_- \cup [-1,0] \times N)} + \|\psi_i\|_{L^2([0,1] \times N \cup M_+)})$$

where C is some constant independent of i. Note that the transverse operator on the cross-section has no kernel. By passing to a subsequence (and possibly interchanging the $\pm$-parts) we can assume that there is a $c > 0$ such that

$$\|\psi_i\|_{L^2(M_- \cup [-1,0] \times N)} \geq c, \quad \forall i.$$

Let f_i be a sequence of cut-off functions on M_{λ_i} with $f = 1$ on $M_- \cup [-1,0] \times N$, $f = 0$ on $[0,1] \times N \cup M_+$ and $sup|df_i| \leq 2/\lambda_i$. We define the manifold with a cylindrical end $M_c := M_- \cup [-1, \infty) \times N$ with the complex $(C^\cdot_{M_c}, d^\cdot_c)$ and associated operators $D_c := d^\cdot_c + (d^\cdot_c)^*$, $\Delta^\cdot_c := D_c^2$. The metrics and the volume are given by the choices made above. Then $f_i \psi_i$ lives on M_c and $\|f_i \psi_i\|_{L^2(M_c)} \geq c$. Moreover

$$\|D_c f_i \psi_i\|_{L^2(M_c)} \leq \|f_i D_c \psi_i\|_{L^2(M_c)} + C\; sup_{M_c}\; |df_i| \|\psi_i\|_{L^2(M_c)} \leq (\mu_i + 2C/\lambda_i).$$

Thus

$$\frac{\|D_c f_i \psi_i\|_{L^2(M_c)}}{\|f_i \psi_i\|_{L^2(M_c)}} \to 0$$

and $0 \in spec \Delta_c$ by the variation principle. This is in contradiction with Proposition 6.7. $\square$

7 The torsion of locally conformally flat manifolds

7.1 Locally conformally flat Tori

Let $L \subset \mathbf{R}^n$ be a lattice isomorphic to $\mathbf{Z}^n$. Then $\mathbf{R}^n/L$ is a flat Torus T_L. Especially it is locally conformally flat. $\mathbf{R}^n$ acts on itself by conformal automorphisms and we equip the complex $C^{\cdot}_{\mathbf{R}^n}$ with a translation invariant metric. It induces a translation invariant Riemannian metric on $\mathbf{R}^n$ and gives also a volume. These metrics and the volume descend to all factors T_L.

Assume that n is odd. Then we can define the local torsion $\tau(C^{\cdot}_{\mathbf{R}^n})(x) =: \tau$ which is just a constant. Let $C^{\cdot}_{T_L}$ be the complex on T_L with the induced metric. We consider its torsion $\tau(C^{\cdot}_{T_L})$.

Proposition 7.1 *1.* $\tau = 1$

2. $\tau(C^{\cdot}_{T_L}) = 1$ *for any lattice* $L \subset \mathbf{R}^n$.

Proof: The idea is to use the fact that T_L is conformally equivalent to $T_{\lambda L}$ for $0 \neq \lambda \in \mathbf{R}$. We use the translation group in order to trivialize the bundles over $\mathbf{R}^n$. Let $\Delta^{\cdot}_{\mathbf{R}}$ be the Laplacian on $\mathbf{R}^n$. Then the heat kernel $e^{-t\Delta^{\cdot}_{\mathbf{R}}}(x,y)$ is just a matrix valued function. The pull back to $\mathbf{R}^n$ of the heat kernel of the Laplacian Δ_L on the Torus T_L is given by averaging:

$$e^{-t\Delta^{\cdot}_L}(x,y) = \sum_{r\in L} e^{-t\Delta^{\cdot}_{\mathbf{R}}}(r+x,y).$$

We split the sum as

$$e^{-t\Delta^{\cdot}_L}(x,y) = e^{-t\Delta^{\cdot}_{\mathbf{R}}}(x,y) + R^{\cdot}_L(t,x,y)$$

with

$$R^{\cdot}_L(t,x,y) := \sum_{0\neq r\in L} e^{-t\Delta^{\cdot}_{\mathbf{R}}}(r+x,y).$$

There is the uniform estimate

$$|e^{-t\Delta^{\cdot}_{\mathbf{R}}}(x,y)| < Ct^{-n/2}e^{-ct}e^{-c\frac{|x-y|^2}{t}}.$$

where the long time part comes from Proposition 6.4. Thus on the diagonal we have

$$\begin{aligned} |R_L(t,x,x)^{\cdot}| &\leq Ce^{-ct}\sum_{0\neq r\in L} e^{-c|r|^2/t} \\ &\leq Ce^{-ct}e^{-cl^2/t} \end{aligned}$$

where l is the minimal displacement of L (the values of C, c may change inside the computation while they are independent of L, t). Thus defining

$$F_L(s,x) := -\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} tr_x z N R^{\cdot}_L(t,x,x)dt$$

(compare Section 4.1 for the notation) we have absolute convergence at 0 and ∞ for all s and we can compute

$$ln\tau_L(x) := \frac{d}{ds}_{s=0} F_L(s,x) = \int_0^\infty t^{-1} tr_x z N R(t,x,x) dt.$$

Of course $ln\tau_L(x)$ is a constant $ln\tau_L$ independent of $x \in \mathbf{R}^n$. We have the estimate

$$\begin{aligned} ln\tau_L &\leq \int_0^\infty t^{-1} C e^{-ct} e^{-cl^2/t} dt \\ &\leq C \int_0^\infty e^{-ct} e^{-cl^2/t} \\ &\leq Ce^{-cl} \int_0^l e^{-ct} dt + C \int_l^\infty e^{-ct} dt \\ &\leq Ce^{-cl}. \end{aligned}$$

The log of the torsion of $C^{\cdot}_{T_L}$ is then given by the integral of $ln\tau + ln\tau_L$ over some fundamental domain of L. Hence

$$ln\tau(C^{\cdot}_{T_L}) = vol(T_L)(ln\tau + ln\tau_L).$$

We have

$$vol(T_L)(ln\tau + ln\tau_L) = ln\tau(C^{\cdot}_{T_L}) = ln\tau(C^{\cdot}_{T_{\lambda L}}) = \lambda^n vol(T_L)(ln\tau + ln\tau_{\lambda L}). \quad (4)$$

Inserting the estimate of $\tau_{\lambda L}$ we get

$$\lambda^n ln\tau \leq ln\tau + C\lambda^n e^{-c\lambda l} + ln\tau_L, \quad \forall \lambda \geq 0.$$

This is only possible when $ln\tau = 0$. Then taking $\lambda \to \infty$ in (4) one gets also $ln\tau_L = 0$. This finishes the proof. $\square$

7.2 Torsion on cylinders

Let N be a compact manifold with a metric of constant sectional curvature and $\tilde{M} = \mathbf{R} \times N$. Then $\tilde{M}$ is locally conformally flat. The group $\mathbf{R}$ acts on $\tilde{M}$ by translations. For any $a \in \mathbf{R}$ we have a lattice $L = a\mathbf{Z} \subset \mathbf{R}$. Let $M_L := \tilde{M}/L$. M_L is a locally conformally flat manifold. We choose translation invariant metrics and volumeon $\tilde{M}$. They descend to all factors M_L. Let $\Delta^{\cdot}$, $\Delta^{\cdot}_L$ be the Laplacians on $\tilde{M}$ and M_L respectively. Again we can obtain the heat kernel of $\Delta^{\cdot}_L$ by averaging. For the pull back to $\tilde{M}$ we have

$$e^{-t\Delta^{\cdot}_L}(x,y) = \sum_{r\in L} e^{-t\Delta^{\cdot}}(r+x,y)$$

where we use a suggestive notation. Again we have the estimate

$$|e^{-t\Delta}(x,y)| \leq Ct^{-n/2} e^{-ct} e^{-c\frac{dist(x,y)^2}{t}}$$

where the long time estimate is due to Proposition 6.6. Let $\tau(x)$ be the local torsion on $\tilde{M}$. It is $\mathbf{R}$-invariant. Define the off-diagonal part as above

$$R_L(t,x,y) := \sum_{0\neq r\in L} e^{-t\Delta^\cdot}(r+x,y).$$

Let

$$ln\tau_L(x) := \int_0^\infty t^{-1} tr_x zNR_L(t,x,x)dt$$

As in the proof of Proposition 7.1 one has

$$ln\tau_L \leq Ce^{-ca}$$

for some constants C, c independent of a. Then

$$ln\tau(C^\cdot_{M_L}) = a\int_N (\tau(x)+\tau_L(x)).$$

Here N is identified with some crossection of $\tilde{M}$. Thus we have

Proposition 7.2 *For large $a\in\mathbf{R}$*

$$\tau(C^\cdot_{M_L})/a = \int_N \tau(x) + O(e^{-ca}).$$

7.3 Blowing up cylinders and the almost additivity of the torsion

Let N be a manifold with constant sectional curvature and M be a locally conformally flat manifold which has a (conformal) decomposition $M = M_- \cup [-1,1]\times N \cup M_+$. Then there is a family on conformally flat manifolds obtained by making the cylinder longer. Let

$$M_\lambda := M_- \cup [-1,0]\times N \cup [-\lambda,\lambda]\times N \cup [\lambda,\lambda+1]\times N \cup M_+.$$

We choose metrics and a volume such that they are translation invariant on the cylindrical part and induce there a product metric giving the metric of constant sectional curvature on N. We assume that the transverse metric does not depend on λ and that the middle cylindrical part has length 2λ. Let $\tilde{M} := \mathbf{R}\times N$ as in the preceeding section. We choose the metrics and the volume compatible with the ones on cylindrical part of M_λ. We study the difference of $ln\tau(C^\cdot_{M_\lambda})(x)$ and $ln\tau(C^\cdot_{\tilde{M}})(x)$ on the part $[-\lambda,\lambda]\times N$. The finite propagation speed method and Propositions 6.6 and 6.8 give the estimate of the difference of the heat kernels

$$|e^{-t\Delta^\cdot_{M_\lambda}}(x,x) - e^{-t\Delta^\cdot_{\tilde{M}}}(x,x)| \leq Ce^{-ct}e^{-c\frac{|x+\lambda|^2\wedge|x-\lambda|^2}{t}}.$$

where e.g.$|x-\lambda|$ is a short notation for the distance of x from the section $\{\lambda\}\times N$. Thus by the same technique as in the proof of Proposition 7.1 we get for $x\in[-\lambda,\lambda]\times N$

$$|ln\tau(C^{\cdot}_{M_\lambda})(x)-ln\tau(C^{\cdot}_{\tilde{M}})(x)|\leq Ce^{-c(|x+\lambda|+|x-\lambda|)}.$$

Let $M_\infty := M_-\cup[-1,\infty)\times N\cup(-\infty,\lambda+1]\times N\cup M_+$. Then by an analogous argument

$$|ln\tau(C^{\cdot}_{M_\lambda})(x)-ln\tau(C^{\cdot}_{M_\infty})(x)|\leq Ce^{-c|x-\lambda|}$$

for $x\in M_-\cup[-1,\lambda]\times N$ which can be identified with the corresponding subset of M_∞ and

$$|ln\tau(C^{\cdot}_{M_\lambda})(x)-ln\tau(C^{\cdot}_{M_\infty})(x)|\leq Ce^{-c|x+\lambda|}$$

for $x\in[0,\lambda+1]\times N\cup M_+$. Let

$$ln\tau := \int_{\{0\}\times N} ln\tau(C^{\cdot}_{\tilde{M}})(x).$$

We can now compute

$$\begin{aligned}
& ln\tau(C^{\cdot}_{M_\lambda})\\
:= & \int_{M_-\cup([-1,0]\cup[-\lambda,-\lambda/2])\times N} ln\tau(C^{\cdot}_{M_\lambda})(x)+\int_{[-\lambda/2,\lambda/2]\times N} ln\tau(C^{\cdot}_{M_\lambda})(x)\\
+ & \int_{([\lambda/2,\lambda]\cup[0,1])\times N\cup M_+} ln\tau(C^{\cdot}_{M_\lambda})(x)\\
= & \int_{M_-\cup([-1,0]\cup[-\lambda,-\lambda/2])\times N} ln\tau(C^{\cdot}_{M_\infty})(x)+\int_{[-\lambda/2,\lambda/2]\times N} ln\tau(C^{\cdot}_{\tilde{M}})(x)\\
+ & \int_{([\lambda/2,\lambda]\cup[0,1])\times N\cup M_+} ln\tau(C^{\cdot}_{M_\infty})(x)+O(e^{-c\lambda})\\
= & \lambda ln\tau+\int_{M_-(\cup[-1,0]\cup[-\lambda,-\lambda/2])\times N} ln\tau(C^{\cdot}_{M_\infty})(x))+\int_{([\lambda/2,\lambda]\cup[0,1])\times N\cup M_+} ln\tau(C^{\cdot}_{M_\infty})(x)\\
+ & O(e^{-c\lambda})\\
= & 2\lambda ln\tau+\int_{M_-\cup[-1,0]\times N} ln\tau(C^{\cdot}_{M_\infty})(x))+\int_{[0,1]\times N\cup M_+} ln\tau(C^{\cdot}_{M_\infty})(x)+O(e^{-c\lambda})\\
+ & 2O(\int_0^{\lambda/2} e^{-cs}ds)\\
= & 2\lambda ln\tau+O(1).
\end{aligned}$$

Thus we have

Proposition 7.3 *For large λ*

$$ln\tau(C^{\cdot}_{M_\lambda})/\lambda = 2ln\tau+O(\lambda^{-1}).$$

Let $M_+ = D^n$ be the conformal ball. Then $N = S^{n-1}$ and M_λ is conformally equivalent to M_0. Thus taking the limit $\lambda\to\infty$ and using $ln\tau(C^{\cdot}_{M_\lambda}) = ln\tau(C^{\cdot}_{M_0})$ one gets

Corollary 7.4 *On* $\mathbf{R} \times S^{n-1}$ *the local torsion* $ln\tau$ *vanishes.*

If $N = S^{n-1}$ then
$$ln\tau_2(C^{\cdot}_{M_\infty}) := \int_{M_\infty} ln\tau(C^{\cdot}_{M_\infty})(x)$$
converges. It is some sort of regularized L^2-torsion. We have the almost additivity of the torsion for conformal connected sums

Corollary 7.5 *If* $N = S^{n-1}$ *then*
$$ln\tau(C^{\cdot}_{M_\lambda}) = ln\tau_2(C^{\cdot}_{M_\infty}) + O(\lambda^{-1}).$$

References

[1] A.L.Besse. *Einstein Manifolds.* Springer-Verlag, 1987.

[2] J. M. Bismut and W. Zhang. Reidemeister, Milnor and Ray-Singer metrics: an extension of a theorem of Cheeger and Müller. *Asterisque*, 250, 1992.

[3] D. Burghelea, L. Friedlander, and T. Kappeler. On the determinant of elliptic differential and finite difference operators in vector in vector bundles over S^1. *Communications in Mathematical Physics*, 138:1–18, 1991.

[4] D. Burghelea, L. Friedlander, and T. Kappeler. Analytic torsion equals Reidemeister torsion, a new proof. Preprint, 1992.

[5] R. D. Canary, D. B. A. Epstein, and P. Green. Notes on notes of Thurston. *London Math. Soc. Lecture Notes Ser. 111*, 1985.

[6] J. Cheeger. On the Hodge theory of Riemannian pseudomanifolds. *Proc. of Symp. in pure Math.*, 36:91–146, 1980.

[7] P. Chernoff. Essential selfadjointness of powers of generators of hyperbolic equations. *Journal of Functional Analysis*, 12:401–414, 1973.

[8] J. Eichhorn. Elliptic differential operators on noncompact manifolds. In *Seminar Analysis of the Karl-Weierstraß-Institute 1986/87*, pages 4–169. Teubner Leipzig, 1988.

[9] R. S. Kulkarni and U. Pinkall. Uniformizations of geometric structures with applications to conformal geometry. In *LNM1209*, pages 190–209. Springer Verlag, 1986.

[10] W. Müller. Analytic torsion and R-torsion for unimodular representations. *Preprint, MPI 91-50*, 1991.

[11] D. B. Ray and I. M. Singer. R-torsion and the Laplacian on Riemannian manifolds. *Adv.Math.*, 7:145–210, 1971.

[12] P. Scott. The geometries of 3-manifolds. *Bull.London math.Soc.*, 15(5):401–487, 1983.

[13] W. P. Thurston. The geometry and topology of 3-manifolds. *lecture notes distributed by diffusion*, 1980.

[14] E. Witten. Supersymmetry and Morse theory. *Journal of Differential Geometry*, 17:661–692, 1982.

æ