Andrejewski Lectures
by

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The Bethe Ansatz*

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January 1993

*supported by the Walter and Eva Andrejewski Foundation

1 Lecture 1

1.1 Motivation of the Bethe Ansatz

For many years the structure of $1+1$-dimensional systems (one space + one time dimension) has been of great interest for mathematical physics.

For mathematicans these systems are attractive because of their beautiful and rich structures.

On the other hand these systems are connected to many different questions of real physics.

The Bethe Ansatz is a unifying method connecting a wide range of different problems in these areas. It has been applied to the magnetic chains in solid state physics as well as to problems of (relativistic) field theory and elementary particle physics (Sine-Gordon equation, solitons, integrable models).

After twenty years of experience in this field we have the feeling that $1+1\,D$ systems are of importance for four-dimensional quantum field theory, too.

In his last adress to the international conference Feynman mentions the Bethe Ansatz and advocates the picture that high energy scattering is mainly $1+1$-dimensional with finitely many transversal modes only.

An application to gravity (and gauge theory) has been found by Verlinde and others [1] , when they separated the $4\,D$-Einstein equations into two sets of $2\,D$ equations. In general we beleive, that the search for the $2\,D$ structures in $4\,D$ problems will become more active in the near future.

In this lectures we shall study in detail the following models:

- magnetic chains
- non-linear sigma model
- Sine-Gordon equation.

and shell make some comments on conformal field theory.

All the models are studied from the quantum mechanical point of view. That is, we use the language of Hilbert space, states and vacuum. The language of classical statistical mechanics or the terminology of quantum groups could be used as well, but we will not do so.

1.2 Spin Chain

The simplest model where the Bethe Ansatz gives an exact solution is presented in the first lecture. It is the spin $\frac{1}{2}$ XXX magnetic chain. The exposition follows a paper of L.Takhtadzan and L. Faddeev, where more details can be found [2].

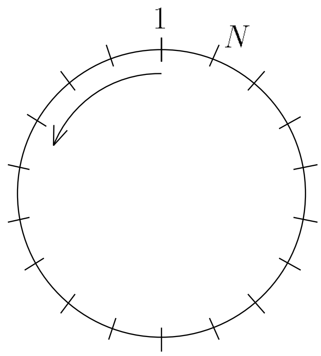


Figure 1: Heisenberg chain with periodic boundary conditions

1.2.1 Definition of the Model

Consider the one–dimensional quantum spin chain with N sites. Each site has a local spin- $\frac{1}{2}$ variable:

$$\vec{s}_n = \tfrac{1}{2}\vec{\sigma}_n, \qquad n \in \{1, 2, 3, \cdots, N\} \tag{1.1}$$

where components of the vector $\vec{\sigma}$ are Pauli matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.2}$$

acting on a local Hilbert space $h_n := \mathbb{C}^2$. The full (physical) Hilbert space is

$$\mathcal{H} := \bigotimes_{n=1}^{N} h_n = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N-\text{times}} \tag{1.3}$$

and has dimension $\dim(\mathcal{H}) = 2^N$. The operators $\vec{s}_n$, acting on $\mathcal{H}$, are defined as follows

$$\vec{s}_n = 1 \otimes \cdots \otimes \tfrac{1}{2}\vec{\sigma} \otimes \cdots \otimes 1. \tag{1.4}$$

From the physical point of view the thermodynamic limit $N \to \infty$ is most interesting. It leads to a kind of infinite tensor product and a model with infinitely many degrees of freedom.

The dynamics of the chain is given by a Hamilton operator with nearest neighbour interaction:

$$H := \sum_{n=1}^{N} \left(\vec{s}_n \cdot \vec{s}_{n+1} - \frac{1}{4} \right) \qquad \text{(Heisenberg Hamiltonian).} \tag{1.5}$$

Since $\vec{s}_n \cdot \vec{s}_{n+1} \leq \frac{1}{4}$ the normalization of H is chosen such that $H \leq 0$. The periodic boundary conditions $\vec{s}_n = \vec{s}_{n+N}$ are imposed on H.

In the following we shall sometimes add the length N of the chain as an index to the Hilbert space $\mathcal{H}_N$ and the Hamiltonian H_N.

Introducing the permutation operator P on $h \otimes h = \mathbb{C}^2 \otimes \mathbb{C}^2$ given by

$$P := \tfrac{1}{2}(1 \otimes 1 + \vec{\sigma} \otimes \vec{\sigma}) = \tfrac{1}{2} 1 \otimes 1 + 2\vec{s} \otimes \vec{s} \tag{1.6}$$

where $\vec{\sigma} \otimes \vec{\sigma} := \sum_{a=1}^{3} \sigma^a \otimes \sigma^a$, we have for any state $a \otimes b \in h \otimes h$

$$P(a \otimes b) = b \otimes a \tag{1.7}$$

or in the basis $\{|++>, |+->, |-+>, |-->\}$ of $h \otimes h$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.8}$$

Using P, we can write H as

$$H = \frac{1}{2} \sum_{n=1}^{N} (P_{n,n+1} - 1). \tag{1.9}$$

1.2.2 Short History of the Heisenberg Spin Chain

- 1928, W. Heisenberg: Introduction of the model [4]
- 1931 H. Bethe: Exact solution of the XXX-model [5]
- 1938 Hulthén: Construction of antiferromagnetic ground state with energy $E_0 = -N \ln(2)$, and thermodynamic limit [6]
- 1962 Cloizeaux and Pearson: Construction of excitations for antiferromagnetic system and thermodynamic limit [7]
- 1966 Yang and Yang: Generalization of Bethe Ansatz to the XXZ model [8]

- 1972 Baxter: Solution of the XYZ model via the transfer matrix method [9]

Here the Bethe Ansatz wil be presented in its (modern) algebraic form. This form resembles the algebraic solution of the harmonic oscillator in quantum mechanics and does not make use of explicit representations of the Schrödinger equation.

The method appeared as a development of inverse scattering methods, which was first applied to the KdV equation in classical physics [10], [11]. But the quantum version appears to be much easier to handle. Without the background of the inverse scattering method the operators introduced in the next section are somewhat artificial. But it will show up soon that they are very useful.

1.3 Fundamental Commutation Relations – Local View

We introduce locally defined Lax operators, one for each lattice site, and derive its (fundamental-) commutation relations. In order to do so we consider the space $h_n \otimes V_a$, where the auxilliary space V_a is just another copy of the $\mathbb{C}^2$, in our example. We define for each lattice site $n \in \mathbb{N}$ the Lax operator by

$$L_{n,a} := \lambda(1_n \otimes 1_a) + i\vec{s_n} \otimes \vec{\sigma_a} \quad \lambda \in \mathbb{C}^2 \tag{1.10}$$

The index "a" refers to the auxilliary space while "n" denotes the n-th site in the physical space. $\vec{\sigma_a}$ are the usual Pauli-matrices acting as the natural operators on $V_a = \mathbb{C}^2$. 1_a and 1_n are the units in h_n and V_a respectively. In a basis of $V = \mathbb{C}^2$ the Lax operator has the form of a matrix with operator valued entries. It can be written in the following way :

$$L_n(\lambda) := \begin{pmatrix} \lambda + is_n^{(3)} & is_n^- \\ is_n^+ & \lambda + is_n^{(3)} \end{pmatrix} \tag{1.11}$$

where $s_n^\pm := s_n^{(1)} \pm is_n^{(2)}$, $\quad i = \sqrt{-1}$.

When using the matrix form of an L–operator, we shall drop the index "a". Using the permutation operators $P_{n,a} = \frac{1}{2} 1_n \otimes 1_a + \vec{s_n} \otimes \vec{\sigma_a}$ we can write:

$$L_n(\lambda) = \left(\lambda - \frac{i}{2}\right) 1_n \otimes 1_a + iP_{n,a}. \tag{1.12}$$

Interpreting $h_n \otimes V_a = \mathbb{C}^2 \otimes \mathbb{C}^2$ as the state space for a two–particle scattering process, the Lax operator can be seen as the scattering matrix of such a model. In a diagram 2 the interaction is expressed as a crossing of the state spaces represented by lines with indices "a" and "n". Note that the part $P_{n,a}$ in the Lax operator simply permutes "n" and "a".

Considering more than one scattering process related to matrices L_{n,a_1} and L_{n,a_2} on a state space $h_n \otimes V_{a_1} \otimes V_{a_2} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ we introduce a commutation relation for matrix elements of L_{n,a_1} and L_{n,a_2}. It is convenient to express it by a diagram 3. Since the sequence of scattering processes is given by the matrix

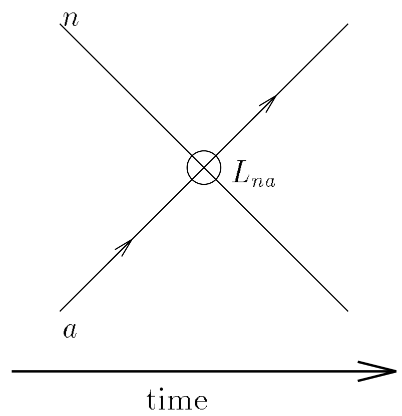


Figure 2: Single Scattering matrix $L_{n,a}$

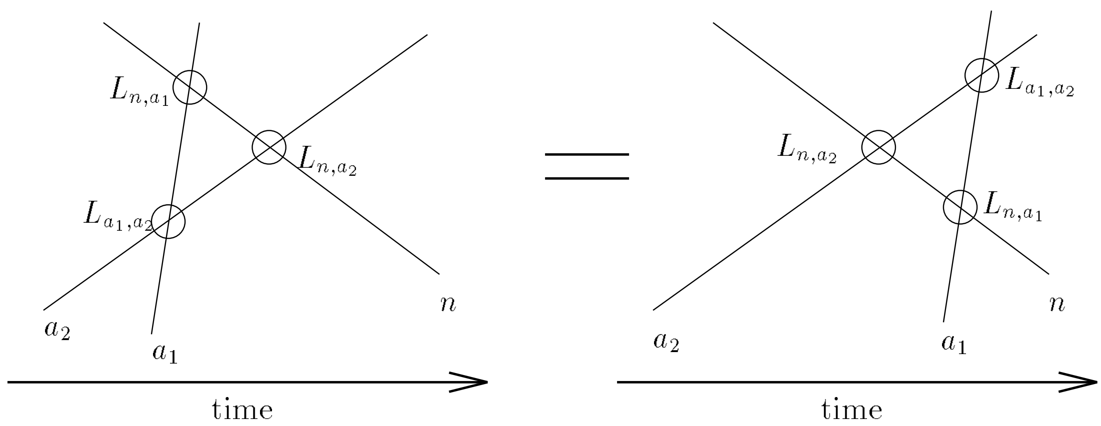


Figure 3: Commutation relations expressed in a 3 state scattering diagram: The order of scattering processes can be reversed.

product of the related scattering operators, we deduce from the diagram the commutation relations:

$$R(\lambda-\mu)\cdot L_{n,a_1}(\lambda)\cdot L_{n,a_2}(\mu)=L_{n,a_2}(\mu)\cdot L_{n,a_1}(\lambda)R(\lambda-\mu)\quad \text{(FCR)} \tag{1.13}$$

These relations are called Fundamental – Commutation Relations (FCR). The matrix $R(\lambda-\mu)$ is given explicitely by i.e.

$$\begin{aligned} R(\lambda) &= \frac{1}{\lambda+i}(\lambda 1 + iP_{a_1,a_2}) = b(\lambda)1 + c(\lambda)P, \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

where $b(\lambda):=\frac{\lambda}{\lambda+i}$ and $c(\lambda):=\frac{i}{\lambda+i}$.

Thus we have the relation

$$R(\lambda)=\frac{1}{\lambda+i}L_{a_1,a_2}\left(\lambda+\frac{i}{2}\right). \tag{1.14}$$

Note that the Fundamental–Commutation Relations are given for each single lattice site: they are local. But they have rather strong consequences for the whole chain. This will be seen in the next section.

1.4 Monodromy Matrix – Global View

After we have defined a Lax operator at each site of the chain, we will interpret the set of all these operators (in the 2×2 matrix form) as a kind of gauge-field generating a parallel transport (or holonomy) along the chain. So we can define the monodromy matrix $T_{N,a}$ for a transport once arround the chain

$$T_{N,a}(\lambda):=L_{N,a}(\lambda)\cdot L_{N-1,a}(\lambda)\cdots L_{1,a}(\lambda) \tag{1.15}$$

The matrix product is the one in the auxilliary space V_a. $T_{N,a}$ is defined on $\mathcal{H}_N\otimes V_a$, thus it is a matrix with entries that are operators on $\mathcal{H}_N$. As we will see later, the Hamiltonian and other useful operators can be expressed through the monodromy matrix.

Now from the (local) FCR's one immediately derives the related global commutation relations for the monodromy matrix

$$R(\lambda-\mu)\cdot T_{N,a_1}(\lambda)\cdot T_{N,a_2}(\mu)=T_{N,a_2}(\mu)\cdot T_{N,a_1}(\lambda)R(\lambda-\mu)\quad (CR) \tag{1.16}$$

by substitution and local commutation.

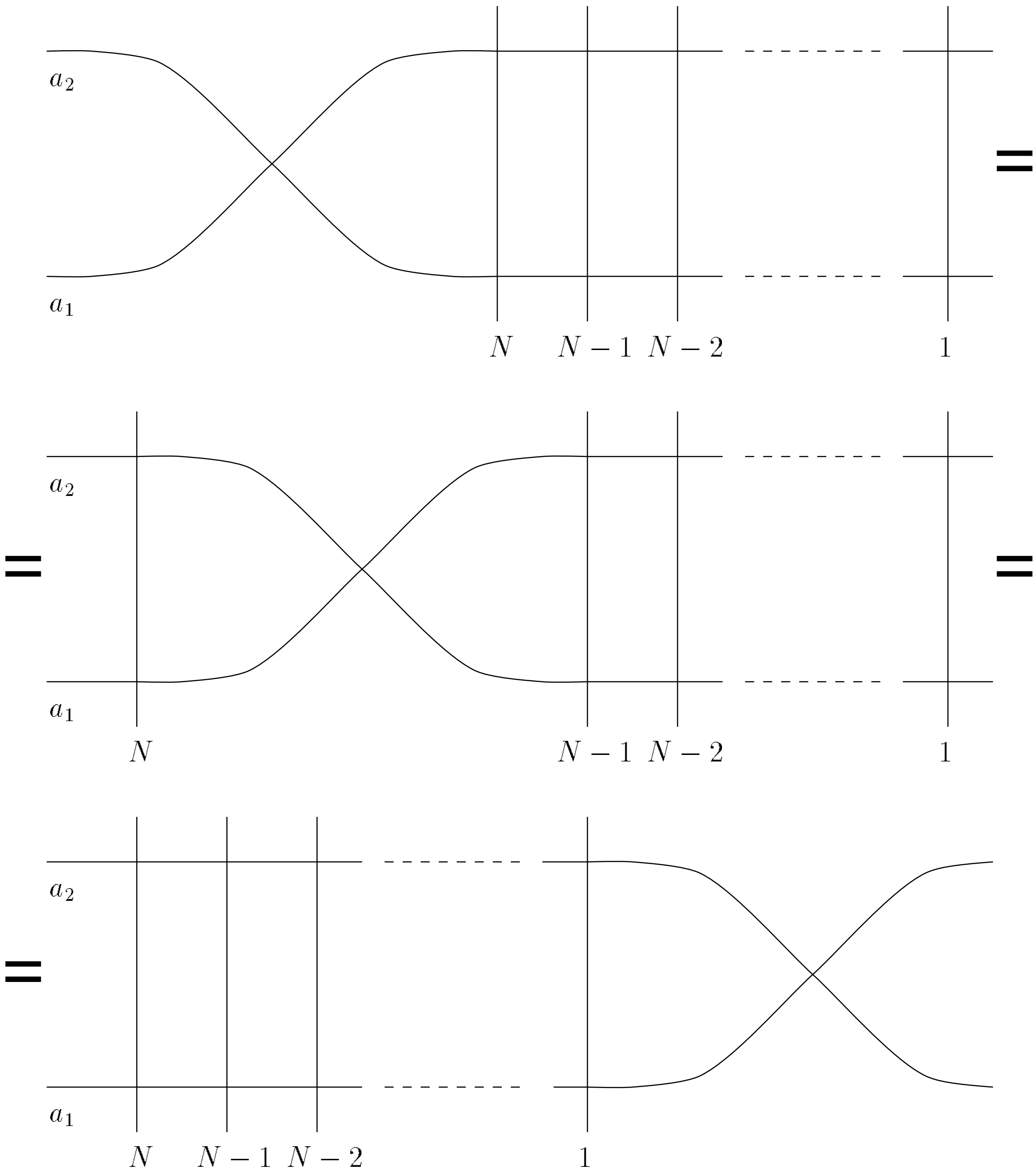


Figure 4: Scattering diagram for global commutation relations on the transfer matrices

Note that the global commutation relations and the local ones are of the same form. In a scattering diagram this procedure is expressed as in fig.4 To see the relation between the monodromy matrix and the Hamiltonian or permutation operator we introduce a family of operators:

$$F(\lambda) := \mathrm{tr}_a \left(T_{N,a}(\lambda)\right) \quad (\lambda \in \mathbb{R}). \tag{1.17}$$

here the trace is taken with respect to the auxilliary space. In the matrix form of the monodromy matrix we have

$$T(\lambda) := \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{1.18}$$

and

$$F(\lambda) := \mathrm{tr}\left(T(\lambda)\right) = A(\lambda) + D(\lambda). \tag{1.19}$$

From the commutation relations (CR) we find that

$$[F(\lambda), F(\mu)] = 0. \tag{1.20}$$

The $F(\lambda)$'s form a family of commuting operators on the (physical) Hilbert space $\mathcal{H}_N$. Together, the CR mean 16 different commutation relations for the operators $A(\lambda)$,$B(\lambda)$,$C(\lambda)$ and $D(\lambda)$.

Three of them are given in the next lines, since we will need them for further computations

$$\begin{aligned} A(\lambda)B(\lambda) &= \alpha(\lambda-\mu)B(\mu)A(\lambda) + \beta(\lambda-\mu)B(\lambda)A(\mu) && (1.21) \\ D(\lambda)B(\lambda) &= \gamma(\lambda-\mu)B(\mu)D(\lambda) + \delta(\lambda-\mu)B(\lambda)D(\mu) && (1.22) \\ B(\lambda)B(\mu) &= B(\mu)B(\lambda) && (1.23) \end{aligned}$$

where $\alpha(\lambda) := \frac{\lambda-i}{\lambda}$, $\gamma(\lambda) := \frac{\lambda+i}{\lambda}$.$\beta$ and δ are irrelevant at this point.

Next we consider the special case $\lambda = \frac{i}{2}$. From

$$L_{n,a}\left(\lambda = \frac{i}{2}\right) = \left(\frac{i}{2} - \frac{i}{2}\right) 1_n \otimes 1_a + iP_{n,a} = iP_{n,a}. \tag{1.24}$$

we find

$$F\left(\lambda = \frac{i}{2}\right) = i^N \mathrm{trace}_a \left(P_{N,a} \cdot P_{N-1,a} \cdots P_{1,a}\right). \tag{1.25}$$

Using $P_{n,a}P_{n-1,a} = P_{n-1,n}P_{n,a}$ and $P_{N-1,N} = P_{N,N-1}$ we get

$$\begin{aligned} F\left(\frac{i}{2}\right) &= i^N \mathrm{trace}_a \left(P_{1,a} \cdot P_{1,2} \cdots P_{N-1,N}\right) \\ &= i^N P_{1,2}\dot{P}_{2,3} \cdots P_{N-1,N} =: i^N U. \end{aligned}$$

U acts as the shift operator on the chain. For each local operator x_n we have

$$Ux_nU^{-1} = x_{n+1} \tag{1.26}$$

The momentum operator P is given by definition by equation $U = exp(iP)$. Hence it will belong to the family of operators $F(\lambda)$. To prove the analogous statements for the Hamiltonian consider the derivative of $F(\lambda)$ at $\lambda = \frac{i}{2}$, which is well defined due to the commutativity (1.20).

By construction $\frac{d}{d\lambda}F(\lambda)\|_{\lambda=\frac{i}{2}}$ is a sum of products of permutations at all but one site. So

$$\frac{d}{d\lambda}F(\lambda)F(\lambda)^{-1}\|_{\lambda=\frac{i}{2}} = \left(\sum_{n=1}^{N} P_{n,n-1}\right) \cdot \text{constant factor} \tag{1.27}$$

is the Hamiltonian up to some constant. More exactly

$$\frac{i}{2}\frac{d}{d\lambda}F(\lambda)\,F(\lambda)^{-1}\|_{\lambda=\frac{i}{2}} = \frac{i}{2}\frac{d}{d\lambda}\left(\ln F\left(\frac{i}{2}\right)\right) = \left(H_N + \frac{N}{2}\right). \tag{1.28}$$

1.5 Diagonalisation of $F(\lambda)$ – the “Bethe-Ansatz”

After finding the Hamiltonian as a member of the family of operators $F(\lambda)$ we will look for eigenstates and eigenvalues of $F(\lambda)$.

Note that the monodromy matrix $T_{N,a}(\lambda)$ is a polynomial of degree N in λ. So $F(\lambda)$ provides N commuting operators.

Consider the local highest weight state

$$\omega_n := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.29}$$

on the Hilbert space h_n. Then $s_n^{+}\omega_n = 0, s^{(3)}\omega_n = \frac{1}{2}\omega_n$ and

$$L_n\omega_n = \underbrace{\begin{pmatrix} a(\lambda) & \dots \\ 0 & d(\lambda) \end{pmatrix}}_{\text{triangular matrix}} \omega_n \tag{1.30}$$

with $a(\lambda) = \lambda + \frac{i}{2}$ and $d(\lambda) = \lambda - \frac{i}{2}$. The global state with all spins up:

$$\Omega := \bigotimes_{n=1}^{N} \omega_n \tag{1.31}$$

is a global highest weight state $f_0 2$ under the total spin operator $\vec{S} = \sum_{n=1}^{N} \vec{s}_n$,

$$\begin{aligned} S^{(+)}\Omega &= 0, \\ S^{(3)}\Omega &= \frac{N}{2}\Omega. \end{aligned}$$

As a product of triangular matrices the monodromy $T_{N,a}$ is also of triangular form on the state Ω

$$T_{N,a}\Omega = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \Omega \tag{1.32}$$

Explicit computation gives for A and D

$$\begin{aligned} A(\lambda) \quad \Omega &= (\lambda + \tfrac{i}{2})^N \Omega \\ D(\lambda) \quad \Omega &= (\lambda - \tfrac{i}{2})^N \Omega \end{aligned}$$

Thus Ω is an eigenstate of $F(\lambda) = A(\lambda) + D(\lambda)$. The role of Ω for the XXX-chain is similar to that of the vacuum state for the harmonic oscillator. To find other eigenstates of $A + D$ we make the "Bethe-Ansatz" :

$$\phi(\{\lambda_l\}) := \underbrace{B(\lambda_1)B(\lambda_2)\cdots B(\lambda_\ell)\Omega}_{\ell\text{-times}} \tag{1.33}$$

$B(\lambda)$ plays the role of a particle creation operator. Using the CR for the A,B and D,B given above, we can compute the action of $F(\lambda)$ on this state and get

$$\begin{aligned} F(\lambda)\phi(\{\lambda_k\}) &= \left[\prod_{k=1}^{l} \alpha(\lambda - \lambda_k)\Big(\lambda + \frac{i}{2}\Big)^N + \prod_{k=1}^{l} \gamma(\lambda - \lambda_k)\Big(\lambda - \frac{i}{2}\Big)^N\right] \phi(\{\lambda_k\}) \\ &\quad + \text{other terms} \qquad (1.34) \\ &= \left[\prod_{k=1}^{l} \left(\frac{\lambda - \lambda_k - i)}{\lambda - \lambda_k}\right) \left(\lambda + \frac{i}{2}\right)^N + \prod_{k=1}^{l} \left(\frac{\lambda - \lambda_k + i}{\lambda - \lambda_k}\right) \left(\lambda - \frac{i}{2}\right)^N\right] \phi(\{\lambda_k\}) \\ &\quad + \text{other terms} \qquad (1.35) \end{aligned}$$

The other terms are generated by the second terms in the RHS of (1.23). Removing this unwanted terms by a choice of $\lambda_1, \ldots \lambda_l$,makes $\phi(\{\lambda_k\})$ an eigenstate of $F(\lambda)$. In order to do so we have solve a transcendental equation.

Note that the left hand side of the eigenvalue equation is a polynomial in λ_k, while the given part of the rhs has poles. So the unwanted terms have poles, too.

At this point a miracle happens: in fact there exists one system of equations, which removes the poles in (1.35),as well as the unwanted terms.

$$\left(\frac{\lambda_k + \frac{i}{2}}{\lambda_k - \frac{i}{2}}\right)^N = \prod_{j=1, j\neq k}^{l} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i} \tag{1.36}$$

As a result we have the following statement :

$$\phi(\{\lambda_1, \ldots \lambda_l\})\text{is eigenvector of } F(\lambda) \Leftrightarrow \lambda_1, \ldots \lambda_l \text{ solve equation (1.36)} \tag{1.37}$$

which gives in principle the solution of the isotropic Heisenberg chain.

Consider the case $\lambda = \frac{i}{2}$. Here the contributions of $\gamma(\lambda)$ in the eigenvalue equation do not appear.

For $U = F(\lambda = \frac{i}{2})$ we get immediately the spectrum of the shift operator.

$$U\phi(\{\lambda_k\}) = A\left(\frac{i}{2}\right)\phi(\{\lambda_k\}) = \prod_{k=1}^{l}\left(\frac{\lambda_k + \frac{i}{2}}{\lambda_k - \frac{i}{2}}\right)\phi(\{\lambda_k\}) \tag{1.38}$$

Thus we see that the spectrum of U is multiplicative. We can interprete the function

$$p(\lambda) := \frac{1}{i} \ln \left(\frac{\lambda + \frac{i}{2}}{\lambda - \frac{i}{2}} \right) \tag{1.39}$$

as the momentum of a "pseudo particle" generated by the creating operator $B(\lambda)$ acting on the vacuum Ω. In the sense of the field theory λ can be interpreted as the "rapidity". To get the energy spectrum we compute the derivative:

$$h(\lambda) = \frac{d}{d\lambda} p(\lambda) = -\tfrac{1}{2} \tfrac{1}{\lambda^2 + \frac{1}{4}} \tag{1.40}$$

and get the additive pseudo particle spectra:

$$H\phi(\{\lambda_1, \ldots \lambda_l\}) = \left(\sum_{k=1}^{l} h(\lambda_k) \right) \phi(\{\lambda_1, \ldots \lambda_l\}) \tag{1.41}$$

We finish the first lecture with some remarks on the spin of Bethe states.

1. It can be proven, that if all λ's are finite then all states are states of highest weight (i.e $S^{(+)}\phi = 0$) (see [2])

2. Each action of $B(\cdot)$ on the state Ω of all spins up turns one spin down.Thus if

   $$\phi = \underbrace{B(\lambda_1) \ldots B(\lambda_k)}_{l\text{-terms}} \Omega$$

   then

   $$S^{(3)}\phi = \left(\frac{N}{2} - l \right) \phi \tag{1.42}$$

   Thus we get a restriction on l, $l \leq \frac{N}{2}$. The degeneracy for a state of l generators is $N - 2l + 1$. Note that N even gives integer spins states, while N odd gives $\frac{1}{2}$–spin . So both cases have to be considered differently.

3. For λ large we find $B(\lambda) \approx \lambda^{N-1} S^{(-)}$. If we add the point $\{\infty\}$ to the space $\mathbb{C}$ as an allowed value for λ, we get all eigenstates of the Heisenberg chain from the Bethe-Ansatz.

We continue to discuss the properties of Bethe states $\phi(\{\lambda_l\})$ and Bethe – Ansatz equations. It is clear from the commutativity of $B(\lambda)$, that $\phi(\{\lambda_l\})$ are symmetric functions of $\lambda_j \quad j = 1, 2, \cdots, l$, Moreover , among λ_j there must not be two equal. Indeed, if two λ in the set $\{\lambda_l\}$ coincide, then the RHS of (1.35) has a double pole and the requirement of regularity leads to one extra equation besides (1.36), this equation can be reduced to the following form [19]:

$$N \frac{1}{\lambda_d^2 + \frac{1}{4}} = 4 + 2 \sum_{j \neq d} \frac{1}{((\lambda_j - \lambda_d)^2 + 1} \tag{1.43}$$

where λ_d is a degenerate λ. The last equation is in general noncompatible with the main Bethe-Ansatz equations.

Thus the structure of Bethe states has "Fermi" character in spirit of the symmetry of eigenvectors. This allows to enumerate Bethe states as well as to use the construction similar to "Fermi sea". Now turn to more detailed investigations of equation (1.36).

2 Second Lecture

In case l is equal to 1 the Bethe ansatz equations decouple and reduce to the set of equations

$$\left(\frac{\lambda_k + i/2}{\lambda_k - i/2}\right)^N = 1 \quad k = 1, \ldots, l.$$

The solution for each λ_k is

$$\lambda_k = \frac{1}{2}\cot\left(\frac{\omega_k}{2}\right), \; \omega_k = \exp\left(\frac{2\pi i k}{N}\right), \; 0 \le k \le N-1.$$

In the limit $N \to \infty$ λ_k fill the whole real line. The same is true in the limit $N \to \infty$ for the BA equations for real solutions when l is fixed. Indeed, the phase factor in the RHS is smooth in comparison with the rapidely oscillating LHS.

However, the BAE have also complex solutions. Due to the reality of the equations, if $(\lambda_1, ..., \lambda_l)$ is a tuple of solutions then $(\bar\lambda_1, ..., \bar\lambda_l)$ is also a tuple of solutions. So we can assume that both λ and $\bar\lambda$ enter the tuple. If λ_k is not real, in the limit $N \to \infty$ the left hand side of the Bethe ansatz equation tends either to infinity or to zero. It turns out that the only possible way to compensate it is to have poles or zeros on the right hand side of the Bethe ansatz equation. More precisely, for large N and finite l the solutions decompose in tuples of λ's, which all have the same real part and are arranged symmetric with respect to the real line. The distance along the imaginary axes between adjacent λ is 1. Such configurations are called strings. They are classified by half integers or integers M. Each string of length $2M+1$ consists of the $(2M+1)$-tuple $(\lambda - iM, \lambda - iM + i, ..., \lambda + iM - i, \lambda + iM)$, where λ is the real part of all the λ 's of the string, which sometimes is called the center of the string. This picture describes all solutions when $N \to \infty$ and l is fixed. In the limit $N \to \infty$ such that l/N remains finite, which is relevant for our case, this decomposing into strings is not proven completely. In this case it is called the string hypothesis.

The momentum of the string of length $M = 1/2$ can be calculated

$$\exp(ip_{1/2}) = \left(\frac{\lambda + i/2 + i/2}{\lambda + i/2 - i/2}\right)\left(\frac{\lambda - i/2 + i/2}{\lambda - i/2 - i/2}\right) = \frac{\lambda + i}{\lambda - i}.$$

The energy has the form

$$h_{1/2}(\lambda) = h(\lambda + i/2) + h(\lambda - i/2) = -\frac{1}{2}\left(1 - \cos\left(p_{1/2}(\lambda)\right)\right).$$

Analogously

$$\exp(ip_M) = \left(\frac{\lambda + i(M+1/2)}{\lambda - i(M+1/2)}\right) \quad \text{and} \quad h_M(\lambda) = -\frac{1}{2M+1}\left(1 - \cos\left(p_M(\lambda)\right)\right).$$

Now we want to consider the limit $N \to \infty$. If l is finite we can introduce the incomplete tensor product of infinite copies of the Hilbert space $\mathbb{C}^2$, which consists of the tensor product of vectors with the exception of a finite number all being equal to $\binom{1}{0}$. V. Neumann was the first to consider this infinite tensor product. The operators S_3, S_+ and S_- can not be defined on this infinite tensor product, but the operator $Q = -S_3 + N/2$ does exist in the limit $N \to \infty$. Hence, the symmetry breaks down and only some part of the symmetry group reminds in the limit $N \to \infty$. For l going to ∞ together with N the Hilbert space is more complicated (see below).

The corresponding scattering theory factorizes. This means, that all multiple scattering matrices are products of two particle scattering matrices. It turns out that the Bethe ansatz equations can be written as

$$\exp(ip) = \prod_{p'} S(p,p').$$

More precisely let

$$S_{0,0}(\lambda) = \frac{\lambda + i}{\lambda - i}$$

be an S-matrix of quasiparticle, corresponding to a real root (0-particle). Then S-matrices for 0,M and M,M' string scattering are given by

$$S_{0,M}(\lambda) = \prod_{m=-M}^{M} S_{0,0}(\lambda + im) = S_{0,0}\left(\frac{\lambda}{M}\right) S_{0,0}\left(\frac{\lambda}{M}\right) \quad \text{with } S_{0,0}\left(\frac{\lambda}{0}\right) = 1,$$

and

$$S_{M,M'}(\lambda) = \prod_{m=|M-M'|}^{M+M'} S_{0,m}(\lambda).$$

The last formula looks as if M-strings correspond to the irreducible representation of spin M of $SU(2,\mathbb{C})$; one associates the factor $S_{0,M}(\lambda)$ with it and then $S_{M,M'}(\lambda)$ corresponds to the tensor product.

Introduction of strings allows to reduce the Bethe ansatz equations to ones with real roots only - namely, to the centers of the strings. Let a configuration of l roots be given by a combination of ν_M strings of length $2M+1$; $\lambda_{j,M}$, $j = 1, ..., \nu_M$ being the real parts of the parameters λ, which belong to the j -th string. We call the total number of particles charge $Q = \sum_M (2M+1)\nu_M$ and the total number of strings $q = \sum_M \nu_M$. Then the Bethe ansatz equations can be written as

$$\left(\frac{\lambda_{k,M} + i(M+1/2)}{\lambda_{k,M} - i(M+1/2)}\right)^N = \prod_{L,\ \nu_L \neq 0} \prod_{\substack{j=1 \\ (L,j)\neq(k,M)}}^{\nu_L} S_{M,L}(\lambda_{k,M} - \lambda_{L,j}).$$

All the parameters are now real. To parametrize the solutions we use a trick. Choose a branch of the logarithm in the form

$$\frac{1}{i} \ln\left(\frac{\lambda + ia}{\lambda - ia}\right) = \pi - 2\arctan\left(\frac{\lambda}{a}\right).$$

Taking the logarithm of both sides of the Bethe ansatz equations we obtain the equations

$$2N\arctan\left(\frac{\lambda_{k,M}}{M+1/2}\right)=2\pi Q_{k,M}+\sum_L\sum_{\substack{j=1\\(L,j)\neq(k,M)}}^{\nu_L}\Phi(\lambda_{k,M}-\lambda_{L,j}),\ \text{where}$$

$$\Phi(\lambda)=2\sum_{l=|M-L|}^{M+L}\left(\arctan\left(\frac{\lambda}{l}\right)+\arctan\left(\frac{\lambda}{l+1}\right)\right)$$

and $Q_{k,M}$ are arbitrary integers, which parametrize the different branches of the logarithm. Now we use just these integers to parametrize the solutions of the Bethe ansatz equations. To each tuple of such integers $Q_{k,M}$ there corresponds a tuple of solutions of the Bethe ansatz equations. We assume, that this map is one to one and furthermore that this map is monotonous. Moreover, there exist some finite $Q_{k,M}^{critical}$, such that $\lambda_{k,M}\to\infty$ as $Q_{k,M}\to Q_{k,M}^{critical}$. The really allowed maximum is suggested to be 'one' below this critical point. We can calculate $Q_{k,M}^{critical}$ by inserting $\lambda_{k,M}=\infty$ in the BA equation to get

$$2N\frac{\pi}{2}=2\pi Q_{k,M}^{critical}+2\pi\sum_L\left(\nu_L J(M,L)-J(M,M)\right)$$

with

$$J(M,L)=\begin{cases}2\min(M,L)+1 & \text{for } M\neq L\\ 2M+1/2 & \text{for } M=L.\end{cases}$$

We recall that $\lambda_{k,M}$ parametrizes the center of a string of length $2M+1$, so to push this string to infinity it is natural to assume the maximal admissible value Q_M^{max} is equal to $Q_M^{critical}-2M-1$. Hence, we have

$$Q_M^{max}=\frac{N}{2}-\sum_L\nu_L J(M,L)-\frac{1}{2}.$$

Now we use a conjecture, which is not completely proven: If all Q 's are pairwise different, then all λ's are also pairwise different. Then all the integers $-Q_M^{max}\leq Q_{1,M},...,Q_{\nu_M,M}\leq Q_M^{max}$ have to be pairwise different. Each of them can take one of the P_M integers

$$-Q_M^{max},-Q_M^{max}+1,...,Q_M^{max},$$

with

$$P_M=2Q_M^{max}+1=N-2\sum_L\nu_L J(M,L).$$

Now we can state the main hypothesis: Every solutions of the Bethe ansatz equation are in one-to-one correspondence to admissible collections of integers $Q_{k,M}$. Let us show, that this is consistent with the relation $\dim(H)=2^N$. To a given configuration of strings $(\nu_0,\nu_{1/2},...)$ there corresponds the number of states

$$Z\left(N,(\nu_0,\nu_{1/2},...)\right)=\prod_M\frac{P_M!}{\nu_M!(P_M-\nu_M)!}.$$

By a combinatorial calculation the total number $Z(M,l)$ of states to a fixed number l is equal to $\frac{N!}{l!(N-l)!} - \frac{N!}{(l-1)!(N-l+1)!}$. The solutions of the Bethe ansatz equations correspond only to highest weight vectors. The total number of states is given by

$$\sum_{l=0}^{N/2}(N-2l+1)\left(\frac{N!}{l!(N-l)!} - \frac{N!}{(l-1)!(N-l+1)!}\right) = 2^N.$$

This is written for N even. For N odd the sum goes from $l=0$ to $(N-1)/2$.

Now consider some representative examples.

1. As many strings of length 1 as possible: $\nu_0 = N/2, \nu_M = 0$ for all $M > 0$. Then P_0 is equal to $N/2$ and there is only one combination of the integers $Q_{1,0} < ... < Q_{N/2,0}$ corresponding to this configuration. Hence, this state is non-degenerated. It corresponds to the lowest energy state - the vacuum.

2. One string of length 1 less than before: $\nu_0 = N/2-1, \nu_M = 0$ for all $M > 0$. Then we have $P_0 = N/2+1$. Thus the integers $Q_{1,0} < ... < Q_{N/2-1,0}$ have two vacancies and this configuration corresponds to a 2-dimensional family of two particle states.

3. One string of length 2 and as many strings of length 1 as possible: $\nu_0 = N/2-2, \nu_{1/2} = 1, \nu_M = 0$ for all $M > 1/2$. Then we have $P_0 = N/2$ and $P_{1/2} = 1$. Here again the integers $Q_{1,0} < .. < Q_{N/2-2,0}$ have two vacancies and the integer $Q_{1,1/2}$ is equal to zero.

We shall see below, that the second and third configurations have the same energy. The full spectrum can be characterized by spin 1/2 particles. Indeed, the Bethe ansatz can be thought of as a new way of reducing spin 1/2 particles in multiplets with multiplicities. This shows that the low energy part of the antiferromagnetic and the ferromagnetic cases are completely different.

3 Lecture 3

3.1 Physical Vacuum and the Low-lying Excitations

The physical vacuum is given by the configuration:

$$\nu_o = \frac{N}{2}, \quad \nu_M = 0, \quad M > 0 \tag{3.1}$$

while the solutions with:

$$\begin{array}{l} \nu_o = N/2 - 1, \quad \nu_M = 0, \quad M > 0 \\ \nu_o = N/2 - 2, \quad \nu_{\frac{1}{2}} = 1, \quad \nu_M = 0, \quad M > \frac{1}{2} \end{array} \tag{3.2}$$

correspond to the triplet and singlet states, respectively.

To find the spectrum for the ground state (vacuum), one has to find the $\{\lambda\}$. We have the equations $(M = 0)$:

$$\arctan(2\lambda_k) = \frac{\pi Q_{k,0}}{N} + \frac{1}{N} \sum_{j=k}^{N/2} \arctan(\lambda_k - \lambda_j) \tag{3.3}$$

with

$$-\frac{N}{4} + \frac{1}{2} \leq Q_{k,o} \leq \frac{N}{4} - \frac{1}{2} = Q_o^{max} \tag{3.4}$$

There is no place left between two adjacent Q's so that $Q_{k,0} = k$.

Let $N \to +\infty$ and then we make the following notation: $\frac{k}{N} = x$ where $-\frac{1}{4} \leq x \leq \frac{1}{4}$.

Then, in the limit $N \to +\infty$:

$$\lambda_k \to \lambda(x) \quad \text{with} \left\{ \begin{array}{ll} \lambda(-\frac{1}{4}) & = -\infty \\ \lambda(\frac{1}{4}) & = +\infty \end{array} \right. \tag{3.5}$$

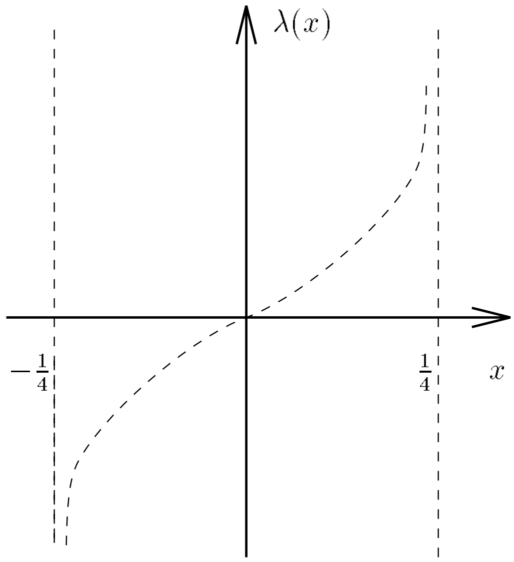


Figure 3.1: Graph of the function $\lambda(x)$.

Under the appropriate regularity assumptions for the function $\lambda(x)$, the sum in (3.3) may be replaced by an integral within an $O\left(\frac{1}{N^2}\right)$ error:

$$\arctan(\lambda(x)) = \pi x + \int_{-\frac{1}{4}}^{\frac{1}{4}} \arctan\left(\lambda(x) - \lambda(y)\right)\, dy \tag{3.6}$$

Thus for the ground state energy we have:

$$E = \sum_k h(\lambda_k) = N \sum h(\lambda(x))\, dx \to N \int h(\lambda)\rho(\lambda)\, d\lambda \tag{3.7}$$

The function $\rho(\lambda)$ defined by the equality $\lambda'(x) = \frac{1}{\rho(\lambda)}$ plays the role of density of λ's. Differentiating (3.6) with respect to x, we obtain the linear integral equation:

$$\frac{2}{1+4\lambda^2}\, \lambda' \quad = \quad \pi + \lambda' \int \frac{1}{1 + \left(\lambda(x) - \lambda(y)\right)^2}\, dy$$

or equivalently,

$$\frac{2}{1+4\lambda^2} \quad = \quad \pi\rho(\lambda) + \int_{-\infty}^{+\infty} \frac{1}{1 + (\lambda - \mu)^2}\, \rho(\mu)\, d\mu \tag{3.8}$$

This equation is solved via a Fourier transform. The ground state energy is thus:

$$E_o = -N \int h(\lambda)\rho(\lambda)\, d\lambda = -N \ln 2 \tag{3.9}$$

The same passage to the limit can be done for the triplet state but with:

$$Q_k = k + \theta(k - k_1) + \theta(k - k_2) \quad \text{with } \theta \text{ the Heaviside function} \tag{3.10}$$

because there are 2 vacancies (holes) and Q_k has two jumps at k_1 and k_2. Instead of (3.6) and (3.8) we have:

$$\arctan\left(2\lambda(x)\right) = \pi x + \frac{1}{N}[\theta(x - x_1) + \theta(x - x_2)] + \int_{-\frac{1}{4}}^{+\frac{1}{4}} \arctan\left(\lambda(x) - \lambda(y)\right) \, dy$$

and

$$\frac{2}{1 + 4\lambda^2} = \pi\rho(\lambda) + \int_{-\infty}^{+\infty} \frac{1}{1 + (\lambda - \mu)^2} \, \rho(\mu) \, d\mu + \frac{\pi}{N} \left(\delta(\lambda - \lambda_1) + \delta(\lambda - \lambda_2)\right)$$

where $\lambda_i = \lambda(x_i), \quad i = 1, 2$ are the parameters of the holes.
We get:

$$\rho_{tr}(\lambda) = \rho_{vac}(\lambda) + \frac{1}{N} \left[\sigma(\lambda - \lambda_1) + \sigma(\lambda - \lambda_2)\right] \tag{3.11}$$

where $\sigma(\lambda)$ satisfies the equation:

$$\pi\sigma(\lambda) + \int_{-\infty}^{+\infty} \frac{\sigma(\mu)}{1 + (\lambda - \mu)^2} \, d\mu = -\pi\delta(\lambda) \tag{3.12}$$

The result is:

$$E_{tr} = E_o \, N + \epsilon(\lambda_1) + \epsilon(\lambda_2) \tag{3.13}$$

where the relative energy $\epsilon(\lambda)$ is given by:

$$\epsilon(\lambda) = \int_{-\infty}^{+\infty} h(\mu) \, \sigma(\mu - \lambda) \, d\mu \tag{3.14}$$

and the relative momentum is

$$k(\lambda) = \int p(\mu) \, \sigma(\mu - \lambda) \, d\mu \tag{3.15}$$

The relative energy and momentum are connected by the dispersion law:

$$0 \leq k \leq \pi, \qquad \epsilon(k) = \frac{\pi}{2} \sin(k) \tag{3.16}$$

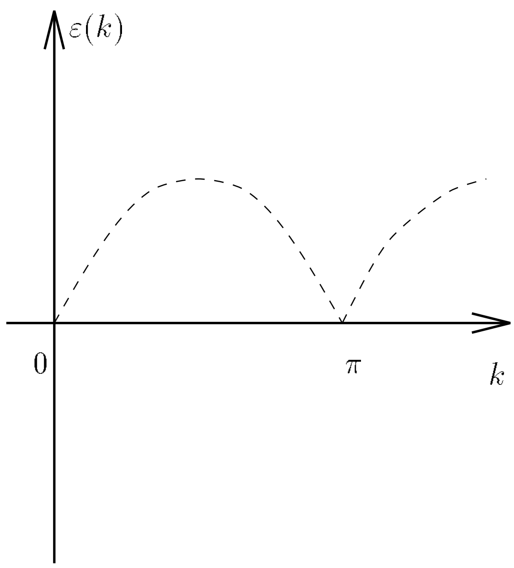


Figure 3.2: Dispersion law for the spin wave.

In the singlet case we have also equations for string $M = \frac{1}{2}$. As $Q_{\frac{1}{2}}^{max} = 0$ we make the notation $\lambda_s = \lambda_{0,1/2}$ for the only existing $\lambda_{j,1/2}$. Then,

$$\arctan(2\lambda_k) = \frac{\pi Q_k}{N} + \frac{1}{N}\sum_{j\neq k}^{N/2}\arctan(\lambda_k - \lambda_j) + \frac{1}{N}\Phi(\lambda_k - \lambda_s) \tag{3.17}$$

$$\frac{2}{1+4\lambda^2} = \pi\rho(\lambda) + \int_{-\infty}^{+\infty}\frac{1}{1+(\lambda-\mu)^2}\,\rho(\mu)\,d\mu + \Phi(\lambda-\lambda_s) + \frac{\pi}{N}(\delta(\lambda-\lambda_1)+\delta(\lambda-\lambda_2)) \tag{3.18}$$

where in the second equation we have put $\lambda_s = \frac{\lambda_1+\lambda_2}{2}$. The computation shows that the energy of the strings cancels,

$$E_{sing} = E_{tr} \tag{3.19}$$

More generally, all the strings give zero contribution to relative energy and momentum. The only difference with respect to the triplet state is the value of the total spin: $S = 1$ for the triplet and $S = 0$ for the singlet.

3.2 Scattering of Spin Waves

Using the method of [13] we find that the two-particle scattering reduces to a pure phase change. Starting from the S–matrix on $\mathbb{C}^2 \otimes \mathbb{C}^2$ one can decompose it according to spin into a triplet and a singlet part. $S = S_{oo} + S_{tr}$.For the triplet the phase is given by the equation:

$$\ln S_{tr}(\lambda) = \int \ln S_{oo}(\mu)\sigma(\mu - \lambda)\, d\mu \quad \text{where} \quad S_{oo}(\lambda) = \frac{\lambda + i}{\lambda - i} \tag{3.20}$$

The solution is:

$$\begin{aligned} S_{tr}(\lambda) &= \frac{\Gamma\left(\frac{1+i\lambda}{2}\right)\Gamma\left(1 - \frac{i\lambda}{2}\right)}{\Gamma\left(\frac{1-i\lambda}{2}\right)\Gamma\left(1 + \frac{i\lambda}{2}\right)} = \frac{f(\lambda)}{f(-\lambda)} \\ \text{with} & \\ f(\lambda) &= \frac{\Gamma\left(\frac{1+i\lambda}{2}\right)}{\Gamma\left(1 - \frac{i\lambda}{2}\right)} \end{aligned} \tag{3.21}$$

it is interesting to note that the function $\mid f(\lambda) \mid^2$ appears also as a Harish-Chandra factor in the representations of $SL(2, \mathbb{R})$.
For the singlet case we get:

$$S_{sing}(\lambda) = \frac{\lambda + i}{\lambda - i} S_{tr}(\lambda) \tag{3.22}$$

We can write the S matrix for two $\frac{1}{2}$-spin waves as:

$$S = S_{tr}(\lambda + iP) \tag{3.23}$$

using the projection operators on $\mathbb{C}^2 \otimes \mathbb{C}^2$ for the triplet and the singlet:

$$\frac{1}{2}(1 + P) = P_{tr} \qquad \frac{1}{2}(1 - P) = P_{sing} \tag{3.24}$$

Here P is the permutation operator.

3.3 Generalisation to Higher Spin S

The FCR (1.13), lying in the foundation of the solution of spin $\frac{1}{2} - XXX$ model, can be rewritten as:

$$R_{a_1,a_2}(\lambda - \mu)R_{a_1,n}(\lambda - \sigma)R_{a_2,n}(\mu - \sigma) = R_{a_2,n}(\mu - \sigma)R_{a_1,n}(\lambda - \sigma)R_{a_1,a_2}(\lambda - \mu) \tag{3.25}$$

Here indices a_1, a_2 and n denote three spaces: n-quantum, associated with the lattice site n and two auxilliary spaces. All these spaces are the same - they correspond to the spin $\frac{1}{2}$ representation, i.e. coincide with $\mathbb{C}$. The L-matrix:

$$L = \begin{pmatrix} \lambda + iS^3 & iS_- \\ iS_+ & \lambda - iS^3 \end{pmatrix} = \lambda I + i\vec{S}\vec{\sigma} = \left(\lambda - \frac{i}{2}\right) I + iP_{na} \tag{3.26}$$

coincides with $R_{a1,n}(\lambda - \frac{i}{2})$ and R in (1.12) is just $R_{a_1,a_2}(\lambda - \mu)$.

Now the analogue of (3.25) is true for other representations of $SU(2)$. In particular:

$$L(\lambda) = \lambda I + i\vec{S_n}\vec{\sigma} \tag{3.27}$$

for $\vec{S_n}$ being the spin S representation also satisfies (1.13) with the same R-matrix and can be used to introduce the Bethe Ansatz. The Bethe Ansatz equations now look as follows:

$$\left(\frac{\lambda_i + iS}{\lambda_i - iS}\right)^N = \prod_{i \neq k} \left(\frac{\lambda_i - \lambda_k + i}{\lambda_i - \lambda_k - i}\right) \tag{3.28}$$

and differ from (1.36) only in the LHS.
They give the eigenstates of the family of operators:

$$F(\lambda) = \mathrm{tr}_a T_a(\lambda).$$

However, $F_a(\lambda)$ can not be used now as a generating function of the local conservation laws.
Indeed, in deriving them in the first Lecture we used the fact that $L_{a,n}(\lambda = \frac{i}{2}) = P_{n,a}$ where P is a permutation. It can be true only if the spaces n and a are identical. So we need the L matrices for which:

- the quantum space and the auxilliary space have the same dimensions;
- for a certain value of the parameter λ the L matrix becomes a permutation,

We shall call such L-matrices fundamental.
The algebraic construction underlining different types of L-operators and R-matrices can be summarized as follows: we start considering an associative algebra $\mathcal{A}$, an operator R on $\mathcal{A} \otimes \mathcal{A}$ and introduce the notation:

$$R_{12} = R \otimes I \text{ and further } R_{13}, R_{23} \ldots \tag{3.29}$$

for operators defined on $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$.
Suppose then that $\mathcal{A}$ has representations parametrized by λ, s: $\rho(\lambda, s)$ where λ is continuous and s is discrete. This is the case for loop algebras for instance. The algebra corresponding to XXX models is its deformation and is called Yangian [14].
Let us define:

$$(\rho(\lambda, s) \otimes \rho(\mu, t))\, R = R_{s,t}(\lambda, \mu) \tag{3.30}$$

and suppose that $R_{s,t}(\lambda, \mu)$ depends only on $\lambda - \mu$. We see that the FCR:

$$R_{a_1,a_2}(\lambda-\mu) R_{a_1,n}(\lambda-\sigma) R_{a_2,n}(\mu-\sigma) = R_{a_2,n}(\mu-\sigma) R_{a,n}(\lambda-\sigma) R_{a_1,a_2}(\lambda-\mu) \tag{3.31}$$

correspond to the Yang-Baxter relations:

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \tag{3.32}$$

Let us now take the same L operator, but with the understanding that the auxiliary space is the spin-S representation space and that the quantum space is the spin-$\frac{1}{2}$ ($\mathbb{C}^2$) space.
In analogy with the Yang-Baxter relations:

$$R_{12}R_{32}R_{31} = R_{31}R_{32}R_{12} \tag{3.33}$$

we have

$$R_{n_1,n_2}(\lambda-\mu)R_{a,n_2}(\sigma-\mu)R_{a,n_1}(\sigma-\lambda) = R_{a,n_1}(\sigma-\lambda)R_{a,n_2}(\sigma-\mu)R_{n_1,n_2}(\lambda-\mu) \tag{3.34}$$

Equation (3.34) may be used to find $R_{s,s}(\lambda)$ as soon as $R_{\frac{1}{2},s}$ is known. We have: $L_{n_1,n_2}(\lambda=0)=P$ so this L is fundamental.

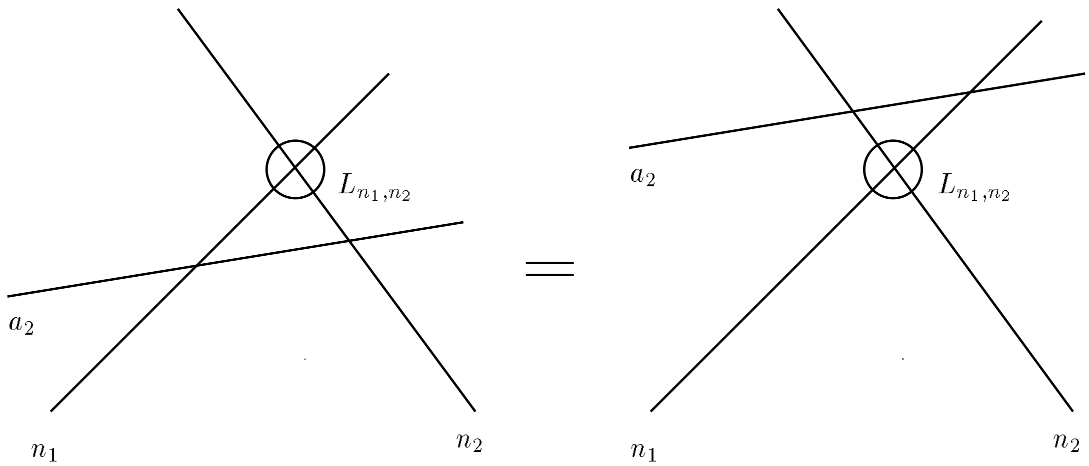


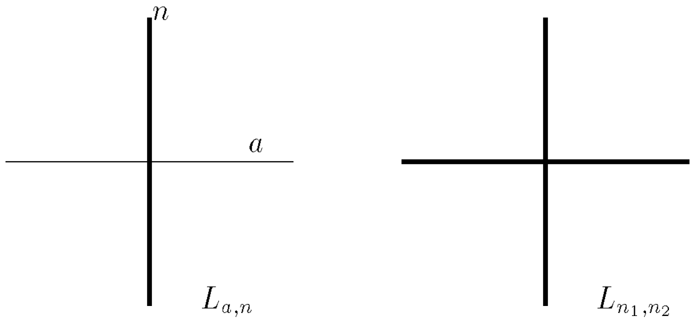
Figure 3.3

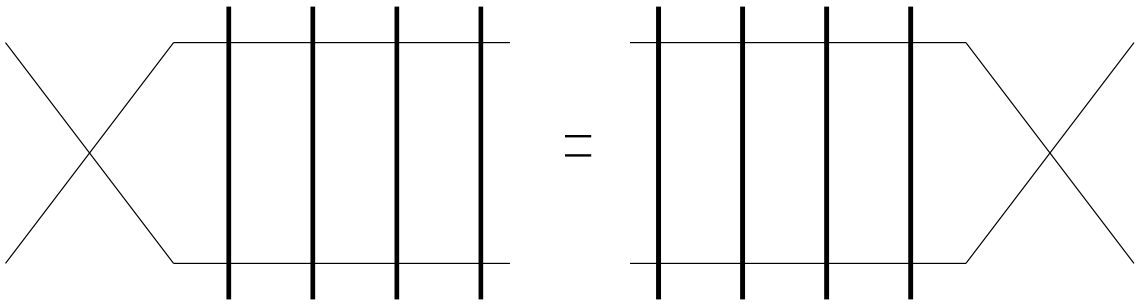
Introducing the two monodromies given by the $L_{a,n}$ and $L_{f,n}$ operators we get:

$$0 = [\mathrm{tr}_a T_a(\lambda), \mathrm{tr}_f T_f(\mu)] \tag{3.35}$$

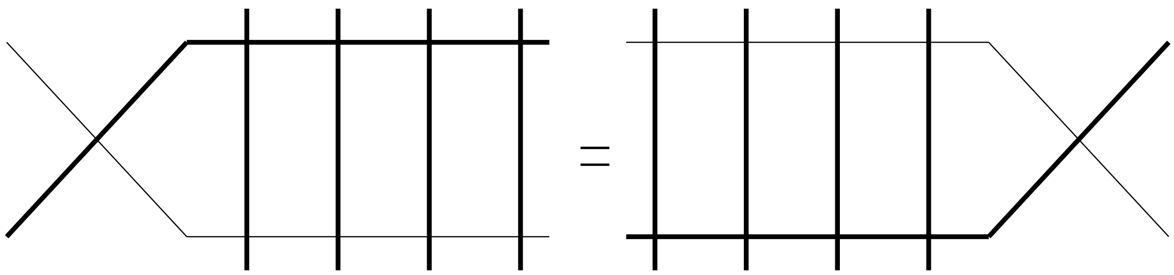
where the first term is convenient for using the Bethe Ansatz and the second term gives the energy:

$$H = \sum H_{n,n+1} \quad \text{with} \quad H_{n,n+1} = \frac{1}{i}\frac{d}{d\lambda}L_{n,n+1}(\lambda)P_{n,n+1}\mid_{\lambda=0} \tag{3.36}$$





$L_{a_1,a_2}(\lambda-\mu)T_{a_1}(\lambda)T_{a_2}(\mu) = T_{a_2}(\mu)T_{a_1}(\lambda)L_{a_1,a_2}(\lambda-\mu)$



$L_{a,f}(\lambda-\mu)T_a(\lambda)T_f(\mu) = T_f(\mu)T_a(\lambda)L_{a,f}(\lambda-\mu)$

Figure 3.4

For spin $S_{\frac{1}{2}}$, $p(\lambda)$ and $\epsilon(\lambda)$ were given by the equations:

$$e^{ip(\lambda)} = \frac{a(\lambda)}{d(\lambda)}$$

$$\epsilon(\lambda) \;=\; \frac{d}{d\lambda}p(\lambda) \tag{3.37}$$

where $a(\lambda)$ and $d(\lambda)$ are eigenvalues of diagonal elements of $L_{n,a}(\lambda)$ on the highest weight state ω_n.

We claim that with P and H as given by $tr_f T_f(\lambda)$ and $a(\lambda), d(\lambda)$ defined by $L_{n,a}$, the same relations (37) hold for $p(\lambda)$ and $\epsilon(\lambda)$ for any spin. This is based on the relation:

$$[H_{n_1,n_2}, L_{n+1}(\lambda)L_n(\lambda)] = \frac{1}{i}\left[L'_{n+1}(\lambda)L_n(\lambda), L_{n+1}(\lambda)L'_n(\lambda)\right] \tag{3.38}$$

which one can get differentiating (3.34), [15].

Here is the result for spin S:

$$e^{ip(\lambda)} = \frac{\lambda + iS}{\lambda - iS} \quad \text{and} \quad \epsilon(\lambda) = -\frac{2S}{\lambda^2 + S^2} \tag{3.39}$$

This result extends to the case of the nonlinear Schrödinger equation. The discrete version of NSE is nothing else but the XXX model for spin value $S \sim \frac{1}{\Delta}$ where Δ is the lattice constant.

The continuous case is therefore given by the limit $S \to \infty$. In this limit we get:

$$e^{ip(\lambda)} \overset{S\to\infty}{\to} 1 - \frac{i\lambda}{2S}$$

$$p(\lambda) = i\ln\left(1 - \frac{i\lambda}{2S}\right) \to \frac{\lambda}{S} + \cdots \tag{3.40}$$

and

$$\epsilon(\lambda) = -\frac{2}{S} + \frac{2\lambda^2}{S^3} + \cdots \tag{3.41}$$

which go into usual expressions $p(\lambda)$ and $\epsilon(\lambda)$ after evident rescaling.

4 Fourth Lecture

4.1 Nonlinear σ-model

In the following we want to investigate relations between the so called $O(3)$ nonlinear σ-model and the XXX-chain Heisenberg model.

The nonlinear σ-model is given by the the Lagrangian $\mathcal{L} = \frac{1}{2\gamma}\int(\partial_\mu \vec{n})^2 dx$ with coupling constant γ and field $\vec{n} : M \mapsto S^2$, where M denotes 1+1 Minkowski spacetime, so in particular $\vec{n}^2 = 1$. Taking the time derivative of the field doesn't seem to be a too good definition for the corresponding momenta, since one would like to have momenta, which are tangent to S^2, so let's define:

$$\vec{l} = \frac{1}{\gamma}\partial_0\, \vec{n} \times \vec{n}, \text{ with } (\vec{n}, \vec{l}) = 0.$$

We introduce quantisation by the following commutation relations, which emerge naturally from the poisson structure of the above system:

$$\begin{aligned} [l^a(x), l^b(y)] &= i\epsilon^{abc} l^c(x)\delta(x-y), \\ [l^a(x), n^b(y)] &= i\epsilon^{abc} n^c(x)\delta(x-y), \\ [n^a, n^b] &= 0 \end{aligned} \tag{4.1}$$

The commutation relations (4.1) reminds us of the commutation relations for the group of motion of $\mathbb{R}^3$, i.e. $E(3)$. So we will now look for an algebra from which the above relations (4.1) can be obtained by deformation (contraction). For getting rid of divergencies we consider the field $\vec{n}$ on a lattice. At one lattice point $\vec{n}$ can be realized as acting on L^2-functions on the sphere. A suggestive hint is now the decomposition:

$$L^2(S^2) = \sum_{J=0}^{\infty} \oplus \mathcal{D}_J = \lim_{s\to\infty} \sum_{0}^{2s} \oplus \mathcal{D}_J = \lim_{s\to\infty} \mathcal{D}_s \otimes \mathcal{D}_s$$

So introduce a lattice of length $2N$ with spin S variables and define:

$$\begin{aligned} l_k &= S_{2k} + S_{2k-1} & \qquad S_{2k} &= \tfrac{l_k}{2} + \sqrt{s(s+1)}\, n_k \\ n_k &= \tfrac{1}{2\sqrt{s(s+1)}}(S_{2k} - S_{2k-1}) & \qquad S_{2k-1} &= \tfrac{l_k}{2} - \sqrt{s(s+1)}\, n_k \end{aligned}$$

We get

$$[n_k^a, n_l^b] = \epsilon^{abc}\frac{l_k^c}{4s(s+1)}\delta_{kl}, \qquad \vec{n}^2 = 1 - \frac{l^2}{4s(s+1)} \tag{4.2}$$

which goes in the limit $S \to \infty$ to the commutation relation (4.1), if we stick to small eigenvalues of l.

For recovering the Hamiltonian of the nonlinear sigma model

$$H = \frac{1}{2}\int(\gamma l^2 + \frac{1}{\gamma} n'^2)dx \tag{4.3}$$

we need to do a suitable gradient expansion in the XXX Hamiltonian [16] [17]. The Hamiltonian of the spin S XXX model is a function of pairwise products of the spinvariables [18].

$$H_{XXX} = \sum_n f(\vec{S}_{n+1}\vec{S}_n) = \sum_n \Big(f(\vec{S}_{2n+1}\vec{S}_{2n}) + f(\vec{S}_{2n}\vec{S}_{2n-1}) \Big) \tag{4.4}$$

and we expand the fields at the lattice point k with respect to the lattice spacing $\Delta = x_{k+1} - x_k$ and calculate the products. We have

$$l_k = \Delta l(x), \qquad n_k = n(x), \qquad x = k\Delta,$$

$$n_{k+1} = n(x) + \Delta n'(x) + \frac{1}{2}\Delta^2 n''(x) + \mathcal{O}(\Delta^3)$$

and

$$l_{k+1} = \Delta l(x) + \mathcal{O}(\Delta^2).$$

So we get

$$\begin{aligned} n_{k+1}n_k &= n^2(x) - \frac{1}{2}n'^2(x)\Delta^2 + \mathcal{O}(\Delta^3) \\ l_{k+1}n_k - n_{k+1}l_k &= -\Delta^2 l(x)n'(x) + \mathcal{O}(\Delta^3) \\ l_{k+1}l_k &= \Delta^2 l^2(x) + \mathcal{O}(\Delta^3) \end{aligned}$$

where we used $(\vec{n}, \vec{n}) = 1 + \mathcal{O}(\Delta^2)$ and $(\vec{l}, \vec{n}) = 0$. From this follows (using (4.2))

$$\begin{aligned} S_{2k}S_{2k-1} &= \Big(\frac{l_k}{2} + \sqrt{s(s+1)}n_k\Big)\Big(\frac{l_k}{2} - \sqrt{s(s+1)}n_k\Big) \\ &= \frac{1}{4}\Delta^2 l^2 - s(s+1)n^2 = \frac{1}{2}\Delta^2 l^2 - s(s+1) \\ S_{2k+1}S_{2k} &= \Big(\frac{l_{k+1}}{2} - \sqrt{s(s+1)}n_{k+1}\Big)\Big(\frac{l_k}{2} + \sqrt{s(s+1)}n_k\Big) \\ &= \frac{1}{4}\Delta^2 l^2 - \Delta^2 l n'\sqrt{s(s+1)} + \frac{1}{2}\Delta^2 n'^2 s(s+1) - s(s+1)n^2 + \mathcal{O}(\Delta^3) \\ &= \Delta^2\Big(\frac{1}{2}l^2 - ln'\sqrt{s(s+1)} + \frac{1}{2}n'^2 s(s+1)\Big) - s(s+1) + \mathcal{O}(\Delta^3) \end{aligned}$$

Inserting in (4.4) finally gives

$$\begin{aligned} H_{XXX} &= \text{const}_1 + \text{const}_2\Delta^2 \int \Big(l^2 - ln'\sqrt{s(s+1)} + \frac{1}{2}n'^2 s(s+1)\Big)\, dx \\ &= \text{const}_1 + \text{const}_2\Delta^2 \int \Big(\underbrace{(l - \frac{1}{2}\sqrt{s(s+1)}n')}_{\tilde{l}}{}^2 + \frac{s(s+1)}{4}n'^2\Big)\, dx. \end{aligned}$$

The transition $l \mapsto \tilde{l}$ is a canonical transformation since $\tilde{l}$ and n obey the same commutation relations as l and n. So one can interpret the additional term in $\tilde{l}$ as a topological charge. Comparison with (4.3) gives us:

$$\gamma^2 = \frac{4}{s(s+1)} \tag{4.5}$$

and $s \to \infty$ correspondes to $\gamma \to 0$, which is compatible with the asymptotic freedom of the σ-model.

We can try to apply the machinery of the Bethe ansatz for calculating the spectrum. In the XXX-model, we had the Lax operator

$$L_n = \begin{pmatrix} \lambda + iS_3 & iS_- \\ iS_+ & \lambda - iS_3 \end{pmatrix}$$

and used a local state ω, $S_+\omega = 0$ to define a reference state. This is not suitable for our case, because the condition $l_k = \mathcal{O}(\Delta)$ shows, that large spins S_{2n} and S_{2n-1} cancel each other, so the product of the highest weights lies very far from the physical ground state. The new observation is that the product of two Lax operators

$$L_{2n,a}(\lambda)L_{2n-1}(\lambda) \tag{4.6}$$

has a new local vacuum in the limit $S \to \infty$.

The lower left matrix element in the product (4.6) is given by

$$c = i\lambda l_+ + \frac{\sqrt{s(s+1)}}{2}(l_3 n_+ - n_3 l_+)$$

Applying c to the reference state Ω has to give zero. So we have two equations:

$$l_+\Omega = 0 \quad \text{and} \quad (l_3 n_+ - n_3 l_+)\Omega = 0$$

If we look at the continuum limit $S \to \infty$ the operators l_3 and l_+ become first order differential operators, namely we have:

$$n_+ = \sin\theta \exp i\phi, \quad l_3 = \partial_\phi \quad \text{and} \quad l_+ = \exp i\phi(i\cot\theta\partial_\phi + \partial_\theta)$$

This gives the solution:

$$\Omega = \frac{1}{n_+}.$$

We are here not concerned about Ω not being in $\mathcal{L}^2$. Ω is an exact eigenfunction of a and d, we get with

$$l_3\Omega = -\Omega : \tag{4.7}$$

that

$$a(\lambda)\Omega = (\lambda + iS)(\lambda - i(S+1))\Omega, \quad \text{and} \quad d(\lambda) = \bar{a}(\bar{\lambda})$$

This gives us the formula for the momentum

$$e^{ip} = \frac{a}{d}$$

and energy $\epsilon(\lambda) = \frac{d}{d\lambda} p(\lambda)$ (see fig. 4.1). It is indefinite which makes the application of the Bethe Ansatz eqs. more complicated.

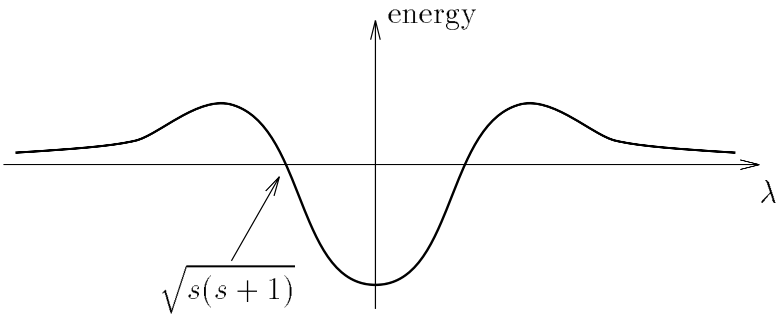


Figure 4.1

The work on this problem is not finished yet.

4.2 Zero curvature condition

Here we discuss an alternative way to introduce the momentum P and the Hamiltonian H trough the monodromy $T_f(\lambda)$. The relevant Lax operators will be

$$L_{n,a}(\lambda) = R_{a,n}(\lambda)$$

and

$$L_{n,f}(\lambda) = R_{f,n}(\lambda)$$

where n correspondes to the quantum space at the lattice point n, a auxillary space ($\mathbb{C}^2$ in our examples) and f is the same as n. The relations (cf lecture 3) are as follows

$$\begin{aligned} R_{f_1,f_2}(\lambda-\mu) L_{n,f_1}(\lambda) L_{n,f_2}(\mu) &= L_{n,f_2}(\mu) L_{n,f_1}(\lambda) R_{f_1,f_2}(\lambda-\mu) && (4.8) \\ R_{f_1,f_2}(\lambda-\mu) L^{-1}_{f_1,a}(-\lambda) L^{-1}_{f_2,a}(-\mu) &= L^{-1}_{f_2,a}(-\mu) L^{-1}_{f_1,a}(-\lambda) R_{f_1,f_2}(\lambda-\mu). && (4.9) \end{aligned}$$

Consider a chain of length 2N. Replace the entry of the Lax operator λ by $\lambda+\omega$ on even sites of the chain and by $\lambda-\omega$ on odd sites. We had already seen in the last lecture, that the traces of Monodromies:

$$\begin{aligned} T_a(\lambda) &= L_{N,a}(\lambda)\dots L_{1,a}(\lambda) && (4.10) \\ T_f(\lambda) &= L_{N,f}(\lambda)\dots L_{1,f}(\lambda) && (4.11) \end{aligned}$$

commute for all values of the spectral parameter:

$$[\mathrm{tr}_a T_a(\lambda), \mathrm{tr}_f T_f(\mu)] = 0. \tag{4.12}$$

So we define analogously:

$$\begin{aligned} L_{2N,a}(\lambda+\omega)\dots L_{2n,a}(\lambda+\omega)L_{2n-1,a}(\lambda-\omega)\dots L_{1,a}(\lambda-\omega) &\stackrel{\text{def}}{=} T_a(\lambda,\omega) \\ L_{2N,f}(\lambda+\omega)\dots L_{2n,f}(\lambda+\omega)L_{2n-1,f}(\lambda-\omega)\dots L_{1,f}(\lambda-\omega) &\stackrel{\text{def}}{=} T_f(\lambda,\omega) \end{aligned}$$

and $tr_f T_f(\lambda,\omega) = F_f(\omega)$. Previously F was interpreted as a shift along the chain

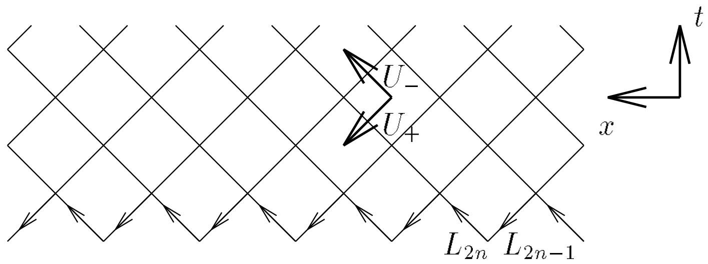


Figure 4.2

Now we interpret the operators $U_\pm \stackrel{\text{def}}{=} F(\pm\omega)$ in some natural way as the shift along the lightcone directions in Minkowski spacetime:

$$U_- = e^{-i\frac{P+H}{2}}, \qquad U_+ = e^{-i\frac{P-H}{2}}$$

The propagation of the Lax operator

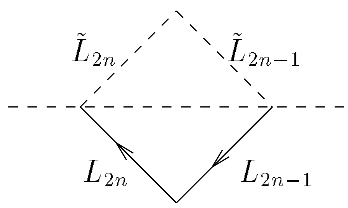


Figure 4.3

where

$$\tilde{L}_{2n-1} = U_+^{-1} L_{2n} U_+ \quad \text{and} \quad \tilde{L}_{2n} = U_- L_{2n-1} U_-^{-1}$$

should lead to the zero curvature condition:

$$L_{2n}L_{2n-1} = \tilde{L}_{2n-1}\tilde{L}_{2n} \tag{4.13}$$

Indeed one can show, that (4.13) is a consequence of the equations (4.9). The proof goes as follows:

Define monodromies, where there is between the "quantum Lax operators" $L_{2n}(\sigma+\omega)$ and $L_{2n-1}(\sigma-\omega)$ an "auxillary Lax operator" $L_{n,a}^{-1}(\lambda-\sigma)$

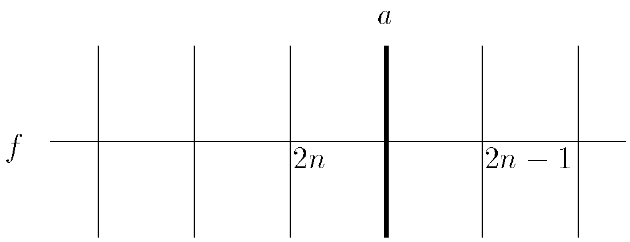


Figure 4.4

so that

$$F(\sigma, n, a, \lambda, \omega) = tr_f(\ldots L_{2n,f}(\sigma+\omega) L_{f,a}(\lambda-\sigma)^{-1} L_{2n-1,f}(\sigma-\omega)\ldots)$$

Using (4.8), (4.9) gives the commutation property:

$$[F(\sigma, n, a, \lambda, \omega), F(\nu, n, a, \lambda, \omega)] = 0 \tag{4.14}$$

for any σ and ν. Remembering, that $L_{2n-1,f}(0) = P_{2n-1,f}$, i.e. the permutation operator between V_{2n-1} and V_f we get:

$$F(\omega, n, a, \lambda, \omega) = tr_f(\ldots L_{2n,f}(2\omega) \underbrace{L_{f,a}(\lambda-\omega)^{-1} P_{2n-1,f}}_{P_{2n-1,f} L_{2n-1,a}(\lambda-\omega)^{-1}} \ldots)$$

Since $L_{2n-1,a}(\lambda-\omega)^{-1}$ commutes with the remaining Lax operators on the right of the above expression it follows, that:

$$F(\omega, n, a, \lambda, \omega) = U_+ L^{-1}_{2n-1,a}(\lambda-\omega)$$

In the same way we obtain:

$$F(-\omega, n, a, \lambda, \omega) = L^{-1}_{2n,a}(\lambda+\omega) U_-.$$

Inserting this now into

$$[F(\omega, n, a, \lambda, \omega), F(-\omega, n, a, \lambda, \omega)] = 0$$

gives (4.13).

Now the laboratory propagators $\exp(-iP)$ and $\exp(-iH)$ could be obtained as $U_- U_+$ and $U_- U_+^{-1}$, correspondingly.

5 Fifth Lecture

5.1 Translation in space and time

We defined in the last lecture the operators:

$$T_f(\lambda,\omega) \stackrel{\text{def}}{=} L_{2N,f}(\lambda+\omega)\dots L_{2n,f}(\lambda+\omega)L_{2n-1,f}(\lambda-\omega)\dots L_{1,f}(\lambda-\omega)$$

$$U_\pm = tr_f T_f(\lambda,\omega)|_{\pm\omega}$$

where the Lax operators fulfilled the conditions:

$$L_{2n,f}(0) = P_{2n,f}$$

Let's introduce a variant of L by the equation:

$$L_{2n,f}(\lambda) = P_{2n,f}l_{2n,f}(\lambda), \quad \text{with} \quad l_{2n,f}(0)=1 \quad \text{and} \quad l_{2n,f}(-\lambda) = l^{-1}_{2n,f}(\lambda)$$

From $l_{2n,f}P_{2n-1,f} = P_{2n-1,f}l_{2n,2n-1}$ we get

$$U_+ = tr(...L_{2n,f}(2\omega)P_{2n,f}....) = V\prod_{n=1}^{N} l_{2n,2n-1}(2\omega).$$

where V is a shift of the index n by one. Analogously we have :

$$U_- = \prod_{n=1}^{N} l_{2n,2n-1}(-2\omega)V.$$

So the product $U_+U_- = V^2$ defines a translation by $2\cdot latticespacing$, i.e. refers to a translation on the lattice in space direction: $V^2 = \exp(-iP)$. In the same way we have:

$$\begin{aligned} \exp(-iH) &= U_-U_+^{-1} = \prod_{n=1}^{N} l_{2n,2n-1}(-2\omega)V\prod_{n=1}^{N} l_{2n,2n-1}(-2\omega)V^{-1} && (5.15)\\ &= \prod_{n=1}^{N} l_{2n,2n-1}(-2\omega)\prod_{n=1}^{N} l_{2n+1,2n}(-2\omega) && (5.16)\end{aligned}$$

5.2 The XXZ-model

By deforming the XXX-model with respect to a parameter γ it is possible to get the more general so called XXZ model, where the Z indicates an anisotropy in direction of the z-variable of the Heisenberg model. The deformed Lax operator looks as follows:

$$L_{n,a}(\lambda) = \frac{1}{\sin\gamma}\begin{pmatrix} \sinh(\lambda+i\gamma S_n^3) & i\sin\gamma S_n^- \\ i\sin\gamma S_n^+ & \sinh(\lambda+i\gamma S_n^3)\end{pmatrix} \qquad (5.17)$$

Here the auxillary space is again $\mathbb{C}^2$. One regains, while rescaling λ by $\lambda \cdot \gamma$, the XXX model in the limit $\gamma \to 0$. The Lax operator $L_{n,a}(\lambda)$ fulfills the Fundamental Commutation Relations (FCR), provided that the generators are now obeying the deformed commutation relations [18]:

$$e^{i\gamma S^3} S_\pm = e^{\pm i\gamma} S_\pm e^{i\gamma S^3} \tag{5.18}$$

$$S_+ S_- - S_- S_+ = \frac{e^{2i\gamma S^3} - e^{-2i\gamma S^3}}{e^{i\gamma} - e^{-i\gamma}} \tag{5.19}$$

After inserting the deformation: $\frac{\lambda+i}{\lambda-i} \leftrightarrow \frac{\sinh(\lambda+i\gamma)}{\sinh(\lambda-i\gamma)}$, the transcendental Bethe ansatz equations read :

$$\left(\frac{\sinh(\lambda_k + i\gamma/2)}{\sinh(\lambda_k - i\gamma/2)}\right)^N = \prod_{j \neq k} \frac{\sinh(\lambda_k - \lambda_j + i\gamma)}{\sinh(\lambda_k - \lambda_j - i\gamma)} \tag{5.20}$$

Since one has to consider the periodicity in the imaginary part of λ (5.20), the picture of the strings will get more complicated.

We choose a representation of (5.19), which draws the connection to the well known quantum field theoretic sine-Gordon model: $\Box\Phi + m^2 sin2\omega = 0$ [19].

Let Π and Φ be canonical variables fulfilling the commutation relation: $[\Pi_n, \Phi_k] = -i\gamma\delta_{nk}$ (where $\hbar = 1$ here), then S_+ and S_- can be written w.r.t. these variables as:

$$\begin{aligned} S_- &= \tfrac{1}{2m\sin\gamma} e^{-i\Pi/2}(1 + m^2 e^{-2i\Phi}) e^{-i\Pi/2} &= \tfrac{1}{2m\sin\gamma} h_-(\Phi) e^{-i\Pi} \\ S_+ &= \tfrac{1}{2m\sin\gamma} e^{+i\Pi/2}(1 + m^2 e^{+2i\Phi}) e^{+i\Pi/2} &= \tfrac{1}{2m\sin\gamma} h_+(\Phi) e^{+i\Pi} \end{aligned}$$

$with H_\pm = 1 + m^2 e^{\pm 2i\Phi + i\gamma}$ The Lax operator (5.17) still fulfills the FCR if one multiplies it with $-2mi\sin\gamma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ from the right:

$$\begin{aligned} \tilde{L}_{n,a}(\lambda) &= \begin{pmatrix} h_+(\Phi_n) e^{i\Pi_n} & -2mi\sinh(\lambda + i\Phi_n) \\ -2mi\sinh(\lambda - i\Phi_n) & h_-(\Phi_n) e^{-i\Pi_n} \end{pmatrix} \\ &= \mathbb{1} + i\triangle \begin{pmatrix} \tilde{\Pi} & -2\sinh(\lambda + i\tilde{\Phi}) \\ -2\sinh(\lambda - i\tilde{\Phi}) & \tilde{-\Pi} \end{pmatrix} + O(\triangle^2) \end{aligned}$$

The last expression is valid in the limit $\triangle \to 0$, when $m \sim \triangle$, $\Pi_n = \triangle\Pi(x) + ...$ and $\Phi \sim \Phi(x)$ and $\delta_{nk} \to \delta(x-y)\triangle$ and coincides with the classical Lax operator of the SG-model [20].

5.3 The Bethe ansatz for the SG-model

In the following we want to apply the Bethe ansatz to the SG-model. It is impossible to find a local vacuum for the Lax operator $\tilde{L}_{n,a}$. Fortunately, there exists one for the Lax operator of two adjacent lattice points:

$$\tilde{L}_{2,a}\tilde{L}_{1,a} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

The annihilator looks as:

$$C = e^{\lambda}(e^{-i\Phi_2}h_+(\Phi_1)e^{i\Pi_1} + h_-(\Phi_2)e^{-i\Pi_2}e^{-i\Phi_1}) + e^{-\lambda}(\text{similar term})$$

We make the following ansatz for the vacuum:

$$\psi = f(q_1)\delta(q_1 - q_2 - \gamma + \pi)$$

and see, that if $C\psi = 0$, then $f(q)$ has to obey the functional equation:

$$h_+(q_1)f(q_1 + \gamma) - h_-(q_1 + \pi)f(q) = 0 \tag{5.21}$$

For rational $\gamma = \frac{2\pi p}{q}$ there exist finite dimensional representations of the commutation relations on $\mathbb{C}^q$.Then (5.21) becomes an equation for the vectors f in $\mathbb{C}^q$ One still has to show, that the vacuum ψ is an eigenstate of A and D. This is exactly the case, if on makes use of the functional equation (5.21). So $A(\lambda)\psi = a(\lambda)\psi$, where:

$$a(\lambda) = 1 + m^4 + 2m^2 \cosh(2\lambda - i\gamma)$$

The LHS of the transcendental Bethe ansatz equations are therefore:

$$\frac{a(\lambda)}{d(\lambda)} = \frac{\cosh(\lambda + \omega - i\gamma/2)\cosh(\lambda - \omega - i\gamma/2)}{\cosh(\lambda + \omega + i\gamma/2)\cosh(\lambda - \omega + i\gamma/2)}$$

where,

$$2\cosh 2\omega = m^2 + \frac{1}{m^2}.$$

Looking at the above equations the sine-Gordon model looks like an inhomogenous spin chain with spin $-1/2$.

To get the dynamics one needs a fundamental operator, corresponding to $\tilde{L}_{n,a}(\lambda)$. It was not found yet. However the following trick works: let us double the lattice again, introducing the canonical variables π_k, ϕ_k at each lattice site, and introduce the Lax operator:

$$\tilde{L}_{n,a}(\lambda) = L^V_{2n,a}(\lambda + \omega)L^V_{2n-1,a}(\lambda - \omega)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.22}$$

where the $L^V_{n,a}(\lambda)$ have the form:

$$L^V(\lambda) = \begin{pmatrix} e^{i\pi} & ie^{\lambda}e^{i\phi} \\ ie^{\lambda}e^{-i\phi} & e^{-i\pi} \end{pmatrix} \tag{5.23}$$

It coincides with $\tilde{L}_{n,a}(\tilde{\lambda})$ in the limit $m = 0$, $me^{\tilde{\lambda}} = e^{\lambda}$. Introducing constraints

$$\chi \overset{\text{def}}{=} \pi_{2n} + \pi_{2n-1} + \phi_{2n} - \phi_{2n-1} \equiv 0$$

which commute

$$[\chi_n, \chi_k] = 0.$$

we effectively reduce the number of degrees of freedoms and can return to variables Φ_n and Π_n. The $\tilde{L}_{n,a}(\lambda)$, from (5.22) then will coincide with (5.21).

For $L^V(\lambda)$ in (5.23) the fundamental L-operator was found by Volkov [21]. It looks as follows:

$$L^V_{n,f}(\lambda) = P_{n,f}\, l(\pi_n + \pi_f - \phi_n + \phi_f, \lambda)$$

where the function $l(w, \lambda)$ satisfies the equation

$$\frac{l(w+\gamma,\lambda)}{l(w-\gamma,\lambda)} = \frac{1+\lambda e^w}{\lambda + e^w}$$

With these data we can use the formalism of Lecture 4 to introduce the expressions for $\exp(-iP)$ and $\exp(-iH)$ in terms of products of the operators $l(w,\lambda)|_{\lambda=\pm w}$.

5.4 Remarks on conformal field theory

If one makes the substitution $\Phi \to \Phi + ia$ in the equation of motion of the sine-Gordon model:

$$\Box\Phi + m^2\left(\frac{e^{2i\Phi} - e^{-2i\Phi}}{2i}\right) = 0$$

and considers the formal limit $m \to 0$ and $a \to \infty$, such that $m^2 e^{2a} = -2$, then the result looks as the field equation for the conformal quantum field theory:

$$i\Box\Phi + e^{-2i\Phi} = 0.$$

The LHS of the Bethe ansatz equations reduce to:

$$\frac{a(\lambda)}{d(\lambda)} \longrightarrow e^{i\gamma}\frac{cosh(\lambda - i\gamma/2)}{cosh(\lambda + i\gamma/2)}.$$

The phase factor $e^{i\gamma}$ is the only difference with the usual XXZ momentum. It leads to a modification of the investigations of the Bethe ansatz equations. As a result one gets a correspondence with the minimal models of CFT for $\gamma = \frac{\pi\nu}{\nu+1}$, ν - integer (see [22] as one of the indications of this). Unfortunately I have no time to discuss this topic now.

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