Adiabatic Invariance and the Regularity of Perturbations

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Abstract

A loss in smoothness of a switching process decreases the accuracy of an adiabatic invariant. Here we show that for classical Hamiltonian systems the degree of smoothness can be observed in the signal.

1 Introduction

A parameter dependent family of integrable dynamical systems can be perturbed by making the parameter time dependent. This perturbation normally destroys the integrability. If one changes the parameters very slowly, the adiabatic theorem tells us that the motion of the perturbed system stays for most of the initial data nearby the unperturbed motion. The parameters change on a time scale $1/\varepsilon$. $\varepsilon \to 0$ is the adiabatic limit and $\tau = \varepsilon t$ the so called slow time.

One interesting situation arises when a parameter is switched on and off. Switching means that we have a time dependent parameter $\omega(\tau)$ which is constant up to a time $\tau_0 = 0$, then changes its value and finally remains constant again for $\tau \geq 1$. (See figure 1.)

We want to examine the influence of the smoothness of the switching on the dynamical behaviour of the system. First we have to distinguish different classes of smoothness:

- Finite differentiability: $\omega \in C^n(\mathbb{R})$, $\omega \notin C^{n+1}(\mathbb{R})$ for $n \in \mathbb{N}$.
- Infinite differentiability: $\omega \in C^\infty(\mathbb{R})$. Here it is possible to distinguish different classes by the growth of the derivatives of ω. These classes are called Gevrey classes and are indexed by $\alpha \in [1, \infty)$. Section 2 discusses these classes in detail.

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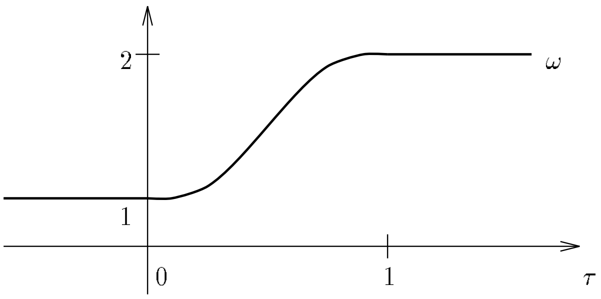

Figure 1: A typical switching function

- Analyticity. In this case ω cannot be constant on any interval. The switching times $\tau = 0$ and $\tau = 1$ are pushed to $-\infty$ resp. $+\infty$.

The known results (stated very roughly) are as follows:

- For time dependent systems of classical mechanics with one degree of freedom the action variable is an adiabatic invariant in the following sense [1], [2], [6]:
$$\max_{\tau\in[0,1]} |J(\tau) - J(0)| = \mathcal{O}(\varepsilon)\,.$$
If we only consider $\Delta J := J(1) - J(0)$ there exist better results depending on the smoothness of ω:
  - For $\omega \in C^n(\mathbb{R})$ one has $\Delta J = \mathcal{O}(\varepsilon^{n+1})$. (See [19], [15], [13].)
  - For the special case of the harmonic oscillator Stengle showed in [24]: If $\|\omega^{(n)}\|_1 = \mathcal{O}(\exp h(n))$ where $n \log n = o(h(n))$, then for any $\delta > 0$ one has
$$\frac{J(\infty)}{J(-\infty)} - 1 = \mathcal{O}(\exp(-h^*(\log \varepsilon^{-1+\delta})))\,,$$
where h^* is the Legendre transform of h. Remark: If ω is Gevrey the optimal choice of h concerning the growth of the derivatives of ω does not meet the assumption $n \log n = o(h(n))$.
  - For analytic ω one has $\Delta J = \mathcal{O}(\exp(-\frac{c}{\varepsilon}))$. (See [19].)

- For time dependent systems of classical mechanics with more than one degree of freedom the situation is qualitatively different since one cannot avoid resonances [2]. They do not allow any better result (even for ΔJ) than the following [20], [7]: If one restricts the initial data to an appropriate set of measure $1 - \sqrt{\varepsilon}$ one gets
$$\max_{\tau\in[0,1]} |J(\tau) - J(0)| = \mathcal{O}(\sqrt{\varepsilon})\,.$$

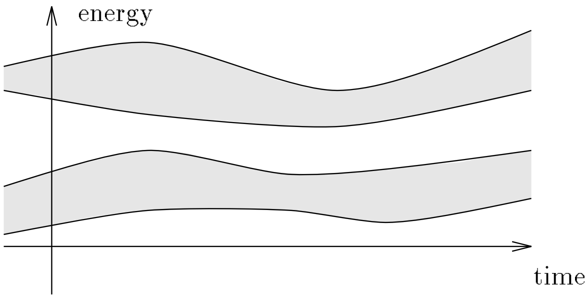

Figure 2: Gap condition in quantum mechanics

- For time dependent quantum mechanical systems one has the following result: If the system satisfies a gap condition (see figure 2), then the tunneling probability $\mathcal{P}$ between one energy band at time $-\infty$ and another band at time $+\infty$ is an adiabatic invariant:
  - If the time dependence is C^n one has $\mathcal{P} = \mathcal{O}(\varepsilon^{n+1})$. (See [3], [14], [21], [11].)
  - In the analytic case one has $\mathcal{P} = \mathcal{O}(\exp(-\frac{c}{\varepsilon}))$. (See [22], [12], [10].)
  - For the Gevrey class case Nenciu showed in [22]: $\mathcal{P} = \mathcal{O}(\exp(-\frac{c}{\sqrt[\alpha]{\varepsilon}}))$.

The gap condition in quantum mechanics can be compared to a non-resonance condition in classical mechanics. If one has more than one degree of freedom in quantum mechanics, one cannot in general avoid level crossings [8] [23]. The parallelism of the quantum mechanical situation and the classical situation with one degree of freedom leads to the conjecture that there should be a similar behaviour also concerning Gevrey classes.

Conjecture *For a system of classical mechanics with one degree of freedom and a perturbation of Gevrey class α the action should be an adiabatic invariant of the form*

$$\Delta J = \mathcal{O}(\exp(-\frac{c}{\sqrt[\alpha]{\varepsilon}}))\,. \tag{1}$$

This conjecture is also motivated by the known relation between ΔJ and the differentiability ($C^n(\mathbb{R})$) of the switching function. The aim of this paper is to support this conjecture by a numerical analysis in the simple case of the harmonic oscillator with a switching function chosen in a Gevrey class. The numerical analysis clearly indicates that there is a measurable effect due to the change of smoothness of the switching function. Therefore it may be of importance for concrete physical experiments to control the smoothness of switching functions.

1.1 The perturbed harmonic oscillator

For our numerical analysis we consider the 1-dimensional harmonic oscillator with slowly varying frequency ω, described by the Hamiltonian

$$E(p,q,\tau) = \frac{1}{2}(p^2 + \omega^2(\tau)q^2), \qquad \tau = \varepsilon t\,,$$

where $\varepsilon \to 0$ in the adiabatic limit. Using the standard canonical transformation generated by

$$S = \frac{1}{2}\omega q^2 \cot\varphi$$

we transform E to action angle variables:

$$H(J,\varphi,\tau) = J\omega + \varepsilon J \frac{\omega'}{2\omega}\sin(2\varphi)\,. \tag{2}$$

The motion is described by the solutions of the canonical equations

$$\varepsilon J' = -\frac{\partial H}{\partial \varphi}, \qquad \varepsilon\varphi' = \frac{\partial H}{\partial J}\,, \tag{3}$$

where the prime denotes the derivative with respect to τ.

2 Switching functions

The perturbation is switched on and off by using a C^∞ function ω which is constant outside $[0,1]$. (See figure 1.) The smoothness of the switching function in the C^∞ class is controlled by the growth of its derivatives. The Gevrey classes interpolate between the pure C^∞ case and the analytic case [5], [9], [16].

2.1 Gevrey classes

Definition 1 *For $\alpha \in [1,\infty)$ a function $f \in C^\infty(\mathbb{R})$ is called of Gevrey class α, if there exist constants $c_1, c_2 > 0$, such that for all $n \in \mathbb{N}_0$ we can estimate the n-th derivative of f by*

$$\|f^{(n)}\|_\infty \le c_1 c_2^n n^{\alpha n}\,.$$

The Gevrey class is denoted by $\Gamma^\alpha(\mathbb{R})$.

The Gevrey classes have the following properties:

1. For $\alpha = 1$ we are in the analytic case.

2. $\Gamma^{\alpha_1} \subset \Gamma^{\alpha_2}$ for $\alpha_1 \le \alpha_2$.

3. $\Gamma^\alpha \cap C_0 \ne \varnothing$ for $\alpha > 1$.

4. Γ^α is an algebra, stable under differentiation, translation and composition.

5. Let f be of Gevrey class α and $F(x) := \int_0^x f(t)\,dt$. If $\|F\|_\infty < \infty$ then $F \in \Gamma^\alpha$.

6. The remainder of the Taylor series expansion of a function of Gevrey class $\alpha > 1$ can be made exponentially small in a sense described in Lemma 1 below.

Remarks

- The properties 1, 2, 4 and 5 are easily verified. See also [5], [16].
- Because of property 3 the Gevrey classes are useful to construct switching functions.
- In the analytic case ($\alpha = 1$) the Taylor series expansion converges to the function in a neighbourhood of the expansion point. For $\alpha > 1$ this is not necessarily the case, moreover the Taylor series can diverge. But it is still possible to make the remainder exponentially small by an appropriate truncation of the series:

Lemma 1 *For $f \in \Gamma^\alpha$ there exist constants $M, c > 0$ such that for all $\varepsilon > 0$ one can find $N_\varepsilon \in \mathbb{N}$ such that the remainder R_{N_ε} of the Taylor series expansion becomes small:*

$$\sup_{x \in U_\varepsilon(x_0)} |R_{N_\varepsilon}(f, x_0; x)| \le M \exp(-\frac{c}{\sqrt[\alpha-1]{\varepsilon}}).$$

Proof. For $f \in \Gamma^\alpha$, $x \in U_\varepsilon(x_0)$ and $N \in \mathbb{N}$ there exists $t \in U_\varepsilon(x_0)$ such that

$$|R_N(f, x_0; x)| = |\frac{f^{(n)}(t)}{N!}(x - x_0)^N| \le |f^{(n)}(t)|\,\frac{\varepsilon^N}{N!}.$$

Using the Stirling formula $n! = \left(\frac{n}{\mathrm{e}}\right)^n \sqrt{2\pi n}\ (1 + \mathcal{O}(\frac{1}{n})) \ge \text{const.}\left(\frac{n}{\mathrm{e}}\right)^n$ and the fact that f is of Gevrey class α we get

$$\begin{aligned} |R_N(f, x_0; x)| &\le c_1(c_2\varepsilon)^N \frac{N^{\alpha N}}{N!} \le c_3(c_2\mathrm{e}\,\varepsilon)^N N^{(\alpha-1)N} \\ &= c_3 \exp\Big(N\Big(\ln(c_2\mathrm{e}\,\varepsilon) + (\alpha - 1)\ln N\Big)\Big). \end{aligned}$$

We minimize the exponent by choosing

$$N_\varepsilon := \left[\frac{1}{\mathrm{e}\,\sqrt[\alpha-1]{c_2\mathrm{e}\,\varepsilon}}\right]$$

where the square brackets denote the integer part. Finally we get

$$|R_N(f, x_0; x)| \le c_3 \exp\left(-\frac{\text{const.}}{\sqrt[\alpha-1]{\varepsilon}}\right).$$

■

2.2 A switching function in the Gevrey class α

For the numerical analysis we need a switching function with adjustable Gevrey class. The switching function will be build using the following "prototype" function:

Lemma 2 *For $\beta > 0$ the function*

$$f_\beta(x) := \begin{cases} \exp(-\frac{1}{x^\beta}) & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

is of Gevrey class $1 + \frac{1}{\beta}$. Moreover, $f_\beta \notin \Gamma^\alpha$ for any $\alpha < 1 + \frac{1}{\beta}$.

To prove Lemma 2 we have to estimate the derivatives of f_β. Lemma 3 gives us an explicit formula for all derivatives. The proof of Lemma 3 is by induction and will not be stated here.

Lemma 3 *f_β is C^∞ and for all $n \in \mathbb{N}$ we have*

$$f_\beta^{(n)} = \sum_{k=0}^{n-1} a_{n,k}\, g_{n,k} \tag{4}$$

with

$$a_{n,0} = \beta^n \tag{5}$$

$$a_{n,k} = (-1)^k \beta^{n-k} \sum_{1 \le j_1 \le \ldots \le j_k \le n-k} \prod_{i=1}^{k} \Big(j_i(\beta+1) + i - 1 \Big) \tag{6}$$

and

$$g_{n,k}(x) = x^{(k-n)\beta - n} f_\beta(x)\,. \tag{7}$$

Proof of Lemma 2. First we estimate the functions $g_{n,k}$. Obviously $g_{n,k}$ tends to zero as $x \to 0$ or $x \to \infty$. The maximum of $g_{n,k}$ is given by

$$\max_{x \in \mathbb{R}} g_{n,k}(x) = \gamma^\gamma \exp(-\gamma)$$

with $\gamma = n(1 + \frac{1}{\beta}) - k$. We get

$$\|g_{n,k}\|_\infty \le \Big(n(1 + \frac{1}{\beta}) \Big)^{n(1+\frac{1}{\beta})-k}. \tag{8}$$

In (6) we estimate each factor of the product by $(n-k)(\beta+1) + k - 1 \le n(\beta+2)$. The sum in (6) has $\binom{n-1}{k}$ terms. So we get

$$|a_{n,k}| \le \beta^{n-k} \binom{n-1}{k} (n(\beta+2))^k \le \tilde{\beta}^n \binom{n-1}{k} (n(\beta+2))^k \tag{9}$$

with $\tilde{\beta} = \max\{\beta, 1\}$. Using (8) and (9) we finally get for $n \in \mathbb{N}$

$$\|f_\beta^{(n)}\|_\infty \leq \sum_{k=0}^{n-1} \binom{n-1}{k} \tilde{\beta}^n (nb)^{n(1+\frac{1}{\beta})} \leq \text{const.}^n n^{n(1+\frac{1}{\beta})}$$

with $b = \max\{1 + \frac{1}{\beta}, \beta + 2\}$. Together with the fact that f_β is bounded we have shown that $f_\beta \in \Gamma^{1+\frac{1}{\beta}}$. We will see now that $f_\beta \notin \Gamma^\alpha$ for any $\alpha < 1 + \frac{1}{\beta}$. The Taylor series expansion of f_β at the point $x_0 = 0$ is identically zero, thus Lemma 1 gives us

$$\exp(-\frac{1}{\varepsilon^\beta}) = \sup_{0<x<\varepsilon} \exp(-\frac{1}{x^\beta}) = \sup_{x \in U_\varepsilon(0)} |f_\beta(x)| \leq M \exp(-\frac{c}{\varepsilon^{\frac{1}{\alpha-1}}}) .$$

This implies $\beta \geq \frac{1}{\alpha-1}$ or $\alpha \geq 1 + \frac{1}{\beta}$. ∎

The derivative of f_β itself has not compact support, but we can use f_β to construct a switching function of the desired type.

$$\omega_\beta(\tau) := c \int_0^\tau f_\beta(s) f_\beta(1-s)\, ds + 1 \,, \tag{10}$$

where the constant c is chosen in such a way that $\omega_\beta(1) = 2$. It follows from Lemma 2 and the properties 4 and 5 that ω_β is of Gevrey class $1 + \frac{1}{\beta}$.

3 Numerical analysis

3.1 Time evolution of the action variable

To calculate the time evolution of the action variable J we have to solve the canonical equations (3) with the Hamilton function (2). This is done numerically by use of the Runge-Kutta algorithm with an error term of order 5. Since we consider C^∞-smooth perturbations and are interested in the behaviour of J for small values of ε, we have to deal with very small differences between $J(\tau)$ and the initial value J_0. So special care has to be taken in the numerical calculations not to compute differences of large values (compared to the difference itself).

A typical time evolution $J(\tau)$ is shown in figure 3. J is oscillating with a finally decreasing amplitude. The interesting point is that the difference $\Delta J = J(1) - J(0)$ is small compared to the maximal amplitude of the oscillation. This behaviour is typical for the switching of a perturbation. ΔJ becomes exponentially small with respect to ε, but $\max_{\tau \in [0,1]} |J(\tau) - J_0|$ scales with $\mathcal{O}(\varepsilon)$.

This can be understood with the help of the perturbation theory, which can be done for example by use of the Lie formalism [4] [17] [13]. In the C^∞ case it

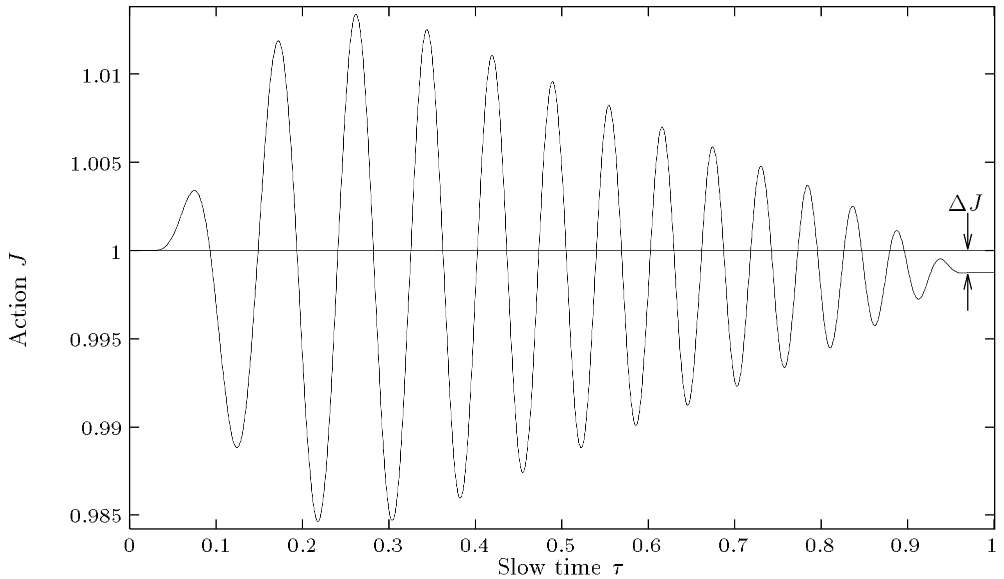

Figure 3: Time evolution of the action variable ($\varepsilon = 0.2$, $\beta = 1$)

gives us an approximation of $J(\tau)$ to any desired order. (See e.g. [13].)

$$J(\tau) = J_0[1 - \varepsilon\frac{1}{2}\frac{\omega'}{\omega^2}\sin 2\tilde{\theta}(\tau) + \varepsilon^2(-\frac{1}{4}\frac{1}{\omega}\left(\frac{\omega'}{\omega^2}\right)'\cos 2\tilde{\theta}(\tau) + \frac{1}{8}\left(\frac{\omega'}{\omega^2}\right)^2)] + \mathcal{O}(\varepsilon^3) \tag{11}$$

with

$$\tilde{\theta}(\tau) = \theta_0 + \frac{1}{\varepsilon}\int_0^\tau \bar{\omega}(s)\,ds, \qquad \bar{\omega} = \omega(1 - \frac{\varepsilon^2}{8}\left(\frac{\omega'}{\omega^2}\right)^2) \tag{12}$$

Obviously $\max_{\tau\in[0,1]}|J(\tau) - J_0|$ is of order $\mathcal{O}(\varepsilon)$, but at $\tau = 0$ and $\tau = 1$ all derivatives of ω vanish (since $\omega \in C^\infty$ and is constant outside $[0,1]$), such that terms of order ε and higher become zero.

3.2 ε-dependence of ΔJ

Since we are interested in the ε-dependence of ΔJ and in the dependence of $\Delta J(\varepsilon)$ on the smoothness of the switching function, computations as shown in figure 3 have been done for a large number of ε values and for many switching functions.

To present the well known situation of a switching function in $C^n(\mathbb{R})$, the numerical analysis has been done for $\tilde{\omega} \in C^1(\mathbb{R})$, $\tilde{\omega}|_{(0,1)} \in C^\infty((0,1))$ and $\tilde{\tilde{\omega}} \in C^2(\mathbb{R})$, $\tilde{\tilde{\omega}}|_{(0,1)} \in C^\infty((0,1))$. In this case formula (11) is still valid for $\tau \in [0,1]$ if we define the derivatives of ω at the boundary points $\tau = 0$ and $\tau = 1$ to be the right hand side (resp. left hand side) limit of the derivatives. Since terms of

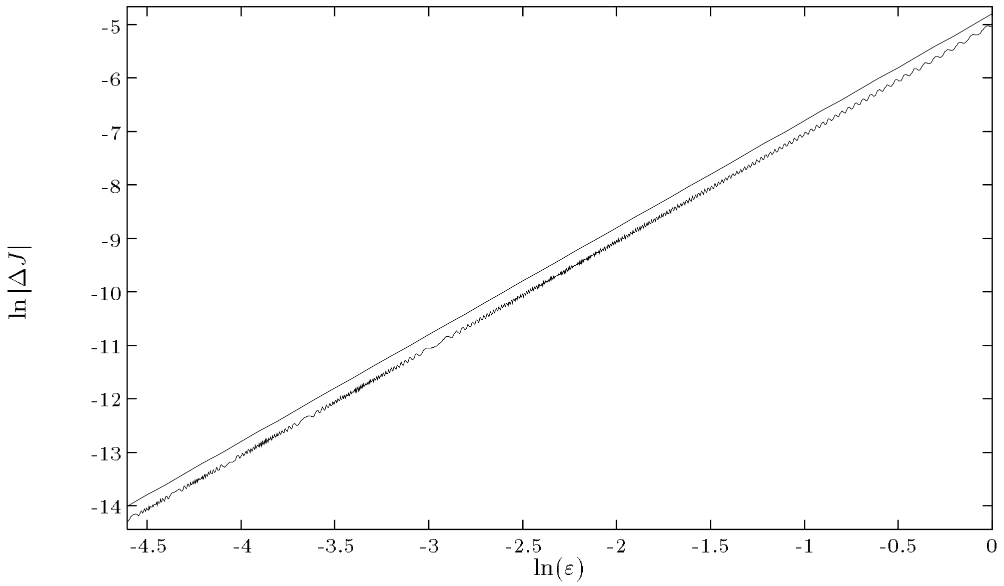

Figure 4: Deviation of the action for $\omega \in C^1(\mathbb{R})$ (including a straight line with slope 2)

order ε^n contain the n-th derivative of ω it is clear that we get $\Delta J = \mathcal{O}(\varepsilon^2)$ for $\tilde{\omega}$ and $\Delta J = \mathcal{O}(\varepsilon^3)$ for $\tilde{\tilde{\omega}}$.

This behaviour can be easily verified in a double logarithmic plot of the numerical data. See figures 4 and 5. The straight lines of slope 2 (resp. 3) indicate this behaviour.

For a switching function in a Gevrey class the numerical analysis in order to verify the conjectured behaviour (1) is more complicated. Since

$$|\Delta J| \approx M \exp(-\frac{c}{\sqrt[\alpha]{\varepsilon}}) \tag{13}$$

implies

$$\ln|\Delta J| \approx \ln M - \frac{c}{\sqrt[\alpha]{\varepsilon}} \tag{14}$$

and

$$\ln|\ln|\Delta J| - \ln M| \approx \ln c - \frac{1}{\alpha}\ln\varepsilon \tag{15}$$

it is a good idea to plot $\ln|\ln|\Delta J||$ over $\ln\varepsilon$. If $|\Delta J| \ll M$ and the action variable has the assumed behaviour the plot should be approximately a straight line with slope $-\frac{1}{\alpha}$ for small ε. Figure 6 shows such a plot. In this scaling larger values of the y-axis correspond to smaller values of ΔJ.

One can see that there are singularities which depend on the chosen initial angle φ_0. (The initial action is always $J_0 = 1$). To eliminate these singularities

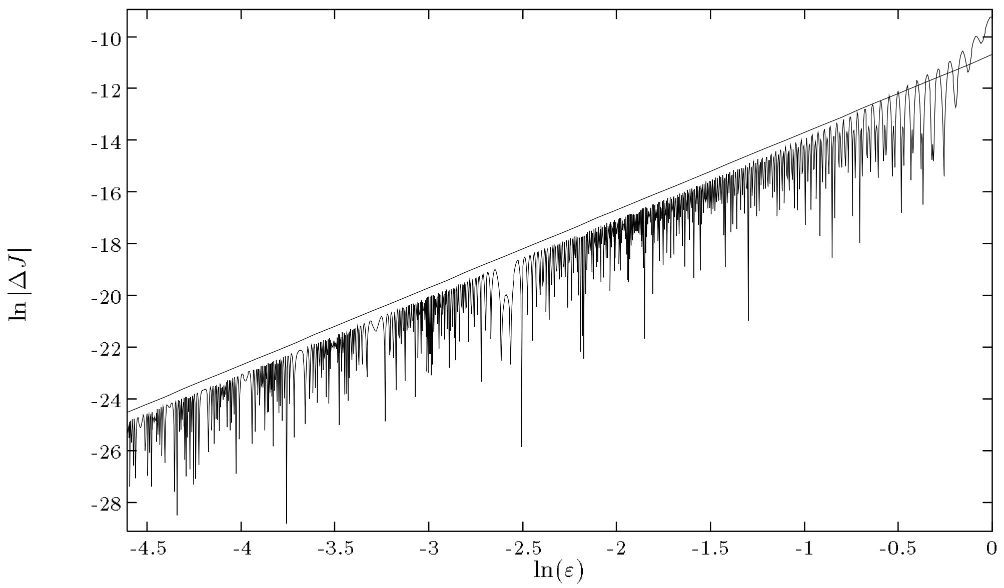

Figure 5: Deviation of the action for $\omega \in C^2(\mathbb{R})$ (including a straight line with slope 3)

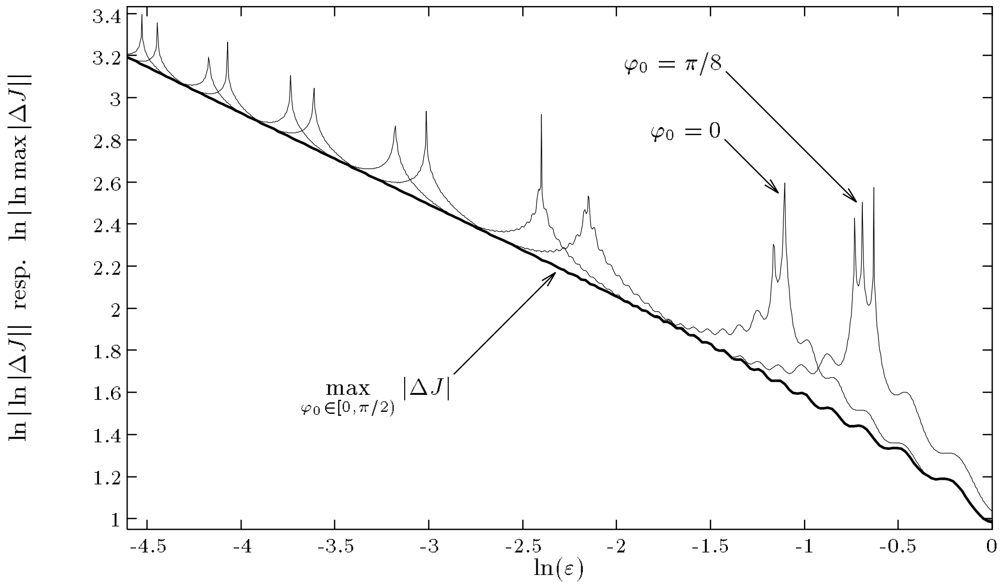

Figure 6: Deviation of the action for different initial angles φ_0 ($\beta = 1$)

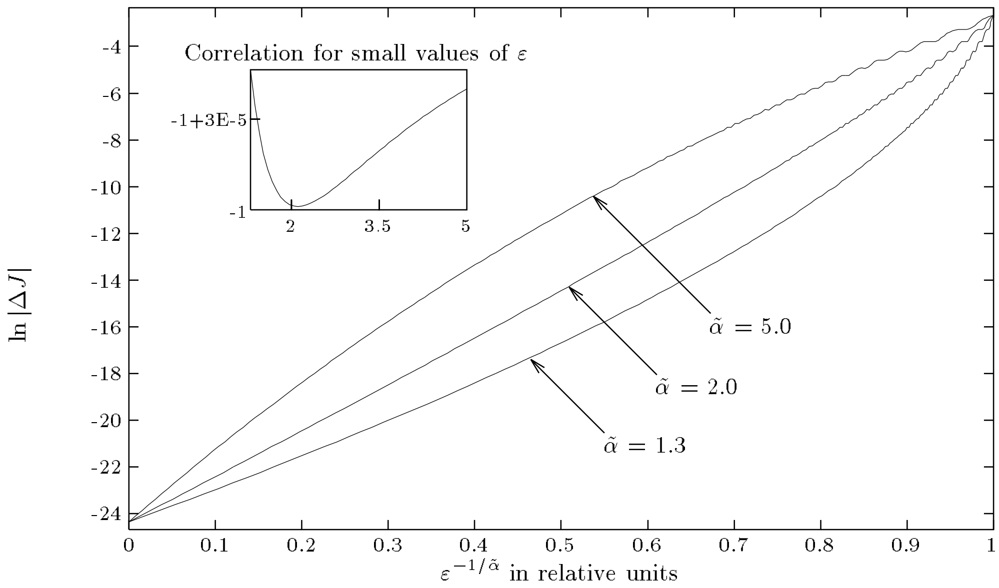

Figure 7: Deviation of the action in different plots ($\beta = 1$)

the computations are done for several different initial angles. For each ε value the greatest $|\Delta J|$ value is plotted. (See also figure 6). This gives a nearly straight line for small values of ε. The numerical analysis of the slope (putting $\ln M = 0$) differs from the conjectured by approximately 12%. To improve this one has to take into consideration the value $\ln M$.

3.3 Gevrey class dependence of $\Delta J(\varepsilon)$

To see the dependence of $\Delta J(\varepsilon)$ from the Gevrey class of the switching function, the parameter β of the switching function ω_β is varied. The idea is to compute a value $\tilde{\alpha}_{\mathrm{opt}}$ from the numerical data, such that the data best fits to the assumed behaviour $\mathcal{O}(\exp(-\frac{c}{\sqrt[\tilde{\alpha}]{\varepsilon}}))$. In other words, for each Gevrey class $\alpha = 1 + \frac{1}{\beta}$ of the perturbation ω_β we plot $\ln |\Delta J(\varepsilon)|$ over $\varepsilon^{-1/\tilde{\alpha}}$ (see eq. (14)) for several values of $\tilde{\alpha}$.

Figure 7 shows such plots. The x-axis is scaled in such a way, that small values of ε are plotted on the left hand side. The optimal value $\tilde{\alpha}_{\mathrm{opt}}(\alpha)$ is defined to be that value $\tilde{\alpha}$ for which the plot at small values of ε is best approximated by a straight line. The choice is made by optimizing the correlation coefficient of a linear regression of the data. (See small picture inside figure 7.) A correlation coefficient with absolute value of 1 indicates a straight line.

If ΔJ has the conjectured behaviour (1), $\tilde{\alpha}_{\mathrm{opt}}$ should equal the Gevrey class $\alpha = 1 + \frac{1}{\beta}$ of the switching function. Figure 8 compares this Gevrey class with the numerically computed "best fit" value $\tilde{\alpha}_{\mathrm{opt}}$. Numerically the two values coincide up to an error of 8% for $\beta = 1$.

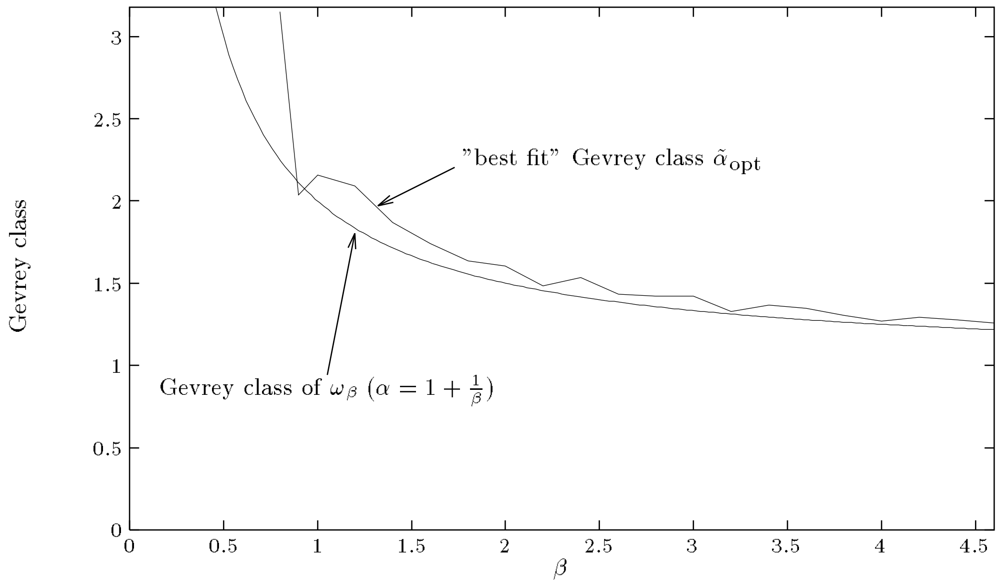

Figure 8: Comparison of numerical result and theoretical conjecture

3.4 Conclusion

The derivation of the action ΔJ depends on the regularity of the switching function. This functional dependence is well understood for switching functions with only finitely many derivatives. For the case of switching functions in a Gevrey class a conjecture is formulated which is close to a result recently shown in the frame work of quantum mechanics.

In this article, conjecture (1) is verified numerically for a special model and switching functions in a Gevrey class with $\alpha > 1$. The case $\alpha = 1$, i.e. the case of analytic switching functions, can be treated analytically and is consistent with conjecture (1), too. For switching functions with parameter $\beta > 1.2$ the best fit value $\tilde{\alpha}_{\mathrm{opt}}$ deviates from the prediction of the conjecture by less than 10%.

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