Explicit Computation of the Discriminant for the Harper Equation with Rational Flux

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Abstract

A central role in the spectral analysis of the Harper equation with rational flux is played by its discriminant which is computed explicitely in this paper. Its coefficients are sums of trigonometric polynomials. The result applies directly to the formula for the density of states. Furthermore we compute the nullspace of the Hofstadter matrix for all rational fluxes and all Bloch vectors.

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1 Introduction

Many properties of the Harper equation

$$f(n+1) + f(n-1) + 2\cos(\Phi n + \theta_2) f(n) = z f(n) \qquad (1)$$

($f : \mathbf{Z} \to \mathbf{C}$; Φ, $\theta_2 \in \mathbf{R}$) have been studied since its introduction by Peierls in 1933 [17] . For a recent review of results see e.g. [2].

The difference equation (1) is the eigenvalue equation for the tight-binding Hamiltonian of an electron in a one-dimensional periodic potential $V_{\theta_2}(n) := 2\cos(\Phi n + \theta_2)$. For the rational flux case, namely for $\Phi = 2\pi\frac{p}{q}$ with p, q relative prime integers, the potential is periodic and the eigenvalue equation (1) reduces to a matrix problem.

Using the following $q \times q$-matrices

$$W_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ & \ddots & \ddots & 0 \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix} \quad , \quad W_2 := \mathrm{diag}\left(e^{i\Phi}, e^{i2\Phi}, \ldots, e^{iq\Phi}\right). \qquad (2)$$

we define the Hofstadter Hamiltonian [11]

$$H(\theta)\psi := (W_1 e^{i\theta_1} + W_1^\star e^{-i\theta_1} + W_2 e^{i\theta_2} + W_2^\star e^{-i\theta_2})\psi = z\psi \qquad (3)$$
$$\psi = (\psi_1, \psi_2, \ldots, \psi_q) \in \mathbf{C}^q, \;\; \theta = (\theta_1, \theta_2) \in \mathbf{R}^2.$$

Setting $f(n) = \psi_{n \bmod q}$ for $n \in \mathbf{Z}$ and $\theta = (\theta_2, 0)$ we recover the Harper equation for rational flux.

A central role in the spectral analysis of (1) is played by the discriminant $\Delta_{p/q}(\theta, z)$ [8, 9, 10, 13]. In terms of $H(\theta)$ it is defined as the characteristic polynomial

$$\Delta_{p/q}(\theta, z) := \det\left(H(\theta) - z\right).$$

Due to Chamber's relation [5] we know that the discriminant decomposes into a polynomial $P_{p/q}(z)$ independent of θ and an offset function $h_{p/q}(\theta)$ independent of z

$$\Delta_{p/q}(\theta, z) = P_{p/q}(z) + h_{p/q}(\theta).$$

The offset function is know explicitly (see below).

Our main result in this paper is

Theorem 1 *The polynomial*

$$P_{p/q}(z) = \sum_{l=0}^{q} c_{p/q}(l)\, z^l$$

in the discriminant has coefficients $c_{p/q}(l) = 0$ for $q-l$ odd and

$$c_{p/q}(l) = (-1)^{\frac{q+l}{2}} \sum_{\pi \in \mathcal{S}(q,l)} \beta_{\pi(1)+q-l-1}\, \beta_{\pi(2)+q-l-3} \cdots \beta_{\pi(\frac{q-l}{2})+1}$$

for $q-l$ even where

$$\beta_k := 4\sin^2(\frac{\Phi}{2}k) \quad (k = 1, 2, \ldots, q)$$

and $\mathcal{S}(q,l)$ is the set of ordered $\frac{q-l}{2}$-tuples $\left(\pi(1), \pi(2), \ldots, \pi(\frac{q-l}{2})\right)$ with $\pi(l) \in \{0, 1, 2, \ldots, l\}$ and $\pi(l) \geq \pi(l+1)$.

For some coefficients the summation can be carried out and yields simple integer expressions:

$$\begin{aligned} c_{p/q}(q) &= (-1)^q \\ c_{p/q}(q-2) &= 2q\,(-1)^{q-1} \\ c_{p/q}(0) &= (-1)^{q/2}\,(1 + (-1)^q) \end{aligned}$$

These are the only coefficients which are integer for all allowed values of p and q.

The proof of Theorem 1 is given in Section 2 of this paper.

In the appendix we present examples of the polynomials and their roots for cases $q \leq 10$. They have been computed with the help of the symbolic computation environment "Mathematica".

Since the spectrum of Harpers model and especially the shape of the Hofstadter "butterfly" are not yet fully understood in analytic terms, we hope that Theorem 1 gives insight into global properties of the spectrum.

One immediate application of Theorem 1 to a result of Wannier, Obermair and Ray [16] leads to an explicit analytic expression for the density of states $g(\lambda)$ in the Hofstadter model.

Let $\chi_{[a,b]}$ be the characteristic function of the interval $[a,b] \subset \mathbf{R}$ then

$$N(E) := \int_0^{2\pi}\int_0^{2\pi} \frac{d^2\theta}{4\pi^2}\, \mathrm{Trace}\left(\chi_{[-4,E]}\,(H(\theta))\right) \tag{4}$$

is the integrated density of states for the Hofstadter model. It is related to the density of states (DOS) $g(\lambda)$ at energy λ by

$$N(E) = \int_{-4}^{E} d\lambda\, g(\lambda).$$

Using Chamber's relation to carry out the integration over θ_2 in the integral (4) it has been shown in [16]

$$g(\lambda) = \begin{cases} \frac{1}{2\pi^2 q} \frac{dP_{p/q}(\lambda)}{d\lambda} K'\left(\frac{1}{4}|P_{p/q}(\lambda)|\right) & \text{for } |P_{p/q}(\lambda)| \leq 4 \\ 0 & \text{else} \end{cases}$$

Here

$$K'(x) = K(1-x) = F(\frac{\pi}{2}, x) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - x\sin^2 y}} dy$$

is the complete elliptic integral of the first kind and $P_{p/q}(\lambda)$ is the characteristic polynomial. In Theorem 1 we have computed all coefficients of $P_{p/q}(\lambda)$ analyticaly and thus know the derivative $\frac{dP_{p/q}}{d\lambda}(\lambda)$ explicitely, too.

Thus combining Theorem 1 with this result gives an analytic expresssion for the density of states in the Hofstadter model.

An other application of Theorem 1 is the determination of the nullspace of the Hofstadter Hamiltonian. Depending on the parity of q the nullspace is one or two dimensional for odd and even q respectively and for θ chosen properly. A complete discussion of the different cases arising for the nullspace of the Hofstadter matrix is given in Theorem 2 of Section 3.

We will hencforth assume that $\Phi = 2\pi\frac{p}{q}$ with $p, q \in \mathbf{Z}$ and p, q relative prime. To simplify notation we omit the index p/q when no confusion arises.

2 Computation of the Discriminant

In this Section we prove Theorem 1. Let us begin with a short sketch of the strategy.

The proof consists of five steps. We start by transforming the Hofstadter matrix $H(\theta)$ into a unitarily equivalent matrix $\tilde{H}(\theta)$. The new matrix $\tilde{H}(\theta)$ has non-vanishing coefficents on the first off-diagonal and in the upper right and lower left corner only.

Next we derive Chamber's relation for the new matrix $\tilde{H}(\theta)$. Since the offset function $\tilde{h}(\theta)$ is known explicitely, it is enough to compute the discriminant for one value $\theta = \theta_0$,

$$\begin{aligned}\det\left(\tilde{H}(\theta) - z\right) &= P(z) + \tilde{h}(\theta) \\ &= \det\left(\tilde{H}(\theta_0) - z\right) - \tilde{h}(\theta_0) + \tilde{h}(\theta).\end{aligned} \tag{5}$$

One of the technicaly important steps is to find $\tilde{H}(\theta_0)$ such that the upper right and the lower left corner vanish for the special value of θ_0.

In this case the expansion of the determinant is sufficently simple and allows a recursive construction of a sequence of polynomials, which leads in the q-th step to the characteristic polynomial $P(z)$ of the matrix $\tilde{H}(\theta_0)$.

This recursion of polynomials translates into a recursion of their coefficients. Thus we get a recursion for the q coefficients $c(1), c(2), \ldots, c(q)$ of the characteristic polynomial

$$P(z) = \sum_{l=0}^{q} c(l)\, z^l.$$

In a final step we solve this recursion and thus derive an explicit formula for the coefficients

The coefficients of the highest and of the second highest order as well as the lowest order term in $P(z)$ can be simplified using trigonometric identities and are shown to be integers.

In the Appendix we list some polynomials for $q \leq 10$ together with their roots.

We begin working through this program with

Definition 1 *(Rotational Algebra [2])*
Let T_1 and T_2 be two unitaries with

$$T_1 T_2 = e^{i\Phi} T_2 T_1. \tag{6}$$

The $C^\star$-Algebra

$$\mathcal{A}_\Phi := \overline{\left\{ A = \sum_{m \in \mathbf{Z}} a(m) T_1^{m_1} T_2^{m_2},\ a : \mathbf{Z}^2 \to \mathbf{C} \text{ rapidly decreasing} \right\}}^{\|\cdot\|}$$

generated by T_1 and T_2 is called rotational Algebra.

The Hofstadter Hamiltonian can be viewed as an element of the rotational algebra $\mathcal{A}_\Phi$ if we choose
$T_1, T_2 \in \mathcal{B}\left(\int_{[0,2\pi/q]^2}^{\oplus} d^2\theta\, \mathbf{C}^q\right)$

$$T_j(\theta) := e^{i\theta_j}\, W_j \quad (j = 1, 2).$$

Since

$$W_1 W_2 = e^{i\Phi} W_2 W_1$$

this is a representation of $\mathcal{A}_\Phi$. Thus

$$H = T_1 + T_2 + T_1^\star + T_2^\star.$$

In this $C^\star$–algebraic setting the spectrum of an element in $\mathcal{A}_\Phi$ can be defined purely algebraicly [6]. Thus it is invariant under any algebra-automorphism.

Proposition 1 *(New representation for the Hofstadter matrix)*

1. *The map $\pi : \mathcal{A}_\Phi \to \mathcal{A}_\Phi$ generated by*

$$\begin{aligned} \pi(T_1) &= e^{i\Phi/2} T_2 T_1 \\ \pi(T_2) &= T_1^\star \end{aligned}$$

and extended to the whole Algebra is an algebra-automorphism of the rotational algebra.

2.

$$\tilde{H}(\theta) := \pi\left(H(\theta)\right) = \begin{pmatrix} 0 & a_1 & 0 & \cdots & \bar{a}_q \\ \bar{a}_1 & 0 & a_2 & & \\ 0 & \bar{a}_2 & 0 & \ddots & 0 \\ \vdots & & \ddots & \ddots & a_{q-1} \\ a_q & & 0 & \bar{a}_{q-1} & 0 \end{pmatrix}$$

with

$$\begin{aligned} a_r &= (1 + e^{i\Phi(r+\frac{1}{2})} + i\theta_2)\, e^{i\theta_1} \\ &= 2\cos\left(\frac{\theta_2}{2} + \frac{\Phi}{2}(r+\tfrac{1}{2})\right)\, e^{i\theta_1 + i\frac{\theta_2}{2} + i\frac{\Phi}{2}(r+\frac{1}{2})}. \end{aligned}$$

By $\bar{a}_k$ we denotes the complex conjugate of a_k. The matrix $\tilde{H}(\theta)$ will be called the new representation of the Hofstadter Hamiltonian.

The proof of Proposition 1 is straightforward. Part 1 follows from

$$\pi(T_1)\,\pi(T_2) = e^{i\Phi/2}T_2T_1T_1^\star = e^{i\Phi}\pi(T_2)\,\pi(T_1).$$

To show part 2 we rewrite $\tilde{H}$ in the matrix representation $T_j := e^{i\theta_j}W_j$ $(j = 1, 2)$ introduced above.

$$\tilde{H}(\theta) = \left(\mathbf{1} + W_2 e^{i\Phi/2 + i\theta_2}\right) W_1 e^{i\theta_1} + W_1^\star e^{-i\theta_1}\left(\mathbf{1} + W_2^\star e^{-i\Phi/2 + -i\theta_2}\right)$$

Using the definitions of W_1 and W_2 (see Section 1, equation (2)) we can read off the coefficients of $\tilde{H}$.

Remarks:

1. In the new representation Harper's equation reads

$$a_n f(n+1) + \overline{a_{n-1}} f(n-1) = z f(n)$$

   Note that this is a second order difference equation without a potential term.

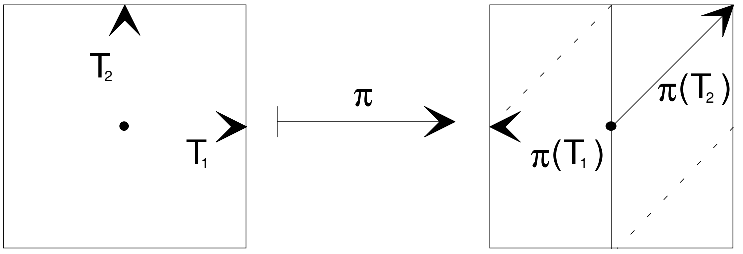

Figure 1: Representation of the algebra automorphism π on $\mathbf{Z}^2$

2. The algebra automorphism π is implemented by the unitary transformation of the standard basis of $\mathbf{C}^q$ into a new ONB given by the basis vectors

$$\phi_\alpha(k) := \frac{1}{\sqrt{q}}(-1)^{pk} e^{i\Phi(\alpha k - \frac{k^2}{2})} \quad (k, \alpha \in \{1, 2, \ldots, q\})$$

As a function of $k \in \mathbf{Z}$ we have choosen $\phi_\alpha(k)$ to be q-periodic in k. Note that the $\phi_\alpha\,(\alpha = 1, 2, \ldots, q)$ are eigenfunctions of $W_1 W_2$. A straightforward computation of the matrix elements $< \phi_\alpha | W_j \phi_\beta >$, $(j = 1, 2)$ shows that

$$\begin{aligned} < \phi_\alpha | W_2 \phi_\beta > &= (W_1^\star)_{\alpha,\beta} \\ < \phi_\alpha | W_1 \phi_\beta > &= \left(W_2 W_1 e^{i\Phi/2 + i\pi p}\right)_{\alpha,\beta} \end{aligned}$$

Here $W_{\alpha,\beta}$ denotes the matrix elements of a matrix W in the standard basis.

3. If we consider T_1 and T_2 to be the magnetic translations along the unit directions of $\mathbf{Z}^2$, the algebra automorphism π represents a rotation of the e_2-axis by an angle of $\pi/4$ (Fig.2). This rotation is visible in the classical phase space pictures, too, as we can see in Fig. 3.

For the new representation of the Hofstadter matrix we have again a Chamber's relation.

Proposition 2 *(Chamber's relation)*

$$\det\left(\tilde{H}(\theta)-z\right)=P(z)+\tilde{h}(\theta)$$

where

$$\tilde{h}(\theta)=(-1)^{q-1}2\,\cos(q\theta_1)+(-1)^p 2\,\cos\left(q(\theta_1+\theta_2)\right)$$

The proof of Proposition 2 will not be given in full detail. It is analogous to the one for Chamber's relation for the original Hofstadter matrix $H(\theta)$ or for one of the other models like the triangular lattice [3] or the Quantum Pendulum [12]. It is based on the following consequence of the commutation relations (2)

$$W_1^l\,W_2^{-m}\,\tilde{H}(\theta_1,\theta_2)\,W_2^m\,W_1^{-l}=\tilde{H}(\theta_1+m\Phi,\theta_2+l\Phi)\quad(l,m\in\mathbf{Z}). \tag{7}$$

Note that W_1 and W_2 are unitary matrices and $\tilde{H}(\theta)$ is 2π-periodic in θ_1 and θ_2. Furthermore for a given pair of relative prime integers p,q there always exists, according to Euklid's Theorem, solutions $l,n\in\mathbf{Z}$ of the diophantine equation $pl+qn=1$. Thus we know that $\det\left(\tilde{H}(\theta)-z\right)$ is a $2\pi/q$ -periodic function in θ_1 and θ_2.

On the other hand it is a polynomial of degree q in its coefficents. Therefore $\det\left(\tilde{H}(\theta)-z\right)$ has a Fourier series in $q\theta$ with five non-vanishing coefficients, which can be determined using the method described in [3].

$P(z)$ is the polynomial of degree q we have to determine for proving Theorem 1. Since it is independent of θ we can, according to (5), evaluate the characteristic polynomial for any value of $\theta=\theta_0$ to get $P(z)$. Our prefered choice is $\theta_0=(\theta_1,\theta_2)=\left(0,-\pi(1+\frac{p}{q})\right)$.

In this case we have $\tilde{h}(\theta_0)=0$. Since also $a_q(\theta_0)=0$, the matrix elements in the upper right and lower left corner disappear and

$$\tilde{H}(\theta_0)=\begin{pmatrix} 0 & \alpha_1 & 0 & \cdots & 0\\ \bar{\alpha}_1 & 0 & \alpha_2 & & \\ 0 & \bar{\alpha}_2 & 0 & \ddots & 0\\ \vdots & & \ddots & \ddots & \alpha_{q-1}\\ 0 & & 0 & \bar{\alpha}_{q-1} & 0 \end{pmatrix} \tag{8}$$

where

$$\alpha_r := a_r(\theta_0) = 2\cos\left(\frac{\Phi}{2}r - \frac{\pi}{2}\right) e^{i\frac{\Phi}{2}r - i\frac{\pi}{2}}$$

Thus the computation of the discriminant for Harper's equation is reduced to finding the characteristic polynomial of the tri-diagonal matrix $\tilde{H}(\theta_0)$ with vanishing diagonal terms.

Here the following formulas apply

Proposition 3 *(recursion for the characteristic polynomial)*

Let M be a tri-diagonal $q \times q$-matrix of the form (8) with coefficients $\alpha_i\,(i = 1, 2, \ldots, q)$ and $\beta_i := |\alpha_i|^2$ for $i = 1, 2, 3, \ldots, q$.

Let $p_0(z), p_1(z), \ldots, p_q(z)$ be a sequence of polynomials fulfilling the recursive equations :

$$\begin{aligned} p_0(z) &= 1 && (9) \\ p_1(z) &= -z && (10) \\ p_i(z) &= -z\,p_{i-1}(z) - \beta_{i-1}\,p_{i-2}(z) \quad (i = 2, 3, \ldots, q) && (11) \end{aligned}$$

Then

$$\det(M - z) = p_q(z).$$

In our case we have

$$\beta_l = 4\sin^2\left(\frac{\Phi}{2}l\right) = 2 - 2\cos(\Phi l) \quad (l = 1, 2, \ldots, q).$$

Proposition 3 is proven using the standard expansion of the determinant by neglecting rows and columns. We do not give details here.

To continue our proof of Theorem 1 we have to determine the coefficients $c(l)$ of the characteristic polynomial $P(z) = p_q(z) = \sum_{l=0}^{q} c(l)\,z^l$.

Let $c_i(1), c_i(2), \ldots, c_i(i)$ denote the coefficients of the i-th polynomial

$$p_i(z) = \sum_{l=0}^{i} c_i(l)\,z^l,$$

such that $c(l) = c_q(l)$. Inserting this expansion into the recursion equations (9)-(11) we get by induction a new set of recursion relations for the coefficents $c_i(l)$, $(i = 0, 1, \ldots, q;\ l = 0, 1, \ldots, i)$.

For the further computation it is useful to renumber the coefficients by defining $\gamma_n(l) := c_{n+l}(l)$, $(l = 0, 1, \ldots, q; n = 0, 1, \ldots, (q - l))$.

Then the recursion relations for the coefficients of the i-th polynomial

$$p_i(z) = \sum_{l=0}^{i} \gamma_{i-l}(l)\, z^l$$

read :

$$\begin{aligned}
\gamma_0(0) &= 1\,, \gamma_1(0) = 0 && (12)\\
\gamma_n(0) &= -\beta_{n-1}\gamma_{n-2}(0) \quad (n = 2, 3, \ldots, q) && (13)\\
\gamma_n(l) &= -\gamma_n(l-1) - \beta_{n-1+l}\, \gamma_{n-2}(l) && (14)\\
& \qquad (l = 1, 2, \ldots, q; n = 2, 3, \ldots, (q - l)) &&\\
\gamma_1(l-1) &= -\gamma_1(l-2) \quad (l = 2, 3, \ldots, q) && (15)\\
\gamma_0(l-1) &= -\gamma_0(l-2) \quad (l = 2, 3, \ldots, q) && (16)
\end{aligned}$$

The recursion (12) - (16) can be solved and we derive

Proposition 4 *(Coefficents of the polynomials $p_i(z)$)*

Let $\gamma_n(l)$ for $(l = 0, 1, \ldots, q;\ n = 0, 1, \ldots, (q - l))$ as above.
Then for n even

$$\gamma_n(l) = (-1)^{l+\frac{n}{2}} \sum_{k^{(0)}=0}^{l} \sum_{k^{(1)}=0}^{k^{(0)}} \cdots \sum_{k^{(\frac{n}{2}-1)}=0}^{k^{(\frac{n}{2}-2)}} \beta_{k^{(0)}+n-1}\, \beta_{k^{(1)}+n-3} \cdots \beta_{k^{(\frac{n}{2}-1)}+1}$$

and $\gamma_n(l) = 0$ for n odd.

Proof:

To prove Proposition 4 we consider the cases n even and n odd independently. If $n = 2k + 1, (2k + 1 \leq q, k \in \mathbf{N})$ we have $\gamma_1(0) = 0$ by equation

(12). Using equations (15) and (14) immediately gives $\gamma_{2k+1}(l) = 0$ for $(l = 0, 1, \ldots, q - (2k+1))$.

The case $n = 2k$, $(2k \le q, k \in \mathbf{N})$ is more complicated. Let us first prove two Lemmata.

Lemma 1 *1.*

$$\gamma_0(l) = (-1)^l \quad (l = 0, 1, \ldots, q)$$

2.

$$\gamma_{2k}(0) = (-1)^k \prod_{m=1}^{k} \beta_{2m-1} \quad (2k \le q)$$

Part one of the Lemma follows immediately from equation (12) if we apply equation (15) to $\gamma_0(0) = 1$. To prove part 2 we start from equation (12) and apply equation (13) to $\gamma_1(0) = 0$.

Note that the equations (12) - (16) define a two step recursion, one in the index l and another one in the index n. In the following Lemma we remove the first of these recursions.

Lemma 2

$$\gamma_n(l) = (-1)^{l+1} \sum_{k=0}^{l} (-1)^k \beta_{n+k-1}\, \gamma_{n-2}(k)$$

To prove this Lemma we define

$$A_{n,k} := \beta_{n+k-1}\, \gamma_{n-2}(k)$$

Thus equation (14) reads

$$\gamma_n(l) = -\gamma_n(l-1) - A_{n,l}$$

and the Lemma follows by holding n fixed and using induction in the index l.

Finally we prove what remains of Proposition 4 by induction in the index n. First we see from Lemma 2 by inserting the first part of Lemma 1

$$\begin{aligned}\gamma_2(l) &= (-1)^{l+1}\sum_{k=0}^{l}(-1)^k\beta_{k+1}\,\gamma_0(k)\\ &= (-1)^{l+1}\sum_{k=0}^{l}\beta_{k+1} \qquad (l=0,1,\ldots,(q-2)).\end{aligned}$$

This is precisely what Proposition 4 gives, if we evaluate $\gamma_n(l)$ for $n=2$.

Now we fix some even $N \in \mathbf{N}$ and assume that $\gamma_n(l)$ is known according to Proposition 4 for all $n \leq N$, $l \leq n$. We get for the next non-vanishing coefficients $\gamma_{N+2}(l), (l \leq N)$ using Lemma 2 and our assumption

$$\begin{aligned}&\gamma_{N+2}(l) =\\ = \;& (-1)^{l+1}\sum_{k=0}^{l}(-1)^k\beta_{N+2+k-1}\gamma_N(k)\\ = \;& (-1)^{l+1}\sum_{k=0}^{l}(-1)^k\beta_{k+(N+2)-1}(-1)^{k+\frac{N}{2}}\sum_{k^{(0)}=0}^{k}\beta_{k^{(0)}+(N+2)-1}\cdots\sum_{k^{(\frac{N}{2}-1)}=0}^{k^{(\frac{N+2}{2}-2)}}\beta_{k^{(\frac{N}{2}-1)}+1}\end{aligned}$$

The last line is in agreement with our result for the coefficients $\gamma_{N+2}(l)$. Thus the proof of Proposition 4 is finished.

Let $q-l$ even. Then the l-th coefficient $c(l)$ of the characteristic polynomial is

$$\begin{aligned}c(l) &= \gamma_{q-l}(l) =\\ &= (-1)^{l+\frac{q-l}{2}}\sum_{k^{(0)}=0}^{l}\sum_{k^{(1)}=0}^{k^{(0)}}\cdots\sum_{k^{(\frac{q-l}{2}-1)}=0}^{k^{(\frac{q-l}{2}-2)}}\beta_{k^{(0)}+q-l-1}\,\beta_{k^{(1)}+q-l-3}\cdots\beta_{k^{(\frac{q-l}{2}-1)}+1}\\ &= (-1)^{\frac{q+l}{2}}\sum_{\pi\in\mathcal{S}(q,l)}\beta_{\pi(1)+q-l-1}\,\beta_{\pi(2)+q-l-3}\cdots\beta_{\pi(\frac{q-l}{2})+1}\end{aligned}$$

Here $\mathcal{S}(q,l)$ is the set of ordered $\frac{q-l}{2}$-tuples $\left(\pi(1),\pi(2),\ldots,\pi(\frac{q-l}{2})\right)$ with $\pi(l)\in\{0,1,2,\ldots,l\}$ and $\pi(l)\geq\pi(l+1)$.

Thus the proof of Theorem 1 is finished.

Some of the coefficients can be computed explicitely as we see in

Proposition 5 *(coefficients known explicitly)*

1.

$$c(q-2) = (-1)^{q-1} \sum_{k=0}^{q-2} \beta_{k+1} = 2q(-1)^{q-1}$$

2.

$$c(0) = \begin{cases} 0 & \textit{for } q \textit{ odd} \\ (-1)^{q/2} \prod_{k=1}^{q/2} \beta_{2k-1} = 4(-1)^{q/2} & \textit{for } q \textit{ even} \end{cases}$$

Part 1 is proven using the following very helpful fact

$$\sum_{k=1}^{q} e^{\frac{2\pi}{q} m\, k} = q\delta_{m,0} \quad (q, m \in \mathbf{Z}). \tag{17}$$

If we take the real of (17), use the definition of β_k and $\cos(\Phi q) = 1$ we immediately have

$$\sum_{k=0}^{q-2} \beta_{k+1} = 2\sum_{k=1}^{q-1}(1+\cos(\Phi k)) = 2(q-1) - 2\sum_{k=1}^{q-1}\cos(\Phi k) = 2q.$$

Proving part 2 of Proposition 5 is more complicate. We need the following Lemma which is in fact Chamber's relation for the free Hofstadter Hamiltonian $H_{\text{free}} = T_2 + T_2^\star = W_2 \exp(i\theta_2) + W_2^\star \exp(-i\theta_2)$.

Lemma 3 *The secular determinant of $H_{free}(\theta_2)\,(\theta_2 \in \mathbf{R})$ is*

$$\begin{aligned} \det\left(H_{free}(\theta_2) - z\mathbf{1}\right) &= (-1)^q 2\, T_q\left(\frac{z}{2}\right) + (-1)^{q-1}\left(e^{i\theta_2 q} + e^{-i\theta_2 q}\right) \\ &= \prod_{k=1}^{q} \left(2\cos\left(\Phi k + \theta_2\right) - z\right) \end{aligned}$$

where $T_q(x) = \cos\left(q \arccos(x)\right)$ is the q-th Chebyshev polynomial.

A proof of this Lemma is given in [12].
Corollary : Evaluation of Lemma 3 at $z = 0$ and $\theta_2 = \frac{\Phi+\pi}{2}$ gives for q even

$$\prod_{k=1}^{q} 2\cos\left(\Phi k + \frac{\Phi+\pi}{2}\right) = 4(-1)^{q/2}$$

since we have $T_q(0) = (-1)^{q/2}$ for Chebyshev polynomials with q even.

To finish the proof of Proposition 5 we proceed as follows : Due to the assumption q even and p and q relative prime we have p odd and for any $\theta_2 \in \mathbf{R}$

$$\cos\left(\Phi k + \theta_2\right) = -\cos\left(\Phi(k + \frac{q}{2}) + \theta_2\right)$$

Thus

$$\begin{aligned}
\beta_{2k+1} &= 2 - 2\cos(\Phi(2k+1)) = 4\cos^2\left(\Phi k + \frac{\Phi + \pi}{2}\right) \\
&= -\cos\left(\Phi k + \frac{\Phi + \pi}{2}\right)\cos\left(\Phi(k + \frac{q}{2}) + \frac{\Phi + \pi}{2}\right)
\end{aligned}$$

and finally together with the corollary of Lemma 3 we have

$$(-1)^{q/2}\prod_{k=1}^{q/2}\beta_{2k+1} = \prod_{k=1} q\cos\left(\Phi k + \frac{\Phi + \pi}{2}\right) = 4(-1)^{q/2}$$

3 The Zero-Energy Modes

The determinant of the Hofstadter matrix can be computed using Theorem 1 and Chamber's relation. It is the sum of the zero order coefficient $c_{p/q}(0)$ in the discriminant and the offset function $\tilde{h}(\theta)$

$$\begin{aligned}
\det\left(H(\theta)\right) &= \det\left(\tilde{H}(\theta)\right) \\
&= P(z=0) + \tilde{h}(\theta) = c_{p/q}(0) + \tilde{h}(\theta) \\
&= 2(-1)^{q/2}(1+(-1)^q) + 2(-1)^{q-1}\cos(q\theta_1) + 2(-1)^p \cos\left(q(\theta_1+\theta_2)\right)
\end{aligned}$$

Solving

$$\det\left(H(\theta_0)\right) = 0$$

allows to determine all $\theta_0 \in \mathbf{R}^2$ for which $0 \in \mathrm{Spectrum}\left(H(\theta_0)\right)$ and thus the nullspace has non-vanishing dimension

$$\mathrm{Kernel}\left(H(\theta_0)\right) \neq \{0\}.$$

Furthermore we can explicitly solve the eigenvector problem

$$\tilde{H}(\theta_0)\,\psi_0 = 0$$

and compute the zero energy modes ψ_0 of the Hofstadter Hamiltonian.

An other way to find the zero energy modes of the Hofstadter matrix in the case $\theta = (0,0)$ has been obtained by U. Pinkall [18] using a connection between the double discrete quantum pendulum and the square of the Hofstadter matrix.

In the rest of this Section we give explicit formulas for the vectors in the nullspace of $\tilde{H}(\theta)$ in Theorem 2 and show phase space plots of this states using the Husimi-transformation.

Theorem 2 *(zero-modes for the Hofstadter model)*

Let $\Phi = 2\pi\frac{p}{q}$, $(p,q \in \mathbf{Z}$, p,q relative prime $)$ and $\tilde{H}(\theta)$ as above. Then

$$Kernel\left(\tilde{H}(\theta)\right) = \begin{cases} span\{\psi_0,\psi_1\} & \text{for } q \text{ even, } \theta_1 = \pi/2, \theta_2 = \pi\frac{p+1}{q} \\ span\{\psi_2\} & \text{for } q \text{ odd, } \theta_1 \in \mathbf{R}, \theta_2 = -\pi(1+\frac{p}{q}) \end{cases}$$

where

$$\psi_0(n) = \begin{cases} e^{-i\frac{\Phi}{4}(n^2-n)} \prod_{l=1}^{\frac{n-1}{2}} \frac{\sin\left(\Phi(l-\frac{1}{2})\right)}{\sin(\Phi l)} & n \ odd \\ 0 & n \ even \end{cases}$$

$$\psi_1(n) = \begin{cases} 0 & n \ odd \\ e^{-i\frac{\Phi}{4}(n^2+n-6)-i\frac{\pi}{2q}(n-2)} \prod_{l=1}^{\frac{n}{2}-1} \frac{\cos\left(\Phi(l+\frac{1}{2})+\frac{\pi}{2q}\right)}{\cos\left(\Phi(l+1)+\frac{\pi}{2q}\right)} & n \ even \end{cases}$$

$$\psi_2(n) = \begin{cases} e^{-i\frac{\Phi}{4}(n^2+n-2)-i\frac{\pi}{2q}(n-1)} \prod_{l=1}^{\frac{n}{2}} \frac{\cos\left(\Phi l+\frac{\pi}{2q}\right)}{\cos\left(\Phi(l+\frac{1}{2})+\frac{\pi}{2q}\right)} & n \ odd \\ 0 & n \ even \end{cases}$$

and

$$n = 1, 2, 3, \ldots, q.$$

Remark:

1. In the discussion of Chamber's relation (Proposition 2 in Section 2) we have seen that the spectrum of $\tilde{H}(\theta)$ is $2\pi/q$ - periodic in θ_1 and θ_2. The eigenvectors $\psi(\theta)$ are 2π-periodic, only. According to equation (7) a unitary transformation $W_1^{l_1} W_2^{l_2} \psi(\theta) = \psi(\theta + \frac{2\pi}{q})$ connects the eigenvectors for θ and $\theta + \frac{2\pi}{q} m$, $(m \in \mathbf{Z}^2)$. $l_1, l_2 \in \{1, 2, \ldots, q\}$ are uniquely defined through a diophantine equation for any fixed m_1 and m_2. Thus the values of θ in Theorem 2 are determined up to an integer multiple of $\frac{2\pi}{q}$ only. Similarily the eigenvectors have to be transformed by $W_1^{l_1} W_2^{l_2}$.

2. For all θ not covered by Theorem 2 or Remark 1 the nullspace of $H(\theta)$ is zero, since the determinant does not vanish.

In preparation for the proof of Theorem 2 we show the following Proposition about the supersymmetric structure of $\tilde{H}$.

Proposition 6 *(Supersymmetry of $\tilde{H}(\theta)$)*
Let $\psi = (\psi(1), \psi(2), \ldots, \psi(n), \ldots, \psi(q)) \in \mathbf{C}^q$ and

$$\begin{aligned} U : \mathbf{C}^q &\rightarrow \mathbf{C}^q : \\ \psi(n) &\rightarrow (-1)^n \psi(n) \quad (n = 1, 2, \ldots, q). \end{aligned}$$

For q even and any $\theta_0 \in \mathbf{R}^2$ or for q odd and θ_0 such that $a_q(\theta_0) = 0$ we have

$$U\,\tilde{H}(\theta_0)\,U^{-1} = -\tilde{H}(\theta_0). \tag{18}$$

Furthermore:

For q odd any vector ψ in the nullspace of $\tilde{H}(\theta_0)$ has the property

$$\psi = (\psi(1), 0, \psi(3), 0, \ldots, \psi(q)) \Rightarrow U\psi = -\psi \in span\{\psi\}$$

For even q the nullspace is, if it exists, at least two dimensional and spanned by two eigenvectors ϕ, ψ of the form

$$\begin{aligned} \psi &= (\psi(1), 0, \psi(3), \ldots, \psi(q-1), 0) \\ \phi &= (0, \phi(2), 0, \phi(4), \ldots, 0, \phi(q)) \end{aligned}$$

Proof:

The supersymmetric structure of $\tilde{H}(\theta_0)$ is due to the bipartitness of the original Hofstadter problem in its 2-dimensional representation. The property (18) can be proved immediately by matrix multiplication.

The eigenvalue equation for the zero energy is

$$\tilde{H}(\theta_0)\psi = 0 \Leftrightarrow \begin{pmatrix} \bar{a}_q\psi(q) + a_1\psi(2) & = & 0 \\ \bar{a}_1\psi(1) + a_2\psi(3) & = & 0 \\ & \vdots & \\ \bar{a}_n\psi(n) + a_{n+1}\psi(n+2) & = & 0 \\ & \vdots & \\ \bar{a}_{q-2}\psi(q-2) + a_{q-1}\psi(q) & = & 0 \\ \bar{a}_{q-1}\psi(q-1) + a_q\psi(1) & = & 0 \end{pmatrix}. \tag{19}$$

If q is odd and $a_q = 0$ we immediately see that $\psi(n) = 0$ for n even. If q is even the system of equations (19) decomposes into two subsystems, for even ond odd sites respectively, generating the two eigenvectors ψ, ϕ.

Corollary :

For θ_0 as in Proposition 6 we have, using Theorem 1,

$$\sum_{l=1}^{q} \lambda_l^2 = \mathrm{Trace}(\tilde{H}^2(\theta_0)) \tag{20}$$

$$= \sum_{k,j=1}^{q} |\left(\tilde{H}(\theta_0)\right)_{(k,j)}|^2 = 4q \tag{21}$$

where $\lambda_1, \lambda_2, \dots, \lambda_q$ are the eigenvalues of $\tilde{H}(\theta_0)$.

To see this write for q even

$$\begin{aligned}
\det\left(\tilde{H}(\theta_0) - z\right) &= \prod_{l=1}^{q}(\lambda_l - z) = \prod_{l=1}^{(q/2)}(\lambda_l - z)(-\lambda_l - z) \\
&= \prod_{l=1}^{(q/2)}(\lambda_l^2 - z^2) \\
&= z^q + - \left(\tfrac{1}{2}\textstyle\sum_{l=1}^{q}\lambda_l^2\right) z^{q-2} + \dots + \sqrt{\prod_{l=1}^{q}\lambda_l^2}.
\end{aligned}$$

The Corollary follows immediately if we compare the coefficents in this polynomial with Theorem 1. For q odd the Corollary follows analogously.

In the rest of this Section we complete the proof of Theorem 2.

First we consider the case q odd. From equation (3) we have computed $\theta_0 = (\theta_1, -\pi(1 + \frac{p}{q}))$ such that $z = 0$ is in the spectrum of $\tilde{H}(\theta_0)$ and $a_q = 0$. From the eigenvalue equation (19) we get the following recursion between the odd components of the eigenvector $\psi = (\psi(1), 0, \psi(3), \dots, 0, \psi(q))$. (The even components are zero as we have seen in Proposition 6.)

$$\begin{aligned}
\psi(2k+1) &= -\frac{\bar{a}_{2k-1}}{a_{2k}}\psi(2k-1) \quad (k = 1, 2, \dots, \frac{q-1}{2}) \\
\Rightarrow\ \psi(2k+1) &= \prod_{l=1}^{k} -\frac{\bar{a}_{2l-1}}{a_{2l}}\psi(1)
\end{aligned}$$

Setting $\psi(1) = 1$ and

$$\frac{\bar{a}_{2l-1}(\theta_0)}{a_{2l}(\theta_0)} = -e^{-2i\Phi l + i\frac{\Phi}{2}}\frac{\sin\left(\Phi(l - \frac{1}{2})\right)}{\sin(\Phi l)}$$

gives the eigenvector ψ_0 which is unique since $a_{2l}(\theta_0) \neq 0$ for all l.

For q even the situation is more complicated and we proceed as follows. In a first step we construct two mutually orthogonal vectors ψ and ϕ such that each of them solves $q-1$ of the q equations (19). Unfortunately the last of this equations can not be shown to hold automatically for these vectors, since the involved trigonometric relations are too complicated. On the other hand all vectors fulfilling the first $q-1$ equations of (19) are in the subspace spanned by ψ and ϕ. Thus there can not exist any other eigenvector. Furthermore we know that the nullspace is of dimension two at least by Proposition 6 and thus the vectors ψ and ϕ span the eigenspace completely.

To construct ψ and ϕ we consider

$$a_r(\theta_0) = e^{i\frac{\pi}{2}(1+\frac{1}{q})+i\frac{\Phi}{2}(r+1)} \cos\left(\frac{\Phi}{2}(r+1) + \frac{\pi}{2q}\right)$$

We get the vector ψ, which has non vanishing odd components, from

$$\begin{aligned} \psi(2k+1) &= -\frac{\bar{a}_{2k-1}}{a_{2k}}\psi(2k-1) = e^{-2i\Phi l-\frac{1}{2}i\Phi-i\frac{\pi}{q}} \frac{\cos\left(\Phi k + \frac{\pi}{2q}\right)}{\cos\left(\Phi(k+\frac{1}{2}) + \frac{\pi}{2q}\right)}\psi(2k-1) \\ &= \prod_{l=1}^{k} -\frac{\bar{a}_{2l-1}}{a_{2l}}\psi(1). \end{aligned}$$

The other vector ϕ with zero in the odd components is generated by

$$\begin{aligned} \phi(2k) &= -\frac{\bar{a}_{2k}}{a_{2k+1}}\phi(2k-1) = e^{-2i\Phi k-\frac{3}{2}i\Phi-i\frac{\pi}{q}} \frac{\cos\left(\Phi(k+\frac{1}{2}) + \frac{\pi}{2q}\right)}{\cos\left(\Phi(k+1) + \frac{\pi}{2q}\right)}\phi(2k-1) \\ &= \prod_{l=1}^{k} -\frac{\bar{a}_{2l}}{a_{2l+1}}\phi(1). \end{aligned}$$

Thus the proof of Theorem 2 is finished.

To visualize the zero-modes we use the Husimitransfomation. The quantum-mechanical wave function $\psi \in \mathbf{C}^q$ is mapped by the Husimi-transformation into

$$F_\psi(P,Q) := |<\psi, W_1^P W_2^q \Omega>|^2 \quad (P,Q) \in \{1,2,\ldots,q\}^2 .$$

Here $\Omega \in \mathbf{C}^q$ is a "discrete gaussian" (a fourier invariant positive vector in $\mathbf{C}^q$). Here we choose for Ω the ground state of the Hofstadter Hamiltonian $H(0,0)$ [14, 4]. The Husimitransformation gives a phase space picture of a state if we interprete Q and P as the position and momentum variable. To visualize F_ψ we use a two dimensional density plot where the grayscale at a point (P,Q) codes the value $F_\psi(P,Q)$. White color denotes a large number and black color a small number.

In Fig. 3 we show the Husimi transform of ψ_0 and ψ_1 in the even case $q = 30$. In Fig. 3 we show the state ψ_2 for the odd case $q = 31$.

By semi-classical methods [8] [9] [10] the phase space pictures are related to the contour level at energy zero of the classical symbol $\tilde{H}_{\text{class}}$ of the Hamiltonian $\tilde{H}$.

$$\tilde{H}_{\text{class}}(P, Q) = 2\cos(P + Q) + 2\cos(Q)$$

For comparison we show these contour levels in Fig. 3.

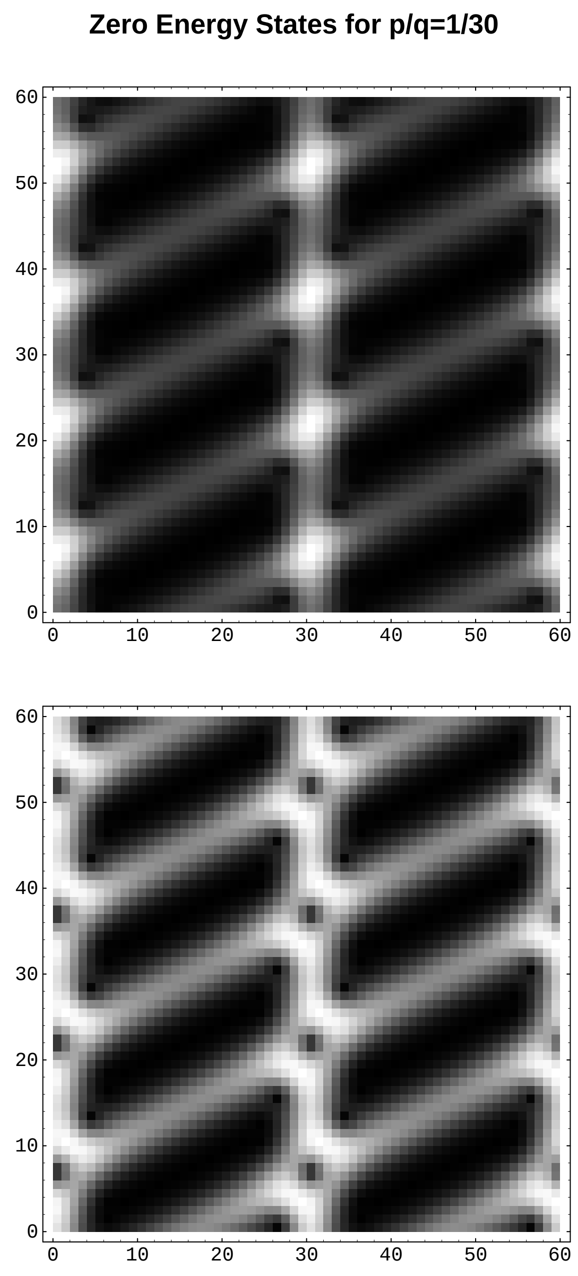

Figure 2: Zero-modes of transformed Hofstadter matrix for $q = 30$

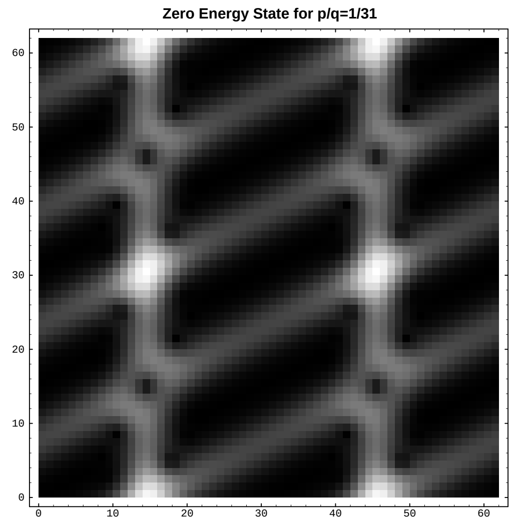

Figure 3: Zero-mode of transformed Hofstadter matrix for $q = 31$

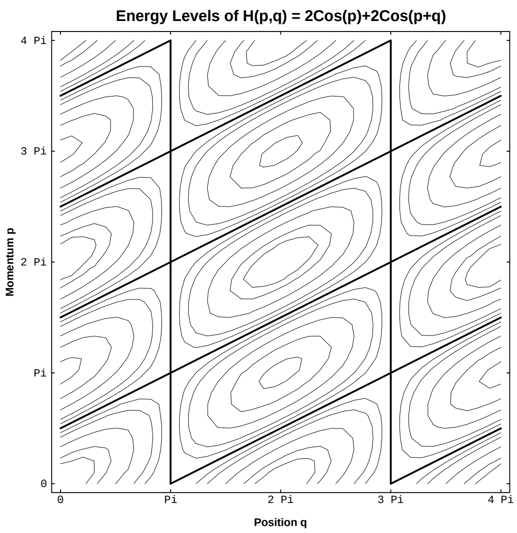

Figure 4: Contour levels of the classical symbol $\tilde{H}_{\text{class}}(P,Q)$

4 Conlusion

In this paper we give an explicit expression for the discriminant of Harper's equation in terms of a sum over products of trigonometric polynomials in Theorem 1.

The proof of this Theorem involves a unitary transformation that transforms Harper's equation into a difference equation without potential term. Then we can expand the secular determinant into a recursive sequence of polynomials. Finally by induction we derive explicit formulas for the coefficents of the characteristic polynomial.

The result of Theorem 1 enables us to compute the characteristic polynomial for any rational flux $\Phi = 2\pi\frac{p}{q}$. Some examples for q up to 10 are given together with their roots in the Appendix.

As an application of Theorem 1 we complete the explicite formula of [16] for the density of states in the Hofstadter model.

Furthermore we compute the nullspace of the Hofstadter matrix in Theorem 2. This is done by using Theorem 1 to compute the determinant of the Hofstadter matrix. The simple form of the transfomed Hofstadter matrix used in the proof of Theorem 1 allows us to solve the eigenvalue equation for energy zero explicitely.

We expect that our results give new insight in the spectral analysis of Hofstadter like models for rational flux $\phi = 2\pi\frac{p}{q}$, which is still not completely solved.

A deeper analysis of the structure of the descriminant should allow a better understanding of the nesting properties [11] of the Hofstadter butterfly.

It has to be checked wether similar methods can be applied to the Mathieu-equation for $\mu \neq 2$ or to the triangular lattice model.

Furthermore the explicte knowledge of the discriminant might allow some progress in the study of the flux phase problem for square lattices in a constant magnetic field [15].

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5 Appendix : Polynomials and roots for some rational fluxes

In the following list we give the discriminats for $p = 1$ and $q = 1, 2, \ldots, 10$ and the roots for the characteristic polynomials at the band edges. Namely the solutions of $P_{p/q}(z) = +4$ and $P_{p/q}(z) = -4$.

$$\begin{aligned}
P_{1/2}(z) &= -4 + z^2 \\
P_{1/2}(z) &= 4 : z = 2\sqrt{2} \vee z = -2\sqrt{2} \\
P_{1/2}(z) &= -4 : z = 0 \vee z = 0
\end{aligned}$$

$$\begin{aligned}
P_{1/3}(z) &= 6\,z - z^3 \\
P_{1/3}(z) &= 4 : z = 2 \vee z = \frac{-2 - 2\sqrt{3}}{2} \vee z = \frac{-2 + 2\sqrt{3}}{2} \\
P_{1/3}(z) &= -4 : z = -2 \vee z = \frac{2 - 2\sqrt{3}}{2} \vee z = \frac{2 + 2\sqrt{3}}{2}
\end{aligned}$$

$$\begin{aligned}
P_{1/4}(z) &= 4 - 8\,z^2 + z^4 \\
P_{1/4}(z) &= 4 : z = 0 \vee z = 0 \vee z = 2\sqrt{2} \vee z = -2\sqrt{2} \\
P_{1/4}(z) &= -4 : z = \sqrt{4 - 2\sqrt{2}} \vee z = -\sqrt{4 - 2\sqrt{2}} \vee \\
&\quad z = \sqrt{4 + 2\sqrt{2}} \vee z = -\sqrt{4 + 2\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
P_{1/5}(z) &= z\left(-14 - 2\cos(\frac{1\,\pi}{5}) + 12\cos(\frac{2\,\pi}{5})\right) + z^3\,10 - z^5 \\
P_{1/5}(z) &= 4 : \text{zeros where not found by mathematica} \\
P_{1/5}(z) &= -4 : \text{zeros where not found by mathematica}
\end{aligned}$$

$$\begin{aligned}
P_{1/6}(z) &= -4 + 24\,z^2 - 12\,z^4 + z^6 \\
P_{1/6}(z) &= 4 : z = \sqrt{2} \vee z = -\sqrt{2} \vee \\
&\quad z = \sqrt{5 - \sqrt{21}} \vee z = -\sqrt{5 - \sqrt{21}} \vee \\
&\quad z = \sqrt{5 + \sqrt{21}} \vee z = -\sqrt{5 + \sqrt{21}}
\end{aligned}$$

$$
\begin{aligned}
P_{1/6}(z) &= -4 : z = 0 \vee z = 0 \vee \\
&\quad z = \sqrt{6 - 2\sqrt{3}} \vee z = -\sqrt{6 - 2\sqrt{3}} \vee \\
&\quad z = \sqrt{6 + 2\sqrt{3}} \vee z = -\sqrt{6 + 2\sqrt{3}}
\end{aligned}
$$

$$
\begin{aligned}
P_{1/7}(z) &= z\left(36 + 12\cos((\frac{\pi}{7}) - 40\cos((\frac{2\pi}{7}) - 16\cos((\frac{3\pi}{7})\right) + \\
&\quad z^3\left(-44 - 10\cos((\frac{\pi}{7}) + 24\cos((\frac{2\pi}{7}) - 10\cos((\frac{3\pi}{7})\right) + \\
&\quad z^5\,14 - z^7 \\
P_{1/7}(z) &= 4 : \text{zeros where not found by mathematica} \\
P_{1/7}(z) &= -4 : \text{zeros where not found by mathematica}
\end{aligned}
$$

$$
\begin{aligned}
P_{1/8}(z) &= 4 + \left(-96 + 32\sqrt{2}\right) z^2 + \left(72 - 8\sqrt{2}\right) z^4 - 16\,z^6 + z^8 \\
P_{1/8}(z) &= 4 : z = 2 \vee z = -2 \vee z = 0 \vee z = 0 \vee \\
&\quad z = \sqrt{6 - 2\sqrt{3 + 2\sqrt{2}}} \vee z = -\sqrt{6 - 2\sqrt{3 + 2\sqrt{2}}} \vee \\
&\quad z = \sqrt{6 + 2\sqrt{3 + 2\sqrt{2}}} \vee z = -\sqrt{6 + 2\sqrt{3 + 2\sqrt{2}}}
\end{aligned}
$$

$$
\begin{aligned}
P_{1/9}(z) &= 6\left(-58 - 10\cos(\frac{\pi}{9}) + 34\cos(\frac{2\pi}{9}) + 16\cos(\frac{3\pi}{9}) + \cos(\frac{4\pi}{9})\right) z + \\
&\quad 2\left(122 + 48\cos(\frac{\pi}{9}) - 102\cos(\frac{2\pi}{9}) - 58\cos(\frac{3\pi}{9}) - 30\cos(\frac{4\pi}{9})\right) z^3 + \\
&\quad \left(-109 - 34\cos(\frac{\pi}{9}) + 52\cos(\frac{2\pi}{9}) + 20\cos(\frac{3\pi}{9}) + 34\cos(\frac{4\pi}{9})\right) z^5 + 18\,z^7 - z^9 \\
P_{1/9}(z) &= 4 : \text{zeros where not found by mathematica} \\
P_{1/9}(z) &= -4 : \text{zeros where not found by mathematica}
\end{aligned}
$$

$$
\begin{aligned}
P_{1/10}(z) &= -4 + 2\left(175 - 245\cos(\frac{\pi}{5}) + 220\cos(\frac{2\pi}{5})\right) z^2 + \\
&\quad 2\left(-172 + 104\cos(\frac{\pi}{5}) - 44\cos(\frac{2\pi}{5})\right) z^4 + \\
&\quad 2\left(66 - 12\cos(\frac{\pi}{5}) + 2\cos(\frac{2\pi}{5})\right) z^6 - 20\,z^8 + z^{10} \\
P_{1/10}(z) &= 4 : \text{zeros where not found by mathematica}
\end{aligned}
$$

$$P_{1/10}(z) = -4 : \text{zeros where not found by mathematica}$$

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