Quantum Group Invariant Integrable n-State Vertex Models with Periodic Boundary Conditions

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Abstract

An $U_q(sl(n))$ invariant transfer matrix with periodic boundary conditions is analysed by means of the algebraic nested Bethe ansatz for the case of q being a root of unity. The transfer matrix corresponds to a 2-dimensional vertex model on a torus with topological interaction w.r.t. the 3-dimensional interior of the torus. By means of finite size analysis we find the central charge of the corresponding Virasoro algebra as $c = (n-1)\,[1 - n(n+1)/(r(r-1))]$.

1 Introduction

Since Bethe's pioneering work [1] on the isotropic XXX-Heisenberg model more than sixty years ago, the Bethe ansatz method has become one of the most important tools in analysing one dimensional integrable quantum chains or equivalently 2-dimensional statistical models. Moreover, it turned out that there is a deep connection between the Bethe ansatz and the underlying symmetry group of the model. This has been stressed by Faddeev and Takhadzhyan [2] investigating the Heisenberg model. For general simple Lie groups this algebraic structure has been summarized in ref. [3].

For the case of the anisotropic XXZ-Heisenberg model and generalizations of it the underlying Yang-Baxter algebras, which guarantee the integrability of the system, are related to new mathematical structures. Drinfeld [4] and Jimbo [5] have formulated these new structures as quantum groups. Therefrom the question arises whether quantum groups should serve as a generalization of symmetry concepts in physics.

However, deforming the $SU(2)$-invariant XXX-Heisenberg model with periodic boundary conditions in the traditional [6] way one obtains an XXZ-Hamiltonian which is not

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$U_q(sl(2))$-invariant [7], [8]. The reason for this puzzle is the non-cocommutativity of quantum groups. This means, roughly speaking, for tensor products one has to distinguish between left and right. So it is not obvious how to identify the most left lattice point with most right one in order to have periodic boundary conditions.

One possibility to obtain a quantum group invariant XXZ-Hamiltonian is to consider open boundary conditions. Such Hamiltonians have been investigated by several authors (see e.g. [9], [7], [10])[1]. For open boundary conditions one has to apply Bethe ansatz techniques introduced by Sklyanin [11] using Cherednik's [12] 'reflection property'. By this method the XXZ-Heisenberg model (see e.g. [13]), the $spl_q(2,1)$ invariant supersymmetric t-J model [14] and the $U_q(sl(n))$ invariant generalization of the XXZ-chain [15] have been solved for open boundary conditions. It turns out that these computations are quite involved, especially for the nested Bethe ansatz case.

In the following we present a type of models with periodic boundary conditions which are in addition quantum group invariant. We consider an n-state vertex model on an $N \times M$-square lattice on a torus or cylinder as shown in Fig. 1. As usual the partition

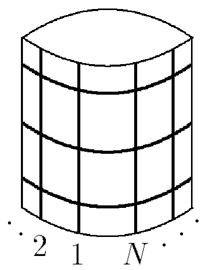


Figure 1: *Square lattice on a cylinder considered as a part of a torus*

function may be written as

$$Z = \mathrm{Tr}\ \tau^M, \tag{1}$$

where the transfer matrix τ maps one cyclic chain $N, N-1, \ldots, 2, 1$ to the next. As is well known, equivalently to these types of model one may analyse one dimensional quantum chain models. We consider a cyclic chain $N, N-1, \ldots, 2, 1, N, \ldots$ where a fundamental representation space $V^{\Lambda_1} = \mathbf{C}^n$ of $U_q(sl(n))$ is associated to each lattice point i, $(i = N, \ldots, 1)$. As a generalization of the Heisenberg model with periodic boundary conditions one can write an integrable $U_q(sl(n))$-symmetric Hamiltonian

$$H = P^{\Lambda_2}_{NN-1} + \ldots + P^{\Lambda_2}_{21} + P^{\Lambda_2}_{1N} \tag{2}$$

acting in the tensor product space

$$\Omega = V^{\Lambda_1}_N \otimes \cdots \otimes V^{\Lambda_1}_1. \tag{3}$$

[1]Sometimes these models are formulated such that the open boundary conditions are not quite obvious.

The $P_{ik}^{\Lambda_2}$ in eq. (2) project onto the representation Λ_2 contained in the product $\Lambda_1 \otimes \Lambda_1$ associated to the lattice points i and k according to

$$\Lambda_1 \otimes \Lambda_1 = 2\Lambda_1 \oplus \Lambda_2. \tag{4}$$

For the case of $q = 1$ ($SU(n)$-symmetry) these projectors $P_{ik}^{\Lambda_2}$ are symmetric under the exchange of i and k. Therefore it is obvious how to define $P_{1N}^{\Lambda_2}$. This changes for the quantum group case. The q-projectors can be written in terms of the unit matrices in $M_{n,n}(\mathbf{C})$ for $i > k$ (lattice point i on the left of lattice point k) (see also [10])

$$P_{ik}^{\Lambda_2} = (q + q^{-1})^{-1} \sum_{\alpha \neq \beta} \left(q^{\mathrm{sign}(\alpha-\beta)} E_{\alpha\alpha}^{(i)} \otimes E_{\beta\beta}^{(k)} - E_{\alpha\beta}^{(i)} \otimes E_{\beta\alpha}^{(k)} \right). \tag{5}$$

Due to the non-cocommutative coproduct of the quantum group $U_q(sl(n))$ the q-projectors $P_{ik}^{\Lambda_2}$ are no longer symmetric with respect to i and k. Therefore it is not obvious how $P_{1N}^{\Lambda_2}$ is defined or how the periodicity for the transfer matrix in eq. (1) has to be formulated. The traditional answer [6] which means symmetrizing the projectors as given by eq. (5) breaks quantum group invariance, whereas the projectors given by eq. (5) are quantum group invariant. Therefore one should also have a quantum group invariant projector $P_{1N}^{\Lambda_2}$. One possibility to get a quantum group invariant Hamiltonian is to cancel the projector $P_{1N}^{\Lambda_2}$ in eq. (3) which means open boundary conditions, as mentioned above.

For models with periodic boundary conditions one has to take care of the nontrivial topology of the space, i.e. of the graph formed by the square lattice on the cylinder of Fig. 1. In ref. [16] one of the authors of the present paper and Schrader gave a definition of invariants of graphs on Riemann surfaces using the language of topological quantum field theories. The surfaces are considered as boundaries of 3-manifolds. These invariants are defined for the quantum group case if q is equal to a root of unity. We use this definition of invariants of graphs in order to define the vertex model of eq. (1) and Fig. 1. Using the techniques as formulated in ref. [16] we can write the partition function (1) in terms of invariants of planar graphs. As a result we obtain for the transfer matrix of eq. (1) as well as for the Hamiltonian eq. (2) expressions in terms of planar graphs which are quantum group invariant and belong to periodic boundary conditions. This transition to planar graphs preserves the cyclic invariance of the models which is obvious from Fig. 1. It turns out that it is much easier to solve the nested Bethe ansatz for the periodic case, compared to the open one. We should add two remarks:

i. The invariants of 3-manifolds and therefore also these vertex models are defined only for quantum groups where q is a root of unity.[2]

ii. The partition function (1) does not only describe a two dimensional vertex model on the torus (or cylinder) but in addition there is a local interaction of the vertices with the interior of the 3-manifold. However, this interaction is of topological nature. This model is similar to σ-models with Chern-Simons term or the WZNW-models [17][18]. This will be explained in more details in Appendix A.

[2]However, the transfer matrix and the Hamiltonian may formally be extended to generic values of q.

In the present paper we solve the eigenvalue equation of the transfer matrix and the Hamiltonian by means of the algebraic nested Bethe ansatz and obtain the Bethe ansatz equations. As an application we calculate the finite size corrections of the ground state energy using the techniques developed in [19] and [8]. Therefrom we obtain the central charge of the Virasoro algebra of the corresponding conformal quantum field theory

$$c = (n-1)\left(1 - \frac{n(n+1)}{r(r-1)}\right) \tag{6}$$

for the $U_q(sl(n))$-model with $q = \exp(i\pi/r)$, $(r = n+2, n+3, \ldots)$. This formula coincides with that obtained from the extended coset construction for A_{n-1} [20] [21]. It has also been obtained in [22] for the $U_q(sl(n))$-RSOS model using Baxter's [23] corner transfer matrix method. For approaches to quantum group symmetric models with periodic boundary conditions which use Baxter's SOS-picture of the models see e.g. refs. [8], [22] and [24].

This paper is organized as follows. In Section 2 we recall the trigonometric $U_q(sl(n))$-solution of the Yang-Baxter equation. We use a graphical notation which is useful in this context to clarify the complicated algebraic structures. We write some fundamental relations as unitarity, crossing relations, Markov properties and Cherednik's reflection relation. In Section 3 we present the transfer matrix of the n-state vertex model with periodic boundary conditions of eq. (1) and Fig. 1. We define monodromy matrices and derive commutation rules from the Yang-Baxter relations. Using these we solve the eigenvalue equation of the transfer matrix by means of the algebraic nested Bethe ansatz and obtain the Bethe ansatz equations. In Section 4 we show how the $U_q(sl(n))$-invariant Hamiltonian (2) for periodic boundary conditions is obtained from the transfer matrix. We perform the finite size analysis of the ground state energy and obtain the central charge. In Appendix A we sketch the derivation of the transfer matrix investigated in this paper. We define the partition function of the vertex model on a torus using techniques of topological quantum field theory as developed in ref. [16]. Finally in Appendix B the relation between Yang-Baxter algebra and quantum groups is used to prove the quantum group invariance of the transfer matrix and the Hamiltonian.

2 Yang-Baxter Equation

Both sides of the Yang-Baxter equation

$$R_{12}(x_{12})R_{13}(x_{13})R_{23}(x_{23}) = R_{23}(x_{23})R_{13}(x_{13})R_{12}(x_{12}) \tag{7}$$

act on the tensor product space $V_1 \otimes V_2 \otimes V_3$. The matrix $R_{ik}(x_{ik})$ depends on the spectral parameter $x_{ik} = x_i/x_k$ and acts on $V_i \otimes V_k$. The "trigonometric" $U_q(sl(n))$-solution [25], [26] can be written in terms of the "constant R-matrix"

$$R(x) = xR - x^{-1}PR^{-1}P \tag{8}$$

where P is the permutation operator $P(\alpha \otimes \beta) = \beta \otimes \alpha$ for $\alpha, \beta \in V$. The constant R-matrix acting on the tensor product of two fundamental representation spaces $V^{\Lambda_1} \otimes V^{\Lambda_1}$

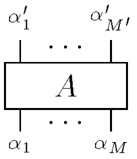
is

$$R = R^{\Lambda_1\Lambda_1} = \sum_{\alpha\neq\beta} E_{\alpha\alpha}\otimes E_{\beta\beta} + q\sum_{\alpha} E_{\alpha\alpha}\otimes E_{\alpha\alpha} + (q-q^{-1})\sum_{\alpha>\beta} E_{\alpha\beta}\otimes E_{\beta\alpha}, \tag{9}$$

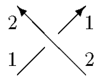
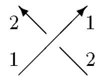
where the $E_{\alpha\beta}$ are the unit matrices in $M_{n,n}(\mathbf{C})$. From eqs. (8) and (9) one easily derives the relation

$$R(1) = R - PR^{-1}P = (q-q^{-1})P. \tag{10}$$

For later convenience we use a graphical notation for matrices (see e.g. ref. [27])

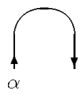
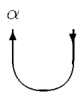
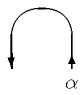
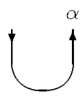
$A \equiv$  $: \left\{ \begin{array}{ccc} V_1\otimes\cdots\otimes V_M & \to & V_1'\otimes\cdots\otimes V_{M'}' \\ |\alpha_1,\ldots,\alpha_M\rangle & \mapsto & |\alpha_1',\ldots,\alpha_{M'}'\rangle \end{array}\right.$. (11)

For example the matrices R_{12} and R_{12}^{-1}: $V_1\otimes V_2 \to V_2\otimes V_1$ defined by eq. (9) are depicted by

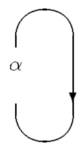
$R_{12} \equiv$  and $R_{12}^{-1} \equiv$  . (12)

The up-arrows denote the representation Λ_1. As another example we consider the intertwiners [27])

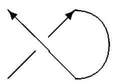
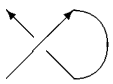
$V^{\Lambda_1}\otimes V^{\Lambda_1^*} \leftrightarrow V^{\Lambda_0}$ $\qquad$ $V^{\Lambda_1^*}\otimes V^{\Lambda_1} \leftrightarrow V^{\Lambda_0}$

 or  $\equiv q^{(n+1)/2-\alpha}$,  or  $\equiv q^{\alpha-(n+1)/2}$. (13)

The states of V^{Λ_1} are labeled by $\alpha\in\{1,2,\ldots,n\}$. The down-arrows denote the conjugate representation Λ_1^* and the trivial representation Λ_0 is depicted by no line. From the intertwiners (13) we obtain the "Markov trace"

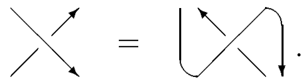
 $\equiv q^{n+1-2\alpha}$ (14)

with the Markov property

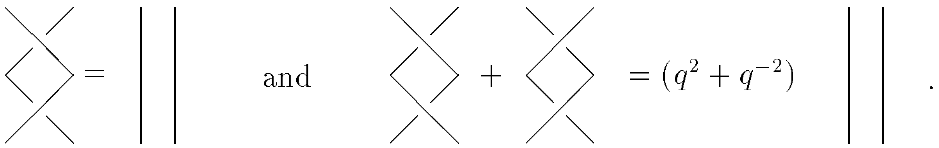
 $= q^n$ and  $= q^{-n}$, (15)

where over the states of the internal lines is summed with the weights (14). In the following summation over internal lines is always assumed.

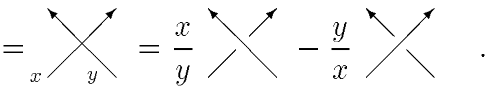
As another application of the intertwiners (13) we define R-matrices acting on other products of V^{Λ_1} and $V^{\Lambda_1^*}$ using the R-matrix (12) and crossing relations, eg.

$$R^{\Lambda_1\Lambda_1^*} =$$  (16)

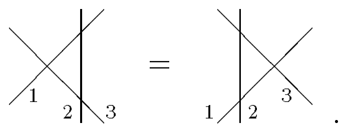
With these notations the R-matrix inversion and Skein relations following from eqs. (9) and (12-16) read for all choices of arrows

 (17)

In addition to this type of graphs considered so far we use graphs [28] where to each line there belongs not only a representation space V of $U_q(sl(n))$ but also a "spectral parameter" x. For example the spectral parameter dependent R-matrix given by eq. (8) is denoted by

$$R(x/y) =$$  (18)

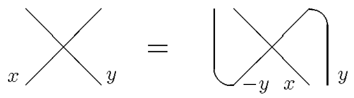
and the graphical notation of the Yang-Baxter equation (7) is

 (7')

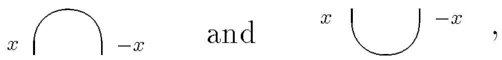
Analogously to eq. (8), we have spectral parameter dependent R-matrices for other representations, e.g.

$$R^{\Lambda_1\Lambda_1^*}(x/y) = \frac{x}{y} R^{\Lambda_1\Lambda_1^*} - \frac{y}{x} P \left(R^{\Lambda_1^*\Lambda_1}\right)^{-1} P. \tag{19}$$

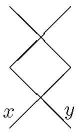
As a spectral parameter dependent version of eq. (16) we have crossing relations like

 (20)

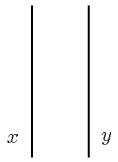
for all choices of arrows, if we introduce, as an extension of rel. (13), spectral parameter dependent intertwiners by

 (21)

where the spectral parameter x changes sign. Using again intertwiners and crossing relations one derives from eq. (18) for all choices of arrows the inversion relation for the spectral parameter dependent R-matrix

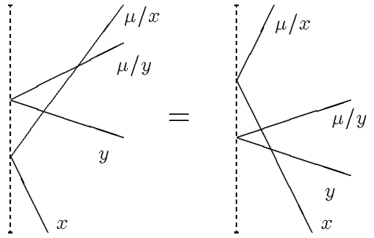


$$= (q^2 + q^{-2} - x^2/y^2 - y^2/x^2)$$



$$. \tag{22}$$

In addition we will make use of Cherednik's [12] reflection property. We only need the case of the reflection matrix $K = \mathbf{1}$ (see e.g. [29]). For all choices of arrows one derives from eqs. (17) and (18)



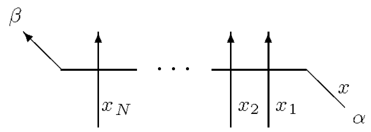
(23)

where the spectral parameter at the reflection point (at the dotted line, later denoted by a bar) changes from x to μ/x and y to μ/y for an arbitrary constant μ. In Sect. 3 we will make use of this arbitrariness and take the limit $\mu \to \infty$ which simplifies the nested Bethe ansatz.

3 The algebraic nested Bethe ansatz

In order to solve the eigenvalue equation for the Hamiltonian (2) we introduce the n-state vertex model of eq. (1). As usual [2] we introduce a monodromy matrix as a product of spectral parameter dependent R-matrices as follows

$$T_\alpha^\beta(x, \{x_i\}) := [R(x_N/x) \dots R(x_2/x) R(x_1/x)]_\alpha^\beta =$$

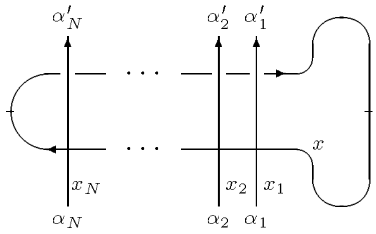


(24)

where to all lines (the horizontal one and the vertical ones) the fundamental representation Λ_1 is associated. We have omitted the indices which belong to the vertical space

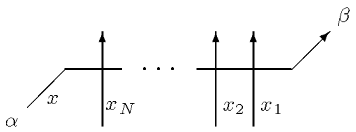
$$\Omega = V_N^{\Lambda_1} \otimes \dots \otimes V_1^{\Lambda_1}. \tag{25}$$

For the rational case $q = 1$, one obtains an $SU(n)$-invariant transfer matrix for periodic boundary conditions as the trace over the horizontal space of the monodromy matrix (24): $\tau|_{q=1} = \sum_{\alpha=1}^n T_\alpha^\alpha$. For the quantum group case this trace does not yield an $U_q(sl(n))$-invariant transfer matrix. However, the transfer matrix of eq. (1) associated to the vertex model depicted in Fig. 1 corresponds to periodic boundary conditions and should be an $U_q(sl(n))$ invariant. We show in Appendix A using the techniques of topological quantum field theory developed in [16] that this transfer matrix is given by the graph

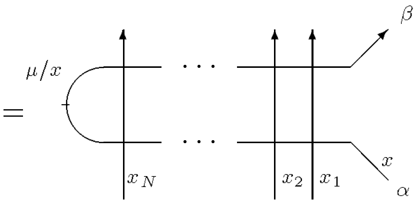
$$\tau_{\underline{\alpha}}^{\underline{\alpha}'}(x, \{x_i\}) =$$  $$, \qquad (26)$$

where in the lower row the monodromy matrix (24), in the upper one a product of constant R-matrices and on the right the Markov trace (14) appear. As in eq. (23) the two bars denote the transition of the spectral parameter from x to $x' = \mu/x = \infty$ (note that $R(x' \to \infty) \approx x'R$). Obviously, for the rational case $q = 1$ the transfer matrix (26) agrees with the conventional one mentioned above, since for $q = 1$ the Markov trace as well as the R-matrix ($R|_{q=1} = \mathbf{1}$) are trivial.

In order to perform the algebraic (nested) Bethe ansatz for the transfer matrix (26) we introduce two more monodromy matrices

$$\tilde{T}_{\alpha}^{\beta}(x, \{x_i\}) =$$  $$, \qquad (27)$$

and

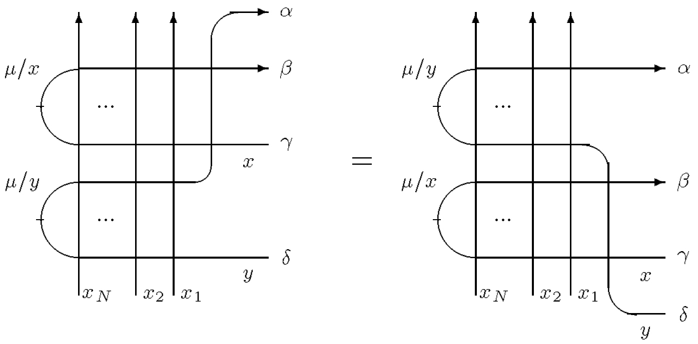
$$\mathcal{T}_{\alpha}^{\beta}(x, \{x_i\}; \mu) := \sum_{\gamma=1}^{n} \tilde{T}_{\gamma}^{\beta}(\mu/x, \{x_i\}) T_{\alpha}^{\gamma}(x, \{x_i\}) =$$  $$, \qquad (28)$$

where the bar again denotes the transition of the spectral parameter x to μ/x as in eq. (23). (Note that because of eq. (22) $\tilde{T} \propto T^{-1}$.) The monodromy matrix (28) has been used in refs. [11] [13][15][14] for the case $\mu = 1$ which corresponds to open boundary conditions. We shall see below that the nested Bethe ansatz simplifies drastically for $\mu \to \infty$. This corresponds to periodic boundary conditions.

The Yang-Baxter equation (2) and the reflection property (23) imply the following Yang-Baxter relation for the monodromy matrix $\mathcal{T}$ given by eq. (28):

$$R_{\alpha'\beta'}^{\alpha\beta}(y/x)\mathcal{T}_{\gamma'}^{\beta'}(x;\mu)R_{\gamma\delta'}^{\gamma'\delta}(\mu/xy)\mathcal{T}_{\delta}^{\delta'}(y;\mu) = \mathcal{T}_{\alpha''}^{\alpha}(y;\mu)R_{\delta''\beta''}^{\alpha''\beta}(\mu/xy)\mathcal{T}_{\gamma''}^{\beta''}(x;\mu)R_{\gamma\delta}^{\gamma''\delta''}(y/x), \qquad (29)$$

graphically expressed by



Compared to the case of the monodromy matrix T these commutation relations are more complicated. The usual decomposition into "wanted" and "unwanted" terms does not appear (see refs. [15][14]). We obtain much simpler commutation rules in the limit $\mu \to \infty$. The contributions from $\tilde{T}$ reduce to constant R-matrices. Also two R-matrices in eq. (29) become constant. So the commutation relations simplify in an essential way.

We write the monodromy matrix $\mathcal{T}$ (up to a factor) in the limit $\mu \to \infty$ in block form as a matrix in the horizontal space

$$\mathcal{T}(x) = \begin{pmatrix} \mathcal{A}(x) & \mathcal{B}(x) \\ \mathcal{C}(x) & \mathcal{D}(x) \end{pmatrix} = \lim_{\mu \to \infty} \Big(\prod_{i=1}^{N} \frac{x x_i}{\mu} \Big) \mathcal{T}(x;\mu) \tag{30}$$

where $\mathcal{A} \in M_{1,1}(End(\Omega))$, $\mathcal{B} \in M_{1,n-1}(End(\Omega))$, etc.. From eq. (29) we get the commutation relations of the operator valued entries of $\mathcal{T}$

$$\mathcal{A}(x)\mathcal{B}_\gamma(y) = \underbrace{q^{-1}a(x/y)\mathcal{B}_\gamma(y)\mathcal{A}(x)}_{\text{wanted term}} - \underbrace{q^{-1}\{c_-(x/y)\mathcal{B}_\gamma(x)\mathcal{A}(y) + (q-q^{-1})\mathcal{B}_\alpha(x)\mathcal{D}^\alpha_\gamma(y)\}}_{\text{unwanted term}} \quad , \tag{31}$$

$$\mathcal{D}^\beta_\gamma(x)\mathcal{B}_\delta(y) = \underbrace{\mathcal{B}_{\beta'}(y)\mathcal{D}^\alpha_{\gamma'}(x)R^{\beta'\beta}_{\delta'\alpha}R^{\gamma'\delta'}_{\gamma\delta}(y/x)}_{\text{wanted term}} - \underbrace{c_-(y/x)\mathcal{B}_{\beta''}(x)\mathcal{D}^{\alpha'}_\delta(y)R^{\beta''\beta}_{\gamma\alpha'}}_{\text{unwanted term}} \quad , \tag{32}$$

where the entries $a(x)$ and $c_-(x)$ of $R(x)$ are obtained from eqs. (9) and (18)

$$a(x) = \frac{x^2 q - q^{-1}}{x^2 - 1} \quad \text{and} \quad c_-(x) = \frac{q - q^{-1}}{x^2 - 1} \ . \tag{33}$$

The transfer matrix (26) is the Markov trace (14) of the monodromy matrix $\mathcal{T}$ defined by eq. (30)

$$\tau = \mathrm{tr}_q \mathcal{T} = \sum_{\alpha=1}^{n} \mathcal{T}^\alpha_\alpha q^{n+1-2\alpha} = q^{n-1}\mathcal{A} + \sum_{\alpha=2}^{n} q^{n+1-2\alpha}\mathcal{D}^\alpha_\alpha. \tag{34}$$

Using crossing, inversion and reflection relations (16, 22 and 23) one proves that eq. (29) implies that transfer matrices (34) commute for different spectral parameters. In Appendix B we show in addition that the transfer matrix commutes with the generators of the quantum group $U_q(sl(n))$. These generators are derived from the monodromy matrices T or $\mathcal{T}$ in the limits $x \to 0$ or ∞.

We now apply the algebraic nested Bethe ansatz method to the eigenvalue problem of the transfer matrix

$$\tau(x)\Psi = \Lambda(x)\Psi \tag{35}$$

for $\Psi \in \Omega$. The action of $\mathcal{T}$ on the reference state $\Phi := \bigotimes_{i=1}^N |1\rangle_i \in \Omega$ which is one of the ferromagnetic ground states is given by

$$\begin{aligned}
\mathcal{A}(x,\{x_i\})\Phi &= q^N \prod_{i=1}^N a(x_i/x)\Phi \\
\mathcal{D}^\alpha_\beta(x,\{x_i\})\Phi &= \delta^\alpha_\beta \Phi \\
\mathcal{B}_\beta(x,\{x_i\})\Phi &\neq 0, \not\propto \Phi \\
\mathcal{C}^\alpha(x,\{x_i\})\Phi &= 0.
\end{aligned} \tag{36}$$

We construct a Bethe ansatz vector by repeated action (r times) of creation operators $\mathcal{B}$ on Φ

$$\Psi(\{\hat{x}_i\}) := \left\{ \sum_{\epsilon_1 \ldots \epsilon_r = 2}^n \mathcal{B}_{\epsilon_1}(\hat{x}_1) \ldots \mathcal{B}_{\epsilon_r}(\hat{x}_r) \hat{\Psi}^{\epsilon_1 \ldots \epsilon_r} \right\} \Phi \quad , \tag{37}$$

where $\hat{\Psi}$ is an element of the reduced quantum space $\hat{\Omega}$ representing a chain of length r and admitting only states $|2\rangle, \ldots, |n\rangle$. To compute the action of $\mathcal{A}$ and $\mathcal{D}$ in the eigenvalue eq. (35) we commute them through all $\mathcal{B}$'s to the right and apply them to Φ. The wanted terms in eqs. (31) and (32) together with (36) yield

$$\begin{aligned}
q^{n-1}\mathcal{A}(x)\Psi &= q^{n-1}\mathcal{A}(x) \left\{ \sum_{\epsilon_1 \ldots \epsilon_r = 2}^n \mathcal{B}_{\epsilon_1}(\hat{x}_1) \ldots \mathcal{B}_{\epsilon_r}(\hat{x}_r) \hat{\Psi}^{\epsilon_1 \ldots \epsilon_r} \right\} \Phi \\
&= q^{N-r+n-1} \prod_{i=1}^r a(x/\hat{x}_i) \prod_{j=1}^N a(x_j/x)\Psi + \mathrm{uwt}
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
\sum_{\alpha=2}^n q^{n+1-2\alpha}\mathcal{D}^\alpha_\alpha(x)\Psi &= \sum_{\alpha=2}^n q^{n+1-2\alpha}\mathcal{D}^\alpha_\alpha(x) \left\{ \sum_{\epsilon_1 \ldots \epsilon_r = 2}^n \mathcal{B}_{\epsilon_1}(\hat{x}_1) \ldots \mathcal{B}_{\epsilon_r}(\hat{x}_r) \hat{\Psi}^{\epsilon_1 \ldots \epsilon_r} \right\} \Phi \\
= \sum_{\epsilon_1 \ldots \epsilon_r = 2}^n & \mathcal{B}_{\beta_1}(\hat{x}_1) \ldots \mathcal{B}_{\beta_r}(\hat{x}_r) \\
& \times \left\{ \sum_{\alpha=2}^n q^{n+1-2\alpha} R^{\beta_r \alpha_{r-1}}_{\delta_r \alpha_r} R^{\gamma_r \delta_r}_{\gamma_{r-1} \epsilon_r}(\hat{x}_r/x) \ldots R^{\beta_1 \alpha}_{\delta_1 \alpha_1} R^{\gamma_1 \delta_1}_{\alpha \epsilon_1}(\hat{x}_r/x) \hat{\Psi}^{\epsilon_1 \ldots \epsilon_r} \right\} \mathcal{D}^{\alpha_r}_{\gamma_r}\Phi + \mathrm{uwt} \\
= \prod_{\epsilon_1 \ldots \epsilon_r = 2}^n & \mathcal{B}_{\beta_1}(\hat{x}_1) \ldots \mathcal{B}_{\beta_r}(\hat{x}_r) \left\{ q^{-1} \hat{\tau} \hat{\Psi} \right\} \Phi + \mathrm{uwt},
\end{aligned} \tag{39}$$

where we have introduced an $U_q(sl(n-1))$ monodromy matrix $\hat{\mathcal{T}}$ and a transfer matrix

$$\hat{\tau} = \sum_{\alpha=1}^{n-1} \hat{\mathcal{T}}_\alpha^\alpha q^{n+1-2\alpha}. \tag{40}$$

The first term (wanted term) on the right hand side of eq. (39) is proportional to Ψ if the eigenvalue equation

$$\hat{\tau}\hat{\Psi} = \hat{\Lambda}\hat{\Psi} \tag{41}$$

is fulfilled. This problem agrees with the original one (35) where n is replaced by $n-1$. Thus iterating the procedure described above from level n to 1 we solve the eigenvalue equation (35). In the following, all operators, Bethe ansatz parameters etc. will be labeled by the number of the corresponding Bethe ansatz level, in particular $r_n = N,\ r_{n-1} = r,\ r_0 = 0,\ x_i^{(n)} = x_i,\ x_i^{(n-1)} = \hat{x}_i$.

In case all unwanted terms cancel, the eigenvalue equation of the transfer matrix eq. (35) is solved and the eigenvalue $\Lambda(x)$ consists of the wanted coefficients λ_k

$$\Lambda(x) = \sum_{k=1}^{n} \lambda_k(x), \tag{42}$$

where

$$\lambda_k(x) = q^{2k-n-1+r_k-r_{k-1}} \prod_{i=1}^{r_{k-1}} a(x/x_i^{(k-1)}) \prod_{j=1}^{r_k} a(x_j^{(k)}/x), \qquad (k=1,\ldots,n). \tag{43}$$

The Bethe ansatz equations are equivalent to the vanishing of all unwanted terms. They can be obtained from the property that $\Lambda(x)$ must have finite values when the spectral parameter x approaches one of the Bethe ansatz parameters $x_i^{(k)}$, because $\tau(x)$ is an analytical function in x. Writing $x_m^{(k)} = \exp\theta_m^{(k)}$ and $q = \exp i\gamma$ we obtain the coupled system of Bethe ansatz equations:

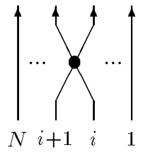
$$\begin{aligned} q^{2+\eta_{n-k}} \prod_{i_1=1}^{r_k} \frac{\sinh\left(\theta_m^{(k)} - \theta_{i_1}^{(k)} + i\gamma\right)}{\sinh\left(\theta_m^{(k)} - \theta_{i_1}^{(k)} - i\gamma\right)} &= - \prod_{i_2=1}^{r_{k+1}} \frac{\sinh\left(\theta_m^{(k)} - \theta_{i_2}^{(k+1)}\right)}{\sinh\left(\theta_m^{(k)} - \theta_{i_2}^{(k+1)} - i\gamma\right)} \\ &\quad \times \prod_{i_3=1}^{r_{k-1}} \frac{\sinh\left(\theta_m^{(k)} - \theta_{i_3}^{(k-1)} + i\gamma\right)}{\sinh\left(\theta_m^{(k)} - \theta_{i_3}^{(k-1)}\right)}\ , \quad (k=1,\ldots,n-1), \end{aligned} \tag{44}$$

where $\eta_{n-k} = r_{k+1} - 2r_k + r_{k-1}$ are the eigenvalues of the $U_q(sl(n))$ Cartan elements $H_{n-k},\ (k=1,\ldots,n-1)$ of the Bethe ansatz vector Ψ (see Appendix B).

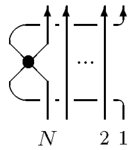
4 The Hamiltonian and finite size analysis

The Hamiltonian (2) may be obtained from the transfer matrix (26) as follows

$$H = \text{const.} \frac{d}{dx} \ln \tau(x, \{x_i\}\Big|_{x=x_i=1} = \text{const.} \left(\sum_{i=1}^{N-1} \right.$$



$$+$$



$$\left. \right) . \qquad (45)$$

The dotted crossings mean the matrix (see eq. (8))

$$\left(R^{-1}(x)\frac{d}{dx}R(x)\right)_{x=1} = \frac{1}{q-q^{-1}}(PR + R^{-1}P) = \frac{q+q^{-1}}{q-q^{-1}}(\mathbf{1} - 2P^{\Lambda_2}), \qquad (46)$$

where the following decomposition (4) of the R-matrix [30] and the completeness of the projectors has been used

$$PR = qP^{2\Lambda_1} - q^{-1}P^{\Lambda_2}. \qquad (47)$$

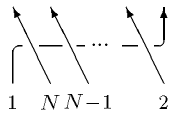
Therefore we get (with const. $= -1/2(q-q^{-1})/(q+q^{-1})$) up to terms proportional to the unity operator the Hamiltonian of eq. (2) as a sum of projectors P^{Λ_2}.

In Appendix A we derive the transfer matrix τ from a cyclic invariant vertex model on a torus. Therefrom it is obvious that the transfer matrix as well as the Hamiltonian (45) describe models with periodic boundary conditions. However, the derivation in Appendix A relies on methods of topological quantum field theory and here we can give only a short sketch of this methods. Therefore it seems worthwhile to present a direct proof of this fact. We denote the Hamiltonian of eq. (45) by $H_{N\ldots21}$ and by $H_{1N\ldots2}$ that one obtained by the cyclic permutation $(N\ldots21) \mapsto (1N\ldots2)$. The physical content is invariant under this permutation since both Hamiltonians are equivalent

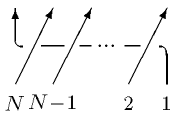
$$R^{-1}_{(N\ldots2)1} H_{N\ldots21} R_{1(N\ldots2)} = H_{1N\ldots2} \qquad (48)$$

where, as a generalization of eq. (12), we have introduced the R-matrices

$$R_{1(N\ldots2)} \equiv$$



and $R^{-1}_{(N\ldots2)1} \equiv$



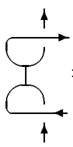
. (49)

For all terms of eq. (45), except for that one where $i=1$, the claim follows from eq. (17) and the special Yang-Baxter relation

$$R_{12}R_{13}(PP^{\Lambda_2})_{23} = (PP^{\Lambda_2})_{23}R_{13}R_{12} \qquad (50)$$

which is a consequence of the general Yang-Baxter relation (7) for $x_{23} = 1/q$ and $x_1 \to \infty$. For the remaining term we use the identity

$$R^{-1}P^{\Lambda_2}R = RP^{\Lambda_2}R^{-1} \quad \text{or}$$



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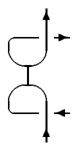
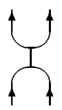
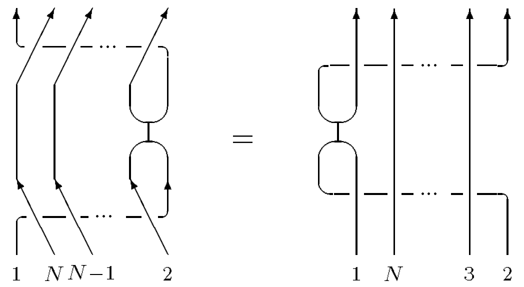


where



$= P^{\Lambda_2}$. (51)

This relation may be obtained directly from eq. (47). It also follows from the defining relation $\Delta_{21}R_{12} = R_{12}\Delta_{12}$ for the R-matrix of quasitriangular Hopf algebras with the coproduct Δ. For the $(i=1)$-term in eq. (45) we obtain with eqs. (50) and (51)



(52)

which finishes the proof of eq. (48).

We now use the results of Sect. 3, especially, the Bethe ansatz equations (44) to investigate the finite size behaviour of the ground state energy. In the thermodynamic limit conformal invariance of the system is expected. As shown by Cardy [31] conformal invariance implies for the maximal eigenvalue $\Lambda_{\max}$ of the transfer matrix

$$\Lambda_{\max} \approx \exp\Big(-Nf + \frac{1}{N}\frac{\pi}{6}c\Big), \qquad (N \to \infty) \tag{53}$$

where f is the free energy per site and c is the central charge of the corresponding Virasoro algebra. Using the techniques developed in [19] the central charge can be calculated from the finite size behaviour of $\Lambda_{\max}$ belonging to the antiferromagnetic ground state.

Taking the logarithm of the Bethe ansatz equation (44) we obtain

$$z_k(u_j^{(k)}) = 2\pi I_j^{(k)}, \quad \begin{cases} I_j^{(k)} \in (\mathbf{Z}+\frac{1}{2}) \cap \,]z_k(-\infty)/2\pi, z_k(\infty)/2\pi[\\ j=1,\ldots,r_k, \;\; k=1,\ldots,n-1, \end{cases} \tag{54}$$

where we have introduced the "rapidities" $u_i^{(k)} = i\gamma(k-n) + 2\theta_i^{(k)}$ and the phase functions

$$z_k(u) = N\delta_{k,n-1}p(u) + \sum_{l\in L_k}\sum_i p(u-u_i^{(l)}) + \sum_i \Phi(u-u_i^{(k)}) + (2+\eta_k)\gamma, \tag{55}$$

where $L_k = \{l = k \pm 1;\; 0 < l < n\}$. The inhomogeneities x_i have been taken to be equal to one. The functions $p(u)$ and $\Phi(u)$ are given by

$$e^{ip(u)} = \frac{\sinh\frac{1}{2}(u-i\gamma)}{\sinh\frac{1}{2}(u+i\gamma)} \quad \text{and} \quad e^{i\Phi(u)} = \frac{\sinh\frac{1}{2}(u+2i\gamma)}{\sinh\frac{1}{2}(u-2i\gamma)}. \tag{56}$$

The Fourier transforms $\tilde{f}(x) = \int du/(2\pi) e^{iux} f(u)$ of the derivatives p' and Φ' are

$$\tilde{p}'(x) = \frac{\sinh(\pi-\gamma)x}{\sinh \pi x} \quad \text{and} \quad \tilde{\Phi}'(x) = 1 - 2\cosh\gamma x \; \tilde{p}'(x). \tag{57}$$

The eqs. (55-57) may be generalized to include string solutions as well. However, since we want to analyse the finite size behaviour of the ground state eigenvalue, we are only looking for real roots of the Bethe ansatz equations (54). Taking the derivative of eq. (55) we find the following matrix equation

$$z_k' = \rho_k + \varphi_k = N\delta_{k,n-1}p' + \sum_l (p' + \Phi')_{kl} * \rho_l \tag{58}$$

where the matrices p' and Φ' are given by $p'_{kl} = \sum_{l' \in L_k} \delta_{l'l}\, p'$ and $\Phi'_{kl} = \delta_{kl}\Phi'$, respectively. The convolution is defined by $(f*g)(u) = \int du'/(2\pi) f(u-u')g(u')$. The densities of roots $\rho_k(u) = 2\pi\sum_{i=1}^{r_k}\delta(u-u_i^{(k)})$ may be written in terms of $\varphi_k(u)$, which describes the density of Bethe ansatz holes as well as finite size corrections. For this purpose one has to invert the matrix $(1-p'-\Phi')_{kl}$ appearing in the integral equation (58). Its Fourier transform is given by the symmetric matrix

$$\widetilde{(1 - p' - \Phi')}_{kl}^{-1}(x) = \frac{\sinh\pi x\ \sinh(n-k)\gamma x\ \sinh l\gamma x}{\sinh(\pi-\gamma)x\ \sinh n\gamma x\ \sinh\gamma x}, \quad k \le l. \tag{59}$$

For large lattice size N and $x \approx 1$ the eigenvalue $\Lambda(x)$ (see eq. (42)) of the transfer matrix is dominated by the term $\lambda_n(x)$. The following calculation may be easily performed also for excitations but for simplicity we restrict here to the ground state. From eqs. (43), (56) and (58) we obtain

$$\begin{aligned}\log\lambda_n(\theta,\gamma) &= N\log a - \int\frac{du}{2\pi}\, ip(u-2\theta)\ \rho_{n-1}(u) - i\gamma \\ &= -Nf_\infty + \int\frac{du}{2\pi}\int\frac{du'}{2\pi}\sum_{k=1}^{n-1} ip(u-2\theta)\,(1-p'-\Phi')^{-1}_{n-1,k}(u-u')\,\varphi_k(u) - i\gamma,\end{aligned} \tag{60}$$

where f_∞ is the free energy per site in the thermodynamic limit. Using the techniques of ref. [19] we find with eq. (53) for the central charge of the Virasoro algebra the formula

$$c = \sum_{k,l=1}^{n-1}\Big(\delta_{kl} - \frac{12}{r^2}\ \widetilde{(1-p'-\Phi')}_{kl}^{-1}(0)\Big). \tag{61}$$

Taking eq. (59) at $x = 0$ we can perform the sums and obtain

$$c = (n-1)\left(1 - \frac{n(n+1)}{r(r-1)}\right), \qquad q = e^{i\pi/r}, \quad r = n+2, n+3, \ldots. \tag{62}$$

This formula has previously been obtained in ref. [22] by means of Baxters [23] corner transfer matrix method for the A_{n-1} RSOS-models. Note that the matrix $\widetilde{(1-p'-\Phi')}(0)$ is just $\frac{r}{r-1}$ times the Cartan matrix A (see e.g. [3]). We expect the central charge of models for general simply laced q-Lie algebras (A, D, E) of rank l to be

$$c = \sum_{i,j=1}^{l}\Big(\delta_{ij} - \frac{12}{r(r-1)}A^{-1}_{ij}\Big). \tag{63}$$

One easily can calculate the sums and finds

$$c = l\left(1 - \frac{g(g+1)}{r(r-1)}\right), \qquad q = e^{i\pi/r}, \quad r = g+2, g+3, \ldots, \tag{64}$$

where g is the dual Coxeter number (for A_l, D_l, E_6, E_7, E_8: $g = l+1, 2l-2, 12, 18, 30$, respectively). This formula coincides (see [21]) with the formula for the central charges of the extended coset algebras constructions of ref. [20] for general simply laced Lie algebras.

In a forthcoming paper we will in addition to the ground state also discuss the excitation spectrum, also for other models related to other quantum groups. As mentioned in Appendix A the quantum group representation of the states are determined by the topology of the interior of the 3-manifold, on whose boundary the vertex model is defined. For the case of trivial topology of the interior of a torus there exist only states which transform trivially under the quantum group. We will analyse more general situations to obtain higher representations. In this paper we have not mentioned questions about positivity, unitarity e.t.c., these will be investigated elsewhere.

Appendix A

In this appendix we define the vertex model with periodic boundary conditions of eq. (1) depicted in Fig. 1. The transfer matrix of eq. (26) is shown to belong to this model. We use the techniques of topological quantum field theory in terms of coloured graphs as developed in ref. [16]. Here we restrict the derivation to the quantum group $Sl_q(2)$ for $q = \exp(i\pi/r)$, $(r = 3, 4, 5, \ldots)$. The construction may easily be generalized to other quantum groups (see e.g. [32]). The model is only defined for q equal to roots of unity, whereas the transfer matrix (26) is of course also meaningful for generic values of q.

Let M be a 3-manifold, a solid torus or more general a cylindric part of a handle of an arbitrary handle body. On the boundary ∂M of M we consider a graph $G(\underline{x})$ ($\underline{x} = \{x_1, \ldots, x_N, x\}$) which forms a square lattice as in Fig. 1 or on the left hand side of Fig. 2. To each line of the graph we associate the fundamental representation Λ_1 and a spectral parameter x_i or x which coincide on opposite legs of the 4-vertices. The 4-vertices are given by eq. (8) in terms of spectral parameter independent R-matrices. For simplicity we take the spectral parameters of the horizontal lines all equal to x. Formula (3.2) of ref. [16] defines a partition function

$$Z(M, G(\underline{x})) = \sum_{\underline{j}, \underline{\tilde{J}}} W(\underline{j}, \underline{\tilde{J}})(X, G(\underline{x})). \tag{65}$$

The right hand side is defined in terms of a triangulation X of the 3-manifold M inducing a triangulation ∂X of ∂M. The sum of eq. (65) runs over all set of colours $\underline{j}$ and $\underline{\tilde{J}}$, where the colours are the irreducible representations ($j = 0, 1/2, 1, \ldots, r/2 - 1$) of $Sl_q(2)$. The set $\underline{j}$ is a colouring of all 1-simplexes of X and $\underline{\tilde{J}}$ a colouring of all plaquettes obtained from the graph $\widetilde{\partial X} \cup G(\underline{x})$, where $\widetilde{\partial X}$ is the dual graph of ∂X. In ref. [16] it is explained

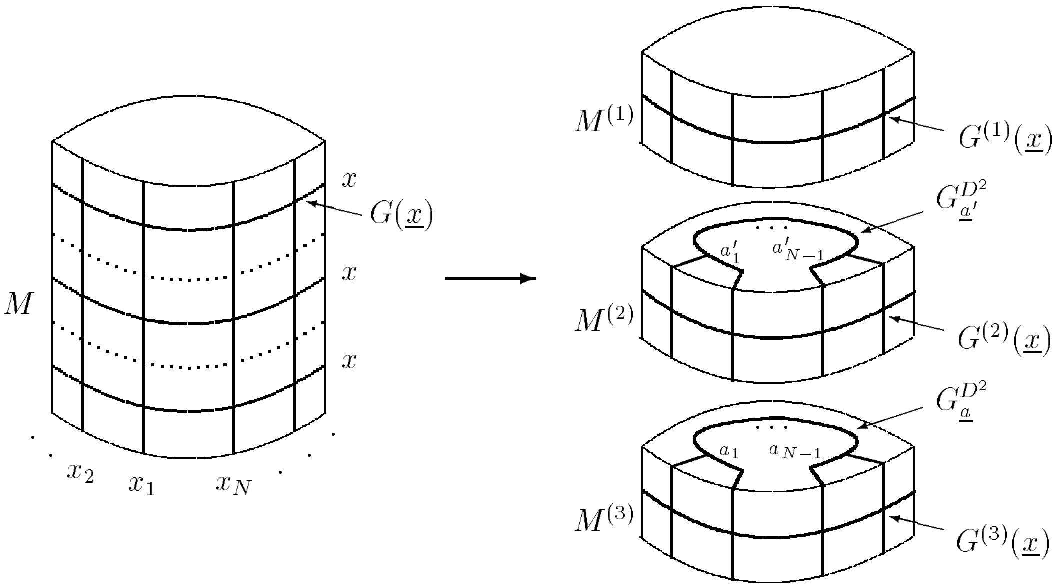


Figure 2: *Surgery along the dotted lines*

how the weights $W(\underline{j}, \underline{\tilde{J}})(X, G(\underline{x}))$ are given in terms of q-dimensions, 6j-symbols and R-matrices. Moreover it is shown that the r.h.s. of eq. (65) does not depend on the triangulation X of M and therefore defines an invariant of the 3-manifold M equipped with a vertex model on its boundary ∂M given by the graph $G(\underline{x})$.

Note that the partition function (65) does not only describes a two dimensional vertex model on the torus or cylinder but in addition there is a local interaction of the vertices with the interior of the 3-manifold M. However, this interaction is of topological nature. Thus this model is similar to σ-models with Chern-Simons term or the WZNW-models [17][18].

We decompose M and the square lattice represented by the graph $G(\underline{x})$ on the boundary as follows

$$\begin{aligned} M &= M^{(1)} \cup_{D^2} M^{(2)} \cup_{D^2} M^{(3)} \\ G(\underline{x}) &= G^{(1)}(\underline{x}) \cup G^{(2)}(\underline{x}) \cup G^{(3)}(\underline{x}) \end{aligned} \tag{66}$$

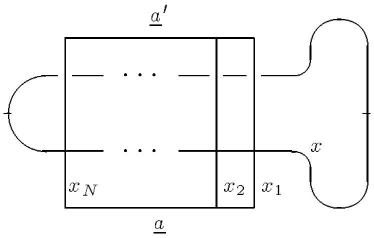
along two discs D^2 as shown in Fig. 2. Applying the general surgery formula (7.3) of ref. [16] we have

$$\begin{aligned} Z(M, G(\underline{x})) &= \sum_{\underline{a}\underline{a}'} W_{\underline{a}}^{D^2} W_{\underline{a}'}^{D^2} \; Z(M^{(1)}, G^{(1)}(\underline{x}) \cup G_{\underline{a}'}^{D^{2*}}) \\ &\quad \times Z(M^{(2)}, G^{(2)}(\underline{x}) \cup G_{\underline{a}'}^{D^2} \cup G_{\underline{a}}^{D^{2*}}) \; Z(M^{(3)}, G^{(3)}(\underline{x}) \cup G_{\underline{a}}^{D^2}). \end{aligned} \tag{67}$$

where $G_{\underline{a}}^{D^2}$ as depicted in Fig. 2, the canonical graph of the disc D^2, is defined in ref. [16]. On the bottoms of $M^{(1)}$ and $M^{(2)}$ there are the mirror graphs $G_{\underline{a}'}^{D^{2*}}$ and $G_{\underline{a}}^{D^{2*}}$, respectively.

The (finite) summation is over all colourings $\underline{a}$ and $\underline{a}'$ of the canonical graphs. Note that the following also holds, if $M^{(1)}$ and $M^{(3)}$ stay connected after the surgery. The piece $M^{(2)}$ is topological equivalent to the ball D^3 with boundary $\partial D^3 = S^2$ and the graph $G^{(2)}(\underline{x}) \cup G^{D^2}_{\underline{a}'} \cup G^{D^{2*}}_{\underline{a}}$ is planar. In ref. [16] is shown that for planar graphs the interaction with the interior of the 3-manifold disappears and

$$Z(M^{(2)}, G^{(2)}(\underline{x}) \cup G^{D^2}_{\underline{a}'} \cup G^{D^{2*}}_{\underline{a}}) = \tau^{\underline{a}'}_{\underline{a}}(x, \{x_i\}) \equiv \tag{68}$$



,

where the 3-vertices are given by intertwiners $V^{a_i} \otimes V^{1/2} \to V^{a_{i+1}}$ as explained in ref. [16]. The transfer matrix $\tau^{\underline{a}'}_{\underline{a}}(x, \{x_i\})$ given by eq. (68) is represented in the path basis. It is equivalent to the transfer matrix $\tau^{\underline{\alpha}'}_{\underline{\alpha}}(x, \{x_i\})$ given by eq. (26), represented in the tensor basis, projected to the sector of total spin $J = 0$. We remark that the other sectors are obtained for nontrivial topology of the interior of the 3-manifold M. In a forthcoming paper we will discuss this more general situation. We stress again that the invariant $Z(M^{(2)}, G^{(2)}(\underline{x}) \cup G^{D^2}_{\underline{a}'} \cup G^{D^{2*}}_{\underline{a}})$ of eq. (68) is defined only for q equal to roots of unity, whereas the transfer matrices $\tau^{\underline{a}'}_{\underline{a}}(x, \{x_i\})$ and $\tau^{\underline{\alpha}'}_{\underline{\alpha}}(x, \{x_i\})$ are also meaningful for generic values of q.

Appendix B

We define the matrices $L^{\pm}$ by the limits x to ∞ or 0 of the monodromy matrix T given by eq. (24) with $\alpha := q - q^{-1}$

$$L^+ = \lim_{x\to\infty} x^{-N} T(x) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \alpha E_1 & 1 & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ * & & \alpha E_{n-1} & 1 \end{pmatrix} \begin{pmatrix} q^{W_1} & 0 & \cdots & 0 \\ 0 & q^{W_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & q^{W_n} \end{pmatrix} \tag{69}$$

$$L^- = \lim_{x\to 0} x^{N} T(x) = \begin{pmatrix} q^{-W_1} & 0 & \cdots & 0 \\ 0 & q^{-W_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & q^{-W_n} \end{pmatrix} \begin{pmatrix} 1 & -\alpha F_1 & & * \\ 0 & 1 & \ddots & \\ & \ddots & \ddots & -\alpha F_{n-1} \\ 0 & & 0 & 1 \end{pmatrix}. \tag{70}$$

These forms of the matrices L^+ and L^- follow from the triangular form of the R-matrix in eq. (9) (see also ref. [33]). The entries E_i, F_i and $q^{\pm W_i}$ of $L^{\pm}$ are given by the N-fold coproduct of the generating elements $q^{\pm h_i/2}$, e_i and f_i, $(i = 1, \ldots, n-1)$ in the fundamental representation Λ_1 of $\hat{U} = U_q(sl(n))$ associated to each lattice site:

$$
\begin{aligned}
E_i &= \Delta^{(N)}(e_i) = \sum_{l=1}^{N} q^{-h_i} \otimes \ldots \otimes q^{-h_i} \otimes \underbrace{e_i}_{l\mathrm{th}} \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}, \\
F_i &= \Delta^{(N)}(f_i) = \sum_{l=1}^{N} \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \underbrace{f_i}_{l\mathrm{th}} \otimes q^{h_i} \otimes \ldots \otimes q^{h_i}, \\
q^{\pm H_i/2} &= \Delta^{(N)}(h_i) = q^{\pm h_i/2} \otimes \ldots \otimes q^{\pm h_i/2},
\end{aligned}
\tag{71}
$$

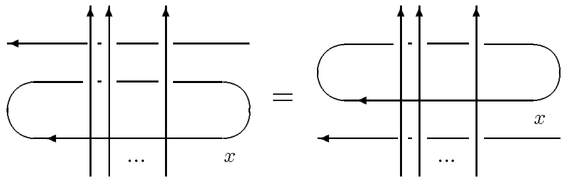
The $H_i = W_i - W_{i+1}$ are the Cartan elements. The Yang-Baxter equation for monodromy matrices implies the commutation rules

$$
R L_1^{\pm} L_2^{\pm} = L_2^{\pm} L_1^{\pm} R \quad \text{and} \quad R L_1^{+} L_2^{-} = L_2^{-} L_1^{+} R. \tag{72}
$$

These commutation rules are equivalent [34] [33] to the defining relations of $U_q(sl(n))$ for the elements E_i, F_i and $q^{\pm H_i/2}$. $\Delta^{(N)}$ is an algebra homomorphism $\hat{U} \to \hat{U} \otimes \ldots \otimes \hat{U}$. Quantum group invariance of the transfer matrix is now shown by applying $L^{\pm}$, e.g.

$$
L^{+} \tau(x) = \lim_{y \to \infty} \left(\prod_{i=1}^{N} \frac{-y}{x_i} \right) T(y)\ \tau(x). \tag{73}
$$

Using the Yang-Baxter (7), crossing (16) and inversion (7) relations one obtains

$L^{+}\tau(x) =$  $= \tau(x) L^{+}$. (74)

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