Corrections to Quantized Charge Transport in Quantum Hall Systems

V. Jakšić * K. Jung ** M. Klein ** R. Seiler **

January 11, 1994

1 Introduction

Still today – more than 60 years after the discovery of the new quantum mechanics – the derivation of Ohm's law from the Schrödinger equation is a great challenge. Some progress has been made for quantum Hall systems: They have (by definition) a gap in the spectrum of their Hamiltonian separating the ground state from the rest. Hence these models are non dissipative, i.e. there is a Hall current only and no direct resistance $\rho_{||}$. It has been shown that the Hall current I is linearly related to the Hall voltage V

$$I = \sigma V + \text{correction terms} \tag{1}$$

for small V with a correction term of size $\mathcal{O}(V^\infty)$ [1], [2], [10].

For ordinary quantum Hall systems the conductivity σ is an integer up to a factor of $\frac{1}{2\pi}$ (in atomic units).

Integrality of Hall conductivity is a consequence of gauge invariance and the gap condition. Under these conditions the adiabatic theorem of quantum mechanics, which originated with the work of Born and Fock [4], is applicable. This comes as follows: Firstly voltage is generated – according to Faraday's law – by a time dependent magnetic flux $\phi(t)$ which enters the Hamiltonian as a gauge field and has a derivative of size V,

$$\dot{\phi}(t) = -V \qquad (t \in [t_{\text{switch on}}, t_{\text{switch off}}]).$$

Hence $1/V$ plays the role of a time scale and the limit $V \to 0$ is the adiabatic limit. Secondly adiabatic dynamics is inherently geometric [8], [3], [15], [2]. The geometric object in question is a line bundle over the 2-torus of two fluxes. It

*Department of Mathematics, University of Toronto, Toronto, M5S-1A1, Canada
**Technische Universität Berlin, Fachbereich Mathematik, Sekr. MA 7-2, 10623 Berlin

turns out that the degree of the line bundle (Chern number) is given by Kubo's formula [14], [2].

The correction term in (1) was investigated experimentally and theoretically [5], [10]. In this note we shall argue that it is exponentially small in V. More precisely, we shall merely analyze the time integral of the current, which is the charge transport, and prove that its correction term is exponentially small in an averaged sense. We do this by combining the arguments of [10] - which actually show that the correction term is of the order of the correction term in the adiabatic theorem - with an improved version of the adiabatic theorem [12] yielding exponential error terms in the analytic category. This adds to the understanding of the experimental discovery of v. Klitzing in 1980.

It is a great pleasure to dedicate this note to Ludwig Faddeev on the occasion of his sixtieth birthday.

2 Adiabatic Dynamics of Quantum Mechanics

Adiabatic dynamics has two aspects: The first one is purely dynamical, the second one is geometric. The centerpiece of the dynamical part is the adiabatic theorem. The setting is as follows: Let $H(s)$, $s \in \mathbb{R}$ be a family of selfadjoint Hamiltonians with fixed domain $\mathcal{D}$ in a Hilbert space $\mathcal{H}$ with the properties

1. $H(s)$ is smooth as a mapping from $\mathcal{D}$ equipped with the graph norm to $\mathcal{H}$ and bounded from below.

2. $H(s)$ has a spectral band separated from the rest of the spectrum by a gap $\Delta > 0$.

The time evolution is defined through the time dependent Schrödinger equation

$$i\partial_s U_\tau(s, s_0) = \tau H(s) U_\tau(s, s_0), \qquad U_\tau(s_0, s_0) = \mathbb{1}\,, \qquad \tau > 0\,. \tag{2}$$

τ plays the role of a time scale, $s = t/\tau$ is the scaled time and t is the physical time. The initial value problem has a unique solution (see [16], [11]).

The adiabatic theorem expresses the fact that physical evolution wants to preserve spectral bands or, as it was put by Born and Fock, that quantum numbers (local bands) are preserved in the adiabatic limit $\tau \to \infty$ or again, in other words, that there is no tunneling out of a spectral band:

Theorem 1 *Under the conditions 1,2 of smoothness and the existence of a gap the following estimates hold:*

1. *For* $s_0, s_1 \in \mathbb{R}$

$$\|P^\perp(s_1) U_\tau(s_1, s_0) P(s_0)\| + \|P(s_1) U_\tau(s_1, s_0) P^\perp(s_0)\| = \mathcal{O}(\tau^{-1})\,. \tag{3}$$

2. For supp$\dot{H}(s) \subset (0,1)$ and $s_0, s_1 \in \mathbb{R}\backslash(0,1)$

$$\|P^{\perp}(s_1)U_\tau(s_1,s_0)P(s_0)\| + \|P(s_1)U_\tau(s_1,s_0)P^{\perp}(s_0)\| = \mathcal{O}(\tau^{-\infty})\,. \qquad (4)$$

This statement has recently been translated [6], [7], [12], [13] from the C^∞ to the analytic category where it reads in the formulation of Martinez as follows:

Let $H(s)$ be a family of operators, defined in some complex strip $S(a) := \{z \,|\, |\mathrm{Im} z| \le a\}, a > 0$, with the properties

A1 $H(s)$ is type A analytic in the sense of Kato and selfadjoint for real s. Furthermore there exist operators $H^\pm$ such that

$$(1+|z|)^{1+\varepsilon}\partial_z^k(H(z) - H^\pm) = \mathcal{O}(1) \qquad (z \in S(a), \pm\mathrm{Re} z > 0)$$

as operators from their common domain $\mathcal{D}$ equipped with the graph norm to $\mathcal{H}$. $\mathrm{Re}H(s) = \frac{1}{2}(H^*(s) + H(s))$ is bounded from below.

A2 $H(s)$ has a spectral gap $\Delta(s)$ uniformly bounded away from zero for $s \in S(a)$.

For technical reasons one assumes in addition:

A3 For $s \in \mathbb{R}$ there are functions $e_1(s) < e_2(s)$ with values inside the spectral gap of $H(s)$ such that

$$\inf_{s\in\mathbb{R}}(e_2(s) - e_1(s)) =: \Gamma_0 > 0$$

and $e(s) = e_1(s) + e_2(s)$ extends to an analytic function of $s \in S(a)$ satisfying

$$(1+|z|)^{1+\varepsilon}\partial^k(e(z) - \alpha^\pm) = \mathcal{O}(1) \qquad (z \in S(a), \pm\mathrm{Re} z > 0)$$

for some $\alpha^\pm \in \mathbb{R}$.

Theorem 2 *Assuming A1, A2 and A3 the following estimates hold:*

$$\lim_{\substack{s_1\to+\infty \\ s_0\to-\infty}} P^{\perp}(s_1)U_\tau(s_1,s_0)P(s_0) = \mathcal{O}(\exp(-D\tau)) \qquad (5)$$

$$\lim_{\substack{s_1\to+\infty \\ s_0\to-\infty}} P(s_1)U_\tau(s_1,s_0)P^{\perp}(s_0) = \mathcal{O}(\exp(-D\tau)) \qquad (6)$$

The constant $D > 0$ depends on the width of the gap $\Delta(s)$ in the complex strip $S(a)$: $D = 2\Sigma - \delta$ for arbitrarily small $\delta > 0$, where

$$\Sigma = \int_{-\Gamma_0/2}^{\Gamma_0/2} \kappa(\tau)d\tau$$

and

$$\kappa(\tau) = \sup\{\kappa \in [0,a) \mid (H - e/2)(s) + \tau \text{ is invertible for all } s \in S(\kappa)\}\,.$$

This concludes the part about the dynamic aspect of the adiabatic time evolution.

The geometric aspect of adiabatic dynamics originates from the question: What are the propagators on Hilbert space which give rise to formulae of type (3) without correction term:

$$\begin{aligned} P^{\perp}(s)V_\tau(s,s_0)P(s_0) &= 0 \\ P(s)V_\tau(s,s_0)P^{\perp}(s_0) &= 0 \end{aligned}$$

These equations are equivalent to

$$P(s)V_\tau(s,s_0) = V_\tau(s,s_0)P(s_0)$$

So $V_\tau(s,s_0)$ maps range $P(s_0)$ onto range $P(s)$. Quite surprisingly this property is shared by the solution of any of the initial value problems

$$i\partial_s V_\tau^f(s,s_0) = \tau\Big(f(H(s)) + \frac{i}{\tau}[\dot{P}(s),P(s)]\Big)V_\tau^f(s,s_0), \qquad V_\tau^f(s_0,s_0) = \mathbb{1}$$

with $f \in C^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Two of them are of particular importance: For $f(x) = x$ we get the adiabatic time evolution $U_\tau^A(s,s_0)$. It has the property that $U_\tau^A(s,s_0)$ is close in an optimal sense to the physical time evolution $U_\tau(s,s_0)$ [2]. Notice that the adiabatic theorem implies for the wave operator comparing the adiabatic with the full evolution that

$$\Omega_\tau(s,s_0) := (U_\tau^A)^*(s,s_0)U_\tau(s,s_0) = \mathbb{1} + \mathcal{O}(\tau^{-1})$$

Corollary 1

$$[\Omega_\tau(s,s_0),P(s_0)] = \mathcal{O}(\tau^{-1}), \qquad \textit{(see (3))} \tag{7}$$

$$[\Omega_\tau(s,s_0),P(s_0)] = \mathcal{O}(\tau^{-\infty}), \qquad \textit{(see (4))} \tag{8}$$

$$\lim_{\substack{s_1\to+\infty \\ s_0\to-\infty}} [\Omega_\tau(s,s_0),P(s_0)] = \mathcal{O}(\exp(-D\tau)), \qquad \textit{(see (5))} \tag{9}$$

The second example is Kato's and is defined by the trivial function $f(x) = 0$ (see [8], [9]). It is a useful tool in perturbation theory of linear operators.

3 Exponential Corrections to the Kubo formula

The multiparticle Hamiltonian $H(\phi)$ modeling the Quantum Hall System is defined by

$$\tilde{H}(\phi) = \sum_{i=1}^{N}\Big(\frac{1}{2}(v_i - \phi_1 a_1 - \phi_2 a_2)^2 + W(x_i)\Big) + \sum_{i<j}\frac{1}{|x_i - x_j|} \tag{10}$$

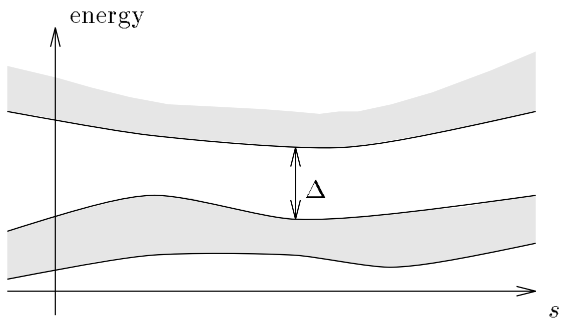

Figure 1: Spectral gap

with periodic boundary conditions on the antisymmetric N-particle space $\bigwedge_{i=1}^{N} L^2(\Lambda)$ where $\Lambda = [0, L_1] \times [0, L_2]$ is a rectangle in $\mathbb{R}^2$. $v_i = p_i - A(x_i)$ is the velocity operator of particle i and $A(x_i)$ the vector potential of the external magnetic field $B = \mathrm{rot} A$. $a_i = \frac{1}{L_i} e_i$ for $i = 1, 2$ are the vector potentials used for the battery (Hall voltage) and the definition of the current operator. (See [1] and [10] for details.) W stands for the impurities and the last term for the electron-electron interaction.

$\tilde{H}(\phi)$ is type A analytic for $\phi \in C^2$ on $\mathcal{D} = \mathrm{domain}(\tilde{H}(0))$. This follows from the proof of Lemma 2.1 in [10] by simply allowing complex values of ϕ. Furthermore, up to a unitary gauge transformation, $\tilde{H}(\phi)$ is periodic in both variables

$$\hat{H}(\phi) = e^{i(\phi_1 F_1 + \phi_2 F_2)} H(\phi) e^{-i(\phi_1 F_1 + \phi_2 F_2)} = \hat{H}(\phi + g) \quad (g \in 2\pi\mathbb{Z} \times 2\pi\mathbb{Z})$$

$$F_j(x_1, \dots, x_N) := \sum_{l=1}^{N} (a_j, x_l)$$

In the following the gap condition will be imposed: The ground state is separated from the rest of the spectrum by a gap Δ (see figure 1) and gives rise to a spectral projection $\tilde{P}(\phi)$. It is assumed that $\dim \tilde{P}(\phi) < \infty$. The gap condition and gauge invariance of the theory are at the heart of this approach.

The Hall voltage across the sample in the one direction is generated by a time dependent flux (Faraday's law):

$$\phi_1 = f(s = t/\tau)$$

where t denotes physical time, τ is a fixed the time scale, f is the switching function, with the typical graph shown in figure 2 and

$$-V(t) = \dot{\phi}_1(t) = \frac{1}{\tau} f'(t/\tau)$$

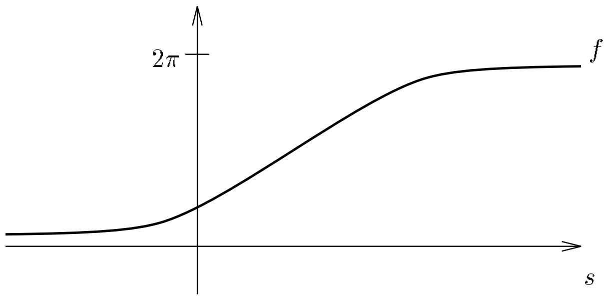

Figure 2: Typical switching function

is voltage. The time evolution of the system is defined by the time dependent Hamiltonian $H(s,\varphi) := \tilde{H}(\phi_1 = f(s), \phi_2 = \varphi)$ and the Schrödinger equation

$$i\partial_s U_\tau(s,s_0,\varphi) = \tau H(s,\varphi) U_\tau(s,s_0,\varphi), \qquad U_\tau(s_0,s_0,\varphi) = \mathbb{1}\ .$$

It gives rise to the (not normalized) physical state

$$P_\tau(s,s_0,\varphi) := U_\tau(s,s_0,\varphi) P(s_0,\varphi) U_\tau^*(s,s_0,\varphi)$$

where $P(s_0,\varphi) := \tilde{P}(\phi_1 = f(s_0), \phi_2 = \varphi)$. The Hall current is defined by

$$I(s,s_0,\varphi) := \frac{1}{q}\,\mathrm{Trace} P_\tau(s,s_0,\varphi)\partial_\varphi H(s,\varphi)$$

where q denotes dim $P(s_0,\varphi)$. The Hall current can be written in terms of a time derivative

$$I(s,s_0,\varphi) = \frac{i}{\tau}\partial_s\,\mathrm{Trace} P(s_0,\varphi) U_\tau^*(s,s_0,\varphi)\partial_\varphi U_\tau(s,s_0,\varphi)\,.$$

Hence the charge transported by the Hall current is

$$\begin{aligned} Q_\tau(s_1,s_0,\varphi) &:= \tau\int_{s_0}^{s_1} ds\, I(s,s_0,\varphi) \\ &= i\,\mathrm{Trace} P(s_0,\varphi) U_\tau^*(s_1,s_0,\varphi)\partial_\varphi U_\tau(s_1,s_0,\varphi)\,. \end{aligned} \tag{11}$$

In the following we shall prove that the average of Q_τ is an integer up to exponential terms in the limit $s_0 \to -\infty$, $s_1 \to +\infty$ if appropriate assumptions on the switching function are met.

Let the switching function $f(s)$ be chosen in such a way that $H(s,\varphi)$ satisfies the assumptions A1, A2 and A3 required for the adiabatic theorem in the formulation of Martinez to hold. This is in fact always possible if $\tilde{H}(\phi)$ satisfies the gap condition: Integrating e.g. $(1+s^2)^{-N}$ and normalizing we obtain an analytic function f satisfying

$$|\partial^k f(z)| \le C_k(1+|z|)^{-N-k+1}, \quad (z \in S(1/2))$$

and

$$\begin{aligned}\lim_{\mathrm{Re}z\to-\infty} f(z) &= 0 \qquad (z\in S(1/2))\,,\\ \lim_{\mathrm{Re}z\to+\infty} f(z) &= 2\pi \qquad (z\in S(1/2))\,.\end{aligned}$$

Furthermore $\mathrm{Im}f(z)$ is bounded in terms of $\mathrm{Im}z$. In view of the analyticity of $\tilde{H}(\phi)$ and the periodicity of its spectrum for real values of ϕ, the spectral gap of $\tilde{H}(\phi)$ extends to some complex neighborhood of the real values of ϕ. Thus $H(s,\varphi)$ defined as above by use of the switching function $f(s)$ satisfies the conditions A1 and A2 of the (analytic) adiabatic theorem in any strip $S(a)$ for a sufficiently small.

To check the technical condition A3, we observe that the supremum $\gamma_1(\phi)$ of the lower spectral band and the infimum $\gamma_2(\phi)$ of the upper spectral band of $H(s)$ are doubly-periodic continuous functions of $\phi\in\mathbb{R}^2$.

Approximating $\gamma_1(\phi)$ from above and $\gamma_2(\phi)$ from below on $[0,2\pi]\times[0,2\pi]$ by polynomial functions $\tilde{e}_j(\phi)$ with $|\gamma_j(\phi)-\tilde{e}_j(\phi)|<\epsilon$, we see that

$$e(s,\varphi)=\tilde{e}_1(f(s),\varphi)+\tilde{e}_2(f(s),\varphi)$$

satisfies A3 in some strip $S(a)$, taking ϵ sufficiently small.

Then we know from Theorem 2 that the wave operator

$$\Omega_\tau(s,s_0,\varphi)=(U^A_\tau)^*(s,s_0,\varphi)U_\tau(s,s_0,\varphi)$$

commutes with $P(s_0,\varphi)$ up to an exponentially small term:

$$R_\tau(s_1,s_0,\varphi):=[\Omega_\tau(s_1,s_0,\varphi),P(s_0,\varphi)]$$

satisfies

$$\lim_{\substack{s_1\to+\infty\\ s_0\to-\infty}} R_\tau(s_1,s_0,\varphi)=\mathcal{O}(\exp(-D\tau))\,, \tag{12}$$

uniformly in $\varphi\in[0,2\pi]$ where $D=2\Sigma_f-\delta>0$ is given by Theorem 2, i.e.

$$\Sigma_f=\int_{-\Gamma_0/2}^{\Gamma_0/2}\kappa_f(\tau)d\tau\,, \tag{13}$$

$$\begin{aligned}\kappa_f(\tau) \;=\; &\sup\{\kappa\in[0,1/2)\mid(\tilde{H}-\tilde{e}/2)(\phi)+\tau \text{ is invertible for all}\\ &\phi\in\mathrm{range}f(S(\kappa))\times[0,2\pi]\}\end{aligned} \tag{14}$$

for a fixed switching function specified above. We shall now show that by scaling the switching function the exponential decay parameter D can be made arbitrarily large. Replacing f by

$$f_\mu(s):=f(\frac{s}{\mu}) \qquad (s\in S(\mu/2),\mu>0)$$

amounts to replacing s by $\tilde{s} = s/\mu$ in the Schrödinger equation (2). Since $\partial_s = \frac{1}{\mu}\partial_{\tilde{s}}$ this changes τ to $\tilde{\tau} = \mu\tau$. More precisely, the solution operator $U_\tau(s, s_0; \mu)$ of the Schrödinger equation (2) with switching function f_μ satisfies

$$U_\tau(s, s_0; \mu) = U_{\mu\tau}(\frac{s}{\mu}, \frac{s_0}{\mu}; 1) .$$

An analogous equation holds for the adiabatic evolution U_τ^A since it is generated by $H^A = H + \frac{i}{\tau}[\dot{P}, P]$ and $\dot{P}$ scales as ∂_s. Thus

$$\Omega_\tau(s, s_0, \varphi; \mu) = \Omega_{\mu\tau}(\frac{s}{\mu}, \frac{s_0}{\mu}, \varphi; 1) .$$

Hence the remainder term in (12) with switching function f_μ scales as

$$\lim_{\substack{s_1 \to +\infty \\ s_0 \to -\infty}} R_\tau(s_1, s_0, \varphi; \mu) = \lim_{\substack{s_1 \to +\infty \\ s_0 \to -\infty}} R_{\mu\tau}(\frac{s_1}{\mu}, \frac{s_0}{\mu}, \varphi; 1) = \mathcal{O}(\exp(-D\mu\tau)) .$$

Thus $\tilde{D} = D\mu$ can be made arbitrarily large.

We remark that the explicit definition of D by (13) and (14) reproduces the same scaling behavior: Since $\mathrm{range} f_\mu(S(\kappa)) = \mathrm{range} f(S(\kappa/\mu))$ we get

$$\kappa_{f_\mu}(\tau) = \mu \cdot \kappa_f(\tau)$$

and therefore

$$\Sigma_{f_\mu} = \mu \cdot \Sigma_f .$$

We now replace U_τ by $U_\tau^A \Omega_\tau$ in (11),

$$\begin{aligned} Q_\tau(s_1, s_0, \varphi) &= i \,\mathrm{Trace} P(s_0, \varphi)(U_\tau^A)^*(s_1, s_0, \varphi)\partial_\varphi U_\tau^A(s_1, s_0, \varphi) \\ &+ \mathrm{Trace} P(s_0, \varphi)\Omega_\tau^*(s_1, s_0, \varphi)\partial_\varphi \Omega_\tau(s_1, s_0, \varphi) \\ &+ \mathcal{O}(\|R_\tau(s_1, s_0, \varphi)\|) \end{aligned}$$

and follow step by step the analysis in [10]. The first term gives rise to an integer. The second term is an integer too, but has to be $\mathcal{O}(\tau^{-1})$ by (7). So it is zero. By Corollary 1 the last term is exponentially small in the limit $s_0 \to -\infty$, $s_1 \to +\infty$. Hence we have shown that

Theorem 3 *Let $\tilde{H}(\phi)$ be defined by (10) and satisfy the gap condition. Then for any $D > 0$ there exists a class of switching functions which are analytic in some strip $S(a)$ such that*

$$\langle Q_\tau \rangle := \lim_{\substack{s_1 \to +\infty \\ s_0 \to -\infty}} \int_0^{2\pi} \frac{d\varphi}{2\pi} Q_\tau(s_1, s_0, \varphi) = Chern(\hat{P}) + \mathcal{O}(\exp(-D\tau)) ,$$

where $Chern(\hat{P})$ denotes an integer.

Let us finish with a remark about the integral term in $\langle Q_\tau \rangle$. It has a natural interpretation as a first Chern number of a vector bundle or alternatively as a winding number of a circle map. It is straight forward to see that the first term can conveniently be rewritten as

$$\mathrm{Chern}(\hat{P}) = \frac{i}{2\pi i} \int\limits_{T^2} \mathrm{Trace} \hat{P} d\hat{P} d\hat{P}$$

(see Lemma 4.2 in [10]). This is the first Chern number of the vector bundle

$$E = \left\{ \langle \phi, \chi \rangle \mid \phi \in T^2, \chi \in \mathrm{range} \hat{P}(\phi) \right\}$$

on the torus T^2 with canonical connection $\nabla = \hat{P} d$ and curvature $\Omega = \hat{P}(d\hat{P})(d\hat{P})\hat{P}$.

Acknowledgment: It is a pleasure to thank J.E. Avron for a longstanding cooperation which led to the setup used in this note. V.J. acknowledges support of the Deutsche Forschungsgemeinschaft SFB 288 Differentialgeometrie und Quantenphysik and NSERC under grant OGP 0007901.

References

[1] J. E. Avron and R. Seiler. Quantization of the Hall conductance for general multiparticle Schrödinger Hamiltonians. *Phys. Rev. Lett.*, 54:259–262, 1985.

[2] J. E. Avron, R. Seiler, and L. G. Yaffe. Adiabatic theorems and applications to the quantum Hall effect. *Comm. Math. Phys.*, 110:33–49, 1987.

[3] M.V. Berry. Quantal phase factors accompanying adiabatic changes. *Proc. R. Soc. Lond. A*, 392:45–57, 1984.

[4] M. Born and V. Fock. Beweis des Adiabatensatzes. *Z. Phys.*, 51:165–169, 1928.

[5] R.J. Haug. Untersuchungen zum Magnetotransport in zweidimensionalen Elektronensystemen. *Dissertation, Fak.Physik Univ.-Stuttgart*, 1988.

[6] V. Jakšić and J. Segert. Exponential approach to the adiabatic limit and the Landau-Zener formula. *Rev. Math. Phys.*, 4:529–574, 1992.

[7] A. Joye and C.-E. Pfister. Exponentially small adiabatic invariant for the Schrödinger equation. *Commun. Math. Phys.*, 140:15–41, 1991.

[8] T. Kato. On the adiabatic theorem of quantum mechanics. *J. Phys. Soc. J.*, 5:435–439, 1950.

[9] T. Kato. *Perturbation theory of linear operators.* Springer, Berlin, Heidelberg, New York, 1966.

[10] M. Klein and R. Seiler. Power law corrections to the Kubo formula vanish in quantum Hall systems. *Commun.Math.Phys.*, 128:141–160, 1990.

[11] S.G. Krein. *Linear Differential Equations in Banach Space.* Number 27 in Transl. Math. Mon. AMS, 1972.

[12] A. Martinez. Precise exponential estimates in adiabatic theory. *Prépublication Mathématiques de l'Université Paris-Nord*, 93-01, 1993.

[13] G. Nenciu. Linear adiabatic theory: Exponential estimates. *Commun. Math. Phys.*, 152:479–496, 1993.

[14] Q. Niu and D.J. Thouless. Quantised adiabatic charge transport in the presence of substrate disorder and many body interactions. *J. Phys. A*, 17:30–49, 1984.

[15] B. Simon. Holonomy, the quantum adiabatic theorem and Berry's phase. *Phys. Rev. Lett.*, 51(2167-2170), 1983.

[16] K. Yoshida. *Functional Analysis.* Number 123 in Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. Springer-Verlag New York Inc., 1968.